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Planar structure for inclusions of finite von Neumann algebras

by

David Signorielli Penneys

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in

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University of California, Berkeley

Committee in charge:

Professor Vaughan F. R. Jones, Chair
Professor Marc A. Rieffel
Professor Ori Ganor

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Planar structure for inclusions of finite von Neumann algebras

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Abstract

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Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Vaughan F. R. Jones, Chair

This dissertation consists of three self-contained papers from my graduate work at UC Berkeley. The chapters increase in complexity from the annular Temperley-Lieb category to strongly Markov inclusions of finite von Neumann algebras to infinite index $II_1$-subfactors.

In Chapter 2, we discuss how two copies of the cyclic category generate the annular Temperley-Lieb category. In the process, we give a presentation of the annular Temperley-Lieb category via generators and relations, and we see the cyclic category evolve from the simplicial and semi-simplicial categories.

Chapter 3 is joint work with Vaughan F. R. Jones. First, we define a canonical planar $*$-algebra associated to a strongly Markov inclusion of finite von Neumann algebras (the notion of such an inclusion is defined within). Second, we show for an inclusion of finite dimensional $C^*$-algebras with the Markov trace, the canonical planar algebra is isomorphic to the graph planar algebra of the Bratteli diagram of the inclusion. We use this fact to show that a subfactor planar algebra embeds into the graph planar algebra of its principal graph.

In Chapter 4, we expand upon Burns’ work on rotations for infinite index $II_1$-subfactors. We start with a $II_1$-factor bimodule, and we construct a tower of centralizer algebras and a sequence of central $L^2$-vectors. In the finite index setting, the centralizer algebras and central $L^2$-vectors agree, but in the infinite index setting, these spaces can differ dramatically. We develop planar calculi for both sequences which are compatible. Interestingly, we obtain planar structure without Jones’ basic construction or the resulting Jones projections! We also generalize Burns work on extremality and the existence of rotations to the bimodule setting, and we recover his main theorem. Along the way, we prove some results about relative tensor products of extended positive cones, and we give an example of an infinite index subfactor with finite dimensional higher relative commutants.
# Contents

1. **Introduction** 1
   1.1 Chapter synopses 3

2. **A cyclic approach to the annular Temperley-Lieb category** 4
   2.1 Introduction 4
   2.2 The Category $\text{Atl}$ 6
   2.3 The Category $\text{a}_\Delta$ 21
   2.4 The Isomorphism of Categories $\text{a}_\Delta \cong \text{Atl}$ 27
   2.5 The Annular Category from Two Cyclic Categories 29
   2.6 Annular Objects 37

3. **The embedding theorem for finite depth subfactor planar algebras** 43
   3.1 Introduction 43
   3.2 The canonical planar $*$-algebra of a strongly Markov inclusion of finite von Neumann algebras 45
   3.3 The planar algebra isomorphism for finite dimensional $C^*$-algebras 62
   3.4 The Embedding Theorem 73

4. **A planar calculus for infinite index subfactors** 76
   4.1 Introduction 76
   4.2 Preliminaries 80
   4.3 Planar calculus for bimodules 90
   4.4 Extremality and rotations 101
   4.5 Examples 110
   4.6 Relative tensor products of extended positive cones 115
   4.7 The action of $\mathbb{BP}$ is well-defined 124
   4.8 Extended positive cones 131

Bibliography 135
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Chapter 1

Introduction

Finite index subfactors

Mathematicians are taxonomists; we classify species of mathematical objects into types. Herein, the species are factors, von Neumann algebras with trivial centers, first defined by von Neumann in his study of quantum mechanics. Murray and von Neumann classified factors into three types, and constructed examples of each. All factors in this subsection are type $II_1$.

Sometimes distinct species share common traits. Fields and $II_1$-factors are algebraically simple, so we study maps in these categories by studying inclusions, i.e., subfields or subfactors. Nakamura and Takeda strengthened this connection with their Galois correspondence for the intermediate subfactor lattice for $M \subset M \rtimes G$ for a finite group $G$ [NT60a, NT60b]. Hence some refer to subfactor theory as “noncommutative Galois theory.”

In his pioneering paper [Jon83], Jones defined an index for a subfactor $M_0 \subset M_1$, showed

$$[M_1 : M_0] \in \{4 \cos^2(\pi/n) \mid n = 3, 4, 5, \ldots \} \cup [4, \infty],$$

and constructed an example with each allowed index. To do so, he used the “basic construction” which constructs a tower of factors $M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots$. The subfactors in this subsection are assumed to be finite index.

Just as topologists study a complicated topological space by its homology groups, we study a subfactor by its standard invariant, two sequences of finite dimensional $C^*$-algebras $P_{n,+} = M_0' \cap M_n$ and $P_{n,-} = M_1' \cap M_{n+1}$ [Jon83, Jon86]. The standard invariant has been axiomatized in three similar ways, each emphasizing slightly different structure: Ocneanu’s paragroups [Ocn88, EK98], Popa’s $\lambda$-lattices [Pop95], and Jones’ planar algebras [Jon99]. Given a standard invariant $P_\bullet$, one can construct a subfactor whose standard invariant is $P_\bullet$ [Pop95, GJS07].

The rich structure of a planar algebra provides connections between subfactor theory, combinatorics, quantum algebra, and tensor categories. Given a subfactor $N \subset M$, its planar algebra $P_\bullet$ encodes two simpler invariants: the index, and the principal graphs, which are bipartite induction-restriction graphs associated to the representation theory of the sub-
factor. The two “even parts” of $P_*$ form two $C^*$-tensor categories of $N - N$ bimodules and $M - M$ bimodules respectively. If there are only finitely many isomorphism classes of such bimodules, the subfactor is called finite depth, and the “even parts” are fusion categories [ENO05]. In this case, the two fusion categories are Morita equivalent [Müg03] via the two “odd parts” of $P_*$, which are module categories of $N - M$ and $M - N$ bimodules.

Subfactors and groups also share traits. For an outer action of a finite group $G$ on a factor $M$ and a subgroup $H \subset G$, the planar algebra of the fixed point subfactor $M^G \subset M^H$ encodes the induction-restriction data of $H \subset G$. If $H$ is trivial, one “even part” of $P_*$ is the fusion category of representations of $G$. This also works for actions of quantum groups.

Jones proved that every finite group has a unique outer action on the hyperfinite $II_1$-factor $R$ [Jon80]. Popa extended this result in his classification of amenable subfactors [Pop94] where he shows that each amenable standard invariant has a unique “action” on $R$.

### Infinite index subfactors

Some finite index results generalize to infinite index subfactors, such as discrete, irreducible, “depth 2” subfactors correspond to outer (cocycle) actions of Kac algebras [HO89, EN96], and the classical Galois correspondence still holds for outer actions of infinite discrete groups and minimal actions of compact groups [ILP98]. We ask:

**Question.** What is a suitable standard invariant for infinite index subfactors?

There are several candidates for the standard invariant, each with its pros and cons. For example, we could take the towers $P_{n,\pm}$ as in the introduction since Enock and Nest showed

$$M_i' \cap M_j \cong M_{i+2}' \cap M_{j+2}$$

for all $i, j \geq 0$ in [EN96]. In his Ph.D. thesis [Bur03], Burns studied rotations and extremality for infinite index subfactors, and he initiated the search for planar structure. He crucially observed that for finite index, the centralizer algebras $M_0' \cap M_n$ and the central $L^2$-vectors

$$M_0' \cap L^2(M_n) = \{ \xi \in L^2(M_n) \mid x\xi = \xi x \text{ for all } x \in M_0 \}$$

coincide. As this is no longer true for infinite index, he focused on the spaces $M_0' \cap L^2(M_n)$, and he showed $M_0 \subset M_1$ is (approximately) extremal if and only if a (non-)unitary rotation operator exists on the $M_0' \cap L^2(M_n)$.
1.1 Chapter synopses

This dissertation consists of three self-contained papers from my graduate work at UC Berkeley. The chapters increase in complexity from the annular Temperley-Lieb category to inclusions of finite von Neumann algebras to infinite index $II_1$-subfactors.

Chapter 2: A cyclic approach to the annular Temperley-Lieb category

This paper was published in J. Knot Theory and its Ramifications [Pen12a]. Its abstract is as follows:

In [Jon00], Jones found two copies of the cyclic category $c\Delta$ in the annular Temperley-Lieb category $\mathcal{A}_\mathcal{T}$. We give an abstract presentation of $\mathcal{A}_\mathcal{T}$ to discuss how these two copies of $c\Delta$ generate $\mathcal{A}_\mathcal{T}$ together with the coupling constants and the coupling relations. We then discuss modules over the annular category and homologies of such modules, the latter of which arises from the cyclic viewpoint.

Chapter 3: The embedding theorem for finite depth subfactor planar algebras

This joint paper with Vaughan F. R. Jones was published in Quantum Topology [JP11]. Its abstract is as follows:

We define a canonical planar $*$-algebra from a strongly Markov inclusion of finite von Neumann algebras. In the case of a connected unital inclusion of finite dimensional $C^*$-algebras with the Markov trace, we show this planar algebra is isomorphic to the bipartite graph planar algebra of the Bratteli diagram of the inclusion. Finally, we show that a finite depth subfactor planar algebra is a planar subalgebra of the bipartite graph planar algebra of its principal graph.

Chapter 4: A planar calculus for infinite index subfactors

This paper was accepted to Communications in Mathematical Physics on May 8, 2012; it can be found at arXiv:1110.3504 [Pen12b]. Its abstract is as follows:

We develop an analog of Jones’ planar calculus for $II_1$-factor bimodules with arbitrary left and right von Neumann dimension. We generalize to bimodules Burns’ results on rotations and extremality for infinite index subfactors. These results are obtained without Jones’ basic construction and the resulting Jones projections.
Chapter 2

A cyclic approach to the annular Temperley-Lieb category

2.1 Introduction

The Temperley-Lieb algebras have been studied extensively beginning with Temperley and Lieb's first paper in statistical mechanics regarding hydrogen bonds in ice-type lattices [TL71]. Since, these algebras have been instrumental in many areas of mathematics, including subfactors [Jon83] and knot theory [Jon85]. The well known diagrammatic representation of these algebras was introduced by Kauffman in [Kau87] in his skein theoretic definition of the Jones polynomial. From these diagrams, we get the Temperley-Lieb category whose objects are $n$ points on a line, morphisms are diagrams with non-intersecting strings, and composition is stacking tangles vertically (we read bottom to top).

Historically, the (affine/annular) Temperley-Lieb algebras have been presented as quotients of the (affine) Hecke algebras [Jon94]. Graham and Lehrer define cellular structures for these algebras in [GL96], and they give the representation theory for affine Temperley-Lieb in [GL98]. Jones' definition of the annular Temperley-Lieb category (see [Jon99], [Jon01]), which we will denote $\text{Atl}$, differs slightly Graham and Lehrer's. First, $\text{Atl}$-tangles have a checkerboard shading, so each disk has an even number of boundary points. Second, the rotation is periodic in $\text{Atl}$, similar to the rotation in Connes' cyclic category $c\Delta$, studied by Connes [Con83], [Con94], Loday and Quillen [LQ83], [Lod98], and Tsygan [Tsy83]. Jones found a connection between $\text{Atl}$ and $c\Delta$ in [Jon00], and raised the question we now address: how does $\text{Atl}$ arise from the interaction of two copies of the cyclic category?

In answering this question, we see $\text{Atl}$ evolve from simple categories. The opposite of the simplicial category $s\Delta^{\text{op}}$ (see 2.5.4) has a well known pictorial representation much like the Temperley-Lieb category: objects are $2n + 2$ points on a line, morphisms are rectangular planar tangles with only shaded caps and unshaded cups, and composition is stacking. In fact, these diagrams closely resemble the string diagrams arising from an adjoint functor pair.
 CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

An asymmetry is present in the above tangles: all shaded regions can be “capped” by applying a face map, but not every unshaded region can be “cupped” by applying a degeneracy. This asymmetry can be corrected by closing the rectangular tangles into annuli, still enforcing the same shading requirements. Jones showed the resulting category is isomorphic to $c\Delta^\text{op}$ in [Jon00]. Of course the category with the reverse shading is also isomorphic to $c\Delta$ (and $c\Delta^\text{op}$), and these two subcategories generate $\text{Atl}^\rightarrow\ast\ast$.

Outline

In Section 2.2, we will define $\text{Atl}$ and offer candidates for generators and relations. We will then prove some uniqueness results which will be crucial to our approach. In Section 2.3, we will take these candidates and define an abstract category $a\Delta$, the annular category, via
generators and relations. We then prove existence of a standard form for words. In Section 2.4, we prove Theorem 2.4.8, which says there is an isomorphism of involutive categories $\text{Atl} \cong a\Delta$ (the isomorphism preserves an involution).

After we have our description of $\text{Atl}$ in terms of abstract generators and relations, we recover the result of Jones in [Jon00] in 2.5, i.e. two isomorphisms from $c\Delta^{\text{op}}$ to subcategories $c\text{Atl}^\pm$ of $\text{Atl}$. After a note on augmentation of the cyclic category in 2.5, we prove the main result of the paper, Theorem 2.5.27, which shows $\text{Atl}$ is a quotient of the pushout of augmented copies of $c\Delta$ and $c\Delta^{\text{op}}$ over a groupoid $T$ of finite cyclic groups:

\[
\begin{array}{ccc}
T & \xrightarrow{} & c\Delta^{\text{op}} \\
\downarrow & & \downarrow \\
\tilde{c}\Delta & \xrightarrow{} & \text{PO} \\
\downarrow & & \downarrow \\
\text{Atl}. & \xrightarrow{} & \text{} \\
\end{array}
\]

In Section 2.6, we define the notion of an annular object in a category $\mathcal{C}$. As $c\Delta^{\text{op}}$ lives inside $a\Delta$ (in two ways), we will have notions of Hochschild and cyclic homology of annular objects in abelian categories. We define these notions and give some easy results in 2.6.

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2.2 The Category $\text{Atl}$

Notation 2.2.1. All categories will be denoted by capital letters in the following sans-serif font: $\text{ABC}$... The categories we discuss will be small, and we will write $X \in A$ to denote that $X \in \text{Ob}(A)$, the set of objects of $A$. We will write $A(X,Y)$ to denote the set of morphisms $\varphi: X \to Y$ where $X,Y \in A$, and we will write $\text{Mor}(A)$ to denote the collection of all morphisms in $A$. In the sequel, objects of our categories will be the symbols $[n]$ for $n \in \mathbb{Z}_{\geq 0} \cup \{0, \pm, \pm\}$. For simplicity and aesthetics, we will write $A(m,n)$ instead of $A([m],[n])$.

Definition 2.2.2. A category $A$ is called involutive if for all $X,Y \in A$, there is a map $* : A(X,Y) \to A(Y,X)$ called the involution such that

1. $\text{id}_X^* = \text{id}_X$ for all $X \in A$, 

(2) $(T^*)^* = T$ for all $T \in A(X,Y)$, and

(3) for all $X,Y,Z \in A$ and all $T \in A(X,Y)$ and $S \in A(Y,Z)$, $(S \circ T)^* = T^* \circ S^*$.

In other words, there is a contravariant functor $\ast : A \rightarrow A$ of period two which fixes all objects.

**Definition 2.2.3.** Suppose $A$ and $B$ are categories and $F : A \rightarrow B$ is a functor.

1. $F$ is called an isomorphism of categories if there is a functor $G : B \rightarrow A$ such that $F \circ G = \text{id}_B$ and $G \circ F = \text{id}_A$, the identity functors. In this case, we say categories $A$ and $B$ are isomorphic, denoted $A \cong B$.

2. If $A$ and $B$ are involutive, we say $F$ is involutive if it preserves the involution, i.e. $F(\varphi^*) = \varphi^*$ for all $\varphi \in A(X,Y)$ for all $X,Y \in A$.

3. An isomorphism of involutive categories is an involutive isomorphism of said categories.

**Remark 2.2.4.** It is clear that if $A$ is involutive, then $A \cong A^{\text{op}}$.

**Annular Tangles**

We provide a definition of an annular $(m,n)$-tangle which is a fusion of the ideas in [Jon99] and [KS04].

**Definition 2.2.5.** An annular $(m,n)$-pretangle for $m, n \in \mathbb{Z}_{\geq 0}$ consists of the following data:

1. The closed unit disk $D$ in $\mathbb{C}$,

2. The skeleton of $T$, denoted $S(T)$, consisting of:

   a) the boundary of $D$, denoted $D_0(T)$,

   b) the closed disk $D_1$ of radius $1/4$ in $\mathbb{C}$, whose boundary is denoted $D_1(T)$,

   c) $2m$, respectively $2n$, distinct marked points on $D_1(T)$, respectively $D_0(T)$, called the boundary points of $D_i(T)$ for $i = 0, 1$. Usually we will call the boundary points of $D_0(T)$ external boundary points of $T$ and the boundary points of $D_1(T)$ internal boundary points.

   d) inside $D$, but outside $D_1$, there is a finite set of disjointly smoothly embedded curves called strings which are either closed curves, called loops, or whose boundaries are marked points of the $D_i(T)$’s and the strings meet each $D_i(T)$ transversally, $i = 0, 1$. Each marked point on $D_i(T)$, $i = 0, 1$ meets exactly one string.

3. The connected components of $D \setminus S(T)$ are called the regions of $T$ and are either shaded or unshaded so that regions whose closures meet have different shadings.
Definition: If there are boundary points of $D_i(T)$, then an interval of $D_i(T)$, $i = 0, 1$, is a connected arc on $D_i(T)$ between two boundary points of $D_i(T)$. A simple interval of $D_i(T)$, $i = 0, 1$, is an interval of $D_i(T)$ in $T$ which touches only two (adjacent) boundary points. If there are no boundary points of $D_i(T)$, then a (simple) interval of $D_i(T)$ is $D_i(T)$ itself.

(4) For each $D_i(T)$, $i = 0, 1$, there is a distinguished simple interval of $D_i(T)$ denoted $*^i_i(T)$ whose interior meets an unshaded region. Starting at $*^i_i(T)$ on $D_i(T)$, we order the marked points of $D_i(T)$ clockwise. This numbering, along with the shading, induces an orientation on the pre-tangle.

Figure 2.4: An example of an annular tangle

Remarks 2.2.6. (1) If $m = 0$, there are two kinds of annular $(0, n)$-pretangles depending on whether the region meeting $D_1(T)$ is unshaded or shaded. If the region meeting $D_1(T)$ is unshaded, we call $T$ an annular $(0+, n)$-pretangle, and if the region is shaded, we call $T$ an annular $(0-, n)$-pretangle. Likewise, when $n = 0$, there are two kinds of annular $(m, 0)$-pretangles. If the region meeting $D_0(T)$ is unshaded, we call $T$ an annular $(m, 0+)$-pretangle, and if the region is shaded, we call $T$ an annular $(m, 0-)$-pretangle. Additionally, we have annular $(0\pm, 0\pm)$-pretangles and annular $(0\pm, 0\mp)$-pretangles.

(2) Loops may be shaded or unshaded.

Definition 2.2.7. An annular $(m, n)$-tangle is an orientation-preserving diffeomorphism class of an annular $(m, n)$-pretangle for $m, n \in \mathbb{N} \cup \{0\}$. The diffeomorphisms preserve (but do not necessarily fix!) $D_0$ and $D_1$.

Definition 2.2.8. Given an annular $(m, n)$-tangle $T$, and an annular $(l, m)$-tangle $S$, we define the annular $(l, m)$-tangle $T \circ S$ by isotoping $S$ so that $D_0(S)$, the marked points of $D_0(S)$, and $*^0_0(S)$, coincide with $D_1(T)$, the marked points of $D_1(T)$, and $*^1_1(T)$ respectively. The strings may then be joined at $D_1(T)$ and smoothed, and $D_1(T)$ is removed to obtain $T \circ S$ whose diffeomorphism class only depends on those of $T$ and $S$. 
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Definition 2.2.9. If $T$ is an annular $(m, n)$-tangle, we define $T^*$ to be the annular $(n, m)$-tangle obtained by reflecting $T$ about the circle of radius $3/4$, which switches $D_i(T)$ and $*_i(T)$, $i = 0, 1$. Clearly $(T^*)^* = T$ and $(T \circ S)^* = S^* \circ T^*$ for composable $S$ and $T$.

Definition 2.2.10. Let $T$ be an annular $(m, n)$-tangle.

Caps: A cap of $T$ is a string that connects two internal boundary points. The set of caps of $T$ will be denoted $\text{caps}(T)$.

$\partial \Lambda$: If $\Lambda \in \text{caps}(T)$, there is a unique interval of $D_1(T)$, denoted $\partial \Lambda$, such that $\Lambda \cup \partial \Lambda$ is a closed loop (which is not smooth at two points) which does not contain $D_1$ in its interior. Using $\partial \Lambda$, the cap $\Lambda$ inherits an orientation as $D_1(T)$ is oriented clockwise. Denote this orientation by an arrow on $\Lambda$.

Index: We define the cap index of $\Lambda$, denoted $\text{ind}(\Lambda)$, to be the number of the marked point to which the arrow points. The set of cap indices of $T$ forms an increasing sequence, which we denote $\text{capind}(T)$.

$B(\Lambda)$: For $\Lambda \in \text{caps}(T)$, we let $B(\Lambda) = \{\Lambda' \in \text{caps}(T)|\partial \Lambda' \subseteq \partial \Lambda\}$, and we say an element $\Lambda' \in B(\Lambda)$ is bounded by $\Lambda$ or that $\Lambda$ bounds $\Lambda'$.

Definition 2.2.11. Let $T$ be an annular $(m, n)$-tangle.

Cups: A cup $V$ of $T$ is a string that connects two external boundary points. The set of cups of $T$ will be denoted $\text{cups}(T)$.
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Figure 2.7: An example of cap indices

\[ \text{capind} \left( \begin{array}{c} \ast \\ \ast \ast \ast \ast \end{array} \right) = \{1, 4, 7\} \]

\[ \partial V: \text{If } V \in \text{cups}(T), \text{there is a unique interval of } D_0(T), \text{denoted } \partial V, \text{such that } V \cup \partial V \text{ is a closed loop (which is not smooth at two points) which does not contain } D_1 \text{ in its interior. Using } \partial V, \text{the cup } V \text{ inherits an orientation as } D_0(T) \text{ is oriented clockwise. Denote this orientation by an arrow on } V. \]

\[ \text{Index: } \text{We define the cup index of } V, \text{denoted } \text{ind}(V), \text{to be the number of the marked point to which the arrow points. The set of cup indices of } T \text{ forms an increasing sequence, which we denote cupind}(T). \]

\[ B(V): \text{For } V \in \text{cups}(T), \text{we let } B(V) = \{V' \in \text{cups}(T)|\partial V' \subseteq \partial V\}, \text{and we say an element } V' \in B(V) \text{ is bounded by } V \text{ or that } V \text{ bounds } V'. \]

\[ \text{Remark 2.2.12. Note } \text{capind}(T) = \text{cupind}(T^*) \text{ for all annular tangles } T. \]

\[ \text{Definition 2.2.13. Suppose } T \text{ is an annular } (m,n)-\text{tangle.} \]

\[ t_0(T): \text{A through string is a string of } T \text{ which connects an internal boundary point of } T \text{ to an external boundary point of } T. \text{ The set of through strings is denoted } t_0(T). \text{ Note that } |t_0(T)| \in 2\mathbb{Z}_{\geq 0}. \text{ We order } t_0(T) \text{ clockwise starting at } *_0(T), \text{so each through string of } T \text{ has a number.} \]

\[ t_1(T): \text{Suppose } T \text{ has a through string. Using } *_0(T) \text{ as our reference, we go counterclockwise along } D_0(T) \text{ to the first through string, which is denoted } t_0(T). \text{ Note the number of } t_0(T) \text{ is } |t_0(T)|. \]

\[ t_1(T): \text{Suppose } T \text{ has a through string. Using } *_1(T) \text{ as our reference, we go counterclockwise along } D_1(T) \text{ to the first through string, which is denoted } t_1(T). \text{ We denote the number of } t_1(T) \text{ by } \# t_1(T). \]

\[ \text{rel}_*(T): \text{We define the relative star position of } T, \text{denoted } \text{rel}_*(T), \text{as follows:} \]

\[ 1) \text{Suppose } T \text{ has an odd number } k \text{ of non-contractible loops. Then there is a unique region } R \text{ which touches both a non-contractible loop and } D_1(T). \text{ If } R \text{ is unshaded, we define } \text{rel}_*(T) \text{ to be the symbol } \pm(k), \text{ and if } R \text{ is shaded, we define } \text{rel}_*(T) \text{ to be the symbol } \mp(k). \text{ This notation signifies the shading switches from unshaded to shaded, respectively shaded to unshaded, as we read } T \text{ from } D_1(T) \text{ to } D_0(T). \]
(2) Suppose $T$ has an even number $k$ of non-contractible loops. If $k = 0$, then there is a unique region $R$ which touches both $D_0(T)$ and $D_1(T)$. If $k \geq 1$, then there is a unique region $R$ which touches both a non-contractible loop and $D_1(T)$. If $R$ is unshaded, we define $\text{rel}_*(T)$ to be the symbol $+(k)$, and if $R$ is shaded, we define $\text{rel}_*(T)$ to be the symbol $-(k)$.

(3) Suppose $T$ has a through string. We define

$$\text{rel}_*(T) = \left\lfloor \frac{\# \text{ts}_1(T)}{2} \right\rfloor \mod \left( \frac{|\text{ts}(T)|}{2} \right) \in \left\{ 0, 1, \ldots, \frac{|\text{ts}(T)|}{2} - 1 \right\}.$$ 

Figure 2.8: An example of relative star position

“Generators and Relations” of $\text{Atl}$

**Definition 2.2.14.** Suppose $T$ is an annular tangle. A loop of $T$ is called contractible if it is contractible in $D \setminus D_1$. Otherwise it is called non-contractible.

**Definition 2.2.15 (Atl Tangle).** An annular $(m, n)$-tangle $T$ is called an Atl $(m, n)$-tangle if $T$ has no contractible loops.

**Definition 2.2.16.** Let $\text{Atl}$ be the following small category:

- **Objects:** $[n]$ for $n \in \mathbb{N} \cup \{0\pm\}$
- **Morphisms:** Given $m, n \in \mathbb{N} \cup \{0\pm\}$, $\text{Atl}(m, n)$ is the set of all triples $(T, c_+, c_-)$ where $T$ is an Atl $(m, n)$-tangle and $c_+, c_- \in \mathbb{Z}_{\geq 0}$.
- **Composition:** Given $(S, a_+, a_-) \in \text{Atl}(m, n)$ and $(T, b_+, b_-) \in \text{Atl}(l, m)$, we define $(S, a_+, a_-) \circ (T, b_+, b_-) \in \text{Atl}(l, n)$ to be the triple $(R, c_+, c_-)$ obtained as follows: let $R_0$ be the annular $(l, n)$-tangle $S \circ T$. Let $d_+$, respectively $d_-$, be the number of shaded, respectively unshaded, contractible loops. Let $R$ be the Atl $(l, n)$-tangle obtained from $R_0$ by removing all contractible loops, and set $c_\pm = a_\pm + b_\pm + d_\pm$. 
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Remark 2.2.17. For simplicity and aesthetics, we write $T$ for the morphism $(T, 0, 0) \in \text{Mor}(\text{Atl})$.

Figure 2.9: An example of composition in $\text{Atl}$

Definition 2.2.18. We give the following names to the following distinguished $\text{Atl} (n, m)$-tangles:

(A) Let $a_1$ be the only $\text{Atl} (1, 0+)$.tangle with no loops, and let $a_2$ be the only $\text{Atl} (1, 0−)$-tangle with no loops. For $n \geq 2$ and $i \in \{1, \ldots, 2n\}$, let $a_i$ be the $\text{Atl} (n, n−1)$-tangle whose $i^{\text{th}}$ and $(i+1)^{\text{th}}$ (modulo $2n$) internal boundary point are joined by a string and all other internal boundary points are connected to external boundary points such that

(i) If $i = 1$, then the first external point is connected to the third internal point.
(ii) If $1 < i < 2n$, then the first external point is connected to the first internal point.
(iii) If $i = 2n$, then the first external point is connected to the $(2n−1)^{\text{th}}$ internal point.

(B) Let $b_1$ be the only $\text{Atl} (0+, 1)$-tangle with no loops, and let $b_2$ be the only $\text{Atl} (0−, 1)$-tangle with no loops. For $n \geq 1$ and $i \in \{1, \ldots, 2n+2\}$, let $b_i$ be the $\text{Atl} (n, n+1)$-tangle whose $i^{\text{th}}$ and $(i+1)^{\text{th}}$ (modulo $2n+2$) external boundary point are joined by a string and all other internal boundary points are connected to external boundary points such that
Figure 2.11: $a_1, a_2, \ldots, a_{2n} \in \text{Atl}(n, n-1)$. (without the dots, $n = 3$)

Figure 2.12: $b_1 \in \text{Atl}(0+, 1)$ and $b_2 \in \text{Atl}(0-, 1)$

(i) If $i = 1$, then the third external point is connected to the first internal point.
(ii) If $1 < i$, then the first external point is connected to the first internal point.
(iii) If $i = 2n + 2$, then the first internal point is connected to the $(2n + 1)^{\text{th}}$ external point.

Figure 2.13: $b_1, b_2, \ldots, b_{2n+2} \in \text{Atl}(n, n+1)$ (without the dots, $n = 3$)

(T) For $n = 1$, let $t$ be the identity $(1,1)$-tangle. For $n \geq 2$, let $t$ be the Atl $(n,n)$-tangle where all internal points are connected to external point such that the third external point is connected to the first internal point.

**Theorem 2.2.19.** The following relations hold in Atl:

1. $a_i a_j = a_{j-2} a_i$ for $i < j - 1$ and $(i,j) \neq (1,2n),$
2. $b_i b_j = b_{j+2} b_i$ for $i \leq j$ and $(i,j) \neq (1,2n + 2),\)
(3) $t^n = \text{id}_{[n]}$,  
(4) $a_it = t\alpha_{i-2}$ for $i \geq 3$,  
(5) $b_it = t\beta_{i-2}$ for $i \geq 3$,  
(6) $(\text{id}_{[0^+]}, 1, 0) = a_1b_1 \in \text{Atl}(0^+, 0^+)$ and $(\text{id}_{[0^+]}, 0, 1) = a_2b_2 \in \text{Atl}(0^-, 0^-)$. If $a_ib_j \in \text{Atl}(n, n)$ with $n \geq 1$, then

\[
\begin{align*}
    a_ib_j = \begin{cases} 
        t^{-1} & \text{if } (i, j) = (1, 2n + 2) \\
        b_{j-2}a_i & \text{if } i < j - 1, (i, j) \neq (1, 2n + 2) \\
        \text{id}_{[n]} & \text{if } i = j - 1 \\
        (\text{id}_{[n]}, 1, 0) & \text{if } i = j \text{ and } i \text{ is odd} \\
        (\text{id}_{[n]}, 0, 1) & \text{if } i = j \text{ and } i \text{ is even} \\
        \text{id}_{[n]} & \text{if } i = j + 1 \\
        b_ja_{i-2} & \text{if } i > j + 1, (i, j) \neq (2n + 2, 1) \\
        t & \text{if } (i, j) = (2n + 2, 1)
    \end{cases}
\end{align*}
\]

(7) $(\text{id}_{[n]}, 1, 0)$ and $(\text{id}_{[n]}, 0, 1)$ commute with all $(T, c_+, c_-) \in \text{Atl}(n, n)$ where $n \in \mathbb{N} \cup \{0 \pm \}$.

**Proof.** These relations can be easily verified by drawing pictures. \hfill \square

**Involution and Tangle Type**

**Proposition 2.2.20.** The map $\ast : \text{Atl} \to \text{Atl}$ given by $[n]^{\ast} = [n]$ for all $n \in \mathbb{N} \cup \{0 \pm \}$ and $(T, c_+, c_-)^{\ast} = (T^*, c_+, c_-)$ defines an involution on $\text{Atl}$.

**Corollary 2.2.21.** We have an isomorphism of categories $\text{Atl} \cong \text{Atl}^{\text{op}}$.

**Proposition 2.2.22.** The involution on $\text{Atl}$ satisfies

$A/B: a_i^* = b_i$ for $i = 1, 2$ if $a_1 \in \text{Atl}(1, 0^+)$ and $a_2 \in \text{Atl}(1, 0^-)$. For $n \geq 2$ and $a_i \in \text{Atl}(n, n - 1)$, so $i \in \{1, \ldots, 2n\}$, $a_i^* = b_i \in \text{Atl}(n - 1, n)$. 

\[
\begin{align*}
    t \in \text{Atl}(n, n) \text{ (without the dots, } n = 3) \\
    \text{Figure 2.14:}
\end{align*}
\]
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

T: For $n \in \mathbb{N}$ and $t \in \text{Atl}(n, n)$, $t^* = t^{-1}$.

D: For $n \in \mathbb{N} \cup \{0\}$, $(\text{id}_{[n]}, 1, 0)^* = (\text{id}_{[n]}, 1, 0)$ and $(\text{id}_{[n]}, 0, 1)^* = (\text{id}_{[n]}, 0, 1)$.

Proof. Obvious. \qed

Definition 2.2.23. An Atl $(m, n)$-tangle $T$ is said to be of

Type I: if $T$ is either $\text{id}_{[n]}$ for some $n \in \mathbb{N} \cup \{\pm 0\}$, or $T$ has no cups, at least one cap, and no non-contractible loops, with the limitation on $*_{0}(T)$ that exactly one of the following occurs:

I-1: There are no through strings, so $*_{0}(T)$ is uniquely determined. Note that if $n = 0-$, then there is no $*_{0}(T)$.

I-2: There are through strings. Using $*_{1}(T)$ as our reference, we go counterclockwise to the first through string, and travel outward until we reach a marked point $p$ of $D_{0}(T)$. The simple interval meeting $p$ whose interior touches an unshaded region is $*_{0}(T)$.

Type II: if $T$ has no cups or caps, so $T$ is a power of the rotation (including the identity tangle) or an annular $(0, 0)$-tangle with $k$ non-contractible loops (here we do not specify $0\pm$).

Type III: if $T$ is either $\text{id}_{[n]}$ for some $n \in \mathbb{N} \cup \{\pm 0\}$, or $T$ has no caps, at least one cup, and no non-contractible loops, with the limitation on $*_{1}(T)$, that exactly one of the following occurs:

III-1: There are no through strings, so $*_{1}(T)$ is uniquely determined. Note that if $m = 0-$, then there is no $*_{1}(T)$.

III-2: There are through strings. Using $*_{0}(T)$ as our reference, we go counterclockwise to the first through string, and travel outward until we reach a marked point $p$ of $D_{1}(T)$. The simple interval meeting $p$ whose interior touches an unshaded region is $*_{1}(T)$.

Denote the set of all tangles of Type $i$ by $T_{i}$, and denote the set of all $(m, n)$-tangles of Type $i$ by $T_{i}(m, n)$ for $i \in \{I, II, III\}$.

Remark 2.2.24. Note that

(1) the $a_{i}$’s are all Type I, and

(2) the $b_{i}$’s are all Type III.

Notation 2.2.25. We will use the notation $s_{+} = a_{2}b_{1} \in \text{Atl}(0+, 0-)$ and $s_{-} = a_{1}b_{2} \in \text{Atl}(0-, 0+)$. 

Remark 2.2.26. For the case \( a_i b_j : [0] \to [0] \) (where we do not specify \( \pm \)), a suitable version of relation (6) reads

\[
    a_i b_j = \begin{cases} 
        s_- & \text{if } (i, j) = (1, 2) \\
        (id_{[0+]}, 1, 0) & \text{if } i = j = 1 \\
        (id_{[0-]}, 0, 1) & \text{if } i = j = 2 \\
        s_+ & \text{if } (i, j) = (2, 1). 
    \end{cases}
\]

Note that we replace \( t^{\pm 1} \) with \( s_\pm \), which supports Graham and Lehrer’s reasoning that the rotation converges to the non-contractible loop as \( n \to 0 \) in [GL98].

Lemma 2.2.27. Let \( m, n \in \mathbb{N} \cup \{0\pm\} \). Types are related to the involution as follows:

(1) \( T \in \mathcal{T}_I(m, n) \) if and only if \( T^* \in \mathcal{T}_{III}(m, n) \), and

(2) If \( T \in \mathcal{T}_{II}(n, n) \), then \( T^* \in \mathcal{T}_{II}(n, n) \).

Proof. Obvious. \( \Box \)

Proposition 2.2.28. Let \( m, n \in \mathbb{N} \cup \{0\pm\} \).

Type I: Any \( T \in \mathcal{T}_I(m, n) \) is uniquely determined by \( \text{capind}(T) \). Moreover, \( \text{rel}_*(T) \in \{0, +(0), -(0)\} \).

Type II: Suppose \( m = n \in \mathbb{N} \) or \( m, n \in \{0+, 0-\} \). Any \( T \in \mathcal{T}_{II}(m, n) \) is uniquely determined by \( \text{rel}_*(T) \).
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Type III: Any \( T \in \mathcal{T}_{III}(m, n) \) is uniquely determined by \( \text{cupind}(T) \). Moreover, \( \text{rel}_*(T) \in \{0, +(0), -(0)\} \).

Proof.

Type I: Suppose \( T_1, T_2 \in \mathcal{T}_I(m, n) \) with \( \text{capind}(T_1) = \text{capind}(T_2) \). If \( \Lambda_i \in \text{caps}(T_i) \) for \( i = 1, 2 \) with \( \text{ind}(\Lambda_1) = \text{ind}(\Lambda_2) \), note that \( |B(\Lambda_1)| = |B(\Lambda_2)| \), so the \( \Lambda_i \)'s must end at the same points. Hence all caps of \( T_i \) start and end at the same points for \( i = 1, 2 \). Now note that all other points on \( D(T_i) \) for \( i = 1, 2 \) (if there are any) are connected to through strings, and recall \( *_0(T_i) \) is uniquely determined by \( *_1(T_i) \) for \( i = 1, 2 \). Hence \( T_1 = T_2 \). The statement about \( \text{rel}_*(T) \) follows immediately from conditions (I-1) and (I-2).

Type II: Note that exactly one of the following occurs:

1. \( m = n \) and \( T = \text{id}_{[n]} \), in which case \( \text{rel}_*(T) \in \{0, +(0), -(0)\} \),
2. \( m = n \) and \( T = t^k \) where \( 0 < k < n \), in which case \( \text{rel}_*(T) = k \),
3. \( m = n = 0 \pm \) and \( T = (s_\pm s_\pm)^k \) for some \( k \in \mathbb{N} \), in which case \( \text{rel}_*(T) = \pm(2k) \), or
4. \( m = 0 \pm \) and \( n = 0 \mp \) and \( T = (s_\pm s_\mp)^k s_\pm \) for some \( k \in \mathbb{Z}_{\geq 0} \), in which case \( \text{rel}_*(T) = \pm(2k + 1) \).

Type III: This follows immediately from the Type I case and Lemma 2.2.27.

Lemma 2.2.29. Tangle type is preserved under tangle composition for tangles.

Proof.

Type I: Suppose \( S, T \in \mathcal{T}_I \) such that \( R = S \circ T \) makes sense. Certainly \( R \) has no cups or loops. It remains to verify that \( *_0(R) \) is in the right place. A problem could only arise in the case where both \( S \) and \( T \) have through strings, but we see that if \( S \) and \( T \) both satisfy condition (I-2), then so does \( R \).

Type II: Obvious.

Type III: Suppose \( S, T \in \mathcal{T}_{III} \) such that \( R = S \circ T \) makes sense. Then by Lemma 2.2.27, we have \( T^*, S^* \in \mathcal{T}_I \) and \( R^* = T^* \circ S^* \) makes sense, so by the Type I case, \( R^* \in \mathcal{T}_I \), and once more by 2.2.27, \( R \in \mathcal{T}_{III} \).

Corollary 2.2.30. By 2.2.24 and Proposition 2.2.29,

(1) any composite of \( a_i \)'s is in \( \mathcal{T}_I \), and

(2) any composite of \( b_i \)'s is in \( \mathcal{T}_{III} \).
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB
CATEGOR Y

Unique Tangle Decompositions

For this section, we use the convention that if \( n = 0 \pm \) and \( z \in \mathbb{Z} \), then \( n + z = z \).

**Definition 2.2.31.** A tangle \( T \in T_I \) is called irreducible if there is at most one cap bounding \( *_1(T) \), and if there is a cap \( \Lambda \) bounding \( *_1(T) \), then all other caps of \( T \) are bounded by \( \Lambda \).

**Remark 2.2.32.** If \( T \in T_I(m, n) \) for \( m \geq 1 \) is irreducible, then \( T \) has a unique representation as follows:

**Case 1:** if there is no cap bounding \( *_1(T) \), then \( T = a_{i_k} \cdots a_{i_1} \) with \( i_j > i_{j+1} \) for all \( j \in \{1, \ldots, k-1\} \) and \( i_j < 2(m-j) + 2 \) for all \( j \in \{1, \ldots, k\} \).

**Case 2:** If there is a cap bounding \( *_1(T) \), then \( T = a_q a_{i_k} \cdots a_{i_1} a_{j_1} \cdots a_{j_1} \) where \( k, l \geq 0 \) and

(i) \( q = 2n + 2 \),

(ii) \( i_r > i_{r+1} \) for all \( r \in \{1, \ldots, k-1\} \), \( i_1 < j_1 \), and \( j_s > j_{s+1} \) for all \( s \in \{1, \ldots, l-1\} \), and

(iii) \( i_r \leq 2(k-r) + 1 \) for all \( r \in \{1, \ldots, k\} \) and \( j_s \geq 2(m-s) + 1 \) for all \( s \in \{1, \ldots, l\} \).

Uniqueness follows by looking at the cap indices which are given as follows:

**Case 1:** If there is no cap bounding \( *_1(T) \), then \( \text{capind}(T) = \{i_k, \cdots, i_1\} \).

**Case 2:** If there is a cap \( \Lambda \) bounding \( *_1(T) \), then \( \text{ind}(\Lambda) = 2(m-l) \) and \( \text{capind}(T) = \{i_k, \cdots, i_1, 2(m-l), j_1, \cdots, j_1\} \).

**Remarks 2.2.33.** Suppose \( T \in T_I(m, n-1) \) with \( m > n - 1 \geq 1 \) is irreducible such that \( *_1(T) \) is bounded. Let \( T = a_q a_{i_k} \cdots a_{i_1} a_{j_1} \cdots a_{j_1} \) be the representation afforded by the above remark. If \( S \in T_I(n-1, p) \) and \( R = S \circ T \), then

(1) there is a cap \( \Lambda \) of \( R \) bounding \( *_1(R) \), of index \( 2(m-l) \). All other caps of \( R \) bounding \( *_1(R) \) have smaller index than \( \Lambda \).

(2) \( |B(\Lambda)| = k + l + 1 \).

(3) \( \text{capind}(R) = \{i_k, \cdots, i_1, c_1, \cdots, c_s, 2(m-l), j_1, \cdots, j_1\} \) for some \( c_1, \ldots, c_s \in \mathbb{N} \) and \( s = m - p - k - l - 1 \).

**Lemma 2.2.34.** Suppose \( T_1 \in T_I(m, m-u-1) \) and \( T_2 \in T_I(m, m-v-1) \) with \( m-u, m-v \geq 2 \) are irreducible and each has one cap bounding \( *_1 \). Suppose \( S_1 \in T_I(m-u-1, w) \) and \( S_2 \in T_I(m-v-1, w) \) such that \( S_1 \circ T_1 = S_2 \circ T_2 \). Then \( T_1 = T_2 \).

**Proof.** Set \( R = S_1 \circ T_1 = S_0 \circ T_0 \). We have that \( *_1(R) \) is bounded by a cap \( \Lambda \) with index \( 2(m-u) = 2(m-v) \), so \( u = v \). Now we have unique irreducible decompositions

\[
T_1 = a_p a_{i_k} \cdots a_{i_1} a_{j_1} \cdots a_{j_1} \quad \text{and} \quad T_2 = a_q a_{g_r} \cdots a_{g_s} a_{h_s} \cdots a_{h_l},
\]
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Figure 2.17: $R = S \circ T$, zoomed in near $*_1(R)$ where $T = a_{2n+2}a_1a_2a_4a_{2m-1} \in T_I(m,n)$ is irreducible

and as the cap indices of $R$ are unique, we have

$$\text{capind}(R) = \{i_k, \cdots, i_1, c_1, \cdots, c_s, 2(m-u), j_1 \cdots, j_1\}$$

$$= \{g_r, \cdots, g_1, c_1, \cdots, c_s, 2(m-v), h_s, \cdots, h_1\}.$$

Hence we must have equality of the two sequences:

$$\{i_k, \cdots, i_1, 2(m-u), j_1 \cdots, j_1\} = \{g_r, \cdots, g_1, 2(m-v), h_s, \cdots, h_1\},$$

and $T_1 = T_2$ by Proposition 2.2.28. \hfill $\square$

**Proposition 2.2.35.** Each $T \in T_I(m,n)$ where $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\pm\}$ has a unique decomposition $T = W_r \cdots W_1$ such that $W_i$ is irreducible for all $i = 1, \ldots, r$.

**Proof.**

**Existence:** The existence of such a decomposition will follow from Algorithm 3.2 below.

**Uniqueness:** We induct on $r$. Suppose $r = 1$. Then uniqueness follows from Remark 2.2.32. Suppose now that $r > 1$ and the result holds for all concatenations of fewer irreducible words. Suppose we have another decomposition

$$T = W_r \cdots W_1 = U_s \cdots U_1.$$

Then by the induction hypothesis, we must have $s \geq r$. As $W_1$ and $U_1$ are irreducible, we apply Lemma 2.2.34 with

1. $T_1 = W_1$ and $S_1 = W_r \cdots W_2$, and
2. $T_2 = U_1$ and $S_2 = U_s \cdots U_2$

to see that $W_1 = U_1$. We may now apply appropriate $b_i$’s to $T$ (on the right) to get rid of $W_1 = U_1$ to get

$$W' = W_r \cdots W_2 = U_s \cdots U_2,$$

where $W'$ is equal to a concatenation of fewer irreducible words. By the induction hypothesis, we can conclude $r = s$ and $U_i = W_i$ for all $i = 2, \ldots, r$. We are finished. \hfill $\square$
Algorithm 2.2.36. The following algorithm expresses a Type I tangle $T \in \mathcal{T}_I(m,n)$ where $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\pm\}$ as a composite of $a_i$’s in the form required by Proposition 2.2.35. Set $T_0 = T$, $m_0 = m$, and $r = 1$.

**Step 1:** Let $S_1 = \{\Lambda \in \text{caps}(T_0)|_*^1(T_0) \subset \partial \Lambda \text{ and } \text{ind}(\Lambda) \in 2\mathbb{N}\}$. Let $S_0$ be the set of all caps that are not in $B(\Lambda)$ for some $\Lambda \in S_1$. If $S_1 = \emptyset$, proceed to Step 4.

**Step 2:** Suppose $|S_1| \geq 1$. Select the cap $\Lambda \in S_1$ with the largest index. There are two cases:

- **Case 1:** $B(\Lambda) = \{\Lambda\}$. Set $W_r = a_{\text{ind}(\Lambda)}$. Proceed to Step 3.
- **Case 2:** $B(\Lambda) \setminus \{\Lambda\} \neq \emptyset$. List the cap indices for all caps $\Lambda' \in B(\Lambda) \setminus \{\Lambda\}$ in decreasing order from right to left, $i_k, \ldots, i_1$ where $i_j > i_{j+1}$ for all $j \in \{1, \ldots, k-1\}$. where $k = |B(\Lambda) \setminus \{\Lambda\}|$. Set $q = \text{ind}(\Lambda) - 2k$ and $W_r = a_q a_k \cdots a_{i_1}$.

**Step 3:** Note that $W_r$ is irreducible. Move $_*^1(T_0)$ counterclockwise to the closest simple interval outside of $\Lambda$ whose interior touches an unshaded region (which is necessarily 2 regions counterclockwise), and remove all caps in $B(\Lambda)$ from $T_0$ to get a new tangle, called $T_1$. Note that $T_0 = T_1 W_r$. Set $m_1$ equal to half the number of internal boundary points of $T_1$, and set $r_1 = r$. Now set $T_0 = T_1$, $m_0 = m_1$, and $r = r_1 + 1$. Go back to Step 1.

**Step 4:** List the cap indices for all caps $\Lambda \in S_0$ in decreasing order from right to left, $i_k, \ldots, i_1$ where $i_j > i_{j+1}$ for all $j \in \{1, \ldots, k-1\}$. There are two cases:

- (i) There are fewer than $m_0$ caps. Set $W_r = a_{i_k} \cdots a_{i_1}$. Note that $W_r$ is irreducible and $T_0 = W_r$. We are finished.
- (ii) There are $m_0$ caps. Proceed to Step 5.

**Step 5:** There are two cases:

- (i) If the region touching $D_0(T_0)$ is unshaded, set $W_r = a_1 a_{i_{k-1}} \cdots a_{i_1}$. Note that $W_r$ is irreducible and $T_0 = W_r$. We are finished.
- (ii) If the region touching $D_0(T_0)$ is shaded, set $W_r = a_2 a_{i_{k-1}} \cdots a_{i_1}$. Note that $W_r$ is irreducible and $T_0 = W_r$. We are finished.

Note that $T = W_r \cdots W_1$ satisfies the conditions of Proposition 2.2.35.

The following Theorem is merely a strengthening of Corollary 1.16 in [Jon94].

**Theorem 2.2.37 (Atl Tangle Decomposition).** Each Atl $(m,n)$-tangle $T$ can be written uniquely as a composite $T = T_{\text{III}} \circ T_{\text{II}} \circ T_{\text{I}}$ where $T_i \in \mathcal{T}_i$ for all $i \in \{I, II, III\}$.

