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Distortion-Energy Tradeoff for Zero Delay Transmission in a Coding Scheme with High Resolution Quantization

A Thesis submitted in partial satisfaction of the requirements for the degree of

Master of Science

in

Electrical Engineering

by

Goksu Yamac

August 2016

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To my parents Kadri and Deniz,

and my brother Gökhan.
ABSTRACT OF THE THESIS

Distortion-Energy Tradeoff for Zero Delay Transmission in a Coding Scheme with High Resolution Quantization

by

Goksu Yamac

Master of Science, Graduate Program in Electrical Engineering
University of California, Riverside, August 2016
Professor Ertem Tuncel, Chairperson

This thesis focuses on the communication of a low bandwidth Gaussian source with specific energy restrictions through an AWGN channel. Since the source has low bandwidth, it is assumed that the transmitter is capable of delivering many dimensions per symbol over the channel. The offered schemes use high resolution quantization and strive to minimize the end-to-end MSE distortion. Several practical schemes with ML estimator and orthogonal modulation are used with different types of (high resolution) quantizers so that the distortion expression can be derived analytically. Due to the cumbersome nature of analytical distortion expression for MMSE estimation, MMSE estimator is offered under suboptimal conditions. To increase the performance of the MMSE estimator, a search algorithm that transitions between receiver and transmitter is introduced. The algorithm starts by assuming that the decoder is applying ML rule and it optimizes the scheme accordingly. The MLE optimized parameters are used to run Monte Carlo simulations for an MMSE based scheme in search of a lower distortion value.
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Chapter 1

Introduction

Data communication has become a casual thing taking place in the lives of most people everyday. Let this either be the communication via phones, computer networks, satellites or sensors, it is all over the world and it is increasing and spreading rapidly. At the very least, continuous communication between phones and satellites to operate Global Positioning System (GPS) conveys the size and importance of the subject. As a consequence of this enormous data flowing across the world everyday, engineers and scientists seek cheaper and faster ways of communication. It is a fact that digital communication does not provide the fastest transmission and optical communication is faster. However, in some scenarios such as sensor networks, transmission speed is not the main concern and it is more cost efficient to prefer digital communication as long as fidelity is the concern. Some sensor networks are special in the sense that the input is slowly-varying. One great example for such scenarios is the temperature sensor networks. By nature these sensors do not have very rapidly changing inputs and outputs, and this makes them low-bandwidth
sources. On the other hand, these sensors supply their energy needs from batteries which puts them under a strict energy limitation. This thesis aims to provide alternative communication methods by analyzing the *distortion-energy tradeoff* [6] for such sensor networks with low bandwidth sources and strict energy limitations. The primary objective of the proposed methods is to achieve high fidelity without creating a huge computational expense.

Prior work [3], [5], [2], on joint source-channel coding with energy constraints show that optimality is reached when source signal is uncoded and passed through a linear encoder. When the bandwidth of the source and channel are matched, i.e., if sending one channel symbol per source symbol is allowed. *Bandwidth mismatch* is the condition when the channel bandwidth used for transmission is not equal to the source bandwidth. In the case of low bandwidth sources, this means the bandwidth expansion ratio between the source and the channel can be made arbitrarily high. In other words, many channel symbols can be used for transmission of a single source symbol. Same case is discussed in other work, [1], where upper and lower bounds on distortion is offered as well as showing the suboptimality of uncoded transmission. It will be used to test the success of the methods proposed in this work.

High resolution quantization [4] is one of the tools that will be used to provide an optimal mapping by taking into consideration the probability distribution of source signal. During the modulation, quantized signals are mapped onto an orthogonal signal space where each quantization cell represents a different dimension, such as FSK [7]. The communication scheme is illustrated with all of its components in Figure 1.1.
Estimation of the received signal is done in two different methods; ML and MMSE. Although MMSE is known to be the best estimator in terms of minimizing MSE distortion, the difficulties in deriving analytical expressions make it computationally inefficient. Therefore, a search algorithm is proposed that tries to increase the performance of a sub-optimal MMSE estimator.

![Communication scheme](image)

**1.1 Overview of the Thesis**

In the following chapter, some preliminary information is provided on subjects such as quantization, orthogonal signaling and estimation methods such as ML and MMSE. These subjects are important to have a thorough understanding of the following chapters. In Chapter 3, the distortion expressions are derived for point-to-point and broadcast communication schemes, and then they are optimized to find the fitting point density function. Third chapter discusses the receiver end of the communication schemes. It includes ML and MMSE estimation, as well as introducing a search algorithm for the MMSE estimation. In Chapter 5, MATLAB simulations and their results are offered. The results are discussed and it is compared with other similar work, [1]. Chapter 6 is the conclusion and summary of the entire thesis.
Chapter 2

Preliminaries

2.1 Quantization

Necessary Definitions of a Quantizer

Quantization is the process of mapping a large set of incoming values to a smaller set of discrete values. This mapping is done by creating a partition according to the specifications of the quantizer. Illustration of an 8-level quantizer is given in Figure 2.1. As seen in the figure, each interval \([x_i, x_{i+1})\) is represented as region \(R_i\), and these sequential pairs form the mapping boundaries of the incoming values. Assuming that there are \(N\) levels of quantization, the mapping can be represented as

\[ Q(x) = y_i \text{ for } x_i \leq x < x_{i+1}, \text{ where } i = 1, 2, ..., N \]
with $x_1 = -\infty$ and $x_N = \infty$, where $y_i$ is the reconstruction point of region $R_i$. The reconstruction points form the codebook or the output set of the quantizer.

$$
\mathbb{C} \equiv \{y_1, y_2, \ldots, y_N\}
$$

More information will be provided in terms of the optimality of the codebook.

\textbf{Necessary Conditions for Optimality}

The main purpose of quantizer design is to obtain the minimum possible average distortion for a fixed number of levels, $N$. In that manner, the designer is responsible for selecting the reconstruction levels, $y_i$, and the partition regions, $x_i$. With $f_X(x)$ representing
the pdf of $X$, the objective function that will be minimized is

$$D = \sum_{i=1}^{N} \int_{R_i} d(x, y_i) f_X(x) dx$$

where $d(x, y)$ is the distance measure that defines how distortion is calculated. In this work, distortion will be defined in mean squared error fashion, and therefore

$$d(x, y_i) = (x - y_i)^2$$

To find the conditions for optimality, one has to fix the encoder and seek for the optimal decoder. Likewise, the decoder needs to be fixed to find the conditions for an optimal encoder. The quantizer can be optimal only if both the encoder and the decoder are optimal.

**The Optimal Encoder for a Given Decoder**

The encoder part of the quantizer is responsible for mapping the input values to the best output value. In this case, 'best' refers to the output that causes minimum possible average distortion. Thus, the mapping that satisfies *nearest neighbor condition* provides the optimal encoder.

