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AN ITERATIVE METHOD FOR SOLVING A CLASS
OF INTEGRAL EQUATIONS IN POTENTIAL AERODYNAMICS*

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Abstract

An optimum multiparameter iterative method for solving a class of linear algebraic systems is formulated. The systems are approximants of regularized integral equations proposed earlier for subsonic panel methods. The optimum parameters are determined explicitly for the convex, non-lifting case. The corresponding method is approximately twice as fast as the non-extrapolated basic method while the additional computational labor is negligible.
1. Introduction

Boundary integral equation methods (panel methods) are a standard tool for the prediction of subsonic aerodynamic characteristics of general 3-dimensional configurations. However, it is still very desirable to increase the efficiency of these methods if they are to be used in time-stepping computations for 3-dimensional unsteady problems where the discretized integral equations have to be solved at each time step.\textsuperscript{1-4}

Maskew\textsuperscript{5} has recently shown that two particular low-order formulations\textsuperscript{5-10} are remarkably robust with respect to body shape and more efficient than the previous higher order formulations. A problem with the simpler and most efficient formulation\textsuperscript{5,8} has been eliminated.\textsuperscript{10,11} For both formulations, regularized integral equations and a new low-order implementation have been proposed.\textsuperscript{10}

In this note—which is closely related to Ref. 10—we formulate an iterative method for solving the linear algebraic systems which approximate the singular equations and their regularized versions. The method exploits the particular structure of the coefficient matrices and does not destroy the structure of the regularized equations. It is about twice as fast as ordinary iteration. These features are realized at the expense of generality, i.e., the parameters for the "optimum" method could be determined explicitly for strictly convex, nonlifting bodies only. The derivation of the method, however, suggests possible generalizations. (Also, the method could be based on the Gauss-Seidel iteration instead of the ordinary iteration.)

The method is similar to the optimum extrapolated methods\textsuperscript{12,13} with the difference that the optimum set of parameters \(\{\beta_1^*, \ldots, \beta_n^*\}\) minimizes the uniform norm (row-norm) of the iteration matrix rather than its spectral radius. The advantage of this approach is that no estimates of the eigenvalues are
required. Also, we are led to introduce a set of \( n \) parameters \((n = \text{number of panels})\) instead of a single parameters \( \beta \) considered in Refs. 12, 13. The optimum values of the parameters are related to the "singularity" of the discretized kernel of the integral equation and the corresponding method converges in the uniform norm. Ignoring this "singularity" corresponds to ordinary iteration of the singular equation which is divergent. In contrast, ordinary iteration of the regularized equations is convergent. The proposed optimum extrapolated method accelerates the convergence by a factor of two \((n \to \infty)\) at negligible additional computational cost.

2. The Integral Equations and Their Approximants

It is convenient to write down the equations involved in the analysis for steady, nonlifting potential flow in the region \( R+ \) around a rigid, impermeable, convex body \( R_- \) bounded by the surface \( S \) (see Ref. 10). If \( \Phi_* \) denotes the potential of the freestream velocity \( V = \nabla \Phi_* \) then the disturbance potential \( \varphi \) is the solution of the exterior Neumann problem

\[
\Delta \varphi(P) = 0 \quad (P \in R_+), \\
\frac{\partial \varphi_+}{\partial n_p}(P) = - \frac{\partial \Phi_*(P)}{\partial n_p} \quad (P \in S), \\
\lim_{|P| \to \infty} \varphi(P) = 0 \quad (P \in S).
\]

In the simpler formulation, \( \varphi \) is written as the potential of a double layer on \( S \):

\[
\varphi(P) = \frac{1}{4\pi} \int_S \Phi_+(q) K(q,p) \, dS_q \quad (P \in R_+). \tag{1}
\]

The kernel is defined by

\[
K(q,p) = \frac{\partial}{\partial n_q} \left( \frac{1}{|P - q|} \right) = n_q \cdot \nabla_q \left( \frac{1}{|P - q|} \right) \quad (q \in S)
\]

and has a singularity \( O(|P - q|^{\alpha - 2}) \) for \( P \to q \), where \( \alpha(0 < \alpha \leq 1) \) is the Lyapunov
exponent of $S$. $K$ has the property

$$\int_S K(q,p) dS_q = -2\pi \quad (p \in S), \quad (2)$$

for closed surfaces $S$ since $K(q,p) dS_q$ is the negative solid angle subtended by $dS_q$ at $P$. The density $\Phi_+$ of the doublet distribution (1) is the value of the total velocity potential $\Phi = \Phi_v + \Phi$ on $S$ and is the unique solution of the integral equation:

$$\Phi_+(p) - \frac{1}{2\pi} \int_S \Phi_+(q) K(q,p) dS_q = 2\Phi_v(p) \quad (p \in S) \quad (3a)$$

which is mathematically equivalent to the regularized equation:

$$\Phi_+(p) = \frac{1}{4\pi} \int_S \left[ \Phi_+(q) - \Phi_+(p) \right] K(q,p) dS_q + \Phi_v(p) \quad (3b)$$

