Title
Gyrokinetic Theory of the Lower-Hybrid Drift Instability in a Current Sheet with a Guide Field

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Gyrokinetic Theory of the Lower-Hybrid Drift Instability in a Current Sheet with a Guide Field

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Physics

by

Kurt Tummel

Dissertation Committee:

Professor Liu Chen, Chair
Professor Zhihong Lin
Professor Roger McWilliams

2014
DEDICATION

This is dedicated to my parents. They will always be the most important teachers in my life.

"Nothing in this world can take the place of persistence. Talent will not: nothing is more common than unsuccessful men with talent. Genius will not; unrewarded genius is almost a proverb. Education will not: the world is full of educated derelicts. Persistence and determination alone are omnipotent." –Calvin Coolidge
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ABSTRACT OF THE DISSERTATION

Gyrokinetic Theory of the Lower-Hybrid Drift Instability in a Current Sheet with a Guide Field

By

Kurt Tummel

Doctor of Philosophy in Physics

University of California, Irvine, 2014

Professor Liu Chen, Chair

This thesis presents an investigation of the lower-hybrid drift instability (LHDI) in a thin Harris current sheet with a guide field. This includes three-dimensional analytical and numerical analyses using the gyrokinetic electron, fully-kinetic ion (GeFi) description, which are compared with results from the Vlasov approach and simulations. Previous fully-kinetic studies solve the electron Vlasov equation by integrating along the unperturbed phase-space orbits, including the complete electron-cyclotron motion. The LHDI satisfies $\omega \ll \omega_{ce}$ and $k_\perp \rho_e \sim 1$, where $\omega_{ce}$ and $\rho_e$ are the electron cyclotron frequency and Larmor radius, respectively, and $k_\perp$ is the wavevector perpendicular to the equilibrium magnetic field. By treating the electron response with gyrokinetic theory, the fast cyclotron motion is removed which greatly simplifies the derivation of the LHDI eigenvalue equations. This allows a more comprehensive LHDI analysis, which is carried out over the entire domain of unstable wavevectors. To our knowledge, an extensive scan of the operative domain of the LHDI in current sheets with a guide field has never been done.

The results will show that two types of electromagnetic LHDI are active in the current sheet. The Type A LHDI is generally consistent with the existing theoretical descriptions of the LHDI, namely, quasi-electrostatic modes localized near the current sheet edges.
with $k\rho_e \sim 1$, $k_\parallel = 0$, and $\omega^2 \sim \omega_{pe}^2/(1 + \omega_{pe}^2/\omega_{ce}^2)$, where $\omega_{pe}$ and $\omega_{pi}$ are the electron and ion plasma frequencies, respectively. However, we will show that in sufficiently thin current sheets, i.e. strong equilibrium drifts, the Type A LHDI is destabilized by finite $k_\parallel$ in the short wavelength domain, $k\rho_e \gtrsim 0.5$. This destabilization increases the range of propagation angles, $k_\parallel/k_\perp$, for which the modes are operative, which reduces the localizing and stabilizing effects of magnetic shear. The dominant Type A modes are localized near the current sheet edge, $z \sim 1.5L$, due to the stabilizing resonant dissipation of the electron $\nabla B$ drift, which is strongest near the current sheet center. At longer wavelengths, $k\sqrt{\rho_e\rho_i} \sim 1$, a second group of instabilities arise, which we define as Type B LHDIs. The Type B LHDIs are operative for a large range of propagation angles, $k_\parallel/k_\perp$, and are suppressed when the electron $\nabla B$ drift is removed. The dominant Type B instabilities are localized near the current sheet center, $z \sim 0.2L$, with moderate $k_\parallel/k_\perp$ and significant magnetic field perturbations. The Type B LHDIs resemble fluctuations observed in nature\cite{31, 62}, and simulations\cite{51, 12}, which have undergone limited analytical analysis.\cite{30, 52} These modes are of general relevance to the evolution of thin current sheets, and may contribute to collisionless magnetic reconnection theory as a source of anomalous resistivity.
Chapter 1

Introduction

Current sheets are plasma structures prevalent in space and laboratory plasmas that are relevant to a broad range of research efforts including magnetic propulsion, magnetic reconnection, and fusion devices.[43, 5, 10, 59, 54] Plasma currents and the associated pressure gradients can serve as free energy sources that drive instabilities like the tearing mode, modified two stream, drift kink, drift sausage, Kelvin-Helmholtz, Ion-Weibel, and lower-hybrid drift instability (LHDI).[57] In collisionless plasmas, the scattering of charged particles by suprathermal electromagnetic fluctuations generated by current sheet instabilities may lead to effects such as anomalous resistivity and cross-field particle transport. The LHDI is a pervasive current sheet instability that has been thoroughly examined in anti-parallel magnetic field configurations. In this work, the LHDI is analyzed in a current sheet with a guide field to investigate the possibility of LHDI activity near the current sheet center, which may generate anomalous resistivity during magnetic reconnection.
1.1 Magnetic Reconnection

Magnetic reconnection is a widely studied plasma phenomenon that involves the breaking and reconnecting of magnetic field lines in a way that connects initially isolated regions of the plasma. This alters the magnetic field topology to a lower energy configuration, which is accompanied by a rapid release of magnetic energy, and consequent plasma heating and transport. This process has been observed in a variety of space and laboratory plasmas, including solar flares, coronal mass ejections, the interaction of solar wind with the Earths magnetic field, sawtooth disruptions in tokamaks, and reconnection experiments like the MRX and SSX.[4, 8, 6] In the earths magnetic field, magnetic reconnection and the consequent geomagnetic storms can damage satellites and disrupt power transmission grids, telecommunication cables, and oil or gas pipelines, especially at high latitudes.[23] The scientific appeal of this phenomenon, along with these commercial implications, have motivated extensive research efforts to improve the understanding and prediction of magnetic reconnection in space and laboratory plasmas.

1.1.1 Sweet-Parker Reconnection

One of the earliest successful theories of magnetic reconnection was developed by Sweet[46] and Parker[40] for highly collisional, incompressible plasmas with uniform resistivity. This model is depicted in Figure 1.1, which shows antiparallel magnetic fields approaching each other at velocity $v_{\text{rec}}$, forming a dissipation region of width $2\delta$ and length $2L$. The magnetic field lines reconnect in this dissipation region, joining initially isolated magnetic field domains, and emerge at velocity $u$. In plasmas with high conductivity, Ohm’s law is given by

$$\mathbf{E} + \frac{v}{c} \times \mathbf{B} = 0,$$  

(1.1)
Figure 1.1: The Sweet-Parker model of magnetic reconnection.

and is referred to as the frozen-in condition, where the magnetic field lines are carried along with the plasma. It can be shown that Eq(1.1) guarantees the conservation of magnetic flux,

\[ \partial_t \int B \cdot dA = 0, \quad (1.2) \]

which is violated during magnetic reconnection. When antiparallel magnetic fields approach each other, the rotation of the magnetic field generates a large current density along the magnetic null. Even for highly conducting plasmas, Ohm’s law must then be modified to the following;

\[ E + \frac{v}{c} \times B = \frac{\eta}{c} J, \quad (1.3) \]

where \( \eta \) is the plasma resistivity. This resistive term can cause a change in magnetic flux, which is found using Faraday’s law and Eq(1.3),

\[ \partial_t \int B \cdot dA = - \int \nabla \times \frac{\eta}{c} J \cdot dA. \quad (1.4) \]
Assuming the resistivity, $\eta$ is constant, we introduce Ampere’s law to find

$$\frac{\partial}{\partial t} \int B \cdot dA = \frac{c\eta}{4\pi} \int \nabla^2 B \cdot dA \approx \frac{c\eta}{4\pi\delta^2} \int B \cdot dA. \tag{1.5}$$

Here, we neglect the displacement current in Ampere’s Law, and approximate the spatial derivative as the scale of the dissipation region, $\delta$, which is shown in Figure 1.1. The time scale of the magnetic flux change is thus

$$\tau_D = \frac{4\pi\delta^2}{c\eta}, \tag{1.6}$$

which is referred to as the magnetic diffusion time scale. It would appear that the magnetic reconnection time scale decreases as $\delta$ decreases, but Sweet and Parker showed that this is not the case for steady-state reconnection.

The frozen in condition is valid away from the magnetic null, so the flow of magnetic flux into and out of the dissipation region is carried by the plasma. The reconnection time scale depends on these flow velocities, which for incompressible plasmas are related as follows;

$$v_{rec}L = u\delta, \tag{1.7}$$

The outflow velocity, $u$, can be approximated as

$$\frac{1}{2} \rho_m u^2 = \delta P, \tag{1.8}$$

where $\delta P$ is the difference in plasma pressure between the magnetic null, and the asymptotic reconnected magnetic field. Setting this plasma pressure difference equal to the magnetic pressure difference, we find $u$ is the Alfven velocity in the asymptotic reconnected magnetic field, $v_A^2 = B^2/4\pi\rho_m$, where $\rho_m$ is the mass density of the plasma. Because $u$, given in Eq(1.8), is insensitive to the size of the dissipation region, the flow of magnetic flux into the
dissipation region increases with $\delta/L$, as seen in Eq(1.7). This inflow of magnetic flux is a measure of the rate of magnetic reconnection, which requires a sufficiently small $\delta$ for the resistive term $\eta\mathbf{J}$ to facilitate magnetic diffusion, but sufficiently large $\delta/L$ to sustain a fast flow of magnetic flux.

Attributing the increase in plasma pressure in the dissipation region to Ohmic heating, we may introduce the following expression;

$$
\frac{1}{2} \rho_m u^2 = \frac{1}{c} \eta J^2 L \frac{L}{u} = \frac{c}{(4\pi)^2} \eta B_0^2 \frac{L}{u \delta^2},
$$

(1.9)

where we relate $J$ and $B_0$ through Ampere’s law and assume $u \gg v_{rec}$. Combining Eq(1.7) and Eq(1.9) we find

$$
\frac{\delta}{L} = \frac{v_{rec}}{v_A} = \sqrt{\frac{\eta c}{2\pi v_A L}} = \frac{\sqrt{2}}{\sqrt{S}},
$$

(1.10)

where $S$ is the Lundquist number, and the ratio of $v_{rec}/v_A$ is the Alfvenic Mach number, which is used as a measure of the reconnection rate.

The Sweet-Parker model has been confirmed in collisional laboratory plasmas[32, 33], but at low collisionality, the predicted reconnection time scale is orders of magnitude too long. The plasma collisionality is described through the Coulomb collision frequency, which is proportional to $nT^{-3/2}$, where $n$ is the density, and the relevant temperature $T$ depends on the colliding particle species. Thus collisions have a negligible influence in high temperature dilute plasmas which are achievable in laboratory experiments[32, 33], and typical in space environments[43, 3, 60]. In the Earths magnetotail, the Lundquist number is $S \sim 10^8$, and the Sweet-Parker model predicts a reconnection time scale $\tau_{rec} \sim 10^5$ seconds, while the observed substorm time scale is $\sim 600$ seconds.[2] Similarly, in the solar corona $S \sim 10^{12}$ and the predicted reconnection time scale is $10^7$ seconds, while the observed solar flare time scale is $\sim 10^3$ seconds.[19] Developing a more accurate model for magnetic reconnection
in collisionless plasmas is an ongoing process, and several alternatives to the Sweet-Parker model are under consideration.

1.1.2 Collisionless Reconnection

The Sweet-Parker model is derived using resistive MHD theory, which treats the plasma as a single fluid. In space plasmas, current sheet scale widths commonly fall below the ion skin depth \([43, 3, 60, 7]\), \(c/\omega_{pi}\), where \(\omega_{pi}\) is the ion plasma frequency, \(\omega_{pi} = 4\pi n e^2/m_i\), with ion mass \(m_i\), and elementary charge \(e\). At this scale, the electron and ion motion becomes decoupled and the plasma must be treated with two fluid theory. The two fluid collisionless Ohms law is given by\([18]\)

\[
\mathbf{E} + \frac{\mathbf{v}_i}{c} \times \mathbf{B} = \frac{\eta}{c} \mathbf{J} + \frac{1}{en} \left( \frac{1}{c} \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P}_e \right) - \frac{m_e}{e} \frac{d\mathbf{v}_e}{dt},
\]

(1.11)

where \(\mathbf{P}_e\) is the electron pressure tensor, \(\mathbf{v}_i, \mathbf{v}_e\) are the ion and electron fluid velocities, respectively, and \(m_e\) is the electron mass. The terms on the right hand side of Eq(1.11) may contribute to the violation of the frozen-in condition and are under investigation as potential mechanisms of the magnetic flux change that occurs during reconnection.

One active topic of collisionless reconnection research is investigating the role of current sheet instabilities on the reconnection rate. Reconnection studies often equate the plasma resistivity with the classical Spitzer result, \(\eta_S\), which is proportional to \(T_e^{-3/2}\), and arises from the momentum exchange between electrons and ions through binary Coulomb collisions. In space and high-temperature laboratory plasmas with infrequent Coulomb collisions, momentum exchange can occur through wave-particle interactions with suprathermal electromagnetic fluctuations excited, e.g., via current sheet instabilities. This instability-driven momentum exchange gives rise to an so-called anomalous resistivity, which may enhance the rate of reconnection directly by increasing magnetic diffusion, or indirectly by generating Petschek-type
shocks.[41, 49, 22] In the MRX, as plasma collisionality was reduced, the effective resistivity was inferred from the ratio of the current density to the reconnection electric field, $\eta_{\text{eff}} = E_\theta / J$.\[33\] At low collisionality, this effective resistivity reached $\sim 10\eta_s$, and restored the accuracy of the Sweet-Parker model. Electromagnetic fluctuations are often observed in reconnection experiments[8, 31], and magnetotail current sheets,[60, 62, 3, 50] and several instabilities, including the ion-acoustic mode, Buneman instability, and LHDI, have attracted attention as potential sources of anomalous resistivity.[39, 14, 42] In typical magnetotail current sheets, $T_i \gg T_e$, and the drift velocities associated with the current density are $v_d \sim v_i$. These conditions generally stabilize ion-acoustic modes and Buneman instabilities, but the LHDI is operative in this domain and has been observed in the magnetotail.[62, 20]

### 1.2 Lower-Hybrid Drift Instability

The first theoretical analyses of the LHDI characterized the most unstable modes as quasi-electrostatic waves restricted to the low $\beta$ current sheet edges with $k\rho_e \sim 1$, $\mathbf{k} \cdot \hat{b} = k_\parallel = 0$, and $\omega^2 \simeq \omega_{pe}^2/(1 + \omega_{pe}^2 / \omega_{ce}^2)$.\[13, 16, 34, 57\] Here $\beta = nT/(B^2/8\pi)$, $\mathbf{k}$ is the wave vector, $\rho_e = \sqrt{2T_e/m_e/\omega_{ce}}$, $\hat{b}$ is the unit vector along the equilibrium magnetic field, $\omega_{pe}$ is the electron plasma frequency, and $\omega_{ce}$ is the electron cyclotron frequency. Anomalous resistivity from an instability localized away from the peak plasma current and current sheet center has questionable relevance to magnetic reconnection. However, Cluster observations[62] and MRX results[31] have raised interest in drift instabilities near the lower-hybrid frequency that penetrate into the current sheet center. The observed fluctuations have finite $k_\parallel$, and recent consideration of the LHDI with finite $k_\parallel$ in equilibria with magnetic shear has revealed a broader range of mode properties that may explain the observed activity at the current sheet center.[12, 51]
Cluster observations of a thin current sheet in the magnetotail report quasi-electrostatic fluctuations near the separatrix and electromagnetic fluctuations near the current sheet center, both in the lower-hybrid frequency range.[62] The magnetic fluctuations propagated obliquely, i.e. \( k_\parallel \neq 0 \), with a non-negligible \( \delta B_\perp \) and \( \lambda \sim 3.5\sqrt{\rho_e \rho_i} \), where \( \lambda \) is the mode wavelength, and \( \delta B_\perp \) is the oscillating magnetic field component perpendicular to the equilibrium field. The quasi-electrostatic fluctuations propagated at \( \theta > 70^\circ \), where \( \theta \) is the angle between \( k \) and \( B \). The current sheet properties were consistent with the Harris model with no evidence of magnetic shear, though they observed a magnetic Hall field that may have played the role of a guide field.[21]

Studies of magnetic reconnection at low collisionality at the MRX have identified a quasi-electrostatic LHDI near the current sheet edge.[8] The fluctuations were measured in the absence of a guide field and had the appropriate scaling \( B/\sqrt{m_i} \). The mode coherence lengths were approximately \( 10\rho_e \), but the propagation direction was not clear. The MRX also reports strong electromagnetic fluctuations near the current sheet center with frequencies reaching the lower-hybrid frequency.[31] These modes were measured at low collisionality with a weak guide field, and the fluctuations were correlated with enhanced reconnection rates. The magnetic fluctuations included substantial \( \delta B_\perp \), and the wave vector had an oblique orientation, \( k \cdot \hat{b} \neq 0 \), at the current sheet edge but it was not successfully measured at the current sheet center.

The MRX observations prompted a local theoretical analysis which suggests the fluctuations are caused by an unstable coupling of whistler waves and ion sound waves.[30] However, these modes do not have strong growth rates, and the equilibrium magnetic field had no guide field component. Simulations consistently show the LHDI dominates current sheet fluctuations in the linear phase, but the LHDI is not an obvious candidate for the observed fluctuations because of its suppression by plasma \( \beta \) and finite \( k_\parallel \).[57] Huba et al.[25] and Davidson et al.[13] investigated the stabilization of the LHDI by increasing \( \beta \) using a local dispersion
relation derived from fluid equations and in a nonlocal kinetic analysis. They determined that the dominant electromagnetic effects associated with increasing $\beta$ are resonant dissipation caused by $\nabla B$ drift orbit modifications, and disrupted electron flow along the field lines due to the oscillating parallel magnetic vector potential $\delta A_\parallel = \delta A \cdot \hat{b}$. The overall influence of $\beta$ was stabilizing, despite the destabilization due to $\delta A_\parallel$, but an interesting exception was found when magnetic shear was included in the equilibrium.

A nonlocal kinetic analysis of the LHDI in equilibria with magnetic shear found that at low $T_e/T_i$, sufficient shear allows the destabilizing influence of $\delta A_\parallel$ to overwhelm the stabilizing influence of the $\nabla B$ drift resonance, and the growth rate is larger at higher $\beta$. This result alone is not sufficient support of potential LHDI activity near a current sheet neutral because the derivation expands around $k_\parallel = 0$, and $\mathbf{J} \cdot \mathbf{B} = 0$, where $\mathbf{J}$ is the equilibrium current. This expansion was justified by the well-established stabilization of the LHDI by finite $k_\parallel$ in local and nonlocal kinetic studies. This stabilization was interpreted as a consequence of a stabilizing electron flow along the field lines, but these works did not address magnetic equilibria with a guide field. In the absence of a guide field, finite $k_\parallel$ orients the mode away from the equilibrium current direction, which reduces wave-particle resonance with the equilibrium drift. This is not the case for current sheets with a guide field, where the resonance is strongest among current-aligned modes, for which $k_\parallel$ is never zero. The high $\beta$ environment of interest in a current sheet with a guide field is near the current sheet center, where the guide field dominates, and modes with $k_\parallel = 0$ do not resonate with the equilibrium current. If the wave-particle resonance condition dominates the $k_\parallel$ stabilization, and $\delta A_\parallel$ facilitates high $\beta$ penetration, the LHDI may be active near the center of a current sheet with a guide field, with an oblique propagation and substantial $\delta B_\perp$. 

9
1.2.1 LHDI with finite $k_\parallel$

The effects of $k_\parallel$ on the LHDI in sheared equilibria have not been extensively addressed in the literature. The destabilization of the LHDI by finite $k_\parallel$ was reported in a nonlocal two-fluid analysis of electrostatic drift instabilities in a current sheet, but the equilibrium field had no shear.[63] Yoon et al. derived a nonlocal electrostatic eigenvalue equation for a Harris equilibrium with a guide field using the two-fluid formulation.[61] This was part of a successful benchmark of a linearized version of the gyrokinetic electron, fully-kinetic ion(GeFi) simulation scheme developed by Lin et al.[36, 35] However, the guide field was removed during the comparison, and $k_\parallel/k$ was held constant. Yoon et al. subsequently published a revised eigenvalue equation with an electron response derived by integrating the electron Vlasov equation along unperturbed orbits.[55] A local electrostatic two-dimensional analysis was compared with GeFi simulation results, but the strongest growth rates occurred when $k_\parallel = 0$. This result treated current sheets with thickness $L \sim \rho_i$. Simulations of the LHDI in thinner current sheets have produced promising results for potential obliquely-propagating LHDI activity at the current sheet center.[61, 12]

Daughton investigated the LHDI in a thin Harris current sheet with a guide field using a two-dimensional fully-kinetic Vlasov PIC simulation.[12] The electron response was derived by integrating along unperturbed particle orbits with $m_i/m_e = 512$. The modes had edge-localized electrostatic fluctuations along with strong magnetic fluctuations near the current sheet neutral with finite $\delta B_\perp$. The magnetic fluctuations near the current sheet center were suppressed at larger current sheet thicknesses $L \geq \rho_i$. A thinner current sheet introduces stronger equilibrium drift velocities which reportedly increases the resilience of LHDI to shear-stabilization, and improves high $\beta$ penetration.[27] Unlike the MRX results, the strongest modes satisfied $k \cdot B = 0$ at the current sheet edge, but a thorough investigation of wavelengths and propagation angles was not feasible.
Wang et al. used a linear version of the GeFi simulation model[36, 35] to analyze current-aligned LHDIs in a thin Harris current sheet with a guide field.[51] Quasi-electrostatic edge-localized modes were observed, along with long-wavelength electromagnetic modes with substantial $\delta B_\perp$ near the current sheet center. The equilibrium included a stationary background plasma, and transition to the more penetrating, electromagnetic modes was observed after reducing the background density and increasing the guide field strength.

The theoretical result of Huba et al.[26] suggests that sufficient shear is necessary for the destabilization of the LHDI by plasma $\beta$, which could explain LHDI activity near the neutral sheet. However, this model did not address finite $k_\parallel$ effects, and in the simulation results by Wang et al.[51], it is unclear whether a larger guide field, or the larger $k \cdot \hat{b}$ throughout the mode width is responsible for the onset of the electromagnetic modes near the current sheet center. It is also unclear whether these modes can be explained as a variety of LHDI, or if they have an entirely different instability mechanism. This thesis aims to resolve these issues with an investigation of the entire domain of unstable wavevectors of the LHDI in a Harris current sheet with a guide field. Although the magnetic shear must be treated with nonlocal theory, a guide field can sustain the magnetization of the electrons near the current sheet center, which is highly advantageous from the standpoint of theoretical analysis. The resulting analytical model permits a thorough investigation of the properties and thresholds of these instabilities and the opportunity for a physical explanation of these modes.