**Nearest Neighbor Condition**

For a given codebook $C$, the partitioning satisfies

$$R_i \subset \{ x : d(x, y_i) \leq d(x, y_j); \text{ for all } j \neq i \};$$
which also means

\[ Q(x) = y_i \text{ only if } d(x, y_i) \leq d(x, y_j) \text{ for all } j \neq i \]

Therefore,

\[ Q(x) = \arg \min_{y_i \in C} d(x, y_i) \quad (2.1) \]

Next, it will be shown that nearest neighbor condition is sufficient for the encoder to be optimal for a given decoder.

*Proof:* It is stated in (2.1) that nearest neighbor mapping provides the minimum distance among the input-output couples. Making use of this in the average distortion gives,

\[ D = \int d(x, Q(x)) f_X(x) dx \geq \int [\min_{y_i \in C} d(x, y_i)] f_X(x) dx \]

The lower bound in this inequality is reached when the nearest neighbor mapping is used. Hence, if the quantizer performs nearest neighbor mapping for a given codebook \( C \), the optimal encoder is obtained.

**The Optimal Decoder for a Given Encoder**

The same approach is used to find the conditions for optimal decoding. It is assumed that the encoder is fixed with a nondegenerate partition, and the necessary optimization condition for the decoder is being looked for.
Centroid Condition

Given an appropriate partition, the codebook is optimized in terms of the average distortion if

\[ y_i = \mathbb{E}[X|X \in R_i] \]

Proof: The average distortion expression for a scalar quantizer is

\[ D = \sum_{i=1}^{N} \int_{R_i} (x - y_i)^2 f_X(x) dx \]

For the sake of simplicity, only one region of the partitioning, say \( y_j \), is considered. Then the expression boils down to

\[
\int_{R_j} (x - y_j)^2 f_X(x) dx = P_j \int_{-\infty}^{\infty} (x - y_j)^2 f_X(x) | R_j(x) dx \\
= P_j \mathbb{E}[(X - y_j)^2 | X \in R_j],
\]

The minimum value that can be attained here is when \( y_j = \mathbb{E}[X|X \in R_j] \). This expression refers to the centroid, or the center of mass, of \( X \) in region \( R_j \). Thus, to reach the optimal decoder given the encoder, the quantizer needs to map the input values to the centroid of each region \( R_i \).

Proof:
\[ E[(X - a)^2 | X \in R_i] = E[(X - \bar{X}^* + \bar{X}^* - a)^2 | X \in R_i] \]

where \( \bar{X}^* = E[X | X \in R_i] \). Then,

\[ E[(X - \bar{X}^* + \bar{X}^* - a)^2 | X \in R_i] = E[(X - \bar{X}^*)^2] + E[(\bar{X}^* - a)^2] \quad (2.2) \]

\[ = E[(X - \bar{X}^*)^2] + (\bar{X}^* - a)^2 \quad (2.3) \]

The second term of the square expansion is excluded because

\[ E[X - \hat{X}^* | X \in R_i] E[X - a | X \in R_i] = (E[X | X \in R_i] - \bar{X}^*)E[X - a | X \in R_i] = 0 \]

Back to (2.3),

\[ E[(X - \hat{X}^*)^2] + (\hat{X}^* - a)^2 \geq E[(X - \bar{X}^*)^2] \]

Thus, the minimum value can be reached when \( a = E[X | X \in R_i] \).

**Implications of Optimal Quantization**

1. If a quantizer satisfies the optimality conditions with a squared error distortion measure, then it is regular.

2. If the codebook of a quantizer satisfies the centroid condition, then
\[ \mathbb{E}[Q(X)] = \mathbb{E}[X], \]

Moreover,

\[ \mathbb{E}[Q(X)(Q(X) - X)]= 0, \]

meaning that the quantization error and the quantizer output are uncorrelated. Lastly,

\[ \mathbb{E}[(X - Q(X))^2] = \mathbb{E}[X^2] - \mathbb{E}[Q(X)^2] = \sigma_X^2 - \sigma_{Q(X)}^2 \]

as given in [4].

\section{2.2 High Resolution Quantization and Companding}

\subsection{2.2.1 Comandor Modeling}

Let \( X \) be a random variable with a known PDF and to be passed through a quantizer in a communication scheme to output \( \hat{X} \). In cases where PDF of the source to be quantized is known, this statistical information on the source can be used to improve the performance of the quantizer. In other words, the quantizer can be built in a way that assigns more levels to more frequent values, and less levels to less frequent values. This way, the dynamic range of the quantizer can be increased and better SNR values can be achieved. The general model for a nonuniform quantizer is formed of three stages, as in Figure 2.2. First, input \( x \) is mapped to \( y \) through the memoryless monotonic nonlinear function...
Figure 2.2: Compandor model

\( y = G(x) \). With this transformation, any two adjacent centroids are mapped to equidistant values. Then \( y \) is quantized using a uniform quantizer and it outputs \( \hat{y} \). Lastly, \( \hat{y} \) is passed through \( G^{-1}(x) \) to form \( \hat{x} \).

Let the boundary points of the quantizer be \( \{x_1, x_2, \ldots, x_{N-1}\} \), and the output points, or the centroids be \( \{y_1, y_2, \ldots, y_N\} \). According to these data points,

\[
G(x_i) = i\Delta + \Delta/2 + \kappa
\]

\[
G(y_i) = i\Delta + \kappa
\]

with the condition that \( G(0) = 0 \), and \( \kappa \) need to be selected accordingly.

Seen in Figure 2.3, by connecting these \( 2N - 1 \) points with straight line segments, continuous monotonic curve \( G(x) \) is obtained. It is assumed that \( G(x) \) is differentiable everywhere in the interval \( (y_1, y_N) \). Note that the step size \( \Delta_i \equiv x_i - x_{i-1} \), is not a constant and is different from the \( \Delta \) of \( G(x) \). The derivative is given by

\[
G'(y_i) \approx \frac{G(x_i) - G(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta}{\Delta_i}
\] (2.4)
2.2.2 High Resolution Quantization

High resolution quantization comes with two essential properties; \( N \) is very large and quantization cells are very small. These properties allow the analysis to be mathematically more tractable by allowing useful approximations.

The average distortion of a nonuniform quantizer is calculated by

\[
D = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (x - y_i)^2 f_X(x) \, dx
\]

where \( x_0 = -\infty \) and \( x_N = +\infty \) for an unbounded pdf.