**Proof.** We begin by proving the uniqueness of such a decomposition as it will tell us how to find such a decomposition.
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB
CATEGORY

Uniqueness: Suppose we have a decomposition $T = T_{III} \circ T_{II} \circ T_I$ where $T_I \in T_I(m,l)$, $T_{II} \in T_{II}(l,k)$, and $T_{III} \in T_{III}(k,n)$ for some $l, k \in \mathbb{N} \cup \{0\pm\}$. Note that $l, k$ are uniquely determined by $|\text{ts}(T)|$ and the shading of $T$. Note further that $\text{capind}(T_I) = \text{capind}(T)$, $\text{rel}_s(T_{II}) = \text{rel}_s(T)$, and $\text{cupind}(T_{III}) = \text{cupind}(T)$. Hence $T_i$ is uniquely determined for $i \in \{I, II, III\}$ by Proposition 2.2.28.

Existence: Let $l = k$ be the number of through strings of $T$. If $l = k = 0$, set $l = 0+$, respectively $l = 0-$ if the region meeting $D_1(T)$ is unshaded, respectively shaded, and set $k = 0+$, respectively $k = 0-$ if the region meeting $D_0(T)$ is unshaded, respectively shaded. Let $T_I \in T_I(m,l)$ be the unique tangle with $\text{capind}(T_I) = \text{capind}(T)$. Let $T_{II} \in T_{II}(l,k)$ be the unique tangle with $\text{rel}_s(T_{II}) = \text{rel}_s(T)$. Let $T_{III} \in T_{III}(k,n)$ be the unique tangle with $\text{cupind}(T_{III}) = \text{cupind}(T)$. It is now obvious that $T = T_{III} \circ T_{II} \circ T_I$.

![Figure 2.18: Decomposition of an ATL tangle into $T_{III} \circ T_{II} \circ T_I$](image)

2.3 The Category $\mathfrak{a} \Delta$

Generators and Relations

Definition 2.3.1. Let $\mathfrak{a} \Delta$, the annular category, be the following small category:

- **Objects**: $[n]$ for $n \in \mathbb{N} \cup \{0\pm\}$, and
Morphisms: generated by

\[ \alpha_1: [1] \to [0^+] \text{, } \alpha_2: [1] \to [0^-], \text{ and } \]
\[ \alpha_i: [n] \to [n-1] \text{ for } i = 1, \ldots, 2n \text{ and } n \geq 2; \]
\[ \beta_1: [0^+] \to [1], \beta_2: [0^-] \to [1], \text{ and } \]
\[ \beta_i: [n] \to [n+1] \text{ for } i = 1, \ldots, 2n+2 \text{ and } n \geq 1; \]
\[ \tau: [n] \to [n] \text{ for all } n \in \mathbb{N}; \text{ and } \]
\[ \delta_{\pm}: [n] \to [n] \text{ for all } n \in \mathbb{N} \cup \{0\pm\} \]

subject to the following relations:

1. \( \alpha_i \alpha_j = \alpha_{j-2} \alpha_i \) for \( i < j - 1 \) and \( (i, j) \neq (1, 2n) \),
2. \( \beta_i \beta_j = \beta_{j+2} \beta_i \) for \( i \leq j \) and \( (i, j) \neq (1, 2n+2) \),
3. \( \tau^n = \text{id}_{[n]} \),
4. \( \alpha_i \tau = \tau \alpha_{i-2} \) for \( i \geq 3 \),
5. \( \beta_i \tau = \tau \beta_{i-2} \) for \( i \geq 3 \),
6. \( \delta_{\pm} = \alpha_1 \beta_1 \in a\Delta(0^+, 0^+) \) and \( \delta_{-} = \alpha_2 \beta_2 \in a\Delta(0^-, 0^-) \). If \( \alpha_i \beta_j: [n] \to [n] \) with \( n \geq 1 \), then

\[ \alpha_i \beta_j = \begin{cases} 
\tau^{-1} & \text{if } (i, j) = (1, 2n + 2) \\
\beta_{j-2} \alpha_i & \text{if } i < j - 1, (i, j) \neq (1, 2n + 2) \\
\text{id}_{[n]} & \text{if } i = j - 1 \\
\delta_{+} & \text{if } i = j \text{ and } i \text{ is odd} \\
\delta_{-} & \text{if } i = j \text{ and } i \text{ is even} \\
\text{id}_{[n]} & \text{if } i = j + 1 \\
\beta_j \alpha_{i-2} & \text{if } i > j + 1, (i, j) \neq (2n + 2, 1) \\
\tau & \text{if } (i, j) = (2n + 2, 1)
\end{cases} \]

(7) \( \delta_{\pm} \) commutes with all other generators (including \( \delta_{\mp} \)).

### Involution and Word Type

**Definition 2.3.2.** A morphism \( h \in \text{Mor}(a\Delta) \) will be called primitive if \( h \) is equal to \( \alpha_i, \beta_i, t, \delta_{\pm}, \) or \( \text{id}_{[n]} \) for \( n \in \mathbb{N} \cup \{0\pm\} \). A word on \( a\Delta \) is a sequence \( h_r \cdots h_1 \) with \( r \geq 1 \) of primitive morphisms in \( a\Delta \). We say the length of such a word is \( r \in \mathbb{N} \). By convention, we will say a word has length zero if and only if \( r = 1 \) and \( h_1 = \text{id}_{[n]} \) for some \( n \in \mathbb{N} \cup \{0\pm\} \).

**Definition 2.3.3.** We define a map \( * \) on \( \text{Ob}(a\Delta) \) and on primitive morphisms in \( \text{Mor}(a\Delta) \):
(Ob) For \( n \in \mathbb{N} \cup \{0, \pm\} \), define \([n]^* = [n]\).

(I) For all \( n \in \mathbb{N} \cup \{0, \pm\} \), define \( \text{id}_{[n]}^* = \text{id}_{[n]} \).

(A) For \( \alpha_1 \in a\Delta(1, 0+) \), define \( \alpha_1^* = \beta_1 \in a\Delta(0+, 1) \). For \( \alpha_2 \in a\Delta(1, 0-1) \), define \( \alpha_2^* = \beta_2 \in a\Delta(0-, 1) \). For \( n \geq 2 \) and \( \alpha_i \in a\Delta(n, n-1) \), so \( i \in \{1, \ldots, 2n\} \), define \( \alpha_i^* = \beta_i \in a\Delta(n-1, n) \).

(B) For \( \beta_1 \in a\Delta(0+, 1) \), define \( \beta_1^* = \alpha_1 \in a\Delta(1, 0+) \). For \( \beta_2 \in a\Delta(0-, 1) \), define \( \beta_2^* = \alpha_2 \in a\Delta(1, 0-) \). For \( n \geq 1 \) and \( \beta_i \in a\Delta(n, n+1) \), so \( i \in \{1, \ldots, 2n+2\} \), define \( \beta_i^* = \alpha_i \in a\Delta(n+1, n) \).

(T) For \( n \in \mathbb{N} \) and \( \tau \in a\Delta(n, n) \), define \( \tau^* = \tau^{-1} \).

(D) For \( n \in \mathbb{N} \cup \{0, \pm\} \) and \( \delta_{\pm} \in a\Delta(n, n) \), define \( \delta_{\pm}^* = \delta_{\pm} \).

Proposition 2.3.4. The following extension of \(*\) to \( \text{Mor}(a\Delta) \) is well defined:

- If \( h_r \cdots h_1 \) is a word on \( a\Delta \), then we define \( (h_r \cdots h_1)^* = h_1^* \cdots h_r^* \).

Hence \(*\) extends uniquely to an involution on \( a\Delta \).

Proof. We must check that \(*\) preserves the relations of \( a\Delta \). Note that relations (3), (6), and (7) are preserved by \(*\), and the following pairs are switched: (1) & (2) and (4) & (5).

Corollary 2.3.5. We have an isomorphism of categories \( a\Delta \cong a\Delta^{\text{op}} \).

Proposition 2.3.6. The following additional relations hold in \( a\Delta \):

1. \( \alpha_1 \tau = \alpha_{2n-1} \) and \( \alpha_2 \tau = \alpha_{2n} \),
2. \( \tau \beta_{2n+1} = \beta_1, \tau \beta_{2n+2} = \beta_2 \), and
3. \( \beta_1 \tau = \tau^2 \beta_{2n-1} \) and \( \beta_2 \tau = \tau^2 \beta_{2n} \).

Proof. (1) By relations (4) and (5), we have

\[
\alpha_{2n-1} = \alpha_{2n-1} \tau^n = \tau \alpha_{2n-3} \tau^{n-1} = \cdots = \tau^{n-1} \alpha_3 \tau = \tau^n \alpha_1 = \alpha_1.
\]

The proof of the other relation is similar.

(2) These relations are merely \(*\) applied to (1).

(3) By relations (4) and (6), we have

\[
\tau^2 \beta_{2n-1} = \tau^2 \beta_{2n-1} \tau^n = \tau^2 \tau \beta_{2n-3} \tau^{n-1} = \cdots = \tau^2 \tau^{n-1} \beta_1 \tau = \tau^{n+1} \beta_1 \tau = \beta_1 \tau.
\]

The proof of the other relation is similar.
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Notation 2.3.7. (1) If \( h \in \text{Mor}(a\Delta) \), we write \( h \in A_1 \) if \( h = \alpha_i \in a\Delta(1, 0\pm) \) where \( i \in \{1, 2\} \). We write \( h \in A_n \) where \( n \geq 2 \) if \( h = \alpha_i \in a\Delta(n, n-1) \) for some \( i \in \{1, \ldots, 2n\} \). We write \( h \in A \) if \( h \in A_n \) for some \( n \geq 1 \). Similarly we define \( B_n \) for \( n \in \mathbb{N} \cup \{0\pm\} \) and \( B \).

(2) For convenience, we will use the notation \( \sigma_+ = \alpha_2\beta_1 \in a\Delta(0+, 0-) \) and \( \sigma_- = \alpha_1\beta_2 \in a\Delta(0-, 0+) \).

Definition 2.3.8. A word \( w = h_r \cdots h_1 \) on \( a\Delta \) is called

Type I: if \( w \) has length zero or if \( h_i \in A \) for all \( i \in \{1, \ldots, r\} \).

Type II: if either

1. \( w \) has length zero,
2. \( r > 0 \) and \( h_i = \tau \) for all \( i \in \{1, \ldots, r\} \), or
3. \( r = 2s \) for some \( s > 0 \) and \( h_ih_{i+1} = \sigma_\pm \) for all odd \( i \) so that

\[
w = \begin{cases} 
  (\sigma_\pm^k\sigma_\pm) & \text{if } s = 2k + 1 \text{ is odd, or} \\
  (\sigma_\pm^k) & \text{if } s = 2k \text{ is even.}
\end{cases}
\]

Type III: if \( w \) has length zero or if \( h_i \in B \) for all \( i \in \{1, \ldots, r\} \).

Denote the set of all words of Type \( i \) by \( W_i \), and denote the set of all words of Type \( i \) with domain \([m]\) and codomain \([n]\) by \( W_i(m, n) \) for \( i \in \{I, II, III\} \).

Lemma 2.3.9. Let \( m, n \in \mathbb{N} \cup \{0\pm\} \). Types are related to the involution as follows:

1. \( w \in W_I(m, n) \) if and only if \( w^* \in W_{III}(n, m) \), and
2. If \( w \in W_{II}(n, n) \), then \( w^* \in W_{II}(n, n) \).

Proof. Obvious.

Standard Forms

Notation 2.3.10. if we replace \( j \) with \( j + 2 \) in the statement of relation (1), we get the equivalent relation

(1') \( \alpha_j\alpha_i = \alpha_i\alpha_{j+2} \) for all \( j \geq i \) with \( (j, i) \neq (2n, 1) \)

as maps \([n+1] \to [n-1]\).

Definition 2.3.11. A word \( w \in W_I(m, n) \) with \( m \geq 1 \) is called irreducible if either

1. \( w = \alpha_{i_k} \cdots \alpha_{i_1} \) where \( i_r > i_{r+1} \) for all \( r \in \{1, \ldots, k-1\} \) and \( i_r < 2(m-r) + 2 \) for all \( r \in \{1, \ldots, k\} \), in which case we also say \( w \) is ordered, or
(2) $w = \alpha_q \alpha_{i_k} \cdots \alpha_{i_1} \alpha_{j_l} \cdots \alpha_{j_1} \in W_I(m, n)$ where $m \geq 1$ and $l, k \geq 0$ such that

(i) $q = 2n + 2$,

(ii) $i_r > i_{r+1}$ for all $r \in \{1, \ldots, k-1\}$, $i_1 < j_l$, and $j_s > j_{s+1}$ for all $s \in \{1, \ldots, l-1\}$, and

(iii) $i_r \leq 2(k-r)+1$ for all $r \in \{1, \ldots, k\}$ and $j_s \geq 2(m-s)+1$ for all $s \in \{1, \ldots, l\}$.

Remark 2.3.12. If $\alpha_q \alpha_{i_k} \cdots \alpha_{i_1} \alpha_{j_l} \cdots \alpha_{j_1}$ is irreducible as in (2) of 2.3.11, then so are

$$\alpha_q \alpha_{i_k} \cdots \alpha_{i_1} \alpha_{j_l} \cdots \alpha_{j_r} \quad \text{and} \quad \alpha_q \alpha_{i_k} \cdots \alpha_{i_s}$$

for all $r \in \{1, \ldots, l\}$ and $s \in \{1, \ldots, k\}$. In particular, if $l > 0$, then $j_l = 2(m-l)+1$, and if $k > 0$, then $i_k = 1$.

Algorithm 2.3.13. Suppose $w = \alpha_{i_k} \cdots \alpha_{i_1} \in W_I(m, n-1)$ is ordered where $n-1 > 0$. The following algorithm gives words $u_1, u_2$ where $u_1$ is irreducible and $\alpha_{2n}w = u_2u_1$. Set $u_1 = \alpha_{2n}w$ and $u_3 = \text{id}_{[n-1]}$.

**Step 1**: If $u_1$ is irreducible, set $u_2 = u_3$. We are finished. Otherwise, proceed to Step 2.

**Step 2**: There is a $j \in \{1, \ldots, k\}$ such that $2(k-j)+1 < i_j < 2(m-j)+1$. Pick $j$ minimal with this property. Use relation (1) to push $\alpha_{i_k} \cdots \alpha_{i_{j+1}}$ past $\alpha_{i_j}$ to get

$$w = \alpha_{2n} \alpha_{i_j-2(k-j)+2} \alpha_{i_{k-1}} \cdots \alpha_{i_{j+1}} \alpha_{i_{j-1}} \cdots \alpha_{i_1}.$$ 

Note that

$$1 < i_j - 2(k-j) < 2(m-j)+1 - 2(k-j) = 2(m-k)+1 = 2n+1,$$

as $m-k = n$, so we may use relation $(1')$ to get

$$\alpha_{i_j-2(k-j)} \alpha_{2n+2} \alpha_{i_{k-1}} \cdots \alpha_{j+1} \alpha_{j-1} \cdots \alpha_{i_1}.$$ 

Set $u_2 = \alpha_{i_j-2(k-j)+2} u_3$. Now set $u_3 = u_2$. Set

$$u_1 = \alpha_{2n+2} \alpha_{i_{k-1}} \cdots \alpha_{j+1} \alpha_{j-1} \cdots \alpha_{i_1}.$$ 

Go back to Step 1.

**Proof.** We need only prove the above algorithm terminates. Note one of the $\alpha_i$’s increases in index each iteration, which cannot happen indefinitely. \qed

Proposition 2.3.14. Suppose $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$. Each $w \in W_I(m, n)$ has a decomposition $w = w_r \cdots w_1$ where each $w_i \in W_I$ is irreducible. Such a decomposition of $w$ is called a standard decomposition of $w$. 

Suppose $\tau$ to get $w'$ where each $w_i$ is ordered and for each $s \in \{1, \ldots, r - 1\}$. If $w_s = \alpha_i \cdots \alpha_i$ and $w_{s+1} = \alpha_j \cdots \alpha_j$, then $i_a = 1$, $j_1 = 2k$, so $\alpha_j \alpha_i = \alpha_{2k} \alpha_1 \in a\Delta(k + 1, k - 1)$ for some $k \geq 2$. There are two cases.

Case 1: $r = 1$. Then $w = w_1$ is ordered, hence irreducible, and we are finished.

Case 2: Suppose $r > 1$. As $w_2 = \alpha_i \cdots \alpha_i$, where $\alpha_i = \alpha_{2k} \in a\Delta(k, k - 1)$, we apply Algorithm 2.3.13 to the word $\alpha_{2k} w_1$ to obtain $u_1, u_2$ with $u_1$ irreducible such that $u_2 u_1 = \alpha_{2k} w_1$. We now note that $w = w' u_1$ where

$$w' = w_r \cdots w_3 \alpha_i \cdots \alpha_{i_2}$$

is a word of strictly smaller length. Applying the induction hypothesis to $w'$ gives us the desired result. \hfill \Box

**Theorem 2.3.15** (Standard Forms). Suppose $w = h_r \cdots h_1$ is a word on $a\Delta$ in $a\Delta(m, n)$ for $m, n \in \mathbb{N} \cup \{0\}$. Then there is a decomposition $w = \delta^+ \delta^- w_{11} w_{II} w_I$ where $w_i \in W_i$ for all $i \in \{I, II, III\}$, $c_+ \geq 0$, and $w_I$ and $w_{III}$ are in the form afforded by Proposition 2.3.14.

**Proof.** Note that it suffices to find $v_i \in W_i$ for $i \in \{I, II, III\}$ and $c_+ \geq 0$ such that $w = \delta^+ c^- v_{11} w_{II} v_I$, as we may then set $w_{II} = v_{II}$ and apply Proposition 2.3.14 to $v_I$ and $v_{III}$ to get $w_I$ and $w_{III}$ respectively. We induct on $r$. The case $r = 1$ is trivial. Suppose $r > 1$ and the result holds for all words of shorter length. Apply the induction hypothesis to $w' = h_{r-1} \cdots h_1$ to get

$$w' = \delta^+ c^- u_{11} u_{II} u_I.$$

There are 4 cases.

(D) Suppose $h_r = \delta_\pm$. Set $c_+ = c'_\pm + 1$, $c_- = c'_\mp$, and $v_i = u_i$ for all $i \in \{I, II, III\}$. We are finished.

(B) Suppose $h_r \in B$. Set $c_+ = c'_\pm$ and $v_i = u_i$ for $i \in \{I, II, III\}$. We are finished.

(T) Suppose $h_r = \tau$. Set $c_+ = c'_\pm$ and $w_I = u_I$. As we push $\tau$ right using relation (5) and Proposition 2.3.6, only two extraordinary possibilities occur:

Case 1: $\tau$ meets $\beta_{2n+1}$ or $\beta_{2n+2}$ in $a\Delta(n, n + 1)$, so $\tau$ disappears when using Proposition 2.3.6, or

Case 2: $\tau$ meets $\beta_1 \in a\Delta(0+, 1)$ or $\beta_2 \in a\Delta(0-, 1)$, so $\tau$ disappears as $id_{[1]} = \tau \in a\Delta(1, 1)$.

Hence we get that $w = v_{III} \tau^s$ where $v_{III} \in W_{III}$ and $s \in \{0, 1\}$. If $s = 0$, set $v_{II} = u_{II}$, and if $s = 1$, set $v_{II} = \tau u_{II}$. We are finished.
(A) Suppose \( h_r = \alpha_q \) for some \( q \in \mathbb{N} \). Use relation (6) to push \( \alpha_q \) to the right of the \( \beta \)'s. There are five cases.

Case 1: We use the relation \( \alpha_i \beta_j = \tau^{\pm 1} \). Arguing as in Case (T) we are finished.

Case 2: We use the relation \( \alpha_i \beta_\pm = id[k] \) for some \( k \in \mathbb{N} \), so \( \alpha_q u_{III} = v_{III} \) for some \( v_{III} \in W_{III} \). Set \( c_\pm = c'_\pm \) and \( v_i = u_i \) for \( i \in \{ I, II \} \). We are finished.

Case 3: We use the relation \( \alpha_i \beta_i = \delta ^{\pm} \), so \( \alpha_q u_{III} = \delta ^{\pm} v_{III} \) for some \( v_{III} \in W_{III} \). Set \( c_\pm = c'_\pm + 1, c_\mp = c'_{\mp}, \) and \( v_i = u_i \) for \( i \in \{ I, II \} \). We are finished.

Case 4: \( \alpha_q \) can be pushed all the way to the right of \( u_{III} \) to obtain \( \alpha_q u_{III} = v_{III} \alpha_p \) for some \( p \in \mathbb{N} \) and \( v_{III} \in W_{III} \). Then necessarily \( u_{II} = \tau^s \) for some \( s \in \mathbb{Z}_{\geq 0} \), so we use relation (4) and 2.3.6 to push \( \alpha_q \) to the right of the \( \tau \)'s. Hence we obtain \( \alpha_q u_{II} = v_{II} \alpha_k \) for some \( k \in \mathbb{N} \) and \( v_{II} \in W_{II} \). Set \( c_\pm = c'_\pm \) and \( v_I = u_I \). We are finished.

Case 5: \( \alpha_q \) can be pushed all the way to the right except for the last \( \beta_i \). This means \( \alpha_q u_{III} = v_{III} \alpha_i \beta_j \) for some \( v_{III} \in W_{III} \) where \( \alpha_i \beta_j = \sigma^\pm \). Set \( v_{II} = \sigma^\pm u_{II} \), \( c_\pm = c'_\pm \), and \( v_I = u_I \). We are finished.

\[ \square \]

Definition 2.3.16. If \( w \in \text{Mor}(\Delta) \), a decomposition of \( w \) as in Theorem 2.3.15 is called a standard form of \( w \).

Remark 2.3.17. It will be a consequence of Theorem 2.4.8 that a word \( w \in \Delta \) has a unique standard form.

2.4 The Isomorphism of Categories \( \Delta \cong \text{Atl} \)

Proposition 2.4.1. The following defines an involutive functor \( F: \Delta \rightarrow \text{Atl} \):

**Objects:** \( F([n]) = [n] \) for all \( n \in \mathbb{N} \cup \{0\pm\} \),

**Morphisms:**

(A) Set \( F(\alpha_i) = a_i \),

(B) Set \( F(\beta_i) = b_i \),

(T) Set \( F(\tau) = t \), and

(D) Set \( F(\delta_+ \in \Delta(n,n)) = (id_{[n]}, 1, 0) \) and \( F(\delta_- \in \Delta(n,n)) = (id_{[n]}, 0, 1) \) for \( n \in \mathbb{N} \cup \{0\pm\} \).

Proof. We must check that \( F(id_{[n]}) = id_{[n]} \) for all \( n \in \mathbb{N} \cup \{0\pm\} \) and that \( F \) preserves composition, but both these conditions follow from Theorem 2.2.19. It is clear \( * \) preserves the involution by Proposition 2.2.22. \[ \square \]
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Remark 2.4.2. We construct a functor $G: \text{Atl} \to \Delta$ as follows: we create a function $G: \text{Atl} \to \Delta$ taking objects to objects (this part is easy as objects in both categories have the same names) and $\text{Atl}(m,n) \to \Delta(m,n)$ bijectively such that $F \circ G = \text{id}_{\text{Atl}}$. It will follow immediately that $G$ is a functor and $G \circ F = \text{id}_{\Delta}.$

Theorem 2.4.3. Let $m, n \in \mathbb{N} \cup \{0\pm\}$. Then $F_i = F|_{W_i(m,n)}: W_i(m,n) \to T_i(m,n)$ is bijective for all $i \in \{I, II, III\}$, i.e. there is a bijective correspondence between words of Type $i$ and Atl tangles of Type $i$ for all $i \in \{I, II, III\}$.

Proof.
Type I: Note that $\text{im}(F_I) \subset T_I(m,n)$. We construct the inverse $G_I$ for $F_I$. Note that by Proposition 2.2.35, each $T \in T_I(m,n)$ can be written uniquely as $T = \prod_i W_{a_i}$, which can further be expanded as

$$T = a_{i_p} \cdots a_{i_1} a_{j_q} \cdots a_{j_1} \cdots a_{k_r} \cdots a_{k_1}$$

satisfying 2.2.35. Set

$$G_I(T) = \alpha_{i_p} \cdots \alpha_{i_1} \cdots \alpha_{j_q} \cdots \alpha_{j_1} \alpha_{k_r} \cdots \alpha_{k_1}.$$ 

It follows $F_I \circ G_I = \text{id}$. Now by Proposition 2.3.14, every word of Type I can be written in this form. Hence we see $G_I$ is in fact the inverse of $F_I$.

Type II: Obvious.

Type III: From the Type I case and the involutions in $\Delta$ and $\text{Atl}$, we have the following bijections:

$$T_{III}(m,n) \leftrightarrow T_I(n,m) \leftrightarrow W_I(n,m) \leftrightarrow W_{III}(m,n).$$

Definition 2.4.4. We define $G: \text{Atl} \to \Delta$ as follows:

Objects: $G([n]) = [n]$ for all $n \in \mathbb{N} \cup \{0\pm\}$.

Morphisms: First define $G(T, 0, 0)$ for a $T \in T_i$ for $i \in \{I, II, III\}$ by the bijective correspondences given in Theorem 2.4.3. Then for an arbitrary Atl $(m,n)$-tangle $T$, define $G(T, 0, 0)$ by

$$G(T, 0, 0) = G(T_{III}, 0, 0) \circ G(T_{II}, 0, 0) \circ G(T_I, 0, 0)$$

where $T_i$ for $i \in \{I, II, III\}$ is defined for $T$ as in the Atl Decomposition Theorem 2.2.37. Finally, define $G(T, c_+, c_-) = \delta^c_+ \delta^- G(T, 0, 0)$ for an arbitrary morphism $(T, c_+, c_-) \in \text{Mor}(\text{Atl})$. Note that $G$ is well defined by the uniqueness part of 2.2.37.

Proposition 2.4.5. If $T$ is an Atl $(m,n)$-tangle of Type $i$ for $i \in \{I, II, III\}$, then $F \circ G(T) = T$.

Proof. This is immediate from the definition of $G$ and Theorem 2.4.3.
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Corollary 2.4.6. \( F \circ G = \text{id}_{\text{Atl}} \), so \( G \) restricted to \( \text{Atl}(m,n) \) is injective into \( \text{a}\Delta(m,n) \) for all \( m, n \in \mathbb{N} \cup \{0, \pm\} \).

Proof. This follows immediately from Theorem 2.2.37 and the definition of \( G \) as \( F \) is a functor.

Proposition 2.4.7. \( G \) restricted to \( \text{Atl}(m,n) \) is surjective onto \( \text{a}\Delta(m,n) \).

Proof. We have that every word \( w \in \text{Mor}(\text{a}\Delta) \) is equal to a word \( \delta^c_+ \delta^c_- w_{III} w_{II} w_I \) in standard form where \( w_i \) is of Type \( i \) for \( i \in \{I, II, III\} \). By 2.4.3 there are unique \( \text{Atl} \) tangles \( T_i \) of Type \( i \) such that \( w_i = G(T_i) \) for all \( i \in \{I, II, III\} \). Set \( T = T_{III} \circ T_{II} \circ T_I \), and note this decomposition into a composite of \( \text{Atl} \) tangles of Types I, II, and III is unique by 2.2.37. It follows that

\[ G(T, c_+, c_-) = \delta^c_+ \delta^c_- w_{III} w_{II} w_I = w \]

by the definition of \( G \).

Theorem 2.4.8. \( F : \text{a}\Delta \to \text{Atl} \) is an isomorphism of involutive categories. Hence \( \text{a}\Delta \) is a presentation of \( \text{Atl} \) via generators and relations.

Proof. Obvious from Corollary 2.4.6 and Proposition 2.4.7.

Corollary 2.4.9. Each word \( w \in \text{Mor}(\text{a}\Delta) \) has a unique standard form.

Proof. Each \( \text{Atl} \) tangle has a unique decomposition as \( T_{III} \circ T_{II} \circ T_I \). Note \( T_{III} \) and \( T_I \) have unique decompositions as in Proposition 2.2.35 which correspond under the isomorphism of categories to decompositions as in Proposition 2.3.14. We are finished.

2.5 The Annular Category from Two Cyclic Categories

The Cyclic Category

In this subsection, we recover Jones’ result in \cite{Jon00} that there are two copies of (the opposite of) the cyclic category \( \text{c}\Delta^{\text{op}} \) in \( \text{a}\Delta \cong \text{Atl} \). We will recycle the notation \( t \) from Section 1. The definitions from this section are adapted from \cite{Lod98}.

Definition 2.5.1. Let \( \text{cAtl}^+ \) be the subcategory of \( \text{Atl} \) with objects \([n]\) for \( n \in \mathbb{N} \) such that for \( m, n \in \mathbb{N} \), \( \text{cAtl}(m,n) \) is the set of annular \((m,n)\)-tangles with no loops, only shaded caps, and only unshaded cups. Let \( \text{cAtl}^- \) be the image of \( \text{cAtl}^+ \) under the involution of \( \text{Atl} \), i.e. \( \text{cAtl}^-(m,n) \) is the set of annular \((m,n)\)-tangles with no loops, only unshaded caps, and only shaded cups.

Remark 2.5.2. Clearly \( \text{cAtl}^+ \cong \text{cAtl}^- \).
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Figure 2.19: Examples of morphisms in $c\text{Atl}^+$ and $c\text{Atl}^-$ respectively.

**Definition 2.5.3.** The opposite of the cyclic category $c\Delta^\text{op}$ is given by

- **Objects:** $[n]$ for $n \in \mathbb{Z}_{\geq 0}$ and
- **Morphisms:** generated by
  
  - $d_i: [n] \to [n-1]$ for $i = 0, \ldots, n$ where $n \geq 1$
  - $s_i: [n] \to [n+1]$ for $i = 0, \ldots, n$ where $n \geq 0$
  - $t: [n] \to [n]$ where $n \geq 0$

subject to the relations

1. $d_id_j = d_{j-1}d_i$ for $i < j$.
2. $s_is_j = s_{j+1}s_i$ for $i \leq j$,
3. $d_is_j = \begin{cases} s_{j-1}d_i & \text{if } i < j \\ \text{id}_{[n]} & \text{if } i = j, j+1 \\ s_jd_{i-1} & \text{if } i > j+1, \end{cases}$
4. $t^{n+1} = \text{id}_{[n]}$,
5. $d_it = td_{i-1}$ for $1 \leq i \leq n$, and
6. $s_it = ts_{i-1}$ for $1 \leq i \leq n$.

**Remark 2.5.4.** The opposite of the simplicial category $s\Delta^\text{op}$ is the subcategory of $c\Delta^\text{op}$ generated by the $d_i$’s and the $s_i$’s subject to relations (1)-(3).

**Remark 2.5.5.** Similar to Proposition 2.3.6, we have the additional relations in $c\Delta^\text{op}$ that $d_0t = d_n$ and $s_0t = t^2s_n$.

**Definition 2.5.6.** For $n \in \mathbb{Z}_{\geq 0}$, we define $s_{-1}: [n] \to [n+1]$ by $s_{-1} = ts_n$. This map is called the extra degeneracy.
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Remark 2.5.7. In [Lod98], Loday names this map \( s_{n+1} \). However, we will use the name \( s_{-1} \) considering Proposition 2.5.8, Corollary 2.5.16, and the fact that if \( R \) is a unital commutative ring, \( A \) is a unital \( R \)-algebra, and \( C_\bullet \) is the cyclic \( R \)-module (see Section 2.6) arising from the Hochschild complex with coefficients in \( A \), then \( C_n = A^{\otimes n+1} \), and

\[
\begin{align*}
s_{-1}(a_0 \otimes \cdots \otimes a_n) &= 1 \otimes a_0 \otimes \cdots \otimes a_n, \\
s_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \text{ for } 0 \leq i \leq n-1, \text{ and} \\
s_n(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_n \otimes 1.
\end{align*}
\]

Proposition 2.5.8. The following additional relations hold for \( s_{-1} \in c\Delta^{\text{op}}(n, n+1) \):

1. \( s_{-1}s_i = s_{i+1}s_{-1} \) for all \( i \geq 0 \),
2. \( d_is_{-1} = \begin{cases} 
\text{id}_n & \text{if } i = 0 \\
-1d_i^{-1} & \text{if } 1 \leq i \leq n \\
t & \text{if } i = n+1, \text{ and}
\end{cases} 
\)
3. \( s_0t = ts_{-1} \).

Proof. (1) Using relations (2) and (6), we get

\[
s_{-1}s_i = ts_{n+1}s_i = ts_is_n = s_{i+1}ts_n = s_{i+1}s_{-1}.
\]

(2) Using Remark 2.5.5, we have \( d_0s_{-1} = d_0ts_n = d_n = \text{id}_n \). If \( 1 \leq i \leq n \), then using relations (3) and (5), we have

\[
d_is_{-1} = d_i(ts_n = t\text{id}_n = t.
\]

Finally, \( d_{n+1}s_{-1} = d_{n+1}ts_n = td_n = t\text{id}_n = t \).

(3) Using Remark 2.5.5, we have \( s_0t = t^2s_n = ts_{-1} \).

Remark 2.5.9. We may now add \( s_{-1} \) to the list of generators of \( c\Delta^{\text{op}} \) after appropriately altering relations (3) and (6).

Proposition 2.5.10. Suppose \( w = h_r \cdots h_1 \) is a word on \( c\Delta^{\text{op}} \) in \( c\Delta^{\text{op}}(m, n) \) for \( m, n \in \mathbb{Z}_{\geq 0} \). Then there is a decomposition \( w = w_{I_{II}I_{II}}w_{I_{II}}w_I \) such that

1. \( w_I = d_{i_0} \cdots d_{i_1} \) with \( i_j > i_{j+1} \) for all \( j \in \{1, \ldots, a-1\} \).
2. \( w_{I_{II}} = t^k \) for some \( k \geq 0 \), and
3. \( w_{I_{III}} = s_{i_a} \cdots s_{i_1} \) with \( i_j < i_{j+1} \) for all \( j \in \{1, \ldots, b-1\} \).
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Proof. The proof is similar to Theorem 2.3.15, but much easier. We proceed by induction on \( r \). If \( r = 1 \), the result is trivial. Suppose \( r > 1 \) and the result holds for all words of shorter length. Apply the induction hypothesis to \( w' = h_{r-1} \cdots h_1 \) to get

\[
w' = u_{III}u_{II}u_I
\]

satisfying (1)-(3). There are three cases.

(T) Suppose \( h_r = t \). Set \( w_I = u_I \). Use relation (6) and Remark 2.5.5 to push \( t \) to the right of the \( s_i \)’s. Either it makes it all the way, or it disappears in the process. Define \( w_{II} \) accordingly. Order the \( s_i \)’s using relation (2) to get \( w_{III} \). We are finished.

(D) Suppose \( h_r = d_i \). Use relation (3) to push \( d_i \) to the right of the \( s_j \)’s. One of three possibilities occurs:

1. We only use the relation \( d_i s_j = s_j d_i \). Thus we can push \( d_i \) all the way to the right. Now push \( d_i \) right of the \( t \)’s using relation (5) and Remark 2.5.5. Order the \( s_j \)’s using relation (2) to get \( w_{III} \), define \( w_{II} \) in the obvious way, and reorder the \( d_i \)’s using relation (1) to get \( w_I \). We are finished.

2. We use the relation \( d_i s_j = id \), and \( d_i \) disappears. Set \( w_i = u_i \) for \( i \in \{I, II\} \), and order the \( s_j \)’s using relation (2) to get \( w_{III} \). We are finished.

3. We use the relation \( d_{n+1}s_{-1} = t \). We are now argue as in Case (T). We are finished.

(S) Suppose \( h_r = s_i \). Order \( s_iu_{III} \) using relation (2) to get \( w_{III} \), and set \( w_i = u_i \) for \( i \in \{I, II\} \). We are finished.

\( \square \)

Theorem 2.5.11. The following defines an injective functor \( H^+: c\Delta^{op} \to a\Delta \):

**Objects:** \( H^+([n]) = [n + 1] \) for \( n \in \mathbb{Z}_{\geq 0} \), and

**Morphisms:** Let \( n \in \mathbb{Z}_{\geq 0} \).

(D) For \( d_j \in c\Delta^{op}(n, n-1) \), set \( H^+(d_j) = \alpha_{2j+1} \in a\Delta(n+1, n) \).

(T) For \( t \in c\Delta^{op}(n, n) \), set \( H^+(t) = \tau \in a\Delta(n+1, n+1) \).

(S) For \( s_j \in c\Delta^{op}(n, n+1) \), set \( H^+(s_j) = \beta_{2j+2} \in a\Delta(n+1, n+2) \).

Proof. Clearly \( H^+ \) is a functor as the relations are satisfied. Injectivity follows immediately from Corollary 2.4.9 and Proposition 2.5.10. \( \square \)

Remark 2.5.12. Note that \( H^+(s_{-1}) = H^+(t s_n) = H^+(t) H^+(s_n) = \tau \beta_{2n+2} = \beta_{2n+4} \tau \).

Corollary 2.5.13. The image of \( F \circ H^+: c\Delta^{op} \to Atl \) is \( c\text{Atl}^+ \). Hence \( c\Delta^{op} \cong c\text{Atl}^+ \).
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Proof. It is clear $F \circ H^+$ is injective and lands in $\mathbf{cAtl}^+$ as all generators of $\mathbf{c}\Delta^{\text{op}}$ land in $\mathbf{cAtl}^+$. Surjectivity follows from Theorem 2.2.37. □

Corollary 2.5.14. A decomposition $w = w_{II}w_Iw_1$ as in Proposition 2.5.10 is unique.

Theorem 2.5.15. The following defines an injective functor $H^- : \mathbf{c}\Delta^{\text{op}} \rightarrow \mathbf{a}\Delta$:

Objects: $H^-(n) = [n + 1]$ for $n \in \mathbb{Z}_{\geq 0}$, and

Morphisms: Let $n \in \mathbb{Z}_{\geq 0}$.

(D) For $d_j \in \mathbf{c}\Delta^{\text{op}}(n, n - 1)$, set $H^-(d_j) = \alpha_{2j+2} \in \mathbf{a}\Delta(n + 1, n)$.

(T) For $t \in \mathbf{c}\Delta^{\text{op}}(n, n)$, set $H^-(t) = \tau \in \mathbf{a}\Delta([n + 1], [n + 1])$.

(S) For $s_j \in \mathbf{c}\Delta^{\text{op}}(n, n + 1)$, set $H^-(s_j) = \beta_{2j+3} \in \mathbf{a}\Delta(n + 1, n + 2)$.

Proof. Clearly $H^-$ is a functor as the relations are satisfied. Injectivity follows immediately from Corollary 2.4.9 and Proposition 2.5.10. □

Remark 2.5.16. Note that $H^-(s_{-1}) = H^-(s_n) = H^-(t)H^-(s_n) = \tau \beta_{2n+3} = \beta_1$.

Corollary 2.5.17. The image of $F \circ H^- : \mathbf{c}\Delta^{\text{op}} \rightarrow \mathbf{Atl}$ is $\mathbf{cAtl}^-$. Hence $\mathbf{c}\Delta^{\text{op}} \cong \mathbf{cAtl}^-$. 

Remark 2.5.18. $\mathbf{cAtl}^+$ and $\mathbf{cAtl}^-$ are exactly the two copies of $\mathbf{c}\Delta^{\text{op}}$ in $\mathbf{Atl}$ found by Jones in [Jon00].

Corollary 2.5.19. There is an isomorphism $\mathbf{c}\Delta \cong \mathbf{c}\Delta^{\text{op}}$.

Proof. We have $\mathbf{cAtl}^- \cong \mathbf{c}\Delta^{\text{op}} \cong \mathbf{cAtl}^+$. Note the involution in $\mathbf{Atl}$ is an isomorphism $\mathbf{cAtl}^+ \cong (\mathbf{cAtl}^-)^{\text{op}}$. The result follows. □

Augmenting the Cyclic Category

Recall from algebraic topology that the reduced (singular, simplicial, cellular) homology of a space $X$ is obtained by inserting an augmentation map $\varepsilon : C_0(X) \rightarrow \mathbb{Z}$ where $C_0(X)$ denotes the appropriate zero chains. In the language of the semi-simplicial category, we see that this is the same thing as looking at an augmented semi-simplicial abelian group, i.e., a functor from the opposite of the augmented semi-simplicial category, which is obtained from the opposite of the semi-simplicial category (see 2.5.4) by adding an object $[-1]$ and the generator $d_0 : [0] \rightarrow [-1]$ subject to the relation $d_id_j = d_{j-1}d_i$ for $i < j$.

$[-1] \leftarrow^{d_0} [0] \leftarrow^{d_0,d_1} [1] \leftarrow^{d_0,d_1,d_2} [2] \leftarrow^{d_0,d_1,d_2,d_3} \cdots$

This immediately raises the question of how one should augment the opposite of the cyclic category. The surprising answer comes from the symmetry arising from the extra degeneracy
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

We should add two objects, $[+]$ and $[-]$, and maps $d_0: [0] \to [+]$ and $s_{-1}: [-] \to [0]$ subject to the relations $d_i d_j = d_{j-1} d_i$ for $i < j$ and $s_i s_j = s_{j+1} s_i$ for $i \leq j$:

As $t: [0] \to [0]$ is the identity, we need not worry about the other relations. Under the isomorphism $\text{c}\Delta^{\text{op}} \cong \text{cAtl}^+$ described in the previous subsection, these maps should be represented by the following diagrams:

![Maps d_0: [0] \to [+] and s_{-1}: [-] \to [0]](image)

Figure 2.20: Maps $d_0: [0] \to [+]$ and $s_{-1}: [-] \to [0]$

Note that these morphisms satisfy the shading convention of $\text{cAtl}^+$ once we add $[0\pm]$ to the objects of $\text{cAtl}^+$. We cannot use just one object as we would then violate the shading convention and closed loops would arise. We will denote the augmented opposite of the cyclic category by $\widetilde{\text{c}\Delta^{\text{op}}}$. For our main result, we will also need to consider the augmented cyclic category $\widetilde{\text{c}\Delta}$, which is just the category $\widetilde{\text{c}\Delta}^{\text{op}}$ with the arrows switched.

**Pushouts of Small Categories**

Let $\text{Cat}$ be the category of small categories. Note that pushouts exist in $\text{Cat}$.

**Definition 2.5.20.** Suppose $A, B_1, B_2$ are small categories and $F_i: A \to B_i$ for $i = 1, 2$ are functors. Then the pushout of the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{F_1} & B_1 \\
| & F_2 | & \downarrow \\
B_2
\end{array}
$$

is the small category $C$ defined as follows:
Objects: \( \text{Ob}(C) \) is the pushout in \( \text{Set} \) of the diagram

\[
\begin{array}{ccc}
\text{Ob}(A) & \xrightarrow{F_1} & \text{Ob}(B_1) \\
\downarrow F_2 & & \downarrow G_1 \\
\text{Ob}(B_2) & \xrightarrow{G_2} & \text{Ob}(C)
\end{array}
\]

This defines maps \( G_i: \text{Ob}(B_i) \to \text{Ob}(C) \) for \( i = 1, 2 \).

Morphisms: For \( X, Y \in \text{Ob}(C) \), \( \text{Mor}(X,Y) \) is the set of all words of the form \( \varphi_n \circ \cdots \circ \varphi_1 \) such that

1. \( \varphi_i \in \text{Mor}(B_1) \cup \text{Mor}(B_2) \) for all \( i = 1, \ldots, n \),
2. the source of \( \varphi_1 \) is in \( G_1^{-1}(X) \cup G_2^{-1}(X) \) and the target of \( \varphi_n \) is in \( G_1^{-1}(Y) \cup G_2^{-1}(Y) \),
3. for all \( i = 1, \ldots, n-1 \), either
   i. the target of \( \varphi_i \) is the source of \( \varphi_{i+1} \), or
   ii. the target of \( \varphi_i \) is \( Z_i \in \text{im}(F_j) \subseteq B_j \) for some \( j \in \{1, 2\} \), and the source of \( \varphi_{i+1} \) is in \( F_k(F_j^{-1}(Z_i)) \) where \( k \neq j \).

subject to the relation \( F_1(\psi) = F_2(\psi) \) for every morphism \( \psi \in \text{Mor}(A) \).

Notation 2.5.21. In the sequel, we will need to discuss \( \tilde{c}\Delta \), the augmented cyclic category. In order that no confusion can arise, we will add a * to morphisms to emphasize the fact that they compose in the opposite order. For example, we have generators \( d^*_i \) satisfying the relation \( d^*_i d^*_i = d^*_i d^*_j \) for \( i < j \).

Definition 2.5.22. Define the small category/groupoid \( T \) by

Objects: \([n]\) for \( n \in \mathbb{Z}_{\geq 0} \cup \{\pm\} \)

Morphisms: Generated by \( t: [n] \to [n] \) subject to the relation \( t^{n+1} = \text{id}_{[n]} \) for \( n \in \mathbb{Z}_{\geq 0} \).

Definition 2.5.23. Let \( \text{PO} \) be the pushout in \( \text{Cat} \) of the following diagram:

\[
\begin{array}{ccc}
T & \xrightarrow{c\Delta^{\text{op}}} & \tilde{c}\Delta^{\text{op}} \\
\downarrow F_2 & & \downarrow G_1 \\
\tilde{c}\Delta & \xrightarrow{F_1} & \text{Ob}(C)
\end{array}
\]

where \( F_i([n]) = [n] \) for \( n \in \mathbb{Z}_{\geq 0} \cup \{\pm\} \) for \( i = 1, 2 \) and \( F_1(t) = t \) and \( F_2(t) = (t^*)^{-1} = (t^{-1})^* \).

Note that if \( c\Delta^{\text{op}} \) has generators \( d_i, s_i, t \) and \( c\Delta \) has generators \( d^*_i, s^*_i, t^* \), then \( \text{PO} \) is the category given by

Objects: \([n]\) for \( n \in \mathbb{Z}_{\geq 0} \cup \{\pm\} \) and
Morphisms: generated by

\[ d_0: [0] \to [-] \quad \text{and} \quad s_{-1}^*: [0] \to [-] \]
\[ s_{-1}: [-] \to [0] \quad \text{and} \quad d_0^*: [0] \to [-] \]
\[ d_i, s_{i-1}^*: [n] \to [n-1] \quad \text{for} \quad i = 0, \ldots, n \quad \text{where} \quad n \geq 1 \]
\[ s_i, d_{i+1}^*: [n] \to [n+1] \quad \text{for} \quad i = -1, \ldots, n \quad \text{where} \quad n \geq 0 \]
\[ t: [n] \to [n] \quad \text{where} \quad n \geq 0 \]

subject to the relations

1. \[ d_i d_j = d_{j-1} d_i \quad \text{and} \quad s_i^* s_j^* = s_{j-1}^* s_i^* \quad \text{for} \quad i < j, \]
2. \[ s_i s_j = s_{j+1} s_i \quad \text{and} \quad d_i^* d_j^* = d_{j+1}^* d_i^* \quad \text{for} \quad i \leq j, \]
3. \[ d_i s_j = \begin{cases} s_{j-1} d_i & \text{if} \quad i < j \\ \text{id}_n & \text{if} \quad i = j, j+1 \end{cases} \quad \text{and} \quad s_{i-1}^* d_j^* = \begin{cases} d_{j-1}^* s_{i-1}^* & \text{if} \quad i < j-1 \\ \text{id}_n & \text{if} \quad i = j, j+1 \\ d_j^* s_{i-2}^* & \text{if} \quad i > j+1, \end{cases} \]
4. \[ t'^{n+1} = \text{id}_n, \]
5. \[ d_i t = t d_{i-1} \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad s_i^* t = t s_{i-1}^* \quad \text{for} \quad 0 \leq i \leq n, \quad \text{and} \]
6. \[ s_i t = t s_{i-1} \quad \text{for} \quad 0 \leq i \leq n \quad \text{and} \quad d_i^* t = t d_{i-1}^* \quad \text{for} \quad 1 \leq i \leq n. \]

Note that \( t = (t^*)^{-1} \) as \( PO \) is the pushout, so \( t^* \) does not appear in the above list.

**Remark 2.5.24.** Note that \( PO \) is involutive using the obvious involution as hinted by the \( * \)-notation.

**Definition 2.5.25.** Let \( PO(\delta_+, \delta_-) \) be the small category obtained from \( PO \) by adding generating morphisms \( \delta^\pm: [n] \to [n] \) for all \( n \in \mathbb{Z}_{\geq 0} \cup \{\pm\} \) which commute with all other morphisms. The maps \( \delta^\pm \) are called the coupling constants.

**Remark 2.5.26.** Note that \( PO(\delta_+, \delta_-) \) is involutive if we define \( (\delta^\pm)^* = \delta^\pm \).

**Theorem 2.5.27.** \( a\Delta \) is isomorphic to the category \( Q \) obtained from \( PO(\delta_+, \delta_-) \) with the additional relations

1. \[ d_i s_j^* = \begin{cases} s_{j-1}^* d_i & \text{if} \quad i < j \\ s_j^* d_{i+1}^* & \text{if} \quad j > i \end{cases} \]
2. \[ d_i d_j^* = \begin{cases} d_{j-1}^* d_i & \text{if} \quad i < j \\ \delta_- & \text{if} \quad i = j \end{cases} \]
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

37

(3) \( s_i^* s_j = \begin{cases} 
  s_{j-1} s_i^* & \text{if } i < j \\
  \delta_+ & \text{if } i = j 
\end{cases} \)

Proof. Define a map \( \Psi : a\Delta \rightarrow Q \) by

Objects: Define \( \Psi([0\pm]) = [\pm] \). For \( n \geq 1 \), define \( \Psi([n]) = [n-1] \).