1.3 Thesis Outline

In the following Chapter, equilibrium conditions are defined, and the linear three-dimensional nonlocal electromagnetic eigenmode equations are derived using the GeFi description. The analytical and numerical results in the electrostatic limit are discussed in Chapter 3, and it is shown that the electrostatic LHDI is destabilized by finite $k_\parallel$ at short wavelengths,
\( k_p \geq 0.5 \). This destabilization is unique to current sheets with a strong equilibrium drift, and is shown directly in the local analysis, and confirmed in the nonlocal analysis through the evaluation of \( \mathbf{k} \cdot \hat{b} \) at the location of the peak eigenfunction amplitudes. In Chapter 4, an analysis of the electromagnetic LHDI is presented, and two types of electromagnetic LHDI are discussed. The Type A electromagnetic LHDI experiences a similar destabilization by finite \( k_\parallel \) in the short-wavelength domain, \( k_\rho \geq 0.5 \). As \( k_\parallel / k \) increases, the electromagnetic fluctuations are enhanced, and the localization moves towards the current sheet center, however, the dominant Type A modes are localized near the current sheet edge, \( z \sim 1.5L \), at all wavelengths. In the long-wavelength domain, \( k^2 \rho_e \rho_i \sim 1 \), a second branch of instabilities arise, which we define as Type B LHDI, which are destabilized by finite \( k_\parallel \) and exhibit significant magnetic field fluctuations. Unlike the Type A LHDI, these modes are localized at the position where the electron \( \nabla B \) drift is strongest, i.e. near the current sheet center, and are suppressed when the \( \nabla B \) drift is removed. A summary and conclusions are given in Chapter 5.
Chapter 2

Theoretical Derivation of Electromagnetic LHDI Eigenmode Equations

In this Chapter we derive a system of eigenmode equations for the electromagnetic LHDI in a thin Harris current sheet[21] with a guide field using the fully-kinetic ion, gyrokinetic electron model[1, 44, 48]. We begin with a description of the equilibrium conditions, and the validity of the model equilibrium in the Earth’s magnetotail. We then derive the fully-kinetic ion response, including the perturbed ion current density, employing the assumption of unmagnetized ions. Finally, we derive the electron response using gyrokinetic theory, and collect the results into a closed system of eigenmode equations derived from Poisson’s equation, and Ampere’s law. In Chapters 3 and 4, these eigenmode equations are solved to carry out a three dimensional, nonlocal analysis of the LHDI in a thin Harris current sheet with a guide field.


2.1 Equilibrium

We adopt the well known Harris current sheet model [21] with a guide field and small Maxwellian background plasma. This model is appropriate for current sheets in the MRX[53], though in this thesis, we use magnetotail parameters. A survey of thousands of magnetotail current sheet crossings by the CLUSTER satellite found 23 percent contain a substantial guide field.[3, 43] These current sheets are most prevalent near \( Y = 0 \) in GSM coordinates, and for current sheet thicknesses of the scale of the ion Larmor radius, the aspect ratio is large enough to justify a slab geometry model.

The current and magnetic field profile of the Harris current sheet with guide field are given by:

\[
\vec{B} = B_0 \Tanh(z/L) \hat{x} + B_g \hat{y},
\]

and

\[
n(z) = n_H \text{Sech}^2(z/L),
\]

and are depicted in Figure 2.1. Here, \( B_g \) is the guide field, \( L \sim O(\rho_i) \) is the current sheet width, and \( \rho_i \) is the ion Larmor radius. A self-consistent current sheet equilibrium distribution function can be constructed from constants of motion and is given by [21]

\[
f(x, v) = \frac{n_0}{\pi^{3/2} \rho_i^3} e^{-\epsilon/T} e^{v_y P_y/T}
\]

\[
= \frac{n_0}{\pi^{3/2} \rho_i^3} e^{-(v_y^2 + (v_y - v_d)^2 + v_z^2)/\rho_i^2} e^{v_d q A_y(z)/Tc},
\]

where \( T = \frac{1}{2} m v_i^2 \), \( \epsilon = \frac{1}{2} m v_i^2 \),

and \( P_y = m v_y + \frac{q}{c} A_y = m v_y - \frac{q B_0 L}{c} \ln(\text{Cosh}(z/L)) \).  

(2.3)
Here, the total energy $\epsilon$, and canonical momentum $P_y$ are the constants of motion. Note $f$ describes both ion and electron equilibria, and the condition $v_{de}/T_e = -v_{di}/T_i$ guarantees equilibrium charge neutrality.

### 2.2 Fully Kinetic Ion Response

We begin with the ions, which are composed of a current sheet and a background component with the same temperature. Thus $f_i(x, v) = f_{ics}(z, v) + f_{ib}(v)$, where

$$f_{ics}(z, v) = \frac{n_{cs}}{\pi^{3/2} v_i^3} \exp\left[-\frac{v_x^2 + (v_y - v_{di})^2 + v_z^2}{v_i^2} + \frac{v_{di} q A_y(z)}{T_i c}\right],$$

(2.5)

and

$$f_{ib}(v) = \frac{n_b}{\pi^{3/2} v_i^3} \exp\left[-\frac{v_x^2}{v_i^2}\right].$$

(2.6)
Typical LHDIs satisfy $|\omega| \gg \omega_{ci}$, and $|k \rho_i| \gg 1$, so we can assume unmagnetized ions and derive the perturbed ion distribution function from the following linearized Vlasov equation;

$$
(\partial_t + v \cdot \nabla) \delta f_i(x, v) = \frac{e}{m_i} \left( \nabla \delta \phi(x) + \frac{1}{c} \partial_t \delta A(x) - \frac{v}{c} \times [\nabla \times \delta A(x)] \right) \cdot \partial_v f_i(z, v).
$$

(2.7)

We seek solutions of the form $\hat{\delta f}(z)e^{ik_z x + ik_y y - i\omega t}$, and Fourier transform from $x$ to $k$. We then obtain

$$
(-\omega + k \cdot v) \hat{\delta f}_i(k_z, v)
= \frac{e}{m_i} \int dk'_z \left( k' \delta \phi(k'_z) - \frac{\omega}{c} \delta A(k'_z) - \frac{v}{c} \times [k' \times \delta A(k'_z)] \right) \cdot \partial_v f_i(k_z - k'_z, v),
$$

(2.8)

where $k = k_\parallel \hat{x} + k_y \hat{y} + k_z \hat{z}$. We recover $\hat{\delta f}_i(z, v)$ with an inverse Fourier transform of $\hat{\delta f}_i(k_z, v)$ which gives

$$
\hat{\delta f}_i(z, v) = \frac{e}{m_i} \int dk_z e^{ik_z z} \frac{1}{-\omega + k \cdot v} \int dk'_z k' \delta \phi(k'_z) \cdot \partial_v f_i(z + k'_z, v)
- \frac{e}{m_i} \int dk_z e^{ik_z z} \frac{1}{-\omega + k \cdot v} \int dk'_z \frac{\omega}{c} \delta A(k'_z) \cdot \partial_v f_i(z + k'_z, v)
- \frac{e}{m_i} \int dk_z e^{ik_z z} \frac{1}{-\omega + k \cdot v} \int dk'_z \left( \frac{v}{c} \times [k' \times \delta A(k'_z)] \right) \cdot \partial_v f_i(z + k'_z, v).
$$

(2.9)

To continue deriving the ion response, we introduce the equilibrium ion distribution, beginning with the current sheet population.
2.2.1 Perturbed Distribution of Current Sheet Ions

We transform the velocity coordinates of the current sheet ions from $v$ to $u = (v - v_{di} \hat{\gamma}) / v_i$, and make the following substitutions:

\[
\frac{1}{-\omega + k \cdot v} = \frac{1}{-\omega + k_y v_{di} + k \cdot u_i};
\]

\[
\partial_v f_{ics}(k_z - k_z', v) = \frac{2u}{v_i} f_{ics}(k_z - k_z', u),
\]

\[
\text{and} \quad -\frac{v}{c} \times [k' \times \delta A(k_z')] \cdot \partial_v f_{ics}(k_z - k_z', v)
\]

\[
= \left( \frac{k' v_{di} \delta A_y(k_z')}{c} - \frac{k_y v_{di} \delta A(k_z')}{c} \right) \cdot \frac{2u}{v_i} f_{ics}(k_z - k_z', u).
\]

In these velocity coordinates, the perturbed distribution of current sheet ions becomes

\[
\delta f_{ics}(z, u)
\]

\[
= -\frac{e}{T_i} \int e^{ik_z z} \frac{k' \cdot u_i}{\frac{1}{-\omega + k_y v_{di} + k \cdot u_i} \left( \hat{\phi}(k_z') - \frac{v_{di}}{c} \hat{A}_y(k_z') \right)} f_i(k_z - k_z', u) dk_z' dk_z
\]

\[
+ \frac{e}{T_i} \int e^{ik_z z} \frac{\delta A(k_z') \cdot u_i}{\frac{1}{-\omega + k_y v_{di} + k \cdot u_i} \frac{\omega - k_y v_{di}}{c} \left( \hat{\phi}(k_z') - \frac{v_{di}}{c} \hat{A}_y(k_z') \right)} f_i(k_z - k_z', u) dk_z' dk_z .
\]

Perturbed Density of Current Sheet Ions

We derive the perturbed density of current sheet ions by integrating the perturbed current sheet distribution function over velocity space. We begin this integration with the terms proportional to $\hat{\phi}(k_z') - \hat{A}_y(k_z') v_{di} / c$ in Eq(2.13), which can be expressed as

\[
-\frac{e}{T_i} \int e^{ik_z z} e^{ik_z' z} \frac{u_s + k_z' u_z / k_s}{u_s + k_z' u_z / k_s + k_z'' u_z / k_s - \xi_{ics}} \hat{\phi}(k_z') f_i(k_z', u) dk_z' dk_{z''} d^3u
\]

\[
+ \frac{e}{T_i} \int e^{ik_z z} e^{ik_z' z} \frac{u_s + k_z' u_z / k_s}{u_s + k_z' u_z / k_s + k_z'' u_z / k_s - \xi_{ics}} \frac{v_{di}}{c} \hat{A}_y(k_z') f_i(k_z', u) dk_z' dk_{z''} d^3u
\]

where $k_{z''} = k_z - k_z', k_z \hat{x} + k_y \hat{y} = k_s, u \cdot k_s = u_s k_s$, and $\xi_{ics} = (\omega - k_y v_{di}) / k_s v_i$. 
To simplify these terms, we assume $|k_z/k_s| \sim \rho_e/L \ll 1$, to be justified a posteriori, and expand to second order in $|k_z v_z/k_s v_s|$. This expansion yields the following expression;

$$
\frac{e}{T_i} \int e^{ik_z z} e^{ik_z z} \frac{u_s + k_z u_z/k_s}{u_s - \xi_{ics}} \sum_{n=0}^2 \left( \frac{-k_z u_z/k_s - k'' u_z/k_s}{u_s - \xi_{ics}} \right)^n 
\times \left( -\delta \phi(k_z') f_i(k_z', u) + \frac{v_{di}}{c} \delta A_y(k_z') f_i(k_z', u) \right) dk_z' \, dk''_z \, d^3 u.
$$

(2.15)

Evaluating this integral yields

$$
\frac{e}{2T_i} \left( Z'(\xi_{ics}) \left( \delta \phi(z) n_{cs}(z) + \frac{1}{k_s^2} \frac{\partial \delta \phi(z)}{\partial z} \frac{\partial n_{cs}(z)}{\partial z} \right) + \frac{1}{2} \xi_{ics} Z''(\xi_{ics}) \frac{\partial^2 \delta \phi(z) n_{cs}(z)}{\partial z^2} \right) 
- \frac{e}{2T_i} \frac{v_{di}}{c} \left( Z'(\xi_{ics}) \left( \delta A_y(z) n_{cs}(z) + \frac{1}{k_s^2} \frac{\partial \delta A_y(z)}{\partial z} \frac{\partial n_{cs}(z)}{\partial z} \right) 
+ \frac{1}{2} \xi_{ics} Z''(\xi_{ics}) \frac{\partial^2 \delta A_y(z) n_{cs}(z)}{\partial z^2} \right),
$$

(2.16)

where $Z(x)$ is the plasma dispersion function[17], and we ignore terms proportional to $\partial_z^2 n_{cs}(z)$.

The remaining terms in Eq(2.13) are proportional to $\delta \mathbf{A}(k_z') \cdot \mathbf{u}$, and are evaluated with the same procedures. We expand around $|k_z/k_s| \sim 0$, which gives

$$
\frac{e}{T_i} \frac{\omega - k_y v_{di}}{k_s c} \int e^{ik_z z} e^{ik_z z} \frac{\delta \mathbf{A}(k_z') \cdot \mathbf{u}}{u_s - \xi_{ics}} \sum_{n=0}^3 \left( \frac{-k_z u_z/k_s - k'' u_z/k_s}{u_s - \xi_{ics}} \right)^n f_i(k_z', u) \, dk_z' \, dk''_z \, d^3 u.
$$

(2.17)

By extending the summation in Eq(2.17) to include $n = 3$, we take advantage of extensive cancellation through the Coulomb gauge condition, $\delta \mathbf{A} \cdot \mathbf{k} = 0$. Integrating the resulting expression yields

$$
\frac{e}{2T_i} \frac{\omega - k_y v_{di}}{c k_s} Z'(\xi_{ics}) \delta A_z(z) \frac{i}{k_s} n_{cs}'(z),
$$

(2.18)
where \( n'_{cs}(z) \) signifies \( \partial_z n_{cs}(z) \). Collecting the terms given in Eq(2.16) and Eq(2.18) gives the total perturbed density of current sheet ions;

\[
\begin{align*}
\delta n_{ics}(z) &= \frac{e}{2T_i} Z' (\xi_{ics}) \left( \delta \phi(z) n_{cs}(z) - \frac{v_{di}}{c} \delta A_y(z) n_{cs}(z) \right) \\
&+ \frac{1}{k_s^2} \delta \phi'(z) n'_{cs}(z) - \frac{1}{k_s^2} \frac{v_{di}}{c} \delta A'_y(z) n'_{cs}(z) \\
&+ \frac{e}{2T_i} \left( \frac{1}{2} \xi_{ics} Z''(\xi_{ics}) \frac{\partial^2}{\partial z^2} \delta \phi(z) n_{cs}(z) - \frac{v_{di}}{c} \delta A_y(z) n_{cs}(z) \right) \\
&+ \frac{v_i}{c} \xi_{ics} Z'(\xi_{ics}) \delta A_z(z) \left( \frac{i}{k_s} n'_{cs}(z) \right),
\end{align*}
\]

(2.19)

Perturbed Current Density of Current Sheet Ions

Analyses of LHDIs typically ignore the perturbed ion current density, but it was shown that this current density can affect the growth rate of LHDIs when \( k_\parallel \neq 0 \) through electromagnetic effects.[9] Such modes are the subject of this work, so we extend our ion response to include the perturbed current density. We begin with the current sheet ion population and define the perturbed current density as

\[
\delta J_{ics}(x) = \int e v \delta f_{ics}(x, v) d^3v = \hat{y} e v_{di} \delta n_{cs}(z) + \int e u v_i \delta f_{ics}(z, u) d^3u.
\]

(2.20)

Substituting the perturbed distribution function given in Eq(2.13) into this expression yields

\[
\begin{align*}
\delta J_{ics}(x) &= \hat{y} e v_{di} \delta n_{cs}(z) \\
- &\frac{e^2}{T_i} v_i \int e^{ik_z z} \frac{u (k' \cdot u) v_i}{-\omega + k_y v_{di} + k \cdot u v_i} \left( \delta \phi(k'_z) - \frac{v_{di}}{c} \delta A_y(k'_z) \right) f_i(k_z - k'_z, u) dk'_z dk_z \\
+ &\frac{e^2}{T_i} v_i \int e^{ik_z z} \frac{u (\delta \mathbf{A}(k'_z) \cdot u) v_i}{-\omega + k_y v_{di} + k \cdot u v_i} \frac{\omega - k_y v_{di}}{c} f_i(k_z - k'_z, u) dk'_z dk_z.
\end{align*}
\]

(2.21)
As with the perturbed density, we begin with the terms proportional to \( \delta \phi(k'_z) - \delta A_y(k'_z) v_{di} / c \) in Eq(2.21) and expand to second order in \(|k_z v_z / k_s v_s|\). These terms become

\[
e^2 v_i \int e^{ik'_z z} e^{ik''_z z} \frac{u(u_s + k'_z u_z / k_s)}{u_s - \xi_{ics}} \sum_{n=0}^{2} \left( \frac{-k'_z u_z / k_s - k''_z u_z / k_s}{u_s - \xi_{ics}} \right)^n (-\delta \phi(k'_z) f_i(k'_z, u) + \frac{v_{di}}{c} \delta A_y(k'_z) f_i(k'_z, u) \right) dk'_z dk''_z d^3 u. \tag{2.22}
\]

We evaluate the integral in Eq(2.22) to arrive at the following expression;

\[
e^2 v_i \frac{2}{T_i} \left( \xi_{ics} Z'(\xi_{ics}) \delta \phi(z) n_{cs}(z) + \frac{1}{2} Z''(\xi_{ics}) \frac{1}{k_s^2} \partial_z n_{cs}(z) \partial_z \delta \phi(z) \right) \hat{k}_s + e^2 v_i \frac{2}{T_i} \left( -\frac{1}{2} Z''(\xi_{ics}) \frac{\partial^2}{k_s^2} \delta \phi(z) n_{cs}(z) - \frac{1}{4} \xi_{ics} Z'''(\xi_{ics}) \frac{\partial^2}{k_s^2} \delta \phi(z) n_{cs}(z) \right) \hat{k}_s + e^2 v_i \frac{2}{T_i} \left( i Z(\xi_{ics}) \frac{1}{k_s} \delta \phi'(z) n_{cs}(z) + \frac{i}{2} Z''(\xi_{ics}) \frac{\partial}{k_s} \delta \phi(z) n_{cs}(z) \right) \hat{z} - e^2 v_i \frac{v_{di}}{2T_i c} \left( \xi_{ics} Z'(\xi_{ics}) \delta A_y(z) n_{cs}(z) + \frac{1}{2} Z''(\xi_{ics}) \frac{1}{k_s^2} \partial_z n_{cs}(z) \partial_z \delta A_y(z) \right) \hat{k}_s - e^2 v_i \frac{v_{di}}{2T_i c} \left( -\frac{1}{2} Z''(\xi_{ics}) \frac{\partial^2}{k_s^2} \delta A_y(z) n_{cs}(z) - \frac{1}{4} \xi_{ics} Z'''(\xi_{ics}) \frac{\partial^2}{k_s^2} \delta A_y(z) n_{cs}(z) \right) \hat{k}_s - e^2 v_i \frac{v_{di}}{2T_i c} \left( i Z(\xi_{ics}) \frac{1}{k_s} \delta A'_y(z) n_{cs}(z) + \frac{i}{2} Z''(\xi_{ics}) \frac{\partial}{k_s} \delta A_y(z) n_{cs}(z) \right) \hat{z}. \tag{2.23}
\]

To evaluate the terms proportional to \( u \left( \delta \dot{A}(k'_z) \cdot u \right) \) in Eq(2.21), we define a right-handed coordinate system \((\hat{k}_s, \hat{z}, \hat{c})\), where \( k_s = k_s \hat{k}_s \). After expanding to third order in \(|k_z v_z / k_s v_s|\), these terms become

\[
e^2 v_i \omega - k_y v_{di} \int e^{ik'_z z} e^{ik''_z z} \frac{u(u_s + k'_z u_z / k_s)}{u_s - \xi_{ics}} \sum_{n=0}^{3} \left( \frac{-k'_z u_z / k_s - k''_z u_z / k_s}{u_s - \xi_{ics}} \right)^n f_i(k'_z, u) dk'_z dk''_z d^3 u. \tag{2.24}
\]
Integrating the expression given in Eq(2.24) gives

\[
\frac{e^2 v_i \omega - k_y v_{di}}{2T_i} \left( Z(\xi_{ics}) \frac{i \partial_z}{k_s} \delta A_z(z) n_{cs}(z) + \xi_{ics} Z'(\xi_{ics}) \frac{i}{k_s} \delta A_z(z) n'_{cs}(z) \right) \hat{k}_s \\
+ \frac{e^2 v_i \omega - k_y v_{di}}{2T_i} \frac{1}{k_s} \left( Z(\xi_{ics}) \delta A_z(z) n_{cs}(z) - \frac{1}{2} Z''(\xi_{ics}) \frac{1}{k_s^2} \delta A'_z(z) n'_{cs}(z) \right) \hat{Z}_s \\
- \frac{1}{4} Z''(\xi_{ics}) \frac{\partial^2}{k_s^2} \delta A_z(z) n_{cs}(z) \right) \hat{Z}_s \\
+ \frac{e^2 v_i \omega - k_y v_{di}}{2T_i} \left( Z(\xi_{ics}) \delta A_z(z) n_{cs}(z) - \frac{1}{4} Z''(\xi_{ics}) \frac{\partial^2}{k_s^2} \delta A_z(z) n_{cs}(z) \right) \hat{c} 
\]  
(2.25)