In high resolution case, it is convenient to use the fundamental theorem of calculus to approximate the sum by an integral. The idea is, for very large \( N \) the distribution of \( X \) in
each interval $R_i$ is assumed to be uniform. Therefore, the sum over each quantization cell can accurately be approximated as an integration of the whole pdf.

$$f_X(x) \approx f_i; \ x \in R_i$$

and

$$P_i = \Pr(X \in R_i) = \int_{x_{i-1}}^{x_i} f_X(x) dx \approx f_i \Delta_i$$

Therefore,

$$D = \sum_{i=1}^{N} P_i \int_{x_{i-1}}^{x_i} \frac{(x - y_i)^2}{\Delta_i} dx.$$

The integral may be approximated as the variance of a uniform random variable with a variance of $\Delta^2/12$, hence the distortion becomes

$$D \approx \frac{1}{12} \sum_{i=1}^{N} P_i \Delta_i^2$$

(2.5)

Next, the distortion integral will be derived for the high resolution by the help of a new function, $\lambda(x)$. It is defined as

$$\lambda(x) \Delta x = \lim_{N \to \infty} \frac{N(x)}{N}$$

where

$N(x)$ : Number of quantization levels in the region $[x, x + \Delta x)$

$N$ : Number of quantization levels
$\lambda(x)$ : Point density function

Note that the sum of $N(x)$ over the whole partition is $N$. Thus,

$$\int_{-\infty}^{\infty} \lambda(x)dx = 1.$$ 

This shows that $\lambda(x)$ is a legitimate density function.

Now, the size of partition regions $\Delta_i$ will be expressed in terms of the point density function.
\[ \Delta_i \equiv x_i - x_{i-1} \]

\[ = \frac{\text{length of interval}}{\text{number of levels in the interval}} \]

\[ = \frac{\Delta x}{N\lambda(x)\Delta x} \approx \frac{1}{N\lambda(y_i)} \]  \hspace{1cm} (2.6)

where \( y_i \) is the centroid of region \( R_i \).

Using (2.6) in (2.5) gives

\[ D \approx \frac{1}{12N} \sum_{i=1}^{N} \frac{P_i}{(N\lambda(y_i))^2} \]  \hspace{1cm} (2.7)

\[ \approx \frac{1}{12N} \sum_{i=1}^{N} \frac{f_X(y_i)\Delta_i}{N^2\lambda(y_i)^2} \]  \hspace{1cm} (2.8)

\[ \approx \frac{1}{12N^2} \int_{x_1}^{x_{N-1}} \frac{f_X(x)\lambda(x)}{\lambda(x)^2} dx. \]  \hspace{1cm} (2.9)

where (2.8) is acquired using the fact that \( P_i \approx f_X(y_i)\Delta_i \) for the high resolution case, and (2.9) is obtained by using Riemann Sum in the reverse sense.

Next task is to find the appropriate compandor function \( G(x) \). In order to do that, \( \Delta_i \) of \( G'(x) \) in (2.4) is substituted with (2.6). Therefore

\[ G'(x) = \frac{\Delta}{\Delta_i} \approx \Delta N\lambda(x) \]
Thus,

\[ G(x) = \int \Delta N \lambda(x) dx \]  \hspace{1cm} (2.10)

Note that \( \Delta = 1/N \) and (2.10) reduces to

\[ G(x) = \int_{-\infty}^{x} \lambda(x) dx \]  \hspace{1cm} (2.11)

### 2.3 M-ary Orthogonal Signaling

In orthogonal signaling, main idea is to transmit waveforms that are mutually orthogonal. One simple use of orthogonal signaling is through frequency-shift keying. In FSK, different frequencies are used to transmit the waveforms over the same time period. Let the duration of the bit interval be \( T_b \), and the lowest frequency of transmission be \( f_1 \). Also, the frequency separation between each successive frequency is constant and it is represented as \( \Delta f \). As the frequency separation is \( \Delta f \), frequency of each waveform can be represented in a general form,

\[ f_i = f_1 + (i - 1) \Delta f \quad i = 1, 2, \ldots, N \]

This way, the waveforms may also be expressed in a general form,
\[ u_i(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t) \quad 0 \leq t \leq T_b, i = 1, 2, \ldots, N \]

\[ = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi (f_1 + (i-1)\Delta f)t) \quad 0 \leq t \leq T_b, i = 1, 2, \ldots, N \]

where \( E_b \) is the energy per bit and \( N \) is the number of waveforms used. The geometrical representation of each waveform in the \( N \)-dimensional space is

\[
U_t = \begin{cases} 
\sqrt{E_s} & t = K \\
0 & t \neq K 
\end{cases}
\] (2.12)

where \( E_s = mE_b \) is the energy per symbol and \( m = \log_2 N \). It should be noted that the basis functions of these waveforms are \( b_i(t) = \sqrt{2/T} \cos(2\pi (f_1 + (i-1)\Delta f)). \)

**Probability of Error in M-ary Orthogonal Signaling**

Assume that the orthogonal waveforms are represented as in (2.12) and the signal that is received at the receiver side of the communication scheme is represented as \( V \). \( V \) is nothing but any one of the \( N \) orthogonal waveforms \( U_i \) added with \( N \)-dimensional channel noise \( W \). The components of \( W^N \) are mutually independent Gaussian random variables with \( \sigma_{w_i}^2 = 1 \) and \( \mu_{w_i} = 0 \). Suppose that \( U_1 \) is the signal to be transmitted.

\[
V = U_1^N 
\] (2.13)

\[
V = (\sqrt{E_s} + w_1, w_2, w_3, \ldots, w_N) 
\] (2.14)

In orthogonal signaling, crosscorrelation is the key tool for detecting the received signal.
The received signal $V$ is crosscorrelated with $N$ waveforms and the detector selects the signal resulting in the largest crosscorrelation. The calculation of $P_e$ will evolve around this.

$$C(V, U_i) = V \cdot U_i = \sum_{k=1}^{N} v_k u_{ik}, \quad i = 1, 2, \ldots, N$$

Note that $u_{ik}$ refers to the $k^{th}$ entry of waveform $U_i$. Assuming that $V = U_1 + W$ as in (2.13),

$$C(V, U_1) = \sqrt{E_s} (\sqrt{E_s} + w_1)$$

$$C(V, U_2) = \sqrt{E_s} w_2$$

$$\vdots$$

$$C(V, U_N) = \sqrt{E_s} w_N$$

As $\sqrt{E_s}$ is nothing but a scaling factor, it can be removed from each correlator output.