Morphisms: We define \( \Psi \) on primitive morphisms:

(A) Define \( \Psi(\alpha_1 \in a\Delta(1,0+)) = d_0 \in Q(0,+) \) and \( \Psi(\alpha_2 \in a\Delta(1,0-)) = s_{-1}^* \in Q(0,-) \).

For \( n \geq 2 \), define

\[
\Psi(\alpha_i \in a\Delta(n,n-1)) = \begin{cases} 
  s_{(i-3)/2}^* \in Q(n-1,n-2) & \text{if } i \text{ is odd} \\
  d_{(i-2)/2} \in Q(n-1,n-2) & \text{if } i \text{ is even.}
\end{cases}
\]

(B) Define \( \Psi(\beta_1 \in a\Delta(0+,1)) = s_{-1} \in Q(-,0) \) and \( \Psi(\beta_2 \in a\Delta(0-,1)) = d_0^* \in Q(+,0) \).

For \( n \geq 1 \), define

\[
\Psi(\beta_i \in a\Delta(n,n+1)) = \begin{cases} 
  s_{(i-3)/2} \in Q(n-1,n) & \text{if } i \text{ is odd} \\
  d_{(i-2)/2}^* \in Q(n-1,n) & \text{if } i \text{ is even.}
\end{cases}
\]

(T) For \( n \geq 1 \), define \( \Psi(\tau \in a\Delta(n,n)) = t \in Q(n-1,n-1) \).

(D) Define \( \Psi(\delta_\pm) = \delta_\pm \).

One checks \( \Psi \) is a well defined isomorphism by showing the relations match up.

\[ \square \]

Remarks 2.5.28. (1) The above relations are called the coupling relations.

(2) Usually we study representations of \( c\Delta \) and \( a\Delta \) in abelian categories and the coupling constants are multiplication by scalars. These scalars can be built into the coupling relations in our abelian category without first defining \( \text{PO}(\delta_+, \delta_-) \). Hence an annular object in an abelian category (see Section 2.6) is obtained from the pushout of two cyclic objects over a \( \text{T} \)-object and then quotienting out by the coupling relations.

(3) Another way to skip passing to \( \text{PO}(\delta_+, \delta_-) \) is to take the linearization of all our categories over some unital commutative ring \( R \) (make the morphism sets \( R \)-modules) and choose scalars \( \delta_\pm \) for the coupling relations.

2.6 Annular Objects

Definition 2.6.1. An annular object in an arbitrary category \( C \) is a functor \( a\Delta \rightarrow C \). A cyclic object is a functor \( c\Delta^{op} \rightarrow C \). If \( C \) is an abelian category and \( X_\bullet \) is an annular, respectively cyclic, object, we replace \( X_\bullet(\tau \in a\Delta(n,n)) \) with \( (-1)^{n-1} X_\bullet(\tau) \), respectively we replace \( X_\bullet(t \in c\Delta^{op}(n,n)) \) with \( (-1)^n X_\bullet(t) \), to account for the sign of the cyclic permutation.
CHAPTER 2. A CYCLIC APPROACH TO THE ANNULAR TEMPERLEY-LIEB CATEGORY

Remarks 2.6.2. Each annular object has two restrictions to cyclic objects.

Notation 2.6.3. Usually such a functor is denoted with a bullet subscript, e.g. $X_\bullet$. If $X_\bullet$ is such a functor, we will use the following standard notation:

1. $X_\bullet([n]) = X_n$ for $n \in \mathbb{Z}_{\geq 0}$ and $X_\bullet([0\pm]) = X_\pm$ where applicable.

2. $X_\bullet(\varphi) = \varphi$, i.e. we will use the same notation for the images of the morphisms in the category $C$.

Note 2.6.4. For an annular object in an abelian category, relations (4), (5), and (6) become

1. $\alpha_i \tau = -\tau \alpha_{i-2}$ for $i \geq 3$,

2. $\beta_i \tau = -\tau \beta_{i-2}$ for $i \geq 3$, and

3. if $\alpha_i \beta_j: [n] \to [n]$ with $n \geq 2$ and $(i, j) = (1, 2n + 2), (2n + 2, 1)$, then $\alpha_1 \beta_{2n+2} = (-1)^{n-1} \tau^{-1}$ and $\alpha_{2n+2} \beta_1 = (-1)^{n-1} \tau$.

Proposition 2.3.6 becomes

1. $\alpha_1 \tau = (-1)^{n-1} \alpha_{2n-1}$ and $\alpha_2 \tau = (-1)^{n-1} \alpha_{2n}$

2. $\tau \beta_{2n+1} = (-1)^n \beta_1$, $\tau \beta_{2n+2} = (-1)^n \beta_2$, and

3. $\beta_1 \tau = (-1)^{n-1} \tau^2 \beta_{2n-1}$ and $\beta_2 \tau = (-1)^{n-1} \tau^2 \beta_{2n}$.

Note 2.6.5. For a cyclic object in an abelian category, relations (5) and (6) become

1. $d_i t = -t d_{i-1}$ for $i \geq 1$ and

2. $s_i t = -t s_{i-1}$ for $i \geq 1$.

Following Remark 2.5.5, we have

(i) $d_0 t = (-1)^n d_n$ and

(ii) $s_0 t = (-1)^n t^2 s_n$.

Definition 2.5.6 becomes $s_{-1} = (-1)^{n+1} t s_n$. Parts (2) and (3) of Proposition 2.5.8 become

1. $d_{n+1} s_{-1} = (-1)^n t$ and

2. $s_0 t = -t s_{-1}$.

Remark 2.6.6. The necessity of this sign convention becomes apparent in calculations with Connes’ boundary map (see 2.6.19 and 2.6.20).

Definition 2.6.7. Let $\mathcal{C}$ be an involutive category and suppose $X_\bullet : a \Delta \to \mathcal{C}$ is an annular object in $\mathcal{C}$. Then $X_\bullet^* : a \Delta \to \mathcal{C}$ is also an annular object in $\mathcal{C}$ where
Objects: $X^*([n]) = X_n$ for all $n \in \mathbb{N} \cup \{0\pm\}$, and
Morphisms: $X^*(w) = X^*(w^*)^*$ for all $w \in \text{Mor}(a\Delta)$.

If $C$ is abelian, then $X^*$ still satisfies the sign convention.

**Remark 2.6.8.** The representation theory of $\text{Atl}$ was studied extensively by Graham and Lehrer in [GL98] and Jones in [Jon01]. In Definition/Theorem 2.2 in [Pet10], Peters gives a good summary of the case of an annular $C^*$-Hilbert module where $\delta_{\pm}$ is given by multiplication by $\delta > 2$.

### Homologies of Annular Modules

As the semi-simplicial, simplicial, and cyclic categories live inside $a\Delta$, we can define Hochschild and cyclic homologies of annular objects in abelian categories. We will focus on annular modules and leave the generalization to an arbitrary abelian category to the reader. Fix a unital commutative ring $R$.

**Definition 2.6.9.** Given a semi-simplicial $R$-module $M_\bullet$, define the Hochschild boundary $b: M_n \to M_{n-1}$ for $n \geq 1$ by

$$b = \sum_{i=0}^{n} (-1)^i d_i.$$ 

The Hochschild homology of $M_\bullet$ is

$$HH_n(M_\bullet, b) = \ker(b) / \text{im}(b)$$

for $n \geq 0$, where we set $M_{-1} = 0$, and $b: M_0 \to M_{-1}$ is the zero map.

**Remark 2.6.10.** As an annular $R$-module is a semi-simplicial $R$-module in two ways, we will have two Hochschild boundaries.

**Definition 2.6.11.** Suppose $X_\bullet$ is an annular $R$-module. Let $X_{\pm}^\bullet$ be the cyclic object obtained from $X_\bullet$ by restricting $X_\bullet$ to $G(\text{cAtl}^\pm)$. For $n \geq 1$, define

$$HH_n^\pm(X_\bullet) = HH_{n-1}^\pm(X_{\pm}^\bullet).$$

**Remark 2.6.12.** The Hochschild boundaries of $X_{\pm}^\bullet$ for $n \geq 2$ are

$$b_+ = \sum_{i=0}^{n-1} (-1)^i \alpha_{2i+1} \quad \text{and} \quad b_- = \sum_{i=0}^{n-1} (-1)^i \alpha_{2i+2}.$$

**Definition 2.6.13.** The above definition does not take into account $X_{\pm}$. We may define the reduced Hochschild homology by looking at the corresponding augmented cyclic objects
(see Subsection 2.5). Define $b_\pm : X_1 \to X_\pm$ by $b_+ = \alpha_1 : X_1 \to X_+$ and $b_- = \alpha_2 : X_1 \to X_-$. Define the reduced Hochschild homology of $X_\bullet$ as follows:

$$\widetilde{HH}_n^\pm(X_\bullet) = HH_n^\pm(X_\bullet) \text{ for } n \geq 2,$$

$$\widetilde{HH}_1^\pm(X_\bullet) = \ker(b_\pm)/\text{im}(b_\pm), \text{ and}$$

$$\widetilde{HH}_0^\pm(X_\bullet) = \text{coker}(b_\pm).$$

Remark 2.6.14. The content of the next proposition was found by Jones in [Jon00].

Proposition 2.6.15. Let $X_\bullet$ be an annular $R$-module. Then for all $n \geq 1$,

$$\beta_1 b_+ + b_+ \beta_1 = \delta_+ \text{id}_{X_n} \text{ and}$$

$$\beta_2 b_- + b_- \beta_2 = \delta_- \text{id}_{X_n},$$

and when $n = \pm$,

$$b_+ \beta_1 = \delta_+ \text{id}_{X_+} \text{ and}$$

$$b_- \beta_2 = \delta_- \text{id}_{X_-}.$$

Proof. This follows immediately from relation 6.

Corollary 2.6.16. If $\delta_\pm$ is multiplication by an element of $R^\times$, the group of units of $R$, then $\widetilde{HH}_n^\pm(X_\bullet) = 0$ for all $n \geq 0$.

Corollary 2.6.17. Let $N \subset M$ be an extremal, finite index $II_1$-subfactor, and let $X_\bullet$ be the annular $\mathbb{C}$-module given by its tower of relative commutants (see [Jon99], [Jon01]). Then $\widetilde{HH}_n^\pm(X_\bullet) = 0$ for all $n \geq 0$.

Example 2.6.18 ($TL_\bullet(\mathbb{Z},0)$). When $\delta_\pm \notin R^\times$, we can have non-trivial homology. For example, for $n \in \mathbb{N} \cup \{0\}$, let $TL_n(\mathbb{Z},0)$ be the set of $\mathbb{Z}$-linear combinations of planar $n$-tangles with no input disks and no loops (adjust the definition of an annular $n$-tangle so that there is no $D_1$). The action of $T \in a\Delta(m,n)$ on $S \in TL_m(\mathbb{Z},0)$ is given by tangle composition $F(T) \circ S$ with the additional requirement that if there are any closed loops, we get zero. We then extend this action $\mathbb{Z}$-linearly. Then $HH_n^\pm(TL_\bullet(\mathbb{Z},0)) \neq 0$ for all $n \geq 0$. In fact, the class of the planar $n$-tangle with only shaded, respectively unshaded, cups is a nontrivial element in $HH_n^\pm(TL_\bullet(\mathbb{Z},0))$ respectively. Clearly all such tangles are in $\ker(b_\pm)$. However, it is only possible to get an even multiple of this tangle in $\text{im}(b_\pm)$. If a shaded region is capped off by an $\alpha_i$ to make a cup, there must be two ways of doing so. Using MAGMA [BCP97], the author has calculated the first few ($+$) reduced Hochschild homology groups of $TL_\bullet(\mathbb{Z},0)$ to be

$$\widetilde{HH}_0^+ = \widetilde{HH}_1^+ = \mathbb{Z},$$
$$HH_2^+ = HH_3^+ = \mathbb{Z}/2,$$
$$HH_4^+ = HH_5^+ = \mathbb{Z}/6, \text{ and}$$
$$HH_6^+ = HH_7^+ = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$
For $n \geq 2$, the class of the tangle described above contributes a copy of $\mathbb{Z}/2\mathbb{Z}$. The question still remains whether this parity continues.

**Definition 2.6.19.** Given a cyclic $R$-module $C\bullet$, define the cyclic bicomplex $BC^{\ast\ast}(C\bullet)$ be the bicomplex

$$
\begin{array}{cccccccc}
  & b & & b & & b & & b & \\
  & C_3 & B & C_2 & B & C_1 & B & C_0 \\
  b & & b & & b & & b & \\
  B & C_2 & B & C_1 & B & C_0 \\
  b & & b & & b & \\
  B & C_1 & B & C_0 \\
  b & \\
  B & C_0 \\
  b & \\

  C_0 \\
\end{array}
$$

where $b$ is the Hochschild boundary obtained by looking at the corresponding semi-simplicial $R$-module, $B = (1 - t)s_{-1}N : C_n \rightarrow C_{n+1}$ is Connes’ boundary map, and

$$
N = \sum_{i=0}^{n} t^i.
$$

Recall $s_{-1} = (-1)^{n+1}ts_n$ is the extra degeneracy. The cyclic homology of $C\bullet$ is given by $HC_n(C\bullet) = H_n(\text{Tot}(BC^{\ast\ast}(C\bullet)))$.

**Remark 2.6.20.** In order for $BC^{\ast\ast}(C\bullet)$ to be a bicomplex, we need $b^2$, $B^2$, and $bB + Bb$ to equal zero. While the first two are trivial, we must use Loday’s sign convention to get the third. Setting

$$
b' = \sum_{i=0}^{n-1} (-1)^id_i : C_n \rightarrow C_{n-1},
$$
we have \( b(1 - t) = (1 - t)b' \), \( b's_{-1} + s_{-1}b' = \text{id} \), and \( b'N = Nb \), so

\[
bB + Bb = b(1 - t)s_{-1}N + (1 - t)s_{-1}Nb = (1 - t)(b's_{-1} + s_{-1}b')N = (1 - t)N = 0.
\]

Without this sign convention, we no longer have \( bB + Bb = 0 \).

**Definition 2.6.21.** Suppose \( X\) is an annular \( R \)-module. Then \( X \) becomes a cyclic module in two ways, so we have two cyclic homologies to study. For \( n \geq 1 \), define \( HC^\pm_n(X) = HC_{n-1}^\pm(X) \).

**Remark 2.6.22.** For \( n \geq 1 \), \( B^\pm : X_n \to X_{n+1} \) is given by

\[
B^+_n = (-1)^n(1 - \tau)(\tau\beta_{2n}) \sum_{i=0}^{n-1} \tau^i = (-1)^n(1 - \tau)(\beta_{2n+2}) \sum_{i=0}^{n-1} \tau^i \\
B^-_n = (-1)^n(1 - \tau)\beta_1 \sum_{i=0}^{n-1} \tau^i
\]

as the two extra degeneracies for \( G(cAt^\pm) \) are \((-1)^n\tau\beta_{2n}\) and \((-1)^n\beta_1\) respectively.

**Corollary 2.6.23.** If \( \delta^\pm \) is multiplication by an element of \( R^\times \), the group of units of \( R \), then \( HC^\pm_n(X) = R \) for all odd \( n \geq 1 \) and \( HC^\pm_n(X) = 0 \) for all even \( n \geq 2 \).

**Corollary 2.6.24.** Let \( N \subset M \) be an extremal, finite index \( II_1 \)-subfactor, and let \( X \) be the annular \( C \)-module given by its tower of relative commutants. Then \( HC^\pm_n(X) = C \) for all odd \( n \geq 1 \) and \( HC^\pm_n(X) = 0 \) for all even \( n \geq 2 \).

**Example 2.6.25.** Once again using MAGMA [BCP97], the author has calculated the first few (+) cyclic homology groups of \( TL\) to be

\[
HC^+_1 = \mathbb{Z} \\
HC^+_2 = \mathbb{Z}/2 \\
HC^+_3 = \mathbb{Z}/2 \oplus \mathbb{Z} \\
HC^+_4 = \mathbb{Z}/2 \oplus \mathbb{Z}/6 \\
HC^+_5 = \mathbb{Z}/2 \oplus \mathbb{Z}/6 \oplus \mathbb{Z}, \text{ and} \\
HC^+_6 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6.
\]
Chapter 3

The embedding theorem for finite depth subfactor planar algebras

3.1 Introduction

A powerful method of construction of subfactors is the use of commuting squares, which are systems of four finite dimensional von Neumann algebras

\[ A_{1,0} \subset A_{1,1} \]
\[ \cup \quad \cup \]
\[ A_{0,0} \subset A_{0,1} \]

included as above, with a faithful trace on \( A_{1,1} \) so that \( A_{1,0} \) and \( A_{0,1} \) are orthogonal modulo their intersection \( A_{0,0} \).

One iterates the basic construction of [Jon83] for the inclusions \( A_{i,j} \subset A_{i,j+1} \) and \( A_{i,j} \subset A_{i+1,j} \) to obtain a tower of inclusions \( A_{0,n} \subset A_{1,n} \). By a lovely compactness argument of Ocneanu [JS97],[EK98], the standard invariant, or higher relative commutants, of the inductive limit inclusion \( A_{0,\infty} \subset A_{1,\infty} \) are the algebras \( A_{0,1} \cap A_{n,0} \). Thus once bases have been chosen, the calculation of the relative commutants is a matter of elementary linear algebra.

It was to formalise this calculation that planar algebras were first introduced [Jon99]. Finite dimensional inclusions are given by certain graphs (Bratteli diagrams), and in [Jon00], a planar algebra associated purely combinatorially to a bipartite graph was introduced so that it is rather obviously the tower of relative commutants for an inclusion \( B_0 \subset B_1 \) having the graph as its Bratteli diagram. But because Ocneanu’s notion of connection was never completely formalised in [Jon99], it was NOT proved that the planar algebra coming from a commuting square via Ocneanu compactness is a planar subalgebra of the one defined in [Jon00] for the graph of the inclusion \( A_{0,0} \subset A_{1,0} \).

Meanwhile the theory of planar algebras grew in its own right and a new method of constructing subfactors evolved by looking at planar subalgebras of a given planar algebra.
Now if a subfactor is of finite depth, then by [Pop90], there is a commuting square that constructs a hyperfinite model of it. Moreover the inclusion $A_{0,0} \subset A_{1,0}$ for this canonical commuting square has Bratteli diagram given by the so-called principal graph, which is a powerful subfactor invariant. Thus if the result of the previous paragraph had been proved, it would have implied the following theorem, which is the main result of this paper:

**Theorem.** A finite depth subfactor planar algebra is a planar subalgebra of the bipartite graph planar algebra of its principal graph.

(See [MPS10] for the definition of the principal graph of a planar algebra.)

We prove this result with the interesting twist of not using connections. In particular, our proof does not invoke the dual principal graph, which is perhaps rather surprising.

There are three steps to our proof. The first step, Section 3.2, is to define a canonical planar $*$-algebra structure on the tower of relative commutants from a connected unital inclusion of finite dimensional $C^*$-algebras whose Bratteli diagram is a given graph. We call this the canonical planar $*$-algebra associated to the inclusion. We do this in more generality, replacing finite dimensionality by a strong Markov property (see Definition 3.2.8), because it is no harder and should have applications.

The second step, Section 3.3, is to identify the canonical planar $*$-algebra with the bipartite graph planar algebra of [Jon00] in the finite dimensional case. Loops on the Bratteli diagram for the inclusion give bases for the relative commutants, so the isomorphism is constructed by choosing bases for the vector spaces in the canonical planar $*$-algebra.

Finally, in Section 3.4, we construct the embedding map as follows: given a finite depth subfactor planar algebra $Q_\bullet$, pick $2r$ suitably large so that the inclusion $Q_{2r,+} \subset Q_{2r+1,+) \subset (Q_{2r+2,+}, e_{2r+1})$ is standard, i.e., isomorphic to the basic construction. Set $M_0 = Q_{2r,+}$ and $M_1 = Q_{2r+1,+}$, and let $P_\bullet$ be the canonical planar $*$-algebra $P_\bullet$ associated to the inclusion $M_0 \subset M_1$. We prove in Theorem 3.4.1 that the map $Q_\bullet \rightarrow P_\bullet$ given by adding $2r$ or $2r + 1$ strings on the left, depending on whether we are in $Q_{n,+}$ or $Q_{n,-}$ respectively, is an inclusion of planar algebras.

While this paper was being written, Morrison and Walker in [MW10] produced a totally different proof which constructs an embedding directly from the planar algebra $Q_\bullet$ without the use of algebra towers and centralisers. Their method also has the advantage that it applies to infinite depth subfactor planar algebras without alteration!
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3.2 The canonical planar ∗-algebra of a strongly Markov inclusion of finite von Neumann algebras

After defining the notion of a strongly Markov inclusion of finite von Neumann algebras, we show the basic construction is also strongly Markov with the same (Watatani) index. We then define the canonical planar ∗-algebra associated to a strongly Markov inclusion.

Many results of this section can be found in [Jon83], [PP86], [Wat90], [Jol90], [Pop94], [Bis97], and [Bur03], but our treatment differs slightly, so we provide some proofs for the reader’s convenience.

Bases, traces, and strongly Markov inclusions

Notation 3.2.1. Throughout this paper, a trace on a finite von Neumann algebra means a faithful, normal, tracial state unless otherwise specified. We will write $M_0 \subset (M_1, \text{tr}_1)$ to mean $M_0 \subset M_1$ is an inclusion of finite von Neumann algebras where $\text{tr}_1$ is a trace on $M_1$. We set $\text{tr}_0 = \text{tr}_1|_{M_0}$.

Let $M_0 \subset (M_1, \text{tr}_1)$. Let $M_2 = \langle M_1, e_1 \rangle = JM_0^*J \subset B(L^2(M_1, \text{tr}_1))$ be the basic construction, where $e_1$ is the Jones projection with range $L^2(M_0, \text{tr}_0)$, and $J: L^2(M_1, \text{tr}_1) \to L^2(M_1, \text{tr}_1)$ is the antilinear unitary given by the antilinear extension of $x\Omega \mapsto x^*\Omega$, where $\Omega \in L^2(M_1, \text{tr}_1)$ is the image of $1 \in M_1$.

Recall that there is a unique trace-preserving conditional expectation $E_{M_0}: M_1 \to M_0$ determined by $\text{tr}_1(xy) = \text{tr}_0(E_{M_0}(x)y)$ for all $x \in M_1$ and $y \in M_0$, i.e., $E_{M_0}$ is the (Banach) adjoint of the inclusion of preduals $(M_0)_* \to (M_1)_*$ [Tak02]. The conditional expectation satisfies $e_1(x\Omega) = E_{M_0}(x)\Omega$ for all $x \in M_1$.

The following proposition is straightforward:

Proposition 3.2.2. The following are equivalent for a finite subset $B = \{b\} \subset M_1$:

(i) $1 = \sum_{b \in B} be_1b^*$,

(ii) $x = \sum_{b \in B} bE_{M_0}(b^*x)$ for all $x \in M_1$, and

(iii) $x = \sum_{b \in B} E_{M_0}(xb)b^*$ for all $x \in M_1$.

Definition 3.2.3. A Pimsner-Popa basis for $M_1$ over $M_0$ is a finite subset $B = \{b\} \subset M_1$ for which the conditions in Proposition 3.2.2 hold.
CHAPTER 3.  THE EMBEDDING THEOREM FOR FINITE DEPTH SUBFACTOR PLANAR ALGEBRAS

We refer the reader to [Wat90] for the proof of the following:

Proposition 3.2.4.  The following are equivalent:

(i) There is a Pimsner-Popa basis for $M_1$ over $M_0$,
(ii) $M_1 \otimes_{M_0} M_1 \rightarrow M_2$ by $x \otimes y \mapsto xe_1y$ is an $M_1 - M_1$ bimodule isomorphism, and
(iii) $M_2 = M_1 e_1 M_1$.

Remark 3.2.5.  $M_1 \otimes_{M_0} M_1$ is a $*$-algebra with multiplication $(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 \otimes E_{M_0}(y_1 x_2)y_2$ and adjoint $(x \otimes y)^* = y^* \otimes x^*$. If there is a Pimsner-Popa basis for $M_1$ over $M_0$, the sum $\sum_{b \in B} b \otimes b^*$ is independent of the choice of basis and is the identity. (We will renormalize in Proposition 3.2.25.)

Definition 3.2.6 ([Wat90]).  If there is a Pimsner-Popa basis $B = \{b\}$ for $M_1$ over $M_0$, then we define the (Watatani) index $[M_1 : M_0] = \sum_{b \in B} bb^*$, which is independent of the choice of basis.

Definition 3.2.7.  Recall from [Pop94] that $M_2$ has a canonical faithful, normal, semifinite trace $\text{Tr}_2$ which is the extension of the map $xe_1y \mapsto \text{tr}_1(xy)$ for $x, y \in M_1$.

Definition 3.2.8.  An inclusion $M_0 \subset (M_1, \text{tr}_1)$ of finite von Neumann algebras is called Markov if it satisfies the Markov property:

(1) $\text{Tr}_2$ is finite with $\text{Tr}_2(1^{-1} \text{Tr}_2 |_{M_1} = \text{tr}_1$.

A Markov inclusion is called strongly Markov if

(2) there is a Pimsner-Popa basis for $M_1$ over $M_0$.

Remark 3.2.9.  Markov inclusions have been studied by Jolissaint [Jol90], Pimsner, Popa [PP86], [Pop94], and more. In [Jol90], Jolissaint showed that condition (1) implies condition (2) when the centers are atomic and the inclusion is connected, i.e., $Z(M_0) \cap Z(M_1) = M'_1 \cap M_0$ is one dimensional. It is unknown to the authors at this point whether condition (1) implies condition (2) for connected inclusions with diffuse centers.

The adjective “strongly” in the term “strongly Markov” comes from Definition 3.6 in [BDH88], where they define the notion of “fortement d’indice fini” for a conditional expectation. This notion translates as the existence of a finite Pimsner-Popa basis.

Remark 3.2.10.  Recall from [Pop94] that $\text{Tr}_2(1^{-1} \text{Tr}_2$ extends $\text{tr}_1$ if and only if $\text{Tr}_2(1) = [M_1 : M_0] \in [1, \infty)$.

Examples 3.2.11.  (1) A finite Jones index inclusion of $II_1$-factors with the unique trace is strongly Markov, and the Watatani index is equal to the Jones index.
CHAPTER 3. THE EMBEDDING THEOREM FOR FINITE DEPTH SUBFACTOR
PLANAR ALGEBRAS

(2) A connected, unital inclusion of finite dimensional $C^*$-algebras with the Markov trace is strongly Markov, and the index is equal to $\|A^TA\|$ where $A$ is the bipartite adjacency matrix for the Bratteli diagram of the inclusion.

Suppose $M_0 \subset (M_1, \text{tr}_1)$ is strongly Markov. Then $M_2$ is finite and $\text{tr}_2 = [M_1 : M_0]^{-1} \text{tr}_2$ extends $\text{tr}_1$, so we may iterate the basic construction for $M_1 \subset (M_2, \text{tr}_2)$. Let $M_3 = \langle M_2, e_2 \rangle \subset B(L^2(M_2, \text{tr}_2))$, where $e_2$ is the Jones projection with range $L^2(M_1, \text{tr}_1)$. Let $\text{Tr}_3$ be the canonical faithful, normal, semifinite trace on $M_3$ (see Definition 3.2.7). The following lemma is straightforward:

Lemma 3.2.12. (1) The conditional expectation $E_{M_1}: M_2 \to M_1$ is given by $E_{M_1}(xe_1y) = xy$.

(2) $e_1e_2e_1 = [M_1 : M_0]^{-1}e_1$ and $e_2e_1e_2 = [M_1 : M_0]^{-1}e_2$, and

(3) if $B$ is a Pimsner-Popa basis for $M_1$ over $M_0$, then $\{[M_1 : M_0]^{1/2}be_1 | b \in B \}$ is a Pimsner-Popa basis for $M_2$ over $M_1$.

Theorem 3.2.13. $M_1 \subset (M_2, \text{tr}_2)$ is strongly Markov and $[M_2 : M_1] = [M_1 : M_0]$.

Proof. Note $M_3 = M_2 e_2 M_2$ by Proposition 3.2.4 and Lemma 3.2.12, so the canonical trace $\text{Tr}_3$ on $M_3$ is finite. By Definition 3.2.7 and Lemma 3.2.12, if $x \in M_2$,

$$\text{Tr}_3(x) = [M_1 : M_0] \sum_{b \in B} \text{Tr}_3(xbe_1e_2e_1b^*) = [M_1 : M_0] \sum_{b \in B} \text{tr}_2(xbe_1b^*) = [M_1 : M_0] \text{tr}_2(x).$$

Hence $[M_2 : M_1] = \text{Tr}_3(1) = [M_1 : M_0]$, and $\text{tr}_3 = [M_1 : M_0]^{-1} \text{Tr}_3$ extends $\text{tr}_2$. \qed

Definition 3.2.14. Suppose $P \subset B(L^2(M_1, \text{tr}_1))$ is a von Neumann algebra containing $M_1$, $\text{tr}_P$ is a trace on $P$ extending $\text{tr}_1$, and $p$ is a projection in $P$. We say the inclusion $M_0 \subset M_1 \subset (P, \text{tr}_P, p)$ is standard if there is an isomorphism of von Neumann algebras $\varphi: P \to M_2$ such that $\varphi|_{M_1} = \text{id}_{M_1}$, $\text{tr}_P = \text{tr}_2 \circ \varphi$, and $\varphi(p) = e_1$.

The following lemma, which is an alteration of Lemma 5.8 of [Jol90] and uses ideas from Lemma 5.3.1 in [JS97], allows us to identify when inclusions are standard:

Lemma 3.2.15. Suppose $M_0 \subset M_1 \subset (P, \text{tr}_P, p)$ such that

(1) $pmp = E_{M_0}(m)p$ for all $m \in M_1$, and

(2) $E_{M_1}(p) = [M_1 : M_0]^{-1}$.

Then $\psi: M_1 \otimes_{M_0} M_1 \to M_1 p M_1$ by $x \otimes y \mapsto xpy$ is an $M_1$-bilinear isomorphism of $*$-algebras. Hence $\varphi: M_1 e_1 M_1 \to M_1 p M_1$ by $xe_1 y \mapsto xpy$ is an isomorphism of $*$-algebras. Moreover, if

(3) $P = M_1 p M_1$,  

then $M_0 \subset M_1 \subset (P, \text{tr}_P, p)$ is standard via $\varphi$. Conversely, if $M_0 \subset M_1 \subset (P, \text{tr}_P, p)$ is standard, then (1), (2), and (3) hold.

Proof. First, note that $px = xp$ for all $x \in M_0$ by (1), and the map $M_1 \to M_1 p$ by $y \mapsto yp$ is injective by (2). Clearly $\psi$ is surjective and preserves the $*$-algebra structure. Suppose

$$\psi \left( \sum_{i=1}^{k} x_i \otimes y_i \right) = \sum_{i=1}^{k} x_i py_i = 0.$$

Then for all $x, y \in M_1$,

$$px \left( \sum_{i=1}^{k} x_i py_i \right) yp = \left( \sum_{i=1}^{k} E_{M_0}(xx_i)E_{M_0}(y_iy) \right) p = 0 \implies \sum_{i=1}^{k} E_{M_0}(xx_i)E_{M_0}(y_iy) = 0.$$

If $B = \{ b \}$ is a Pimsner-Popa basis for $M_1$ over $M_0$, by Remark 3.2.5,

$$\sum_{i=1}^{k} x_i \otimes y_i = \sum_{a \in B} a \otimes a^* \left( \sum_{i=1}^{k} x_i \otimes y_i \right) \sum_{b \in B} b \otimes b^* = \sum_{a,b \in B} \sum_{i=1}^{k} a \otimes E_{M_0}(a^*x_i)E_{M_0}(y_i b) b^* = 0.$$

The remaining claims follow as in [Jol90].

The Jones tower and tensor products

We give the background necessary to define the canonical planar $*$-algebra associated to a Markov inclusion and to prove its uniqueness. Many facts stated without proof in Subsection 3.2 rely on the results of this subsection. In particular, the multistep basic construction described in this subsection helps us understand tangles which cap off on the left (see Proposition 3.2.47), which are crucial to the proof of Theorem 3.4.1, the main result of this paper.

For the rest of this section, let $M_0 \subset (M_1, \text{tr}_1)$ be a strongly Markov inclusion of finite von Neumann algebras, and set $d = [M_1 : M_0]^{1/2}$. For $n \in \mathbb{N}$, inductively define the basic construction

$$M_{n+1} = \langle M_n, e_n \rangle = M_n e_n M_n = J_n M_{n-1} J_n \subset B(L^2(M_n, \text{tr}_n))$$

with canonical trace $\text{tr}_{n+1}$ extending $\text{tr}_n$ and satisfying $\text{tr}_{n+1}(xe_n) = d^{-2} \text{tr}_n(x)$ for all $x \in M_n$ where $e_n \in B(L^2(M_n, \text{tr}_n))$ is the Jones projection with range $L^2(M_{n-1}, \text{tr}_{n-1})$. For $n \in \mathbb{N}$, set $E_n = de_n$.

Fact 3.2.16. The $E_i$’s satisfy the Temperley-Lieb relations:

(i) $E_i^2 = dE_i = dE_i^*$,

(ii) $E_i E_j = E_j E_i$ for $|i - j| > 1$, and
Proposition 3.2.17. Suppose \( N \subset (M, \text{tr}_M) \) and \( M \subset (P, \text{tr}_P) \) such that \( \text{tr}_P|_M = \text{tr}_M \). Suppose \( A = \{a\} \) is a Pimsner-Popa basis for \( P \) over \( M \) and \( B = \{b\} \) is a Pimsner-Popa basis for \( M \) over \( N \). Then

1. \( AB = \{ab|a \in A \) and \( b \in B\} \) is a Pimsner-Popa basis for \( P \) over \( N \),
2. \( [P: N] = [P: M][M: N] \), and
3. \( \sum_{b \in B} b e_N^P b^* = e_M^P \in B(L^2(P, \text{tr}_P)), \) where \( e_N^P \) is the projection \( L^2(P, \text{tr}_P) \to L^2(N, \text{tr}_N) \) and \( e_M^P \) is the projection \( L^2(P, \text{tr}_P) \to L^2(M, \text{tr}_M) \).

Proof. (1) For all \( x \in P \),

\[
\sum_{a \in A} abE_N^P (b^* a^* x) = \sum_{a, b} abE_N^M (E_M^P (b^* a^* x)) = \sum_{a, b} abE_N^M (b^* E_M^P (a^* x)) = \sum_a a E_M^P (a^* x) = x.
\]

(2) Immediate from (1).

(3) If \( p \in P \) and \( \Omega \in L^2(P, \text{tr}_P) \) is the image of \( 1 \in P \), then

\[
\sum_{b \in B} b e_N^P b^* p \Omega = \sum_{b \in B} b E_N^P (b^* p) \Omega = \sum_{b \in B} b E_N^M (b^* (E_M^P (p)) \Omega = E_M^P (p) \Omega = e_M^P p \Omega.
\]

\( \square \)

Corollary 3.2.18. \( M_k \subset (M_n, \text{tr}_n) \) is strongly Markov for all \( 0 \leq k \leq n \).

The following technical lemma will be used to define the multistep basic construction in Proposition 3.2.20.

Lemma 3.2.19. For all \( 0 \leq k \leq n \), let

\[
f_{n-k}^n = d^{k(k-1)} (e_n e_{n-1} \cdots e_{n-k+1}) (e_{n+1} e_n \cdots e_{n-k+2}) \cdots (e_{n+k-1} e_{n+k-2} \cdots e_n) \in M_{n+k}.
\]

If \( 0 \leq j \leq k \leq n \) and \( B \) is a Pimsner-Popa basis for \( M_{n-j} \) over \( M_{n-k} \), then \( \sum_{b \in B} b f_{n-k}^n b^* = f_{n-j}^n \).

Proof. For \( j + 1 \leq i \leq k \), let \( A_i \) be a Pimsner-Popa basis for \( M_{n-i+1} \) over \( M_{n-i} \). Then \( A = A_{j+1} \cdots A_k \) is a Pimsner-Popa basis for \( M_{n-j} \) over \( M_{n-k} \) by Proposition 3.2.17, and

\[
\sum_{a_i \in A_i} a_{j+1} \cdots a_k f_{n-k}^n a_k^* \cdots a_{j+1}^* = \sum_{a_i \in A_i} a_{j+1} \cdots a_k f_{n-k+1}^n a_k^* \cdots a_{j+1}^* = \cdots = \sum_{a_{j+1} \in A_{j+1}} a_{j+1} f_{n-j-1}^n a_{j+1}^* = f_{n-j}^n.
\]
For $B$ another Pimsner-Popa basis for $M_{n-j}$ over $M_{n-k}$, define $U \in \text{Mat}_{|A| \times |B|}(M_{n-k})$ by $U_{a,b} = F_{M_{n-k}}^a(a^*b)$. If we consider $A$ as a row vector in $\text{Mat}_{1 \times |A|}(M_{n-j})$, then $B = AU$ and $A = BU^*$. For $\ell \in \mathbb{N}$, let $F_\ell = f_{n-k}^\ell I_\ell \in \text{Mat}_{\ell \times \ell}(M_{n+k})$, i.e., $F_\ell$ is the $\ell \times \ell$ diagonal matrix with all diagonal entries equal to $f_{n-k}^\ell$. Then since $f_{n-k}^\ell$ commutes with $M_{n-k}$, we have
\[
\sum_{b \in B} b f_{n-k}^n b^* = BF_B |B| B^* = AU F_B |U^* A^* = AUU^* F_A |A^* = AF_A |A^* = \sum_{a \in A} a f_{n-k}^n a^* = f_{n-j}^n.
\]

Forms of the next proposition appear in [PP88], [Jol90], and [Bis97].

**Proposition 3.2.20** (Multistep Basic Construction). The inclusion
\[ M_{n-k} \subset M_n \subset (M_{n+k}, \text{tr}_{n+k}, f_{n-k}^n) \]

is standard. (See Remark 3.2.45).

**Proof.** Let $B$ be a Pimsner-Popa basis for $M_n$ over $M_{n-k}$. Then by Lemma 3.2.19,
\[
\sum_{b \in B} b f_{n-k}^n b^* = 1,
\]

so $M_n f_{n-k}^n M_n = M_{n+k}$. It is straightforward to check $f_{n-k}^n x f_{n-k}^n = E_{M_{n-k}}(x)f_{n-k}^n$ for all $x \in M_n$ and $E_{M_n}(f_{n-k}^n) = d^{-2k}$, and the result follows by Lemma 3.2.15.

**Remark 3.2.21.** Note that $L^2(M_n, \text{tr}_n)$ has left and right actions of $M_0, \ldots, M_{2n}$, where as usual, the right action of $M_i$ is the left action of $J_n M_i J_n \cong M^{\text{op}}_i$. Note that $M'_i = J_n M_{2n-i} J_n$, so we define a canonical trace on $M'_i \cap B(L^2(M_n, \text{tr}_n))$ by $\text{tr}'_i(x) = \text{tr}_{2n-i}(J_n x^* J_n)$ for all $x \in M'_i \cap B(L^2(M_n, \text{tr}_n))$.

**Proposition 3.2.22** (Shifts). For all $0 \leq k \leq n$, there is a canonical isomorphism $M'_k \cap M_n \cong M'_{k+2} \cap M_{n+2}$.

**Proof.** On $B(L^2(M_n, \text{tr}_n))$, the map $x \mapsto J_n x^* J_n$ gives an anti-isomorphism $M'_k \cap M_n \cong M'_{k+2} \cap M_{2n-k}$. On $B(L^2(M_{n+1}, \text{tr}_{n+1}))$, the map $y \mapsto J_{n+1} y^* J_{n+1}$ gives an anti-isomorphism $M'_n \cap M_{2n-k} \cong M'_{k+2} \cap M_{n+2}$.

**Proposition 3.2.23.** The canonical trace-preserving conditional expectation $M_{n+k} \to M_{n+k-i}$ is given by $x f_{n-k}^n y \mapsto d^{-2i} x f_{n-k+i}^n y$ where $x, y \in M_n$. The canonical trace-preserving conditional expectation $M'_{n-k} = J_n M_{n+k} J_n \to J_n M_{n+k-i} J_n = M'_{n-k+i}$ is given by the same formula, only with $x, y \in M'_n = J_n M_n J_n$.

**Proof.** We prove the first statement, as the second is similar. By the Markov property, for all $x, y \in M_n$,
\[
\text{tr}_{n+k} (x f_{n-k}^n y) = d^{-2k} \text{tr}_n (xy) = d^{-2i} \text{tr}_{n+k-i} (x f_{n-k+i}^n y),
\]
so the map is trace-preserving. Now $M_{n+k-i}$-bilinearity follows from the following two facts:
(i) for all \(1 \leq i \leq k\), \(M_{n-k} \subset M_{n-k+i}\), so \(f_{n-k+i}^n f_{n-k}^n = f_{n-k}^n\), and

(ii) \(E_{M_{n+k}^M}(f_{n-k}^n) = d^{-2i} f_{n-k+i}^n\).

We can now strengthen Proposition 2.7 from [Bis97], versions of which also appear in [Bur03]. This is the main proposition describing left-capping tangles.

**Proposition 3.2.24.** Let \(0 \leq k \leq \ell \leq n\), and let \(B\) be a Pimsner-Popa basis for \(M_\ell^M\) over \(M_k^M\). The conditional expectation \(E_{M_\ell^M}^{M_k^M} : (M_k^M \cap B(L^2(M_n, \text{tr}_n)), \text{tr}_{k}) \to (M_\ell^M \cap B(L^2(M_n, \text{tr}_n)), \text{tr}_\ell)\) is given by

\[
E_{M_\ell^M}^{M_k^M}(x) = \frac{1}{d^{2(\ell-k)}} \sum_{b \in B} bxb^*.
\]

In particular, this map is independent of \(n\) and the choice of basis.

**Proof.** The result follows from Lemma 3.2.19 and Proposition 3.2.23, since for \(x, y \in J_n M_n J_n \subset M_\ell^M\),

\[
\sum_{b \in B} bx f_k^n y b^* = \sum_{b \in B} xb f_k^n b^* y = x f_\ell^n y.
\]

To define our planar ∗-algebra in Subsection 3.2, we need the following fact, which follows from Proposition 3.2.4 and a simple induction argument.

**Proposition 3.2.25.** For \(k \in \mathbb{N}\), let \(v_k = E_k E_{k-1} \cdots E_1\). For all \(n \in \mathbb{N}\), there are isomorphisms of \(M_1 - M_1\) bimodules

\[
\theta_n : \bigotimes_{M_0}^n M_1 \longrightarrow M_n \text{ by }
\]

\[
x_1 \otimes \cdots \otimes x_n \longmapsto x_1 v_1 x_2 v_2 \cdots v_{n-1} x_n.
\]

**Remark 3.2.26.** Recall that \(L^2(M_n, \text{tr}_n)\) is the completion of \(M_n\) with inner product \(<x, y> = \text{tr}_n(y^* x)\). As usual, \(\theta_n\) gives an isomorphism of Hilbert-bimodules

\[
\bigotimes_{M_0}^n L^2(M_1, \text{tr}_1) \longrightarrow L^2(M_n, \text{tr}_n)
\]

where the tensor product on the left is Connes’ relative tensor product with inner product given inductively by

\[
<x_1 \otimes u, y_1 \otimes v>_n = <E_{M_0}(y_1^* x_1) u, v>_{n-1}
\]

\[
<u \otimes x_n, v \otimes y_n>_n = <u, v E_{M_0}(y_n x_n^*)>_{n-1}.
\]
CHAPTER 3. THE EMBEDDING THEOREM FOR FINITE DEPTH SUBFACTOR PLANAR ALGEBRAS

The following operators will be useful in the definition of the rotation operators in Subsections 3.2 and 3.2.

**Definition 3.2.27.** Given \(x \in M_1\), we get

1. left and right multiplication operators
   \[
   L(x), R(x): \bigotimes_{n=0}^M L^2(M_1, tr_1) \rightarrow \bigotimes_{M_0}^n L^2(M_1, tr_1)
   \]
   by \(L(x)(v) = xv\) and \(R(x)(v) = vx\), and
2. left and right creation operators
   \[
   L_x, R_x: \bigotimes_{n=0}^M L^2(M_1, tr_1) \rightarrow \bigotimes_{M_0}^{n+1} L^2(M_1, tr_1)
   \]
   by \(L_x(v) = x \otimes v\) and \(R_x(v) = v \otimes x\).

**Fact 3.2.28.** For \(x \in M_1\), we have

\[
L_x^*(y_1 \otimes \cdots \otimes y_{n+1}) = E_{M_0}(x^*y_1)y_2 \otimes \cdots \otimes y_{n+1} \text{ and } \]
\[
R_x^*(y_1 \otimes \cdots \otimes y_{n+1}) = y_1 \otimes \cdots \otimes y_n E_{M_0}(y_{n+1}x^*).
\]

The following lemma will be instrumental in defining the action of tangles.

**Lemma 3.2.29.** If \(A\) is a \(C\)-algebra, \(V_1\) is a right \(A\)-module, \(V_2\) is an \(A-A\) bimodule, and \(V_3\) is a left \(A\)-module, then for each \(A\)-invariant \(v_2 \in V_2\), the map

\[
v_1 \otimes v_3 \mapsto v_1 \otimes v_2 \otimes v_3
\]

defines a linear map \(\phi_{v_2}: V_1 \otimes_A V_3 \rightarrow V_1 \otimes_A V_2 \otimes_A V_3\). Moreover, the map \(v \mapsto \phi_v\) on \(A' \cap V_2 = \{ v \in V_2 | av = va \text{ for all } a \in A \}\) is \(C\)-linear.

**Proof.** Middle \(A\)-linearity is satisfied as \(v_2\) is \(A\)-invariant. \(\square\)

**Remark 3.2.30.** This lemma gives an alternate proof that the map \(E_{M_0}^{M_1}\) is well defined in Proposition 3.2.24. By Remark 3.2.5, \(d^{-2} \sum_{b \in B} b \otimes b^*\) is independent of the choice of Pimsner-Popa basis \(B\), so the composite map

\[
x \mapsto \phi_x \mapsto \phi_x \left( d^{-2} \sum_{b \in B} b \otimes b^* \right) = d^{-2} \sum_{b \in B} b \otimes x \otimes b^* \mapsto d^{-2} \sum_{b \in B} bxb^*
\]
on \(M_0' \cap B(L^2(M_n, tr_n))\) is independent of the choice. Moreover, the result is \(M_1\)-invariant, since for any unitary \(u \in M_1\), \(\{ ub | b \in B \}\) is another Pimsner-Popa basis for \(M_1\) over \(M_0\).
CHAPTER 3. THE EMBEDDING THEOREM FOR FINITE DEPTH SUBFACTOR PLANAR ALGEBRAS

Definition of the canonical planar $*$-algebra

The definition of a planar $*$-algebra has evolved since its inception in [Jon99]. We use the definition of [Jon10] (see also [Pet10]), but we do not reproduce it here.

In [Jon99], it was shown how to endow the tower of relative commutants of an extremal, finite index $II_1$-subfactor with the structure of a subfactor planar algebra, i.e., a planar $*$-algebra $Q_* = \{Q_{n,\pm}\}$ with $\dim(Q_{n,\pm}) < \infty$ for all $n \geq 0$ which is

- Spherical: $\dim(Q_{0,\pm}) = 1$ and any fully labelled 0-tangle is invariant under spherical isotopy. This implies shaded and unshaded contractible loops count for the same multiplicative factor of $d$, called the modulus of $Q_*$. 

- Positive-definite: The bilinear form on $Q_{n,\pm}$ given by $\langle a, b \rangle = d^{-n} \text{tr}(b^*a)$ is positive definite.

The only essential ingredient to the construction of [Jon99] is a Pimsner-Popa basis, so the same construction applies to a strongly Markov inclusion $M_0 \subset (M_1, \text{tr}_1)$. As we do not require the algebras to be factors or the inclusion to be extremal, the resulting planar algebra need not be spherical nor positive-definite nor have finite dimensional $n$-box spaces.

Below, we define a planar $*$-algebra structure on the vector spaces $P_{n,\pm} (n \geq 0)$ given by $P_{n,+} = \theta^{-1}_{n}(M'_0 \cap M_n)$ and $P_{n,-} = \theta^{-1}_{n}(M'_1 \cap M_{n+1})$. This planar algebra is independent of any choices, so we call it the canonical planar $*$-algebra associated to $M_0 \subset (M_1, \text{tr}_1)$.

We define the action of a planar tangle in standard form:

1. all the input and output disks are horizontal rectangles with all strings (that are not closed loops) emanating from the top edges of the rectangles,

2. all the input disks are in disjoint horizontal bands and all maxima and minima of strings are at different vertical levels, and not in the horizontal bands defined by the input disks, and

3. the distinguished (starred) intervals of all the disks are at the left edges of the rectangles. (In the sequel, we will assume this convention and omit the $*$'s.)

We do not provide the proof of isotopy invariance, i.e., that the action is independent of the choice of standard form, as this proof is identical to that in [Jon99]. However, in Subsection 3.2, we provide Burns' elegant proof that the rotation operator is well-defined.

Suppose we have a $(k, \pm)$-tangle $T$ in standard form with $s$ input rectangles, and input rectangle $j$ has $2r_j$ strings emanating from the top. We define the action of $T$ on an $s$-tuple $\xi = (\xi_1, \cdots, \xi_s)$ where $\xi_j \in P_{r_j,\pm_j}$ and $\pm_j = \pm$ if the region just below input rectangle $j$ is unshaded or shaded respectively.