The low-order implementation\(^{10}\) of (3a,b) leads to the linear systems

$$\tilde{\Phi}_j - \frac{1}{2\pi} \sum_{i=1}^{3} \tilde{\omega}_j^{i} \tilde{\Phi}_i = 2\Phi_v(p_j), \quad (4a)$$

$$\tilde{\Phi}_j - \frac{1}{4\pi} \sum_{i=1}^{3} \tilde{\omega}_j^{i} (\tilde{\Phi}_i - \Phi_j) + \Phi_v(p_j), \quad (4b)$$

respectively ($j = 1, \ldots, n$).

We are interested in the iterative solution of these equations and of equations of this form (which occur in the more general and improved formulations\(^{10,11}\)) in order to compare their efficiency and robustness.

We expect the combination (3b), (4b) to be numerically superior to the combination (3a), (4a) for the following reason. The matrix elements $\tilde{\omega}_j^{i}$ are approximations of (negative) solid angles

$$\omega_j^{i} = \int_{S_i} K(q,p_j) dS_q$$

subtended by $S_i$ at the centroid $p_j$ of $S_j$. Although the approximations $\tilde{\omega}_j^{i} \approx \omega_j^{i}$, as defined in Appendix 2 of Ref. 10, are usable and non-singular for $|p_i - p_j| \rightarrow 0$, \ldots
their accuracy increases with increasing distance. It is therefore reasonable that they are multiplied in (4b) by a weight \((\Phi_i - \Phi_j)\) which is small in absolute value for small distances \(d(p_i, p_j)\). Observe that \(d(p_i, p_j)\) is the distance measured along the surface \(S\). In other words, \(|p_i - p_j|\) may be small and \(d(p_i, p_j)\) large if \(p_i\) and \(p_j\) happen to be "antipodal" points, e.g., points on the upper and lower side of a thin wing such that \(|p_i - p_j|\) is small. The corresponding weight \((\Phi_i - \Phi_j)\) is then not necessarily small. Difficulties associated with such configurations are therefore not necessarily eliminated by the regularization.

On the other hand, the analysis in Appendix 4 of Ref. 10 suggests that the error \(\tilde{\omega}_j - \omega_j\) will not be amplified by the weights \((\Phi_i - \Phi_j)\), even not near the trailing edge of a wing. (The rough approximation in Appendix 4 can be eliminated in the case of Joukowski airfoils). Furthermore, effects of roundoff errors in \((\Phi_i - \Phi_j)\) are harmless since \(|\tilde{\omega}_j| < 2\pi\) and \(n \approx 1000\).

Observe that the matrix \((\tilde{\omega}_j)\) is typically a full matrix satisfying the rowsum conditions

\[
\sum_{t=1}^{n} \tilde{\omega}_j = -2\pi \quad (j = 1, \ldots, n),
\]

the discretized version of (2). These conditions will be exploited in the following iterative methods.

3. The \(\beta\)-Extrapolated Method

The method described in this section solves equations of the type (4b) in such a way that the weights \((\Phi_i - \Phi_j)\) are not destroyed, and the method has a natural analogon for equation of the type (4a) (Section 6).

It is convenient to write (4b) as:

\[
x_j = \frac{1}{4\pi} \sum_{t=1}^{n} \omega_{ij} (x_j - x_t) + b_j \quad (j = 1, \ldots, n).
\]
Since \( \omega_{ji} = -\omega_{ij} \), Eqs. (5) correspond to
\[
\sum_{i=1}^{n} \omega_{ji} = 2\pi \quad (j = 1, \ldots, n). \tag{7}
\]

The iterative method has the form:
\[
x^0 = \text{given starting vector,} \quad x^{k+1} = x^k + \beta^* r^k \quad (k = 0, 1, 2, \ldots), \quad x^k = (x^k_1, \ldots, x^k_n)^T, \quad r^k = (r^k_1, \ldots, r^k_n)^T, \tag{8a}
\]
\[
r^k_j = \frac{1}{4\pi} \sum_{(i \neq j)} \omega_{ji} (x^k_i - x^k_j) + b_j - x^k_j.
\]