At this point, it is possible to express the PDF of the correlator outputs. The transmitted waveform, $U_1$ in this example, causes the first correlator output to be $v_1 = \sqrt{E_s} + w_1$. The PDF of them are,

$$f_V(v_1) = \frac{1}{\sqrt{2\pi\sigma^2_w}}e^{-(v_1-\sqrt{E_s})^2/\sigma}$$

and the rest $(N - 1)$ outputs,
\[ f_V(v_i) = \frac{1}{\sqrt{2\pi \sigma^2_w}} e^{-v_i^2/\sigma^2_w}, \quad i = 2, 3, \ldots, M \]

For correct detection, \( C(V, U_1) \) has to be largest among \( N \) correlator outputs. Hence,

\[
P(\text{Correct Detection}) = P(w_2 < v_1, w_3 < v_1, \ldots, w_N < v_1|v_1)P(v_1)
\]

where

\[
P(w_2 < v_1, w_3 < v_1, \ldots, w_N < v_1|v_1)
= P(w_2 < v_1|V_1)P(w_3 < v_1|V_1)\ldots P(w_N < v_1|V_1)P(v_1) \tag{2.15}
\]

\[
= \int_{-\infty}^{\infty} f_W(w_2 < v_1|V_1)f_W(w_3 < v_1|V_1)\ldots f_W(w_N < v_1|V_1)f_V(v_1)dv_1 \tag{2.16}
\]

\[
= \int_{-\infty}^{\infty} (1 - \Phi(v_1))^{N-1} f_V(v_1)dv_1 \tag{2.17}
\]

where \( \Phi(v_1) \) represent the CDF of a normal distribution. In obtaining (2.17) the property of independence between correlators is used. The reason for integrating over \( v_1 \) is to go over all possibilities in calculating the probability of correct detection.

\[
P_e = 1 - P_c
\]

\[
= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 - \Phi(\sqrt{v_1}))^{N-1} e^{-(v_1 - \sqrt{E_s})^2}dv_1
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 - (1 - \Phi(\sqrt{v_1}))^{N-1})e^{-(v_1 - \sqrt{E_s})^2}dv_1 \tag{2.18}
\]
2.4 Lagrange Multiplier Method

Lagrange multiplier is a very well-known optimization method in search of a local minima or local maxima of a function that is subject to some constraints.

Let $g(x, y) = b$ be the constraint function. This represents a contour on the x-y space and on this contour $g(x, y)$ takes the value $b$. The extrema (either minimum or maximum) of the objective function $f(x, y)$ is looked for on this contour of $g(x, y) = b$. At the point where $f(x, y)$ is no longer increasing or decreasing (depending on the type of optimization), the gradient functions of $f(x, y)$ and $g(x, y)$ become parallel and. This relationship can be expressed by introducing the Lagrange multiplier $\lambda$.

$$\Delta f(x, y) = \lambda \Delta g(x, y)$$

and to convey these in a single equation, an auxiliary function is introduced.

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - b)$$

and by solving

$$\Delta_{x,y,\lambda} L(x, y, \lambda) = 0,$$

the constrained extrema of $f(x, y)$ is reached.
2.5 Signal Estimation

2.5.1 Maximum Likelihood Estimation

Maximum likelihood estimation is a method for parameter extraction using observation data on a non-bayesian statistical model. Let $X$ represent a set of random variables, and $f_X(.)$ be the probability density function of $X$ with unknown parameters $\theta$. Now assume that $(x_1, x_2, ..., x_n)$ forms a set of observations that are samples from identically and independently distributed set of random variables, which will be the point of interest in this thesis. Then $f_X$ can be represented as a product of each random variable and $\theta$ reduces to $\theta$.

$$f(x_1, x_2, ..., x_n|\theta) = f(x_1|\theta)f(x_2|\theta)...f(x_n|\theta) = \prod_{i=1}^{N} f(x_i|\theta)$$

The underlying idea in maximum likelihood estimation is to treat the parameter as a free variable and finding the $\theta$ which maximizes the likelihood function, which is defined as

$$L(x; \theta) = f(x_1, x_2, ..., x_n|\theta)$$

Hence

$$\hat{\theta}_{ML} = \arg \max_{\theta} \prod_{i=1}^{N} f(x_i|\theta) \quad (2.19)$$

2.5.2 Minimum Mean Square Error Estimation

MMSE estimation is a method that is based on minimizing the mean squared error. Let $X$ represent the signal that will be transmitted and be received as $Y$ on the receiver side.
of a communication scheme. In this case, the MSE is defined as

\[ MSE = \mathbb{E}[(X - \hat{X})^2] \]

where \( \hat{X} \) is the estimator and it is given by

\[ \hat{X} = \mathbb{E}[X|Y] \]
Chapter 3

Distortion

The compandor is responsible for high resolution quantization and as shown in Eqn. (2.11), partition is done using point density function, \( \lambda(x) \). In order to find the optimal point density function, one should decide on a measure to express the distortion of overall transmission and optimize it with respect to \( \lambda(x) \).

In the preliminaries chapter, (2.9), distortion expression for high resolution quantizer is derived in terms of quantization error, where only source coding takes place. However, for the distortion expression to be complete and accurate regarding the involved communication scheme given in Figure 3.1, it also has to include the distortion caused by channel coding. In this chapter, a modified quantizer will be introduced and complete distortion expression for both communication schemes will be derived. These expressions will then be optimized using the method of Lagrange multipliers.
3.1 Distortion Expressions

In this section, three communication schemes with different quantizers will be studied. A modified quantizer will be introduced that has the first quantization cell covering \((-\infty, a)\) and the last cell covering \((a, \infty)\). This modification will bring in a new parameter that has to be taken into account during optimization. The third type of quantizer that will be implemented in the scheme is the uniform quantizer. It should be noted that the high resolution quantization, zero delay transmission and orthogonal signaling is valid for all three schemes.

The source signal that will be is a Gaussian random variable with zero mean and one variance. Distortion is defined in the well-known minimum mean squared error fashion

\[
D = \mathbb{E}[(X - \hat{X})^2]
\]

Since the source signal is quantized, distortion can be expressed as the sum of each cell’s
conditional distortion.

**Regular Quantizer**

This type of quantizer has its quantization cells covering from \((-\infty, \infty)\). Let \(P_e\) represent the probability of error in orthogonal signaling (2.18) with ML decoding. Then by the total probability law,

\[
E[(X - \hat{X})^2] = P_e E[(X - \hat{X})^2 | K \neq \hat{K}] + (1 - P_e) E[(X - \hat{X})^2 | K = \hat{K}] \tag{3.1}
\]

In the above expression, first term refers to the distortion introduced by the channel and the second term is the distortion caused by source coding by itself. Therefore, second expectation term is equal to (2.9), and the first term can be calculated through expansion.

\[
E[(X - \hat{X})^2 | K \neq \hat{K}] = E[X^2 | K \neq \hat{K}] - 2E[X \hat{X} | K \neq \hat{K}] + E[\hat{X}^2 | K \neq \hat{K}] \tag{3.2}
\]

\[
= E[X^2] + 2E[X] E[\hat{X} | K \neq \hat{K}] + E[\hat{X}^2 | K \neq \hat{K}]
\]

\[
= 1 + E[\hat{X}^2 | K \neq \hat{K}]
\]

\[
= 1 + \int f_{\hat{X}|K \neq \hat{K}}(\hat{x}) \hat{x}^2 d\hat{x} \tag{3.3}
\]

The information that \(K \neq \hat{K}\) is insignificant for \(E[X^2 | K \neq \hat{K}]\) since the distribution of \(X\) is independent of transmission, and thus the conditioning is irrelevant and can be dropped. Furthermore, the second term in (3.2) vanishes because \(X\) and \(\hat{X}\) are independent given an error in transmission, and \(X\) has zero mean.