We read the action of $T$ on $\xi$ by sliding a horizontal line through the tangle from bottom to top. For a fixed vertical $y$-value, off the input disks' horizontal bands and away from the relative extrema of the strings, the horizontal line will meet $n_y$ shaded regions from left to right. One should think of the shaded regions along this line as elements of $M_1$ and the
unshaded regions between shaded regions as the symbols $\otimes_{M_0}$. Near the top, the line will meet $k$ or $k+1$ shaded regions depending on whether the left-most region of $T$ is unshaded or shaded respectively. We illustrate a typical $(3, +)$-tangle with the horizontal line about half way through its travel:

For each $y$ coordinate of the horizontal line, one reads off an $M_i$-invariant element $\eta_y \in \bigotimes_{M_0}^{n_y} M_1$, where $i = 0$ if $T$ is a $(k, +)$-tangle and $i = 1$ if $T$ is a $(k, -)$-tangle.

The element $\eta_y$ begins as $1 \in M_i$ near the bottom, and it remains constant as long as the horizontal line meets neither maxima, minima, nor rectangles. If the horizontal line passes input rectangle $j$ for which exactly $t$ shaded regions sit to the left, then we insert $\xi_j$ into $\eta_y$ as in Figure 3.1 by applying Lemma 3.2.29 with $v_2 = \xi_j$,

$$V_1 = \bigotimes_{M_0}^t M_1, \ V_2 = P_{r_j, \pm_j}, \ \text{and} \ V_3 = \bigotimes_{M_0}^{n_y - t} M_1.$$ 

Note that $V_1, V_3$ are considered as $M_{\ell}$-modules and $P_{r_j, \pm_j}$ is an $M_{\ell} - M_{\ell}$ bimodule, where $\ell = 0$ if $\pm_j = +$ and $\ell = 1$ if $\pm_j = -$. Note that inserting $\xi_j$ into $\eta_y$ gives an $M_i$-invariant vector.

As the horizontal line passes a maximum or minimum, $\eta_y$ changes according to Figure 3.2 where the changes indicated on the tensors are to be inserted into the position indicated by the shaded regions on the horizontal (dashed) line. With the exception of one case, each of these maps is an $M_1 - M_1$ bimodule map, so it will preserve $M_i$-invariant elements. The remaining case to consider is when the left-most or right-most shaded region is capped off by applying the third map pictured above, which is an $M_0 - M_0$ bimodule map. But this will
only occur when the distinguished (starred) interval of the external disk meets an unshaded region, so \( i \) would have to be 0 from the beginning.

The action of the tangle on \( \xi \) is the element \( \eta_y \in P_{k,\pm} \) read for horizontal lines sufficiently close to the top. The \(*\)-structure is the same as that of [Jon99].

Example 3.2.31. To calculate

\[
\xi = \sum_{i=1}^{k} x_1^i \otimes \cdots \otimes x_n^i \in \theta_n^{-1}(M_0' \cap M_n),
\]

for
we first isotope the tangle into a standard form. The horizontal line travels upward as shown:

\[ \xi \]

which we read as:

\[
1_{\mathbb{C}} \mapsto 1_{M} \mapsto d^{-1} \sum_{b \in B} b \otimes b^* \mapsto d^{-1} \sum_{b \in B} b \otimes \xi \otimes b^* \mapsto d^{-1} \sum_{b \in B} b \otimes x_1^i \otimes \cdots \otimes x_{n-1}^i \otimes x_n^i b^* \\
\mapsto \sum_{b \in B} \sum_{i=1}^k b \otimes x_1^i \otimes \cdots \otimes x_{n-1}^i E_{M_0}(x_n^i b^*),
\]

the last line giving the output of the tangle applied to \( \xi \).

**Burns’ treatment of the rotation operator on** \( P_{n,+} \)

The key to showing that the \( P_{n,\pm} \)'s define a planar algebra is isotopy invariance, which relies on the existence of the rotation on \( P_{n,\pm} \). A particularly elegant treatment of this is due to Michael Burns, but it only appears in his thesis \cite{Bur03}, so we include a proof below for the reader’s convenience.

**Definition 3.2.32.** Let \( B \) be a Pimsner-Popa basis of \( M_1 \) over \( M_0 \). For

\[ x = x_1 \otimes \cdots \otimes x_n \in \bigotimes_{M_0} M_1, \]

define \( \rho(x) = \sum_{b \in B} L_b R_b^*(x) = \sum_{b \in B} b \otimes x_1 \otimes \cdots \otimes x_{n-1} E_{M_0}(x_n b^*) \) (see Example 3.2.31).

**Proposition 3.2.33.** The map \( \rho \) preserves \( P_{n,+} \), and its restriction to \( P_{n,+} \) is independent of the choice of \( B \).

**Proof.** Middle linearity is respected by \( \rho \), so it is well defined, though it may depend on \( B \). By Lemma 3.2.29 and Remark 3.2.5, for \( M_0 \)-invariant \( x \), the sum

\[
\sum_{b \in B} b \otimes x \otimes b^*
\]

is independent of \( B \). We obtain \( \rho \) by applying an \( M_0 - M_0 \) bilinear map which does not involve \( B \), so the restriction of \( \rho \) is \( M_0 \)-invariant and independent of \( B \).
\( \square \)
Theorem 3.2.34 ([Bur03]). For \( x \in P_{n,+} \) and \( y_1, \ldots, y_n \in M_1 \),
\[
\langle \rho(x), y_1 \otimes \cdots \otimes y_n \rangle = \langle x, y_2 \otimes \cdots \otimes y_n \otimes y_1 \rangle,
\]
so \( \rho^n = \text{id} \) on \( P_{n,+} \).

Proof. As \( \rho(x) = \sum_{b \in B} L_b R_b^*(x) \), we have
\[
\langle \rho(x), y_1 \otimes \cdots \otimes y_n \rangle = \sum_{b \in B} \langle L_b R_b^* x, y_1 \otimes \cdots \otimes y_n \rangle = \sum_{b \in B} \langle x, R_b L_b^* y_1 \otimes \cdots \otimes y_n \rangle = \sum_{b \in B} \langle x, E_{M_0}(b^* y_1) y_2 \otimes \cdots \otimes y_n \otimes b \rangle
\]
\[
= \sum_{b \in B} \langle x, y_2 \otimes \cdots \otimes y_n \otimes b E_{M_0}(b^* y_1) \rangle = \langle x, y_2 \otimes \cdots \otimes y_n \otimes y_1 \rangle.
\]

\[ \square \]

Corollary 3.2.35. The rotation on \( P_{n,+} \) is well defined.

\begin{center}
\begin{tikzpicture}
\end{tikzpicture}
\end{center}

The rotation on \( P_{n,-} \)

We mimic Burns’ treatment of the rotation on \( P_{n,+} \) to define the rotation on \( P_{n,-} \).

Definition 3.2.36. Let \( B \) be a Pimsner-Popa basis of \( M_1 \) over \( M_0 \). For
\[
x = x_1 \otimes \cdots \otimes x_{n+1} \in \bigotimes_{M_0}^{n+1} M_1,
\]
define \( \sigma(x) = \sum_{b \in B} R(b^*) R_1^* L_b(x) = \sum_{b \in B} b \otimes x_1 \otimes \cdots \otimes x_n E_{M_0}(x_{n+1}) b^* \).

Proposition 3.2.37. The map \( \sigma \) preserves \( P_{n,-} \), and its restriction to \( P_{n,-} \) is independent of the choice of \( B \).

Proof. Similar to Proposition 3.2.33. \[ \square \]

Theorem 3.2.38. For \( x \in P_{n,-} \) and \( y_1, \ldots, y_{n+1} \in M_1 \),
\[
\langle \sigma(x), y_1 \otimes \cdots \otimes y_{n+1} \rangle = \langle x, y_2 \otimes \cdots \otimes y_n \otimes y_{n+1} y_1 \otimes 1 \rangle.
\]
Proof. Similar to Theorem 3.2.34.

**Corollary 3.2.39.** $\sigma^n = \text{id}$ on $P_{n,-}$.

Proof. As $\sigma$ preserves $P_{n,-}$, we repeatedly apply Theorem 3.2.38 for $x \in P_{n,-}$ to get

$$\langle \sigma^n(x), y_1 \otimes \cdots \otimes y_{n+1} \rangle = \langle \sigma^{n-1}(x), y_2 \otimes \cdots \otimes y_n \otimes y_{n+1}y_1 \otimes 1 \rangle$$
$$= \langle \sigma^{n-2}(x), y_3 \otimes \cdots \otimes y_n \otimes y_{n+1}y_1 \otimes y_2 \otimes 1 \rangle$$
$$= \cdots = \langle x, y_{n+1}y_1 \otimes y_2 \otimes \cdots \otimes y_n \otimes 1 \rangle.$$

We then invoke Burns’ trick again to get

$$\langle x, y_{n+1}y_1 \otimes y_2 \otimes \cdots \otimes y_n \otimes 1 \rangle = \langle y_{n+1}^*x, y_1 \otimes \cdots \otimes y_n \otimes 1 \rangle$$
$$= \langle xy_{n+1}^*, y_1 \otimes \cdots \otimes y_n \otimes 1 \rangle$$
$$= \langle x, y_1 \otimes \cdots \otimes y_n \otimes y_{n+1} \rangle.$$

**Corollary 3.2.40.** The rotation on $P_{n,-}$ is well defined.

**Uniqueness of the canonical planar $*$-algebra**

We have the following facts whose proofs are similar to those in [Jon99] and will be omitted (they are straightforward from the results in Subsections 3.2 and 3.2). We shade tangles as much as possible, but sometimes we will not have enough information.

**Proposition 3.2.41** (Multiplication). Suppose $x, y \in M_n$ such that

$$\theta_n^{-1}(x) = x_1 \otimes \cdots \otimes x_n \text{ and } \theta_n^{-1}(y) = y_1 \otimes \cdots \otimes y_n$$

Then

$$\theta_n^{-1}(xy) = \begin{cases} 
  x_1 \otimes \cdots \otimes x_k E_{M_0}(x_{k+1}E_{M_0}(x_{k+2}E_{M_0}(x_{k+3}\cdots y_{k-1})y_k) \otimes y_{k+1} \otimes \cdots \otimes y_{2k} & n = 2k \\
  x_1 \otimes \cdots \otimes x_{k+1}E_{M_0}(x_{k+2}E_{M_0}(x_{k+3}\cdots y_{k-1})y_k)y_{k+1} \otimes \cdots \otimes y_{2k+1} & n = 2k + 1.
\end{cases}$$

**Remark 3.2.42.** If $x, y$ as above are in $M'_i \cap M_n$ where $i \in \{0, 1\}$, then

$$\theta_n^{-1}(xy) = \begin{cases} 
  x_1 \otimes \cdots \otimes x_n & \text{where the shading depends on } i \text{ and the parity of } n.
\end{cases}$$
CHAPTER 3. THE EMBEDDING THEOREM FOR FINITE DEPTH SUBFACTOR PLANAR ALGEBRAS

Proposition 3.2.43 ($\ast$-Structure). Suppose $x \in M_n$ such that $\theta_n^{-1}(x) = x_1 \otimes \cdots \otimes x_n$. Then $\theta_n^{-1}(x^*) = x_n^* \otimes \cdots \otimes x_1^*$.

Proposition 3.2.44 (Jones Projections). (1) For $n \geq 1$, the Jones projection $E_n \in P_{n+1,+}$ is given by

(2) For $n \geq 2$, the Jones projection $E_n \in P_{n,-}$ is given by

Remark 3.2.45. The multistep basic construction projection of Proposition 3.2.20 is given by

Proposition 3.2.46 (Inclusions). (1) Let $i_n : M'_0 \cap M_n \to M'_0 \cap M_{n+1}$ be the inclusion. Then the inclusion $\theta_{n+1}^{-1} \circ i_n \circ \theta_n : P_{n,\pm} \to P_{n+1,\pm}$ is given by

(2) If $x \in P_{n,-}$, then

Proposition 3.2.47 (Conditional Expectations). (1) The conditional expectation $\theta_n^{-1} \circ E_{M_n^{-1}} \circ \theta_n : P_{n,+} \to P_{n-1,+}$ is given by

(2) The conditional expectation $\theta_n^{-1} \circ E_{M'_1} \circ \theta_n : P_{n,+} \to P_{n-1,-}$ (see Proposition 3.2.24) is given by

Notation 3.2.48. We use the notation from [Pen12a]:

(1) Denote the annular capping maps $P_{n,+} \to P_{n-1,+}$ by $\alpha_j$ as shown:
i.e., numbering the boundary points clockwise from $\ast$, the $i^{th}$ and $(i + 1)^{th}$ (modulo $2n$) internal boundary points are joined by a string and all other internal boundary points are connected to external boundary points such that

(i) If $i = 1$, then the first external point is connected to the third internal point.

(ii) If $1 < i < 2n$, then the first external point is connected to the first internal point.

(iii) If $i = 2n$, then the first external point is connected to the $(2n - 1)$th internal point.

(2) Denote the annular cupping maps $P_{n-1,+} \to P_{n,+}$ by $\beta_j$ as shown:

\[ \ast \quad \ast \quad \ast \quad \ast \]

\[ \ast \quad \ast \quad \ast \quad \ast \]

\[ \ast \quad \ast \quad \ast \quad \ast \]

\[ \ast \quad \ast \quad \ast \quad \ast \]

i.e., $\beta_j$ is $\alpha_j$ turned inside out.

The following lemma is similar to a result in [KS04]:

**Lemma 3.2.49.** Suppose $P_\bullet$ is a planar $*$-algebra with modulus $d \neq 0$ and $Q_{n,\pm} \subset P_{n,\pm}$ are $*$-subalgebras which are closed under the following operations:

(1) left and right multiplication by tangles $E_n = \begin{array}{c}
\cdots \\
\vdots \\
n-1 \\
\cdots 
\end{array} \in P_{n+1,+}$ for $n \in \mathbb{N}$;

(2) The maps from $P_{n,+}$ as follows:

\[ \alpha_n = \begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots 
\end{array} : P_{n,+} \to P_{n-1,+}, \quad \beta_{n+1} = \begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots 
\end{array} : P_{n,+} \to P_{n+1,+}, \]

\[ \gamma_n^+ = \begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots 
\end{array} : P_{n,+} \to P_{n-1,-}; \quad \text{and} \]

(3) the map $i_{n}^- = \begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots 
\end{array} : P_{n,-} \to P_{n+1,+}$.

Then the $Q_{n,\pm}$ define a planar $*$-subalgebra $Q_\bullet \subset P_\bullet$. 
Proof. As $Q_{n,\pm}$ is closed under multiplication and $\ast$, it suffices to show $Q_\bullet$ is closed under all annular maps. To show this, it suffices to show all $\alpha_j$’s, all $\beta_j$’s, and both rotations by 1 preserve $Q_\bullet$.

First, note that the maps $\gamma_n^{-} : P_{n,-} \to P_{n-1,+}$ and $i_n^+ : P_{n,+} \to P_{n+1,-}$ given by

\[
\gamma_n^-(x) = \frac{1}{d} \alpha_n \alpha_{n+1} (E_n E_{n-1} \cdots E_1 \cdot \beta_{n+2} (i_n^- x)) \cdot E_1 E_2 \cdots E_n \quad \text{and}
\]

\[
i_n^+ (x) = \gamma_{n+2}^+ (E_1 E_2 \cdots E_n) \cdot \beta_{n+2} \beta_{n+1} (x) \cdot (E_{n+1} E_n \cdots E_1)
\]

preserve $Q_\bullet$.

We show all $\alpha_j$’s preserve $Q_\bullet$. For $j < n$ and $x \in Q_n$,

\[
\alpha_j (x) = \frac{1}{d} \alpha_n \alpha_{n+1} ((E_n E_{n-1} \cdots E_j) \cdot \beta_{n+1} (x) \cdot (E_n)).
\]

The case $n < j < 2n$ is similar. It is clear $\alpha_{2n} (x) = \alpha_{2n-1} (i_{n-1}^- (\gamma_n^+ (x)))$.

We show all $\beta_j$’s preserve $Q_\bullet$. If $j < n + 1$, we have

\[
\beta_j (x) = (E_j E_{j-1} \cdots E_n) \cdot \beta_{n+1} (x).
\]

The case $n + 1 < j < 2n + 2$ is similar. It is clear $\beta_{2n+2} (x) = \alpha_2 \gamma_{n+1} \gamma_n^+ (x)$.

We show both rotations by 1 preserve $Q_\bullet$. We have

\[
\frac{1}{d} \gamma_{n+1}^+ \alpha_{2n+2} i_{n+1}^+ i_n^- \alpha_n \beta_{n+1} (x) \quad \text{and}
\]

\[
\alpha_{n+1} \beta_{n+2} \alpha_{2n+1} i_n^- (x).
\]

\[\square\]

**Theorem 3.2.50.** Given a strongly Markov inclusion $M_0 \subset (M_1, \text{tr}_1)$, there is a unique planar $\ast$-algebra $P_\bullet$ of modulus $d = [M_1 : M_0]^{1/2}$ where

\[
P_{n,+} = \theta_n^{-1} (M_0' \cap M_n) \quad \text{and} \quad P_{n,-} = \theta_{n+1}^{-1} (M_1' \cap M_{n+1})
\]

such that the multiplication is given by Remark 3.2.42,

(0) for all tangles $T$ with $n$ input disks, $T (\xi_1^*, \cdots, \xi_n^*) = T^* (\xi_1, \cdots, \xi_n)^*$ where for $\xi_i \in P_{n_i, \pm_i}$, $\xi_i^*$ is as in Proposition 3.2.43 and $T^*$ is the mirror image of $T$;
CHAPTER 3. THE EMBEDDING THEOREM FOR FINITE DEPTH SUBFACTOR
PLANAR ALGEBRAS

(1) for \( n \in \mathbb{N} \), \( E_n = \begin{array}{c}
\ldots \\
\vdots \\
\ldots \\
\end{array} \in P_{n+1,+};
\]

(2) for \( x \in P_{n,+} \) and \( B \) a Pimsner-Popa basis for \( M_1 \) over \( M_0 \),
\[
\begin{array}{c}
\ldots \\
x \\
\ldots \\
\end{array} = dE_{M_{n-1}}(x),
\quad \begin{array}{c}
\ldots \\
x \\
\ldots \\
\end{array} = x \in P_{n+1,+}, \text{ and }
\]

\[
\begin{array}{c}
\ldots \\
x \\
\ldots \\
\end{array} = dE_{M_1^n}^M(x) = d^{-1} \sum_{b \in B} bx b^*; \quad \text{and}
\]

(3) for \( x \in P_{n,-}, \)
\[
\begin{array}{c}
\ldots \\
x \\
\ldots \\
\end{array} = x \in P_{n+1,+}.
\]

Proof. Uniqueness follows from Lemma 3.2.49. Existence follows from the existence of the canonical planar \(*\)-algebra associated to \( M_0 \subset (M_1, \text{tr}_1) \).

Corollary 3.2.51. The canonical planar \(*\)-algebra associated to an extremal, finite index \( II_1 \)-subfactor is the subfactor planar algebra constructed in [Jon99].

3.3 The planar algebra isomorphism for finite dimensional \( C^* \)-algebras

We now restrict our attention to a connected unital inclusion \( M_0 \subset M_1 \) of finite dimensional \( C^* \)-algebras with the Markov trace. We show that in this case, the canonical planar \(*\)-algebra of Theorem 3.2.50 is isomorphic to the bipartite graph planar algebra [Jon00] of the Bratteli diagram.

Many of the results in this section can be found in [GdlHJ89],[JS97],[EK98], but we present them here for completeness and for the reader’s convenience.

Loop algebras

We define loop algebras in the spirit of [Jon00] which are another description of Evans, Ocneanu, and Sunder’s path algebras [GdlHJ89],[JS97],[EK98], with a more GNS (rather than spatial) flavor.

Notation 3.3.1. For this section, let \( \Gamma \) be a finite, connected, bipartite multi-graph. Let \( \mathcal{V}_\pm \) denote the set of even/odd vertices of \( \Gamma \), and let \( \mathcal{E} \) denote the edge set of \( \Gamma \). Usually we will denote edges by \( \varepsilon \) and \( \xi \). All edges will be directed from even to odd vertices, so
we have source and target functions \( s: \mathcal{E} \to \mathcal{V}_+ \) and \( t: \mathcal{E} \to \mathcal{V}_- \). We will write \( \varepsilon^* \) to denote an edge \( \varepsilon \) traversed from an odd vertex to an even vertex, and we define source and target functions \( s: \mathcal{E}^* = \{ \varepsilon^* | \varepsilon \in \mathcal{E} \} \to \mathcal{V}_- \) and \( t: \mathcal{E}^* \to \mathcal{V}_+ \) by \( s(\varepsilon^*) = t(\varepsilon) \) and \( t(\varepsilon^*) = s(\varepsilon) \). Let \( m_+: \mathcal{V}_+ \to \mathbb{N} \) be a dimension (row) vector for the even vertices. For \( v \in \mathcal{V}_- \), define the dimension (row) vector for the odd vertices by

\[
m_-(v) = \sum_{t(\varepsilon)=v} m_+(s(\varepsilon)).
\]

Let \( \Lambda \) be the bipartite adjacency matrix for \( \Gamma \) (\( \Lambda_{i,j} \) is the number of times the \( i^{th} \) vertex in \( \mathcal{V}_+ \) is connected to the \( j^{th} \) vertex in \( \mathcal{V}_- \)).

**Remark 3.3.2.** Given \( (\Gamma, m_+) \), we can associate a connected unital inclusion of finite dimensional C*-algebras \( M_0 \subset M_1 \). We set

\[
M_0 = \bigoplus_{v \in \mathcal{V}_+} M_{m_+(v)}(\mathbb{C}) \quad \text{and} \quad M_1 = \bigoplus_{v \in \mathcal{V}_-} M_{m_-(v)}(\mathbb{C}),
\]

and the inclusion is such that \( \Gamma \) is the Bratteli diagram for the inclusion, and \( \Lambda \) is the inclusion matrix (\( \Lambda_{i,j} \) is the number of times the \( i^{th} \) simple summand of \( M_0 \) is contained in the \( j^{th} \) simple summand of \( M_1 \)). Conversely, given such an inclusion, we get a finite, connected, bipartite multi-graph (the Bratteli diagram) and a dimension vector \( m_+ \) (corresponding to the simple summands of \( M_0 \)).

**Definition 3.3.3.** Let \( G_{0,\pm} \) be the complex vector space with basis \( \mathcal{V}_\pm \) respectively. For \( n \in \mathbb{N} \), \( G_{n,\pm} \) will denote the complex vector space with basis loops of length \( 2n \) on \( \Gamma \) based at a vertex in \( \mathcal{V}_\pm \) respectively.

We discuss the vector spaces \( G_{n,+} \). The spaces \( G_{n,-} \) are similar, and it is clear what the corresponding notation should be and how they will behave.

**Notation 3.3.4.** Loops in \( G_{n,+} \) will be denoted \([\varepsilon_1 \varepsilon_2^* \cdots \varepsilon_{2n-1} \varepsilon_{2n}^*]\). Any time we write such a loop, it is implied that

(i) \( t(\varepsilon_i) = s(\varepsilon_{i+1}) = t(\varepsilon_{i+1}) \) for all odd \( i < 2n \),

(ii) \( t(\varepsilon_i^*) = s(\varepsilon_i) = s(\varepsilon_{i+1}) \) for all even \( i < 2n \), and

(iii) \( t(\varepsilon_{2n}^*) = s(\varepsilon_{2n}) = s(\varepsilon_1) \).

For a loop \( \ell = [\varepsilon_1 \varepsilon_2^* \cdots \varepsilon_{2n-1} \varepsilon_{2n}^*] \in G_{n,+} \) and \( 1 \leq k \leq 2n \), we define the following paths in \( \ell \):

\[
\ell_{[1,k]} = \begin{cases} 
\varepsilon_1 \varepsilon_2^* \cdots \varepsilon_{k-1} \varepsilon_k^* & k \text{ even} \\
\varepsilon_1 \varepsilon_2^* \cdots \varepsilon_{k-1} \varepsilon_k & k \text{ odd}
\end{cases}
\]

\[
\ell_{[k,2n]} = \begin{cases} 
\varepsilon_k \varepsilon_{k+1}^* \cdots \varepsilon_{2n-1} \varepsilon_{2n}^* & k \text{ odd} \\
\varepsilon_k^* \varepsilon_{k+1} \cdots \varepsilon_{2n-1} \varepsilon_{2n} & k \text{ even}
\end{cases}
\]
Definition 3.3.5. Define an antilinear map $*$ on $G_{n,+}$ by the antilinear extension of the map

$$[\varepsilon_1 \varepsilon_2^* \cdots \varepsilon_{2n-1} \varepsilon_{2n}^*]^* = [\varepsilon_{2n}^* \varepsilon_{2n-1}^* \cdots \varepsilon_2^* \varepsilon_1].$$

There is also an obvious notion of taking $*$ of a path $\gamma_{j,k}(\ell)$ for a loop $\ell \in G_{n,+}$. We define a multiplication on $G_{n,+}$ by

$$\ell_1 \cdot \ell_2 = \delta_{j,k}^{(n+1,2n);(2,1)} [((\ell_1)_{1,n+1}(\ell_2)_{1,2})_{n+1,2n+2}].$$

It is clear that $*$ is an involution, i.e., an anti-automorphism of period 2, for $G_{n,+}$ under this multiplication.

Remark 3.3.6. We can think of a loop in $G_{n,+}$ as a path up and down the multi-graph $\Gamma_n$ corresponding to the Bratteli diagram for the inclusions

$$M_0 \subset M_1 \subset \cdots \subset M_n,$$

which is obtained by reflecting $\Gamma$ a total of $n-1$ times, as the inclusion matrix of $M_j \subset M_{j+1}$ is given by $\Lambda$ or $\Lambda^T$ if $j$ is even or odd, respectively [Jon83].

Definition 3.3.7. Let $\tilde{\Gamma}$ be the augmentation of the bipartite graph $\Gamma$ by adding a distinguished vertex $\star$ which is connected to each $v \in \mathcal{V}_+$ by $m_+(v)$ distinct edges. These edges are oriented so they begin at $\star$. We will denote these added edges by $\eta$'s (and $\zeta$'s and $\kappa$'s when necessary).

Definition 3.3.8. For $n \in \mathbb{Z}_{\geq 0}$, let $A_n$ be the algebra defined as follows: a basis of $A_n$ will consist of loops of length $2n+2$ on $\tilde{\Gamma}$ of the form

$$[\eta_1 \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1} \varepsilon_{2n}^* \eta_2^*],$$

i.e., the loops start and end at $\star$, but remain in $\Gamma$ otherwise. Note that we have an obvious $*$-structure on each $A_n$. Multiplication will be given as follows: if one defines the similar path notation as in Notation 3.3.4, then we have

$$\ell_1 \cdot \ell_2 = \delta_{j,k}^{(n+2,2n+2);(2,1)} [((\ell_1)_{1,n+2}(\ell_2)_{1,2})_{n+2,2n+2}].$$

Remark 3.3.9. We can think of a loop in $A_n$ as a path up and down the multi-graph $\tilde{\Gamma}_n$ corresponding to the Bratteli diagram for the inclusions

$$\mathbb{C} \subset M_0 \subset M_1 \subset \cdots \subset M_n.$$

Definition 3.3.10 (Inclusions). The inclusion $A_n \to A_{n+1}$ is given by the linear extension of

$$[\eta_1 \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1} \varepsilon_{2n}^* \eta_2^*] \mapsto \begin{cases} \sum_{s(\varepsilon)=s(\varepsilon_n)} [\eta_1 \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_n \varepsilon^* \varepsilon_{n+1} \cdots \varepsilon_{2n-1} \varepsilon_{2n}^* \eta_2^*] & n \text{ even} \\ \sum_{s(\varepsilon)=t(\varepsilon_n)} [\eta_1 \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_n \varepsilon^* \varepsilon_{n+1} \cdots \varepsilon_{2n-1} \varepsilon_{2n}^* \eta_2^*] & n \text{ odd.} \end{cases}$$

We identify $A_n$ with its image in $A_{n+1}$. 
Remark 3.3.11. The inclusion identifications allow us to define a multiplication $A_m \times A_n \to A_{\max\{m,n\}}$ by including $A_m, A_n$ into $A_{\max\{m,n\}}$ and using the regular multiplication. Explicitly, if $\ell_1 \in A_m$ and $\ell_2 \in A_n$ with $m \leq n$, then

$$\ell_1 \cdot \ell_2 = \delta(\ell_1)_{[m+2,2m+2]}(\ell_2)_{[1,m+1]}[(\ell_1)_{[1,m+1]}(\ell_2)_{[m+2,2n+2]}].$$

The case $m \geq n$ is similar.

Towers of loop algebras

We provide an isomorphism of the tower $(M_n)_{n \geq 0}$ coming from a connected unital inclusion of finite dimensional $C^*$-algebras with the Markov trace and the corresponding tower $(\mathcal{A}_n)_{n \geq 0}$ of loop algebras. Assume the notation of Subsection 3.3.

For $n \geq 0$, if $S_i$ is the $i$th simple summand of of $M_n$, then loops $\ell$ in $A_n$ for which $\ell_{[1,n+1]}$ ends at the corresponding vertex of $\bar{T}_n$ form a system of matrix units for a simple algebra isomorphic to $S_i$. Hence for $n \in \mathbb{Z}_{\geq 0}$, there is a $*$-algebra isomorphism $\mathcal{A}_n \cong M_n$, and $\dim(\mathcal{A}_n) = \dim(M_n)$.

At this point, we only choose such isomorphisms $\varphi_n : A_n \to \mathcal{A}_n$ for $n = 0,1$ which respects the inclusion given in Definition 3.3.10. In Proposition 3.3.17, we will inductively define isomorphisms $\varphi_n : A_n \to M_n$ for $n \geq 2$ to identify the Jones projections.

Definition 3.3.12. Following [Jon83], let $\lambda_i$ be the Markov trace (column) vector for $M_i$ for $i = 0,1$ such that

$$m_+\lambda_0 = 1 = m_-\lambda_1,$$

so $\lambda_i$ gives the traces of minimal projections in the simple summands of $M_i$ for $i = 0,1$. In order for the trace on $M_1$ to restrict to the trace on $M_0$, we must have $\Lambda \lambda_1 = \lambda_0$.

Recall that the inclusion matrix for $M_n \subset M_{n+1}$ is given by $\Lambda$ if $n$ is even and $\Lambda^T$ if $n$ is odd. This means that to extend the trace, we must have $\Lambda \Lambda^T \lambda_0 = d^{-2} \lambda_0$, $\Lambda^T \Lambda \lambda_1 = d^{-2} \lambda_1$, and $\lambda_n = d^{-2} \lambda_{n-2}$ for all $n \geq 2$, where $\lambda_n$ is the Markov trace vector for $M_n$ and

$$d = \sqrt{\| \Lambda \Lambda^T \|} = \sqrt{\| \Lambda^T \Lambda \|}.$$

Definition 3.3.13. Let $\lambda = \begin{pmatrix} \lambda_0 \\ d\lambda_1 \end{pmatrix}$, a Frobenius-Perron eigenvector for $\begin{pmatrix} 0 & \Lambda \\ \Lambda^T & 0 \end{pmatrix}$.

Definition 3.3.14 (Traces). We define a trace on $A_0$ by

$$\text{tr}_n([\eta_1 \eta_2^*]) = \begin{cases} \lambda(t(\eta_1)) = \lambda_0(t(\eta_1)) & \text{if } \eta_1 = \eta_2 \\ 0 & \text{else.} \end{cases}$$

Suppose $\ell = [\eta_1 \varepsilon_1 \varepsilon_2^* \cdots \varepsilon_{2n-1} \varepsilon_{2n}^* \eta_2^*] \in A_n$ with $n \geq 1$. We define a trace on $A_n$ by

$$\text{tr}_n(\ell) = \begin{cases} d^{-n} \lambda(s(\varepsilon_n)) & \text{if } n \text{ is even and } \ell = \ell^* \\ d^{-n} \lambda(t(\varepsilon_n)) & \text{if } n \text{ is odd and } \ell = \ell^* \\ 0 & \text{if } \ell \neq \ell^*. \end{cases}$$
Remark 3.3.15. The isomorphisms $\varphi_n$ for $n = 0, 1$ preserve the trace. Moreover, $\operatorname{tr}_{n+1}|_{A_n} = \operatorname{tr}_n$ for all $n \in \mathbb{N}$ as $\lambda$ is a Frobenius-Perron eigenvector.

Proposition 3.3.16 (Conditional Expectations). If $\ell = [\eta_1 \varepsilon_1 \varepsilon_2^* \cdots \varepsilon_{n-1} \varepsilon_{2n}^* \eta_2^*] \in A_n$, the conditional expectation $A_n \rightarrow A_{n-1}$ is given by

$$E_{A_{n-1}}(\ell) = \begin{cases} 
-\delta_{\varepsilon_n, \varepsilon_{n+1}} \frac{\lambda(t(\varepsilon_n))}{\lambda(t(\varepsilon_{n+1}))} [\eta_1 \varepsilon_1 \varepsilon_2^* \cdots \varepsilon_{n-1} \varepsilon_{2n}^* \eta_2^*] & \text{n even} \\
-\delta_{\varepsilon_n, \varepsilon_{n+1}} \frac{\lambda(t(\varepsilon_n))}{\lambda(t(\varepsilon_{n+1}))} [\eta_1 \varepsilon_1 \varepsilon_2^* \cdots \varepsilon_{n-1} \varepsilon_{2n}^* \eta_2^*] & \text{n odd}.
\end{cases}$$

Proof. We consider the case $n$ even. The case $n$ odd is similar. We must show $\operatorname{tr}_n(xy) = \operatorname{tr}_{n-1}(E_{A_{n-1}}(x)y)$ for all $x \in A_n$ and $y \in A_{n-1}$. It suffices to check when $x$, $y$ are loops. If

$$x = [\eta_1 \varepsilon_1 \varepsilon_2^* \cdots \varepsilon_{n-1} \varepsilon_{2n}^* \eta_2^*]$$

and $y = [\eta_3 \xi_1 \xi_2^* \cdots \xi_{n-3} \xi_{2n-2} \eta_4^*]$, using the formula above, we have

$$\operatorname{tr}_{n-1}(E_{A_{n-1}}(x)y) = d^{-n} \delta_{\varepsilon_n, \varepsilon_{n+1}} \delta_{x, [1, n], [n+2, 2n+2]} \frac{\lambda(s(\varepsilon_n))}{\lambda(t(\varepsilon_{n+1}))} \operatorname{tr}_{n-1}([\eta_1 \varepsilon_1 \cdots \varepsilon_{n-1} \xi_{n+1} \cdots \xi_{2n-2} \eta_4^*])$$

$$= d^{-n} \delta_{\varepsilon_n, \varepsilon_{n+1}} \delta_{x, [1, n], [n+1, 2n-2]} \lambda(s(\varepsilon_n)) = \operatorname{tr}_n(xy).$$

Definition 3.3.17 (Jones Projections). For $n \in \mathbb{N}$, define distinguished elements of $A_{n+1}$ as follows: if $n$ is odd, define

$$F_n = \sum_{\vec{i}} \sum_{t(\eta) = s(\varepsilon_1)} \frac{[\lambda(t(\varepsilon_n))\lambda(t(\varepsilon_{n+1}))]^{1/2}}{\lambda(s(\varepsilon_1))} [\eta \varepsilon_1 \varepsilon_2^* \cdots \varepsilon_{n-1} \varepsilon_n \varepsilon_{n+1} \varepsilon_{n+1} \varepsilon_{n-1} \cdots \varepsilon_{2n} \varepsilon_{2n}^* \eta^*]$$

where the sum is taken over all vectors $\vec{i} = (i_1, i_2, \ldots, i_{n+1})$ such that

$$[\varepsilon_{i_1} \varepsilon_{i_2}^* \cdots \varepsilon_{i_{n-1}} \varepsilon_{i_n} \varepsilon_{i_{n+1}} \varepsilon_{i_{n+1}} \varepsilon_{i_{n-1}} \cdots \varepsilon_{2n} \varepsilon_{2n}^*] \in G_{n+1, +}$$

If $n$ is even, then define

$$F_n = \sum_{\vec{i}} \sum_{t(\eta) = s(\varepsilon_1)} \frac{[\lambda(s(\varepsilon_n))\lambda(s(\varepsilon_{n+1}))]^{1/2}}{\lambda(t(\varepsilon_1))} [\eta \varepsilon_1 \varepsilon_2^* \cdots \varepsilon_{n-1} \varepsilon_n \varepsilon_{n+1} \varepsilon_{n+1} \varepsilon_{n-1} \cdots \varepsilon_{2n} \varepsilon_{2n}^* \eta^*]$$

with a similar limitation on the vectors $\vec{i} = (i_1, i_2, \ldots, i_{n+1})$.

Lemma 3.3.18. (1) $F_n x F_n = dE_{A_{n-1}}(x) F_n$ for all $x \in A_n$ and

(2) $\operatorname{tr}_{n+1}(xF_n) = d^{-1} \operatorname{tr}_n(x)$ for all $x \in A_n$, i.e., $E_{A_n}(F_n) = d^{-1}$.

Proof. We prove the case $n$ odd. The case $n$ even is similar.
CHAPTER 3. THE EMBEDDING THEOREM FOR FINITE DEPTH SUBFACTOR
PLANAR ALGEBRAS

(1) If \( x = \left[ \xi_1 \xi_2^* \cdots \xi_{n-1}^* \xi_n \cdots \xi_{2n-1}^* \xi_{2n}^* \right] \in A_n \), then

\[
F_n x F_n = \sum \sum \frac{[\lambda(t(\varepsilon_{i_m})) \lambda(t(\varepsilon_{i_{m+1}}))]}{\lambda(s(\varepsilon_{i_n}))} \left[ \eta \varepsilon_{i_1} \varepsilon_{i_2}^* \cdots \varepsilon_{i_{n-1}}^* \varepsilon_{i_n}^* \varepsilon_{i_{n+1}}^* \varepsilon_{i_{n+1}} \cdots \varepsilon_{i_{2n-1}}^* \eta^* \right] \times \\
\sum \frac{[\lambda(t(\varepsilon_{i_j})) \lambda(t(\varepsilon_{i_{j+1}}))]}{\lambda(s(\varepsilon_{i_n}))} \left[ \kappa \varepsilon_{j_1} \varepsilon_{j_2}^* \cdots \varepsilon_{j_{n-1}}^* \varepsilon_{j_n}^* \varepsilon_{j_{n+1}}^* \varepsilon_{j_{n+1}} \cdots \varepsilon_{j_{2n-1}}^* \kappa^* \right]
\]

\[
= \sum \frac{[\lambda(t(\varepsilon)) \lambda(t(\varepsilon_{i_{n+1}}))]}{\lambda(s(\varepsilon))} \left[ \xi_1 \xi_2^* \cdots \xi_{n-1}^* \xi_n \cdots \xi_{2n-1}^* \xi_{2n}^* \right] \times \\
\sum \frac{[\lambda(t(\varepsilon_{i_j})) \lambda(t(\varepsilon_{i_{j+1}}))]}{\lambda(s(\varepsilon))} \left[ \kappa \varepsilon_{j_1} \varepsilon_{j_2}^* \cdots \varepsilon_{j_{n-1}}^* \varepsilon_{j_n}^* \varepsilon_{j_{n+1}}^* \varepsilon_{j_{n+1}} \cdots \varepsilon_{j_{2n-1}}^* \kappa^* \right]
\]

\[
= \delta_{\xi_n, \xi_{n+1}} \frac{[\lambda(t(\varepsilon_{i_n}))]}{\lambda(s(\varepsilon))} \sum \frac{[\lambda(t(\varepsilon)) \lambda(t(\varepsilon_{i_{n+1}}))]}{\lambda(s(\varepsilon))} \left[ \xi_1 \xi_2^* \cdots \xi_{n-1}^* \xi_n \cdots \xi_{2n-1}^* \xi_{2n}^* \right] \\
= dE_{A_{n-1}} x F_n.
\]

(2) Another straightforward calculation.

\[
\]

Proposition 3.3.19 (Basic Construction). For \( n \in \mathbb{N} \), the inclusion

\( A_{n-1} \subset A_n \subset (A_{n+1}, \text{tr}_{n+1}, d^{-1} F_n) \)

is standard. Hence for all \( k \geq 0 \), there are isomorphisms \( \varphi_k : A_k \to M_k \) preserving the trace such that \( \varphi_{k+1}|_{A_k} = \varphi_k \) and \( \varphi_m(F_n) = E_n \) for all \( m > n \).

Proof. We construct the isomorphisms \( \varphi_n \) for \( n \geq 1 \) by induction on \( n \). The base case is finished. Suppose we have constructed \( \varphi_n \) for \( n \geq 1 \). We know that \( M_{n+1} = M_n E_n M_n \) and \( A_n \cong M_n \) via \( \varphi_n \). By Lemmata 3.2.15 and 3.3.18, there is an algebra isomorphism \( h_{n+1} : M_{n+1} \to M_n E_n M_n \) \( A_n F_n A_n \subseteq A_{n+1} \) such that \( E_n \mapsto F_n \). But \( \dim(M_{n+1}) = \dim(A_{n+1}) \), so \( A_{n+1} = A_n F_n A_n \), and we set \( \varphi_{n+1} = h_{n+1}^{-1} \), which extends \( \varphi_n \). Finally, note the \( \varphi_m \)'s preserve the trace by construction and the uniqueness of the Markov trace.

Relative commutants are isomorphic to loop algebras

We provide isomorphisms between the relative commutants of the tower \( (A_n)_{n \geq 0} \) and the spaces \( G_{n, \pm} \).

Proposition 3.3.20 (Central Vectors). A basis for the central vectors \( A_0 \cap A_n \) is given by

\[
S_{0,n} = \left\{ \sum_{t(\varepsilon_{i_1})} [\eta \varepsilon_{i_1} \varepsilon_{i_2}^* \cdots \varepsilon_{2n-1}^* \varepsilon_{2n} \eta^*] \in A_n \left| \varepsilon_{i_1} \varepsilon_{i_2}^* \cdots \varepsilon_{2n-1} \varepsilon_{2n}^* \right] \in G_{n, \pm} \right\}.
\]
A basis for the central vectors $A_1' \cap A_{n+1}$ is given by

$$S_{1,n+1} = \left\{ \sum_{t(\eta) = s(\varepsilon)} [\eta \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}^* \eta^*] \in A_{n+1} \mid [\varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}] \in G_{n,-} \right\}. $$

Proof. Note that if $[\zeta_1 \zeta_2^*] \in A_0$, then we have

$$[\zeta_1 \zeta_2^*] \cdot \sum_{t(\eta) = s(\varepsilon)} [\eta \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}^* \eta^*] = \sum_{t(\eta) = s(\varepsilon)} \delta_{\zeta_2, \eta} [\zeta_1 \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}^*\eta^*] $$

$$= [\zeta_1 \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}^* \zeta_2^*] = \sum_{t(\eta) = s(\varepsilon)} \delta_{\eta, \zeta_1} [\eta \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}^* \zeta_2^*] $$

$$= \left( \sum_{t(\eta) = s(\varepsilon)} [\eta \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}^* \eta^*] \right) \cdot [\zeta_1 \zeta_2^*] $$

Hence $S_{0,n} \subseteq A_0' \cap A_n$. Similarly, $S_{1,n+1} \subseteq A_1' \cap A_n$.

Suppose now that $x \in A_0' \cap A_n$. Then since $1_{A_0} = \sum \eta [\eta \eta^*]$, we have

$$x = \left( \sum_{\eta} [\eta \eta^*] \right) x = \left( \sum_{\eta} [\eta \eta^*] \cdot [\eta \eta^*] \right) x = \sum_{\eta} [\eta \eta^*] \cdot x \cdot [\eta \eta^*] \in \text{span}(S_{0,n}). $$

Similarly, $A_1' \cap A_{n+1} \subseteq \text{span}(S_{1,n+1})$. \qed

Corollary 3.3.21. There are $\ast$-algebra isomorphisms

$$\phi_{n,+} : G_{n,+} \to A_0' \cap A_n \quad \text{and} \quad \phi_{n,-} : G_{n,-} \to A_1' \cap A_{n+1}. $$

If $n = 0$, the isomorphisms are given by

$$\phi_{0,+}(v_+) = \sum_{t(\eta) = v_+} [\eta \eta^*] \quad \text{and} \quad \phi_{0,-}(v_-) = \sum_{t(\eta) = s(\varepsilon): t(\varepsilon) = v_-} [\eta \varepsilon \varepsilon^* \eta^*]. $$

For $n \in \mathbb{N}$, the isomorphisms are given by

$$\phi_{n,+}([\varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}^*]) = \sum_{t(\eta) = s(\varepsilon)} [\eta \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}^* \eta^*] \quad \text{and} $$

$$\phi_{n,-}([\varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}^*]) = \sum_{t(\eta) = s(\varepsilon)} [\eta \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}^* \eta^*]. $$

It will be helpful to have an explicit Pimsner-Popa basis for $A_1$ over $A_0$:
CHAPTER 3. THE EMBEDDING THEOREM FOR FINITE DEPTH SUBFACTOR
PLANAR ALGEBRAS

**Proposition 3.3.22** (Pimsner-Popa Bases). For each \( v_+ \in V_+ \), pick a distinguished \( \eta_{v_+} \) with \( t(\eta_{v_+}) = v_+ \). Set

\[
B_1 = \left\{ \left( \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} \right)^{1/2} \sum_{t(\eta)=s(\varepsilon_1)} [\eta \varepsilon_1 \varepsilon_2^* \eta^*] \left| \varepsilon_1 \varepsilon_2^* \right\} \in G_{1,+} \right\}
\]

and

\[
B_2 = \left\{ \left( \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} \right)^{1/2} [\eta_1 \varepsilon_1 \varepsilon_2^* \eta^*] \mid s(\varepsilon_1) \neq s(\varepsilon_2) \right\}
\]

Then \( B = B_1 \Pi B_2 \) is a Pimsner-Popa basis for \( A_1 \) over \( A_0 \).

**Proof.** Suppose \( x = [\xi_1 \xi_2 \xi_3^*] \in A_1 \).

**Case 1:** Suppose that \( s(\xi_1) = s(\xi_2) \), so \( [\xi_1 \xi_3^*] \in G_{1,+} \). If \( b \in B_2 \), then \( E_{A_0}(b^* x) = 0 \) as the formula will have delta functions \( \delta_{\xi_i, \xi_j} \) for \( i = 1, 2 \). Hence we have

\[
\sum_{b \in B} b E_{A_0}(b^* x) = \sum_{b \in B_1} b E_{A_0}(b^* x) = \sum_{b \in B_1} \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} \sum_{t(\eta)=s(\varepsilon_1)} [\eta \varepsilon_1 \varepsilon_2^* \eta^*] E_{A_0}([\xi_2 \varepsilon_2^* \xi_3^*])
\]

\[
= \sum_{b \in B_1} \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} \sum_{t(\eta)=s(\varepsilon_1)} \delta_{\xi_1, \xi_2} [\eta \varepsilon_1 \varepsilon_2^* \eta^*] E_{A_0}([\xi_2 \varepsilon_2^* \xi_3^*])
\]

\[
= \sum_{b \in B_1} \sum_{t(\eta)=s(\varepsilon_1)} \delta_{\xi_1, \xi_2} [\eta \varepsilon_1 \varepsilon_2^* \eta^*] E_{A_0}([\xi_2 \varepsilon_2^* \xi_3^*])
\]

\[
= \sum_{b \in B_1} \sum_{t(\eta)=s(\varepsilon_1)} \delta_{\xi_1, \xi_2} [\eta \varepsilon_1 \varepsilon_2^* \eta^*] \cdot [\xi_1 \xi_2^*] = [\xi_1 \xi_2^*] = x.
\]

**Case 2:** Suppose that \( s(\xi_1) \neq s(\xi_2) \). If \( b \in B_1 \), then similarly, \( E_{A_0}(b^* x) = 0 \). Hence

\[
\sum_{b \in B} b E_{A_0}(b^* x) = \sum_{b \in B_2} b E_{A_0}(b^* x) = \sum_{b \in B_2} \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} [\eta \varepsilon_1 \varepsilon_2^* \eta^*] E_{A_0}([\xi_2 \varepsilon_2^* \xi_3^*])
\]

\[
= [\xi_1 \xi_2^* \eta(s(\varepsilon_2)) \cdot [\eta(s(\varepsilon_2)) \xi_2^*] = [\xi_1 \xi_2^*] = x.
\]

**Remark 3.3.23.** One could also take

\[
B_2 = \left\{ \left( \frac{d\lambda(s(\varepsilon_2))}{m_+(s(\varepsilon_2)) \lambda(t(\varepsilon_2))} \right)^{1/2} [\eta_1 \varepsilon_1 \varepsilon_2^* \eta^*] \mid s(\varepsilon_1) \neq s(\varepsilon_2) \right\}
\]

**Corollary 3.3.24** (Commutant Conditional Expectations). If

\[
x = \sum_{t(\xi)=s(\xi)} [\xi_1 \xi_2^* \cdots \xi_{2n-1} \xi_{2n}^*] \in A_0 \cap A_n,
\]
the conditional expectation $A'_0 \cap A_n \to A'_1 \cap A_n$ is given by

$$E_{A'_1}(x) = d^{-1} \delta_{\xi_1, \xi_2n} \left( \frac{\lambda(s(\xi_1))}{\lambda(t(\xi_1))} \right) \sum_{t(\xi) = s(\xi); t(\xi) = t(\xi_2)} [\eta \xi_2^* \xi_3 \cdots \xi_{2n-2} \xi_{2n-1} \varepsilon^* \eta^*].$$

**Proof.** Let $B$ be as in Proposition 3.3.22. By Proposition 3.2.24, we have

$$d^2 E_{A'_1}(x) = \sum_{b \in B} b b^* + \sum_{b \in B_2} b b^*.$$

We treat each sum separately:

$$\sum_{b \in B_1} b b^* = \sum_{b \in B_1} \left( \frac{\lambda(s(\xi_2))}{\lambda(t(\xi_2))} \right) \sum_{t(\eta) = s(\xi_2); t(\eta) = t(\xi_1)} [\eta \varepsilon^* \xi_2^* \eta^*] \cdot [\xi_1 \xi_2 \cdots \xi_{2n-1} \varepsilon^* \eta^*] \cdot [\kappa \varepsilon^* \xi_2^* \eta^*]$$

$$= d \sum_{s(\xi_2) = s(\xi_2)} \left( \frac{\lambda(s(\xi_2))}{\lambda(t(\xi_2))} \right) \sum_{t(\eta) = s(\xi_2); t(\eta) = t(\xi_1)} \delta_{\eta, \varepsilon^* \xi_2^* \eta^*} [\xi_1 \xi_2 \cdots \xi_{2n-1} \varepsilon^* \eta^*]$$

$$= d \sum_{t(\eta) = s(\xi_2); t(\eta) = t(\xi_1)} \left( \frac{\lambda(s(\xi_1))}{\lambda(t(\xi_1))} \right) \delta_{\xi_1, \xi_2} [\eta \varepsilon^* \xi_2^* \eta^*].$$

Similarly, we have

$$\sum_{b \in B_2} b b^* = d \sum_{t(\eta) = s(\xi_2); t(\eta) = t(\xi_1)} \left( \frac{\lambda(s(\xi_1))}{\lambda(t(\xi_1))} \right) \delta_{\xi_1, \xi_2} [\eta \varepsilon^* \xi_2^* \eta^*].$$

Putting these two together, we get the desired formula for $E_{A'_1}(x)$.