Combining the results given in Eq(2.23) and Eq(2.25) gives the perturbed current density of the current sheet ions,

\[
\hat{J}_{ics}(z) = e v_{di} \delta n_{ics}(z) \hat{y} + \frac{e^2 v_i v_{di}}{2T_i} \xi_{ics} \left( Z(\xi_{ics}) \delta A_z(z) n_{cs}(z) \right) \\
- \frac{1}{4} Z''(\xi_{ics}) \frac{\partial^2}{k_s^2} \delta A_z(z) n_{cs}(z) \right) \hat{c} + \frac{e^2 v_i v_{di}}{2T_i} \left( \xi_{ics} Z'(\xi_{ics}) \left( \delta \phi(z) - \frac{v_{di}}{c} \delta A_y(z) \right) n_{cs}(z) \right) \\
- \frac{1}{2} Z''(\xi_{ics}) \frac{1}{k_s} \left( \delta \phi'(z) - \frac{v_{di}}{c} \delta A'_y(z) n'_{cs}(z) \right) \hat{k}_s \\
+ \frac{e^2 v_i v_{di}}{2T_i} \left( Z(\xi_{ics}) \frac{i}{k_s} \delta A_z(z) n_{cs}(z) + \xi_{ics} Z'(\xi_{ics}) \frac{i}{k_s} \delta A_z(z) n'_{cs}(z) \right) \hat{z} \\
+ \frac{e^2 v_i v_{di}}{2T_i} \frac{1}{c} \xi_{ics} \left( Z(\xi_{ics}) \frac{i}{k_s} \delta A_z(z) n_{cs}(z) - \frac{v_{di}}{c} \delta A'_y(z) n_{cs}(z) \right) \hat{Z}_s \\
+ \frac{e^2 v_i v_{di}}{2T_i} \frac{1}{2} Z''(\xi_{ics}) \frac{i}{k_s} \left( \delta \phi(z) n_{cs}(z) - \frac{v_{di}}{c} \delta A_y(z) n_{cs}(z) \right) \hat{Z}_s \\
- \frac{1}{4} Z''(\xi_{ics}) \frac{\partial^2}{k_s^2} \delta A_z(z) n_{cs}(z) \right) \hat{Z}_s 
\]  
(2.26)
2.2.2 Background Ion Response

The background ion response can be found by setting $v_{di} = 0$, and $\partial_z n_{ib} = \partial^2_z n_{ib} = 0$ in the current sheet ion response. The perturbed density of background ions is given by

$$\delta n_{ib}(z) = \frac{e}{2T_i} \left( Z'(\xi_{ib})\delta \phi(z)n_b + \frac{1}{2} \xi_i Z''(\xi_{ib}) \frac{1}{k_s^2} \delta \phi''(z)n_b \right),$$

(2.27)

where $\xi_{ib} = \omega/k_s v_i$. The perturbed current density of the background ions is

$$\delta J_{ib}(z) = \frac{e^2 v_i}{2T_i} \left( Z(\xi_{ib})\delta A_c(z)n_b - \frac{1}{4} Z''(\xi_{ib}) \frac{1}{k_s^2} \delta A''_c(z)n_b \right) \hat{\epsilon}$$

$$+ \frac{e^2 v_i}{2T_i} \left( \xi_{ib} Z'(\xi_{ib})\delta \phi(z)n_b - \frac{1}{4} \xi_{ib} Z''(\xi_{ib}) \frac{1}{k_s^2} \delta \phi''(z)n_b + \frac{v_i}{c} \xi_{ib} Z(\xi_{ib}) \frac{i}{k_s} \delta A'_c(z)n_b \right) \hat{k}_s$$

$$+ \frac{e^2 v_i}{2T_i} \left( -\xi_{ib} Z'(\xi_{ib}) \frac{i}{k_s} \delta \phi'(z)n_b + \frac{v_i}{c} \xi_{ib} \left( Z(\xi_{ib})\delta A_c(z)n_b - \frac{1}{4} Z''(\xi_{ib}) \frac{1}{k_s^2} \delta A''_c(z)n_b \right) \right) \hat{z}$$

(2.28)

2.3 Gyrokinetic Electron Response

We adopt the gyrokinetic equations to describe the electron dynamics, since $|\omega| \ll |\omega_{ce}| \ll 1$, and $|k_{\parallel}\rho_e| \ll |k_{\perp}\rho_e| \sim 1$, where $\rho_e$ is the electron Larmor radius[1, 44, 48]. First, we need to transform from the particle phase space coordinates, $(x, v)$, to the gyro-center coordinates, $(X, v_{\parallel}, \mu, \alpha, \sigma)$, where $X = x + v_{\perp} \times \hat{b}/\omega_{ce}$, $\mu = v_{\perp}^2/2B$, $\alpha$ is the gyrophase angle, and $\sigma$ is the sign of $v_{\parallel}$. It was shown that, ignoring $O(\rho_e^2/L^2)$ corrections, the electron current sheet guiding-center equilibrium distribution function is given by[51]

$$F_{ecc}(X, V) \simeq \frac{n_0}{\pi^{3/2} v_e^3} \text{Exp}[-\frac{2\mu B + (v_{\parallel} - v_{de})^2}{v_e^2} - \frac{v_{de} e A_y(Z)}{T_e c}].$$

(2.29)

Here $v_{de} = v_{de} B_y/B(Z)$, and we use capitalized variables for expressions in guiding-center coordinates. The background electron guiding-center equilibrium distribution, meanwhile,
The gyrokinetic ordering $|\omega/\omega_{ce}| \sim \rho_e/L \ll 1$ requires a sufficiently strong magnetic field for a given electron temperature. Near the current sheet center, this ordering relies on a sufficient guide field strength. We will address the validity of this ordering a posteriori.

For the electron perturbed responses, we use the linear electromagnetic gyrokinetic equation. [1, 44, 48] Defining local coordinates $(\hat{b}, \hat{z}, \hat{\zeta})$, where $\hat{b}$ is parallel to the local magnetic field, $\hat{z}$ is along the variation of the equilibrium, and $\hat{\zeta} = \hat{b} \times \hat{z}$, we have

$$\dot{\delta f}_e = -\frac{e}{m_e} \left( \delta \phi \left( \frac{1}{v_\parallel} \frac{\partial F_e}{\partial v_\parallel} + \frac{1}{B} \frac{\partial F_e}{\partial \mu} \right) - \delta \psi \frac{k_\parallel v_\parallel}{\omega} \frac{1}{B} \frac{\partial F_e}{\partial \mu} \right) + \delta G_e e^{-i\lambda \sin(\alpha)}$$

$$\dot{\delta G}_e = -\frac{e}{m_e} \frac{1}{B} \frac{\partial F_e}{\partial \mu} \left( J_0(\lambda) \left( -\delta \phi + \frac{k_\parallel v_\parallel}{\omega} \delta \psi \right) - J_1(\lambda) \frac{v_\perp}{ck_\perp} \delta B_\parallel \right) + \dot{H}_e$$

$$\dot{\delta H}_e(\omega - \omega_d - k_\parallel v_\parallel) = -\frac{e}{m_e} \omega \frac{1}{v_\parallel} \frac{\partial F_e}{\partial v_\parallel} + \frac{k_\perp \times \hat{b}}{\omega_{ce}} \cdot \nabla F_e$$

$$\times \left( J_0(\lambda) \left( -\delta \phi + \frac{k_\parallel v_\parallel}{\omega} \delta \psi \right) - J_1(\lambda) \frac{v_\perp}{ck_\perp} \delta B_\parallel \right)$$

Here $\lambda = k_\perp v_\perp/\omega_{ce}$, $\omega_d = k_\perp \cdot v_d$, $L_k = k \cdot (\hat{b} \times v)/\omega_{ce}$, $v_d$ is the $\nabla B$ drift velocity, and the ordering $|k_\parallel| \ll |k_\perp|$ permits the approximation

$$\delta A \simeq \delta A_\parallel \hat{b} + \delta A_\perp (\hat{b} \times \hat{k}_\perp) = \frac{ck_\parallel}{\omega} \delta \psi \hat{b} - i \delta B_\parallel \frac{\hat{b} \times k_\perp}{k^2_\perp}.$$  (2.34)

We will now derive the perturbed distribution functions for the current sheet and background electrons separately, beginning with the current sheet electrons.
2.3.1 Perturbed Distribution of the Current Sheet Electrons

Using the equilibrium distribution function given in Eq(2.29), we evaluate the following expressions;

\[
\frac{1}{v_\parallel} \frac{\partial F_{ecs}}{\partial v_\parallel} = -\frac{2}{v_e^2} \left( 1 - \frac{v_{de||}}{v_\parallel} \right) F_{ecs}, \quad \frac{1}{B} \frac{\partial F_{ecs}}{\partial \mu} = -\frac{2}{v_e^2} \frac{v_{de||}}{v_\parallel} F_{ecs}. \tag{2.35}
\]

\[
\nabla_x F_{ecs} = -\frac{2}{v_e^2} \left( \frac{\omega_d \omega_{ce} v_{de||}}{k_\parallel} - (v_\parallel - v_{de||}) \partial_Z v_{de||} + \omega_{ce} v_{de} \frac{B_x}{B} \right) F_{ecs} \hat{z}. \tag{2.36}
\]

We can approximate the third term in Eq(2.36) as \((v_\parallel - v_{de||}) \partial_Z v_{de||} \sim v_{de||} B_g/LB\), and neglect it relative to the fourth term, \(v_\parallel B_g/L \omega_{ce} B_x \sim \rho_e B_g/L B_x \ll 1\), for ion-scale current sheets with moderate guide fields. Using the expressions given in Eq(2.35) and Eq(2.36), the perturbed distribution function given in Eq(2.31) becomes

\[
\hat{\delta} f_{ecs} = \frac{e}{T_e} \left( \hat{\delta} \phi - \hat{\delta} \psi \frac{k_\parallel v_{de||}}{\omega} \right) F_{ecs} + \hat{\delta} G_{ecs} e^{-i \lambda \sin(\alpha)}. \tag{2.37}
\]

We simplify the expression for \(\hat{\delta} G_{ecs}\) given in Eq(2.32) by noting that

\[
-\frac{e}{m_e} \frac{1}{B} \frac{\partial F_{ecs}}{\partial \mu} = \frac{e}{T_e} \frac{v_{de||}}{v_\parallel} F_{ecs}, \tag{2.38}
\]

and organizing the terms in \(\hat{\delta} H_{ecs}\) as follows;

\[
-\frac{e}{m_e} \left( \omega \frac{1}{v_\parallel} \frac{\partial F_{ecs}}{\partial \mu} + k_\parallel \frac{\mu}{\omega_{ce}} \cdot \nabla_x F_{ecs} \right) = \frac{e}{T_e} \left( - (\omega - \omega_d) \frac{v_{de||}}{v_\parallel} + \omega + k_\parallel v_{de} \frac{B_x}{B} \right) F_{ecs}
\]

\[
= \frac{e}{T_e} \left( - (\omega - \omega_d - k_{\parallel v_\parallel}) \frac{v_{de||}}{v_\parallel} + \omega - k_y v_{de} \right) F_{ecs}. \tag{2.39}
\]
The resulting perturbed distribution function is

\[
\hat{f}_{ecs} = \frac{e}{T_e} \left( \hat{\phi} - \frac{k_v}{\omega} \hat{\psi} \right) F_{ecs} + \hat{G}_{ecs} e^{-i\lambda \sin(\alpha)} \tag{2.40}
\]

\[
\hat{G}_{ecs} = \frac{e}{T_e} \left( \frac{\omega - k_v v_{de}}{\omega - \omega_d - k_v \|v\|} \omega \hat{\psi} \right) F_{ecs}, \tag{2.41}
\]

The operator \( e^{-i\lambda \sin(\alpha)} \) in Eq(2.40) transforms \( \hat{G}_{ecs} \) from gyro-center coordinates to particle phase space coordinates. The equilibrium distribution function in the adiabatic terms needs to be similarly transformed as follows;

\[
F_{ecs}(Z) = \frac{n_0}{\pi^{3/2} v^3_e} \exp\left[ -\frac{2\mu B + (v_\| - v_{de\|})^2}{v^2_e} - \frac{v_{de\|} A_y(Z)}{T_e c} \right] \nonumber
\]

\[
\simeq \frac{n_0}{\pi^{3/2} v^3_e} \exp\left[ -\frac{v_x^2 + v_y^2 + (v_y - v_{de\|})^2}{v^2_e} - \frac{v_{de\|} A_y(z)}{T_e c} - \frac{v_{de\|} A''_y(z)}{2v^2_{ce}} \right] \nonumber
\]

\[
= f_{ecs}(z) \exp\left[ -\frac{v_{de\|} A''_y(z)}{T_e c} \frac{v^2_x}{2v^2_{ce}} \right] \nonumber
\]

\[
\simeq f_{ecs}(z) \left( 1 - \frac{v_{de\|} A''_y(z)}{T_e c} \frac{v^2_x}{2v^2_{ce}} \right) \sim f_{ecs}(z) \left( 1 - \frac{v^2_x}{v^2_e} \frac{v_{de\|}}{v^2_{ce} L} \right). \tag{2.42}
\]

For our parameters, \( v_{de\|}/\omega_{ce} L \sim v^2_{de\|}/v^2_e \ll 1 \), so we ignore the transformation of the adiabatic terms.

### 2.3.2 Perturbed Density of Current Sheet Electrons

We obtain the perturbed density of the current sheet electrons by integrating the perturbed distribution function, given in Eq(2.40) and Eq(2.41), over velocity space. We begin the integration with the adiabatic terms,

\[
\hat{n}_{ecs} = \int \hat{f}_{ecs} d^3 v = \frac{e n_{ecs}}{T_e} \left( \hat{\phi} - \frac{k_v}{\omega} \hat{\psi} \right) + \int \hat{G}_{ecs} e^{-i\lambda \sin(\alpha)} v_\perp dv_\perp dv_\| d\alpha, \tag{2.43}
\]
where \( n_{cs} \) is the equilibrium current sheet density. Integrating \( \delta G_{ecs} e^{-i\lambda\sin(\alpha)} \) over \( v_\parallel \) and gyro angle yields

\[
\int \delta G_{ecs} e^{-i\lambda\sin(\alpha)} d^3v
= \int J_0(\lambda) \frac{en_{cs}}{T_e} \frac{\omega - k_\parallel v_{de}}{k_\parallel v_e} Z(\xi_{ecs} + \Omega_d u_\perp^2) \left( J_0(\lambda) \delta \psi + J_1(\lambda) \frac{v_\perp}{ck_\perp} \delta B_\parallel \right.
- J_0(\lambda) \frac{k_\parallel v_{de}}{\omega} \delta \psi \left. \right) \text{Exp}[-u_\perp^2] 2u_\perp du_\perp
+ \int J_0(\lambda) \frac{en_{cs}}{T_e} \frac{\omega - k_\parallel v_{de} - Z'(\xi_{ecs} + \Omega_d u_\perp^2)}{k_\parallel v_e} \left( -J_0(\lambda) \frac{k_\parallel v_e}{\omega} \delta \psi \right) \text{Exp}[-u_\perp^2] 2u_\perp du_\perp,
\]

(2.44)

where we define

\[
\frac{\omega - k_\parallel v_{de} - \omega_d}{k_\parallel v_e} = \frac{\omega - k_\parallel v_{de}}{k_\parallel v_e} + \Omega_d u_\perp^2 = \xi_{ecs} + \Omega_d u_\perp^2, \quad \text{and} \quad u_\perp = \frac{v_\perp}{v_e}.
\]

(2.45)

To integrate over the \( u_\perp \) dependence in the argument of the plasma dispersion function, we assume \( |\Omega_d/\xi_{ecs}| \ll 1 \), and expand to first order. Because the scale length of the equilibrium inhomogeneity is much larger than the wave scale, \( L_{ks} \sim L/\rho_e \gg 1 \), we approximate \( k_\perp \approx k_\zeta - \partial^2_z/2k_\zeta \) in Eq(2.44) as follows:

\[
J(\lambda) = J(k_\perp \rho_e) \approx J(\lambda_0 u_\perp) + J'(\lambda_0 u_\perp) \lambda_1 \partial^2_z u_\perp,
\]

where \( \lambda_0 = \frac{\bar{k}_\zeta B_0}{B(z)} \), \( \lambda_1 = -\frac{\rho^2_e B_0}{2k_\zeta B(z)} \), and \( \bar{k}_\zeta = k_\zeta \rho_e \).

(2.46)
After substituting the expressions in Eq(2.46) and expanding about $|\Omega_d/\Omega_{ecs}| = 0$, Eq(2.44) becomes

$$\int \delta G_{ecs} e^{-i\lambda \sin(\alpha)} d^3v$$

$$\approx \frac{e}{T_e} \int \left[ J_0 - J_1 \lambda_1 \partial_Z^2 u_\perp \right] n_{cs} \Lambda \left( Z(\xi_{ecs}) \left[ J_0 - J_1 \lambda_1 \partial_Z^2 u_\perp \right] \hat{\delta} \phi \right.$$  

$$+ Z'(\xi_{ecs}) \left[ J_1 + J'_1 \lambda_1 \partial_Z^2 u_\perp \right] \delta a_\perp + \frac{\nu_{de}||}{v_e} Z(\xi_{ecs}) \left[ J_0 - J_1 \lambda_1 \partial_Z^2 u_\perp \right] \hat{\delta} a_\parallel$$  

$$- \frac{1}{2} Z''(\xi_{ecs}) \left[ J_0 - J_1 \lambda_1 \partial_Z^2 u_\perp \right] \delta a_\perp \right) \exp\left[-u_\perp^2/2u_\perp \right] d u_\perp$$

$$+ \frac{e}{T_e} \int \left[ J_0 - J_1 \lambda_1 \partial_Z^2 u_\perp \right] n_{cs} \Lambda \Omega_d \left( Z'(\xi_{ecs}) \left[ J_0 - J_1 \lambda_1 \partial_Z^2 u_\perp \right] \hat{\delta} \phi \right.$$  

$$+ Z'(\xi_{ecs}) \left[ J_1 + J'_1 \lambda_1 \partial_Z^2 u_\perp \right] \delta a_\perp + \frac{\nu_{de}||}{v_e} Z'(\xi_{ecs}) \left[ J_0 - J_1 \lambda_1 \partial_Z^2 u_\perp \right] \hat{\delta} a_\parallel$$  

$$- \frac{1}{2} Z''(\xi_{ecs}) \left[ J_0 - J_1 \lambda_1 \partial_Z^2 u_\perp \right] \delta a_\perp \right) \exp\left[-u_\perp^2/2u_\perp \right] d u_\perp,$$  

(2.47)

where we define

$$\hat{\delta} a_\parallel = - \frac{k_\parallel v_e}{\omega} \hat{\delta} \psi, \quad \hat{\delta} a_\perp = \frac{v_e}{k_\perp c} \hat{\delta} B_\parallel, \quad \text{and} \quad \Lambda = \frac{\omega - k_y v_{de}}{k_\parallel v_e}.$$  

(2.48)

Here the Bessel’s functions at the left end of the integrals in Eq(2.47) execute the transformation from gyro-center coordinates to particle phase-space coordinates. The remaining Bessel’s functions result from averaging the perturbed fields over gyroangle. So the coordinate transformation operator $\partial_Z$ acts on $n_{cs}$, $\Lambda$, $\xi_{ecs}$, $\lambda_0$, and the perturbed fields. Evaluating the integrals in Eq(2.47) we arrive at the following expression for the perturbed density of current sheet electrons;

$$\hat{\delta} n_{ecs} = \chi_{\phi 0} \hat{\delta} \phi + \chi_{\phi 1} \hat{\delta} \phi' + \chi_{\phi 2} \hat{\delta} \phi''$$

$$+ \chi_{a 0} \hat{\delta} a_\parallel + \chi_{a 1} \hat{\delta} a_\parallel' + \chi_{a 2} \hat{\delta} a_\parallel''$$

$$+ \chi_{a 0} \hat{\delta} a_\perp + \chi_{a 1} \hat{\delta} a_\perp' + \chi_{a 2} \hat{\delta} a_\perp'',$$  

(2.49)
where

\[
\begin{align*}
\chi_{\phi 0} &= \frac{e_{\text{en}}}{T_e} \left( 1 + \Lambda Z(\xi_{\text{ecs}}) \Gamma_0 + \Lambda Z'(\xi_{\text{ecs}}) \Omega_d \Gamma_{01} \right), \\
\chi_{\phi 1} &= -2 \lambda_1 \frac{e}{T_e} \left( n_{\text{cs}} \Lambda Z(\xi_{\text{ecs}}) \right)' \Gamma_1 \\
&+ (n_{\text{cs}} \Lambda Z'(\xi_{\text{ecs}}) \Omega_d)' \Gamma_{11}, \\
\chi_{\phi 2} &= -2 \lambda_1 \frac{e_{\text{en}}}{T_e} \left( \Lambda Z(\xi_{\text{ecs}}) \Gamma_1 + \Lambda Z'(\xi_{\text{ecs}}) \Omega_d \Gamma_{11} \right), \\
\chi_{a_{0}} &= \frac{e_{\text{en}}}{T_e} \left( \frac{v_{\text{de}\parallel}}{v_e} + \Lambda Z(\xi_{\text{ecs}}) \Gamma_0 \frac{v_{\text{de}\parallel}}{v_e} + \Lambda Z(\xi_{\text{ecs}}) \Gamma_0 + \Lambda Z'(\xi_{\text{ecs}}) \Omega_d \Gamma_{01} \frac{v_{\text{de}\parallel}}{v_e} \\
&+ \Lambda Z'(\xi_{\text{ecs}}) \Omega_d \Gamma_{01} \right), \\
\chi_{a_{1}} &= -2 \lambda_1 \frac{e_{\text{en}}}{T_e} \left( \Lambda Z(\xi_{\text{ecs}}) \frac{v_{\text{de}\parallel}}{v_e} \right)' \Gamma_1 \\
&+ \left( n_{\text{cs}} \Lambda Z'(\xi_{\text{ecs}}) \Omega_d \frac{v_{\text{de}\parallel}}{v_e} \right)' \Gamma_{11} + (n_{\text{cs}} \Lambda Z'(\xi_{\text{ecs}}))' \Gamma_1 \\
&+ (n_{\text{cs}} \Lambda Z'(\xi_{\text{ecs}}) \Omega_d)' \Gamma_{11} + 2 \lambda_1 \frac{e_{\text{en}}}{T_e} \left( \Lambda Z(\xi_{\text{ecs}}) \frac{v_{\text{de}\parallel}}{v_e} \lambda_0' \Gamma_2 + \Lambda Z'(\xi_{\text{ecs}}) \Omega_d \frac{v_{\text{de}\parallel}}{v_e} \lambda_0' \Gamma_{21} \\
&+ \Lambda Z'(\xi_{\text{ecs}}) \lambda_0' \Gamma_2 + \Lambda Z'(\xi_{\text{ecs}}) \Omega_d \lambda_0' \Gamma_{21} \right), \\
\chi_{a_{2}} &= -2 \lambda_1 \frac{e_{\text{en}}}{T_e} \left( \Lambda Z(\xi_{\text{ecs}}) \Gamma_1 \frac{v_{\text{de}\parallel}}{v_e} \right) \\
&+ \Lambda Z(\xi_{\text{ecs}}) \Gamma_1 + \Lambda Z'(\xi_{\text{ecs}}) \Omega_d \Gamma_{11} \frac{v_{\text{de}\parallel}}{v_e} + \Lambda Z'(\xi_{\text{ecs}}) \Omega_d \Gamma_{11} \right), \\
\chi_{a_{3}} &= \frac{e_{\text{en}}}{T_e} \left( \Lambda Z(\xi_{\text{ecs}}) \Gamma_1 \frac{v_{\text{de\parallel}}}{v_e} + \Lambda Z'(\xi_{\text{ecs}}) \Omega_d \Gamma_{11} \right), \\
\chi_{a_{4}} &= -2 \lambda_1 \frac{e_{\text{en}}}{T_e} \left( \Lambda Z(\xi_{\text{ecs}}) \Gamma_1 \frac{v_{\text{de\parallel}}}{v_e} \right)' \Gamma_2 \\
&+ (n_{\text{cs}} \Lambda Z'(\xi_{\text{ecs}}) \Omega_d)' \Gamma_{21} + 2 \lambda_1 \frac{e_{\text{en}}}{T_e} \left( \Lambda Z(\xi_{\text{ecs}}) \lambda_0' \Gamma_4 + \Lambda Z'(\xi_{\text{ecs}}) \Omega_d \lambda_0' \Gamma_{41} \right), \\
\text{and} \quad \chi_{a_{5}} &= -\lambda_1 \frac{e_{\text{en}}}{T_e} \left( \Lambda Z(\xi_{\text{ecs}}) (\Gamma_2 - \Gamma_3) + \Lambda Z'(\xi_{\text{ecs}}) \Omega_d (\Gamma_{21} - \Gamma_{31}) \right). \quad (2.50)
\end{align*}
\]
Here we define

\[
Z_1(x) = -\frac{1}{2} Z'(x), \quad b = \frac{\lambda_0}{2}, \quad \Gamma_0 = e^{-b} I_0(b), \quad \Gamma_{01} = \frac{1}{2} e^{-b} ((2 - 2b) I_0(b) + 2b I_1(b)),
\]

\[
\Gamma_1 = \frac{\lambda_0}{2} e^{-b} (I_0(b) - I_1(b)), \quad \Gamma_{11} = \frac{\lambda_0}{2} e^{-b} ((2 - 2b) I_0(b) + (2b - 1) I_1(b)),
\]

\[
\Gamma_2 = b e^{-b} (I_0(b) - I_1(b)), \quad \Gamma_{21} = b e^{-b} ((3 - 2b) I_0(b) + (2b - 2) I_1(b)),
\]

\[
\Gamma_3 = \frac{1}{2} e^{-b} ((1 - 2b) I_0(b) + (1 + 2b) I_1(b)),
\]

\[
\Gamma_{31} = \frac{1}{2} e^{-b} ((2 - 6b + 4b^2) I_0(b) + (1 + 4b - 4b^2) I_1(b)),
\]

\[
\Gamma_4 = \frac{\lambda_0}{2} e^{-b} ((1 - 2b) I_0(b) + 2b I_1(b)),
\]

and \( \Gamma_{41} = \frac{\lambda_0}{4} e^{-b} ((6 - 18b + 8b^2) I_0(b) + (14b - 8b^2) I_1(b)). \) (2.51)

