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Some curious involutions of spheres

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Consider an involution $T$ of the sphere $S^n$ without fixed points. Is the quotient manifold $S^n/T$ necessarily isomorphic to projective $n$-space? This question makes sense in three different categories. One can work either with topological manifolds and maps, with piecewise linear manifolds and maps, or with differentiable manifolds and maps.

For $n \leq 3$ the statement is known to be true (Livesay [6]). In these cases it does not matter which category one works with. On the other hand, for $n = 7$, in the differentiable case, the statement is known to be false (Milnor [10]).

This note will show that, in the piecewise linear case, the statement is false for all $n \geq 5$. Furthermore, for $n = 5, 6$, we will construct a differentiable involution $T: S^n \rightarrow S^n$ so that the quotient manifold is not even piecewise linearly homeomorphic to projective space. Our proofs depend on a recent theorem of J. Cerf.

Let us start with the exotic 7-sphere $M^7_1$ as described by Milnor [7]. This differentiable manifold $M^7_1$ is defined as the total space of a certain 3-sphere bundle over the 4-sphere. It is known to be homeomorphic, but not diffeomorphic, to the standard 7-sphere.

Taking the antipodal map on each fibre we obtain a differentiable involution $T: M^7_1 \rightarrow M^7_1$ without fixed points. (The quotient manifold $M^7_1/T$ can be considered as the total space of a corresponding projective 3-space bundle over $S^4$.) The following lemma was pointed out to us, in part, by P. Conner and D. Montgomery.

**Lemma 1.** There exists a differentiably imbedded 6-sphere, $S^6_0 \subset M^7_2$, which is invariant under the action of $T$, and a differentiably imbedded $S^6_0 \subset S^6_0$ which is also invariant.

Thus in this way one constructs a differentiable involution of the standard sphere in dimensions 5, 6.

The proof will depend on the explicit description of $M^7_2$ (or more generally of $M^7_1$) which was given in [7]. Take two copies of $R^4 \times S^3$ and identify the subsets $(R^4 - \{0\}) \times S^3$ under the diffeomorphism

$$(u, v) \rightarrow (u', v') = (u/||u||^2, u'v^j/||u||),$$

using quaternion multiplication, where $h + j = 1, h - j = k$. The involution $T$ changes the sign of $v$ and $v'$. Let $S^5_0$ be the set of all points of $M^7_2$ such that $R(v') = R(uv) = 0$, where $R(uv) = R(vu)$ denotes the real
part of the quaternion $uv$. This set is clearly invariant under $T$. To prove that $S_0^6$ is a manifold diffeomorphic to the standard 6-sphere, consider the function $g: M^7_4 \to R$ which is defined by

$$g = \Re(uv)/(1 + \|u\|^2)^{1/2} = \Re(v')(1 + \|u\|^2)^{1/2}.$$ 

It is easily verified that $g$ is well defined, differentiable, and has only two critical points, both nondegenerate. Hence the set of zeros of $g$ is diffeomorphic to $S^6$. (Compare [7].) But this set of zeros is precisely $S_0^6$.

Similarly let $S_0^8$ be the set of points of $S_0^6$ which satisfy $\Re(v) = \Re(u'(v')^{-1}) = 0$. This is a sphere, since it is the set of zeros of the function $f: S_0^8 \to R$ which is defined (as in [7]) by

$$f = \Re(v)/(1 + \|u\|^2)^{1/2} = \Re(u'(v')^{-1})(1 + \|u\|^2)^{1/2}.$$ 

This function also is nondegenerate, with two critical points, which completes the proof.

**Remark.** It would be interesting to know whether this game could be continued one stage further, however the authors do not know any further suitable functions.

**Lemma 2.** The manifold $S_0^6/T$ is not diffeomorphic to the projective space $P^n$ for $n = 5, 6$.

**Proof.** Note that a tubular neighborhood of $S_0^6/T$ in $M^7_4/T$ can be considered as a twisted line-segment bundle over $S_0^6/T$. The complement of such a neighborhood is a 7-disk. (This is easily proved using the function $g$.) Hence the differentiable manifold $M^7_4/T$ can be reconstructed out of $S_0^6/T$ as follows:

**Step 1.** Take the unique twisted line-segment bundle $B$ over $S_0^6/T$.

**Step 2.** Form a closed 7-manifold by matching the boundary of $B$ with the boundary of a 7-disk under a certain diffeomorphism $h$.

Similarly, if one starts with $P^6$ and applies this construction, using an analogous diffeomorphism $h'$ then one arrives at a manifold diffeomorphic to $P^7$. The only ambiguity here lies in the choice of $h'$. If one uses the wrong diffeomorphism then one will arrive instead at a manifold which is diffeomorphic to the connected sum $P^7 \# \Sigma$ for some twisted 7-sphere $\Sigma$. (Compare for example [9].)

Now suppose that $S_0^6/T$ is diffeomorphic to $P^6$. Proceeding as above, it follows that $M^7_4/T$ is diffeomorphic to some $P^7 \# \Sigma$. Passing to the 2-fold covering space, it follows that $M^8_4$ is diffeomorphic to $S^7 \# \Sigma \# \Sigma$.