**The bipartite graph planar algebra and the isomorphism**

We refer the reader to [Jon00] for the full definition of the planar algebra of a bipartite graph.

Let $G_\ast$ be the planar algebra of the bipartite graph $\Gamma$ with spin vector $\lambda$ as in Subsections 3.3 and 3.3. We briefly recall the action of tangles on the $G_{n, \pm}$, and we calculate some necessary examples.

A state $\sigma$ of a tangle $T$ is a way of assigning the regions and strings of $T$ with compatible vertices and edges of $\Gamma$ respectively, i.e., if a string $S$ of $T$ partitions the unshaded region $R_+$ from the shaded region $R_-$, then for $\sigma(S) \in \mathcal{E}$, $s(\sigma(S)) = \sigma(R_+) \in \mathcal{V}_+$ and $t(\sigma(S)) = \sigma(R_-) \in \mathcal{V}_-.

Define the output loop $\ell_\sigma$ as the loop obtained by reading clockwise around the outer boundary of $T$ once it has been labeled by $\sigma$. 

Suppose now that $T$ has $n$ input disks, and $\ell = \ell_1 \otimes \cdots \otimes \ell_n$ is a simple tensor of loops where $\ell_i$ is a loop in $G_{n_i, \pm_i}$. Then the action of $T$ on $\ell$ is given by

$$T(\ell) = \sum_{\text{states } \sigma} c(\sigma, \ell) \ell_\sigma,$$

where $c(\sigma, \ell)$ is a correction factor defined as follows:

1. First, label the regions and strings of $T$ adjacent to the input disks with the edges and vertices which compose the $\ell_i$’s. If the labeling contradicts $\sigma$, then $c(\sigma, \ell) = 0$.

2. If the labels agree, put the tangle in a standard form similar to Section 3.2, where the only difference is that the half the strings emanate from the top of the input rectangles, and half the strings emanate down, but the $*$ is still on the left side. Let $E(T)$ be the set of local extrema of the strings of the standard form of the tangle. For each $e \in E(T)$, let $\text{conv}(e)$ be the vertex assigned by $\sigma$ to the convex region of the extrema, and let $\text{conc}(e)$ be the vertex assigned to the concave region. Set

$$k_e = \sqrt{\frac{\lambda(\text{conv}(e))}{\lambda(\text{conc}(e))}}.$$

Below is an example of an extrema $e$ on a string $S$ with $\sigma(S) = \varepsilon$, connecting vertices $w, v$:

$$\varepsilon \quad \text{concave} \quad \text{convex} \quad \longrightarrow \quad k_e = \sqrt{\frac{\lambda(w)}{\lambda(v)}}.$$

Note that $\text{conv}(e)$ may be in either $V_+$ or $V_-$. Finally, set

$$c(\sigma, \ell) = \prod_{e \in E(T)} k_e.$$

The $*$-structure on the bipartite graph planar algebra is given as follows: if $T, \ell$ are as above, then

$$T(\ell_1^* \otimes \cdots \otimes \ell_n^*) = T^*(\ell_1 \otimes \cdots \otimes \ell_n)^*$$

where $T^*$ is the mirror image of $T$, and the adjoint of a loop is the loop traversed backwards as in Definition 3.3.5.

**Remark 3.3.25.** Contractible loops are traded for a multiplicative factor of $d$ as $\lambda$ is a Frobenius-Perron eigenvector (see Definition 3.3.13).

**Remark 3.3.26.** Note from Corollary 3.3.21 that there is a natural inclusion identification $G_{n,-} \to G_{n+1,+}$ given by

$$[\varepsilon_1^* \varepsilon_2 \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}] \mapsto \sum_{t(\varepsilon) = s(\varepsilon_1)} [\varepsilon \varepsilon_1^* \varepsilon_2 \cdots \varepsilon_{2n-1}^* \varepsilon_{2n} \varepsilon^*].$$
Examples 3.3.27. (0) If $\ell_1, \ell_2 \in G_{n,\pm}$, then $\ell_1 \cdot \ell_2 = \begin{array}{c} \ell_1 \\ \ell_2 \end{array}$, the shading depending on $n, \pm$.

(1) For $n \in \mathbb{N}$ odd,

$$
\sum_{\vec{i}} \frac{[\lambda(t(\varepsilon_{i_n}))\lambda(t(\varepsilon_{i_{n+1}}))]^{1/2}}{\lambda(s(\varepsilon_{i_n}))} [\varepsilon_{i_1} \cdots \varepsilon_{i_{n-1}} \varepsilon_{i_n}^{*} \varepsilon_{i_{n+1}}^{*} \varepsilon_{i_{n+2}}^{*} \varepsilon_{i_{n+3}}^{*} \cdots \varepsilon_{i_1}^{*}],
$$

where the sum is taken over all vectors $\vec{i} = (i_1, i_2, \ldots, i_{n+1})$ such that

$$
[\varepsilon_{i_1} \varepsilon_{i_2}^{*} \cdots \varepsilon_{i_{n-1}}^{*} \varepsilon_{i_n}^{*} \varepsilon_{i_{n+1}}^{*} \varepsilon_{i_{n+2}}^{*} \varepsilon_{i_{n+3}}^{*} \cdots \varepsilon_{i_1}^{*}] \in G_{n+1, +}.
$$

There is a similar formula for $n$ even. (Compare with Definition 3.3.17.)

(2) Suppose $\ell = [\varepsilon_1 \varepsilon_2^{*} \cdots \varepsilon_{2n-1}^{*} \varepsilon_{2n}] \in G_{n, +}$.

(i) If $n$ is even, then

$$
\begin{array}{c} \ell \end{array} = \frac{\lambda(s(\varepsilon_{i_n}))}{\lambda(t(\varepsilon_{i_n}))} [\varepsilon_1 \varepsilon_2^{*} \cdots \varepsilon_{n-1}^{*} \varepsilon_n + 2 \varepsilon_{2n-1} \cdots \varepsilon_{2n}],
$$

with a similar formula for $n$ odd. (Compare with Proposition 3.3.16.)

(ii) If $n$ is even, then

$$
\begin{array}{c} \ell \end{array} = \sum_{s(\varepsilon) = s(\varepsilon_{i_n})} [\varepsilon_1 \varepsilon_2^{*} \cdots \varepsilon_{n}^{*} \varepsilon_{n+1} \cdots \varepsilon_{2n-1} \varepsilon_{2n}],
$$

with a similar formula for $n$ odd. (Compare with Definition 3.3.10.)

(iii) \begin{array}{c} \ell \end{array} = \frac{\lambda(s(\varepsilon_1))}{\lambda(t(\varepsilon_1))} [\varepsilon_2^{*} \varepsilon_3^{*} \cdots \varepsilon_{2n-1}^{*} \varepsilon_{2n-2}^{*} \varepsilon_{2n-1}].

(Compare with Proposition 3.3.24 and Remark 3.3.26.)

(3) If $\ell = [\varepsilon_1^{*} \varepsilon_2 \cdots \varepsilon_{2n-1}^{*} \varepsilon_{2n}] \in G_{n, -}$, then

$$
\begin{array}{c} \ell \end{array} = \sum_{t(\varepsilon) = s(\varepsilon_1)} [\varepsilon_1^{*} \varepsilon_2 \cdots \varepsilon_{2n-1}^{*} \varepsilon_{2n}^{*}],
$$

which may be identified with $\ell \in G_{n+1, +}$ by Remark 3.3.26.
CHAPTER 3. THE EMBEDDING THEOREM FOR FINITE DEPTH SUBFACTOR PLANAR ALGEBRAS

Theorem 3.3.28. The canonical planar ∗-algebra $P_\bullet$ associated to $M_0 \subset (M_1, \text{tr}_1)$ is isomorphic to the bipartite graph planar ∗-algebra $G_\bullet$ of the Bratteli diagram $\Gamma'$ for the inclusion.

Proof. To show that the ∗-algebra isomorphisms $G_n, + \rightarrow A_0' \cap A_n \rightarrow M_0' \cap M_n \rightarrow P_n, +$ give an isomorphism of planar ∗-algebras $G_\bullet \rightarrow P_\bullet,$ we must check that

1) they map Jones projections in $G_\bullet$ to those in $P_\bullet,$ and
2) they preserve the action of annular tangles.

Both follow immediately from Examples 3.3.27 and the proof of Lemma 3.2.49. □

3.4 The Embedding Theorem

Let $Q_\bullet$ be a finite depth subfactor planar algebra of modulus $d.$ Pick $r \geq 0$ minimal such that $Q_{2r,+} \subset Q_{2r+1,+} \subset (Q_{2r+2,+}, e_{2r+1})$ is standard (with the usual trace). Note this is possible if and only if $Q_\bullet$ has finite depth. In fact, $Q_{k,+} \subset Q_{k+1,+} \subset (Q_{k+2,+}, e_{k+1})$ is standard for all $k \geq 2r.$ For $n \geq 0,$ set $M_n = Q_{2r+n,+}$ and $F_{n+1} = E_{2r+n+1}$ (shifted Jones projections). Let $P_\bullet$ be the canonical planar ∗-algebra associated to the inclusion $M_0 \subset M_1,$ i.e.,

$$P_{n,+} = M_0' \cap M_n = Q_{2r,+}' \cap Q_{2r+n,+} \quad \text{and} \quad P_{n,-} = M_1' \cap M_{n+1} = Q_{2r+1,+}' \cap Q_{2r+n+1,+},$$

where we suppress the isomorphisms $\theta_n$ with the tensor products of $Q_{2r+1,+}$ over $Q_{2r,+}.$

Theorem 3.4.1. Define $\Phi : Q_\bullet \rightarrow P_\bullet$ by adding $2r$ strings to the left for $x \in Q_{n,+}$ and adding $2r + 1$ strings to the left for $x \in Q_{n,-}$.

Then $\Phi$ is an inclusion of planar ∗-algebras.

Proof. We use Lemma 3.2.49. Note that $\Phi(x^*) = \Phi(x)^*$ and $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in Q_{n,\pm}.$
(1) Since $\Phi(E_j) = E_{2r+j} = F_j$ for all $j \in \mathbb{N}$, we have $\Phi(E_jx) = F_j\Phi(x)$ and $\Phi(xE_j) = \Phi(x)F_j$ for all $x \in Q_{n,\pm}$ and all $j \in \mathbb{N}$.

(2) Note that

(i) For $n \in \mathbb{N}$, $\Phi(E_{Q_{n-1,+}}(x)) = E_{P_{n-1,+}}(\Phi(x))$ since $E_{Q_{n-1,+}} \big|_{Q_{2r,+}\cap Q_{2r+n,+}} = E_{P_{n-1,+}} \big|_{P_{n,+}} = E_{P_{n-1,+}}$ (since $Q_{2r,+} \subset Q_{2r+n-1,+}$, we have that $E_{Q_{2r+n-1,+}}$ preserves $Q_{2r,+}$-central vectors as it is $Q_{2r+n-1,+}$-bilinear).

(ii) $\Phi(\beta_{n+1}(x)) = \beta_{n+1}(\Phi(x))$ for all $x \in Q_{n,+}$ since the inclusion $P_{n,+} \rightarrow P_{n+1,+}$ is the restriction of the inclusion $Q_{2r+n,+} \rightarrow Q_{2r+n+1,+}$.

(iii) Let $B = \{b\}$ be a Pimsner-Popa basis for $M_1 = Q_{2r+1,+}$ over $M_0 = Q_{2r,+}$. Since each $b \in B$ is an $(2r+1,+)$-box in $Q_{2r+1,+}$,

\[
\frac{1}{d} \sum_{b \in B} b \Phi(x) b^* = 1_{P_{2r+2,+}}
\]

Then by Proposition 3.2.24 and Theorem 3.2.50, for all $x \in Q_{n,+}$,

\[
\gamma_n^+(\Phi(x)) = \frac{1}{d} \sum_{b \in B} b \Phi(x) b^* = \frac{1}{d} \sum_{b \in B} b \Phi(x) b^* = \frac{1}{d} \sum_{b \in B} b \Phi(x) b^* = 1_{P_{2r+2,+}} = \Phi(\gamma_n^+(x)).
\]
(3) The inclusion $i^-_n : P_{n,-} \to P_{n+1,+}$ is the identity in the canonical planar $*$-algebra. If $x \in Q_{n,-}$, then we have

$$i^-_n(\Phi(x)) = \Phi(x) = \Phi(i^-_n(x)).$$

**Corollary 3.4.2.** Let $N \subset M$ be a finite index, finite depth II$_1$-subfactor, and let $P_\bullet$ be the associated canonical subfactor planar algebra. Let $\Gamma$ be the principal graph of $N \subset M$, and let $G_\bullet$ be the bipartite graph planar algebra of $\Gamma$. Then there is an embedding of planar algebras $P_\bullet \to G_\bullet$. 
Chapter 4

A planar calculus for infinite index subfactors

4.1 Introduction

Jones initiated the modern theory of subfactors in [Jon83]. Given a finite index $II_1$-subfactor $A_0 \subseteq A_1$, he used the basic construction to obtain the Jones tower $(A_n)_{n \geq 0}$, obtained iteratively by adding the Jones projections $(e_n)_{n \geq 1}$ which satisfy the Temperley-Lieb relations. Jones used this structure to show the index lies in the range $\{4 \cos^2(\pi/n) | n \geq 3\} \cup [4, \infty)$, and he found an example for each value.

Much initial subfactor research classified hyperfinite subfactors of small index ($[A_1: A_0] \leq 4$) by studying the standard invariant, i.e., the two towers of higher relative commutants $(A'_i \cap A_j)_{i=0,1,j \geq 0}$ [Ocn88, GdlHJ89, Izu91, Pop94]. This combinatorial data was axiomatized in three slightly different structures: paragroups [Ocn88], $\lambda$-lattices [Pop95], and planar algebras [Jon99]. When combined, these viewpoints produce strong results, e.g., standard invariants with index in $(4, 5)$ are completely classified, excluding the $A_\infty$ standard invariant at each index value [Pop93] (see [MS11, MPPS12, IJMS11, PT12] for more details).

Some finite index results generalize to infinite index subfactors, such as discrete, irreducible, “depth 2” subfactors correspond to outer (cocycle) actions of Kac algebras [HO89, EN96], and the classical Galois correspondence still holds for outer actions of infinite discrete groups and minimal actions of compact groups [ILP98].

In his Ph.D. thesis [Bur03], Burns studied rotations and extremality for infinite index, since the key to isotopy invariance of Jones’ planar calculus in [Jon99] is the rotation operator (also known to Ocneanu). Burns’ essential observation for finite index was that the centralizer algebras $A'_0 \cap A_n$ coincide with the central $L^2$-vectors:

$$A'_0 \cap L^2(A_n) = \{ \zeta \in L^2(A_n) | a\zeta = \zeta a \text{ for all } a \in A_0 \}.$$
Burns found an elegant formula for the rotation on $P_{n,+} = A_0' \cap \bigotimes_{A_0}^n L^2(A_1)$:

$$\rho = \sum_{\beta} L_\beta R^*_\beta$$

where $\{\beta\}$ is a Pimsner-Popa basis for $A_1$ over $A_0$, $L_\beta$ is the left creation operator, and $R^*_\beta$ is the right annihilation operator (see Definition 4.2.4). This approach was generalized in [JP11] to define a canonical planar $\ast$-algebra associated to a strongly Markov inclusion of finite von Neumann algebras. Burns adapted his formula to infinite index, and he showed existence of the rotation on the central $L^2$-vectors is equivalent to approximate extremality of the subfactor.

In infinite index, $A_0' \cap A_n$ and $A_0' \cap L^2(A_n)$ do not coincide. One naturally asks:

**Question 4.1.1.** What is a suitable standard invariant for infinite index subfactors?

A definitive answer to Question 4.1.1 is not yet known. On one hand, we have the two towers of centralizer algebras $(A_i' \cap A_j)_{i=0,1,j \geq 0}$ in which we can multiply (the shift isomorphisms $A_i' \cap A_j \cong A_{i+2}' \cap A_{j+2}$ still hold by [EN96]). On the other hand, we have the central $L^2$-vectors on which we have Burns’ rotation (in the approximately extremal case) and graded multiplication in the sense of [GJS10] (tensoring of central vectors). However, the operator valued weights which replace the conditional expectations do not preserve these spaces and may not be well-defined. All this structure is necessary for a good planar calculus. We ask:

**Question 4.1.2.** What is the strongest planar calculus we can define for infinite index subfactors?

In this paper, we propose an answer to Question 4.1.2 using both centralizer algebras and central $L^2$-vectors. We do so in more generality, starting with a bimodule $A H_A$ over a $II_1$-factor $A$ (one recovers the subfactor case when $A = A_0$ and $H = L^2(A_1)$). First, we set $H^n = \bigotimes_A^n H$, $Q_n = A' \cap (A' \cap B(H^n))$ (the centralizer algebras), and $P_n = A' \cap H^n = \{\zeta \in H^n | a\zeta = \zeta a \text{ for all } a \in A\}$ (the central $L^2$-vectors). As mentioned above, the $P_n$’s naturally form a graded algebra $P_\bullet$ in the sense of [GJS10] under relative tensor product. We represent central vectors in $P_n$ as in [GJS10] by boxes with $n$ strings emanating from the top, and we denote graded multiplication (relative tensor product) of $\zeta_m \in P_m$ and $\zeta_n \in P_n$ by

$$\zeta_m \otimes \zeta_n = \begin{array}{c} m \\ \zeta_m \end{array} \begin{array}{c} n \\ \zeta_n \end{array} \in P_{m+n}.$$

We represent elements of $Q_n$ as boxes with strings emanating from top and bottom. For $\zeta \in P_n$, note that the creation-annihilation operator $L(\zeta)L(\zeta)^* = R(\zeta)R(\zeta)^*$ lies in $Q_n$,
which we represent as

\[ L(\zeta)L(\zeta)^* = \begin{array}{c}
\zeta \\
\zeta \\
\zeta \\
\zeta \\
\end{array} \in Q_n. \]

**Theorem 4.1.3.** The extended positive cones \( \widehat{Q}_n^+ \) (in the sense of [Haa79]) naturally form an algebra \( \widehat{Q}_n^+ \) over the operad \( \mathbb{BP} \) generated by the oriented tangles

![Diagram of oriented tangles]

for \( m, n \geq 0 \) up to planar isotopy. (We suppress external disks, draw one thick string labelled \( n \) for \( n \) individual strings, and orient all strings upward unless otherwise specified.)

Moreover, the \( \mathbb{BP} \)-algebra \( \widehat{Q}_n^+ \) and graded algebra \( P_n \) are compatible: if \( z \in \widehat{Q}_n^+ \) and \( \zeta \in P_n \), then

\[ z(\omega \zeta) = \begin{array}{c}
\zeta \\
z \\
\zeta \\
z \\
\end{array} = \begin{array}{c}
\zeta \\
\zeta \\
z \\
z \\
\end{array} = \text{Tr}_n(L(\zeta)L(\zeta)^* \cdot z) \]

where \( \text{Tr}_n \) is the canonical trace on \( Q_n \) coming from the right A-action on \( H^n \). (Note that the multiplication tangle only makes sense once we take the trace by [Haa79]. See Theorem 4.2.14 for more details.)

We generalize to bimodules Burns’ work on rotations: an operator \( \rho \) on the central \( L^2 \)-vectors \( P_n \) is a Burns rotation if for all left and right bounded vectors \( b_1, \ldots, b_n \in H \) (omitting the subscript \( A \) on the tensors,)

\[ \langle \rho(\zeta), b_1 \otimes \cdots \otimes b_n \rangle = \langle \zeta, b_2 \otimes \cdots \otimes b_n \otimes b_1 \rangle. \]

Note this equation implies the uniqueness and periodicity of \( \rho \) if it exists. We generalize Burns’ notion of (approximate) extremality, and we prove the following theorem:

**Theorem 4.1.4.** Consider the following statements (include all or none of the parenthetical statements):

1. \( H^n \) is (approximately) extremal for some \( n \geq 1 \),
(2) $H^n$ is (approximately) extremal for all $n \geq 1$,

(3) The (possibly non-)unitary $\rho$ exists on $P_{2n}$ for all $n \geq 1$, and

(4) The (possibly non-)unitary $\rho$ exists on $P_{2n}$ for some $n \geq 1$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). If $H$ is symmetric, then (4) $\Rightarrow$ (1).

When $\rho$ exists, we represent it diagrammatically by

$$\rho^n(\zeta) = \begin{array}{c}
\zeta \\
\zeta \\
\zeta
\end{array}$$

for $\zeta \in P_{m+n}$,

(well-defined by Corollary 4.4.16) and these diagrams are compatible with the diagrams above in the sense of Theorem 4.4.17.

Interestingly, we find our planar structure without the use of Jones’ basic construction and resulting Jones projections!

Outline:

In Section 4.2, we give a brief introduction to modules, the relative tensor product, extended positive cones, and operator valued weights. Subsections 4.2 and 4.2 provide some helpful, well-known results for the convenience of the reader.

In Subsection 4.3, starting with our $A - A$ bimodule $H$, we introduce $H^n$ along with two towers of algebras $C_n, C_n^{op}$, a tower of centralizer algebras $Q_n = C_n \cap C_n^{op}$, and the central $L^2$-vectors $P_n$. We then compute formulas for the various canonical maps associated with these towers. In Subsection 4.3, we show the extended positive cones (in the sense of [Haa79]) of the centralizer algebras $\hat{Q}_n^+$ naturally form an algebra over an operad $\mathbb{B}\mathbb{P}$ (we use positive cones so we can “conditionally expect” using operator valued weights). In Subsection 4.3, we show that the vectors in $P_\bullet$ are left and right $A$-bounded and form a graded algebra in the sense of [GJS10]. We then show the compatibility of $\hat{Q}_n^+$ and $P_\bullet$ in Subsection 4.3.

Subsection 4.4 defines extremality for bimodules and Burns rotations. In Subsection 4.4, we show how the Burns rotation fits in our planar calculus, and in Subsection 4.4, we show that (approximate) extremality implies the existence of the Burns rotation (Theorem 4.4.20). A converse of this theorem for symmetric bimodules is obtained in Subsection 4.4, which finishes the proof of Theorem 4.1.4.

Subsection 4.5 discusses centralizer algebras $Q_n$ and central $L^2$-vectors $P_n$ for some basic examples, including the infinite index group-subgroup subfactor, and Subsection 4.5 determines if the examples are (approximately) extremal. In particular, Corollaries 4.5.9, 4.5.11, and 4.5.20 give an extremal infinite index subfactor for which $\dim(Q_n) < \infty$ and $\dim(P_n) = 1$ for all $n \in \mathbb{N}$. This example contrasts Burns’ example of an infinite index subfactor with a type $III$ summand in a higher relative commutant [Bur03].

Throughout the paper, we need some technical results which have been included in the last few sections. Section 4.6 shows that the relative tensor product of extended positive


cones is well-defined and associative, which is necessary for our planar calculus. Section 4.7 discusses the operad $\mathbb{BP}$ which acts on the positive cones $\hat{Q}_n^+$, including results on generating sets of tangles, standard form of tangles, and that the action is well-defined. In Section 4.8, we axiomatize the notion of extended positive cone to make rigorous the idea of a planar algebra over such objects. The main intricacy is that we must make multiplication by $\infty$ well-defined.

**Future research:**

The annular Temperley-Lieb category, especially the rotation, played an important role in the construction of certain exotic finite index subfactors [Pet10, BMPS09]. In a future paper with Jones, we will incorporate the odd Jones projections for infinite index (see [Bur03]) into the planar calculus, and we will give the analog of the annular Temperley-Lieb category for infinite index. We hope this viewpoint will be as fruitful as in the finite index case.

The results of this paper should generalize to bimodules over an arbitrary finite von Neumann algebra. As it requires substantial calculations while obscuring the main new ideas presented here, this generalization will appear in a future paper.

Finally, it would be interesting to try to connect Connes’ results on self-dual positive cones [Con74] to the extended positive cones axiomatized in Section 4.8.

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## 4.2 Preliminaries

**Notation 4.2.1.**

- Throughout this paper, a trace on a finite von Neumann algebra means a faithful, normal, tracial state unless otherwise specified.

- $A$ will always denote a finite von Neumann algebra with trace $\text{tr}_A$.

- We use the notation $\hat{a}$ to denote the image of $a \in A$ in $L^2(A, \text{tr}_A)$. 
• For a semifinite von Neumann algebra $M$ with normal, faithful, semifinite (n.f.s.) trace $\text{Tr}_M$, we write

\[ n_{\text{Tr}_M} = \{ x \in M | \text{Tr}_M(x^* x) < \infty \} \]  
and  
\[ m_{\text{Tr}_M} = n_{\text{Tr}_M}^* n_{\text{Tr}_M} = \text{span} \{ x^* y | x, y \in n_{\text{Tr}_M} \}. \]

**Modules and the relative tensor product**

This exposition follows [Con80, Sau83, Pop94, EN96, Bis97, EV00, Bur03].

**Definition 4.2.2** (Left modules). If $A K$ is a left Hilbert $A$-module, then the set of left $A$-bounded vectors is given by

\[ D(AK) = \{ \eta \in K | \|a\eta\|_2 \leq \lambda \|a\|_2 \text{ for some } \lambda \geq 0 \}, \]

and each $\eta \in D(AK)$ gives a bounded map $R(\eta) : L^2(A) \to H$ by the extension of $\hat{a} \mapsto a\eta$. For $\eta_1, \eta_2 \in D(AK)$, we have an $A$-valued inner product given by

\[ A\langle \eta_1, \eta_2 \rangle = JR(\eta_1)^* R(\eta_2) J \in A \]
satisfying

1. $A\langle a\eta_1 + \eta_2, \eta_3 \rangle = a_A \langle \eta_1, \eta_3 \rangle + \langle \eta_2, \eta_3 \rangle$,
2. $A\langle \eta_1, \eta_2 \rangle^* = A\langle \eta_2, \eta_1 \rangle$, and
3. $A\langle x\eta_1, \eta_2 \rangle = A\langle \eta, x^* \eta_2 \rangle$

for all $a \in A$, $x \in A' \cap B(K)$, and $\eta_1, \eta_2, \eta_3 \in D(AK)$ (note $x\eta_i \in D(AK)$).

An $A K$-basis is a set of vectors $\{ \alpha \} \subset D(AK)$ such that

\[ \sum_{\alpha} R(\alpha) R(\alpha)^* = 1_K \iff \sum_{\alpha} A\langle \eta, \alpha \rangle \alpha = \eta \text{ for all } \eta \in D(AK). \]

$A K$-bases exist by [Con80].

The canonical trace on $A' \cap B(K)$ is given by $\text{Tr}_{A' \cap B(K)}(x) = \sum_\alpha \langle x\alpha, \alpha \rangle$ where $\{ \alpha \}$ is any $A K$ basis.

If $\eta \in D(AK)$, then $\text{Tr}_{A' \cap B(K)}(R(\eta) R(\eta)^*) = \text{tr}_A(A \langle \eta, \eta \rangle) = \|\eta\|_2^2$.

**Definition 4.2.3** (Right modules). A right Hilbert $A$-module is the same as a left Hilbert $A^{\text{op}}$-module. If $H_A$ is a right Hilbert $A$-module, we write $\xi a$ for $a^{\text{op}} \xi$ for all $a^{\text{op}} \in A^{\text{op}}$. We get parallel definitions:

The set of right $A$-bounded vectors is given by

\[ D(H_A) = \{ \xi \in H | \|\xi a\|_2 \leq \lambda \|a\|_2 \text{ for some } \lambda \geq 0 \}. \]
Each $\xi \in D(H_A)$ defines a bounded map $L(\xi): L^2(A) \to H$ by the extension of $a \mapsto \xi a$.

For $\xi_1, \xi_2 \in D(H_A)$, we have an $A$-valued inner product given by

$$\langle \xi_1|\xi_2\rangle_A = L(\xi_1)^*L(\xi_2) \in A$$

satisfying

1. $\langle \xi_1|\xi_2a + \xi_3\rangle_A = \langle \xi_1|\xi_2\rangle_Aa + \langle \xi_1|\xi_3\rangle_A,$
2. $\langle \xi_1|\xi_2\rangle_A^* = \langle \xi_2|\xi_1\rangle_A,$ and
3. $\langle x\xi_1|\xi_2\rangle_A = \langle \xi_1|x^*\xi_2\rangle_A$

for all $a \in A$, $x \in (A^{op})' \cap B(H)$, and $\xi_1, \xi_2, \xi_3 \in D(H_A)$ (note $x\xi_i \in D(H_A)$).

An $H_A$-basis is a set of vectors $\{\beta\} \subset D(H_A)$ such that

$$\sum_\beta L(\beta)L(\beta)^* = 1_H \iff \sum_\beta \langle \beta|\xi\rangle_A = \xi \text{ for all } \xi \in D(H_A).$$

$H_A$-bases exist by [Con80].

The canonical trace on $(A^{op})' \cap B(H)$ is given by $\text{Tr}_{(A^{op})' \cap B(H)}(x) = \sum_\beta \langle x\beta, \beta \rangle$ where $\{\beta\}$ is any $H_A$ basis.

If $\xi \in D(H_A)$, then $\text{Tr}_{(A^{op})' \cap B(H)}(L(\xi)L(\xi)^*) = \text{tr}_A(\langle \xi|\xi\rangle_A) = \|\xi\|_2^2$.

**Definition 4.2.4** (Relative tensor product). The relative tensor product $H \otimes_A K$ is given by one of the three equivalent definitions:

1. the completion of the algebraic tensor product $D(H_A) \otimes_A K$ under the pseudo-norm induced by the sesquilinear form $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \langle \xi_2|\xi_1\rangle_A\eta_1, \eta_2 \rangle$,
2. the completion of the algebraic tensor product $H \otimes_A D(AK)$ under the pseudo-norm induced by the sesquilinear form $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1A\langle \eta_1, \eta_2 \rangle, \xi_2 \rangle_H$, or
3. the completion of the algebraic tensor product $D(H_A) \otimes_A D(AK)$ under the pseudo-norm induced by the sesquilinear form

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1A\langle \eta_1, \eta_2 \rangle, \xi_2 \rangle_H = \langle \langle \xi_2|\xi_1\rangle_A\eta_1, \eta_2 \rangle_K.$$

The image of $\xi \otimes \eta$ in $H \otimes_A K$ is denoted $\xi \otimes \eta$. (This notation avoids confusion with the operators $x \otimes_A y$ as in Lemma 4.6.4.)

Given $\xi \in D(H_A)$ and $\eta \in D(AK)$, we get bounded creation operators $L_\xi: K \to H \otimes_A K$ by $\eta' \mapsto \xi \otimes \eta'$ and $R_\eta: H \to H \otimes_A K$ by $\eta' \mapsto \xi' \otimes \eta$, whose adjoints are the annihilation operators given by $L_\xi^*(\xi' \otimes \eta') = \langle \xi'|\xi\rangle_A\eta'$ and $R_\eta^*(\xi' \otimes \eta') = \xi'\langle \eta', \eta \rangle_A$. 
Definition 4.2.5 (Fiber product, [Sau85, EV00]). Suppose \( A^\text{op} \subset M_1 \subset B(H) \) and \( A \subset M_2 \subset B(K) \). Then we define

\[
M_1' \otimes_A M_2' = \{ x \otimes_A y | x \in M_1' \text{ and } y \in M_2' \} \subset B(H \otimes_A K)
\]

(see Section 4.6 and Lemma 4.6.4), and the fiber product of \( M_1 \) and \( M_2 \) over \( A \) is given by \( M_1 \ast_A M_2 = (M_1' \otimes_A M_2')' \). The fiber product satisfies:

1. \( (M_1 \ast_A M_2) \cap (N_1 \ast_A N_2) = (M_1 \cap N_1) \ast_A (M_2 \cap N_2) \) and
2. \( M_1 \ast_A A = ((A^\text{op})' \cap M_1) \otimes_A 1_K \) and \( A^\text{op} \ast_A M_2 = 1_H \otimes_A (A' \cap M_2) \).

In particular,

\[
(B(H) \ast_A A)' = ((A^\text{op})' \otimes_A 1_K)' = A^\text{op} \ast_A B(K) = 1_H \otimes_A A'.
\]

Some easy facts about the relative tensor product

The following are well-known to experts, but we reproduce them here for the sake of completeness and the reader’s convenience. For this subsection, \( H_A \) is a right Hilbert \( A \)-module, and \( A_K \) is a left Hilbert \( A \)-module unless otherwise stated.

Lemma 4.2.6. Suppose \( \{ \beta \} \) is an \( H_A \)-basis. Then if \( u \in U((A^\text{op})' \cap B(H)) \), \( \{ u \beta \} \) is another \( H_A \)-basis. If \( v \in U(A) \), then \( \{ \beta v \} \) is also an \( H_A \)-basis. A similar result holds for left modules.

Proof. For \( u \in (A^\text{op})' \cap B(H) \), \( L(u \beta)L(u \beta)^* = u L(\beta)L(\beta)^* u^* \). Thus

\[
\sum_{u \beta} L(u \beta)L(u \beta)^* = u \left( \sum_{\beta} L(\beta)L(\beta)^* \right) u^* = 1_H.
\]

If \( v \in U(A) \), then \( L(\beta v)L(\beta v)^* = L(\beta)vv^*L(\beta)^* = L(\beta)L(\beta)^* \), and the result follows. \( \square \)

Lemma 4.2.7. Let \( \xi_1, \xi_2 \in D(H_A) \) and \( \eta_1, \eta_2 \in D(A_K) \). Then \( L_{\xi_1}^* L_{\xi_2} \in B(K) \) is left multiplication by \( \langle \xi_1 | \xi_2 \rangle_A \) and \( R_{\eta_1}^* R_{\eta_2} \in B(H) \) is right multiplication by \( _A \langle \eta_1, \eta_2 \rangle \).

Proof. \( \langle L_{\xi_1}^* L_{\xi_2}, \eta_1, \eta_2 \rangle = \langle \xi_2 \otimes \eta_1, \xi_1 \otimes \eta_2 \rangle = \langle \xi_2 | \xi_1 \rangle_{A} \eta_1, \eta_2 \rangle \). The other is as trivial. \( \square \)

Lemma 4.2.8. If \( \{ \beta \} \) is an \( H_A \)-basis, then \( \sum_{\beta} L_{\beta}^* L_{\beta} = 1_{H \otimes_A K} \). Similarly, if \( \{ \alpha \} \) is an \( A_H \)-basis, then \( \sum_{\alpha} R_{\alpha}^* R_{\alpha} = 1_{H \otimes_A K} \).

Proof. We prove the first statement. Suppose \( \xi \in D(H_A) \) and \( \eta \in D(A_K) \). Then

\[
\sum_{\beta} L_{\beta}^* L_{\beta} (\xi \otimes \eta) = \sum_{\beta} L_{\beta} (L_{\beta}^* L_{\beta}) \eta = \sum_{\beta} \beta(\beta | \xi)_{A} \otimes \eta = \xi \otimes \eta.
\]

\( \square \)
Lemma 4.2.9. Suppose \( \eta \in \mathcal{A}K \) and \( \eta' \in D(\mathcal{A}K) \). Then there is a unique \( \mathcal{A}\langle \eta', \eta \rangle \in L^2(\mathcal{A}) \subset L^1(\mathcal{A}) \) such that \( \langle a\eta, \eta' \rangle_K = \langle a, \mathcal{A}\langle \eta', \eta \rangle \rangle_{L^2(\mathcal{A})} \) for all \( a \in \mathcal{A} \). A similar result holds for right modules.

Proof. If \( \xi \in D(\mathcal{A}K) \), this is just the usual Radon-Nikodym derivative, and

\[
\|\mathcal{A}\langle \eta', \eta \rangle\|_2 = \sup_{a \in \mathcal{A}, \|\hat{a}\|_2 \leq 1} |\langle \hat{a}, \mathcal{A}\langle \eta', \eta \rangle \rangle_{L^2(\mathcal{A})}| = \sup_{a \in \mathcal{A}, \|\hat{a}\|_2 \leq 1} \text{tr}(\mathcal{A}\langle \eta, \eta' \rangle a) = \sup_{a \in \mathcal{A}, \|\hat{a}\|_2 \leq 1} |\langle a\eta, \eta' \rangle_K| \leq \left( \sup_{a \in \mathcal{A}, \|\hat{a}\|_2 \leq 1} \|a^*\eta'\|_2 \right) \|\eta\|_2 \leq \lambda \|\eta\|_2
\]

for some \( \lambda > 0 \) depending only on \( \eta' \) as \( \eta' \in D(\mathcal{A}K) \). Now if \( \eta \notin D(\mathcal{A}K) \), take \( \eta_n \in D(\mathcal{A}K) \) with \( \eta_n \to \eta \) in \( \|\cdot\|_2 \), and define

\[
\mathcal{A}\langle \eta', \eta \rangle = \lim_n \mathcal{A}\langle \eta', \eta_n \rangle
\]

which exists by the above estimate. Now \( \langle a\eta, \eta' \rangle_K = \langle \hat{a}, \mathcal{A}\langle \eta', \eta \rangle \rangle_{L^2(\mathcal{A})} \) for all \( a \in \mathcal{A} \) by construction.

\[
\text{Corollary 4.2.10. Each } \eta \in \mathcal{A}K \text{ gives a closable operator } R(\eta)^0: \hat{\mathcal{A}} \to \mathcal{A}K \text{ by } \hat{a} \mapsto a\eta. \text{ A similar result holds for right modules.}
\]

Proof. We need only show its adjoint is densely defined. If \( \eta' \in D(\mathcal{A}K) \), then

\[
\langle R(\eta)^0 \hat{a}, \eta' \rangle_K = \langle a\eta, \eta' \rangle_K = \langle \hat{a}, \mathcal{A}\langle \eta', \eta \rangle \rangle_{L^2(\mathcal{A})}
\]

by Lemma 4.2.9, and the result follows as \( D(\mathcal{A}K) \) is dense in \( K \).

\[
\text{Corollary 4.2.11. Each } \eta \in \mathcal{A}K \text{ gives a closable unbounded operator } R^0_\eta: D(H_A) \to H \otimes \mathcal{A}K \text{ by } \xi \mapsto \xi \otimes \eta. \text{ A similar result holds for each } \xi' \in H_A.
\]

Proof. Once again, we show its adjoint is densely defined. If \( \xi' \in D(H_A) \) and \( \eta' \in D(\mathcal{A}K) \), then by Lemma 4.2.9,

\[
\langle R^0_\eta \xi', \xi' \otimes \eta \rangle_{H \otimes \mathcal{A}K} = \langle \xi \otimes \eta, \xi' \otimes \eta \rangle_{H \otimes \mathcal{A}K} = \langle \langle \xi' | \xi \rangle_{A\eta}, \eta' \rangle_K = \langle \langle \xi' | \xi \rangle_{A\eta}, \mathcal{A}\langle \eta', \eta \rangle \rangle_{L^2(\mathcal{A})} = \langle L(\xi')^* \xi, \mathcal{A}\langle \eta', \eta \rangle \rangle_{L^2(\mathcal{A})} = \langle \xi, L(\xi') \mathcal{A}\langle \eta', \eta \rangle \rangle_H.
\]

The result now follows as \( D(H_A) \otimes \mathcal{A} D(\mathcal{A}K) \) is dense in \( H \otimes \mathcal{A} K \).

**Haagerup’s extended positive cones and operator valued weights**

For this subsection, \( M \) is a von Neumann algebra acting on a Hilbert space \( H \).

**Definition 4.2.12** (Section 1 of \cite{Haa79}). The extended positive cone of \( M \), denoted \( \hat{M}^+ \), is the set of weights on the predual of \( M \), i.e., maps \( m: M^+_* \to [0, \infty] \) such that
(1) \( m(\lambda \phi + \psi) = \lambda m(\phi) + m(\psi) \) for all \( \lambda \geq 0 \) and \( \phi, \psi \in M_*^+ \), and

(2) \( m \) is lower semicontinuous.

The extended positive cone has additional structure:

- There is a natural inclusion \( M^+ \to \hat{M}^+ \) by \( m \mapsto (\phi \mapsto \phi(m)) \).
- For \( m \in \hat{M}^+ \) and \( a \in M \), we define \( a^*ma \in \hat{M}^+ \) by
  \[
  a^*ma(\phi) = m(a\phi^a) = m(\phi(a^* \cdot a)).
  \]

We write \( \lambda m \) for \( \lambda^{1/2} m \lambda^{1/2} \) for \( \lambda \geq 0 \).

- There is a natural partial ordering on \( \hat{M}^+ \) given by \( m_1 \leq m_2 \) if \( m_1(\phi) \leq m_2(\phi) \) for all \( \phi \in M_*^+ \).
- If \( I \) is a directed set, we say \( (m_i)_{i \in I} \subset \hat{M}^+ \) increases to \( m \in \hat{M}^+ \) if \( i \leq j \) implies \( m_i \leq m_j \) and \( \sup_i m_i(\phi) = m(\phi) \) for all \( \phi \in M_*^+ \). Hence we can define the sum of elements of \( \hat{M}^+ \) pointwise.
- Each \( \phi \in M_*^+ \) extends uniquely to a map \( \hat{M}^+ \to [0, \infty] \) by \( \phi(m) = m(\phi) \).

Remark 4.2.13 (Section 1 of [Haa79]). There are equivalent definitions of \( \hat{M}^+ \):

- Given a projection \( p \in P(M) \) and a densely-defined positive, self-adjoint operator \( S \) in \( K = pH \) affiliated with \( M \), we can define
  \[
  m_{(K,S)}(\omega_\xi) = \begin{cases} 
  \|S^{1/2}\xi\| & \text{if } \xi \in D(S^{1/2}) \\
  \infty & \text{else} 
  \end{cases}
  \quad (4.1)
  \]
  where \( \omega_\xi = \langle \cdot, \xi, \xi \rangle \). Conversely, given \( m \in \hat{M}^+ \), there are unique \( (K, S) \) such that Equation (4.1) holds. In the sequel, we will write \( m = (K, S) \) when we use this bijective correspondence.

- Each \( m \in \hat{M}^+ \) has a unique spectral resolution
  \[
  m(\phi) = \int_0^\infty \lambda d\phi(e_\lambda) + \infty\phi(p)
  \]
  where \( \{e_\lambda\}_{\lambda \in [0, \infty]} \) are increasing family of projections in \( M \) such that:

  (1) \( \lambda \mapsto e_\lambda \) is strongly continuous from the right, and

  (2) \( p = 1 - \lim_{\lambda \to \infty} e_\lambda \)
Moreover,

\[ e_0 = 0 \iff m(\phi) > 0 \text{ for all } \phi \in M^+_s \setminus \{0\} \]
\[ p = 0 \iff \{ \phi \in M^+_s | m(\phi) < \infty \} \text{ is dense in } M^+_s. \]

- Every \( m \in \widehat{M}^+ \) is a pointwise limit of an increasing sequence of operators in \( M^+ \).
- \( \widehat{M}^+ \) is the set of all \( m \in \mathcal{B}(H)^+ \) affiliated to \( M \) (\( umu^* = m \) for all \( u \in U(M') \)).

**Theorem 4.2.14** ([Haa79], Proposition 1.11, Theorem 1.12). Suppose \( M \) is a semifinite von Neumann algebra with n.f.s. trace \( \text{Tr}_M \). For \( x, y \in M^+ \), let \( \text{Tr}_M(x \cdot y) = \text{Tr}_M(x^{1/2}yx^{1/2}) \). Then the map \( (x, y) \mapsto \text{Tr}_M(x \cdot y) \) has a unique extension to \( \widehat{M}^+ \times \widehat{M}^+ \) such that

- \( \text{Tr}_M(x \cdot y) = \text{Tr}_M(y \cdot x) \) for all \( x, y \in \widehat{M}^+ \),
- \( \text{Tr}_M \) is additive and homogeneous in both variables,
- if \( (x_i), (y_j) \subset \widehat{M}^+ \) with \( x_i \nearrow x \) and \( y_j \nearrow y \), then \( \text{Tr}_M(x_i \cdot y_j) \nearrow \text{Tr}_M(x \cdot y) \), and
- \( \text{Tr}_M((a^*xa) \cdot y) = \text{Tr}_M(x \cdot (aya^*)) \) for all \( x, y \in \widehat{M}^+ \) and \( a \in M \).

Moreover

- The map \( x \mapsto \text{Tr}(x \cdot) \) is a homogeneous, additive bijection from \( \widehat{M}^+ \) onto the set of normal weights of \( M \),
- \( x \leq y \iff \text{Tr}(x \cdot) \leq \text{Tr}(y \cdot) \) and \( x_i \nearrow x \iff \text{Tr}(x_i \cdot) \nearrow \text{Tr}(x \cdot) \), and
- If \( x = \int_0^\infty \lambda \, d\varepsilon_\lambda + \infty p \), then \( \text{Tr}(x \cdot) \) is faithful if and only if \( e_0 = 0 \) and semifinite if and only if \( p = 0 \).

**Definition 4.2.15** ([Haa79], Definitions 2.1 and 2.2). Let \( M \) and \( N \) be von Neumann algebras \( N \subseteq M \). An operator valued weight from \( M \to N \) is a map \( T : M^+ \to \widehat{N}^+ \) which satisfies the following conditions:

1. \( T(\lambda x + y) = \lambda T(x) + T(y) \) for all \( \lambda \geq 0 \) and \( x, y \in M^+ \), and
2. \( T(a^*xa) = a^*T(x)a \) for all \( x \in M^+ \) and \( a \in N \).

As in the case of ordinary weights, we set

\[ n_T = \{ x \in M | T(x^*x) \in N^+ \} \]
\[ m_T = n_T^* n_T = \text{span} \{ x^*y | x, y \in n_T \}. \]

Moreover, we say \( T \) is:
• **normal** if \( x_i \nearrow x \Rightarrow T(x_i) \nearrow T(x) \) for all \( x_i, x \in M^+ \),
• **faithful** if \( T(x^*x) = 0 \Rightarrow x = 0 \) for all \( x \in M^+ \), and
• **semifinite** if \( n_T \) is \( \sigma \)-weakly dense in \( M \).

We will abbreviate normal, faithful, semifinite by the acronym n.f.s.

**Remarks 4.2.16.**
(1) \( T \) is a conditional expectation if and only if \( T(1) = 1 \).

(2) If \( T \) is normal, it has a unique extension to \( \hat{M}^+ \) satisfying (1) and (2).

(3) \( n_T \) is a left-ideal and \( n_T, m_T \) are algebraic \( N-N \) bimodules. By polarization, \( T \) extends to a map \( T: m_T \to N \), and \( T(axb) = aTx b \) for all \( x \in m_T \) and \( a, b \in N \).

**Theorem 4.2.17** ([Haa79], Theorem 2.7). Given an inclusion \( N \subseteq M \) of semifinite von Neumann algebras with n.f.s. traces \( \text{Tr}_N, \text{Tr}_M \) respectively. Then there is a unique n.f.s. trace-preserving operator valued weight \( T: M^+ \to \hat{N}^+ \). Moreover, if \( x \in M^+ \), \( T(x) \) is the unique element of \( \hat{N}^+ \) such that
\[
\text{Tr}_M(y \cdot x) = \text{Tr}_N(y \cdot T(x)) \quad \text{for all } y \in N^+
\]  
(4.2)

(where we also write \( \text{Tr}_N \) for the unique extension of \( \text{Tr}_N \) to \( \hat{N}^+ \)).

**Definition 4.2.18.** For \( N \subseteq M \) an inclusion of von Neumann algebras, we write
- \( \mathcal{P}(M, N) \) for the set of n.f.s. operator valued weights \( M^+ \to \hat{N}^+ \), and
- \( \mathcal{P}_0(M, N) \subseteq \mathcal{P}(M, N) \) for the set of operator valued weights whose restriction to \( N' \cap M \) is semifinite.

**Lemma 4.2.19** ([ILP98], Lemma 2.5 and Proposition 2.8, [Yam94], Corollary 28). Let \( N \subset M \) be an inclusion of semifinite von Neumann algebras.

(1) There is a unique central projection \( z \in N' \cap M \) such that
- \( \mathcal{P}_0(pMp, pN) = \emptyset \) for all \( p \in N' \cap M, \ p \leq (1-z) \) and
- \( \mathcal{P}_0(zMz, zN) = \mathcal{P}(zMz, zN) \).