\(f_{\hat{X}|K \neq \hat{K}}(x)\) represents the distribution of \(\hat{X}\) given that there is an error in the transmis-
sion. In cases of erroneous transmission, orthogonal signaling causes the error to be distributed uniformly among the other quantization cells because the modulator maps the centroid values onto an orthogonal set of equidistant vectors. Taking a look at the scheme in Figure 1.1, it is seen that the uniform distribution caused by erroneous transmission is passed through the expander to form $\hat{X}$. It is a known fact that if a random variable is passed through its CDF, the output is distributed uniformly. Because of that, if

$$U(x) \rightarrow G^{-1}(x) \rightarrow f_{\hat{X}|\hat{K} \neq K}(x)$$

then

$$f_{\hat{X}|\hat{K} \neq K}(x) \rightarrow G(x) \rightarrow U(x)$$

and $G(x)$ is the CDF of $\lambda(x)$. So, it can be inferred that $f_{\hat{X}|\hat{K} \neq K}(\hat{x})$ is equal to $\lambda(x)$.

Back to (3.3),

$$\mathbb{E}[\hat{X}^2|K \neq \hat{K}] = \int f_{\hat{X}|\hat{K} \neq K}(y)y^2dy = \int \lambda(y)y^2dy$$

By using (3.3) and (2.9) in (3.1), distortion expression is found as

$$D = P_e \left(1 + \int y^2 \lambda(y)^2dy\right) + \frac{(1 - P_e)}{12N^2} \int f_X(y)\lambda(y)^{-2}dy \quad (3.4)$$

So, (3.4) is the complete distortion expression assuming that the quantizer is not bounded, and high resolution and optimality conditions are satisfied.
Modified Quantizer

This is a regular (high resolution) quantizer with the exception of the first cell being \((-\infty, a]\) and the \(N\)th cell being \([a, \infty)\) for some \(a\). In the regular case, source signals are mapped to the centroids of the corresponding quantization cell. In the modified quantizer case, however, first and \(N\)th cells will be manually mapped on to \(-d\) and \(d\) respectively, and this makes \(d\) a new variable to be considered for optimization. From a different perspective, the modified quantizer is a regular quantizer which violates the centroid condition in the first and the last quantization cells.

\[
Pr[K = 1] = Pr[K = N] = Q(a)
\]

and

\[
Pr[2 \leq K \leq N - 1] = 1 - 2Q(a).
\]

Also,

\[
E[X|K = N] = \int_a^\infty x f_{X|K=N}(x) dx = \frac{\int_a^\infty x f_X(x) dx}{\int_a^\infty f_X(x) dx} = \frac{e^{-a^2/2}}{\sqrt{2\pi}Q(a)} \Delta c(a)
\]

Similarly,

\[
E[X|K = 1] = -c(a)
\]
For all other cells, $\mathbb{E}[X|K = k]$ will be approximated as the midpoint of the cell, as is usually done in high resolution analysis.

Let $\lambda(x)$ be the point density function of the quantizer such that $\lambda(x) > 0$ and

$$\int_{-a}^{a} \lambda(x) = 1$$

Since the error event $K \neq \hat{K}$ is independent of $K$, and due to symmetry, the expected distortion achieved by the quantizer can be expanded using the total expectation law as

$$\mathbb{E}[(X - \hat{X})^2] = 2Q(a)P_e \mathbb{E}[(X - \hat{X})^2|K = 1, \hat{K} \neq K]$$

$$+ 2Q(a)(1 - P_e) \mathbb{E}[(X - \hat{X})^2|K = 1, \hat{K} = K]$$

$$+ [1 - 2Q(a)]P_e \mathbb{E}[(X - \hat{X})^2|2 \leq K \leq N - 1, \hat{K} \neq K]$$

$$+ [1 - 2Q(a)](1 - P_e) \mathbb{E}[(X - \hat{X})^2|2 \leq K \leq N - 1, \hat{K} = K] \quad (3.5)$$

Next, each of the expectation terms above will be further expanded.

- $K = 1, \hat{K} \neq K$: $X$ and $\hat{X}$ are conditionally independent with distributions

$$f_{X|K=1,\hat{K}\neq K}(x) = \begin{cases} 
\frac{f_X(x)}{Q(a)} & x \leq -a \\
0 & x > -a 
\end{cases}$$

and

$$f_{\hat{X}|K=1,\hat{K}\neq K}(\hat{x}) \approx \frac{N - 2}{N - 1} \lambda(\hat{x}) + \frac{1}{N - 1} \delta(\hat{x} - d)$$
Thus,

\[
\mathbb{E}[(X - \hat{X})^2|K = 1, \hat{K} \neq K] = \mathbb{E}[X^2|K = 1] + \mathbb{E}[\hat{X}^2|K = 1, \hat{K} \neq K] - 2\mathbb{E}[X|K = 1] \cdot \mathbb{E}[\hat{X}|K = 1, \hat{K} \neq K]
\]

\[
\approx \frac{1}{\sqrt{2\pi}Q(a)} \int_{-\infty}^{-a} x^2 e^{-\frac{x^2}{2}} \, dx + \frac{N - 2}{N - 1} \int_{-a}^{a} \hat{x}^2 \lambda(\hat{x}) d\hat{x} + \frac{d^2}{N - 1} - 2c(a) \frac{d}{N - 1}
\]

\[
\approx 1 + ac(a) + \frac{N - 2}{N - 1} \int_{-a}^{a} \hat{x}^2 \lambda(\hat{x}) d\hat{x} + \frac{d^2}{N - 1} - 2c(a) \frac{d}{N - 1}
\] (3.6)

• **\( K = 1, \hat{K} = K \):**

\[
\mathbb{E}[(X - \hat{X})^2|K = 1, \hat{K} = K] = \frac{1}{\sqrt{2\pi}Q(a)} \int_{-\infty}^{-a} [x + d]^2 e^{-\frac{x^2}{2}} \, dx
\]

\[
= \frac{1}{\sqrt{2\pi}Q(a)} \int_{-\infty}^{-a} x^2 e^{-\frac{x^2}{2}} \, dx + d^2 - 2dc(a)
\]

\[
= 1 + ac(a) + d^2 - 2dc(a).
\] (3.7)