### 2.3.3 Perturbed Current Density of Current Sheet Electrons

We continue our derivation of the perturbed electron response with the perturbed current density of the current sheet electrons. We begin with the component of the perturbed current density along the equilibrium magnetic field, which is given by

\[
\hat{\delta} J_{ec\parallel} = -e \int v_{ec\parallel} \hat{\delta} f_{ec} \, d^3 v.
\] (2.52)

After transforming the parallel velocity to \( u_{\parallel} = (v_{\parallel} - v_{de\parallel})/v_e \), we evaluate the integral given in Eq(2.52) for the adiabatic terms in \( \hat{\delta} f_{ec} \), which yields

\[
\hat{\delta} J_{ec\parallel} = -e^2 n_{ecs} T_e v_{de\parallel} \left( \hat{\delta} \phi + \frac{v_{de\parallel}}{v_e} \hat{\delta} a_{\parallel} \right) - e v_{de\parallel} \hat{\delta} n_{ecs} - e v_e \int u_{\parallel} \hat{\delta} G_{ec} e^{-i \lambda \sin(\alpha)} \, d^3 u. \] (2.53)

The integration of \( \hat{\delta} G_{ec} \) in Eq(2.53) follows the process used to derive the perturbed density. We integrate over gyrophase and \( u_{\parallel} \) and expand the resulting expression to first order in
$\Omega_d/\xi_{ecs}$ and expand to second order in $\partial_Z/k_\zeta$:

$$
- e v_e \int u_\parallel \delta G_{ecs} e^{i \lambda \sin(\alpha)} v_\perp d v_\perp d v_\parallel d \alpha
$$

$$
\simeq \frac{e^2 v_e}{2T_e} \int [J_0 - J_1 \lambda_1 \partial^2_Z u_\perp] n_{cs} \Lambda \left( Z'(\xi_{ecs}) [J_0 - J_1 \lambda_1 \partial^2_Z u_\perp] \delta \phi 
+ Z'(\xi_{ecs}) [J_1 + J'_1 \lambda_1 \partial^2_Z u_\perp] \delta a_\perp + \frac{v_{de}}{v_e} Z'(\xi_{ecs}) [J_0 - J_1 \lambda_1 \partial^2_Z u_\perp] \delta a_\parallel 
+ \xi_{ecs} Z''(\xi_{ecs}) [J_0 - J_1 \lambda_1 \partial^2_Z u_\perp] \delta a_\parallel \right) \exp[-u_\perp^2] 2u_\perp du_\perp
$$

$$
+ \frac{e^2 v_e}{2T_e} \int [J_0 - J_1 \lambda_1 \partial^2_Z u_\perp] n_{cs} \Lambda \Omega_d \left( Z''(\xi_{ecs}) [J_0 - J_1 \lambda_1 \partial^2_Z u_\perp] \delta \phi 
+ Z''(\xi_{ecs}) [J_1 + J'_1 \lambda_1 \partial^2_Z u_\perp] \delta a_\perp + \frac{v_{de}}{v_e} Z''(\xi_{ecs}) [J_0 - J_1 \lambda_1 \partial^2_Z u_\perp] \delta a_\parallel 
+ (\xi_{ecs} Z''(\xi_{ecs}) + Z'(\xi_{ecs})) [J_0 - J_1 \lambda_1 \partial^2_Z u_\perp] \delta a_\parallel \right) \exp[-u_\perp^2] 2u_\perp^3 du_\perp, \tag{2.54}
$$

Evaluating the integrals in Eq(2.54) completes the derivation of the perturbed current density along the equilibrium magnetic field, which is given by

$$
\delta J_{ecs\parallel} = - e v_{de\parallel} \delta n_{ecs} + D_{\parallel 0} \delta \phi + D_{\parallel 1} \delta \phi' + D_{\parallel 2} \delta \phi''
+ D_{a_{\parallel 0}} \delta a_\parallel + D_{a_{\parallel 1}} \delta a_\parallel' + D_{a_{\parallel 2}} \delta a_\parallel''
+ D_{a_{\perp 0}} \delta a_\perp + D_{a_{\perp 1}} \delta a_\perp' + D_{a_{\perp 2}} \delta a_\perp'', \tag{2.55}
$$
where

\[
D_{\phi 0}^{\parallel} = -\frac{e^2 n_{cs}}{T_e} v_e \left( \frac{v_{de}}{v_e} + \Lambda Z_1(\xi_{e cs}) \Gamma_0 + \Lambda Z_1'(\xi_{e cs}) \Omega_d \Gamma_{01} \right),
\]

\[
D_{\phi 1}^{\parallel} = 2 \lambda_1 \frac{e^2}{T_e} v_e \left( (n_{cs} \Lambda Z_1(\xi_{e cs}))' \Gamma_1 + (n_{cs} \Lambda Z_1'(\xi_{e cs}) \Omega_d)' \Gamma_{11} \right)
- 2 \lambda_1 \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z_1(\xi_{e cs}) \lambda_0 \Gamma_2 + \Lambda Z_1'(\xi_{e cs}) \Omega_d \lambda_0 \Gamma_{21} \right),
\]

\[
D_{\phi 2}^{\parallel} = 2 \lambda_1 \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z_1(\xi_{e cs}) \Gamma_1 + \Lambda Z_1'(\xi_{e cs}) \Omega_d \Gamma_{11} \right),
\]

\[
D_{a, 0}^{\parallel} = -\frac{e^2 n_{cs}}{T_e} v_e \left( \frac{v_{de}}{v_e} + \Lambda Z_1(\xi_{e cs}) \Gamma_0 \right) + \left( n_{cs} \Lambda Z_1'(\xi_{e cs}) \Omega_d \frac{v_{de}}{v_e} \right)' \Gamma_{11} + \left( n_{cs} \Lambda Z_1'(\xi_{e cs}) \Omega_d \frac{v_{de}}{v_e} \right)' \Gamma_1
+ \left( n_{cs} \Lambda Z_1'(\xi_{e cs}) \Omega_d \frac{v_{de}}{v_e} \right)' \Gamma_1
\]

\[
D_{a, 1}^{\parallel} = 2 \lambda_1 \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z_1(\xi_{e cs}) \frac{v_{de}}{v_e} \right)' \Gamma_1 + \left( n_{cs} \Lambda Z_1'(\xi_{e cs}) \Omega_d \frac{v_{de}}{v_e} \right)' \Gamma_{11}
- 2 \lambda_1 \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z_1(\xi_{e cs}) \frac{v_{de}}{v_e} \lambda_0 \Gamma_2 \right) + \Lambda Z_1'(\xi_{e cs}) \Omega_d \lambda_0 \Gamma_{21}
+ \Lambda Z_2(\xi_{e cs}) \lambda_0 \Gamma_2 + \Lambda Z_2'(\xi_{e cs}) \Omega_d \lambda_0 \Gamma_{21} + \Lambda Z_2(\xi_{e cs}) \Gamma_1 + \Lambda Z_2'(\xi_{e cs}) \Omega_d \Gamma_{11}
\]

\[
D_{a, 2}^{\parallel} = 2 \lambda_1 \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z_1(\xi_{e cs}) \frac{v_{de}}{v_e} \right)' \Gamma_1 + \left( n_{cs} \Lambda Z_1'(\xi_{e cs}) \Omega_d \frac{v_{de}}{v_e} \right)' \Gamma_{11}
+ \Lambda Z_2(\xi_{e cs}) \Gamma_1 + \Lambda Z_2'(\xi_{e cs}) \Omega_d \Gamma_{11}
\]

\[
D_{a, 3}^{\parallel} = -\frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z_1(\xi_{e cs}) \Gamma_1 + \Lambda Z_1'(\xi_{e cs}) \Omega_d \Gamma_{11} \right),
\]

\[
D_{a, 1}^{\parallel} = 2 \lambda_1 \frac{e^2}{T_e} v_e \left( (n_{cs} \Lambda Z_1(\xi_{e cs}))' \Gamma_2 + (n_{cs} \Lambda Z_1'(\xi_{e cs}) \Omega_d)' \Gamma_{21} \right)
- 2 \lambda_1 \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z_1(\xi_{e cs}) \lambda_0 \Gamma_4 + \Lambda Z_1'(\xi_{e cs}) \Omega_d \lambda_0 \Gamma_{41} \right),
\]

\[
and \ D_{a, 2}^{\parallel} = \lambda_1 \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z_1(\xi_{e cs})(\Gamma_2 - \Gamma_3) + \Lambda Z_1'(\xi_{e cs}) \Omega_d (\Gamma_{21} - \Gamma_{31}) \right). \quad (2.56)
\]

Here we define

\[
Z_2(x) = -\frac{x}{2} Z'(x).
\] (2.57)

The components of the perturbed current density that are perpendicular to the equilibrium
magnetic field are defined as follows;

\[ \hat{\delta}J_{ecs\perp} = -e \int v_{\perp} \hat{\delta}f_{ecs} d^3v. \] (2.58)

The adiabatic terms in \( \hat{\delta}f_{ecs} \) vanish upon integration, and Eq(2.58) becomes

\[ \hat{\delta}J_{ecs\perp} = -e \int v_{\perp} \hat{\delta}G_{ecs} e^{-i\lambda \sin(\alpha)} d^3v = -e \int v_{\perp} (\cos(\alpha) \hat{k}_{\perp} + \sin(\alpha) \hat{k}_{\perp} \times \hat{b}) \hat{\delta}G_{ecs} e^{-i\lambda \sin(\alpha)} d^3v. \] (2.59)

Integration over gyroangle eliminates the \( \hat{k}_{\perp} \) component of Eq(2.59) leaving

\[ \hat{\delta}J_{ecs\perp} \cdot (\hat{k}_{\perp} \times \hat{b}) = \hat{\delta}J_{ecs\perp} = -e \int v_{\perp} \sin(\alpha) \hat{\delta}G_{ecs} e^{-i\lambda \sin(\alpha)} v_{\perp} dv_{\perp} dv_{\parallel} d\alpha \]

\[ = ie \int J_1 \hat{\delta}G_{ecs} 2\pi v_{\perp}^2 dv_{\perp} dv_{\parallel}. \] (2.60)

Integrating over \( v_{\parallel} \) and expanding to first order in \( \Omega_d/\xi_{ecs} \) and second order in \( \partial Z/k_\xi \) yields

\[ i e^2 v_e \int \left[ J_1 + J'_1 \lambda_1 \partial_z^2 u_{\perp} \right] \left( n_{cs} \Lambda \left\{ Z(\xi_e) + Z'(\xi_e) \Omega_d u_{\perp}^2 \right\} \left[ J_0 - J_1 \lambda_1 \partial_z^2 u_{\perp} \right] \delta \phi \]

\[ + n_{cs} \Lambda \left\{ -Z'(\xi_e) - Z''(\xi_e) \Omega_d u_{\perp}^2 \right\} \left[ J_0 - J_1 \lambda_1 \partial_z^2 u_{\perp} \right] \delta a_{\parallel} \]

\[ + n_{cs} \Lambda \left( \frac{v_{de}}{v_e} \right) \left( Z(\xi_e) + Z'(\xi_e) \Omega_d u_{\perp}^2 \right) \left[ J_0 - J_1 \lambda_1 \partial_z^2 u_{\perp} \right] \delta a_{\parallel} \]

\[ + n_{cs} \Lambda \left( Z(\xi_e) + Z'(\xi_e) \Omega_d u_{\perp}^2 \right) \left[ J_1 u_{\perp} + J'_1 \lambda_1 \partial_z^2 u_{\perp}^2 \right] \delta a_{\perp} \] \[ \text{Exp}\left[ -u_{\perp}^2 \right] 2u_{\perp}^2 u_{\perp} \] (2.61)

Evaluating the integrals in Eq(2.61) gives

\[ \hat{\delta}J_{ecs\perp} = D_{\phi_0}^{\perp} \hat{\delta}\phi + D_{\phi_1}^{\perp} \hat{\delta}\phi' + D_{\phi_2}^{\perp} \hat{\delta}\phi'' \]

\[ + D_{a_{\parallel}}^{\perp} \hat{\delta}a_{\parallel} + D_{a_{\perp}}^{\perp} \hat{\delta}a_{\parallel}' + D_{a_{\parallel}}^{\perp} \hat{\delta}a_{\parallel}'' \]

\[ + D_{a_{\perp}}^{\perp} \hat{\delta}a_{\perp} + D_{a_{\perp}}^{\perp} \hat{\delta}a_{\perp}' + D_{a_{\perp}}^{\perp} \hat{\delta}a_{\perp}'' \] (2.62)
where

\[
D_{\phi 0}^\perp = i \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z (\xi_{ecs}) \Gamma_1 + \Lambda Z' (\xi_{ecs}) \Omega_d \Gamma_{11} \right),
\]

\[
D_{\phi 1}^\perp = 2 \lambda_1 \frac{e^2}{T_e} v_e \left( (n_{cs} \Lambda Z (\xi_{ecs}))' \Gamma_3 + (n_{cs} \Lambda Z' (\xi_{ecs}) \Omega_d)' \Gamma_{31} \right)
- i 2 \lambda_1 \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z (\xi_{ecs}) \lambda_0 \Gamma_4 + \Lambda Z' (\xi_{ecs}) \Omega_d \lambda_0 \Gamma_{41} \right),
\]

\[
D_{\phi 2}^\perp = \lambda_1 \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z (\xi_{ecs}) (\Gamma_3 - \Gamma_2) + \Lambda Z' (\xi_{ecs}) \Omega_d (\Gamma_{31} - \Gamma_{21}) \right),
\]

\[
D_{a \parallel 0}^\perp = i \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z (\xi_{ecs}) \Gamma_1 \frac{v_{de||}}{v_e} + \Lambda Z_1 (\xi_{ecs}) \Gamma_1 + \Lambda Z' (\xi_{ecs}) \Omega_d \Gamma_{11} \frac{v_{de||}}{v_e} \right)
+ \Lambda Z_1' (\xi_{ecs}) \Omega_d \lambda_0 \Gamma_{11}, \quad D_{a \parallel 1}^\perp = i 2 \lambda_1 \frac{e^2}{T_e} v_e \left( (n_{cs} \Lambda Z (\xi_{ecs}))' \Gamma_3 \right)
+ \left( n_{cs} \Lambda Z' (\xi_{ecs}) \Omega_d \frac{v_{de||}}{v_e} \right)' \Gamma_{31} + \left( n_{cs} \Lambda Z_1 (\xi_{ecs}) \Omega_d \right)' \Gamma_{31}
- i 2 \lambda_1 \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z (\xi_{ecs}) \frac{v_{de||}}{v_e} \lambda_0 \Gamma_4 + \Lambda Z' (\xi_{ecs}) \Omega_d \lambda_0 \Gamma_{41} \right)
+ \Lambda Z_1' (\xi_{ecs}) \Omega_d \lambda_0 \Gamma_{41}, \quad D_{a \perp 0}^\perp = i \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z (\xi_{ecs}) \Gamma_2 + \Lambda Z' (\xi_{ecs}) \Omega_d \Gamma_{21} \right),
\]

\[
D_{a \perp 1}^\perp = i 2 \lambda_1 \frac{e^2}{T_e} v_e \left( (n_{cs} \Lambda Z (\xi_{ecs}))' \Gamma_4 + (n_{cs} \Lambda Z' (\xi_{ecs}) \Omega_d)' \Gamma_{41} \right)
+ i 2 \lambda_1 \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z (\xi_{ecs}) \lambda_0 \Gamma_5 + \Lambda Z' (\xi_{ecs}) \Omega_d \lambda_0 \Gamma_{51} \right),
\]

and \( D_{a \perp 2}^\perp = i 2 \lambda_1 \frac{e^2 n_{cs}}{T_e} v_e \left( \Lambda Z (\xi_{ecs}) \Gamma_4 + \Lambda Z' (\xi_{ecs}) \Omega_d \Gamma_{41} \right). \) \quad (2.63)

Here we define

\[
\Gamma_5 = \frac{1}{2} e^{-b} \left( (2 - 2b)^2 I_0(b) + (1 + 2b - 4b^2) I_1(b) \right),
\]

and \( \Gamma_{51} = \frac{1}{4} e^{-b} \left( (6 - 32b + 52b^2 - 16b^3) I_0(b) + (4 + 12b - 44b^2 + 8b^3) I_1(b) \right). \) \quad (2.64)

The perturbed density and perturbed current density given in Eq(2.49), Eq(2.50), Eq(2.55),
Eq(2.56), Eq(2.62), and Eq(2.63) constitute a complete electron response for the current sheet electrons that is sufficient for nonlocal electromagnetic analysis.