Moreover, for all \( T \in \mathcal{P}(M, N) \),
- \( (1-z)(N' \cap M) \cap m_T = \{0\} \), and
- \( T|_{z(N' \cap M)} \) is semifinite.

(2) If \( \mathcal{P}_0(M, N) \neq \emptyset \) and \( \mathcal{P}_0(N', M') \neq \emptyset \), then \( N' \cap M \) is a direct sum of type I factors, and \( pN \subset pMp \) has finite index for every finite rank \( p \in N' \cap M \).
Useful lemmata on extended positive cones

For this subsection, \( M \) is a von Neumann algebra acting on a Hilbert space \( H \).

**Lemma 4.2.20.** For \( m \in \hat{M}^+ \) and \( \eta, \xi \in H \), the parallelogram identity holds:

\[
m(\omega_{\eta+\xi}) + m(\omega_{\eta-\xi}) = 2m(\omega_\eta) + 2m(\omega_\xi).
\]

**Proof.** Take \((x_i) \subset M^+ \) with \( x_i \) increasing to \( m \). Then

\[
m(\omega_{\eta+\xi}) + m(\omega_{\eta-\xi}) = \sup_{i,j} \left( x_i(\omega_{\eta+\xi}) + x_j(\omega_{\eta-\xi}) \right)
\]

\[
\leq \sup_{i,j} \left( \sup_{k \geq i,j} \left( x_k(\omega_{\eta+\xi}) + x_k(\omega_{\eta-\xi}) \right) \right)
\]

\[
= \sup_{i,j} \left( \sup_{k \geq i,j} \left( 2x_k(\omega_\eta) + 2x_k(\omega_\xi) \right) \right)
\]

\[
\leq \sup_{i',j'} \left( 2x_{i'}(\omega_\eta) + 2x_{j'}(\omega_\xi) \right) = 2m(\omega_\eta) + 2m(\omega_\xi).
\]

The other inequality is proved similarly. \( \square \)

**Lemma 4.2.21.**

1. \( m_1 \leq m_2 \) if and only if \( m_1(\omega_\xi) \leq m_2(\omega_\xi) \) for all \( \xi \in H \).

2. \((m_i)_{i \in I}\) increases to \( m \) if and only if \( i \leq j \) implies \( m_i \leq m_j \) and \( \sup_i m_i(\omega_\xi) = m(\omega_\xi) \) for all \( \xi \in H \).

3. If \((m_i)_{i \in I}\) increases to \( m \) and \( a \in M^+ \), then \( a^*m_ia \) increases to \( a^*ma \).

**Proof.** First, note every \( \phi \in M_*^+ \) is a sum of functionals \( \omega_{\xi_k} = \langle \cdot, \xi_k, \xi_k \rangle \) for \( \xi_k \in H \).

1. Follows immediately by lower semicontinuity of \( m \in \hat{M}^+ \).

2. Suppose \( \phi = \sum_k \omega_{\xi_k} \). By lower semicontinuity,

\[
m(\phi) = \sum_k m(\omega_{\xi_k}) = \sum_k \sup_i m_i(\omega_{\xi_k})
\]

\[
\geq \sup_i \sum_k m_i(\omega_{\xi_k}) = \sup_i m_i \left( \sum_k \omega_k \right) = \sup_i m_i(\phi).
\]

There are two cases:

**Case 1:** Suppose \( m(\phi) = \infty \). Then there is a \( \varepsilon > 0 \) such that \( \sup_i m_i(\omega_{\xi_k}) > \varepsilon \) for infinitely many \( k \), say \((k_n)\). Let \( N > 0 \), and let \( M > 0 \) such that \( M\varepsilon > N \). Choose
CHAPTER 4. A PLANAR CALCULUS FOR INFINITE INDEX SUBFACTORS

\[ j_1 \in I \text{ such that } i \geq j_1 \text{ implies } m_i(\omega_{k_1}) > \varepsilon. \]  
For \( n = 2, \ldots, M \), inductively choose \( j_n > j_{n-1} \) such that \( i \geq j_n \) implies \( m_i(\omega_{k_n}) > \varepsilon \). Then for all \( i > j_M \),

\[
\sum_{k} m_i(\omega_{\xi_k}) \geq M \sum_{n=1}^{M} m_i(\omega_{\xi_{k_n}}) \geq M \varepsilon > N.
\]

Since \( N \) was arbitrary, we must have

\[
\sup_i m_i(\phi) = \sup_i m_i(\omega_k) = \sup_i \sum_{k} m_i(\omega_k) = \infty.
\]

**Case 2:** Suppose \( m(\phi) < \infty \). Let \( \varepsilon > 0 \). Then there is an \( N \in \mathbb{N} \) such that \( \sum_{k>N} m(\omega_{\xi_k}) < \varepsilon \). Now as in the proof of Lemma 4.2.20,

\[
m(\phi) - \varepsilon < \sum_{k=1}^{N} \sup_{i} m_i(\omega_{\xi_k}) = \sup_{i} \sum_{k=1}^{N} m_i(\omega_{\xi_k}) \leq \sup_{i} \sum_{k} m_i(\omega_k) = \sup_{i} m_i(\phi),
\]

and the result follows as \( \varepsilon \) was arbitrary.

(3) We use (2). Let \( \xi \in H \).

\[
a^* m_i a(\omega_\xi) = m_i(\omega_{a\xi}) \leq m_j(\omega_{a\xi}) = a^* m_j a(\omega_\xi) \text{ for all } i \leq j \text{ and }
\]

\[
\sup_i a^* m_i a(\omega_\xi) = \sup_i m_i(\omega_{a\xi}) = m(\omega_{a\xi}) = a^* m a(\omega_\xi).
\]

\[ \square \]

**Remark 4.2.22.** Suppose \((x_i)_{i \in I}, (y_i)_{i \in I} \subset M^+ \) are directed families and \( \lambda \geq 0 \). Then by Lemma 4.2.21 and techniques similar to those used in the proof of Lemma 4.2.20,

\[
\sup_i (\lambda x_i + y_i) = \lambda \sup_i x_i + \sup_j y_j.
\]

**Lemma 4.2.23.** Suppose \( F \subset \hat{M}^+ \) is a directed family, i.e., if \( x, y \in F \), then there is a \( z \in F \) with \( z \geq x \) and \( z \geq y \). Then there is a unique \( m_F = (K_F, S_F) \in \hat{M}^+ \) with \( K_F = \text{Dom}(S_F^{1/2}) \) such that

\[
m_F(\omega_\xi) = (S_F^{1/2} \xi, S_F^{1/2} \xi) = \sup_{x \in F} x(\omega_\xi) \text{ for all }
\]

\[
\xi \in \text{Dom}(S_F^{1/2}) = \left\{ \xi \in H \left| \sup_{x \in F} x(\omega_\xi) < \infty \right. \right\}.
\]

We denote \( m_F \) by \( \sup_{x \in F} x \).
Proof. As in [Haa79, Con80, Tak03], one checks that the extended quadratic form $s_F: H \to [0, \infty]$ given by $s_F(\xi) = \sup_{x \in F} x(\omega_\xi)$ satisfies

1. $s_F(\lambda \xi) = |\lambda|^2 s_F(\xi),$
2. $s_F(\eta + \xi) + s_F(\eta - \xi) = 2s_F(\eta) + 2s_F(\xi),$
3. $s_F$ is lower semicontinuous, and
4. $s_F(u \xi) = s_F(\xi)$ for all $u \in M'.$

(1) and (4) are trivial. (3) follows as sups of lower semicontinuous maps are lower semicontinuous. (2) is similar to the proof of Lemma 4.2.20.

Definition 4.2.24. Suppose $M$ is a semifinite von Neumann algebra with n.f.s. trace $\text{Tr}_M$ acting on the right of $H.$ Let $\xi \in D(H_M),$ and suppose $(x_i) \in (M' \cap B(H))^+$ with $x_i \not\to x \in (M' \cap B(H))^+.$ Then each $L(\xi)^* x_i L(\xi) \in M^+$ as it commutes with the right $M$-action on $L^2(M, \text{Tr}_M),$ so we define

$$L(\xi)^* x L(\xi) = \sup_i L(\xi)^* x_i L(\xi) \in \widehat{M}^+.$$ 

Note that if $\kappa \in L^2(M, \text{Tr}_M),$ then

$$\left(L(\xi)^* x L(\xi)\right)(\omega_\kappa) = \sup_i \left(L(\xi)^* x_i L(\xi)\right)(\omega_\kappa) = \sup_i x_i(\omega_{\xi \otimes \kappa}) = x(\omega_{\xi \otimes \kappa}),$$

which is independent of the choice of $(x_i).$ Hence $L(\xi)^* x L(\xi)$ is well-defined by Lemma 4.2.21. Similarly, we may define operators of the form $R(\eta)^* y R(\eta),$ $L_\xi^* x L_\xi,$ and $R_{\eta}^* y R_{\eta}.$

4.3 Planar calculus for bimodules

For this section, let $A$ be a $II_1$-factor, and let $_A H_A$ be an $A - A$ Hilbert bimodule, i.e., $H$ has commuting actions of $A$ and $A^{\text{op}}.$

Centralizer algebras, central $L^2$-vectors, and canonical maps

Definition 4.3.1. For an $A - A$ bimodule $K$ (algebraic or Hilbert), we define

$$A' \cap K = \{ \xi \in K | a \xi = \xi a \text{ for all } a \in A \}.$$ 

Notation 4.3.2. For $n \geq 0,$ let

- $H^n = \bigotimes_A^n H,$ with the convention that $H^0 = L^2(A),$
• $B^n = D(AH^n) \cap D(H^*_A)$, which is dense in $H^n$ by Lemma 1.2.2 of [Pop86]. We also use the convention $B = B^1$. Note $B^0 = A$.

• $\{\alpha\} \subset B$ be an $A$-$H$ basis (possible due to the density of $B$ in $H$), with

\[
\{\alpha^n\} = \{\alpha_1 \otimes \cdots \otimes \alpha_n | \alpha_i \in \{\alpha\} \text{ for all } i = 1, \ldots, n\} \subset B^n
\]

the corresponding $A$-$H^n$ basis (as $R_{\alpha_1 \otimes \cdots \otimes \alpha_n} = R_{\alpha_1} \cdots R_{\alpha_n}$). We let $\{\beta\} \subset B$ be an $H_A$ basis, with $\{\beta^n\} \subset B^n$ the corresponding $H^n_A$ basis.

• (central $L^2$-vectors) $P_n = A' \cap H^n$. Note $P_0 = A' \cap L^2(A) = \mathbb{C}\mathbf{1}$.

• $C_n = (A^\text{op})' \cap B(H^n)$ (the commutant of the right $A$-action on $H^n$) with canonical trace $\text{Tr}_n = \sum_{\beta_n} \langle \cdot, \beta^n \rangle$,

• $C_n^\text{op} = A' \cap B(H^n)$ with canonical trace $\text{Tr}^\text{op}_n = \sum_{\alpha_n} \langle \cdot, \alpha^n, \alpha^n \rangle$,

• (centralizer algebras) $Q_n = C_n \cap C_n^\text{op}$.

Remark 4.3.4. Note that $A \subset C_n$ and $A^\text{op} \subset C_n^\text{op}$.

Definition 4.3.4. $H$ is called symmetric if there is a conjugate-linear isomorphism $J : H \rightarrow H$ such that $J(a\xi b) = b^*(J\xi)a^*$ for all $a, b \in A$ and $\xi \in H$ and $J^2 = \text{id}_H$.

Remark 4.3.5. If $H$ is symmetric, then for $n \geq 1$, $H^n$ is symmetric with conjugate-linear isomorphism $J_n : H^n \rightarrow H^n$ given by the extension of

\[
J_n(\xi_1 \otimes \cdots \otimes \xi_n) = (J\xi_1) \otimes \cdots \otimes (J\xi_n).
\]

for $\xi_i \in B$ for all $i$. Note that $J_n A J_n = A^\text{op}$, $J_n C_n J_n = C_n^\text{op}$, and $J_n B^n = B^n$. On $B(H^n)$, we define $j_n$ by $j_n(x) = J_n x J_n$. Note that $j_n^2 = \text{id}$ and $\text{Tr}_n = \text{Tr}^\text{op}_n \circ j_n$.

If $H$ is not symmetric, then in general, $C_n^\text{op}$ is not the opposite algebra of $C_n$, e.g. $R_{\otimes}L^2(R \otimes R)_{R \otimes R}$ where $R$ is the hyperfinite $II_1$-factor.

Remark 4.3.6. It is clear that $B^n$ is an $A-A$ bimodule. If $\eta \in B^n$ and $c \in C_n$, then $c\xi \in D(H^*_A)$, but in general, $c\xi \notin D(AH^n)$. However, if $c \in Q_n$, then clearly $c\xi \in B^n$.

Proposition 4.3.7. We have natural inclusions:

\[
i_n : C_n \rightarrow C_{n+1} \quad \text{by} \quad x \mapsto x \otimes_A \text{id}_H = (\eta \otimes \xi \mapsto (x\eta) \otimes \xi \text{ for } \eta \in B^n \text{ and } \xi \in B) \quad \text{and}
\]

\[
i_n^\text{op} : C_n^\text{op} \rightarrow C_{n+1}^\text{op} \quad \text{by} \quad y \mapsto \text{id}_H \otimes A y = (\xi \otimes \eta \mapsto \xi \otimes (y\eta) \text{ for } \xi \in B \text{ and } \eta \in B^n).
\]

Both maps include $Q_n \rightarrow Q_{n+1}$.

Proof. If $z \in Q_n$, then $i_n(z) \in Q_{n+1}$ as for all $a, b \in A,$

\[
(z \otimes_A \text{id}_H)[a(\xi \otimes \eta)b] = (z(a\xi)) \otimes (\eta b) = (a(z\xi)) \otimes (\eta b) = a[(z\eta) \otimes \xi]b.
\]

The result is similar for $i_n^\text{op}$.

\[\square\]
Proposition 4.3.8. If \( x \in C_n \), then \( i_n(x) = \sum_{\alpha} R_{\alpha} x R_{\alpha}^* \). If \( y \in C_n^{\text{op}} \), then \( i_{n}^{\text{op}}(y) = \sum_{\beta} L_{\beta} y L_{\beta}^* \).

Proof. We prove the first statement. If \( \xi_1, \ldots, \xi_{n+1} \in B \), we have
\[
\left( \sum_{\alpha} R_{\alpha} x R_{\alpha}^* \right) \xi_1 \otimes \cdots \otimes \xi_n = \sum_{\alpha} R_{\alpha} x (\xi_1 \otimes \cdots \otimes \xi_{n-1} A (\xi_n, \alpha))
= \sum_{\alpha} (x (\xi_1 \otimes \cdots \otimes \xi_{n-1}) \otimes A (\xi_n, \alpha))
= \sum_{\alpha} (x (\xi_1 \otimes \cdots \otimes \xi_{n-1})) \otimes A (\xi_n, \alpha)
= [x (\xi_1 \otimes \cdots \otimes \xi_{n-1})] \otimes \xi_n = i_n(x) (\xi_1 \otimes \cdots \otimes \xi_n).
\]

\[\square\]

Remark 4.3.9. By Definition 4.2.5, \((C_{k} \otimes_A \text{id}_{n-k})' \cap B(H^n) = \text{id}_{k} \otimes_A C_{n-k}^{\text{op}}\).

Lemma 4.3.10. Suppose \( \xi \in H^n \) and \( y \in (C_{n+1}^{\text{op}})^+ \). Recall the operator \( R_{\xi}^0 : B \rightarrow H^{n+1} \) by \( \eta \mapsto \eta \otimes \xi \) is closable by Corollary 4.2.11. Then \( y^{1/2} R_{\xi}^0 : B \rightarrow H^{n+1} \) is also closable.

Proof. Let \( p \) be the range/kernel perp projection of \( y^{1/2} \). By the spectral theorem, there are projections \( p_k \in C_{n+1}^{\text{op}} \) such that \( y^{1/2} p_k = p_k y^{1/2} \) is invertible on \( p_k H^{n+1} \) and \( p_k \not\sim p \) (strongly). Fix \( k \geq 0 \). Vectors of the form \( \zeta = \sum_{i=1}^j \sigma_i \otimes \kappa_i \in p_k H^{n+1} \) where \( \sigma_1, \ldots, \sigma_j \in B \) and \( \kappa_1, \ldots, \kappa_j \in B^n \) are dense in \( p_k H^{n+1} \) by the density of \( B \otimes_A B^n \subset H^{n+1} \). Then for such \( \zeta \) and all \( \eta \),
\[
\langle y^{1/2} R_{\xi}^0 \eta, y^{-1/2} p_k \zeta \rangle = \sum_{i=1}^j \langle \eta \otimes \xi, \sigma_i \otimes \kappa_i \rangle = \sum_{i=1}^j \langle \eta, L_{\sigma_i} (A (\kappa_i, \xi)) \rangle = \left\langle \eta, \sum_{i=1}^j L_{\sigma_i} (A (\kappa_i, \xi)) \right\rangle
\]
(see Corollary 4.2.11). Finally, the span of vectors of the form \( y^{-1/2} p_k \zeta \) where \( \zeta \) is as above and \( k \geq 0 \) is dense in \( pH^{n+1} \).

The following proposition and its proof are similar to Theorem 3.2.26 and Proposition 3.2.27 of [Bur03].

Proposition 4.3.11. Recall from Proposition 4.3.7 that \( i_n(C_n) \subset C_{n+1} \) and \( i_n^{\text{op}}(C_n^{\text{op}}) \subset C_{n+1}^{\text{op}} \). The unique trace-preserving operator valued weight
\[
T_{n+1} : (C_{n+1}^+, \text{Tr}_{n+1}) \rightarrow (\widetilde{C}_n^+, \text{Tr}_n) \text{ is given by } x \mapsto \sum_{\beta} R_{\beta}^* x R_{\beta}.
\]

The unique trace-preserving operator valued weight
\[
T_{n+1}^{\text{op}} : ((C_{n+1}^{\text{op}})^+, \text{Tr}_{n+1}^{\text{op}}) \rightarrow ((C_n^{\text{op}})^+, \text{Tr}_n^{\text{op}}) \text{ is given by } y \mapsto \sum_{\alpha} L_{\alpha}^* y L_{\alpha}.
\]

In particular, \( T_{n+1} \) and \( T_{n+1}^{\text{op}} \) are independent of the choice of basis.
Proof. We prove the result for the second statement.

Suppose \( y \in (C_{n+1}^\text{op})^+ \) and \( \xi \in H^n \). By Lemma 4.3.10, \( y^{1/2} R_\xi^0 \) is closable, so we set \( S = (y^{1/2} R_\xi^0)^* y^{1/2} R_\xi^0 \), which is affiliated with \( C_1^\text{op} \), and define \( m_S \in (C_1^\text{op})^+ \) as in Equation (4.1) by

\[
m_S(\omega_\eta) = \begin{cases} 
\|S^{1/2} \eta\| & \text{if } \eta \in D(S^{1/2}) \supset B \\
\infty & \text{else.}
\end{cases}
\]

Now we calculate that

\[
\text{Tr}_1^\text{op}(m_S) = \sum_\alpha m_S(\omega_\alpha) = \sum_\alpha \|S^{1/2} \alpha\|^2 = \sum_\alpha \|y^{1/2} R_\xi^0 \alpha\|^2 \\
= \sum_\alpha \langle y(\alpha \otimes \xi), (\alpha \otimes \xi) \rangle_{H_{n+1}} = \left\langle \left( \sum_\alpha L_\alpha^* y L_\alpha \right) \xi, \xi \right\rangle_{H^n} = T_{n+1}^\text{op}(y)(\omega_\xi).
\]

As all elements of \( B(H)_+ \) are sums \( \sum_i \omega_\xi_i \), \( T_{n+1}^\text{op} \) is well-defined and independent of the choice of \( \{\alpha\} \).

Note that \( T_{n+1}^\text{op}((C_{n+1}^\text{op})^+) \subset (C_{n}^\text{op})^+ \) as if \( y \in (C_{n+1}^\text{op})^+ \), \( \xi \in H^n \), and \( u \in U(A) \), then

\[
\sum_\alpha L_\alpha^* y L_\alpha (\omega_\xi) = \sum_\alpha \langle y(\alpha \otimes u \xi), (\alpha \otimes u \xi) \rangle = \sum_\alpha \langle y(\alpha u \otimes \xi), (\alpha u \otimes \xi) \rangle \\
= \sum_\alpha L_{\alpha u}^* y L_{\alpha u} (\omega_\xi) = \sum_\alpha L_\alpha^* y L_\alpha (\omega_\xi)
\]

as \( \{\alpha u\} \) is another \( \mathcal{A} \mathcal{H} \) basis by Lemma 4.2.6.

Finally, if \( x \in (C_n^\text{op})^+ \) and \( y \in (C_{n+1}^\text{op})^+ \), then

\[
\text{Tr}_{n+1}^\text{op} \left( [i_n^\text{op}(x^{1/2})] [i_n^\text{op}(x^{1/2})] \right) = \sum_{\alpha^{n+1}} \left\langle [i_n^\text{op}(x^{1/2})] [i_n^\text{op}(x^{1/2})] \alpha^{n+1}, \alpha^{n+1} \right\rangle \\
= \sum_{\alpha, \alpha^n} \langle y(\alpha \otimes (x^{1/2} \alpha^n)), (\alpha \otimes (x^{1/2} \alpha^n)) \rangle \\
= \sum_{\alpha^n} \left\langle \sum_\alpha L_\alpha^* y L_\alpha (x^{1/2} \alpha^n), (x^{1/2} \alpha^n) \right\rangle \\
= \text{Tr}_{n}^\text{op} \left( x^{1/2} T_{n+1}^\text{op}(y)x^{1/2} \right),
\]

so \( T_{n+1}^\text{op} \) is the unique trace-preserving operator valued weight by Equation (4.2) in Theorem 4.2.17.

\[\square\]

Remark 4.3.12. If \( z \in Q_n^+ \), then \( T_{n+1}^\text{op}(z) \in \widehat{Q}_n^+ \) as if \( \xi \in H^n \) and \( u \in U(A) \),

\[
\sum_\alpha L_\alpha^* z L_\alpha (\omega_\xi u) = \sum_\alpha \langle z(\alpha \otimes \xi u), (\alpha \otimes \xi u) \rangle = \sum_\alpha \langle (z(\alpha \otimes \xi) u) u^*, (\alpha \otimes \xi) \rangle = \sum_\alpha L_\alpha^* z L_\alpha (\omega_\xi).
\]

A similar result holds for \( T_{n+1} \).
The following relations hold among the maps.

**Theorem 4.3.15.** The relations given in the next theorem will be important in our approach.

**Corollary 4.3.13.** If \( z \in Q_n^+ \), then \( \sum_\alpha L(\alpha)^* z L(\alpha) = \text{Tr}^{_{\text{op}}} (z) 1_{L^2(A)} \).

Similarly, \( \sum_\alpha R(\beta)^* z R(\beta) = \text{Tr}_1(z) 1_{L^2(A)} \).

**Proof.** We prove the first formula. First, \( \sum_\alpha L(\alpha)^* z L(\alpha) \in \hat{Q}_0^+ = [0, \infty] \). Now

\[
\left( \sum_\alpha L(\alpha)^* z L(\alpha) \right) (\eta) = \sum_\alpha \langle L(\alpha)^* z L(\alpha) \hat{1}, \hat{1} \rangle = \sum_\alpha \langle \alpha z, \alpha \rangle = \text{Tr}^{_{\text{op}}} (z).
\]

**Proposition 4.3.14.** The unique trace-preserving operator valued weight

\[
\hat{T}_{n+1} : (Q_{n+1}^+, \text{Tr}_{n+1}) \to (i_n^{_{\text{op}}} (\hat{Q}_n^+), \text{Tr}_n) \text{ is given by } x \mapsto \sum_\beta L^* \beta x L_\beta.
\]

The unique trace-preserving operator valued weight

\[
\hat{T}_{n+1}^{_{\text{op}}} : (Q_{n+1}^+, \text{Tr}_{n+1}^{_{\text{op}}}) \to \left( i_n (Q_n^+), \text{Tr}_n^{_{\text{op}}} \right) \text{ is given by } y \mapsto \sum_\alpha R^* \alpha y R_\alpha.
\]

In particular, \( \hat{T}_{n+1} \) and \( \hat{T}_{n+1}^{_{\text{op}}} \) are independent of the choice of basis.

**Proof.** Similar to the proof of Proposition 4.3.11 using Remark 4.3.12. Note that if \( u \in U(A) \), then \( \{u \alpha\}, \{\beta u\} \) are also \( A H, H_A \)-bases respectively by Lemma 4.2.6.

**Planar algebra over extended positive cones of centralizer algebras**

In this subsection, we define an operad \( \mathbb{BP} \), and describe a \( \mathbb{BP} \)-algebra of extended positive cones \( \hat{Q}_n^+ \). The proof that the action is well-defined is deferred to Section 4.7 as it is quite technical. The relations given in the next theorem will be important in our approach.

**Theorem 4.3.15.** The following relations hold among the maps \( i_n, i_n^{_{\text{op}}}, T_n, T_n^{_{\text{op}}}, \otimes_A, \text{Tr}_n, \text{Tr}_n^{_{\text{op}}} \) for \( m, n \geq 1 \) (compare with Theorem 4.7.2, Remark 4.7.8, and the proof of Theorem 4.7.13):

1. \( T_n T_{n+1}^{_{\text{op}}} (z) = T_n^{_{\text{op}}} T_{n+1} (z) \) for all \( z \in \hat{Q}_{n+1}^+ \).
2. \( z_1 \otimes_A (z_2 \otimes_A z_3) = (z_1 \otimes_A z_2) \otimes_A z_3 \) for all \( z_i \in \hat{Q}_n^+ \), \( i = 1, 2, 3 \),
3. \( T_{m+n} (z_1 \otimes z_2) = z_1 \otimes_A (T_n z_2) \) and \( T^{_{\text{op}}}_{m+n} (z_1 \otimes z_2) = (T^{_{\text{op}}} m z_1) \otimes_A z_2 \) for all \( z_1 \in Q_m^+ \) and \( z_2 \in \hat{Q}_n^+ \),
4. \( \text{Tr}_n(z_1 \cdot z_2) = \text{Tr}_n(z_2 \cdot z_1) \) for all \( z_1, z_2 \in \hat{Q}_n^+ \), and similarly for \( \text{Tr}_n^{_{\text{op}}} \), and
5. \( \text{Tr}_{n+1}(z_1 \cdot i_n(z_2)) = \text{Tr}_n(T_{n+1}(z_1) \cdot z_2) \) for all \( z_1 \in \hat{Q}_{n+1}^+ \) and \( z_2 \in \hat{Q}_n^+ \), and a similar statement holds with \( ^{_{\text{op}}} \).
Proof. (1) For all $\xi \in H^n$ and $z \in \hat{Q}_{n+1}^+$,

$$
(T_nT_{n+1}^o(z))\langle\omega\rangle = \left(\sum_\beta R_\beta^* \left( \sum_\alpha L_\alpha^* z L_\alpha \right) R_\beta \right)\langle\omega\rangle = \sum_{\alpha,\beta} \left( \sum_\alpha L_\alpha^* \left( \sum_\beta R_\beta^* z R_\beta \right) L_\alpha \right)\langle\omega\rangle = \left( T_n^o T_{n+1}^o(z) \right)\langle\omega\rangle.
$$

(2) This is Corollary 4.6.14.

(3) Suppose $z_{1,j} \in Q_m^+$ increases to $z_1$ and $z_{2,k} \in Q_n^+$ increases to $z_2$. Then

$$
T_{m+n}(z_{1,j} \otimes_A z_{2,k}) = \sum_\beta R_\beta^* (z_{1,j} \otimes_A z_{2,k}) R_\beta = \sum_\beta z_{1,j} \otimes_A (R_\beta^* z_{2,k} R_\beta) = z_{1,j} \otimes_A (T_n z_{2,k}).
$$

Now $T_n z_{2,k}$ increases to $T_n z_2$, and we are finished by Theorem 4.6.16. The other equality is similar.

(4) This is Theorem 4.2.14.

(5) This is Proposition 4.8.11.

\[\Box\]

Corollary 4.3.16. The following relations also hold:

1. $i_{n+1}^o i_n^o(z) = i_{n+1}^o i_n(z)$ for all $z \in \hat{Q}_n^+$.

2. $i_{m+n}(z_1 \otimes_A z_n) = z_1 \otimes_A i_n(z_2)$ and $i_{m+n}^o(z_1 \otimes_A z_2) = i_m^o(z_1) \otimes_A i_n(z_2)$ for all $z_1 \in \hat{Q}_m^+$ and $z_2 \in \hat{Q}_n^+$.

3. $i_{n-1}^o T_n(z) = T_{n+1}^o i_n(z)$ and $i_{n-1} T_n^o(z) = T_{n+1}^o i_n(z)$ for all $z \in \hat{Q}_n^+$.

4. $(T_{m+n} \circ \cdots \circ T_m)(z_1 \otimes_A z_2) = T_n(z_2) z_1$ for all $z_1 \in \hat{Q}_m^+$ and $z_2 \in \hat{Q}_n^+$, and a similar statement holds with $^op$. In particular, $T_{m+n}(z_1 \otimes z_2) = T_m(z_1) T_n(z_2)$ and $T_{m+n}^o(z_1 \otimes z_2) = T_m^o(z_1) T_n^o(z_2)$.

5. $\text{Tr}_{m+n}(z_1 \otimes_A z_2) \cdot (z_3 \otimes_A z_4) = \text{Tr}_m(z_1) \cdot \text{Tr}_n(z_2)$ for all $z_1, z_3 \in \hat{Q}_m^+$ and $z_2, z_4 \in \hat{Q}_n^+$. A similar statement holds for $\text{Tr}_n^o$. 
Definition 4.3.17. The bimodule planar operad $\mathbb{BP}$ is the operad of oriented, unshaded planar tangles (up to planar isotopy) generated by

\[
\begin{align*}
&\quad, \\
&\quad, \\
&\quad, \\
\end{align*}
\]

for $m, n \geq 0$ up to planar isotopy. (We draw all disks as boxes, suppress external disks, draw one thick string labelled $n$ for $n$ individual strings, and orient all strings upward unless otherwise specified.) A topological characterization of $\mathbb{BP}$ tangles is given in Theorem 4.7.9.

A $\mathbb{BP}$-algebra (of extended positive cones) $V_*$ is a sequence $\{V_n\}_{n \geq 0}$ of extended positive cones (defined in Section 4.8) and an action by multilinear maps $Z : \mathbb{BP} \to ML\{V_n\}$

$(Z$ is the partition function$)$ which is well-behaved under composition.

A $\mathbb{BP}$-algebra is called:

- **central** if $V_0 = [0, \infty]$
- **normal** if $Z(T)$ is normal for all $T \in \mathbb{BP}$, and
- **self-dual** if $V_n$ is self-dual for all $n$, and for all annular tangles $T \in \mathbb{BP}$, flipping it inside out gives the adjoint map (see Definitions 4.8.8 and 4.8.10).

Theorem 4.3.18. Given an $A - A$ bimodule $H$, the extended positive cones $Q_n^+$ form a central, normal, self-dual $\mathbb{BP}$-algebra $\hat{Q}_n^+$ such that:

1. $\text{id}_{H^n} = \text{id}_n = \quad$, 
2. $T_{n+1}(z) = \quad$ and $T_{n+1}^{\text{op}}(z) = \quad$ for all $z \in \hat{Q}_{n+1}^+$,
3. $z_1 \otimes_A z_2 = \quad$ (defined in Section 4.6) for all $z_1 \in \hat{Q}_m^+$ and $z_2 \in \hat{Q}_n^+$, and
(4) $\text{Tr}_n(z_1 \cdot z_2) = \frac{z_1}{z_2}^n$ and $\text{Tr}_n^{op}(z_1 \cdot z_2) = \frac{z_1}{z_2}^n$ for all $z_1, z_2 \in \hat{Q}_n^+$. Moreover, the following hold:

(5) $i_n(z) = \frac{z}{n}$ and $i_n^{op}(z) = \frac{z}{n}$ for all $z \in \hat{Q}_n^+$

(6) $\dim_{-A}(H) = T_1(1) = 1$ and $\dim_{A-}(H) = T_1^{op}(1)$.

Note that for $Z$ to be well-defined, any closed diagram must count for a multiplicative factor in $\hat{Q}_0^+ = \hat{Z}(A)^+ = [0, \infty]$.

We call $\hat{Q}_n^+$ the canonical $\mathbb{BP}$-algebra associated to $H$.

Proof. We will show (1)-(4) uniquely determine the action of any $\mathbb{BP}$-tangle. We defer this technical proof to Section 4.7 (Theorem 4.7.13), which uses the important relations given in Theorem 4.3.15 and Corollary 4.3.16. Note that $\hat{Q}_n^+$ is central since $\hat{Q}_0^+ = \hat{Z}(A)^+ = [0, \infty]$, normal by Theorem 4.2.14 and Remark 4.8.7, and self-dual by Proposition 4.8.11.

Remark 4.3.19. Given some operad $\mathbb{P}$ of (shaded, unshaded, oriented, disoriented, etc.) planar tangles, it is not always possible to define an (extended) positive cone planar algebra over $\mathbb{P}$. For example, the rotation does not always map positive elements to positive elements in a subfactor planar algebra.

Graded algebra of central $L^2$-vectors

In this subsection, we define a graded algebra $P_*$ of central $L^2$-vectors.

Lemma 4.3.20. Suppose $K$ is a Hilbert $A - A$ bimodule. Then $A' \cap K \subseteq (D_AK) \cap D(K_A)$.

Proof. Suppose $\zeta \in A' \cap K$, $\zeta \neq 0$. Define $\varphi : A_+ \rightarrow \mathbb{C}$ by $a \mapsto \langle a\zeta, \zeta \rangle$. Note that $\varphi$ is tracial as

$$\varphi(a^*a) = \langle a^*a\zeta, \zeta \rangle = \langle a^*\zeta a, \zeta \rangle = \langle a^*\zeta, a^*\zeta \rangle = \langle aa^*\zeta, \zeta \rangle = \varphi(aa^*)$$

Hence there is a $\lambda \geq 0$ such that $\varphi = \lambda \text{tr}_A$ by the uniqueness of the trace on a $II_1$-factor. Now for all $a \in A$,

$$\|a\zeta\|^2 = \|\zeta a\|^2 = \varphi(a^*a) = \lambda \varphi(a^*a) = \lambda \|a\|^2,$$

and $\zeta$ is left and right $A$-bounded. \qed
Remark 4.3.21. In the sequel, we will confuse elements $\zeta \in P_n$ and the operators $L(\zeta) = R(\zeta) : L^2(A) \to H^n$. We will omit $R(\zeta)$ and only write $L(\zeta)$.

Definition 4.3.22. We represent elements $\zeta \in P_n$ by boxes with $n$ strings emanating from the top

$$\zeta \text{ or } L(\zeta) = \begin{array}{c} n \hline \zeta \end{array}.$$  

By Lemma 4.3.20, the $P_n$’s form a graded algebra $P_\bullet$ in the sense of [GJS10] where the graded multiplication is given by relative tensor product (over $A$) of central vectors. We denote the product of $\zeta_m \in P_m$ and $\zeta_n \in P_n$ by

$$\zeta_m \otimes \zeta_n = \begin{array}{c} m \hline \zeta_m \end{array} \begin{array}{c} n \hline \zeta_n \end{array} \in P_{m+n}.$$  

If $z \in Q_n$ and $\zeta \in P_n$, then $z\zeta \in P_n$, which we denote as:

$$z\zeta \text{ or } L(z\zeta) = \begin{array}{c} z \hline \zeta \end{array}.$$  

The reflections of these diagrams denote the functionals $\langle \cdot, \zeta \rangle$ or adjoints $L(\zeta)^* = \begin{array}{c} \zeta \hline n \end{array}$. 

The inner product $\langle \cdot, \cdot \rangle : P_n \times P_n^* \to \mathbb{C}$ is given by $\langle \xi, \zeta \rangle = \begin{array}{c} \zeta \hline \xi \end{array}$ (see Lemma 4.3.23 (2)).

Compatibility

We now show how the BP-algebra $\widehat{Q}_\bullet^+$ and the graded algebra $P_\bullet$ are compatible.

Lemma 4.3.23.  

1. If $\zeta \in P_n$ and $\xi \in B^n$, then $A\langle \zeta, \xi \rangle = \langle \xi | \zeta \rangle_A$. 

2. If $\zeta, \xi \in P_n$, $A\langle \zeta, \xi \rangle = \langle \xi | \zeta \rangle_A = \langle \zeta, \xi \rangle_{L^2(A)} = \langle \zeta, \xi \rangle_{1_{L^2(A)}} \in \mathbb{C}1_{L^2(A)}$. 

3. For $\zeta \in P_n$, $L(\zeta)L(\zeta)^* = R(\zeta)R(\zeta)^* \in Q_n^+$. We denote the common operator as:

$$\begin{array}{c} n \hline \zeta \end{array} \begin{array}{c} n \hline \zeta \end{array} \in Q_n^+.$$


(4) If $\zeta \in P_n$ and $\|\zeta\|_2 = 1$, $L(\zeta)L(\zeta)^*|_{P_n} = p_\zeta$, the projection onto $\mathbb{C}\zeta$.

Proof. (1) Suppose $a_1, a_2 \in A$. Then
\[
\langle A(\langle \zeta, \xi \rangle \hat{a}_1, \hat{a}_2) \rangle = \langle J R(\zeta)^* R(\xi) J \hat{a}_1, \hat{a}_2 \rangle = \langle \hat{a}_2^* \xi a_1 \zeta, \hat{a}_1^* \zeta a_2 \rangle = \langle \zeta a_2^* a_1^* \xi, \zeta a_1 \zeta a_2 \rangle = \langle L(\zeta) \hat{a}_1, L(\xi) \hat{a}_2 \rangle = \langle \langle \zeta | \xi \rangle_A \hat{a}_1, \hat{a}_2 \rangle.
\]

(2) Since $\zeta, \xi \in P_n$, for all $a, b, a_1, a_2 \in A$,
\[
\langle \langle \xi | \zeta \rangle_A (a \hat{a}_1 b), \hat{a}_2 \rangle = \langle \xi a_1 b, \xi a_2 \rangle = \langle \zeta a_1, \xi a_2 \rangle = \langle \zeta \omega \zeta, \zeta \omega \zeta \rangle = \tau(\zeta) = \tau(\xi) \xi = \xi \xi = \xi \xi.
\]

(3) For $\xi \in B^n$, by (1),
\[
L(\zeta)L(\zeta)^* \xi = \xi L(\zeta)L(\zeta)^* \xi = \langle \zeta | \xi \rangle_A = A(\langle \zeta, \xi \rangle \zeta) = R(\zeta)^* R(\xi) \zeta,
\]
so the two are equal on $H^n$. We have $C_n \ni L(\zeta)L(\zeta)^* = R(\zeta)^* R(\xi) \xi = Q_n^+$, so $L(\zeta)L(\zeta)^* \in Q_n^+$.

(4) Trivial from (2) and (3).

\[\square\]

Theorem 4.3.24. Suppose $\zeta \in P_n$ and $z \in \hat{Q}_n^+$.

(1) $L(\zeta)^* z L(\zeta) = z(\omega \zeta)1_{L^2(A)} = R(\zeta)^* z R(\zeta)$. We denote this diagrammatically by

(2) In the notation of Theorem 4.2.14,
\[
z(\omega \zeta) = \tau_A(L(\zeta)^* z L(\zeta)) = \tau_n(L(\zeta)L(\zeta)^* z) = \tau(\zeta, \xi) = \tau(\xi, \zeta) = \tau(\zeta, \zeta) = \tau(\xi, \xi).
\]

In diagrams,

\[\zeta \quad \hat{z} \quad \hat{\zeta} \quad \zeta \quad \hat{z} = \zeta \quad \hat{z} \quad \zeta \quad \hat{z} \quad \zeta \quad \hat{z} \].
CHAPTER 4. A PLANAR CALCULUS FOR INFINITE INDEX SUBFACTORS

Proof. (1) We show the first equality. If \( z \in Q_n^+ \), this is just (2) of Lemma 4.3.23 with \( \zeta_1 = \zeta_2 = z^{1/2} \). Now for \( z \in \hat{Q}_n^+ \), pick \((z_m) \subset Q_n^+ \) with \( z_m \not\to z \) to get

\[
L(\zeta)^* z L(\zeta) = \lim_{m \to \infty} L(\zeta)^* z_m L(\zeta) = \lim_{m \to \infty} z_m (\omega \zeta)^1 L^2(A) = z (\omega \zeta)^1 L^2(A).
\]

The second equality is similar.

(2) We show the second equality. We may assume \( z \in Q_n^+ \), after which we may take sups to get the full result. Then as \( z^{1/2} \zeta \in P_n \), we have

\[
\text{Tr}_n(z \cdot L(\zeta) L(\zeta)^*) = \text{Tr}_n(z^{1/2} L(\zeta) L(\zeta)^* z^{1/2}) = \text{Tr}_n(L(z^{1/2} \zeta) L(z^{1/2} \zeta)^*)}
\]

\[
= \text{tr}_A(L(z^{1/2} \zeta)^* L(z^{1/2} \zeta)) = \text{tr}_A(L(\zeta)^* z L(\zeta)).
\]

The other equality is similar. \( \square \)

Remark 4.3.25. If \( a \in Q_n, z \in \hat{Q}_n^+ \), and \( \zeta \in P_n \),

\[
\begin{array}{c}
\zeta \\
\downarrow \\
n^*za \\
\downarrow \\
\zeta
\end{array} = (n^*za)(\omega \zeta) = z(\omega a \zeta) = \begin{array}{c}
a^* \zeta \\
\downarrow \\
a \zeta
\end{array}.
\]

Corollary 4.3.26. If \( \zeta_1 \in P_m, \zeta_2 \in P_n, z_1 \in Q_m^+, \text{ and } z_2 \in Q_n^+ \), then

\[
\begin{array}{c}
\zeta_1 \otimes \zeta_2 \\
\downarrow \\
z_1 \otimes_A z_2 \\
\downarrow \\
\zeta_1 \otimes \zeta_2
\end{array} = \langle (z_1 \otimes_A z_2)(\zeta_1 \otimes \zeta_2), (\zeta_1 \otimes \zeta_2) \rangle = \langle z_1 \zeta_1, \zeta_1 \rangle \langle z_2 \zeta_2, \zeta_2 \rangle = \begin{array}{c}
\zeta_1 \\
\downarrow \\
z_1
\end{array} \begin{array}{c}
\zeta_2 \\
\downarrow \\
z_2
\end{array}.
\]

For \( z_1 \in \hat{Q}_m^+ \) and \( z_2 \in \hat{Q}_n^+ \), taking sups gives

\[
\begin{array}{c}
\zeta_1 \otimes \zeta_2 \\
\downarrow \\
z_1 \otimes_A z_2 \\
\downarrow \\
\zeta_1 \otimes \zeta_2
\end{array} = (z_1 \otimes_A z_2)(\omega \zeta_1 \otimes \zeta_2) = z_1(\omega \zeta_1) z_2(\omega \zeta_2) = \begin{array}{c}
\zeta_1 \\
\downarrow \\
z_1
\end{array} \begin{array}{c}
\zeta_2 \\
\downarrow \\
z_2
\end{array}.
\]
CHAPTER 4. A PLANAR CALCULUS FOR INFINITE INDEX SUBFACTORS

Theorem 4.3.27 \((P_\bullet \text{ acts on } \hat{Q}^+_1)\). Given a tangle \(\mathcal{T} \in \mathbb{B}P\) with \(2n\) boundary points and a \(\zeta \in P_n\), we have

\[
\zeta \quad \downarrow \quad \mathcal{T} \\
\quad \downarrow \quad \zeta
\]

\[:= \text{ev}_{\omega_\zeta} \circ \mathcal{T} : V_{i_1} \times \cdots \times V_{i_k} \rightarrow [0, \infty].\]

In this sense, we say \(P_\bullet\) acts as weights on \(\hat{Q}^+_1\). By Theorems 4.3.15 and 4.3.24 and Corollary 4.3.26, we may remove closed subdiagrams and multiply by the appropriate scalar in \([0, \infty]\).

Remark 4.3.28. If \(A \subset (B, \text{tr}_B)\) is an inclusion of \(II_1\)-factors and \(H = L^2(B)\), then one can also define a shaded bimodule planar operad which works similarly to the above construction. This will be explored in a future paper.

4.4 Extremality and rotations

For this section, \(A\) is a \(II_1\)-factor. Assume the notation of the last section.

Extremality

Definition 4.4.1. \(H\) is \textit{approximately extremal with constant} \(\lambda \geq 1\) if on \(Q^n_1\),

\[
\lambda^{-1} \text{Tr}_1 \leq \text{Tr}^{\text{op}}_1 \leq \lambda \text{Tr}_1.
\]

\(H\) is \textit{extremal} if \(\text{Tr}_1 = \text{Tr}^{\text{op}}_1\) on \(Q^n_1\).

The following proposition is almost identical to Proposition 2.8 in [ILP98].

Proposition 4.4.2 (Structure of \(Q_n\)). \(Q_n = a_n \oplus b_n \oplus b^{\text{op}}_n \oplus c_n\) such that

- \(a_n\) is a direct sum of type \(I\) factors, and for each finite rank \(p \in a_n\), \(pA \subset pC_n p\) has finite index.
- \(\text{Tr}_n|_{a_n \oplus b_n}\) and \(\text{Tr}^{\text{op}}_n|_{a_n \oplus b^{\text{op}}_n}\) are semifinite,
- \(b^{\text{op}}_n \oplus c_n \cap m_{\text{Tr}_n} = \{0\} = b_n \oplus c_n \cap m_{\text{Tr}^{\text{op}}_n}\), and
- If \(H^n\) is symmetric, then \(j_n\) fixes \(a_n, c_n\) and \(j_n(b_n) = b^{\text{op}}_n\).
Proof. By Lemma 4.2.19, let \( z_n, z_n^{op} \in Q_n \) be the unique central projections corresponding to \( A \subset C_n \) and \( A^{op} \subset C_n^{op} \). Set

\[
\begin{align*}
a_n &= z_n z_n^{op} Q_n \\
b_n &= z_n (1 - z_n^{op}) Q_n \\
c_n &= (1 - z_n) (1 - z_n^{op}) Q_n,
\end{align*}
\]

and the rest follows immediately. \( \square \)

Proposition 4.4.3. Let \( Q_1 = a_1 \oplus b_1 \oplus b_1^{op} \oplus c_1 \) as in Proposition 4.4.2. The following are equivalent:

1. \( H \) is approximately extremal with constant \( \lambda \geq 1 \), and
2. \( b_1 = b_1^{op} = \{0\} \) and there is a \( \lambda \geq 1 \) such that on \( Q_1^+ \cap a_1 \), \( \lambda^{-1} \text{Tr}_1 \leq \text{Tr}_1^{op} \leq \lambda \text{Tr}_1 \).

A similar result holds for the extremal case.

Proof. 

(1) \( \Rightarrow \) (2): Suppose \( H \) is approximately extremal. We show \( b_1 = \{0\} \). As \( \text{Tr}_1|_{a_1 \oplus b_1} \) is semifinite by Proposition 4.4.2, we choose \( z \in b_1 \) such that \( z \geq 0 \) and \( z \in m_{\text{Tr}_1} \). Then \( z \in m_{\text{Tr}_1^{op}} \), but \( b_1 \cap m_{\text{Tr}_1^{op}} = \{0\} \). Similarly \( b_1^{op} = \{0\} \).

(2) \( \Rightarrow \) (1): \( \text{Tr}_1|_{c_1 \cap Q_1^+} = \text{Tr}_1^{op}|_{c_1 \cap Q_1^+} = \infty. \) \( \square \)

Corollary 4.4.4. \( H \) is extremal if and only if for each Hilbert \( A - A \) bimodule \( K \subset H \), the left and right von Neumann dimensions agree.

Remark 4.4.5. If \( H \) has a two-sided basis \( \{\gamma\} \), then \( H \) is extremal as

\[
\text{Tr}_1 = \sum_{\gamma} \langle \cdot, \gamma \rangle = \text{Tr}_1^{op}.
\]

Remark 4.4.6. If \( H \) is approximately extremal, then there is a \( \lambda \geq 1 \) such that for all \( z \in \hat{Q}_1^+ \),

\[
\lambda^{-1} \sum_{\beta} z(\omega_\beta) \leq \sum_{\alpha} z(\omega_\alpha) \leq \lambda \sum_{\beta} z(\omega_\beta).
\]

If \( H \) is extremal, then \( \lambda = 1 \) works.

Theorem 4.4.7. (1) If \( H \) is (approximately) extremal (with constant \( \lambda \geq 1 \)), then \( H^n \) is (approximately) extremal for all \( n \geq 1 \) (with constant \( \lambda^n \)).

(2) If \( H^n \) is (approximately) extremal for some \( n \geq 1 \), then \( H \) is (approximately) extremal.

Proof. We prove the extremal case, and the approximately extremal case is similar.
(1) We use strong induction on \( n \). Suppose \( H^1 \) and \( H^n \) are extremal. If \( z \in Q_{n+1}^+ \),

\[
\text{Tr}_{n+1}(z) = \begin{array}{c} z \\ n+1 \end{array} = \begin{array}{c} z \\ n \end{array} = \begin{array}{c} z \\ n \end{array} = \begin{array}{c} z \\ n+1 \end{array} = \text{Tr}_{n+1}^{\text{op}}(z).
\]

Hence \( H^{n+1} \) is extremal.