• **\( 2 \leq K \leq N - 1, \hat{K} \neq K \):** Once again, \( X \) and \( \hat{X} \) are conditionally independent with distributions

\[
f_{X|2 \leq K \leq N - 1, \hat{K} \neq K}(x) = \begin{cases} 
\frac{f_X(x)}{1 - 2Q(a)} & -a \leq x \leq a \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
f_{\hat{X}|2 \leq K \leq N - 1, \hat{K} \neq K}(\hat{x}) \approx \frac{N - 3}{N - 1} \lambda(\hat{x}) + \frac{1}{N - 1} \delta(\hat{x} - d) + \frac{1}{N - 1} \delta(\hat{x} + d)
\] (3.8)
Therefore,

\[ \mathbb{E}[(X - \hat{X})^2 | 2 \leq K \leq N - 1, \hat{K} \neq K] \]

\[ = \mathbb{E}[X^2 | 2 \leq K \leq N - 1] + \mathbb{E}[\hat{X}^2 | 2 \leq K \leq N - 1, \hat{K} \neq K] \]

\[ - 2 \mathbb{E}[X | 2 \leq K \leq N - 1] \cdot \mathbb{E}[\hat{X}^2 | 2 \leq K \leq N - 1, \hat{K} \neq K] \]

\[ \approx 1 \left( \frac{1}{\sqrt{2\pi}} \right)^2 \frac{1}{[1 - 2Q(a)]} \int_{-a}^{a} x^2 e^{-\frac{x^2}{2}} dx + \frac{N - 3}{N - 1} \int_{-a}^{a} \hat{x}^2 \lambda(\hat{x}) d\hat{x} + \frac{2d^2}{N - 1} \]

\[ = 1 - \frac{2ac(a)Q(a)\lambda(a)}{[1 - 2Q(a)]} + \frac{N - 3}{N - 1} \int_{-a}^{a} \hat{x}^2 \lambda(x) dx + \frac{2d^2}{N - 1} \] \quad (3.9)

- \( 2 \leq K \leq N - 1, \hat{K} = K \): Bennett Integral is used in the derivation of the following.

\[ \mathbb{E}[(X - \hat{X})^2 | 2 \leq K \leq N - 1, \hat{K} = K] \approx \frac{1}{12(N - 2)^2[1 - 2Q(a)]} \int_{-a}^{a} f_X(x) dx \] \quad (3.10)

In deriving (3.6)-(3.10), following expansion is used.

\[ \int_{x_1}^{x} x^2 e^{-\frac{x^2}{2}} dx = \int_{x_1}^{x} e^{-\frac{x^2}{2}} dx + x_1 e^{-\frac{x_1^2}{2}} - x_1 e^{-\frac{x^2}{2}} \]

**Uniform Quantizer**

In the case of uniform quantization, point density function is a constant and therefore the compandor function \( G(x) \) is linear. Since the unit density function is a legit pdf,

\[ \int_{-a}^{a} \lambda(x) dx = 1 \]
\[2a\lambda(x) = 1 \implies \lambda(x) = \frac{1}{2a}\]

The distortion expression is reached by using this \(\lambda(x)\) in equations (3.6) – (3.10).

\[
\mathbb{E}[(X - \hat{X})^2] = 2Q(a)P_e(1 + ac(a) + \frac{a^2(N - 2)}{3(N - 1)} + \frac{d^2}{N - 1} - 2c(a) \frac{d}{N - 1})
\]
\[
+ 2Q(a)(1 - P_e)(1 + ac(a) + d^2 - 2dc(a))
\]
\[
+ P_e(1 - 2Q(a))(1 - \frac{2ac(a)Q(a)}{1 - 2Q(a)} + \frac{a^2(N - 3)}{3(N - 1)} + \frac{2d^2}{N - 1})
\]
\[
+ \frac{(1 - 2Q(a))(1 - P_e)}{48a^2(N - 2)^2}
\]

(3.11)

### 3.2 Optimizing the Distortion

Optimizing the distortion expression is a crucial step to find the ideal quantizer. In order to optimize the distortion, a Lagrange multiplier will be introduced and it will be used to find the optimal point density function \(\lambda(x)\). For the modified quantizer case, optimization will be done in terms of \(d\) as well.

**Regular Quantizer**

The optimization problem can be represented as

\[
\begin{align*}
\text{minimize} \quad & P_e\left(1 + \int y^2\lambda(y)^2dy\right) + \frac{(1 - P_e)}{12N^2} \int f_X(y)\lambda(y)^{-2}dy \\
\text{subject to} \quad & \int_{-\infty}^{\infty} \lambda(x)dx = 1,
\end{align*}
\]

(3.12)
The Lagrange multiplier will be represented as \( \mu \). Then the auxiliary function is

\[ L(\lambda, \mu) = P_e \left(1 + \int y^2 \lambda(y)^2 dy\right) + \frac{(1 - P_e)}{12N^2} \int f_X(y) \lambda(y)^{-2} dy + \mu \left( \int_{-\infty}^{\infty} \lambda(x) dx - 1 \right) \]  

(3.13)

\[ \frac{\partial L(\lambda, \mu)}{\partial \mu} = 0 \]  

(3.14)

\[ \frac{\partial L(\lambda, \mu)}{\partial \lambda} = 0 \]  

(3.15)

\[ \frac{\partial L(\lambda, \mu)}{\partial \mu} = \int_{-\infty}^{\infty} \lambda(x) dx - 1 = 0 \]

\[ \int_{-\infty}^{\infty} \lambda(x) dx = 1 \]  

(3.16)

\[ \frac{\partial L(\lambda, \mu)}{\partial \lambda} = P_e x^2 - 2(1 - P_e)\lambda(x)^{-3} + \mu = 0 \]

\[ \lambda(x) = \frac{3}{6N^2(\mu + P_e x^2)} \frac{f_x(x)(1 - P_e)}{\sqrt{f_X(y)}} \]  

(3.17)

Equations (3.16) and (3.17) have to be used together in order find the minimum achievable distortion.

**Modified Quantizer**

The optimization of this quantizer will be done in two steps. First, (3.5) will be differentiated with respect to \( d \) to find a critical point, and then method of Lagrange multipliers will be used to optimize it with respect to \( \lambda(x) \).

For the sake of simplicity, each term in (3.5) will be differentiated individually.

First term:
\[
\frac{\partial (2Q(a)P_e E[(X - \hat{X})^2 | K = 1, \hat{K} \neq K])}{\partial d} \\
= 2Q(a)P_e \frac{\partial}{\partial d} \left( 1 + ac(a) + \frac{N - 2}{N - 1} \int_{-a}^{a} x^2 \lambda(x) dx + \frac{d^2}{N - 1} - 2c(a) \frac{d}{N - 1} \right) \\
= 2Q(a)P_e \left( - \frac{2d}{N - 1} - \frac{2c(a)}{N - 1} \right) \\
= \frac{4Q(a)}{N - 1} (d - c(a)) 
\] (3.18)

Second term:

\[
\frac{\partial (2Q(a)(1 - P_e) E[(X - \hat{X})^2 | K = 1, \hat{K} = K])}{\partial d} \\
= 2Q(a)(1 - P_e) \frac{\partial}{\partial d} \left( 1 + ac(a) + d^2 - 2dc(a) \right) \\
= 2Q(a)(1 - P_e)(2d - 2c(a)) \\
= \frac{4Q(a)(1 - P_e)(d - c(a))}{N - 1} 
\] (3.19)