### 2.3.4 Perturbed Distribution of Background Electrons

The background electron response is found from the current sheet electron response by substituting $n_b$ for $n_{cs}$ and setting $v_{de} = 0$. Defining

$$
\xi_{eb} = \frac{\omega}{k\|v_e},
$$

(2.65)

the perturbed distribution of background electrons is then given by

$$
\hat{d}f_{eb} = \frac{e}{T_e} \hat{d}\phi F_{eb} + \hat{d}G_{eb}e^{-i\lambda\sin(\alpha)}
$$

(2.66)

$$
\hat{d}G_{eb} = \frac{e}{T_e} \frac{\omega}{\omega - \omega_d - k\|v\|} \left( J_0(\lambda) \left( -\hat{d}\phi + \frac{k\|v\|}{\omega} \hat{d}\psi \right) - J_1(\lambda) \frac{v_\perp}{ck_\perp} \hat{d}B_\parallel \right) F_{eb}.
$$

(2.67)

### 2.3.5 Perturbed Density of Background Electrons

The perturbed density of background electrons is then

$$
\hat{d}n_{eb} = \chi_{\phi 0} \hat{d}\phi + \chi_{\phi 1} \hat{d}\phi' + \chi_{\phi 2} \hat{d}\phi''
$$

$$
+ \chi_{a_{\| 0}} \hat{d}a_\| + \chi_{a_{\| 1}} \hat{d}a_\|' + \chi_{a_{\| 2}} \hat{d}a_\|''
$$

$$
+ \chi_{a_{\perp 0}} \hat{d}a_\perp + \chi_{a_{\perp 1}} \hat{d}a_\perp' + \chi_{a_{\perp 2}} \hat{d}a_\perp''
$$

(2.68)
where

\[
\chi_{\phi 0} = \frac{e_n}{T_e} \left( 1 + \xi_{eb} Z(\xi_{eb}) \Gamma_0 + \xi_{eb} Z' / (\xi_{eb}) \Omega_d \Gamma_0 \right), \quad \chi_{\phi 1} = -2 \lambda_1 \frac{e}{T_e} \left( (n_b \xi_{eb} Z(\xi_{eb}))' \Gamma_1 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \Gamma_1 \right) + (n_b \xi_{eb} Z'(\xi_{eb}) \Omega_d)' \Gamma_{11} + 2 \lambda_1 \frac{e_n}{T_e} \left( \xi_{eb} Z(\xi_{eb}) \lambda_0 \Gamma_2 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \lambda_0 \Gamma_{21} \right),
\]

\[
\chi_{\phi 2} = -2 \lambda_1 \frac{e_n}{T_e} \left( \xi_{eb} Z(\xi_{eb}) \Gamma_1 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \Gamma_{11} \right),
\]

\[
\chi_{a_{||} 0} = \frac{e_n}{T_e} \left( \xi_{eb} Z(\xi_{eb}) \Gamma_0 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \Gamma_0 \right), \quad \chi_{a_{||} 1} = -2 \lambda_1 \frac{e}{T_e} \left( (n_b \xi_{eb} Z(\xi_{eb}))' \Gamma_1 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \Gamma_1 \right) + (n_b \xi_{eb} Z'(\xi_{eb}) \Omega_d)' \Gamma_{11} + 2 \lambda_1 \frac{e_n}{T_e} \left( \xi_{eb} Z(\xi_{eb}) \lambda_0 \Gamma_2 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \lambda_0 \Gamma_{21} \right),
\]

\[
\chi_{a_{||} 2} = -2 \lambda_1 \frac{e_n}{T_e} \left( \xi_{eb} Z(\xi_{eb}) \Gamma_1 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \Gamma_{11} \right),
\]

\[
\chi_{a_{\perp} 0} = \frac{e_n}{T_e} \left( \xi_{eb} Z(\xi_{eb}) \Gamma_0 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \Gamma_1 \right), \quad \chi_{a_{\perp} 1} = -2 \lambda_1 \frac{e}{T_e} \left( (n_b \xi_{eb} Z(\xi_{eb}))' \Gamma_2 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \Gamma_2 \right) + (n_b \xi_{eb} Z'(\xi_{eb}) \Omega_d)' \Gamma_{21} + 2 \lambda_1 \frac{e_n}{T_e} \left( \xi_{eb} Z(\xi_{eb}) \lambda_0 \Gamma_4 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \lambda_0 \Gamma_{41} \right),
\]

and

\[
\chi_{a_{\perp} 2} = -\lambda_1 \frac{e_n}{T_e} \left( \xi_{eb} Z(\xi_{eb}) (\Gamma_2 - \Gamma_3) + \xi_{eb} Z'(\xi_{eb}) \Omega_d (\Gamma_{21} - \Gamma_{31}) \right). \quad \text{(2.69)}
\]

### 2.3.6 Perturbed Current Density of Background Electrons

The parallel component of the perturbed current density of the background electrons is given by

\[
\hat{J}_{eb||} = D_{\phi 0||} \hat{\delta \phi} + D_{\phi 1||} \hat{\delta \phi}' + D_{\phi 2||} \hat{\delta \phi}'' + D_{a_{||} 0||} \hat{\delta \mathbf{a}} + D_{a_{||} 1||} \hat{\delta \mathbf{a}}' + D_{a_{||} 2||} \hat{\delta \mathbf{a}}'' + D_{a_{\perp} 0||} \hat{\delta \mathbf{a}}_{\perp} + D_{a_{\perp} 1||} \hat{\delta \mathbf{a}}_{\perp}' + D_{a_{\perp} 2||} \hat{\delta \mathbf{a}}_{\perp}'' \quad \text{(2.70)}
\]
where

\[ D_{\phi 0}^\parallel = -\frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z_1(\xi_{eb}) \Gamma_0 + \xi_{eb} Z_1'(\xi_{eb}) \Omega_{d0} \Gamma_0 \right), \quad D_{\phi 1}^\parallel = 2 \lambda_1 \frac{e^2}{T_e} v_e \left( (n_b \xi_{eb} Z_1(\xi_{eb}))' \Gamma_1 \right) + (n_b \xi_{eb} Z_1'(\xi_{eb}) \Omega_d) \Gamma_{11} \right) - 2 \lambda_1 \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z_1(\xi_{eb}) \lambda_0 \Gamma_2 + \xi_{eb} Z_1'(\xi_{eb}) \Omega_d \lambda_0 \Gamma_{21} \right); \]

\[ D_{\phi 2}^\parallel = 2 \lambda_1 \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z_2(\xi_{eb}) \Gamma_1 + \xi_{eb} Z_2'(\xi_{eb}) \Omega_d \Gamma_11 \right), \]

\[ D_{a_\parallel 0}^\parallel = -\frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z_2(\xi_{eb}) \lambda_0 \Gamma_2 + \xi_{eb} Z_2'(\xi_{eb}) \Omega_d \lambda_0 \Gamma_{21} \right), \]

\[ D_{a_\parallel 1}^\parallel = 2 \lambda_1 \frac{e^2}{T_e} v_e \left( (n_b \xi_{eb} Z_2(\xi_{eb}))' \Gamma_1 + (n_b \xi_{eb} Z_2'(\xi_{eb}) \Omega_d) \Gamma_{11} \right) - 2 \lambda_1 \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z_2(\xi_{eb}) \lambda_0 \Gamma_2 + \xi_{eb} Z_2'(\xi_{eb}) \Omega_d \lambda_0 \Gamma_{21} \right), \]

\[ D_{a_\perp 0}^\parallel = -\frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z_1(\xi_{eb}) \Gamma_1 + \xi_{eb} Z_1'(\xi_{eb}) \Omega_d \Gamma_{11} \right), \]

\[ D_{a_\perp 1}^\parallel = 2 \lambda_1 \frac{e^2}{T_e} v_e \left( (n_b \xi_{eb} Z_1(\xi_{eb}))' \Gamma_2 + (n_b \xi_{eb} Z_1'(\xi_{eb}) \Omega_d) \Gamma_{21} \right) - 2 \lambda_1 \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z_1(\xi_{eb}) \lambda_0 \Gamma_4 + \xi_{eb} Z_1'(\xi_{eb}) \Omega_d \lambda_0 \Gamma_{41} \right), \]

and \[ D_{a_\perp 2}^\parallel = \lambda_1 \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z_1(\xi_{eb})(\Gamma_2 - \Gamma_3) + \xi_{eb} Z_1'(\xi_{eb}) \Omega_d (\Gamma_{21} - \Gamma_{31}) \right). \]

The perpendicular component of the perturbed current density of the background electrons is

\[ \delta J_{eb, \perp} = D_{\phi 0}^\perp \delta \phi + D_{\phi 1}^\perp \delta \phi' + D_{\phi 2}^\perp \delta \phi'' + D_{a_\parallel 0}^\perp \hat{\delta} a_{\parallel} + D_{a_\parallel 1}^\perp \hat{\delta} a_{\parallel}' + D_{a_\parallel 2}^\perp \hat{\delta} a_{\parallel}'' + D_{a_\perp 0}^\perp \hat{\delta} a_{\perp} + D_{a_\perp 1}^\perp \hat{\delta} a_{\perp}' + D_{a_\perp 2}^\perp \hat{\delta} a_{\perp}'' , \]
where

\[
D_{\phi 0}^{\perp} = i \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z(\xi_{eb}) \Gamma_1 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \Gamma_{11} \right), \quad D_{\phi 1}^{\perp} = i 2 \lambda_1 \frac{e^2}{T_e} v_e \left( (n_b \xi_{eb} Z(\xi_{eb}))' \Gamma_3 + (n_b \xi_{eb} Z'(\xi_{eb}))' \Gamma_3 \right.
\]
\[+ (n_b \xi_{eb} Z'(\xi_{eb}) \Omega_d)' \Gamma_{31} - i 2 \lambda_1 \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z(\xi_{eb}) \lambda_0' \Gamma_4 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \lambda_0' \Gamma_{41} \right),
\]
\[D_{\phi 2}^{\perp} = i \lambda_1 \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z(\xi_{eb}) (\Gamma_3 - \Gamma_2) + \xi_{eb} Z'(\xi_{eb}) \Omega_d (\Gamma_{31} - \Gamma_{21}) \right),
\]
\[D_{a_\parallel 0}^{\perp} = i \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z(\xi_{eb}) (\Gamma_1) + \xi_{eb} Z'(\xi_{eb}) \Omega_d \Gamma_{11} \right),
\]
\[D_{a_\parallel 1}^{\perp} = i 2 \lambda_1 \frac{e^2}{T_e} v_e \left( (n_b \xi_{eb} Z(\xi_{eb}))' \Gamma_3 + (n_b \xi_{eb} Z'(\xi_{eb}))' \Gamma_{31} \right.
\[+i 2 \lambda_1 \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z(\xi_{eb}) \lambda_0' \Gamma_4 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \lambda_0' \Gamma_{41} \right),
\]
\[D_{a_\parallel 2}^{\perp} = i \lambda_1 \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z(\xi_{eb}) (\Gamma_3 - \Gamma_2) + \xi_{eb} Z'(\xi_{eb}) \Omega_d (\Gamma_{31} - \Gamma_{21}) \right),
\]
\[D_{a_\perp 0}^{\perp} = i \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z(\xi_{eb}) \Gamma_2 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \Gamma_{21} \right),
\]
\[D_{a_\perp 1}^{\perp} = i 2 \lambda_1 \frac{e^2}{T_e} v_e \left( (n_b \xi_{eb} Z(\xi_{eb}))' \Gamma_4 + (n_b \xi_{eb} Z'(\xi_{eb}) \Omega_d)' \Gamma_{41} \right)
\[+i 2 \lambda_1 \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z(\xi_{eb}) \lambda_0' \Gamma_5 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \lambda_0' \Gamma_{51} \right),
\]
\[\text{and} \quad D_{a_\perp 2}^{\perp} = i 2 \lambda_1 \frac{e^2 n_b}{T_e} v_e \left( \xi_{eb} Z(\xi_{eb}) \Gamma_4 + \xi_{eb} Z'(\xi_{eb}) \Omega_d \Gamma_{41} \right). \quad (2.73)
\]

### 2.4 Electromagnetic LHDI Eigenmode Equations

To collect the perturbed responses into a system of nonlocal eigenmode equations, we must resolve the different coordinates used in the ion and electron responses. In the ion response, we have defined

\[
\delta A = \delta A_{k_k} \hat{k}_k + \delta A_{\hat{z}} \hat{z} + \delta A_{c}(\hat{k}_a \times \hat{z}). \quad (2.74)
\]
where \(z\) is the current sheet normal, and \(\delta A_{k_s} = \delta \mathbf{A} \cdot \mathbf{k}_s\). For the electron response, we define

\[
\delta \mathbf{A} \simeq \delta A_{\parallel} \hat{b} + \delta A_{\perp} (\hat{b} \times \hat{k}_\perp) = -\frac{c}{v_e} \delta a_{\parallel} \hat{b} - i \frac{c}{v_e} \delta a_{\perp} (\hat{b} \times \hat{k}_\perp)
\]  

(2.75)

Consistent with our assumption \(|k_z/k_s| \ll 1\), we take \(\delta A_{k_s} \sim 0\), and transform the magnetic vector potential in the ion response to the components used in the electron response derivation;

\[
\delta A_y = -\frac{k_x}{k_\zeta} \frac{c}{v_e} \delta a_{\parallel}, \quad \delta A_x = \frac{k_y}{k_\zeta} \frac{c}{v_e} \delta a_{\parallel}, \quad \delta A_c = \frac{k_s}{k_\zeta} \frac{c}{v_e} \delta a_{\parallel}, \quad \text{and} \quad \delta A_z \simeq i \frac{c}{v_e} \delta a_{\perp}.
\]  

(2.76)

We may now combine the perturbed responses into Poisson’s equation, and Amperes law;

\[
k^2 \hat{k} \delta \phi = -4\pi e (\hat{b} \cdot \delta \mathbf{n} - \hat{b} \cdot \delta \mathbf{n}_i),
\]

(2.77)

\[
-k^2 \frac{c}{v_e} \hat{k} \delta a_{\parallel} = \frac{4\pi}{c} (\hat{b} \cdot \delta \mathbf{J}_e + \hat{b} \cdot \delta \mathbf{J}_i),
\]

(2.78)

and

\[
k^2 \hat{k} \delta A_{\parallel} \cdot (\hat{k}_\perp \times \hat{b}) = ik^2 \frac{c}{v_e} \hat{k} \delta a_{\parallel} = \frac{4\pi}{c} (\hat{b} \cdot \delta \mathbf{J}_e + \hat{b} \cdot \delta \mathbf{J}_i) \cdot (\hat{k}_\perp \times \hat{b}).
\]

(2.79)

Using Eq(2.77), Eq(2.78), and Eq(2.79), we will now perform numerical and analytical analyses of the LHDI in a thin current sheet with magnetic shear. We begin in the electrostatic limit, which is discussed in Chapter 3, and continue with electromagnetic results in Chapter 4.
Chapter 3

The Electrostatic LHDI

In this Chapter, we analyze the LHDI in the electrostatic limit as a preliminary investigation of finite $k_\parallel$ effects and the influence of a guide field on the LHDI, and to benchmark our results with the GeFi simulation of Lin et.al.[35, 36] The observed fluctuations[31, 62] and simulation results[12, 51] in the lower-hybrid frequency range differ from conventional LHDIs by their activity near the current sheet center, significant magnetic field perturbations, and oblique propagation, i.e. $k_\parallel \neq 0$. The stabilization of the LHDI by finite $k_\parallel$ is well established in the literature for antiparallel magnetic field configurations[34, 16], and, consequently, obliquely-propagating LHDIs remain relatively unaddressed. In current sheets with a guide field, the concept of oblique propagation becomes ambiguous, but the present work will address this both in the local limit, where $k \cdot \hat{b}$ is unambiguous, and by evaluating $k \cdot \hat{b}$ over the extent of the mode width, particularly at the location of the peak eigenfunction amplitude in the nonlocal analysis. In Daughton’s simulations[12], the electromagnetic modes near the current sheet center were suppressed when the equilibrium drift was reduced. We will show that, in the electrostatic limit, the LHDI is destabilized by finite $k_\parallel$ in the strong drift domain. In the following sections, we describe the three dimensional theory of the electrostatic LHDI
in a current sheet with a guide field, present numerical and analytical results, and compare our results with GeFi simulations.

### 3.1 Local Stability Analysis

We begin our analysis of the LHDI in a Harris current sheet with a guide field in the electrostatic limit with a local stability analysis. Setting $k_z = -i\partial_z = 0$ in the eigenmode equation Eq(2.77), we arrive at the following local dispersion relation:

\[
1 + \frac{2\omega_{pe}^2}{k_s^2v_e^2} + \frac{2\omega_{pecs}^2}{k_s^2v_e^2} \frac{\omega - k_yv_de}{k_{\parallel}v_e} \left\{ \Gamma_0(\lambda_0)Z(\xi_{ecs}) + \Gamma_{01}(\lambda_0)Z'(\xi_{ecs})\Omega_d \right\} \\
+ \frac{2\omega_{peb}^2}{k_s^2v_e^2} \xi_{eb} \left\{ \Gamma_0(\lambda_0)Z(\xi_{eb}) + \Gamma_{01}(\lambda_0)Z'(\xi_{eb})\Omega_d \right\} - \frac{T_e \omega_{peb}^2}{T_i k_s^2v_e^2} Z'('\xi_{ib}) - \frac{T_e \omega_{pecs}^2}{T_i k_s^2v_e^2} Z'(\xi_{ics}) = 0.
\] (3.1)

In Eq(3.1),

\[
\omega_{pe}^2 = \frac{4\pi n_b e^2}{m_e} + \frac{4\pi n_{cs} e^2}{m_e} \text{Sech}^2(z/L) = \omega_{peb}^2 + \omega_{pecs}^2.
\]

We solve Eq(3.1) at various current sheet positions, $z/L$, and wave vector orientations, $k_{\parallel}/k_s$, to determine the most unstable LHDI as a function of $k_s$. The numerical results for current-aligned, $k_x = 0$, modes are shown in Figure 3.1. The numerical results for modes with finite $k_x$ are shown in Figure 3.2. The adopted parameters are

\[
L = 0.23\rho_i0, \quad \frac{\omega_{peb}^2}{\omega_{ce0}^2} = 100, \quad \frac{n_{cs}}{n_b} = 27.55, \quad \frac{B_y}{B_0} = 0.1, \quad \text{and} \quad \frac{T_e}{T_i} = 0.1,
\]

where $\rho_i0$ and $\Omega_{ce0}$ are defined in the asymptotic field, $B_0$. These parameters are consistent with a thin magnetotail current sheet with a guide field, negligible curvature, and a small background plasma[45, 3]. The results satisfy our ordering $|\omega/\omega_{ce}| \ll 1 \ll |\omega/\omega_{ci}|$ and
\[ |k_\parallel \rho_e| \ll |k_\perp \rho_e| \sim 1 \ll |k_{\rho i}|.\]

Figure 3.1: The strongest mode frequencies (left) and localizations (right), scaled by the asymptotic values of \( \rho_e \) and the LHD frequency, \( \omega_{\text{LH}} \).

Yoon derived a similar eigenvalue equation for electrostatic LHDIs in a Harris current sheet with a guide field[55]. He assumed unmagnetized ions, and evaluated the perturbed electron response by integrating the electron Vlasov equation along unperturbed particle orbits. Yoon’s local dispersion relation is

\[
1 + \frac{2\omega_{\text{pe}}^2}{\kappa_{\rho e}^2 v_e^2} + \frac{2\omega_{\text{pe}}^2 \omega - k_y v_{de} Z(\xi_{\text{ec}}) \Gamma_0(\lambda_0) J_0(\frac{k_y v_{de} B_x^2}{B^2 \omega_{\text{ce}}}) - T_e \omega_{\text{pe}}^2}{T_i k_{\rho e}^2 v_e^2} Z'(\xi_{\text{ies}}) = 0. \tag{3.2}
\]

The solutions of Eq(3.1) and Eq(3.2) are qualitatively similar for our magnetotail parameters. The Bessel’s function, \( J_0 \), that appears in Yoon’s result and not in Eq(3.1) is a consequence of the orbit integration and is approximately 1 for current sheets with weak guide fields and small \( |v_{de}/v_e| \). The terms proportional to \( \Omega_d \) in Eq(3.1) which are not included in Eq(3.2) are the \( \nabla B \) drift effects. These corrections are significant near the neutral sheet, but not at the current sheet edge, where the most unstable LHDIs are localized. The final difference between Eq(3.1) and Eq(3.2) comes from our addition of a background plasma, which, as will be shown, has a significant impact on mode stability.

Based on the insights given by the numerical results, we may develop corresponding analytical theories which will elucidate the underlying physics. We begin with a marginal stability
Figure 3.2: A contour plot of the growth rate at fixed \( z = 1.5L \) is given in (top right), where \( k_\parallel = 0 \) along the dashed line. For the remaining figures we scan over \( z \) to find dominant instabilities. The most unstable propagation angles with respect to the local B are given in (top left), showing the deviation of dominant modes from \( k_\parallel = 0 \) in the short-wavelength domain. The two unstable lobes have similar features and we plot the frequencies and localizations of the \( k_\parallel \geq 0 \) lobe in (bottom left), and (bottom right), respectively.

analysis, which will demonstrate the significance of the background plasma to the LHDI stability. We will then develop analytical theories describing the LHDI in the long-wavelength, and short-wavelength domains.

### 3.1.1 Marginal Stability Analysis

At \( k_\parallel = 0 \), the imaginary part of the local dispersion relation, given in Eq(3.1), is

\[
-\tau \frac{\omega_{pecs}^2}{k_\parallel n_e^2} (-2\sqrt{\pi} \xi_{ics} e^{-\xi_{ics}^2}) - \tau \frac{\omega_{peb}^2}{k_\parallel n_e^2} (-2\sqrt{\pi} \xi_{ib} e^{-\xi_{ib}^2}) = 0, \tag{3.3}
\]
which can be reduced to the following expression;

\[ 1 = \frac{n_{cs}}{n_b} \frac{1 - \alpha}{\alpha} e^{u_i^2(2\alpha - 1)}, \]  

(3.4)

where \( \alpha = \omega / k_y v_{di} \), \( u_i = k_y v_{di} / k_s v_i \), and \( \tau = T_e / T_i \). The right hand side of Eq (3.4) gives the ratio of the inverse Landau damping of the current sheet ions to the Landau damping of the background ions, with marginal stability occurring when these opposing influences are in balance. This ratio shows that the destabilizing inverse Landau damping of the current sheet ions dominates as \( \alpha \to 0 \), while the stabilizing Landau damping of the background ions dominates as \( \alpha \to 1 \). The existence of two marginal stability solutions thus requires

\[
\partial_\alpha \left( \frac{1 - \alpha}{\alpha} e^{u_i^2(2\alpha - 1)} \right) = e^{u_i^2(2\alpha - 1)} 2u_i^2 \left( -\frac{\alpha^2 + \alpha - 1}{2u_i^2} \right) > 0,
\]

i.e. \(-\alpha^2 + \alpha - 1/2u_i^2 > 0\).  

(3.5)

This condition is only possible when \( u_i > \sqrt{2} \), or

\[ v_{di} B_x / v_i B > \sqrt{2}, \]  

(3.6)

where the magnetic fields enter through the assumption \( k || = 0 \).

In the strong drift limit, \( u_i \gg 1 \), the exponential factor on the right hand side of Eq (3.4) dominates, and the two marginal stability solutions are \( \alpha \sim 1 \), and \( \alpha \sim 1/2 \). In this limit, the real part of the dispersion relation can be expressed as

\[
\Gamma_0 \left( 2 + 2 \frac{n_b}{n_{cs}} + 2\tau \frac{1}{\alpha} \right) = 2 + 2 \frac{n_b}{n_{cs}} - \tau \text{Re}[Z'(u_i \alpha - u_i)] - \tau \frac{n_b}{n_{cs}} \frac{1}{u_i^2 \alpha^2},
\]

(3.7)

where \( \text{Re}[x] \) signifies the real part of \( x \). Here we ignore the vacuum term and \( \nabla B \) drift corrections in Eq (3.1) because \( \omega_{pe}^2 \gg \omega_{ce}^2 \), and the dominant modes are localized near the current sheet edge. This expression yields the wavelengths of the marginal stability solutions
through the the $\Gamma_0$ function. Near $\alpha = 1$, $\Gamma_0 \sim 1$, which is the long-wavelength limit of the instability. The short-wavelength limit of the instability is then given by the $\alpha \sim 1/2$ solution, which yields

$$\Gamma_0 \approx 1 - \frac{2\tau}{1 + \frac{n_b}{n_{cs}} + 2\tau}.$$  

(3.8)

In the absence of a background population, a single marginal stability solution occurs for modes with $k_\parallel = 0$, at $\alpha = 1$ and $k_s = 0$. The inclusion of background plasma modifies this solution, introducing a long-wavelength limit of the instability. However, a short-wavelength limit of the instability can not exist without a strong equilibrium drift, and an appropriate background plasma density, which satisfy Eq(3.4) and Eq(3.5).