(2) Suppose \( H^n \) is extremal and \( z \in Q_1^+ \). Then \( z \otimes_A \cdots \otimes_A z \in Q_n^+ \). By the bimodule planar calculus,

\[
\begin{pmatrix} \begin{array}{c} z \\ n \end{array} \end{pmatrix} = \begin{array}{c} z \\ n \end{array} \cdots \begin{array}{c} z \\ n \end{array} = \sum_{\beta} R^{\beta}_A z = \sum_{\alpha} R^{\alpha}_A z.
\]

In equations:

\[
\text{Tr}_1(z)^n = \text{Tr}_n(z \otimes_A \cdots \otimes_A z) = \text{Tr}_{n}^{\text{op}}(z \otimes_A \cdots \otimes_A z) = \text{Tr}_1^{\text{op}}(z)^n.
\]

Taking \( n^{\text{th}} \) roots gives the desired result.

\[\square\]

Proposition 4.4.8. If \( H \) is extremal and \( z \in \hat{Q}_n^+ \), then \( \sum_\beta R^{*}_\beta z R_\beta = \sum_\alpha R^{*}_\alpha z R_\alpha \) and \( \sum_\alpha L^{*}_\alpha z L_\alpha = \sum_\beta L^{*}_\beta z L_\beta \).

Proof. Immediate from Propositions 4.3.11 and 4.3.14.

\[\square\]

**Rotations**

Definition 4.4.9 (Inspired by [Bur03]). A Burns rotation is a map \( \rho : P_n \to P_n \) such that for all \( \zeta \in P_n \) and \( b_1, \ldots, b_n \in B \),

\[
\langle \rho(\zeta), b_1 \otimes \cdots \otimes b_n \rangle = \langle \zeta, b_2 \otimes \cdots \otimes b_n \otimes b_1 \rangle.
\]

(4.3)

An opposite Burns rotation is defined similarly:

\[
\langle \rho^{\text{op}}(\zeta), b_1 \otimes \cdots \otimes b_n \rangle = \langle \zeta, b_n \otimes b_1 \otimes \cdots \otimes b_{n-1} \rangle.
\]
Remark 4.4.10. Note that if such a $\rho$ exists, it is unique, and $\rho^n = \text{id}_{P_n}$. In this case, $\rho^{\text{op}} = \rho^{-1}$.

Theorem 4.4.11 (Essentially due to [Bur03]). If $\rho = \sum_\beta L_\beta R_\beta^*$ converges strongly on $P_n$, then $\rho$ is a Burns rotation. Similarly, if $\rho^{\text{op}} = \sum_\alpha R_\alpha L_\alpha^*$ converges strongly on $P_n^n$, then $\rho^{\text{op}}$ is an opposite Burns rotation.

Proof. We must show that $\rho$ preserves $P_n$ and that $\rho$ satisfies Equation (4.3). The latter follows from:

$$\langle \rho(\zeta), b_1 \otimes \cdots \otimes b_n \rangle = \sum_\beta \langle \zeta, R_\beta L_\beta^*(b_1 \otimes \cdots \otimes b_n) \rangle$$

$$= \sum_\beta \langle \zeta, \langle \beta | b_1 \rangle A b_2 \otimes \cdots \otimes b_n \otimes \beta \rangle$$

$$= \sum_\beta \langle \langle \beta | b_1 \rangle_A^* \zeta, b_2 \otimes \cdots \otimes b_n \otimes \beta \rangle$$

$$= \sum_\beta \langle \zeta \langle \beta | b_1 \rangle^*, b_2 \otimes \cdots \otimes b_n \otimes \beta \rangle$$

$$= \sum_\beta \langle \zeta, b_2 \otimes \cdots \otimes b_n \otimes \beta \langle \beta | b_1 \rangle_A \rangle$$

$$= \langle \zeta, b_2 \otimes \cdots \otimes b_n \otimes \beta \rangle.$$ 

Now $\rho$ is independent of the choice of $\{\beta\}$. In particular, for any $u \in U(A)$, $\{u\beta\}$ is an $H_A$-basis, and

$$u \rho(\zeta) u^* = u \left( \sum_\beta L_\beta R_\beta^* \zeta \right) u^* = \sum_\beta L_{u\beta} R_{u\beta}^* \zeta = \rho(\zeta) \in P_n.$$

Diagrammatic representation of the Burns rotation

For this section, we assume the Burns rotation $\rho$ exists on $P_n$ for all $n \geq 0$. Recall for all $k \geq 0$, $\rho^{-k} = (\rho^{\text{op}})^k$.

Notation 4.4.12. For $\zeta \in P_{m+n}$, we denote $\rho^m(\zeta) = (\rho^{\text{op}})^n(\zeta) \in P_{m+n}$ by moving $m$ strings around the bottom counterclockwise or by moving $n$ strings around the bottom clockwise:
Proposition 4.4.13. If $\eta \in P_m$ and $\xi \in P_n$, then $\rho^n(\eta \otimes \xi) = \xi \otimes \eta$:

![Diagram]

Proof. Suppose $\alpha \in B^m$ and $\beta \in B^n$. Then by (1) of Lemma 4.3.23,

$$\langle \rho^n(\eta \otimes \xi), \beta \otimes \alpha \rangle = \langle \eta \otimes \xi, \alpha \otimes \beta \rangle = \langle \langle \alpha | \eta \rangle_A \xi, \beta \rangle = \langle \xi_A \langle \eta, \alpha \rangle, \beta \rangle = \langle \xi \otimes \eta, \beta \otimes \alpha \rangle.$$

$\square$

Definition 4.4.14. For $0 \leq j < m$, define $\mu_j : P_m \times P_n \to P_{m+n}$ by $\mu_j(\eta, \xi) = \rho^{-j}(\rho^j(\eta) \otimes \xi)$. We represent $\mu_j$ diagrammatically as follows:

$$\mu_j(\eta, \xi) = \begin{array}{c}
\begin{array}{c}
\eta \\
m-j
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\xi \\
j
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\eta \\
m
\end{array}
\end{array}.$$

That this diagram is well-defined relies on the following proposition.

Proposition 4.4.15. The $\mu_i$'s are associative, i.e., if $\sigma \in P_{\ell}$, $\eta \in P_m$, and $\xi \in P_n$, and $i \leq \ell$, $j \leq m$, then

$$\mu_i(\kappa, \mu_j(\eta, \xi)) = \mu_{i+j}(\mu_i(\kappa, \eta), \xi).$$

Proof. Suppose $\alpha \in B^{\ell-i}$, $\beta \in B^{m-j}$, $\gamma \in B^n$, $\delta \in B^j$, and $\varepsilon \in B^i$. Then

$$\langle \mu_i(\kappa, \mu_j(\eta, \xi)), \alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \varepsilon \rangle = \langle \rho^{-i}(\rho^i(\kappa) \otimes \rho^{-j}(\rho^j(\eta) \otimes \xi)), \alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \varepsilon \rangle$$

$$= \langle \rho^i(\kappa) \otimes \rho^{-j}(\rho^j(\eta) \otimes \xi), \varepsilon \otimes \alpha \otimes \beta \otimes \gamma \otimes \delta \rangle$$

$$= \langle \rho^{-j}(\rho^j(\eta) \otimes \xi), (\rho^j(\kappa) | \varepsilon) A \alpha \otimes \beta \otimes \gamma \rangle$$

$$= \langle \rho^j(\eta) \otimes \xi, \delta \otimes (\rho^j(\kappa) | \varepsilon) A \alpha \otimes \beta \rangle$$

$$= \langle \rho^j(\eta), \delta \otimes (\rho^j(\kappa) | \varepsilon) A \alpha \otimes \beta A(\gamma, \xi) \rangle$$

$$= \langle \eta, (\rho^j(\kappa) | \varepsilon) A \alpha \otimes \beta A(\gamma, \xi) \otimes \delta \rangle$$

$$= \langle \rho^i(\kappa) \otimes \eta, \varepsilon \otimes \alpha \otimes \beta A(\gamma, \xi) \rangle$$

$$= \langle \rho^i(\kappa) \otimes \eta, \delta \otimes \varepsilon \otimes \alpha \otimes \beta A(\gamma, \xi) \rangle$$

$$= \langle \rho^j(\kappa) \otimes \eta, \delta \otimes \varepsilon \otimes \alpha \otimes \beta A(\gamma, \xi) \rangle$$

$$= \langle \rho^i(\kappa) \otimes \eta, \delta \otimes \varepsilon \otimes \alpha \otimes \beta A(\gamma, \xi) \rangle$$

$$= \langle \rho^{-i-j}(\rho^{i+j}(\rho^{-i}(\rho^i(\kappa) \otimes \eta)) \otimes \xi), \alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \varepsilon \rangle$$

$$= \langle \mu_{i+j}(\mu_i(\kappa, \eta), \xi), \alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \varepsilon \rangle.$$
Corollary 4.4.16. $P_\bullet$ naturally forms an algebra over the operad generated by the unshaded, oriented tangles

for $m, n \geq 0$ up to planar isotopy.

The Burns rotation is also compatible with the $\mathbb{BP}$-algebra $\hat{Q}_+^\bullet$.

Theorem 4.4.17. (1) For all $\zeta \in P_{m+n}$ and $x \in Q_m$, and $y \in Q_n$, $\rho^n((x \otimes_A y)\zeta) = (y \otimes_A x)\rho^n(\zeta)$:

(2) If $\rho$ is unitary, then for all $\zeta \in P_{m+n}$ and $x \in \hat{Q}_m$, and $y \in \hat{Q}_n$, $(y \otimes_A x)(\omega_{\rho^n}\zeta) = (x \otimes_A y)(\omega_\zeta)$:

Proof. (1) For $\eta \in B^n$ and $\xi \in B^m$,

$$\langle \rho^n((x \otimes_A y)\zeta), \eta \otimes \xi \rangle = \langle (x \otimes_A y)\zeta, \xi \otimes \eta \rangle = \langle \zeta, (x^* \otimes_A y^*)(\xi \otimes \eta) \rangle = \langle \zeta, (x^* \otimes_A y^*)(\xi \otimes \eta) \rangle = \langle (y \otimes_A x)\rho^n(\zeta), \eta \otimes \xi \rangle.$$

(2) Pick $(x_i) \subset Q_m$ and $(y_j) \subset Q_n$ with $x_i \nearrow x$ and $y_j \nearrow y$. Then by (1), for all $i$,

$$(y_j \otimes_A x_i)(\omega_{\rho^n}\zeta) = \|y_j^{1/2} \otimes_A x_i^{1/2})\rho^n\zeta\|_2 = \|\rho^n((x_i^{1/2} \otimes_A y_j^{1/2})\zeta)\|_2$$

We are finished by Theorem 4.6.13, since $x_i \otimes_A y_j \nearrow x \otimes_A y$ and $y_j \otimes_A x_i \nearrow y \otimes_A x$.

Remark 4.4.18. When the operads for $P_\bullet$ and $\hat{Q}_+^\bullet$ interact as in Theorem 4.3.27, we may remove closed subdiagrams and multiply by the appropriate scalar in $[0, \infty]$ by Corollary 4.4.16 and Theorem 4.4.17.
Extremality implies the existence of the Burns rotation

We will show in the next lemma and theorem that (approximate) extremality implies the existence of the Burns rotation. The intuition comes from the bimodule planar calculus. In diagrams, for the extremal case, we have:

Although these diagrams are not yet well-defined, they tell us how to proceed. They become well-defined after the Burns rotation exists by Theorems 4.3.24 and 4.4.17.

**Lemma 4.4.19.** Let $p_n$ be the projection in $B(H^n)$ with range $P_n$.

1. If $H$ is approximately extremal with constant $\lambda \geq 1$, then
   $$\left(\sum_{\beta} p_n R_\beta R^*_\beta p_n\right) \leq \lambda^{n-1} p_n \text{ and } \left(\sum_{\alpha} p_n L_\alpha L^*_\alpha p_n\right) \leq \lambda^{n-1} p_n.$$

2. If $H$ is extremal, then on $P_n$, $\sum_{\beta} p_n R_\beta R^*_\beta p_n = p_n = \sum_{\alpha} p_n L_\alpha L^*_\alpha p_n$.

**Proof.** (1) We prove the first inequality. Note that $R^*_\beta \zeta \in D(AH^{n-1})$, and $R(R^*_\beta \zeta) = R^*_\beta R(\zeta) : L^2(A) \to H^{n-1}$. Since $H$ is (approximately) extremal, so is $H^{n-1}$ with constant $\lambda^{n-1}$, and

$$\left\langle \left(\sum_{\beta} p_n R_\beta R^*_\beta p_n\right) \zeta, \zeta \right\rangle_{P_n} \leq \sum_{\beta} ||R^*_\beta \zeta||^2_{L^2} = \sum_{\beta} \text{tr}_A (A R^*_\beta \zeta, R^*_\beta \zeta) = \sum_{\beta} \text{Tr}^\text{op}_{n-1} (R^*_\beta R(\zeta) R(\zeta)^* R_\beta) = \text{Tr}^\text{op}_{n-1} \text{Tr}_{n-1} (R(\zeta) R(\zeta)^*) \\ \leq \lambda^{n-1} \text{Tr}_{n-1} (L(\zeta) L(\zeta)^*) = \lambda^{n-1} \text{Tr}_n (L(\zeta) L(\zeta)^*) = \lambda^{n-1} ||\zeta||^2 \leq \left\langle (\lambda^{n-1} p_n) \zeta, \zeta \right\rangle_{P_n}.$$

(2) As $\lambda = 1$, by (1),

$$\left\langle \left(\sum_{\beta} p_n R_\beta R^*_\beta p_n\right) \zeta, \zeta \right\rangle = \left\langle \zeta, \zeta \right\rangle$$

for all $\zeta \in P_n$, and the result follows from polarization.
Theorem 4.4.20. Suppose $H$ is approximately extremal. Then $\rho = \sum_{\beta} L_{\beta} R_{\beta}^*$ converges strongly on $P_n$. Moreover if $H$ is extremal, $\rho$ is unitary. A similar result holds for $\rho^\text{op} = \sum_{\alpha} R_{\alpha} L_{\alpha}^*$.

Proof. We begin as in the proof of Proposition 3.3.19 of [Bur03], but as we do not have Jones projections, we use Lemma 4.4.19.

Suppose $\zeta \in P_n$, and enumerate $\{\beta\} = \{\beta_i\}_{i \in \mathbb{N}}$. We will show
\[
\left\| \sum_{i=r}^s L_{\beta_i} R_{\beta_i}^* \zeta \right\|_2^2 \to 0 \text{ as } r, s \to \infty.
\]

First note that the infinite matrix $(L_{\beta_j}^* L_{\beta_i})$ is a projection, so it is dominated by $1 = \delta_{i,j}$. Hence each corner $(L_{\beta_j}^* L_{\beta_i})_{i,j=r}^s$ is dominated by $1 = \delta_{i,j}$, and
\[
\left\| \sum_{i=r}^s L_{\beta_i} R_{\beta_i}^* \zeta \right\|_2^2 \leq \sum_{i,j=r}^s \left\langle (L_{\beta_j}^* L_{\beta_i}) R_{\beta_i}^* \zeta, R_{\beta_j}^* \zeta \right\rangle \leq \sum_{i=r}^s \left\langle R_{\beta_i}^* \zeta, R_{\beta_i}^* \zeta \right\rangle.
\]

We need to show that the right hand side tends to zero, which is certainly true if the infinite sum $\sum_{\beta} \left\| R_{\beta}^* \zeta \right\|_2^2$ converges. But this follows immediately from Lemma 4.4.19. Hence $\rho$ converges and $\|\rho\| \leq \sqrt{\lambda^{n-1}}$ (where $\lambda$ is the approximate extremality constant). If $\lambda = 1$, then $\|\rho\| \leq 1$ and $\rho^n = \text{id}_{P_n}$, so $\rho$ is necessarily isometric and thus unitary. \hfill $\square$

Symmetric bimodules and a converse of Theorem 4.4.20

We prove a converse of Theorem 4.4.20, with some additional structure on $H$.

Remark 4.4.21. For the rest of this section, we assume $H$ is symmetric (see Remark 4.3.5).

Lemma 4.4.22. For all $\eta, \xi \in B^n$, $\left\langle \eta | \xi \right\rangle_A = A \left\langle J\eta, J\xi \right\rangle$.

Proof. Suppose $a_1, a_2 \in A$. Then
\[
\left\langle A \left\langle J\eta, J\xi \right\rangle \hat{a}_1, \hat{a}_2 \right\rangle = \left\langle J R(J\eta)^* R(J\xi) J \hat{a}_1, \hat{a}_2 \right\rangle = \left\langle \hat{a}_2^*, R(J\eta)^* R(J\xi) \hat{a}_1^* \right\rangle = \left\langle a_2^*, a_1^* J\xi \right\rangle = \left\langle J(\eta a_2), J(\xi a_1) \right\rangle = \left\langle \eta a_2, \xi a_1 \right\rangle = \left\langle \left\langle \eta | \xi \right\rangle_A \hat{a}_1, \hat{a}_2 \right\rangle.
\]

Definition 4.4.23. Using Lemma 4.4.22, we define an algebra structure on $B^n \otimes_A B^n$ as follows: if $\eta_1, \eta_2, \xi_1, \xi_2 \in B^n$, then
\[
(\eta_1 \otimes \xi_1)(\eta_2 \otimes \xi_2) = \eta_1 \langle J \xi_1 | \eta_2 \rangle_A \otimes \xi_2 = \eta_{1A} \langle \xi_1, J \eta_2 \rangle \otimes \xi_2.
\]

Proposition 4.4.24 ([Sau83, HO89]). The map $B^n \otimes_A B^n \to C_n$ by $\eta \otimes J_\eta \xi \mapsto L(\eta)L(\xi)^*$ gives a $*$-algebra isomorphism onto its image, and it extends to a $C_n - C_n$ bimodule isomorphism $\theta_n : \mathcal{H}^{2n} \to L^2(C_n, \text{Tr}_n)$. The same result holds swapping $\text{op}$.
Proof. The map is well defined as it is $A$-middle linear:

$$\eta a \otimes J_n \xi \mapsto L(\eta) a L(\xi)^* = L(\eta) L(a \xi)^*$$

and

$$\eta \otimes a J_n \xi \mapsto L(\eta) L(J_n(a J_n \xi))^* = L(\eta) L(\xi a)^*$$

The map clearly preserves the multiplicative structure and is isometric by construction. If $n, k \in \mathbb{N}$, then

$$\langle L(\eta_1) L(\xi_1)^*, L(\eta_2) L(\xi_2)^* \rangle_{L^2(C_n, \Tr_m)} = \Tr_n(\eta_1) L(\xi_2)^* L(\eta_1) L(\xi_1)^*)$$

$$= \Tr_n(\eta_2^A L(\eta_1) L(\xi_1)^*)$$

$$= \langle \xi_2(\eta_2^A, \xi_1) \rangle_{H^n}$$

$$= \langle J_n \xi_1, J_n(\xi_2(\eta_1^A)) \rangle_{H^n}$$

$$= \langle \eta_1 \otimes J_n \xi_1, \eta_2 \otimes J_n \xi_2 \rangle_{H^{2n}}$$

Hence it clearly extends to a $C_n - C_n$ bilinear bimodule isomorphism.

Corollary 4.4.25. $C_{n-k} \subseteq C_n \subseteq C_{n+k}$ is standard (isomorphic to the basic construction) for all $n, k \geq 0$.

Proof. By Remark 4.3.9 and Proposition 4.4.24,

$$J_{2n}(C_{n-k} \otimes_A \id_{n+k})^* J_{2n} = J_{2n}(\id_{n-k} \otimes_A \op) J_{2n} = C_{n+k} \otimes_A \id_{n-k}$$

Lemma 4.4.26 ([Bur03], Theorem 3.3.13). Let $N$ be a von Neumann subalgebra of a semifinite von Neumann algebra $M$ with n.f.s. trace $\Tr_M$. Then

1. $N' \cap L^2(M) = \overline{N' \cap \mathcal{M}_n} \|\|^2$

2. $(N' \cap L^2(M))^\perp = \overline{[N, \mathcal{M}_n]} \|\|^2$, the closure of the span of the commutators in $L^2(M)$.

Remark 4.4.27. By Proposition 4.4.24 and Lemma 4.4.26, $\theta_n$ yields an isomorphism

$$P_{2n} = A' \cap H^{2n} \cong A' \cap L^2(C_n, \Tr_n) = \overline{A' \cap \mathcal{M}_n} \|\|^2 = \overline{\op \cap \mathcal{M}_n} \|\|^2 = L^2(Q_n, \Tr_n)$$

of $Q_n - Q_n$ bimodules. A similar result holds swapping $\op$.

Theorem 4.4.28. If $\rho$ exists on $P_{2n}$, then $H^n$ is approximately extremal. If $\rho$ is unitary, then $H^n$ is extremal.

Proof. The main step is to show the following lemma, whose proof is essentially the same as in [Bur03].
Lemma 4.4.29 (3.3.21.(ii) of [Bur03]). If $\rho$ exists on $P_{2n}$, then for all $x \in C_n^{op} \cap n_{Tr_n}$, $\rho^n(\theta_n^{-1}(\hat{x})) = \theta_n^{-1}(\hat{j}_n(x)) \in C_n^{op} \cap n_{Tr_n}$. In particular, $C_n^{op} \cap n_{Tr_n} = n_{Tr_n^{op}} \cap n_{Tr_n}$. A similar result holds swapping $^{op}$.

Using this lemma, Burns' proof shows $Tr_n^{op} \leq \|\rho^n\| Tr_n$ on $Q_n^+$. Suppose $z \in Q_n$. If $Tr_n(z^*z) = \infty$, we are finished. Otherwise, $z \in C_n^{op} \cap n_{Tr_n} = n_{Tr_n^{op}} \cap n_{Tr_n}$, and

$$Tr_n^{op}(z^*z) = Tr_n \circ j_n(z^*z) = Tr_n(j_n(z)^*j_n(z)) = \langle \hat{j}_n(z), \hat{j}_n(z) \rangle_{L^2(Q_n, Tr_n)}$$

$$= \left\langle \theta_n^{-1}(\hat{j}_n(z)), \theta_n^{-1}(\hat{j}_n(z)) \right\rangle_{P_n} = \left\langle \rho^n(\theta_n^{-1}(z)), \rho^n(\theta_n^{-1}(z)) \right\rangle_{P_n}$$

$$= \|\rho^n(\theta_n^{-1}(z))\|^2_{P_n} \leq \|\rho^n\|^2 \|\theta_n^{-1}(z)\|^2_{P_n} = \|\rho^n\|^2 \|z\|^2_{L^2(Q_n, Tr_n)}$$

Similarly $Tr_n \leq \|\rho^n\|^2 Tr_n^{op}$ on $Q_n^+$, and $H_n$ is approximately extremal. In particular, if $\|\rho\| = 1$, $H_n$ is extremal. \hfill \blacksquare

Remark 4.4.30. Theorem 4.1.4 now follows immediately from Theorems 4.4.7, 4.4.20, and 4.4.28.

4.5 Examples

Centralizer algebras and central $L^2$-vectors

Example 4.5.1 (Bifinite bimodules). In the case that $H$ is a symmetric, bifinite $A - A$ bimodule, then the $\mathbb{B}P$-algebra structure encodes the $C^*$-tensor category whose objects are the sub-bimodules of $H^n$ for some $n$ and whose morphisms are intertwiners.

Example 4.5.2. Suppose $A_0 = A \subset B = A_1$ is an infinite index inclusion of $II_1$-factors. Then $H = L^2(B)$ gives an $A - A$ bimodule. In this case, letting $A_{n+1}$ be the $n^{th}$ iterated basic construction of $A_{n-1} \subset A_n$, we have

- $H_n \cong L^2(A_n, Tr_n)$,
- $C_n, C_n^{op}$ is the left, right action respectively of $A_{2n}$, and
- $Q_n = A_0 \cap A_{2n}$.

Theorem 4.1.4 was proven for this case by [Bur03].

Example 4.5.3. Suppose $A$ is a $II_1$-factor, and $\sigma \in \text{Aut}(A)$. Define $H_\sigma = A L^2(A)_{\sigma(A)}$ by $\hat{abc} = \hat{a} \sigma(b \sigma(c))$ for all $a, b, c, \in A$. Suppose that $\sigma$ is outer and not periodic, and $\sigma^n$ is outer for all $n \in \mathbb{N}$. Then $H_\sigma^n \cong H_{\sigma^n}$ is extremal and $P_n = (0)$ for all $n \geq 1$. 


Example 4.5.4 (Group actions). Suppose $G$ is a countable i.c.c. group, and $\pi : G \to U(K)$ is a unitary representation. We can define two bimodules:

1. $H = K \otimes_c \ell^2(G)$ where the left action is given by the diagonal action $\pi \otimes \lambda$ and the right action is given by $1 \otimes \rho$ where $\lambda, \rho$ are the left, right regular representation of $G$ on $\ell^2(G)$. Hence $K \otimes_c \ell^2(G)$ gives an $A - A$ bimodule where $A = LG$. Then we may identify

$$H^n = K^n \otimes_c \ell^2(G)$$

where we write $K^n = K^{\otimes n}$, and the left action is the diagonal action $\pi^n \otimes \lambda$ and the right action is $1_n \otimes \rho$. It is clear that projections in $Q_n$ correspond to $LG - LG$ invariant subspaces of $H^n$. Every $G$-invariant subspace of $K^n$ yields such a subspace, but in general, they do not exhaust all possible subspaces.

2. To fix this problem, we use an idea of Richard Burstein and add a copy of the hyperfinite $II_1$-factor $R$. Suppose $\alpha : G \to \text{Aut}(R)$ is an outer action, so $A = R \rtimes_\alpha G$ is a $II_1$-factor. Set $H = K \otimes_c L^2(R) \otimes_c \ell^2(G)$, and consider the left and right actions where

$$r_1(k \otimes \hat{r} \otimes \delta_g)r_3 = k \otimes r_1 r_2 \alpha_g(r_3) \otimes \delta_g$$
$$g_1(k \otimes \hat{r} \otimes \delta_{g_2})g_3 = (\pi_{g_1}k) \otimes \alpha_{g_1}(r) \otimes \delta_{g_1 g_2 g_3}$$

for $r, r_i \in R$ and $g, g_i \in G$ for $i = 1, 2, 3$. Hence $g \in G$ acts on the left by $\pi_g \otimes \alpha_g \otimes \lambda_g$ and on the right by $1 \otimes 1 \otimes \rho_g$. Then similarly we may identify

$$H^n = K^n \otimes_c L^2(R) \otimes_c \ell^2(G).$$

Theorem 4.5.5. For $A = R \rtimes_\alpha G$ and $H^n$ as above, $A - A$ invariant subspaces of $H^n$ correspond to $G$-invariant subspaces of $K^n$.

Proof. First, if $L_0 \subset K^n$ is a $G$-invariant subspace, then $L_0 \otimes L^2(A)$ is an $A - A$ invariant subspace of $H^n$.

Now suppose $L \subset H^n$ is an $A - A$ invariant subspace, and let $p \in Q_n$ be the projection onto $L$. Note that

$$p \in \left(1_{K^n} \otimes R\right)' \cap \left(1_{K^n} \otimes \text{op}\right)'$$
$$= \left(B(K^n) \otimes (R' \cap B(L^2(A)))\right) \cap \left(B(K^n) \otimes A\right)$$
$$= B(K^n) \otimes (R' \cap A) = B(K^n) \otimes 1_{L^2(A)}.$$

Hence there is a $q \in B(K^n)$ such that $p = q \otimes 1_{L^2(A)}$. But since $q$ commutes with the left $G$-action on $H^n$, we have $q \in \pi(G)' \cap B(K^n)$. \qed

Corollary 4.5.6. $A - A$ invariant vectors of $H^n$ correspond to $G$-invariant vectors of $K^n$. 

Corollary 4.5.9. The infinite index II$_1$-subfactor $R \times G_0 \subset R \times G_1$ for $G_0 = \text{Stab}(1) \subset S_\infty = G_1$ has finite dimensional higher relative commutants.

Theorem 4.5.10. Suppose $G_0 \subset G_1$ and $K$ are as in Example 4.5.7 such that $[G_1: G_0] = \infty$ and $\#G_0 \setminus G_1 / G_0 = 2$. Then

(1) the space of $G_0$-invariant vectors in $K^n$ is one dimensional, and

(2) zero is the only $G_1$-invariant vector in $K^n$.

Proof. Let $\{g_i\}_{i \geq 0}$ be a set of coset representatives for $G_1 / G_0$ with $g_0 = e$. Since $\#G_0 \setminus G_1 / G_0 = 2$, for $i, j \geq 1$, there are $h_{i,j} \in G_0$ such that $h_{i,j} g_i G_0 = g_j G_0$.

(1) Suppose

$$ \xi = \sum_{i_1, \ldots, i_n} \lambda_{i_1, \ldots, i_n} \delta_{g_{i_1} G_0} \otimes \cdots \otimes \delta_{g_{i_n} G_0} \in K^n $$

is $G_0$-invariant. Then since $\pi_{h_{i,j}} \xi = \xi$ for all $i, j \geq 1$, we must have $\lambda_{i_1, \ldots, i_n} = 0$ unless $i_j = 0$ for all $j = 1, \ldots, n$. (Otherwise, there would be infinitely many coefficients which would be nonzero and equal, a contradiction to $\xi \in K^n \cong \ell^2((G_1 / G_0)^n)$.) Hence $\xi \in \text{span}\{\delta_{G_0} \otimes \cdots \otimes \delta_{G_0}\}$.

(2) Since $\delta_{G_0} \otimes \cdots \otimes \delta_{G_0}$ is not $G_1$-invariant, the result follows from (1).
Corollary 4.5.11. Let $G_0 = \text{Stab}(1) \subset S_\infty = G_1$. Let $A_i = R \rtimes G_i$ for $i = 0, 1$, and let $K = \ell^2(G_1/G_0)$.

1. When we consider $H = K \otimes C L^2(R) \otimes C \ell^2(G_1)$ as an $A_1 - A_1$ bimodule, $P_n = (0)$.

2. When we consider $H = L^2(A_1) = L^2(R) \otimes C \ell^2(G_1)$ as an $A_0 - A_0$ bimodule,

$$H^n \cong L^2(A_n) \cong K^{n-1} \otimes C L^2(R) \otimes C \ell^2(G_1),$$

and for all $n \geq 0$, $P_n$ is one-dimensional and spanned by

$$\hat{1} \otimes \cdots \otimes \hat{1} \in \bigotimes_{A_0}^n L^2(A_1) \cong L^2(A_n).$$

In joint work with Steven Deprez, we have shown an even stronger result:

Theorem 4.5.12. The algebras $Q_n$ for the bimodules in (1) and (2) in Example 4.5.7 are finite dimensional, and the dimensions grow super-factorially.

Corollary 4.5.13. The infinite index $II_1$-subfactor $L G_0 \subset L G_1$ where $G_0 = \text{Stab}(1) \subset S_\infty = G_1$ has finite dimensional higher relative commutants.

(Approximate) Extremality

Example 4.5.14. If $A H_A$ is a bifinite bimodule (e.g., as in Example 4.5.1), then $\dim(Q_1) < \infty$ by [Jon83]. Since any two faithful traces on a finite dimensional von Neumann algebra are comparable, $H$ is approximately extremal.

In the case that $H = L^2(A_1)$ and $A = A_0$ where $A_0 \subset A_1$ is a finite index (not necessarily extremal) $II_1$-subfactor, rotations for $H^n$ were constructed in [JP11].

Example 4.5.15. To get an example of an infinite index approximately extremal bimodule, take any bifinite bimodule $A H_A$ and tensor it with $\ell^2$ over $\mathbb{C}$.

In the subfactor setting, this is equivalent to looking at the infinite index subfactor $A_0 \otimes 1 \subset A_1 \otimes R$ where $A_0 \subset A_1$ is finite index. To get an example which is approximately extremal and not extremal, just take $A_0 \subset A_1$ non-extremal (such examples with principal graph $A_{-\infty,\infty}$ are given in [Jon83]).

Example 4.5.16. The bimodules in Example 4.5.3 and Theorem 4.5.10 (2) are trivially extremal, and the rotation is trivial.

We will now derive necessary and sufficient conditions for the (approximate) extremality for the infinite index group-subgroup subfactor as in Example 4.5.7. For the rest of this subsection, Suppose $G_0 \subset G_1$ is an inclusion of countable groups with $[G_1 : G_0] = \infty$, and $\alpha : G_1 \to \text{Aut}(R)$ is an outer action. Set $A_0 = R \rtimes_\alpha G_0 \subset R \rtimes_\alpha G_1 = A_1$ and $H = L^2(A_1)$, and note that $A_0 \subset A_1$ is an irreducible inclusion of $II_1$-factors, i.e., $A_0' \cap A_1 = \mathbb{C}1$. 
Example 4.5.17 (Two-sided bases). As stated in Remark 4.4.5, any time $H$ has a two-sided basis, $H$ is extremal. For example, if $G_0 = \{e\}$ is trivial, then $H = L^2(A_1) \cong L^2(R) \otimes \ell^2(G_1)$ is extremal, since $\{\hat{1} \otimes \delta_g g \in G_1\}$ is a two-sided basis.

In fact, an $H_A$-basis is obtained from a set of left coset representatives for $G_1 / G_0$, and an $A H$-basis is obtained from a set of right coset representatives. Hence if $G_1$ has a set of simultaneous left and right coset representatives, then $H$ is extremal by Remark 4.4.5. For example, if $G_0 = \text{Stab}(1) \subset S_\infty = G_1$, then such a set of representatives is given by the transpositions $\{(1 \ n) | n \in \mathbb{N}\}$.

Proposition 4.5.18 (Similar to [ILP98], Example 3.5). For $g \in G_1$, let $|\mathcal{O}_{gG_0}|$ denote the size of the orbit of $gG_0$ in the $G_0$-set $G_1 / G_0$. Then

1. $Q_1 \cong \ell^\infty(G_0 \setminus G_1 / G_0)$, where we denote the minimal projection onto $\mathbb{C} \delta_{G_0 gG_0}$ by $p_g$ for $g \in G_1$.
2. $\text{Tr}_1(p_g) = |\mathcal{O}_{gG_0}| = |G_0 : G_0 \cap gG_0g^{-1}|$, and
3. Since $j_1(p_g) = p_{g^{-1}}$,
   $$\text{Tr}_1^{op}(p_g) = |\mathcal{O}_{g^{-1}G_0}| = |G_0 : G_0 \cap g^{-1}G_0g| = [gG_0g^{-1} : G_0 \cap gG_0g^{-1}]$$.

Theorem 4.5.19. Assume the notation of Proposition 4.5.18. Then exactly one of the following occurs:

1. $|\mathcal{O}_{gG_0}| = |\mathcal{O}_{g^{-1}G_0}|$ for all $g \in G_1$ and $H$ is extremal, or
2. there is a $g \in G_1$ for which $|\mathcal{O}_{gG_0}| \neq |\mathcal{O}_{g^{-1}G_0}|$, and $H$ is not approximately extremal.

Proof. If there is a $g \in G$ where exactly one of $|\mathcal{O}_{gG_0}|, |\mathcal{O}_{g^{-1}G_0}|$ is finite, then $H$ is not approximately extremal. Hence we must only consider the case where for all $g \in G$, both $|\mathcal{O}_{gG_0}|, |\mathcal{O}_{g^{-1}G_0}|$ are finite or infinite. Recall that the commensurator

$$\text{Comm}_{G_1}(G_0) = \{g \in G_1 | |\mathcal{O}_{gG_0}|, |\mathcal{O}_{g^{-1}G_0}| < \infty\}$$

is a subgroup of $G_1$, and the map $\varphi: \text{Comm}_{G_1}(G_0) \to \mathbb{Q}_{>0}$ by

$$g \mapsto \frac{|\mathcal{O}_{gG_0}|}{|\mathcal{O}_{g^{-1}G_0}|}$$

is a homomorphism. Hence if there is a $g \in \text{Comm}_{G_1}(G_0)$ with $\varphi(g) > 1$, then for each $n \in \mathbb{N}$, there is a $k_n \in \mathbb{N}$ such that

$$n < \varphi(g)^{k_n} = \varphi(g^{k_n}) = \frac{|\mathcal{O}_{g^{k_n}G_0}|}{|\mathcal{O}_{g^{-k_n}G_0}|} = \frac{\text{Tr}_1(p_{g^{k_n}})}{\text{Tr}_1^{op}(p_{g^{k_n}})},$$

and $H$ is not approximately extremal. \qed
Corollary 4.5.20.  (1) If $H$ is approximately extremal, then $H$ is extremal.

(2) If $\#G_0 \setminus G_1 / G_0 = 2$, then $H$ is extremal.

(3) If there is a $g \in G_1$ such that $gG_0g^{-1} \subsetneq G_0$, then $H$ is not approximately extremal.

Remark 4.5.21. In [ILP98], Izumi, Longo, and Popa give an example of $G_0 \subset G_1$ where there is a $g \in G_1$ such that $gG_0g^{-1} \subsetneq G_0$ (so $|O_g^{-1}G_0| = 1$) and $|O_gG_0| = \infty$. Thus they give an example of an irreducible infinite index subfactor which is not approximately extremal.

Finally, we leave the reader with an open question:

Question 4.5.22. Is there an irreducible infinite index II$_1$-subfactor which is approximately extremal and not extremal?

4.6 Relative tensor products of extended positive cones

Notation 4.6.1. For this section, let $H_A$ be a right Hilbert $A$-module, $AK_B$ be a Hilbert $A - B$ bimodule, and $BL$ be a left Hilbert $B$-module where $A, B$ are finite von Neumann algebras. We write:

- $X = (A^{op})' \cap B(H)$,
- $AK$ when we ignore the right $B$-action,
- $Y_0 = A' \cap B(K)$,
- $Y = A' \cap (B^{op})' \cap B(K)$,
- $Z = B' \cap B(L)$,
- $X \otimes_A Y_0 = \{x \otimes_A y | x \in X \text{ and } y \in Y_0\}$, and
- $X \otimes_A Y \otimes_B Z = \{x \otimes_A y \otimes_B z | x \in X, y \in Y, \text{ and } z \in Z\}$.

The goal of this section is to define the operator $x \otimes_A y \in (X \otimes_A Y_0)^+$ for $x \in X^+$ and $y \in Y_0^+$ such that certain properties, e.g., associativity, are satisfied.

The next three lemmata are straightforward, but we include some proofs for completeness and for the convenience of the reader.

Lemma 4.6.2. Suppose $x \in M^+$ and $(x_i)_{i \in I} \subset M^+$ is a directed net, with $x_i \leq x$ for all $i \in I$. The following are equivalent:

(1) $x_i \rightarrow x$ strongly (if and only if $\sigma$-strongly as $\|x_i\|_\infty \leq \|x\|_\infty$ for all $i$)
(2) \( x_i \to x \) weakly (if and only if \( \sigma \)-weakly as \( \| x_i \|_\infty \leq \| x \|_\infty \) for all \( i \))

(3) \( x_i \nearrow x \), i.e., \( x_i(\omega_\xi) \nearrow x(\omega_\xi) \) for all \( \xi \in H \),

(4) \( x_i(\omega_\xi) \nearrow x(\omega_\xi) \) for all \( \xi \) in a dense subspace \( D \) of \( H \).

**Proof.** Clearly (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4).

(3) \( \Rightarrow \) (1): Suppose \( (x - x_i)(\omega_\xi) \to 0 \) for all \( \xi \in H \). Then \( \| \sqrt{x - x_i}\xi \|_2 \to 0 \), so \( \sqrt{x - x_i} \to 0 \) strongly. Hence \( x_i \to x \) strongly as multiplication is strongly continuous on bounded sets.

(4) \( \Rightarrow \) (3): Choose an orthonormal basis \( \{ e_n \}_{n \geq 1} \subset D \) for \( H \). Suppose \( \xi = \sum_n \lambda_n e_n \in H \setminus \{0\} \), and let \( \varepsilon > 0 \). Then there is an \( N > 0 \) such that

\[
\| \xi_N \|_2^2 = \sum_{n > N} |\lambda_n|^2 < \frac{\varepsilon^2}{16 \| x \|_\infty^2 \| \xi \|_2^2}.
\]

For \( n = 1, \ldots, N \), there are \( i_n \in I \) such that \( i > i_n \) implies

\[
|\langle (x - x_i)\lambda_n e_n, \xi \rangle| \leq \| (x - x_i)\lambda_n e_n \|_2 \| \xi \|_2 < \frac{\varepsilon}{2^{n+1}}.
\]

Now choose \( i' > i_n \) for all \( n = 1, \ldots, N \). We calculate that for \( i > i' \),

\[
\langle (x - x_i)(\omega_\xi), \xi \rangle = \langle (x - x_i)\xi, \xi \rangle
\]

\[
\leq \sum_{n=1}^N |\langle (x - x_i)\lambda_n e_n, \xi \rangle| + |\langle (x - x_i)\xi_N, \xi \rangle|
\]

\[
\leq \sum_{n=1}^N \frac{\varepsilon}{2^{n+1}} + |\langle x\xi_N, \xi \rangle| + |\langle x_i\xi_N, \xi \rangle|
\]

\[
\leq \sum_{n=1}^N \frac{\varepsilon}{2^{n+1}} + 2\| x \|_\infty \| \xi_N \|_2 \| \xi \|_2
\]

\[
< \frac{\varepsilon}{2} + 2\| x \|_\infty \frac{\varepsilon}{4 \| x \|_\infty \| \xi \|_2^2} \| \xi \|_2 = \varepsilon.
\]

As \( \varepsilon \) was arbitrary, we are finished. \( \square \)

**Lemma 4.6.3.** If \( x, y \in M^+ \), and \( (x_i)_{i \in I}, (y_j)_{j \in J} \subset M^+ \) are directed nets of increasing operators such that

- any two elements in \( \{ x, y \} \cup \{ x_i | i \in I \} \cup \{ y_j | j \in J \} \) commute and
- \( x_i \nearrow x \) and \( y_j \nearrow y \),

then \( x_i y_j \nearrow xy \) (and Lemma 4.6.2 applies).
Lemma 4.6.4. Suppose \( x \in X \) and \( y \in Y_0 \). Then \( x \otimes_A y : H \otimes_A K \to H \otimes_A K \) given by the unique extension of \( \xi \otimes \eta \mapsto (x\xi) \otimes (y\eta) \) where \( \xi \in D(H_A) \) and \( \eta \in D(A_K) \) is well-defined and bounded, and \( \|x \otimes_A y\|_\infty \leq \|x\|_\infty \|y\|_\infty \). Hence the \(*\)-algebra map \( x \otimes_C y \mapsto x \otimes_A y \) is a binormal representation of \( X \otimes_C Y_0 \) on \( H \otimes_A K \).

Proof. (1) Fix \( \xi_1, \ldots, \xi_k \in D(H_A) \) and \( \eta_1, \ldots, \eta_k \in D(A_K) \), and let \( \xi = (\xi_1, \ldots, \xi_k) \) and \( \eta = (\eta_1, \ldots, \eta_k) \). Since the matrices \( m = (A\langle y\eta_i, y\eta_j \rangle)_{i,j} \), \( n = (\langle \xi_j, \xi_i \rangle_A)_{i,j} \in M_k(A) \) are positive (see Lemma 1.8 of [Bis97]), we have

\[
\left\| \sum_{i=1}^k (x\xi_i) \otimes (y\eta_i) \right\|_2^2 = \sum_{i,j=1}^k (\langle x\xi_i, (x\xi_j) \otimes (y\eta_j) \rangle) \\
= \sum_{i,j=1}^k (\langle (x\xi_i)A(y\eta_i, y\eta_j) \rangle, (x\xi_j) \rangle) = (\langle x\xi \rangle n^{1/2})^2 \\
\leq \|x\|_\infty^2 \|\xi n^{1/2}\|_2^2 = \|x\|_\infty^2 \sum_{i,j=1}^k \langle \xi_i A(y\eta_i, y\eta_j), \xi_j \rangle \\
= \|x\|_\infty^2 \sum_{i,j=1}^k \langle \xi_j, \xi_i \rangle A(y\eta_i, y\eta_j) = \|x\|_\infty^2 \|m^{1/2}(y\eta)\|_2^2 \\
\leq \|x\|_\infty^2 \|y(m^{1/2}\eta)\|_2^2 \leq \|x\|_\infty^2 \|y\|_\infty^2 \|m^{1/2}\eta\|_2^2 \\
= \|x\|_\infty^2 \|y\|_\infty^2 \left\| \sum_{i=1}^k \xi_i \otimes \eta_i \right\|_2^2 .
\]

(2) That \( x \mapsto x \otimes_A 1_K \) is a normal representation of \( X \) follows from the density of \( D(H_A) \otimes_A K \) and (4) of Lemma 4.6.2. Similar for \( y \mapsto 1_H \otimes_A y \).

\[\square\]

Notation 4.6.5. Let \( \mathcal{B} \) be the Borel \( \sigma \)-algebra of subsets of \( [0, \infty] \). For a spectral measure \( E : \mathcal{B} \to P(H) \), we use the conventions \( E_\lambda = E([0, \lambda]) \), so \( E_\infty = 1 \), and \( E^{\infty} = E(\{\infty\}) \) (in general, our spectral measures on \( \mathcal{B} \) have non-trivial mass at \( \infty \)).

Lemma 4.6.6. Suppose \( E : \mathcal{B} \to P(X) \subset B(H_A) \) is a spectral measure. Suppose \( f : [0, \infty] \to [0, \infty) \) is a bounded Borel-measurable function, and \( (\varphi_n) \) is a sequence of positive simple functions increasing pointwise to \( f \). Then

\[
\int_0^\infty f(\lambda) \, dE_\lambda := \sup_n \int_0^\infty \varphi_n(\lambda) \, dE_\lambda
\]

is well-defined.
CHAPTER 4. A PLANAR CALCULUS FOR INFINITE INDEX SUBFACTORS

Proof. Suppose $\xi \in H$. Then as $\omega_\xi$ is normal, $\omega_\xi \circ E$ is a Borel measure, and

$$\int_0^\infty f(\lambda) \, d(\omega_\xi(E_\lambda)) = \sup_n \int_0^\infty \varphi_n(\lambda) \, d(\omega_\xi(E_\lambda))$$

is independent of the choice of positive simple functions $\varphi_n$ increasing to $f$. \qed

Proposition 4.6.7. Suppose

$$E : \mathcal{B} \rightarrow P(X) \subset B(H_A) \quad \text{and} \quad F : \mathcal{B} \rightarrow P(Y_0) \subset B(A_K)$$

are spectral measures.

(1) The map $E \otimes_A F : \mathcal{B} \otimes \mathcal{B} \rightarrow P(X \otimes_A Y_0)$ by

$$(I_1, I_2) \mapsto \int_{I_1 \times I_2} d(E_\lambda \otimes_A F_\mu) := E(I_1) \otimes_A F(I_2)$$

extends uniquely to a spectral measure by countable additivity.

(2) If $\varphi, \psi : [0, \infty] \rightarrow [0, \infty)$ are positive simple functions, then

$$\int_0^\infty \int_0^\infty \varphi(\lambda)\psi(\mu) \, d(E_\lambda \otimes_A F_\mu) = \left(\int_0^\infty \varphi(\lambda) \, dE_\lambda\right) \otimes_A \left(\int_0^\infty \psi(\mu) \, dF_\mu\right) \in X \otimes_A Y_0.$$ 

(3) If $f, g$ are bounded, $\mathcal{B}$-measurable functions and $(\varphi_m), (\psi_n)$ are sequences of positive simple functions increasing to $f, g$, then

$$\sup_{m,n} \int_0^\infty \int_0^\infty \varphi_m(\lambda)\psi_n(\mu) \, d(E_\lambda \otimes_A F_\mu) = \left(\int_0^\infty f(\lambda) \, dE_\lambda\right) \otimes_A \left(\int_0^\infty g(\mu) \, dF_\mu\right) \in X \otimes_A Y_0.$$ 

Proof. (1) One simply needs to check countable additivity (pointwise on $H \otimes_A K$), which follows from countably additivity on products of intervals, which is straightforward.

(2) Obvious.

(3) Immediate from (2) together with Lemmas 4.6.3 and 4.6.6. \qed

Lemma 4.6.8. The relative tensor product of spectral measures as in Proposition 4.6.7 is associative, i.e., if

$$E : \mathcal{B} \rightarrow P(X) \subset B(H_A),$$

$$F : \mathcal{B} \rightarrow P(Y) \subset B(A_KB),$$

and

$$G : \mathcal{B} \rightarrow P(Z) \subset B(BL)$$

then

$$(E \otimes_A F) \otimes_A G = E \otimes_A (F \otimes_A G).$$
are spectral measures on $B$, then $(E \otimes_A F) \otimes_B G = E \otimes_A (F \otimes_B G)$. Moreover, if $f, g, h : [0, \infty) \to [0, \infty)$ are bounded $B$-measurable functions, and $(\phi_m), (\psi_n), (\gamma_k)$ are positive simple functions increasing to $f, g, h$ respectively, then
\[
\sup_{m,n,k} \int_0^\infty \int_0^\infty \phi_m(\lambda) \psi_n(\mu) \gamma_k(\nu) \, d(E_{\lambda} \otimes_A F_\mu \otimes_B G_\nu) = \left( \int_0^\infty f(\lambda) \, dE_\lambda \right) \otimes_A \left( \int_0^\infty g(\mu) \, dF_\mu \right) \otimes_B \left( \int_0^\infty h(\nu) \, dG_\nu \right) \in X \otimes_A Y \otimes_B Z.
\]

**Proof.** Immediate from associativity of the relative tensor product and Proposition 4.6.7.

**Definition 4.6.9.** Suppose $x \in \hat{X}^+$ and $y \in \hat{Y}_0^+$ have spectral resolutions
\[
x = \int_{[0,\infty)} \lambda \, dE_\lambda + \infty E^\infty \quad \text{and} \quad y = \int_{[0,\infty)} \mu \, dF_\mu + \infty F^\infty
\]
(recall Notation 4.6.5). Then
\[
E : B \longrightarrow P(X) \subset B(H_A) \quad \text{and} \quad F : B \longrightarrow P(Y_0) \subset B(A_K)
\]
are two spectral measures as in Proposition 4.6.7. For $m, n \in \mathbb{N}$, set
\[
x_m = \int_{[0,m]} \lambda \, dE_\lambda + m E^\infty \quad \text{and} \quad y_n = \int_{[0,n]} \mu \, dF_\mu + n F^\infty.
\]

Applying Lemma 4.2.23 to the directed set
\[
\mathcal{F} = \{ x_m \otimes_A y_n | m, n \in \mathbb{N} \} \subset (X \otimes_A Y_0)^+,
\]
we get a positive, self-adjoint operator affiliated to $X \otimes_A Y_0$ and densely-defined in an affiliated subspace of $X \otimes_A Y_0$. We denote this operator as $x \otimes_A y$.