Third term:

\[
\frac{\partial ([1 - 2Q(a)]P_e E[(X - \hat{X})^2 | 2 \leq K \leq N - 1, \hat{K} \neq K])}{\partial d} \\
= [1 - 2Q(a)]P_e \frac{\partial}{\partial d} \left( 1 - \frac{2ac(a)Q(a)}{1 - 2Q(a)} + \frac{N - 3}{N - 1} \int_{-a}^{a} \hat{x} \lambda(\hat{x}) d\hat{x} + \frac{2d^2}{N - 1} \right) \\
= [1 - 2Q(a)]P_e \left( \frac{4d}{N - 1} \right) \\
= \frac{4dP_e (1 - 2Q(a))}{N - 1} 
\] (3.20)
Fourth term:

\[
\frac{\partial}{\partial d} (1 - 2Q(a))(1 - P_e) \frac{\partial}{\partial d} \left( (X - \hat{X})^2 \right)_{2 \leq K \leq N - 1, \hat{K} = K} = \frac{\partial}{\partial d} \left( (X - \hat{X})^2 \right)_{2 \leq K \leq N - 1, \hat{K} = K} = 0
\]

since (3.10) is not a function of \( d \).

Combining (3.18)-(3.20) gives

\[
\frac{\partial\mathbb{E}[(X - \hat{X})^2]}{\partial d} = 4Q(a) \left( \frac{d - c(a)}{N - 1} + (1 - P_e)(d - c(a)) \right) + \frac{4dP_e(1 - 2Q(a))}{N - 1} .
\] (3.21)

Solving (3.21) for \( d \) results with

\[
d = \frac{Q(a)c(a)(2P_e - NP_e + N - 1)}{Q(a)(2P_e - NP_e + N - 1) + P_e(1 - 2Q(a))}
\] (3.22)

Optimization with respect to \( \lambda(x) \) will be done in the same manner as the regular quantizer.

\[
\min_{\lambda(x)} \mathbb{E}[(X - \hat{X})^2] \quad \text{subject to} \quad \int_{-a}^{a} \lambda(x) dx = 1,
\] (3.23)
and the auxiliary function is

$$L(\lambda, \mu) = \mathbb{E}[(X - \hat{X})^2] + \mu\left(\int_{-a}^{a} \lambda(x)dx - 1\right)$$  \hspace{1cm} (3.24)

where $\mathbb{E}[(X - \hat{X})^2]$ is derived throughout (3.18)-(3.20).

$$\frac{\partial L(\lambda, \mu)}{\partial \lambda} = 0,$$

Then

$$2Q(a)P_e \frac{N - 2}{N - 1} x^2 + [1 - 2Q(a)]P_e \frac{N - 3}{N - 1} x^2 - [1 - 2Q(a)](1 - P_e) \frac{f_X(x)\lambda(x)^{-3}}{6(N - 2)^2[1 - 2Q(a)]} = \mu.$$  

After some manipulations, the above expression boils down to

$$\lambda(x) = \frac{f_X(x)(1 - P_e)}{\sqrt{6(N - 2)^2\left(\frac{P_e}{N - 1}(2Q(a)x^2 + N - 3) + \mu\right)}}$$  \hspace{1cm} (3.25)
Chapter 4

Estimation

4.1 Implementation of the ML Estimator

As discussed in the Preliminaries chapter, ML estimator is given by

\[ \hat{\theta}_{ML} = \arg \max_{\theta} p_{Y|\theta}(y|\theta) = \prod_{i=1}^{N} f_{Y|\theta}(y|\theta) \quad (4.1) \]

where \( Y \) is the set of samples from i.i.d. random variables and \( \theta \) is the parameter of this statistical model. (4.1) has to be modified in order to be able to used in a setting where available information is the codebook and the noisy received signal. So,

\[ \hat{V}_{ML} = \arg \min_{\hat{V} \in U} f_{V|U}(y|\hat{U}) \quad (4.2) \]
where $u_j$ is an N-dimensional element of the N-element codebook,

$$
u_j = \begin{cases} \sqrt{E} & j = K \\ 0 & j \neq K \end{cases}$$

and $v$ is the N-dimensional received signal,

$$v = u_j + w$$

while $w$ is the N-dimensional noise vector with $\mathcal{N} \sim (0, I)$.

$$f_{V|U}(v|u_j) = f_W(v - u_j)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(v-u_j)^2/2}$$

$$= \prod_{i=1}^{N} \frac{e^{-(v_i-u_i)^2/2}}{\sqrt{2\pi}}$$

$$= e^{-\sum_{i=1}^{N} (v_i-u_i)^2/2}$$

$$= \frac{e^{-\|v-u_j\|^2/2}}{(2\pi)^{N/2}}$$

(4.3)

So, the ML estimator becomes

$$\arg \max_{\hat{u}_j} f_W(v - u_j) = \arg \max_{\hat{u}_j} e^{-\|v-u_j\|^2/2}$$

(4.4)
It is convenient to apply logarithm to both sides at this point. Then

\[
\ln f_W(v - u_j) = -\frac{||v - u_j||^2}{2}
\]

If the minus sign is removed from the right hand side of the above expression, it turns into a minimization problem.

\[
\arg \max_{u_j} -||v - u_j||^2 = \arg \min_{u_j} ||v - u_j||^2 = \arg \min_{u_j} (||v||^2 - 2||u_jv|| + ||u_j||^2) = \arg \min_{u_j} -2||u_jv|| = \arg \max_{u_j} ||u_jv||
\]

(4.5)

Recall that \(u_j\) has only one non-zero element that is equal to \(\sqrt{E}\). Thus,

\[
||u_jv|| = \sqrt{E}v_j
\]

where \(v_j\) is the \(j\)th element of \(v\). In other words, the search for maximum is done among the elements of \(v\) because for each \(j\), \(||u_jv||\) gives the \(j\)th element of \(v\). So this whole thing boils down to

\[
\hat{V}_{MLE} = \arg \max_j v_j
\]
4.2 Implementation of the MMSE Estimator

MMSE estimator is implemented using joint source-channel decoding instead of the separate source-channel decoding used for MLE, as illustrated in Fig 4.1.