3.1.2 Long-wavelength Domain

We proceed with an analysis of the LHDI in the long-wavelength domain by simplifying the local dispersion relation, given in Eq(3.1), to extract a more instructive expression. We define $\omega = k_y v_{di} + \delta \omega$, and as suggested by Figures 3.1 and 3.2, we assume $|\delta \omega/\omega| \ll 1$, and thus $|\xi_{ics}| \ll 1$. In the strong drift limit, $u_i \gg 1$, we may also assume $|\xi ib| \gg 1$, $|\xi ecs| \gg 1$, and $|\xi eb| \gg 1$. Performing the appropriate expansions on the plasma dispersion functions in Eq(1) yields the following equation for $\delta \omega$;

$$\frac{\delta \omega}{k_y v_{di}} = -\left((2n + 2n_{cs}\tau)(b_e - \frac{1}{2u_e^2})\right)\left(2n_{cs}\left(\frac{1}{u_e^2} + \tau\right) + 2i\sqrt{\pi}n_{cs}\tau u_i\right)^{-1},$$  

(3.9)

where $u_e = k_y v_{di}/k_\parallel v_e$, and $\Gamma_0 \approx 1 - k_\parallel^2 \mu_e^2/2 = 1 - b_e$. This expression is valid in the long-wavelength domain;

$$b_e \ll \frac{\tau u_i \sqrt{\pi}}{1 + n_b/n_{cs} + \tau}.$$  

(3.10)
The growth rate given by Eq(3.9) is reduced as $z \to 0$, since an increasing $k_\parallel$ enhances the stabilizing $u_e$ terms. The instability is also damped as $z \to \infty$ because the current sheet density diminishes. Setting $\partial_z \text{Im}[\delta \omega] = 0$ in Eq(3.9) yields the following equation for the location of the most unstable mode relative to the current sheet neutral plane, $z = 0$;

$$\partial_z \delta \omega \text{Im} \simeq -\frac{n'_e}{n_{cs}} \omega^2_{pe} (b_e - \frac{1}{2u_e^2}) + \omega^2_{pe} (b'_e + \frac{u'_e}{u_e^3}) = 0.$$  \hfill (3.11)

The stabilizing increase of $k_\parallel$ near the neutral sheet dominates this expression. This makes the growth rate larger as $z$ increases, until the background plasma begins to overwhelm the current sheet population. Modes that satisfy $k_\parallel = 0$ near the neutral sheet are strongly suppressed by the $\nabla B$ drift. This drift is essential in determining the dominance of edge-localized modes, but is ultimately neglected in Eq(3.9) and Eq(3.11), because it is insignificant at the current sheet edge. The localization predicted by Eq(3.11) falls within $L \leq z \leq 2L$ for the wavelengths we consider. The exact predictions will be shown in section 3.2, along with the solutions of the nonlocal eigenmode equation.

The growth rate predicted by Eq(3.9) scales with $2i\sqrt{\pi n_{cs} \tau_i}$, which is the inverse Landau damping of the current sheet ions. The modes are driven by wave-particle resonance with the equilibrium current, which is largely carried by ions when $|v_{di}/v_{de}| = T_i/T_e \gg 1$.

The mode stability is very sensitive to the wave vector magnitude and direction, as seen in the electron response terms in Eq(3.9). The terms proportional to $b_e$ represent the electron polarization drift, which reduces the real part of $\omega/k_y v_{di}$ as $k_\perp$ increases. The parallel electron dynamics appear through the $u_e$ terms in Eq(3.9), which increase the real part of $\omega/k_y v_{di}$ as $k_\parallel/k_s$ increases. When

$$b_e - \frac{1}{2u_e^2} > 0,$$

$\omega$ falls below $k_y v_{di}$, and the mode is unstable. As $b_e$ increases further, the real part of $\omega/k_y v_{di}$
continues to fall, and the assumption $\delta \omega / k_y v_{di} \ll 1$ is no longer valid. So we must develop a separate analytical theory to describe the LHDI in the short-wavelength domain.

### 3.1.3 Short-Wavelength Domain

The expression given in Eq(3.9) predicts a maximum growth rate when $k_{\parallel} = 0$ for all wavelengths. This is consistent with previous LHDI analyses[13, 25, 55] and our numerical results in the long-wavelength domain. However, as $k_s$ increases, Figure 3.2 shows modes with finite $k_{\parallel}$ dominate modes with $k_{\parallel} = 0$ at fixed total wavelength. Guided by these numerical results, we develop a separate analytical theory for these short-wavelength modes that demonstrates the destabilization of the LHDI by finite $k_{\parallel}$.

We isolate the influence of $k_{\parallel}$ on the mode frequencies by evaluating

$$
\frac{d\omega_0}{dk_{\parallel}} = -\frac{\partial k_{\parallel}}{\partial D_0} \bigg|_{\omega = \omega_0} (\partial \omega D_0)^{-1},
$$

(3.13)

where $D_0$ is the local dispersion relation given in Eq(3.1), and $\omega_0$ are the local frequency solutions. Assuming $|k_{\parallel} / k_s| \ll 1$, we find

$$
\frac{d\omega_0}{dk_{\parallel}} = \frac{\omega_0}{k_{\parallel}} \Gamma_0 \left( 2n_{cs} \tau \frac{k_y v_{di}}{\omega_0} \frac{k_{\parallel}^2 v_e^2}{\omega_0^2} + 2n_e \frac{k_{\parallel}^2 v_e^2}{\omega_0^2} \right) \left( 2n_{cs} \tau \frac{k_y v_{di}}{\omega_0} \Gamma_0 \right)
$$

$$
+ 3n_{cs} \tau \frac{k_y v_{di}}{\omega_0} \frac{k_{\parallel}^2 v_e^2}{\omega_0^2} \Gamma_0 + 2n_e \frac{k_{\parallel}^2 v_e^2}{\omega_0^2} \Gamma_0 - n_{cs} \tau \xi_{ib} Z''(\xi_{ics}) - n_b \tau \xi_{ib} Z''(\xi_{ib}) \right)^{-1}.
$$

(3.14)

This expression is zero at $k_{\parallel} = 0$, indicating a local extremum in the LHDI growth rate. As seen in Figure 3.2, this extremum is a local maximum at long wavelengths, but becomes a local minimum at short wavelengths. To distinguish these cases we evaluate $d^2 \omega_0 / dk_{\parallel}^2$ at
$k_\parallel = 0$, which is given by

$$\frac{d^2 \omega_0}{dk_\parallel^2} = \omega_0 \Gamma_0 \left( 2 n_{cs} \frac{k_y v_{di} v_e^2}{\omega_0^2} + 2 n_e \frac{v_e^2}{\omega_0^2} \right) \times \left( 2 n_{cs} \frac{k_y v_{di}}{\omega_0} \Gamma_0 - n_{cs} \tau_{\xi_{ics}} \frac{Z''(\xi_{ics})}{\xi_{ics}} - n_b \tau_{\xi_{ib}} \frac{Z''(\xi_{ib})}{\xi_{ib}} \right)^{-1}.$$  \hspace{1cm} (3.15)

The sign of the imaginary part of $d^2 \omega_0 / dk_\parallel^2$, given above, will illustrate the influence of finite $k_\parallel$ on mode stability.

At marginal stability, only the ion response terms in Eq(3.15) are complex, and the condition for the destabilization of LHDIs by finite $k_\parallel$ becomes

$$n_{cs} (2 - 4 \xi_{ics}^2) e^{-\xi_{ics}^2} + n_b (2 - 4 \xi_{ib}^2) e^{-\xi_{ib}^2} < 0.$$  \hspace{1cm} (3.16)

The above expression is greatly simplified by substituting the marginal stability condition given in Eq(3.3), which yields the following;

$$-\alpha^2 + \alpha - 1/2 u_i^2 > 0.$$  \hspace{1cm} (3.17)

As was shown in Eq(3.5), the short-wavelength marginal stability solution must satisfy the condition given above, and will inherently experience further destabilization by finite $k_\parallel$. This condition relies on a strong equilibrium drift, given by Eq(3.5). This is why analyses of LHDIs in current sheets with moderate or weak drift do not find dominant instabilities with finite $k_\parallel$.

In current sheets with large equilibrium drifts, there exists a frequency range, given by Eq(3.6), in which the overall destabilization due to the ion response increases with increasing $\omega_{Re} / k_y v_{di}$, where $\omega_{Re}$ is the real part of the mode frequency. The LHDI does not occupy this range in the long-wavelength domain, but as $k_\perp$ increases, the electron polarization drift
lowers $\omega_{Re}/k_y v_{di}$ into this range. For these short-wavelength modes, the increase in $\omega_{Re}/k_y v_{di}$ by finite $k_\parallel$ is destabilizing.

### 3.2 Nonlocal Stability Analysis

We continue our analysis by returning to Poisson’s equation and relaxing the $k_z = -i\partial_z = 0$ assumption. The nonlocal eigenfunctions and eigenvalues are solved using the Galerkin spectral method. We begin with a coordinate transformation, tanh($z/L$) = $y$, and choose a set of basis functions, $f_n$ that satisfy $f_n(y = \pm 1) = 0$. The resulting eigenfunctions are well-localized at the current sheet edge, so the even and odd solutions have negligible differences. The results for current-aligned modes, $k_x = 0$, and finite $k_x$ modes are shown in Figures 3.3 and 3.4, respectively.

The inhomogeneous scale length defined as $|\partial_z \ln[\delta \phi(z)]|$ is much less than $k_s$ for all solutions with $k_s \rho_{e0} \geq 0.2$. This validates the small argument expansion of $k_z/k_s$ used to derive Eq(2.77), and explains the strong agreement between the results from local theory and the nonlocal eigenvalue problem. At the location of the peak eigenfunction amplitude, $k_s \cdot \hat{b}$ agrees with the $k_\parallel$ predicted by local theory to within a few percent. This is a consequence of the strong agreement between the peak eigenfunction amplitude and the localization predicted by local theory. For $k_s \rho_{e0} \geq 0.2$, the solutions are consistent with the gyrokinetic electrons ordering, the unmagnetized ion ordering, and the small argument expansion of the $\nabla B$ drift corrections. The modes are localized at the current sheet edge, with a shape similar to those shown in Figures 3.3 and 3.4.

We also compare our local and nonlocal solutions for current-aligned modes with results from the GeFi simulation by Lin et.al.[35, 36] In the GeFi simulation, the electrons are treated as gyrokinetic particles, while fully kinetic ions are taken into account. In this way, rapid
electron cyclotron motion is averaged out and thus removed from the calculation, but finite electron Larmor radius effects are retained. This allows for efficient simulation runs for a realistic proton-to-electron mass ratio. The simulation model is suitable for problems in which the wave frequency satisfies $\omega \ll |\omega_{ce}|$.[55]

The initial ion distribution consists of a Harris current sheet component with a background ion population as given in Chapter 2. The initial current sheet electron distribution is given by

$$F_{He,g} = \frac{n_{h0}}{(2\pi T_e/m_e)^{3/2}} (1 - m_e v_{de}^2 dB_z/dz) \times \exp[- \frac{B_z^2 m_e v_{de}^2}{B^2 2T_e}] \times \exp[- \frac{1}{2T_e} [2\mu B + m_e (v_\| - \frac{v_{de} B_0}{B})^2],$$

which includes corrections that are ignored in our derivation.[51, 35]

The equilibrium parameters are consistent with those given in section 3.1. The grid number, $N_z \times N_x \times N_y$, is $128 \times 1 \times 32$, and the particle number per cell is 100 for both electrons and ions. The ion-to-electron mass ratio is $m_i/m_e = 1836$, and the times step is $\Omega_e \Delta t = 0.5$.

The simulated dispersion relation, which depicts the real frequency and growth rate versus wave number, is shown in Figure 3.3. The real frequencies and growth rates obtained from GeFi simulation are in excellent agreement with both the local and nonlocal theory. The eigenfunction of the most unstable mode obtained from theory and simulation is presented in Figure 3.3. The simulated and theoretical eigenfunctions have similar shapes, and peak near $z = 1.5L$, indicating that the instability occurs at the edge of the current sheet.

These solutions are consistent with our ordering, and the expansion of $\omega_d/k_\| v_e$, and $k_z/k_s$ for $k_s \rho_{e0} \geq 0.2$. The modes are localized at the current sheet edge, with a shape similar to those shown in Figures 3.3 and 3.4. We found the modes penetrate closer to the neutral sheet as
the wavelength increases, but this occurs at small \( k_s \), where our small \( k_z/k_s \) assumption is unreliable.

### 3.3 Summary

We have analyzed electrostatic LHDIs in a thin Harris current sheet with a guide field using local theory, nonlocal theory, and simulations. For long-wavelength modes, we derive an analytical expression for the local stability, given in Eq(3.9), which identifies the inverse
Figure 3.4: The real frequency and growth rate of the most unstable, or least damped modes are shown in (left) and (right), respectively. We evaluate $\rho_e$ and $\omega_{LH}$ in the asymptotic magnetic field, and includes results from local theory, and nonlocal theory.

Landau damping of the current sheet ions as the drive of the instability. The growth rate is strongest when $k_\parallel = 0$, and the modes are destabilized by the electron polarization drift. The growth rate given in Eq(3.9) is strongest near the current sheet edge, $z \sim 1.5L$. This location marks the balance between an increasing $k_\parallel$ and $\nabla B$ drift stabilization as $z \to 0$, and an increase in the relative background population as $z \to \infty$. The nonlocal eigenvalue equation and the perturbed responses are derived under the assumption $|\partial_z \ln[\delta\phi]| \ll k_s$, and is less accurate for long-wavelength modes. Nevertheless, the nonlocal eigenvalue solutions and local results agree, as seen in Figures 3.3 and 3.4, and the solutions satisfy our gyrokinetic electron, fully-kinetic ion ordering, as well as our weak $\nabla B$ drift assumption, $\omega_d/k_\parallel v_e \ll 1$.

As the wavelength decreases, modes with finite $k_\parallel$ begin to dominate modes with $k_\parallel = 0$, for small but finite total wavelength. This is shown in Figure 3.2, and is attributed to the existence of a frequency range in which the overall destabilization of the ion response increases with increasing frequency. Within this range, the real part of the mode frequencies are increased by finite $k_\parallel$, which destabilizes the modes. These dominant modes with finite $k_\parallel$ do not arise without sufficiently strong equilibrium drifts, as given by Eq(3.6). This explains the limited attention these modes have received in the literature.[63, 61] The agreement...
between our numerical local analysis and the nonlocal eigenvalue problem is shown in Figures 3.3 and 3.4.

In principle, a guide field could create a compromise between $k_\parallel$ stabilization, and wave-particle resonance, which is strongest when $k_\parallel \neq 0$. Surprisingly, when the equilibrium drift is sufficiently strong, we find modes with $k_\parallel < 0$ that dominate modes with stronger wave-particle resonance and $k_\parallel = 0$, as seen in Figure 3.2. The threshold equilibrium drift, given in Eq(3.6), is very similar to the threshold equilibrium drift found by Daughton[12], for the onset of electromagnetic LHDI's near the current sheet center. This condition was also satisfied in the simulations by Wang et.al.[51], and the observed fluctuations in the magnetotail and MRX.[31, 62] Although the significance of a strong equilibrium drift is clear in the electrostatic limit, the interesting features of the observations and simulation results necessitate an electromagnetic theory. So, with our model convincingly verified by GeFi simulations, we proceed with the electromagnetic analysis of the LHDI in a thin current sheet with a guide field.
Chapter 4

The Electromagnetic LHDI

In this Chapter, we analyze the electromagnetic LHDI in a current sheet with a guide field to investigate the influence of a guide field and finite \( k_\parallel \) on the properties of the instability in the strong drift domain, \( v_{di}/v_i \gg 1 \). Guided by observations\[31, 62\] and simulation results\[12, 51\], we are interested in obliquely-propagating instabilities in the lower-hybrid frequency range with \( k\sqrt{\rho_e\rho_i} \sim 1 \), that are operative near the current sheet center with stronger magnetic fluctuations than those in typical LHDIs\[13, 16, 34, 57\]. Our results will show that there are two types of electromagnetic LHDIs. The Type A electromagnetic LHDI is destabilized by finite \( k_\parallel \) under the same conditions as the electrostatic LHDI, i.e., at short wavelengths, \( k_s\rho_e \gtrsim 0.5 \), in the strong drift domain. However, these Type A modes are quasi-electrostatic and, like the electrostatic LHDI, are localized at the current sheet edge. At locations near the current sheet center, the Type B electromagnetic LHDI arises, and dominates the Type A LHDI modes, which are weakened by the electron \( \nabla B \) drift. Unlike the Type A LHDI, Type B modes are localized near the position where the electron \( \nabla B \) drift is strongest, i.e. near the current sheet center, and are suppressed when the electron \( \nabla B \) drift is removed. In the local limit, the Type B modes are operative over a wide range of \( k_\parallel/k_s \), and exhibit strong penetration into the neutral sheet in the nonlocal analysis. The dominant Type
B modes exhibit features consistent with the simulations and observations\cite{51, 12, 31, 62}; i.e.,
\[ k \sqrt{\rho e \rho_i} \sim 1, \quad k_{\parallel} \ll k_{\perp}, \]
with significant magnetic perturbations including \( \delta B_{\perp} \). In the following
sections, we describe the three dimensional theory of these electromagnetic instabilities in a
current sheet with a guide field, present numerical and analytical results, and compare our
results with available GeFi simulations.

### 4.1 Local Stability Analysis

We begin with a local stability analysis of the electromagnetic LHDI in a Harris current
sheet with a guide field. A system of local equations is obtained by setting \( k_z = -i \partial_z = 0 \)
in the eigenmode equations given in \( \text{Eq}(2.77), \text{Eq}(2.78) \) and \( \text{Eq}(2.79) \). The local equation
derived using Poisson’s equation becomes

\[
0 = A_{\delta \phi} \delta \phi + A_{\delta a_{\parallel}} \delta a_{\parallel} + A_{\delta a_{\perp}} \delta a_{\perp},
\]

where \( A_{\delta \phi} = 1 + \frac{2 \omega_{pe}^2}{k_s^2 v_e^2} \) and \( A_{\delta a_{\parallel}} = 2 \omega_{pe}^2 \frac{\omega - k_y v_{de}}{k_{\parallel} v_e} (Z(\xi_{ecs}) \Gamma_0 + \Gamma') (\xi_{ecs}) \Omega_d \Gamma_01 \)

\[
-\frac{T_e \omega_{pecs}}{\xi_{eb}} k_s^2 v_e^2 Z'(\xi_{ecs}) + \frac{2 \omega_{peb}}{k_s^2 v_e^2} \xi_{eb} (Z(\xi_{eb}) \Gamma_0 + \Gamma' (\xi_{eb}) \Omega_d \Gamma_01) - \frac{T_e \omega_{pecs}}{\xi_{eb}} k_s^2 v_e^2 Z'(\xi_{eb}),
\]

\[
A_{\delta a_{\parallel}} = 2 \omega_{pecs} \frac{v_{de}}{v_e} (Z(\xi_{ecs}) \Gamma_0 + \Gamma') (\xi_{ecs}) \Omega_d \Gamma_01 + 2 \omega_{peb} \frac{\omega - k_y v_{de}}{k_{\parallel} v_e} (1 + \xi_{eb} Z(\xi_{ecs}) \Gamma_0 + Z(\xi_{ecs}) \Gamma_1)
\]

and \( A_{\delta a_{\perp}} = 2 \omega_{pecs} \frac{\omega - k_y v_{de}}{k_{\parallel} v_e} (Z(\xi_{ecs}) \Gamma_1 + \Gamma' (\xi_{ecs}) \Omega_d \Gamma_{11}) + \frac{2 \omega_{peb}}{k_s^2 v_e^2} \xi_{eb} (Z(\xi_{eb}) \Gamma_1 + \Gamma' (\xi_{eb}) \Omega_d \Gamma_{11}). \) (4.1)