**Remark 4.6.10.** Assume the notation of Definition 4.6.9. When we work with $x \otimes_A y$, it helps to consider the following 3 projections:
\[
p_0 = (E_0 \otimes_A 1_K) \lor (1_H \otimes F_0), \quad p_\infty = \left( (1 - E_0) \otimes_A F^\infty \right) + \left( E^\infty \otimes_A (1 - F_0) \right) + E^\infty \otimes_A F^\infty, \quad \text{and} \quad p_f = \sup_{\lambda, \mu < \infty} E_\lambda \otimes_A F_\mu = (1 - E^\infty) \otimes_A (1 - F^\infty),
\]
which we should think of as having the following “supports” given by the shaded areas in $[0_\mathbb{R}, \infty_\mathbb{R}]^2$ below:
These three projections commute with \( x \otimes_A y \).

- Dom\((x \otimes_A y)^{1/2}\) \(\subset (1 - p_\infty)(H \otimes_A K)\), and \((x \otimes_A y)(1 - p_\infty)\) is densely defined on \((1 - p_\infty)(H \otimes_A K)\).

- \((x \otimes_A y)p_f = \sup_{m,n<\infty} \int_{[0,m]} \int_{[0,n]} \lambda \mu \, d(E_{\lambda} \otimes_A F_{\mu})\).

- \((x \otimes_A y)p_0 = 0\).

**Lemma 4.6.11.** Let \( x \in \widehat{X}^+ \) and \( y \in \widehat{Y}_0^+ \), and assume the notation of Definition 4.6.9 and Remark 4.6.10. Suppose \( x' \in X^+ \), \( y' \in Y_0^+ \) with \( x' \leq x \) and \( y' \leq y \). Then

1. \( (x' \otimes_A y')p_0 = p_0(x' \otimes_A y') = 0\),
2. for all \( \xi \in H \otimes_A K \), \((x \otimes_A y)(\omega_\xi) = (x \otimes_A y)(\omega_{(1-p_0)\xi})\), and
3. \( x' \otimes_A y' \leq x \otimes_A y \).

**Proof.**

1. Suppose \( \eta \in D((E_0H)_A) \) and \( \kappa \in D(AK) \) (recall \( E_0 \in X \) and \( F_{\infty} \in Y_0 \)). Then since \( x' \leq x \), we must have

\[
\| (x')^{1/2} \eta \|_H = \langle x' \eta, \eta \rangle = x'(\omega_\eta) \leq x(\omega_\eta) = x(\omega_{E_0}) = xE_0(\omega_\eta) = 0.
\]

But this implies \( x' \eta = 0 \). Hence we have

\[
(x' \otimes_A y')(\eta \otimes \kappa) = 0.
\]

Similarly, for all \( \eta \in D(H_A) \) and \( \kappa \in D(A(F_0K)) \), \((x' \otimes_A y')(\eta \otimes \kappa) = 0\). By density of \( D(H_A) \otimes_A D(AK) \), we have \((x' \otimes_A y')p_0 = 0\). Taking adjoints gives \( p_0(x' \otimes_A y') = 0\).

2. By (1), for all \( m, n > 0 \), \( p_0(x_m \otimes_A y_n) = (x_m \otimes_A y_n)p_0 = 0 \), so

\[
(x \otimes_A y)(\omega_\xi) = \sup_{m,n} (x_m \otimes_A y_n)(\omega_\xi) = \sup_{m,n} \left( (x_m \otimes_A y_n)(\omega_{(1-p_0)\xi}) + \langle (x_m \otimes_A y_n)p_0 \xi, p_0 \xi \rangle + (x_m \otimes_A y_n)p_0 \xi, \xi \rangle \right) = \sup_{m,n} (x_m \otimes_A y_n)(\omega_{(1-p_0)\xi}) = (x \otimes_A y)(\omega_{(1-p_0)\xi}).
\]

3. By (2), it suffices to show that for all \( \xi \in \text{Dom}(x \otimes_A y)^{1/2} \) with \( \xi = p_f \xi \),

\[
(p_f(x' \otimes_A y')p_f)(\omega_\xi) = (x' \otimes_A y')(\omega_\xi) \leq (x \otimes_A y)(\omega_\xi) = \left( p_f(x \otimes_A y)p_f \right)(\omega_\xi).
\]
Fix such a $\xi$, and let $\varepsilon > 0$. As $E_\lambda \otimes_A F_\mu \to p_f$ strongly as $\lambda, \mu \to \infty$ from below, there is an $N > 0$ such that for all $\lambda, \mu > N$, 

$$\left(p_f(x' \otimes_A y')p_f - (E_\lambda x' E_\lambda \otimes_A F_\mu y' F_\mu)\right)(\omega_\xi) < \varepsilon.$$ 

Since $x' \leq x$ and $y' \leq y$, we have $E_N x'E_N \leq xE_N$, $F_N y'F_N \leq yF_N$ by Lemma 4.2.21, so $E_N x'E_N \otimes_A F_N y'F_N \leq xE_N \otimes_A yF_N$ as all these operators mutually commute. Hence 

$$\left(p_f(x' \otimes_A y')p_f\right)(\omega_\xi) = \left(p_0(x_m \otimes_A y_n)p_0 - (E_N x'E_N \otimes_A F_N y' F_N)\right)(\omega_\xi)$$

$$+ (E_N x'E_N \otimes_A F_N y' F_N)(\omega_\xi)$$

$$< \varepsilon + (xE_N \otimes_A yF_N)(\omega_\xi) \leq \varepsilon + (x \otimes_A y)(\omega_\xi).$$

Since $\varepsilon$ was arbitrary, the result follows. \hfill $\square$

**Lemma 4.6.12.** Suppose $(x'_j)_{j \in J} \subset \hat{X}^+$ increases to $x \in \hat{X}^+$. Suppose $p, q \in P(X)$ are spectral projections of $x$ such that $p + q = 1$. Then $(x'_j p \xi, q \xi) \to 0$ for all $\xi \in \mathrm{Dom}(x^{1/2})$.

**Proof.** For $k = 0, 1, 2, 3$, $p \xi + i^k q \xi \in \mathrm{Dom}(x^{1/2}) \subseteq \mathrm{Dom}((x'_j)^{1/2})$ for all $j \in J$. Since $x'_j$ increases to $x$, by polarization

$$\lim_{j \in J} \langle (x'_j)^{1/2} p \xi, (x'_j)^{1/2} q \xi \rangle = \lim_{j \in J} \frac{1}{4} \sum_{k=0}^{3} i^k x'_j (\omega_{p \xi + i^k q \xi}) = \frac{1}{4} \sum_{k=0}^{3} i^k x (\omega_{p \xi + i^k q \xi})$$

$$= \langle x^{1/2} p \xi, x^{1/2} q \xi \rangle = 0$$

as $p, q$ commute with $x^{1/2}$. \hfill $\square$

**Theorem 4.6.13.** Let $x \in \hat{X}^+$ and $y \in \hat{Y}^+$, and assume the notation of Definition 4.6.9 and Remark 4.6.10. Suppose there are sequences $(x'_m) \subset X^+$, $(y'_n) \subset Y^+_0$ which increase to $x, y$ respectively. Then $x'_m \otimes_A y'_n$ increases to $x \otimes_A y$.

**Proof.**

Case 1: Suppose $\xi \notin \mathrm{Dom}((x \otimes_A y)^{1/2})$ and $M > 0$. Since $\sup_{m,n} x_m \otimes_A y_n = x \otimes_A y$, there is an $N_0 \in \mathbb{N}$ such that for all $m, n \geq N_0$, $(x_m \otimes_A y_n)(\omega_\xi) > M$. Since $p_0 \xi \neq \xi$ by Lemma 4.6.11, we must have 

$$(1_H \otimes_A (1_K - F_0))\xi \neq 0$$

and 

$$(1_H - E_0) \otimes_A 1_K)\xi \neq 0.$$ 

**Claim:** There is an $N_1 > N_0$ such that $(x'_m \otimes 1_K)\xi \neq 0 \neq (1_H \otimes_A y'_n)\xi$ for all $m, n > N_1$. 

CHAPTER 4. A PLANAR CALCULUS FOR INFINITE INDEX SUBFACTORS

Proof. We prove the second non-equality. Suppose not. Then for each \( n > 0 \), there is an \( k > n \) such that \( (1 \otimes_A y_k')\xi = 0 \). But then

\[
(1_H \otimes_A y_n')(\omega_\xi) \leq (1_H \otimes_A y_k')(\omega_\xi) = 0,
\]

so \( (1_H \otimes_A y_n')\xi = 0 \) for all \( n \in \mathbb{N} \). Since \( (1_H \otimes_A (1 - F_0))\xi \neq 0 \), and \( D(H_A) \otimes_AD(A((1-F_0)K)) \) is dense in \( H \otimes_A ((1_K - F_0)K) \), there is an \( \eta \in D(H_A) \) such that \( L_\eta^*\xi \in ((1_K - F_0)K) \setminus \{0\} \) and \( L_\eta L_\eta^* \leq 1_H \otimes_A 1_K \). Now since \( y_n' \) increases to \( y \), and \( y(\omega_{L_\eta^*\xi}) > 0 \), there is an \( N' > 0 \) such that for all \( n > N' \),

\[
0 < y_n'(\omega_{L_\eta^*\xi}) = (L_\eta y_n L_\eta^*)(\omega_\xi) = \left( L_\eta L_\eta^*(1_H \otimes_A y_n') \right)(\omega_\xi) \leq (1_H \otimes_A y_n')(\omega_\xi) = 0,
\]
a contradiction. \( \square \)

Choose \( N_1 \) as in the claim, and suppose \( n > N_1 \). Let \( \{\alpha_i\} \subset D(AK) \) be an \( AK \)-basis, and let \( \eta = (1_H \otimes_A (y_{N_1})^{1/2})\xi \neq 0 \), and note \( (x_{N_1} \otimes_A 1_K)(\omega_\eta) > M \). Then

\[
M < (x_{N_1} \otimes 1_K)(\omega_\eta) = \left( (x_{N_1} \otimes_A 1_K) \left( \sum_i R_{\alpha_i} R_{\alpha_i}^* \right) \right)(\omega_\eta) = \sum_i (R_{\alpha_i}(x_{N_1})R_{\alpha_i}^*)(\omega_\eta),
\]

so there is an \( N_2 > 0 \) such that

\[
M < \sum_{i=1}^{N_2} (R_{\alpha_i} x_{N_1} R_{\alpha_i}^*)(\omega_\eta) = \sum_{i=1}^{N_2} x_{N_1}(\omega_{R_{\alpha_i}^*}) \leq \sum_{i=1}^{N_2} x(\omega_{R_{\alpha_i}^*}).
\]

Now as \( x_m' \) increases to \( x \), there is an \( N_3 > N_1 \) such that \( m > N_3 \) implies

\[
M < \sum_{i=1}^{N_2} x_m'(\omega_{R_{\alpha_i}^*}) = \sum_{i=1}^{N_2} (R_{\alpha_i} x_m' R_{\alpha_i}^*)(\omega_\eta) \leq \sum_{i} (R_{\alpha_i} x_m' R_{\alpha_i}^*)(\omega_\eta)
\]

\[
= \left( (x_m' \otimes_A 1_K) \left( \sum_i R_{\alpha_i} R_{\alpha_i}^* \right) \right)(\omega_\eta) = (x_m' \otimes y_{N_1})(\omega_\xi).
\]

Repeating the above argument for \( y_n' \) yields an \( N_4 \) such that \( m, n > N_4 \) implies \( M < (x_m' \otimes_A y_n')(\omega_\xi) \).

Case 2: Suppose \( \xi \in \text{Dom}((x \otimes_A y)^{1/2}) \). Then \( \xi = (1 - p_\infty)\xi \). We want to show

\[
\sup_{m,n} (x_m' \otimes_A y_n')(\omega_\xi) = (x \otimes_A y)(\omega_\xi) = \sup_{m,n} (x_m \otimes_A y_n)(\omega_\xi),
\]

so by Lemma 4.6.11, we may assume \( \xi = (1 - p_0)\xi \), and thus \( \xi = pf\xi \). Let \( \varepsilon > 0 \). Since

\[
pf(x \otimes_A y)pf = \sup_{\lambda,\mu<\infty} x E\lambda \otimes_A y F\mu,
\]
there is an $N_0 \in \mathbb{N}$ such that for all $\lambda, \mu \geq N_0$,

$$\left( (x \otimes_A y) - (xE_\lambda \otimes_A yF_\mu) \right)(\omega_\xi) < \frac{\varepsilon}{4}.$$ 

By Lemma 4.6.11, $x_m' \otimes_A y_n' \leq x \otimes_A y$ for all $m, n$, so using Lemma 4.2.21, we have

$$\left( (x_m' \otimes A y_n') - (E_{N_0}x_m E_{N_0}) \otimes_A (F_{N_0}y_n' F_{N_0}) \right) \leq \left( (x \otimes_A y) - (x E_{N_0} \otimes A y F_{N_0}) \right)$$

and

$$E_{N_0}x_m' E_{N_0} \otimes_A F_{N_0} y_n' F_{N_0} \leq x E_{N_0} \otimes_A y F_{N_0}$$

by multiplying on either side by $1_{H \otimes_A K} - (E_{N_0} \otimes_A F_{N_0})$ and $E_{N_0} \otimes_A F_{N_0}$ respectively. Now since $x_m', y_n'$ increase to $x, y$ respectively, by Lemma 4.2.21, $E_{N_0}x_m' E_{N_0}, E_{N_0} y_n' F_{N_0}$ increases to $xE_{N_0}, yF_{N_0}$ respectively. Thus $E_{N_0}x_m' E_{N_0} \otimes_A F_{N_0} y_n' F_{N_0}$ increases to $x E_{N_0} \otimes_A y F_{N_0}$ by Lemma 4.6.3, and there is an $N_1 > N_0$ such that for all $m, n \geq N_1$,

$$\left( (x E_{N_0} \otimes_A y F_{N_0}) - (E_{N_0}x_m' E_{N_0} \otimes_A F_{N_0} y_n' F_{N_0}) \right)(\omega_\xi) < \frac{\varepsilon}{4}.$$ 

By Lemma 4.6.12, there is an $N_2 > N_1$ such that for all $m, n > N_2$,

$$\left| \langle (x_m' \otimes y_n')(1_{H \otimes_A K} - E_{N_0} \otimes A F_{N_0}) \xi, (E_{N_0} \otimes A F_{N_0}) \xi \rangle \right| < \frac{\varepsilon}{4}.$$ 

Now we calculate that for all $m, n > N_2$,

$$(x \otimes_A y - x_m' \otimes y_n')(\omega_\xi) = (1 - E_{N_0} \otimes A F_{N_0})(x \otimes_A y - x_m' \otimes y_n')(1 - E_{N_0} \otimes A F_{N_0})(\omega_\xi)$$

$$+ (1_{H \otimes_A K} - E_{N_0} \otimes A F_{N_0})(x \otimes_A y - x_m' \otimes y_n')(E_{N_0} \otimes A F_{N_0})(\omega_\xi)$$

$$+ (E_{N_0} \otimes A F_{N_0})(x \otimes_A y - x_m' \otimes y_n')(1_{H \otimes A K} - E_{N_0} \otimes A F_{N_0})(\omega_\xi)$$

$$+ (E_{N_0} \otimes A F_{N_0})(x \otimes_A y - x_m' \otimes y_n')(E_{N_0} \otimes A F_{N_0})(\omega_\xi)$$

$$\leq \left( (x \otimes_A y) - (x E_{N_0} \otimes A y F_{N_0}) \right)(\omega_\xi)$$

$$+ |((1_{H \otimes A K} - E_{N_0} \otimes A F_{N_0})(x_m' \otimes y_n')(E_{N_0} \otimes A F_{N_0})(\omega_\xi)|$$

$$+ |(E_{N_0} \otimes A F_{N_0})(x_m' \otimes y_n')(1 - E_{N_0} \otimes A F_{N_0})(\omega_\xi)|$$

$$+ \left( (x E_{N_0} \otimes A y F_{N_0}) - (E_{N_0}x_m' E_{N_0} \otimes A F_{N_0} y_n' F_{N_0}) \right)(\omega_\xi)$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$ 

\[ \Box \]

**Corollary 4.6.14.** If $x \in \widetilde{X}^+$, $y \in \widetilde{Y}^+$, and $z \in \widetilde{Z}^+$, then $(x \otimes_A y) \otimes_B z = x \otimes_A (y \otimes_B z)$. 
Proof. Take sequences \((x_m) \subset X^+\), \((y_n) \subset Y^+\), and \((z_\ell) \subset Z^+\) which increase to \(x, y, z\) respectively. Then 
\[(x \otimes_A y) \otimes_B z = \sup_{m,n,\ell} (x_m \otimes_A y_n) \otimes_B z_\ell = \sup_{m,n,\ell} x_m \otimes_A (y_n \otimes_B z_\ell) = x \otimes_A (y \otimes_B z).\]

Corollary 4.6.15. If \(x, w \in \hat{X}^+\), \(y \in \hat{Y}^+_0\), and \(\lambda \in [0, \infty]\), then 
\[(\lambda x + w) \otimes_A y = \lambda (x \otimes_A y) + (w \otimes_A y).\]

Proof. Choose \(X^+ \ni x_m, w_n \rightharpoonup x, w \in \hat{X}^+\) respectively and \(\hat{Y}^+_0 \ni y_\ell \rightharpoonup y \in \hat{Y}^+_0\). Then 
\[(\lambda x_m + w_n) \otimes_A y_\ell = \lambda (x_m \otimes_A y_\ell) + (w_n \otimes_A y_\ell),\]
and the result follows by Remark 4.2.22 and Theorem 4.6.13.

By taking sups appropriately, and with a little more care, Lemma 4.6.11 and Theorem 4.6.13 can be generalized to prove:

Theorem 4.6.16. Let \(x \in \hat{X}^+\) and \(y \in \hat{Y}^+_0\). Suppose there are nets \((x_i)_{i \in I} \subset \hat{X}^+\), \((y_j)_{j \in J} \subset \hat{Y}^+_0\) which increase to \(x, y\) respectively. Then \(x_i \otimes_A y_j \rightharpoonup x \otimes_A y\).

4.7 The action of \(\mathbb{B}P\) is well-defined

In this section, we show the action of \(\mathbb{B}P\) is well-defined in Theorem 4.3.18. We do so in two steps. First, we define a sub-operad \(\mathbb{B}P_1 \subset \mathbb{B}P\), define the action of \(\mathbb{B}P_1\) on the extended positive cones \(\hat{\mathbb{Q}}^+_n\), and show the action is well-defined. We show that each connected tangle (see Definition 4.7.1) has a unique standard form (see Algorithm 4.7.4) that behaves well under composition, analogous to the methods of [Pen12a]. Second, we extend the action to \(\mathbb{B}P\) and show it is well-defined by considering the possibilities that occur when inserting connected \(\mathbb{B}P_1\)-tangles into the quadratic pairing tangle \(\tau_n\) or \(\tau_n^{\text{op}}\) (see Definition 4.7.7).

The operad \(\mathbb{B}P_1\)

Definition 4.7.1. We will define \(\mathbb{B}P_1\), an operad of unshaded, oriented tangles up to planar isotopy. First, we require for tangles \(\mathcal{T} \in \mathbb{B}P_1\):

1. \(\mathcal{T}\) has an external disk \(D_0\) and internal disks \(D_1, \ldots, D_s\), each with an even number \(2k_i\) of market boundary points and a distinguished interval marked \(*\). The boundary points of \(D_i\) are numbered \(1, \ldots, 2k_i\) clockwise from \(*\), and and we use the convention that for \(1 \leq n \leq 2k_i\), the \(-n^{\text{th}}\) boundary point is the point numbered \(2k_i - n + 1\).

2. Each boundary point of \(\mathcal{T}\) is connected to exactly one oriented string. Each oriented string is either a closed loop, or it is attached to two distinct boundary points.
(3) For \( i = 1, \ldots, s \), reading counter-clockwise from \( * \), the strings attached to the first \( k_i \) boundary points of \( D_i \) are oriented away from \( D_i \), and the second \( k_i \) strings are oriented toward \( D_i \),

(4) Reading counter-clockwise from \( * \), the strings attached to the first \( k_0 \) boundary points of \( D_0 \) are oriented toward \( D_0 \), and the second \( k_0 \) strings are oriented away from \( D_0 \),

When we draw such a tangle, we draw all disks \( D_i \) \((0 \leq i \leq s)\) as rectangles with \( k_i \) strings connected to the top and bottom, we suppress the external disk, we draw one thick string labelled \( n \) for \( n \) individual strings, and we orient all strings upward unless otherwise specified.

A tangle with disks \( \{D_i\}_{i=0}^s \) and strings \( \{S_j\}_{j=1}^t \) satisfying (1)-(4) is called:

- **connected** if \( \{D_i\}_{i=0}^s \cup \{S_j\}_{j=1}^t \) is connected in \( \mathbb{R}^2 \), and

- **internally connected** if \( T \) has no external boundary points and \( \{D_i\}_{i=1}^s \cup \{S_j\}_{j=1}^t \) is connected in \( \mathbb{R}^2 \).

Let \( \mathbb{BP}_1 \) be the operad generated by the following tangles:

- **Temperley-Lieb**: For \( n \geq 0 \), the “Temperley-Lieb” tangle \( 1_n \) with no inputs and 2\( n \) boundary points:

\[
1_n = \begin{array}{c}
| \hline
\end{array}
\]

Note that \( 1_0 \) is the empty tangle.

- **Partial trace**: For \( n \geq 0 \), the tangles \( t_{n+1}, t_{n+1}^{\text{op}} \) with 2\( n + 2 \) internal boundary points and 2\( n \) external boundary points and only one right, left cap respectively:

\[
t_{n+1} = \begin{array}{c}
\begin{array}{c}
| \hline
\end{array}
\end{array}
\] and \( t_{n+1}^{\text{op}} = \begin{array}{c}
\begin{array}{c}
\hline
\end{array}
\end{array} \),

- **Tensoring**: For \( m, n \geq 0 \), the tangles \( \otimes_{m,n} \) with internal disks \( D_1, D_2 \) with 2\( m \), 2\( n \) internal boundary points and 2(\( m + n \)) external boundary points as follows:

\[
\otimes_{m,n} = \begin{array}{c}
\begin{array}{c}
| \hline
\end{array}
\end{array}
\] .

**Theorem 4.7.2.** The following relations hold in \( \mathbb{BP}_1 \) for \( m, n \geq 0 \) (compare with (1)-(3) in Theorem 4.3.15):

1. \( t_m t_{m+1}^{\text{op}} = t_{m}^{\text{op}} t_{m+1} \),
2. \( \otimes_{t, m+n}(-, \otimes_{m,n}(-, -)) = \otimes_{t, m+n}(-, -, -) \), and
(3) \( t_{m+n}(\otimes_{m,n}(-,-)) = \otimes_{m,n-1}(-,t_n(-)) \) and \( t_{m+n}^{\text{op}}(\otimes_{m,n}(-,-)) = \otimes_{m-1,n}(t_m^{\text{op}}(-),-). \)

Proof. Clear by drawing pictures. \(\square\)

**Theorem 4.7.3.** Suppose \( T \) is an unshaded, oriented tangle which satisfies requirements (1)-(4) in Definition 4.7.1. Then

- (BP0) If boundary points \( m \) and \( n \) of \( D_0 \) are connected by a string, then \( m = -n \) (recall the convention \( -n = 2k_i - n + 1 \) from (1) of Definition 4.7.1).

The tangle \( T \) is in \( \mathbb{B}P_1 \) if and only if the following conditions are satisfied:

- (BP1) No string may connect the input disks \( D_i \) and \( D_j \) for \( i \neq j \).
- (BP2) If the string \( S \) connects the \( n^{\text{th}} \) boundary point of \( D_i \) to the \( m^{\text{th}} \) boundary point of \( D_0 \), then there is a string \( S' \) connecting the \(-n^{\text{th}} \) boundary point of \( D_i \) to the \(-m^{\text{th}} \) boundary point of \( D_0 \), and any other string connected to \( D_i \) must only be connected to \( D_i \) or \( D_0 \).

If (BP1) and (BP2) hold, then the following condition also holds:

- (BP3) If the string \( S \) connects boundary points \( m \) and \( n \) of \( D_i \), then \( m = -n \). Such a string is called an \( i \)-cap of \( T \). We call the \( i \)-cap a left \( i \)-cap if when we connect boundary points \( n \) and \(-n \) by an imaginary string \( S' \) inside \( D_i \), the loop \( S \cup S' \) contains the distinguished interval of \( D_i \). The \( i \)-cap is a right \( i \)-cap otherwise.

Proof. (BP0) follows from (1)-(4) in Definition 4.7.1 by a simple counting argument. Similarly, (BP3) follows from (BP0)-(BP2). Clearly tangles in \( \mathbb{B}P_1 \) satisfy (BP1) and (BP2), since these properties are preserved under composition of the tangles which generate \( \mathbb{B}P_1 \).

Now suppose \( T \) satisfies (BP0)-(BP3). If \( T \) is internally connected, then either \( T \) is a closed loop, or \( T \) has only one input disk \( D_1 \), and we may write \( T \) uniquely as

\[
T = t_1^{\text{op}} \cdots t_\ell^{\text{op}} t_{\ell+1} t_{\ell+2} \cdots t_{\ell+r} \tag{4.4}
\]

where \( \ell \) is the number of left caps and \( r \) is the number of right caps of \( D_1 \) of \( T \). Hence, we may reduce to the case that \( T \) is connected. Now Algorithm 4.7.4 expresses the connected tangle \( T \) in a standard form as a composite of generators of \( \mathbb{B}P_1 \). \(\square\)

**Algorithm 4.7.4** (Standard form of connected \( \mathbb{B}P_1 \)-tangles). Suppose \( T \) satisfies (1)-(4) of Definition 4.7.1 and (BP1)-(BP3) in Theorem 4.7.3, and suppose \( T \) is connected. Then we can use \( \otimes_{m,n} \) to "parenthesize" the \( D_i \)’s (\( i \geq 0 \)) and groups of through strings \( 1_b \) from right to left. Before we give the algorithm we give an example:

\[
\begin{array}{c}
\includegraphics{algorithm_4.7.4.png}
\end{array}
\]

The following algorithm expresses \( T \) in a standard form as a composite of generators of \( \mathbb{B}P_1 \):
(0) If \( T \) is the empty tangle, break.

(1) Start at \( * \) on the external boundary. Going clockwise along \( D_0 \), denote the strings oriented toward \( D_0 \) by \( S_1, \ldots, S_{k_0} \) (note \( k_0 > 0 \)). Set:

- \( a = k_0 \) (\( a \) is the number of strings \( S_1, \ldots, S_{k_0} \) remaining to be examined)
- \( n = 0 \) (\( S_{n+1} \) is the string we are currently examining).

Record a place holder \( ? \) to be replaced.

(2) If \( S_{n+1} \) connects \( D_0 \) to \( D_0 \), find \( b \) maximal such that \( S_{n+1}, \ldots, S_{n+b} \) all connect \( D_0 \) to \( D_0 \). Set \( a = a - b \).

(2a) If \( a = 0 \), replace the last ? with 1 and break.

(2b) If \( a > 0 \) and \( b > 0 \), replace the last ? with \( \otimes_{b,a}(1,?) \), where ? will be replaced later, and set \( n = n + b \).

(3) Now \( a > 0 \), and \( S_{n+1} \) is the first string connecting \( D_0 \) to some input disk \( D_i \). Find \( m_i \) maximal such that \( S_{n+1}, \ldots, S_{n+m_i} \) connect \( D_0 \) to \( D_i \). Set \( a = a - m_i \), let \( \ell_i \) be the number of left caps of \( D_i \), and let \( r_i \) be the number of right caps of \( D_i \).

(3b) If \( a = 0 \), replace the last ? with \( \top_{m_i+1}^{m_i} \cdots \top_{m_i+\ell_i+1}^{m_i} \cdots \top_{m_i+\ell_i+r_i}^{m_i} \) and break.

(3a) If \( a > 0 \), replace the last ? with \( \otimes_{m_i,a}(t_{m_i+1}^{m_i+\ell_i} t_{m_i+\ell_i+1}^{m_i+\ell_i+1} \cdots t_{m_i+\ell_i+r_i}^{m_i+\ell_i+r_i},?) \), where ? will be replaced later, set \( n = n + m_i \), and go to (2).

**Definition 4.7.5** (Action of tangles in \( \mathbb{BP}_1 \)). We may now describe the action of a tangle \( T \in \mathbb{BP} \) on a tuple

\[
(z_1, \ldots, z_s) \in \prod_{i=1}^s \hat{Q}_{n_i}^+.
\]

If \( T \) is connected, we put \( T \) in the standard form afforded by Algorithm 4.7.4, label the inputs with the \( z_i \)'s, and replace \( 1_n \) with \( \text{id}_{H_n} \); \( t_n, t_{n}^{\text{op}} \) with \( T_n, T_n^{\text{op}} \); and \( \otimes_{m,n} \) with \( \otimes_A \).

If \( T \) is not connected, then there are internally connected subtangles which are either closed loops, or which can be uniquely written as in Equation (4.4). These subtangles will act as scalars in \( \hat{Q}_0^+ = Z(A)^+ = [0, \infty]_R \), and the order of scalar multiplication does not matter, so it suffices to define the scalar given by a single internally connected subtangle.

First, closed loops count for a multiplicative factor:

\[
\dim_{-A}(H) = T_1(1) = 1 \quad \text{and} \quad \dim_{-A}(H) = T_1^{\text{op}}(1) = 1.
\]

Suppose \( S \) is a closed, internally connected subtangle of \( T \) with only one input disk. Then we may write \( S \) uniquely as in Equation (4.4), label the tangle by \( z_i \), and replace \( t_n, t_{n}^{\text{op}} \) with \( T_n, T_n^{\text{op}} \).
Theorem 4.7.6. Definition 4.7.5 gives a well-defined action of $\mathbb{BP}_1$.

Proof. The methods of [Pen12a] show that the standard forms of connected and internally connected tangles given in Algorithm 4.7.4 and Equation (4.4) and the maps given in Subsection 4.3 behave the same under composition by Theorems 4.3.15 and 4.7.2. We briefly sketch such an argument.

We need only consider the composites $R \circ S$ and $S \circ T$ where $R, S, T \in \mathbb{BP}_1$ such that $R$ is internally connected with 1 input disk and $S, T$ are connected. That the action is well-defined follows from using the relations in Theorems 4.3.15 and 4.7.2 and (4) in Corollary 4.3.16 to get the standard form of the composite from the composite of the standard forms (push all $\otimes_{m,n}, \otimes_A$ as far to the left as possible, and push all left caps $t^\text{op}, T^\text{op}$ to the left of the right caps $t, T$). Once again, since internally connected tangles act as scalars in $[0, \infty]$, the order in which we remove them and multiply by the scalar does not matter. □

The operad $\mathbb{BP}$

We now include the pairing tangles to get the operad $\mathbb{BP}$ and show its action is well-defined.

Definition 4.7.7. Let $\mathbb{BP}$ be the operad generated by $\mathbb{BP}_1$ and the following tangles:

- **Pairing**: For $n \geq 1$, the tangles $\tau_n, \tau_n^\text{op}$ with two input disks, each with $2n$ internal boundary points, and no external boundary points such that boundary point $m$ of input disk $D_1$ is connected to boundary point $2n - m + 1$ of input disk $D_2$ for each $m = 1, \ldots, 2n$ as follows:

$$
\tau_n = \begin{array}{c}
\phantom{a} \\
\otimes \\
\otimes \\
\phantom{a}
\end{array}
\quad \text{and} \quad
\tau_n^\text{op} = \begin{array}{c}
\otimes \\
\otimes \\
\otimes \\
\phantom{a}
\end{array}.
$$

There are similar notions of connectivity and internal connectivity for tangles $T \in \mathbb{BP}$.

Remark 4.7.8. $\tau_n(T_1(-), T_2(-)) = \tau_n(T_2(-), T_1(-))$ and similarly for $\tau_n^\text{op}$ for all $T_1, T_2 \in \mathbb{BP}$ up to reindexing internal disks.

Theorem 4.7.9. Suppose $T$ is an unshaded, oriented, internally connected tangle which satisfies (1)-(4) in Definition 4.7.1. Then $T \in \mathbb{BP}$ if and only if conditions (BP0), (BP2), and (BP3) from Theorem 4.7.3 are satisfied (we now exclude (BP1)) along with the following conditions:

- (BP4) If the string $S$ connects boundary point $m$ of $D_i$ to boundary point $n$ of $D_j$ where $1 \leq i < j \leq s$, then
  (i) no string of $D_i$ or $D_j$ connects to $D_0$, and
(ii) there is another string $S'$ connecting boundary points $-m$ of $D_i$ and $-n$ of $D_j$.

We call $S \cup S'$ an $i,j$-cap of $T$. In this case, if we connect boundary points $m$ and $-m$ of $D_i$ and boundary points $n$ and $2k_j - n + 1$ of $D_j$ by imaginary strings $S_i, S_j$ inside $D_i, D_j$ respectively, then the loop $S \cup S' \cup S_i \cup S_j$ either

(i) contains the $\ast$’d intervals of $D_i$ and $D_j$, and the $i,j$-cap is a left $i,j$-cap, or

(ii) does not contain the $\ast$’d intervals, and the $i,j$-cap is a right $i,j$-cap.

• (BP5) The $i,j$-caps of $T$ are either all right or all left caps, and they form concentric circles.

Proof. Once again, it is clear that any tangle in $\mathbb{B} \mathbb{P}$ satisfies the desired properties, since these properties are preserved under composition of tangles (the total number of $i,j$-caps can only decrease under composition of connected and internally connected tangles), and the generating tangles satisfy these properties. The other direction follows from Algorithm 4.7.11, which shows how to ‘comb’ the tangle into a unique standard form.

Example 4.7.10. The tangle on the left is in $\mathbb{B} \mathbb{P}$ (see Algorithm 4.7.11), but the tangle on the right is not:

![Two tangles](image)

Algorithm 4.7.11. Suppose $T$ is an internally connected tangle which satisfies (1)-(4) of Definition 4.7.1 and (BP0),(BP2),(BP3) in Theorem 4.7.3 and (BP4),(BP5) in Theorem 4.7.9. Suppose further that $T$ has at least two input disks, so there is an $i,j$-cap. Let $C_1$ be the outermost $i,j$-cap of $T$. Then there is a unique smallest $n \in \mathbb{N}$ and two unique connected tangles $T_1, T_2 \in \mathbb{B} \mathbb{P}_1$ up to swapping such that:

Right: if $C_1$ is a right $i,j$-cap, $T = \tau_n(T_1(-), T_2(-))$, and

Left: if $C_1$ is a left $i,j$-cap, $T = \tau_n^{up}(T_1(-), T_2(-))$.

We give an algorithm for the right-cap case, and the left-cap case is similar. We will build $T_1$ and $T_2$ by partitioning the internal disks of $T$ into two sets $U$ and $L$, standing for “upper” and “lower.” All $i,j$-caps of $T$ will be between a $D_i \in U$ and a $D_j \in L$. We form $T_1$ by putting a box around the $D_i \in U$ together with all “contractible” $i$-caps, and we form $T_2$ by doing the same to the $D_j \in L$. 
Before we describe the algorithm, we give an example:

(1) Start at the * on the external boundary. Set $U = L = \emptyset$. Let $c$ be the number of $i,j$-caps of $T$.

(2) If $c = 0$, then go to (4).

(3) Find the next outermost $i,j$-cap $C$ in $T$, where $i < j$. Set $c = c - 1$.

   (3a) If $U = L = \emptyset$, then set $U = \{D_i\}$ and $L = \{D_j\}$.

   (3b) If $D_i$ or $D_j$ is not in $U \cup L$ (note that at least one of $D_i, D_j$ is in $U \cup L$), put the
   missing one where the other one is not, e.g., if $D_i \notin U \cup L$ and $D_j \in L$, then set
   $U = U \cup \{D_i\}$. (There are 4 cases here.)

   (3c) Isotope the tangle so that
   - all disks in $U$ and $L$ appear on the same horizontal levels, with $L$ below $U$,
   - any string connecting a disk $D_u \in U$ to a disk $D_\ell \in L$ travels upward from
     $D_\ell$ to $D_u$ with no critical points, or travels in a large arc from $D_u$ to $D_\ell$ with
     only two critical points,
   - all $k$-caps which enclose the $i,j$-cap $C$ are large arcs with only two critical
     points,
   - all $k$-caps for $D_k \in U \cup L$ which do not enclose an $a,b$-cap are close to $D_k$.

   (3d) Go to (2).

(4) Put boxes around the disks and caps in $U, L$ as desired. We have $\tau_n(\mathcal{T}_1(-), \mathcal{T}_2(-))$ for
some $n \in \mathbb{N}$ and some connected tangles $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{BP}_1$.

Note that the $n$ is determined by the $i,j$-caps and the $k$-caps which enclose an $i,j$-cap, and
this $n$ is minimal when all other $\ell$-caps are contracted so they are close to $D_\ell$. Moreover, the
only choice we made was the initial choice $U = \{D_i\}$ and $L = \{D_j\}$ with $i < j$, but if we
swapped $U$ and $L$, we would have ended up with $\tau_n(\mathcal{T}_2(-), \mathcal{T}_1(-))$. Hence $\mathcal{T}_1, \mathcal{T}_2$ are unique
up to swapping.
Definition 4.7.12 (Action of tangles in BP). We extend the action of BP to an action of BP. Note that it suffices to define the action of an internally connected tangle with at least 2 input disks (so there is necessarily an i, j-cap), and any such tangle can be written uniquely as \( \tau_n(T_1, T_2) \) (or \( \tau_n^{\text{op}} \)) with \( n \) minimal and \( T_1, T_2 \in \text{BP}_1 \) unique up to swapping by Algorithm 4.7.11. Simply use the action prescribed by Definition 4.7.5 for \( T_1 \) and \( T_2 \), and then the action of \( \tau_n, \tau_n^{\text{op}} \) is given by replacing it with \( \tau_n, \tau_n^{\text{op}} \).

Theorem 4.7.13. Definition 4.7.12 gives a well-defined action of BP.

Proof. We show that for any connected \( S_1, S_2 \in \text{BP}_1 \) and \( m \in \mathbb{N} \), that the action of the composite tangle \( \tau_m(S_1, S_2) \) is the same as the composite of the actions of \( \tau_m \) and the actions of the tangles \( S_1, S_2 \in \text{BP}_1 \). A similar result holds for \( \tau_m^{\text{op}} \).

First, note that (4) and (5) of Corollary 4.3.16 allow us to reduce to the case where \( \tau_m(S_1, S_2) \) is internally connected. If \( \tau_m(S_1, S_2) \) is internally connected, then Algorithm 4.7.11 gives a standard form \( \tau_n(T_1, T_2) = \tau_m(S_1, S_2) \) where \( n \in \mathbb{N} \) is minimal and \( T_1, T_2 \in \text{BP}_1 \) are unique connected tangles up to swapping. If \( m > n \), then setting \( b = m - n \), we must have (up to swapping) that \( T_1 = t_{n+1} \cdots t_{n+b}(S_1) \) and \( S_2 = \otimes_{n,b}(T_2, 1_b) \). A similar statement holds for \( \tau_m^{\text{op}} \) using \( t^{\text{op}} \)'s and \( \otimes_{b,n}(1_b, -) \).

Now the result follows from (5) in Theorem 4.3.15 (which is also Proposition 4.8.11). \( \square \)

4.8 Extended positive cones

For the bimodule planar calculus, we need to make multiplication by \( \infty \) rigorous. We do so by generalizing the notion of an extended positive cone.

Definition 4.8.1. An extended positive cone is a set \( V \) together with a partial order \( \leq \), an addition \( + : V \times V \to V \), and a scalar multiplication \( \cdot : [0, \infty_\mathbb{R}] \times V \to V \) such that

Additivity axioms:

- (Zero) There is a \( 0_V \in V \) such that \( 0_V + v = v + 0_V = v \) for all \( v \in V \).
- (Infinity) There is an \( \infty_V \in V \setminus \{0\} \) such that \( v + \infty_V = \infty_V + v = \infty_V \) for all \( v \in V \).
- (Associativity) \( v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3 \) for all \( v_1, v_2, v_3 \in V \).
• (Commutativity) \( v_1 + v_2 = v_2 + v_1 \) for all \( v_1, v_2 \in V \).

**Multiplicative axioms:**

• (Unit) \( 1_R v = v \) for all \( v \in V \).

• (Associativity) \( (\lambda \mu)v = \lambda (\mu v) \) for all \( \lambda, \mu \in [0_R, \infty_R] \) and \( v \in V \).

• (Zero) \( 0_R v = 0_V \) for all \( v \in V \).

• (Infinity) \( \lambda \infty_V = \infty_V \) for all \( \lambda > 0_R \).

**Distributivity:**

• (Scalars distribute) \( \lambda (v_1 + v_2) = \lambda v_1 + \lambda v_2 \) for all \( \lambda \in [0_R, \infty_R] \) and \( v_1, v_2 \in V \).

• (\( V \) distributes) \( (\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v \) for all \( \lambda_1, \lambda_2 \in [0_R, \infty_R] \) and \( v \in V \).

**Partial order axioms:**

• (Non-degeneracy) \( 0_V \leq x \leq \infty_V \) for all \( x \in V \).

• (Linearity) if \( x_i \leq y_i \) for \( i = 0, 1 \) and \( \lambda \in [0_R, \infty_R] \), then \( \lambda x_0 + x_1 \leq \lambda y_0 + y_1 \).

**Remark 4.8.2.**

1. \( 0_V, \infty_V \in V \) are unique.

2. If \( \lambda v = 0_V \), then \( v = 0_V \) or \( \lambda = 0_R \).

**Examples 4.8.3.**

1. The set \([0_R, \infty_R]\) with the usual ordering and the convention that \( \lambda \infty_R = \infty \lambda = \infty_R \) for all \( \lambda \in \mathbb{R}_{>0} \) and \( 0_R \infty_R = \infty_R 0_R = 0_R \) is an extended positive cone.

2. Let \( X \) be a nonempty set. The space of functions \( \{f : X \to [0_R, \infty_R]\} \) is an extended positive cone with pointwise addition and scalar multiplication, where \( f \leq g \) if \( f(x) \leq g(x) \) for all \( x \in X \). Similarly, the space of extended positive measurable functions on a measure space is an extended positive cone.

3. If \( M \) is a von Neumann algebra, \( \omega(M) \), the set of normal weights \( \omega : M^+ \to [0_R, \infty_R] \), is an extended positive cone where \( \infty_{\omega(M)} \) is the map which sends \( 0_M \) to \( 0_R \) and all other elements of \( M^+ \) to \( \infty_R \), and \( \varphi \leq \psi \) if \( \varphi(x) \leq \psi(x) \) for all \( x \in M^+ \).

4. If \( M \) is a von Neumann algebra, \( \hat{M}^+ \) is an extended positive cone where \( \infty_{\hat{M}^+} \) is the unbounded operator affiliated to \( M \) with domain \( (0) \), and \( m_1 \leq m_2 \) if \( m_1(\phi) \leq m_2(\phi) \) for all \( \phi \in \hat{M}^+ \).

5. If \( V, W \) are extended positive cones, then so is \( V \times W \) where \( (v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \), \( \lambda (v_1, w_1) = (\lambda v_1, \lambda w_1) \), \( 0_V \times W = (0_V, 0_W) \), \( \infty_V \times W = (\infty_V, \infty_W) \), and \( (v_1, w_1) \leq (v_2, w_2) \) if \( v_1 \leq v_2 \) and \( w_1 \leq w_2 \).
**Definition 4.8.4.** Let $V, W$ be extended positive cones. A function $T: V \to W$ is a linear map (of extended positive cones) if

- $T(\lambda u + v) = \lambda Tu + Tv$ for all $u, v \in V$ and $\lambda \in [0, \infty_\mathbb{R}]$, and
- if $u, v \in V$ with $u \leq v$, then $Tu \leq Tv$.

We define a multi-linear map of extended positive cones $V_1 \times \cdots \times V_n \to V_0$ similarly.

**Examples 4.8.5.**

1. For a fixed scalar $\lambda \in [0, \infty_\mathbb{R}]$, multiplication by $\lambda$ is a map of extended positive cones.

2. Suppose $\omega: M^+ \to [0, \infty_\mathbb{R}]$ is a normal weight. Then its unique extension to a normal weight $\hat{\omega}: \hat{M}^+ \to [0, \infty_\mathbb{R}]$ is a map of extended positive cones.

3. If $m \in \hat{M}^+$, then $m: \omega(M) \to [0, \infty_\mathbb{R}]$ given by $\varphi \mapsto m(\varphi)$ is a map of extended positive cones.

4. Suppose $N \subset M$ is an inclusion of von Neumann algebras, $i: \hat{N}^+ \to \hat{M}^+$ is the inclusion (well-defined by Equation (4.1)), and $T: \hat{M}^+ \to \hat{N}^+$ is the unique extension of an operator valued weight $M^+ \to \hat{N}^+$. Then $i, T$ are maps of extended positive cones.

5. Using the notation of Section 4.6, the map $\hat{X}^+ \times \hat{Y}_0^+ \to X \otimes_A Y_0^+$ given by $(x, y) \mapsto x \otimes_A y$ is a multilinear map of extended positive cones by Lemma 4.6.15.

**Definition 4.8.6.** The dual space of $V$, denoted $V^*$, is the set of all normal maps $V \to [0, \infty_\mathbb{R}]$. Note that $V^*$ is a complete extended positive cone with

- $(\lambda \varphi + \psi)(v) = \lambda \varphi(v) + \psi(v)$ for all $v \in V$, $\lambda \in [0, \infty_\mathbb{R}]$, and $\varphi, \psi \in V^*$, with the convention that $0 \cdot \infty_\mathbb{R} = 0$, $0$, $\infty_\mathbb{R}$,

- $0_{V^*}$ is the zero map,

- $\infty_{V^*}(v) = \begin{cases} 0 & \text{if } v = 0 \\ \infty_V & \text{else, and} \end{cases}$

- $(\sup_{i \in I} \varphi_i)(v) := \sup_{i \in I} \varphi_i(v)$.

**Definition 4.8.8.** An increasing net $(x_i)_{i \in I} \subset V$ converges to $x \in V$ if $x$ is the unique least upper bound for $(x_i)_{i \in I}$. We denote this convergence by $\sup_{i \in I} x_i = x$ or $x_i \nearrow x$.

- $V$ is complete if each increasing net $(x_i)_{i \in I}$ has a unique least upper bound.

- A map $T: V \to W$ is normal if $x_i \nearrow x$ implies $Tx_i \nearrow Tx$.

**Remark 4.8.7.** The maps in Examples 4.8.5 are all normal.
• There is a natural inclusion $V \to V^{**}$ by $x \mapsto (\text{ev}_x: \varphi \mapsto \varphi(x))$.

• The completion of $V$ is the set of sups of increasing nets in the image of $V$ in $V^{**}$.

**Theorem 4.8.9.** Let $M$ be a semifinite von Neumann algebra with n.f.s. trace $\text{Tr}_M$. Let $\omega(M)$ be the set of normal weights on $M^+$.

1. $\widehat{M}^+$ is the dual extended positive cone of $\omega(M)$ (the ordering on each is given in Examples 4.8.3).

2. The map $\widehat{M}^+ \ni x \mapsto \text{Tr}_M(x \cdot) \in \omega(M)$ is a normal isomorphism of extended positive cones.

**Proof.** This is a rewording of Theorem 4.2.14 into the language of this subsection.

**Definition 4.8.10.** If $T: V \to W$ is a normal map of extended positive cones, we get a map of dual spaces $T^*: W^* \to V^*$ by $T^*(\phi) = \phi \circ T$ for all $\phi \in W^*$. We can characterize it as the unique map satisfying

$$\langle T(v), \varphi \rangle_W = \varphi(T(v)) = \langle v, T^*(\varphi) \rangle_V$$

for all $v \in V$ and $\varphi \in W$.

**Proposition 4.8.11.** Suppose $N \subset M$ is an inclusion of semifinite von Neumann algebras with n.f.s. traces $\text{Tr}_N$, $\text{Tr}_M$ respectively. Let $i: \omega(N) \cong \widehat{N}^+ \to \widehat{M}^+ \cong \omega(M)$ be the inclusion, and let $T: \widehat{M}^+ \to \widehat{N}^+$ be the unique extension to $\widehat{M}^+$ of the unique trace-preserving operator valued weight. Then $i, T$ are normal and $T = i^*$, $T^* = i$.

**Proof.** Clearly $i, T$ are normal. Suppose $n \in \widehat{N}^+$ and $m \in (\widehat{M}^+)^* = \widehat{M}^+$. Then

$$\langle i(n), m \rangle_{\widehat{M}^+} = \text{Tr}_M(m \cdot n) = \text{Tr}_N(T(m) \cdot n) = \langle n, T(m) \rangle_{\widehat{N}^+},$$

so $T = i^*$. Since $\text{Tr}_M(m \cdot n) = \text{Tr}_M(n \cdot m)$, $i = T^*$. 

\qed
Bibliography