![Figure 4.1: Coding scheme with MMSE decoder](image)

Minimum mean square error estimator is calculated by finding the posterior mean of the parameter to be estimated, that is

\[
\mathbb{E}[X|V] = \int f_{X|V}(x)dx = K^{-1} \int f_{V|X}(x)f_X(x)dx
\]  

(4.6)

where

\[
K = \int f_{V|X}(x)f_X(x)dx
\]
and

\[
f_{Y|X}(x) = f_W(v - u(x)) \left\prod_{i=1}^{N} f_W(v_i - u_i(x)) \right.
\]

Then

\[
\mathbb{E}[X|V] = K^{-1} \int f_W(v - u(x)) f(x) x dx
\]

\[
= K^{-1} \int f_X(x) x \left\prod_{i=1}^{N} f_W(v_i - u_i(x)) \right) dx
\]

\[
= K^{-1} \sum_{i=1}^{N} \left( \int_{x \in R_i} f_X(x) x \prod_{k=1}^{N} f_W(v_k - \delta_{ik} \sqrt{E}) dx \right)
\]

If the expressions are further expanded,

\[
\mathbb{E}[X|V] = \sum_{i=1}^{N} c_i \prod_{k=1}^{N} e^{-\left(v_k - \sqrt{E}\delta_{ik}\right)^2} \sum_{i=1}^{N} p_i \prod_{k=1}^{N} e^{-\left(v_k - \sqrt{E}\delta_{ik}\right)^2}
\]

\[
= \sum_{i=1}^{N} c_i e^{-\sum_{k=1}^{N} \left(v_k - \sqrt{E}\delta_{ik}\right)^2} \sum_{i=1}^{N} p_i e^{-\sum_{k=1}^{N} \left(v_k - \sqrt{E}\delta_{ik}\right)^2}
\]

\[
= \frac{\sum_{i=1}^{N} c_i e^{-b_i}}{\sum_{i=1}^{N} p_i e^{-b_i}}
\]

where

\[
c_i = \frac{1}{\sqrt{2\pi}} \int_{x \in R_i} f_X(x) x dx
\]

\[
p_i = \frac{1}{\sqrt{2\pi}} \int_{x \in R_i} f_X(x) dx
\]
Let
\[ i^* = \arg \min_i \sum_{k=1}^{N} (v_k - \sqrt{E\delta_{ik}})^2. \]

\( i^* \) is used to prevent the numerical errors during the simulation. It is subtracted from the exponentials in both nominator and denominator to reduce the power of the exponentials. So the MMSE estimate becomes
\[ \hat{X} = E[X|V] = \frac{\sum_{i=1}^{N} c_i e^{-(b_i-b_{i^*})}}{\sum_{i=1}^{N} p_i e^{-(b_i-b_{i^*})}} \]  

(4.8)

It is stated that the high resolution quantizer requires a specific point density function in order to operate optimally. The way to derive this point density function is through optimization of the distortion expression, which is discussed in detail in Chapter 3.

While using MMSE estimator it is not possible to find an analytical expression and derive a distortion expression for this communication scheme. Therefore, compandor uses the optimal point density assuming that the estimator is doing MLE, which is not optimal for the MMSE estimator. As a way to improve the performance of MMSE estimator, a search algorithm is offered to search over \((N,a)\) space for the sake of increasing it’s performance.

### 4.3 Search Algorithm for MMSE

Search algorithm for MMSE is an algorithm that optimizes the source coding for MLE, and then searches over the parameter’s space by applying MMSE estimator to improve...
the end-to-end distortion. This proposed algorithm is not a guaranteed solution to achieve
the minimum distortion that can be achieved using MMSE estimator. This is because opti-
timal $\lambda(x)$ can not be found since the distortion expression can not be derived analyti-
cally for MMSE estimation at the decoder.

The way this method works is as following;

**Step 1:** Optimal $(N, a)$ pair for a specific energy is found for ML decoder. Let this pair
of $(N, a)$ be $(N_1, a_1)$.

**Step 2:** Monte Carlo simulation is done assuming MMSE estimation at the receiver, at 9
points including $(N_1, a_1)$. Let the horizontal and vertical distance between neighbor pairs
be called distN and distA. For each pair, a $\lambda(x)$ is found assuming ML estimation at the
decoder. The $(N, a)$ pair that has the least distortion among those 9 pairs is the new cen-
ter for the 9-pair square. Let this pair be $(N_2, a_2)$.

**Step 3:** 8 new pairs are selected according to $(N_2, a_2)$. If $(N_2, a_2) = (N_1, a_1)$, then distN
and distA are reduced so that the search is done in a smaller area (as in the red square
of Figure 4.1). If $(N_2, a_2) \neq (N_1, a_1)$, then the new center of the 9-pair square becomes
$(N_2, a_2)$ and the previous step is repeated.

This loop is repeated for a specific number of iterations or as long as distN and distA
remain above a certain value. The whole method is illustrated in Figure 4.1. Also, note
that distN and distA are not necessarily equal as shown in figure. $N$ has to be an integer,
so distN has to be an integer. There is no such restriction on $a$.  

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Figure 4.2: Search algorithm for MMSE estimation
Chapter 5

Simulation Results

In order to achieve the minimum possible distortion, the quantizer has to be given the optimum point density function, and the optimum point density function is found by optimizing the distortion expression. Since it is cumbersome to find the distortion expression analytically for MMSE estimator, it was not possible to find the associated optimum point density function. Yet, it is not as difficult to find an expression for the posterior mean, $E[X|Y]$, which means that it is implementable even though it is suboptimal. So in the analysis of these results, it is important to be aware of the fact that MMSE results do not offer the best achievable results. On the other hand, optimization with respect to MLE offer the best achievable results since the quantizer can be optimized.
Figure 5.1: MLE with different quantizers

Figure 5.2: MMSE with different quantizers
Figure 5.3: Iterative search algorithm for MMSE and suboptimal MMSE

Figure 5.4: Overall comparison
Chapter 6

Conclusion

This thesis has carried out the optimization and analysis of an energy limited low bandwidth Gaussian source transmission in a zero delay setting by using high resolution quantization for encoding and orthogonal signaling for modulation. The use of high resolution quantizer introduced a density function that had to be optimized in order to reach minimum possible distortion.

On the receiver end of the scheme, MMSE and MLE estimators were used to decode the received signal. Since it was simple to express the distortion in a scheme with ML estimator, the optimization was done for ML estimator. On the other hand, it was not possible to express the distortion analytically using MMSE estimator, and therefore the quantizer could not be optimized for MMSE, and its results are not the best possible results.

The results also showed that in case there is bandwidth mismatch between the source and the channel for low bandwidth sources, it is not optimal to prefer uncoded transmission unless there is an energy scarcity at the transmitter. In all of the proposed schemes, un-
coded transmission is outperformed after a certain energy value.

High resolution quantization has proven itself to be a good source coding method for these settings even under suboptimal conditions, because optimizing it according to MLE and then switching the estimator to MMSE, rather than just applying uniform quantization with MMSE estimation, achieved very satisfying distortion values. The proposed search algorithm surpassed the other proposed schemes and the ones in a similar work, [1]. It can further be developed by increasing the number of iterations and/or searching over a larger area depending on the energy restrictions of the sensor. It should be noted that the design of such coding scheme is not an easy task, but it has to be done once and the implementation is not troublesome. On the other hand, same scheme with ML estimator is very easy to design and implement, and the achieved distortion values are still good, although not best.
Bibliography