Here, the notations are the same as those defined in \( \text{Eq}(2.14), \text{Eq}(2.45), \text{Eq}(2.46), \text{Eq}(2.48), \)
\( \text{Eq}(2.51), \text{Eq}(2.64), \) and \( \text{Eq}(2.65) \). The local equation derived using the parallel component
of Ampere’s law becomes

\[ 0 = A_\delta \phi \delta \phi + A_{\delta a_1} \delta a_\parallel + A_{\delta a_2} \delta a_\perp, \]

where \( A_{\delta \phi} = \frac{2\omega^2_{\text{peccs}} v_{\text{de}}}{k_s^2 v_e^2} \frac{\omega - k_y v_{de}}{k_{||} v_e} \left( [1 + \xi_{eb} Z(\xi_{ecs})] \Gamma_0 + \frac{Z(\xi_{ecs}) + \xi_{eb} Z'(\xi_{ecs})}{{\Omega}_d \Gamma_{01}} \right) \)

\[ + \frac{\omega^2_{\text{peccs}}}{k_s^2 v_e^2} \xi_{eb} Z'(\xi_{ecs}) \Gamma_{01} + \frac{\omega^2_{\text{peccs}}}{k_s^2 v_e^2} \xi_{ib} Z'(\xi_{ib}), \]

\[ A_{\delta a_1} = -\frac{c^2}{v_e^2} + \frac{2\omega^2_{\text{peccs}} v_{\text{de}}}{k_s^2 v_e^2} \frac{\omega - k_y v_{de}}{k_{||} v_e} \left( \frac{v_{\text{de}}^2}{v_e^2} Z(\xi_{ecs}) \Gamma_0 - \frac{v_{\text{de}}^2}{v_e^2} Z'(\xi_{ecs}) \Gamma_0 \right) \]

\[ + \frac{Z_2(\xi_{ecs}) \Gamma_0 + \frac{v_{\text{de}}^2}{v_e^2} Z'(\xi_{ecs}) \Omega_d \Gamma_{01} - \frac{v_{\text{de}}^2}{v_e^2} Z''(\xi_{ecs}) \Omega_d \Gamma_{01} + \frac{Z_2(\xi_{ecs}) \Omega_d \Gamma_{01}}{\Omega_d \Gamma_{01}} + \frac{Z_2(\xi_{ecs}) \Omega_d \Gamma_{01}}{\Omega_d \Gamma_{01}} \right) \]

\[ - \frac{\omega^2_{\text{peccs}} m_e}{k_s^2 v_e^2} \frac{v_{\text{de}}^2}{v_e^2} \frac{k_x B}{k_x B} \xi_{ics} Z'(\xi_{ics}) \left( \xi_{ics} Z'(\xi_{ics}) - \xi_{ics} Z(\xi_{ics}) \right) \]

\[ + \frac{2\omega^2_{\text{peccs}}}{k_s^2 v_e^2} \xi_{eb} Z_2(\xi_{ecs}) \Gamma_0 + \frac{Z_2(\xi_{ecs}) \Omega_d \Gamma_{01}}{\Omega_d \Gamma_{01}} + \frac{m_e \omega_{\text{peccs}}}{m_i} k_{\|} \xi_{ib} Z(\xi_{ib}), \text{ and} \]

\[ A_{\delta a_2} = \frac{2\omega^2_{\text{peccs}} \omega - k_y v_{de}}{k_{||} v_e} \left( [1 + \xi_{eb} Z(\xi_{ecs})] \Gamma_1 + [Z(\xi_{ecs}) + \xi_{eb} Z'(\xi_{ecs})] \Omega_d \Gamma_{11} \right) \]

\[ - \frac{\omega^2_{\text{peccs}}}{k_s^2 v_e^2} \xi_{eb} Z'(\xi_{ecs}) \Gamma_1 + Z''(\xi_{ecs}) \Omega_d \Gamma_{11}; \] (4.2)
and, finally, the local equation derived using the perpendicular component of Ampere’s law becomes

\[
0 = A_{\delta\phi^3} \delta \phi + A_{\delta a^3} \delta a_{\parallel} + A_{\delta a_{\perp}^3} \delta a_{\perp},
\]

where

\[
A_{\delta\phi^3} = \frac{2\omega_p^2 \omega - k_y v_{de}}{k_s v_e^2} (Z(\xi_{ecs}) \Gamma_1 + Z'(\xi_{ecs}) \Omega_d \Gamma_{11})
\]

\[
+ \frac{2\omega_p^2 \xi_{eb}}{k_s v_e^2} (Z(\xi_{eb}) \Gamma_1 + Z'(\xi_{eb}) \Omega_d \Gamma_{11}),
\]

\[
A_{\delta a^3} = \frac{2\omega_p^2 \omega - k_y v_{de}}{k_s v_e^2} \left( [1 + \xi_{eb} Z(\xi_{ecs})] \Gamma_1 + [Z(\xi_{ecs}) + \xi_{eb} Z'(\xi_{ecs})] \Omega_d \Gamma_{11} \right)
\]

\[
- \frac{\omega_p^2 \xi_{eb}}{k_s v_e^2} (Z(\xi_{eb}) \Gamma_1 + Z''(\xi_{eb}) \Omega_d \Gamma_{11}),
\]

and

\[
A_{\delta a_{\perp}^3} = -\frac{c^2}{v_e^2} + \frac{2\omega_p^2 \omega - k_y v_{de}}{k_s v_e^2} (Z(\xi_{ecs}) \Gamma_2 + Z'(\xi_{ecs}) \Omega_d \Gamma_{21}) + \frac{\omega_p^2 m_e \xi_{ics} Z(\xi_{ics})}{k_s v_e^2 m_i}
\]

\[
+ \frac{2\omega_p^2 \xi_{eb}}{k_s v_e^2} (Z(\xi_{eb}) \Gamma_2 + Z'(\xi_{eb}) \Omega_d \Gamma_{21}) + \frac{\omega_p^2 m_e m_i}{k_s v_e^2 m_i} \xi_{ib} Z(\xi_{ib}).
\]  

The corresponding local dispersion relation is then given by

\[
D_0 = \det \begin{pmatrix}
A_{\delta\phi^1} & A_{\delta a^1} & A_{\delta a_{\perp}^1} \\
A_{\delta\phi^2} & A_{\delta a^2} & A_{\delta a_{\perp}^2} \\
A_{\delta\phi^3} & A_{\delta a^3} & A_{\delta a_{\perp}^3}
\end{pmatrix} = 0. \quad (4.4)
\]

Huba et.al. [26] derived a similar result by integrating the electron Vlasov equation along unperturbed orbits, and expanding around \( k_{\parallel} \) and \( \mathbf{J} \cdot \mathbf{B} = 0 \). The system of equations given by Eq(4.1), Eq(4.2), and Eq(4.3) recovers Huba’s result upon removing the background plasma, the perturbed ion current, the terms proportional to \( v_{de\parallel} \), and the term proportional to \( \delta a_{\parallel} \) in the perturbed ion density. Aside from the neglect of a background plasma, which has a significant impact on mode stability, the simplifications in Huba’s eigenmode equations are justified when \( k_{\parallel}/k_s \ll 1 \). However, obliquely-propagating LHDIs are the subject of this thesis, so the more comprehensive system of equations given by Eq(4.1), Eq(4.2), and
Eq(4.3) is more appropriate. We solve Eq(4.4) at various current sheet positions, $z/L$, and wave vector orientations, $k_{∥}/k_s$, to determine the strongest growth rate and localization of the LHDI as a function of wavelength. The numerical results for current-aligned modes, i.e. $k_x = 0$, are shown in Figure 4.1. The numerical results for modes with finite $k_x$ are shown in Figure 4.2 and 4.3. The adopted parameters are

$$L = 0.17 \rho_{i0}, \quad \frac{\omega_{peb}^2}{\omega_{ce0}^2} = 100, \quad \frac{n_{cs}}{n_b} = 2.7, \quad \frac{B_g}{B_0} = 0.1, \quad \text{and} \quad \frac{T_e}{T_i} = 0.1.$$  

These parameters are consistent with a thin magnetotail current sheet with a guide field and negligible curvature.[3, 43]

Figure 4.1: The frequencies(top left), mode localizations(top right), and associated eigenvectors(bottom), of the most unstable current-aligned modes, scaled by the asymptotic values of $\rho_i$ and $\omega_{ci}$. The localization of the strongest modes is given in(top right) along with the localizations found when the $\nabla B$ drift is removed from the dispersion relation. In (bottom) we compare the relative magnitude of the electromagnetic and electrostatic perturbations as well as the relative magnitude of the perturbed magnetic field components as a function of wavelength.
Figure 4.2: A contour plot of the growth rate of the Type A LHDI at fixed $z = 1.3L$ is given in (top right), where $k_\parallel = 0$ along the dashed line. For the remaining figures we scan over $z$ to find dominant instabilities. The most unstable propagation angles with respect to the local B are given in (top left), showing the deviation of dominant modes from $k_\parallel = 0$ in the short-wavelength domain. The two unstable lobes have similar features, and the frequencies and localizations of the $k_\parallel \geq 0$ lobe are given in (bottom left) and (bottom right), respectively.

Guided by our numerical results, we will develop corresponding analytical theories which will elucidate the underlying physics. We begin with the simplified version of the dispersion relation that agrees with the result derived by Huba, which is valid for modes with $k_\parallel/k_s \sim 0$. We will then discuss modes with finite $k_\parallel$, and discuss their relevance to the observations and simulation results. [62, 31, 51, 12]
Figure 4.3: The relative magnitude of the electromagnetic and electrostatic perturbations and the relative magnitude of the magnetic perturbation components of the most unstable modes given in Figure 4.2, as a function of wavelength.

4.1.1 Type A Modes with $k_\parallel = 0$

We proceed with an analysis of the Type A LHDI at $k_\parallel = 0$ by simplifying the local dispersion relation, given in Eq(4.4), to extract a more instructive expression. At $k_\parallel = 0$, we may ignore the perturbed ion current, the terms proportional to $v_{de}\parallel$, and the term proportional to $\delta a_\parallel$ in the perturbed ion density in Eq(4.4). When $|\xi_{eb}| \gg 1$, and $|\xi_{ecs}| \gg 1$, we may expand the plasma dispersion functions in the electron response in Eq(4.4) and reduce the local dispersion relation to the following:

$$D_0 = \left( A_{\delta \phi 1} \left( \frac{c^2}{v_e^2} + 2\Omega_1\Gamma_2 \right) + 4\Omega_1^2\Gamma_1^2 \right) \left( \frac{c^2}{v_e^2} + \Omega_1\Gamma_0 \right),$$

(4.5)

where

$$\Omega_1 = \frac{\omega_{pe}^2}{k_s^2 v_e^2} + \frac{\omega_{pecs}^2 k_y v_{di}}{k_s^2 v_e^2} \frac{\omega}{\omega}.$$  (4.6)

Here we have ignored the $\nabla B$ drift, and neglected $A_{\delta \phi 2}$, and $A_{\delta a_{\perp 2}}$, which are proportional to $k_\parallel$. The dispersion relation, given in Eq(4.5), has one solution given by $c^2/v_e^2 + \Omega_1\Gamma_0 = 0$, where
which is a purely oscillating mode that propagates against the equilibrium current direction. The unstable modes are found in the remaining roots of the dispersion relation;

\[ A_{\delta \phi_1} \left( \frac{c^2}{v_e^2} + 2\Omega_e \Gamma_2 \right) + 4\Omega_e^2 \Gamma_1^2 = 0. \] (4.8)

Based on Figure 4.2, we assume \( \omega = k_y v_{di} + \delta \omega \), where \( |\delta \omega/k_y v_{di}| \ll 1 \), and thus, \( |\xi_{ics}| \ll 1 \).

In the strong drift limit, \( u_i \gg 1 \), we may also assume \( |\xi_{ib}| \gg 1 \), and perform the appropriate expansions to arrive at the following equation for \( \delta \omega \);

\[
\delta \omega/k_y v_{di} = \left( \frac{c^2}{v_e^2} \left[ 2\Omega_e b + 2\tau \Omega_{cs} b - \tau \Omega_b u_i \frac{1}{u_i^2} \right] + 2\tau^2 \Omega_{cs}^2 b + 2\Omega_e b + 4\tau \Omega_{cs} \Omega_b b \\
-2\tau^2 \Omega_{cs} \Omega_b u_i \frac{1}{u_i^2} - 2\tau^2 \Omega_{cs} \Omega_b \frac{1}{u_i^2} \right) \left( -2i \sqrt{\pi} \tau \Omega_{cs} u_i \left[ \frac{c^2}{v_e^2} + 2\Omega_e b + 2\tau \Omega_{cs} b \right] \\
-\frac{c^2}{v_e^2} \left[ 2\tau \Omega_{cs} (1 - b) + 2\tau \Omega_b \frac{1}{u_i^2} \right] - 4\tau \Omega_e \Omega_b u_i \frac{1}{u_i^2} - 6\tau^2 \Omega_{cs} \Omega_b \frac{1}{u_i^2} \right)^{-1}, \] (4.9)

where

\[ \Omega_e = \frac{\omega_{pe}^2}{k_s^2 v_e^2}, \quad \Omega_{cs} = \frac{\omega_{pecs}^2}{k_s^2 v_e^2}, \quad \text{and} \quad \Omega_b = \frac{\omega_{peb}^2}{k_s^2 v_e^2}. \] (4.10)

Here, we have simplified the Bessel’s functions in Eq(4.4) by letting \( b = k_1^2 \rho_e^2/2 \ll 1 \). The growth rate given by Eq(4.9) is proportional to the \( i\sqrt{\pi} u_i \) term, which is the inverse Landau damping of the current sheet ions. As in the electrostatic LHDI, the Type A electromagnetic LHDI is driven by wave particle resonance with the equilibrium current, which is largely carried by the ions when \( \tau \ll 1 \). The modes, given by Eq(4.9), are unstable when the numerator of Eq(4.9) is positive. Removing the common factor of \( 2\Omega_e + 2\tau \Omega_{cs} \) from the
numerator, the criterion for instability, \( \text{Im}[\delta \omega] > 0 \), can be expressed as follows:

\[
 b - \tau \frac{1}{u_i^2} \frac{n_b}{2n + 2\tau n_{cs}} + \frac{1}{2} \beta_e \frac{n_{cs}}{2n} - \tau \beta_e \frac{n_b}{2n} \frac{1}{u_i^2} > 0, 
\]

(4.11)

where the electron plasma beta, \( \beta_e = \frac{8\pi n_e T_e}{B^2} \), is defined locally. Consistent with the results in the electrostatic limit, a background plasma is essential to the marginal stability of the Type A electromagnetic LHDI at \( k_\parallel = 0 \). The terms proportional to \( \beta_e \) in Eq(4.11) demonstrate the destabilizing electrostatic-electromagnetic coupling which originates in the off-diagonal terms of Eq(4.4). However, the overall effect of plasma \( \beta \) is stabilizing, as seen in Figure 4.2, which shows the localization of dominant modes at the current sheet edge, i.e. \( L \leq z \leq 2L \). This is due to the \( \nabla B \) drift, which suppresses modes with \( k_\parallel = 0 \) near the current sheet center. However, this drift is negligible when \( L \leq z \leq 2L \), and is ignored in the derivation of Eq(4.9).

The first-order \( k_\parallel \) effects greatly complicate the expression given in Eq(4.9). It, however, can be shown that finite \( k_\parallel \) stabilizes the modes given by Eq(4.9). This is consistent with our numerical results for long-wavelength modes, but as the wavelength decreases, finite \( k_\parallel \) becomes destabilizing, as seen in Figure 4.2. Although \( \omega/k_y v_{di} \) is relatively constant in Figure 4.2, when \( k_\parallel = 0 \); \( \omega/k_y v_{di} \) decreases considerably with \( k_s \), and the assumption \( |\delta \omega| \ll 1 \) is no longer acceptable in the short-wavelength domain. Therefore, we must develop a separate analytical theory for these short-wavelength modes.

### 4.1.2 Type A Modes in the Short-Wavelength Domain

As in the electrostatic limit, the Type A electromagnetic LHDI is destabilized by finite \( k_\parallel \) in the short-wavelength domain. We will demonstrate this destabilization with the same procedure used in the electrostatic work. Assuming \( |k_\parallel| \ll k_s \), we may simplify the dispersion
relation given by Eq(4.4) to the following:

\[
D_0 = A_{\delta \phi_1} \left( \frac{c^2}{v_e^2} + \Omega_1 \Gamma_0 \left( 1 + \frac{3}{2 \xi_{eb}^2} \right) \right) \left( \frac{c^2}{v_e^2} + 2 \Omega_1 \Gamma_2 \left( 1 + \frac{1}{2 \xi_{eb}^2} \right) \right) - A_{\delta \phi_1} \Omega_1^2 \Gamma_1^2 \frac{1}{\xi_{eb}}
+ \Omega_1 \Gamma_0 \frac{1}{\xi_{eb}^2} \left( -2 \Omega_1^2 \Gamma_1^2 + \Omega_1 \Gamma_0 \left( \frac{c^2}{v_e^2} + 2 \Omega_1 \Gamma_2 \right) \right)
- 2 \Omega_1 \Gamma_1 \left( 1 + \frac{1}{2 \xi_{eb}^2} \right) \left( \Omega_1^2 \Gamma_0 \Gamma_1 \frac{1}{\xi_{eb}^2} - 2 \Omega_1 \Gamma_1 \left( 1 + \frac{1}{2 \xi_{eb}^2} \right) \left( \frac{c^2}{v_e^2} + \Omega_1 \Gamma_0 \left( 1 + \frac{3}{2 \xi_{eb}^2} \right) \right) \right),
\]

where \( \Omega_1 = \frac{\omega_{pe}^2}{k_s v_e^2} + \frac{\omega_{pe}^2 \tau k_i v_{di}}{k_s v_e^2} \). \( (4.12) \)

Here, we have expanded the plasma dispersion function in the electron response and ignored the perturbed ion current, the term proportional to \( \delta a_\parallel \) in the perturbed ion density, and terms proportional to \( v_{de} \parallel \) and the \( \nabla B \) drift. As in the electrostatic limit, we evaluate \( d\omega_0 / dk_\parallel = -\partial_{k_\parallel} D_0 |_{\omega=\omega_0} (\partial_\omega D_0)^{-1} \), where \( \omega_0 \) are the solutions of the local dispersion relation, \( D_0 \). At \( k_\parallel = 0 \), \( d\omega_0 / dk_\parallel = 0 \), indicating a local extremum in the growth rate. To determine where this extremum is a local minimum, we evaluate \( d^2 \omega_0 / dk_\parallel^2 \) at \( k_\parallel = 0 \), which is given by the following expression

\[
\frac{d^2 \omega_0}{dk_\parallel^2} = \frac{v_e^2}{\omega^2} \left( A_{\delta \phi_1} \left[ 2 \Omega_1 \Gamma_2 \left( \frac{c^2}{v_e^2} + \Omega_1 \Gamma_0 \right) + 3 \Omega_0 \Gamma_0 \left( \frac{c^2}{v_e^2} + 2 \Omega_1 \Gamma_2 \right) - 2 \Omega_1^2 \Gamma_1^2 \right] \right)
+ \frac{c^4}{v_e^4} \Omega_1 (-2 \Gamma_0) + \frac{c^2}{v_e^2} \Omega_1^2 (-4 \Gamma_0 \Gamma_2 + 8 \Gamma_1^2) + \Omega_1^2 (12 \Gamma_0 \Gamma_1^2) \left( \partial_\omega D_0 \right)^{-1}. \quad (4.13)
\]

Because we evaluate this derivative at the solutions of \( D_0 \), we may substitute

\[
A_{\delta \phi_1} = 4 \Omega_1^2 \Gamma_1^2 \left( -\frac{c^2}{v_e^2} - 2 \Omega_1 \Gamma_2 \right)^{-1}, \quad (4.14)
\]

from Eq(4.8), which yields

\[
\frac{d^2 \omega_0}{dk_\parallel^2} = \frac{v_e^2}{\omega^2} \left( \frac{c^6}{v_e^6} \Omega_1 (2 \Gamma_0) + \frac{c^4}{v_e^4} \Omega_1^2 (8 \Gamma_0 \Gamma_2 - 8 \Gamma_1^2) \right)
+ \frac{c^2}{v_e^2} \Omega_1^3 \Gamma_2 (8 \Gamma_0 \Gamma_2 - 8 \Gamma_1^2) + \Omega_1^2 \Gamma_1^2 (8 \Gamma_0 \Gamma_2 - 8 \Gamma_1^2) \left( \partial_\omega D_0 \right)^{-1} \left( \frac{c^2}{v_e^2} + 2 \Omega_1 \Gamma_2 \right)^{-1}. \quad (4.15)
\]
The sign of the imaginary part of \( d^2 \omega_0 / dk_{\|}^2 \), given above, will illustrate the influence of finite \( k_{\|} \) on mode stability. At marginal stability, only the ion response terms in \( \partial_\omega D_0 \) are complex, and the imaginary part of Eq(4.15) is contained in the following terms:

\[
(\partial_\omega A_{\delta \phi 1}) \left( \frac{c^2}{v_e^2} + \Omega_1 \Gamma_0 \right) \left( \frac{c^2}{v_e^2} + 2\Omega_1 \Gamma_2 \right).
\tag{4.16}
\]

Noting that \( \Gamma_0 \Gamma_2 - \Gamma_1^2 > 0 \) for all arguments, and that \( \Omega_1 > 0 \) for \( \omega > 0 \), the condition for the destabilization of the Type A electromagnetic LHDI by finite \( k_{\|} \) becomes

\[
\text{Im}[(\partial_\omega A_{\delta \phi 1})] = \frac{1}{\omega} \text{Im} \left[ \frac{\omega_0^2 v_e^2}{k_y^2 v_i^2} \right] \left[ \frac{\omega_{peb}^2}{k_y^2 v_e^2} \xi_{ib} Z''(\xi_{ics}) + \frac{\omega_{peb}^2}{k_y^2 v_e^2} \xi_{ib} Z''(\xi_{ib}) \right] > 0.
\tag{4.17}
\]

Because we ignore the perturbed ion current for modes with \( k_{\|} = 0 \), the condition for the marginal stability of the Type A electromagnetic LHDI is the same as the electrostatic LHDI when \( k_{\|} = 0 \). Thus, we see the perturbed electron current and associated local equations derived from Ampere’s law do not change the condition for the destabilization of the LHDI by finite \( k_{\|} \). Thus, we recover the criteria found in the electrostatic limit;

\[
-\alpha^2 + \alpha - 1/2 u_i^2 > 0,
\tag{4.18}
\]

where \( \alpha = \omega / k_y v_{di} \), and \( u_i = k_y v_{di} / k_s v_i \), which requires

\[
\frac{v_{di} B_x}{v_i B} > \sqrt{2}.
\tag{4.19}
\]

The Type A electromagnetic LHDI is destabilized by finite \( k_{\|} \) in the same conditions and by the same mechanism as the electrostatic LHDI. In current sheets with strong equilibrium drifts, given by Eq(4.19), there exists a frequency range in which the strength of the inverse Landau damping of the current sheet ions, relative to the Landau damping of the background ions, increases with increasing frequency. At short wavelengths, modes with \( k_{\|} = 0 \) enter this
frequency range, and the increase in $\omega$ with $k_\parallel$ is destabilizing. In this argument, we made several approximations to simplify $D_0$ including the neglect of the perturbed ion current and terms proportional to $v_{de\parallel}$ and the $\nabla B$ drift. These approximations are all justified in the limit of $|k_\parallel/k_s| \ll 1$, with the exception of the weak $\nabla B$ drift approximation, which becomes invalid as $z \to 0$.

Upon including the first-order $\nabla B$ drift effects in this analysis, Eq(4.15) becomes

$$
\frac{d^2 \omega_0}{d k_\parallel^2} = \frac{v_e^2}{\omega^2} \left( \frac{c^6}{v_e^6} \Omega_1 (2 \Gamma_0 + 6 \bar{\omega} \Gamma_{01}) + \frac{c^4}{v_e^4} \Omega_1^2 (8 \Gamma_0 \Gamma_2 - 8 \Gamma_1^2 + \bar{\omega} \Gamma_A) 
+ \frac{c^2}{v_e^2} \Omega_1^3 \Gamma_2 (8 \Gamma_0 \Gamma_2 - 8 \Gamma_1^2 + \bar{\omega} \Gamma_B) + \Omega_1^4 \Gamma_2^2 (8 \Gamma_2^2 \Gamma_2 - 8 \Gamma_1^2 + \bar{\omega} \Gamma_C) \right) \left( \partial_\omega D_0 \right)^{-1} 
\left( \frac{c^2}{v_e^2} + 2 \Omega_1 \Gamma_2 + 2 \bar{\omega} \Omega_1 \Gamma_2 \right)^{-1},
$$

(4.20)

where

$$
\Gamma_A = 24 \Gamma_{01} \Gamma_2 - 32 \Gamma_{01} \Gamma_1 + 8 \Gamma_{21} \Gamma_0,
\Gamma_B = \Gamma_{01} (32 \Gamma_1^2 + 4 \Gamma_0 \Gamma_2 + 24 \Gamma_2^2) + \Gamma_{11} (-8 \Gamma_0 \Gamma_1 - 48 \Gamma_1 \Gamma_2) + \Gamma_{21} (8 \Gamma_2^2 + 16 \Gamma_0 \Gamma_2),
\Gamma_C = 72 \Gamma_{01} \Gamma_2^2 \Gamma_2 - 48 \Gamma_{11} \Gamma_1^3 + 24 \Gamma_{21} \Gamma_0 \Gamma_2^2, \quad \text{and} \quad \bar{\omega} = \frac{k_\parallel}{\omega} v_d.
$$

(4.21)

Here, we retain the complete Bessel’s functions to account for the increase in $\rho_e$ as $z \to 0$. The terms proportional to the $\nabla B$ drift in Eq(4.20) are positive for all arguments of the Bessel’s functions, and thus, the conditions given in Eq(4.18) and Eq(4.19) are unchanged.