But the group $\Gamma$, consisting of all oriented diffeomorphism classes of twisted 7-spheres, is cyclic of order 28 (see [5]) and the class of $M^8_4$ can be taken as a generator of this group (see [2]). Thus the class
of $M^2_3$ cannot be divisible by two. This yields a contradiction, and completes the proof for $n = 6$.

Now suppose that $S^6_0/T$ were diffeomorphic to $P^6$. Then a similar argument would show that $S^6_0/T$ must be diffeomorphic to $P^6 \# \Sigma'$ for some twisted 6-sphere $\Sigma'$. But every twisted 6-sphere is diffeomorphic to $S^6$ (see [5]). Therefore we can cancel $\Sigma'$ and obtain a contradiction, which completes the proof of Lemma 2.

Choose $C^1$-triangulations of $S^6_0/T$ and of $P^n$ for $n = 5, 6$. (See for example [12].) The resulting simplicial complexes will be denoted by $S^6_0/T$ and $P^n$ respectively. The two-fold covering complex $S^6_0$ of $S^6_0/T$ is clearly a combinatorial $n$-sphere, and $T: S^6_0 \to S^6_0$ is a fixed point free simplicial involution.

**Theorem 1.** The complex $S^6_0/T$ is not piecewise linearly homeomorphic to $P^n$, for $n = 5, 6$.

**Proof.** Suppose that $P^n$ were piecewise linearly homeomorphic to $S^6_0/T$. Then according to Munkres [11] there would exist a sequence of obstructions

$$\delta_i \in \mathcal{HC}(P^n; \Gamma_{n-i})$$

to finding a diffeomorphism between $S^6_0/T$ and $P^n$. Here $\mathcal{HC}$ denotes homology based on infinite chains, with twisted coefficients in the nonorientable case. The group $\Gamma_m$, consisting of all oriented diffeomorphism classes of twisted $m$-spheres, is known to be zero for $m = 1, 2, 3, 5, 6$. (See [11], [5].) Furthermore, an eagerly awaited paper by J. Cerf will prove that $\Gamma_4 = 0$. Assuming this theorem of Cerf, it follows that all of the groups $\mathcal{HC}(P^n; \Gamma_{n-i})$ are zero for $n \leq 6$. Thus there are no obstructions: the existence of a piecewise linear homeomorphism would imply the existence of a diffeomorphism, and hence would contradict Lemma 2. This completes the proof.

In dimension 7 our result will be somewhat weaker, since $M^7_3$ is not a standard 7-sphere. Choose a $C^1$-triangulation of $M^7_3/T$, thus yielding a simplicial complex $M^7_3/T$.

**Theorem 2.** The complex $M^7_3/T$ is not piecewise linearly homeomorphic to $P^7$. However its 2-fold covering complex $M^7_3$ is piecewise linearly homeomorphic to $S^7$.

The proof is similar to that of Theorem 1, but there is a complication since $\Gamma_7 \neq 0$. To get around this, we first remove a point $x$ from

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1 In general, boldface letters will be used for simplicial complexes, and for piecewise linear maps.
$M^T_3/T$, and note that $M^T_3/T - x$ is not diffeomorphic to $P^T - y$. For if it were diffeomorphic, then the boundary of a small ball around $x$ would correspond to a sphere around $y$ bounding a manifold which, according to Smale [13, §5.1] must be a 7-disk. It would then follow that $M^T_3$ must be diffeomorphic to the connected sum $P^T \# \Sigma$ for some twisted sphere $\Sigma$. But this is impossible, as we have seen during the proof of Lemma 2.

Now suppose that the corresponding complex $M^T_3/T - x$ were piecewise-linearly homeomorphic to $P^T - y$. Since the groups $\pi_n(P^T - y; \Gamma_{T - 1})$ are all zero, this would imply the existence of a diffeomorphism. We have just seen that this is impossible.

It follows a fortiori that $M^T_3/T$ cannot be piecewise-linearly homeomorphic to $P^T$.

Proof that $M^T_3$ is piecewise-linearly homeomorphic to $S^7$ (following [8]). Recall that $M^T_3$ can be expressed as the union of two smooth 7-disks which intersect only along their common boundary. Choosing a suitable $C^1$-triangulation it follows that the resulting simplicial complex $M^T_3$ can be expressed as the union of two combinatorial 7-cells which intersect only along their common boundary. It now follows easily that $M^T_3$ is piecewise-linearly homeomorphic to the combinatorial sphere $S^7$, which completes the proof of Theorem 2.

In still higher dimensions, one can generate examples as follows. Suppose that we start with any piecewise linear manifold $Q^n$ whose 2-fold covering space $Q^n$ is piecewise linearly homeomorphic to $S^n$. Let $Q^{n+1} = Q^n \cup C\hat{Q}^n$ denote the complex formed from $Q^n$ by adjoining the cone over its 2-fold covering space. Then $Q^{n+1}$ is again a piecewise linear manifold, and its 2-fold covering is the suspension of $\hat{Q}^n$. This construction can be iterated ad infinitum.