These obliquely-propagating modes do not dominate in the long-wavelength domain, and as seen in Figures 4.2 and 4.3, they are quasi-electrostatic, and are localized near the current sheet edge. These results are not consistent with the Cluster observations[62] and MRX results[31], which report long-wavelength, electromagnetic modes near the current sheet center. Although the solutions shown in Figures 4.2 and 4.3 are the dominant instabilities throughout the current sheet, we will now consider the possibility of weaker instabilities that
are operative near the current sheet center which may share some of the characteristics found in the simulations[12, 51] and observations.

4.1.3 Type B Modes in the Long-Wavelength Domain

We proceed with a more thorough investigation of the instabilities that are active near the current sheet center. The contour plot of growth rate given in Figure 4.2 is evaluated at $z = 1.3L$. As $z \to 0$, the contour plot of growth rate reveals evidence of a second branch of instabilities; called Type B in contrast to the Type A mode which is strongest at the current sheet edge. In Figure 4.4, we see the emergence of dominant long-wavelength modes with finite $k_\parallel$ at $z = 0.6L$. At $z = 0.3L$, the obliquely-propagating Type B modes become clearly distinct from, and dominant over the long-wavelength Type A LHDIs, which are strongest when $k_\parallel = 0$. The Type B modes are unstable for a wide range of $k_\parallel/k_s$, and the dominant modes propagate at $\theta \sim 40^\circ$ with respect to the local magnetic field.

These obliquely-propagating Type B modes challenge several of the limitations of our model. Near the current sheet center, the $\nabla B$ drift exceeds the equilibrium drift, $v_{di}$, which contradicts the assumption $|k_\zeta v_d/\omega| \ll 1$, that is used to derive the perturbed electron response, as seen in Eq(2.47), Eq(2.54), and Eq(2.61). Additionally, near the current sheet center, the gyrokinetic orderings, $|\omega| \sim |k_\parallel v_e| \ll |\omega_{ce}|$, become dubious. These violations of self-consistency are avoided when the equilibrium guide field is increased, which is done for the remainder of the analysis. Unfortunately, this precludes an assessment of the significance of the guide field to the properties of these modes, since an analysis at a weaker guide field strengths is beyond the limit of the current model.

In the simulations by Wang et al.[51], electromagnetic modes near the current sheet center arose when the background plasma density was reduced, and the guide field strength was increased to $B_g/B_0 = 0.2$. So, we adopt the parameters used in those simulations to
Figure 4.4: A contour plot of growth rate at $z=0.6L$ (top) and $z=0.3L$ (bottom left), and (bottom right). Here $k_\parallel = 0$ along the dashed line, and the dashed arc in (bottom right) shows $k_\sqrt{\rho_e \rho_i} = 1$.

compare our results, and to allow a self-consistent investigation of these modes. The relevant parameters from the simulations are

$$L = 0.12 \rho_i, \quad \frac{\omega_{pe}^2}{\omega_{ce0}^2} = 100, \quad \frac{n_{ca}}{n_b} = 5.2, \quad \frac{B_g}{B_0} = 0.2, \quad \text{and} \quad \frac{T_e}{T_i} = 0.1.$$  

For these parameters, the two branches of unstable solutions are present, and the orderings adopted in our model are valid. The dominant Type B modes are depicted in Table 4.1.
Table 4.1: The properties of dominant Type B modes scaled by the ion Larmor radius, $\rho_i$ and ion cyclotron frequency, $\omega_{ci}$ in the asymptotic magnetic field. The table shows the wavevector, the propagation angle with respect to the local magnetic field, the mode frequencies, mode localizations, the relative strength of the electromagnetic perturbations, and the relative strength of the perturbed magnetic field components.

As the wavelength decreases, the Type B modes merge with the Type A modes, i.e., for $k_s \rho_i \geq 17.5$, the growth rate smoothly increases as $k_\parallel$ approaches zero, and the localization approaches the current sheet edge. As seen in Figure 4.4, there are two groups of Type B modes, one with $k_\parallel > 0$ which are given in Table 4.1, and a group with $k_\parallel < 0$. These two groups have comparable growth rates, but for $k_\parallel < 0$, the dominant instabilities have considerably lower real frequencies. As such, these solutions do not satisfy the assumption that $|k_\parallel v_d/\omega| \ll 1$ used in our model, and are unreliable. However, the instabilities with $-1 \ll k_\parallel/k_s < 0$ do satisfy the assumptions of our model. So, although we cannot comment on the properties of the strongest Type B modes with $k_\parallel < 0$, we can report that the growth rate of these modes increases as $k_\parallel/k_s$ decreases below 0, and the modes have features that are comparable to the analogous instabilities with $k_\parallel > 0$.

The results given in Table 4.1 show that for the parameters used in the simulation by Wang et al.[51], the dominant Type B modes are excellently approximated as current-aligned. Although weaker guide fields could not be examined by our model, we postulate that the orientation of the simulation plane was responsible for the transition between the typical Type A LHDIs and these obliquely-propagating Type B modes that was observed in the simulations.
The Type B modes given in Table 4.1 have features similar to the electromagnetic instabilities found in simulations of a thin current sheet with a guide field by Daughton[12]. Daughton[12] reported the suppression of these modes when the drift velocity was reduced from \( v_{di} = 2v_i \) to \( v_{di} = v_i \). Our results confirm the damping of these modes at lower equilibrium drift velocity, but at long wavelengths, \( k_s \rho_{i0} \sim 1 \), the instability persists at \( v_{di} \ll v_i \), though the dominant propagating angle, \( k_\parallel/k_s \), increases as \( v_{di}/v_i \) decreases. The threshold found by Daughton may be a consequence of the simulation plane, which does not include the wavevectors of the dominant instabilities at lower drift velocities.

Unlike the Type A LHDI, these obliquely-propagating Type B modes are strongest near the location where the electron \( \nabla B \) drift is strongest, i.e., near the current sheet center, and are suppressed when the \( \nabla B \) drift is removed. This is seen in the localization of the dominant current-aligned modes, shown in Figure 4.1, which move away from the current sheet center when the \( \nabla B \) drift is removed. Consequently, an analytical expression which describes the mode frequencies and elucidates the underlying physics of these Type B modes must include the electron \( \nabla B \) drift. Additionally, at moderate \( k_\parallel/k_s \), the electron Landau damping cannot be ignored. So, despite our best efforts, the electron response in Eq(4.4) could not be simplified to a tractable analytical expression that accurately depicted these instabilities, and our analysis must rely on numerical solutions.

4.2 Nonlocal Stability Analysis

We continue our analysis by relaxing the \( k_z = -i\partial_z = 0 \) assumption in Eq(2.77), Eq(2.78) and Eq(2.79). The nonlocal eigenfunctions and eigenvalues are solved using the Tau spectral method. We begin with a coordinate transformation, \( \text{Tanh}(z/L) = y \), and choose a set of basis functions, \( f_n \), that satisfy \( f_n(y = \pm 1) = 0 \). The results for the Type A LHDI, which are localized near the point where \( k_\parallel = 0 \), are shown in Figures 4.5, and 4.6.
As the wavelength decreases, the perturbed magnetic vector potential becomes much weaker than the perturbed electric potential, which is consistent with the local prediction shown in Figure 4.3. The solutions justify the small $\nabla B$ drift approximation, $|k_z v_d/\omega| \ll 1$, as well as the gyrokinetic orderings. The inhomogeneous scale length defined as $|\partial_z \ln[\delta(z)]|$ is much less than $k_s$ for all solutions with $k_s \rho_{i0} \geq 0.15$. This is evaluated for all three eigenfunction components, $\delta(z)$, and justifies the small argument expansion of $k_z/k_s$ used to derive Eq(2.77), Eq(2.78), and Eq(2.79).

Because the obliquely-propagating Type B modes, given in Table 4.1, are so accurately approximated as current-aligned modes, we will investigate the current-aligned modes to allow a direct comparison with the results of the GeFi simulation. These results are given in Figures 4.7 and 4.8.

The solutions justify the approximations and orderings of our model, with the exception of the small argument expansion of $k_z/k_s$. For $k_y \rho_{i0} \geq 10$, the inhomogeneous scale length of the eigenfunction components is $\leq 0.5$, and drops further at larger $k_y \rho_{i0}$. Because the eigenmode equations are solved using the Tau spectral method, a higher order expansions of $k_z/k_s$ in the system of eigenmode equations given by Eq(2.77), Eq(2.78), and Eq(2.79) could
Figure 4.6: The real and imaginary parts of the eigenfunctions of the most unstable mode, at $k_x \rho_{i0} = 0.35$, are shown above. The perturbed electric potential is given in (top left), and the parallel and perpendicular components of the perturbed magnetic vector potential, scaled by $v_e/c$, are given in (top right) and (bottom), respectively. The vertical line shows the localization predicted by local theory.

be undertaken. However, such a modification of the model will not be considered here. For the solutions given in Figure 4.7 and 4.8, the mode localization tends toward the current sheet edge as $k_y \rho_{i0}$ increases, and the amplitude of the perturbed magnetic vector potential decreases relative to the amplitude of the perturbed electric potential. This is consistent with the transition to Type A LHDI properties as the wavelength decreases.

The theoretical results and simulations shown in Figure 4.7 agree on the real frequencies of these modes, but report significantly different growth rates. These simulations were carried out using the GeFi simulation scheme developed by Lin et al.[36], which was subsequently revised to include more nonuniform density and magnetic field effects[35]. The comparison
Figure 4.7: The real frequency and growth rate of the most unstable current-aligned modes, i.e. $k_x = 0$, are shown in (left) and (right), respectively. We evaluate $\rho_{i0}$ and $\omega_{ci}$ in the asymptotic magnetic field, and include results from local theory, nonlocal theory, and the GeFi simulation results[51]. The vertical line in (right) shows where $k\sqrt{\rho_{e0}\rho_{i0}} = 1$.

performed on the electrostatic work, shown in Chapter 2, was done with the new GeFi simulation, and demonstrated strong agreement between the theory and simulations. A new comparison between the theoretical results and the modern GeFi simulation of these electromagnetic modes is underway, and initial results are promising.

4.3 Summary

We have investigated electromagnetic LHDIs in a thin Harris current sheet with a guide field using local and nonlocal theories, and simulations. We have derived an analytical expression for the local stability of Type A modes with $k_\parallel \sim 0$, given in Eq(4.11), which identifies the inverse Landau damping of the current sheet ions as the drive of the instability. The Type A modes are stabilized by finite $k_\parallel$ and the electron $\nabla B$ drift, and are destabilized by electromagnetic-electrostatic coupling. The growth rate, given by Eq(4.11), peaks at the current sheet edge, $z \sim 1.5L$, where a balance exists between an enhanced electron $\nabla B$ drift stabilization as $z \rightarrow 0$, and a stabilizing increase in the relative background population as $z \rightarrow \infty$. The nonlocal eigenvalue equations and the perturbed responses are derived under the
Figure 4.8: The real and imaginary parts of the eigenfunctions of the most unstable mode, at \(k_y \rho_{i0} = 10\), are shown above. The perturbed electric potential is given in (top left), and the parallel and perpendicular components of the perturbed magnetic vector potential, scaled by \(v_e/c\), are given in (top right) and (bottom), respectively. The vertical line shows the localization predicted by local theory.

The assumption of weak inhomogeneity, \(|k_z/k_s| \ll 1\), which is less accurate for long-wavelength modes. Nevertheless, the nonlocal eigenvalue solutions and local results agree well for all wavelengths, as seen in Figures 4.5 and 4.6, and the solutions satisfy our gyrokinetic electron, fully-kinetic ion ordering, as well as our weak \(\nabla B\) drift assumption, \(\omega_d/k||v_e \ll 1\).

As the wavelength decreases, Type A modes with finite \(k||\) begin to dominate Type A modes with \(k|| = 0\), as seen in Figure 4.2. This occurs in a frequency range in which the overall destabilization of the ion response increases with increasing frequency. Within this range, the real part of the mode frequencies increases with increasing \(|k||/k_s|\), which, in turn, destabilizes the modes. These dominant Type A modes with finite \(k||\) do not arise without sufficiently
strong equilibrium drifts, given by Eq(4.19). The agreement between our numerical local analysis and the nonlocal eigenvalue problem is shown in Figure 4.5. The solutions satisfy our gyrokinetic electron, fully-kinetic ion ordering, as well as our weak $\nabla B$ drift assumption, $\omega_d/k_\parallel v_e \ll 1$.

With the exception of the destabilization by finite $k_\parallel$ in the short-wavelength domain, these Type A modes resemble typical LHDIIs investigated extensively by previous authors[13, 16, 34, 57]. The dominant instabilities are localized near the current sheet edge, $z \sim 1.5L$, and the perturbed magnetic field amplitudes are much lower than the perturbed electric field amplitudes. Near the current sheet center, $z \ll L$, the Type B instabilities emerge, which are destabilized by finite $k_\parallel$ and are suppressed when the electron $\nabla B$ drift is removed. These obliquely-propagating Type B instabilities have several features in common with the fluctuations found in simulations[12, 51] and observed in the magnetotail[62] and at the MRX[31]. The modes are unstable for a wide range of $k_\parallel/k_s$, and the dominant modes peak when the propagation angle reaches $\theta \sim 45^\circ$ with respect to the local magnetic field, i.e. $k_\perp \sim k_\parallel$. The dominant Type B instabilities are localized near the current sheet center, $z \sim 0.2L$, with larger perturbed magnetic field amplitudes than perturbed electric field amplitudes, and the strongest instabilities occur at $k_\parallel^2 \rho_{eo} \rho_{i0} \sim 1$. Due to the significance of the electron $\nabla B$ drift, and the finite $k_\parallel$, the local dispersion relation could not be simplified to a tractable analytic expression that explains the detailed instability mechanism and threshold.

The simulations by Daughton[12] suggest these obliquely-propagating modes require a strong equilibrium drift velocity, and in simulations by Wang et al.[51], such modes arose when the guide field strength was increased and the background plasma was reduced. However, our results show that these modes persist in the weak drift domain, $v_{di} \ll v_i$, and although our results for weak guide fields are not reliable due to self consistency violations, we postulate that the threshold conditions reported by Daughton[12] and Wang et al.[51] reflect the limitations of a two dimensional analysis.
Based on the distinction of these Type B modes from typical Type A LHDIas as $\alpha \rightarrow 0$, and the influence of the electron $\nabla B$ drift, we postulate that these modes do not belong to the conventional LHDI branch of solutions. Other instabilities have been considered for their potential activity near the center of a current sheet including the Kelvin-Helmholtz, ion-Weibel, and ion-ion modes[57, 37]. These modes are not expected to operate in equilibria without velocity shear, or with a single ion species[37, 56, 9]. However, the Type B modes persist when the background plasma is removed, which also eliminates the equilibrium velocity shear. Modes like the drift-kink or drift-sausage instability are similarly considered when activity near the current sheet center is observed[57, 37]. However, these drift modes generally peak when $k\rho_i \sim 1$ and $k_\parallel = 0$, and are expected to have weak growth rates, $\omega_{\text{Im}} \lesssim \omega_{ci}$ at realistic mass ratios[58, 11]. The disparity between the properties of these known current sheet instabilities and the Type B modes presented in this thesis is beyond explanation by simple extension or analogy. Unfortunately, at this time, an analytical theory for the instability mechanism and underlying physics of these Type B modes are beyond the reach of this thesis and must be left to future investigations.
Chapter 5

Conclusions

In this thesis, a system of eigenmode equations, given by Eq(2.77), Eq(2.78), and Eq(2.79), is derived using the gyrokinetic electron, fully-kinetic ion model to study drift instabilities in a Harris current sheet with a guide field. After a successful benchmark of the LHDI in the electrostatic limit with GeFi simulations, an analysis of electromagnetic drift instabilities is carried out to investigate potential sources of anomalous resistivity.

5.1 Results of the Local Stability Analysis

An analytical expression is derived for the local stability of LHDIs with $|k_\parallel/k_s| \ll 1$ in the electrostatic limit, Eq(3.9), and for Type A electromagnetic LHDIs, Eq(4.9). These expressions show the Type A LHDI is driven by the inverse Landau damping of the current sheet ions, and is stabilized by finite $k_\parallel$, the electron $\nabla B$ drift, and the background plasma. These expressions are valid at long wavelengths, but in the short-wavelength domain, $k_s\rho_{e0} \gtrsim 0.5$, the phase velocity of the dominant instabilities becomes substantially lower than the equilibrium ion drift velocity, and the dispersion relation cannot be accurately simplified to a
tractable analytical expression. In Chapters 3 and 4, it is shown that in current sheets with strong equilibrium drifts, \( v_{di}B_z/v_iB > \sqrt{2} \), there exists a range of phase velocities in which the overall destabilization of the modes by wave-particle resonance with the ions increases with increasing phase velocity. At short wavelengths, the Type A LHDI enters this phase velocity range, and the increase in phase-velocity with finite \( k_\parallel \) is destabilizing. This is seen in Figure 3.2 and 4.2, though this conclusion is only meaningful in the local limit. In reality, \( k_\parallel = k_s \cdot \hat{b} \) varies across the mode width, but the resilience of these modes to \( k_\parallel \) stabilization can increase the nonlocal mode width, and reduce the shear-stabilization of the instability. This is crucial for instabilities near the current sheet center, where the shear is strongest. However, as seen in Figures 3.2 and 4.2, these short-wavelength, obliquely-propagating Type A modes are localized near the current sheet edge, \( z \sim 1.5L \). This is due to the \( \nabla B \) drift stabilization, which strongly suppresses modes near the current sheet center, and as such, these linear results do not show promise for Type A LHDI-generated anomalous resistivity near a current sheet center.

Near the current sheet center, the Type B LHDI emerges, as shown in Figure 4.4 and Table 4.1. These modes are destabilized by finite \( k_\parallel \) and are supported by the \( \nabla B \) drift and, consequently, are localized near the current sheet center, \( z \sim 0.2L \). The dominant instabilities occur at \( k_s^2 \rho_i \rho_e \sim 1 \), with \( |\mathbf{k} \times \delta \mathbf{E}|/|\mathbf{k} \cdot \delta \mathbf{E}| > 1 \). These instabilities resemble the fluctuations observed in the magnetotail[62] and at the MRX[31], and found in simulations[51, 12], and persist for a wide range of equilibrium drift velocities. The dominant Type B modes challenge several of the limitations of our model, but at large drift velocity and moderate guide field strength, the results maintain self-consistency. At finite \( k_\parallel \), the electron response in the dispersion relation cannot be accurately simplified to a tractable analytical expression, and our results are entirely numerical.
5.2 Results of the Nonlocal Analysis

Guided by the results in the local limit, the system of nonlocal eigenmode equations, given by Eq(2.77), Eq(2.78), and Eq(2.79), is solved for the wavevectors of the dominant instabilities from the local stability analysis. The derivation of the eigenmode equations assumes weak inhomogeneous scale lengths, $|k_z/k_s| \ll 1$, which, in principle, is unreliable at long wavelengths. Nevertheless, the nonlocal analysis is carried out for the entire range of unstable wavelengths, and the results agree well with the local analysis. The results in the electrostatic limit are compared with simulations employing the new GeFi simulation scheme developed by Lin et al.[35]. As seen in Figure 3.3, the agreement is excellent. The electromagnetic results are compared with available simulations using an older version of the GeFi simulation scheme[36]. As seen in Figure 4.7, the agreement between the theory and simulations is excellent for the real mode frequencies, but the growth rates are considerably different. A comparison with the improved GeFi scheme is underway, and initial results are encouraging.

5.3 Future Research

The instabilities of interest are the Type B, obliquely-propagating drift instabilities operative near the current sheet center. These modes challenge the gyrokinetic ordering $|k_\parallel \rho_e| \ll 1$, and the weak $\nabla B$ drift approximation, $k_\zeta v_d/\omega$, used to derive Eq(2.77), Eq(2.78), and Eq(2.79). These challenges are avoided in equilibria with moderate guide fields, $B_g \lesssim B_0$, which reduce the peak values of the $\nabla B$ drift and the electron Larmor radius, $\rho_e$. An improved model, that relaxes the weak $\nabla B$ drift assumption, and includes higher order $k_\parallel$ effects, would allow analyses of this instability over a wider range of guide field strengths to better understand the influence of a guide field on these modes.
The perturbed electric and magnetic fields generated by these instabilities could facilitate momentum exchange between the electrons and waves, giving rise to collisionless anomalous resistivity near the center of a current sheet. The strength, and thus, the relevance of the ensuing anomalous resistivity depends on the amplitude of the perturbations, which cannot be determined with linear theories, like those presented in this thesis. Several nonlinear saturation mechanisms of the Type A LHDI have been proposed, including current relaxation\cite{28}, mode coupling\cite{15}, and electron resonance broadening\cite{29}. These theories, and subsequent simulations studies\cite{47, 24}, are guided by the conventional LHDI properties, namely, $k \simeq k_{\perp} \sim 1/\rho_e$. This treatment is not appropriate for the Type B LHDIs described in this thesis, and new nonlinear analyses are necessary to determine the possible influence of these modes on magnetic reconnection.

Much of the recent collisionless magnetic reconnection research has relied on experiments and simulations, and considers antiparallel magnetic field equilibria.\cite{54} In current sheets with guide field, the electron magnetization persists in the dissipation region, which enables the use of gyrokinetic theory to analyze current sheets analytically. Analytical models offer the potential for a physical interpretation of results, and the ability to perform three dimensional investigations under various equilibrium conditions that are too computationally expensive to perform with simulations. The ability to guide future simulations and offer physical insight make theoretical models a valuable tool for future research on magnetic reconnection.
Bibliography