Now start with $Q^8 = S^6_0/T$. It is easily verified that the corresponding $Q^8$ and $Q^7$ can be identified with $S^6_0/T$ and $M^3_3/T$ respectively. Each of these piecewise linear manifolds can be given a compatible differentiable structure. But if we iterate the construction once more, we obtain a piecewise linear manifold $Q^8$ which cannot be given a compatible differentiable structure. This can be proved using the obstruction theory of Hirsch [4]. In fact the obstruction class in

$$H^8(Q^8; \Gamma_{T}) \cong \Gamma_{T}/2\Gamma_{T}$$

can be identified with the class of the manifold $M^T_3$. Similarly none of the $Q^n$, $n \geq 8$, possess compatible differentiable structures. It follows that no $Q^n$ is piecewise linearly homeomorphic to $P^n$.

For each $n \geq 5$ we have the following:

Unsolved Problem. Is the manifold $Q^n$ homeomorphic to $P^n$?
The existence of such a homeomorphism would contradict the Hauptvermutung for manifolds. Its nonexistence would imply that the corresponding involution of $Q^n \cong S^n$ is not conjugate to the antipodal map even in the group of all homeomorphisms of the $n$-sphere.

In conclusion let us study the extent to which our various imitation projective spaces resemble the true projective space. The following is well known.

**Lemma 3.** For any continuous fixed point free involution of a topological $n$-sphere, the orbit space $S^n/T$ has the homotopy type of $P^n$.

**Proof.** We will construct a map $f : S^n \rightarrow S^n$ of degree one such that $fT(x) = -f(x)$. It is easy to see that such an $f$ gives rise to a map $S^n/T \rightarrow P^n$ which induces isomorphisms of homotopy groups, and hence is a homotopy equivalence.

We think of $S^n$ as the unit sphere in $R^{n+1}$. Define

$$f(x) = (x - Tx)/||x - Tx||.$$

As a parameter $s$ runs from 0 to 1, let $T_s(x)$ run from $T(x)$ to $-x$ along the unique shortest circular arc on $S^n$. This arc avoids $x$ because $T(x) \neq x$. Now define

$$f_s(x) = (x - T_s(x))/||x - T_s(x)||.$$

We have defined a homotopy between $f$ and the identity proving that $f$ has degree 1.

**Remark.** In the piecewise linear case we can even assert that the orbit space $S^n/T$ has the same simple homotopy type as $P^n$. This is true since simple homotopy type and homotopy type coincide for complexes with fundamental group of order $\leq 4$. (Whitehead [14], Higman [3].)

Now let us look at the differentiable cases. Here one has an additional invariant: the tangent bundle.

**Lemma 4.** The homotopy equivalences $P^n \rightarrow S^n/T$ and $P^n \rightarrow M^n/T$ can be extended to bundle maps of the respective tangent bundles.

**Proof.** Let $\tau$ denote the tangent bundle of $P^n$, and let $\tau'$ denote the bundle over $P^n$ induced from the other tangent bundle by the homotopy equivalence. We must prove that $\tau$ is isomorphic to $\tau'$ for $n = 6, 7$.

It follows from Adams [1, §7.4] that a vector bundle over a projective space of dimension $\leq 8$ is determined up to s-isomorphism by its Stiefel-Whitney classes. But, for tangent bundles, these are homotopy type invariants. Thus $\tau$ is s-isomorphic to $\tau'$. 
Restricting to \( P^{n-1} \) it follows easily that \( \tau|_{P^{n-1}} \) is isomorphic to \( \tau'|_{P^{n-1}} \). Choosing a fixed isomorphism \( i \) between these bundles the obstruction to extending \( i \) over \( P^n \) is now a well-defined element of \( H^n(P^n; \pi_{n-1}SO_n) \). For \( n = 7 \) this group is zero, so that there is no problem.

For \( n = 6 \) the group \( H^6(P^6; \pi_5SO_6) \) is infinite cyclic. (The coefficients are twisted.) Furthermore the projection \( p: S^6 \to P^6 \) induces a monomorphism \( H^6(P^6; \pi_5SO_6) \to H^6(S^6; \pi_5SO_6) \). Hence it is sufficient to check that the obstruction becomes zero when we pass to the universal covering space \( S^6 \). But \( p^*\tau \) is clearly isomorphic to \( p^*\tau' \). (Both bundles have Euler number \( \pm 2 \).) Hence, if \( i \) is chosen carefully, the obstruction to extending \( i \) will be zero. This completes the proof.

*Added in proof.* The corresponding statement for \( n = 5 \) is true also. In fact I. M. James and E. Thomas, in a forthcoming paper, show that any vector bundle over an odd dimensional projective space which is \( s \)-isomorphic to the tangent bundle and has the same dimension must actually be isomorphic to the tangent bundle.

**References**