\( \beta^* \) is the optimum value of the parameter \( \beta \), namely that value which minimizes the uniform norm of the iteration matrix \( L_\beta \) of (8a):
\[
\|L_\beta\|_\infty \leq \|L_\beta\|_\infty \quad (\beta \in I). \tag{8b}
\]

The interval \( I \) will be determined by the requirements \( 0 < \|L_\beta\|_\infty < 1 \). The iteration matrix \( L_\beta \) is defined by the equation
\[
x^{k+1} = L_\beta x^k + u_\beta = f_\beta(x^k) \tag{9a}
\]
being equivalent to (8a) identically in \( \beta \). We obtain, using (7):
\[
L_\beta = (1 - \frac{\beta}{2}) I - \frac{\beta}{4\pi} \Omega , \quad I = (\delta_{ji}) , \quad \Omega = (\omega_{ji}). \tag{9b}
\]
\[
u_\beta = \beta b , \quad b = (b_1, \ldots, b_n)^T. \tag{9c}
\]

If \( S \) is strictly convex, all \( \omega_{ji} \) are positive. As a consequence we have:
\[
\|\Omega\|_\infty = \max_j \sum_{i=1}^{n} |\omega_{ji}| = \max_j \sum_{i=1}^{n} \omega_{ji} = 2\pi. \tag{10}
\]

This implies that the convergence of the ordinary iteration for systems of the type (4a)
\[
x^{k+1} = -\frac{1}{2\pi} \Omega x^k + 2b \tag{11}
\]
is not guaranteed. In fact, this iteration is divergent, i.e., the spectral radius of
\((-\Omega/2\pi)\) is at least one since \((1, 1, \ldots, 1)^T\) is an eigenvector corresponding to the eigenvalue \((-1)\). In the next section we can, however, establish the convergence of the ordinary iteration for the regularized systems of the type (4b) or (6): this is the non-extrapolated iteration (9a) with \(\beta = 1\).

4. Instability of the Optimum \(\beta\)-Extrapolated Method

Assuming that \(S\) is strictly convex \((0 < \omega_j < 2\pi)\) we determine the range \(I\) of the parameter \(\beta\) such that \(0 < \|L_{\beta}\| < 1\) for the iteration function \(f_{\beta}(x) = L_{\beta}x + v_{\beta}\).

a) \(\beta < 0:\)

\[\|L_{\beta}\| = \max_j \left\| \sum_{i=1}^n \left| 1 - \frac{\beta}{2} \right| \delta_{ji} - \frac{\beta}{4\pi} \omega_{ji} \right\| = \max_j \left\| \sum_{i=1}^n \left| 1 - \frac{\beta}{2} \right| \delta_{ji} - \frac{\beta}{4\pi} \omega_{ji} \right\| = 1 - \beta > 1.\]

b) \(\beta > 0:\)

\[\|L_{\beta}\| = \max_j \left[ \sum_{(i \neq j)} \left| 1 - \frac{\beta}{2} \right| \omega_{ji} \right] + \left| \left| 1 - \frac{\beta}{2} \right| - \frac{\beta}{4\pi} \omega_{jj} \right|. \quad (12)\]

We define \(c_j = (1 - \frac{\beta}{2}) - \frac{\beta}{4\pi} \omega_{jj}\).

b1) \(c_j \geq 0 \quad (j=1, \ldots, n)\): this is the case if

\[\beta \leq \frac{4\pi}{2\pi + \omega_{jj}} \quad (j=1, \ldots, n),\]

or

\[0 < \beta \leq \frac{4\pi}{2\pi + \omega_M}, \quad \omega_M = \max_j (\omega_{jj}) \quad (13)\]

Now (12) can be written as:

\[\|L_{\beta}\| = \max_j \left[ 1 - \frac{\omega_{jj}}{2\pi} - \beta \right] = 1 - \frac{\omega_m}{2\pi} \beta. \quad (14)\]
where

\[ \omega_m = \min_j (\omega_{jj}) \]  \tag{15} \]

Notice that \( \omega_m \neq 0 \) since \( S \) is strictly convex. In order to minimize \( \|L_{\beta}\|_m \) we insert the maximal value of \( \beta \) in (14).

\[ \beta_m = \frac{4\pi}{2\pi + \omega_m} \]  \tag{16} \]

We obtain:

\[ \|L_{\beta_m}\| = 1 - \frac{\omega_m \beta_m}{2\pi} = 1 - \frac{2\omega_m}{2\pi + \omega_m} < 1 , \]  \tag{17} \]

The requirements \( 0 < \|L_{\beta}\| < 1 \) are satisfied for \( \beta \in \mathcal{I}_c = (0, \beta_m] \).

b2) \( \alpha_j < 0 \) (\( j = 1, \ldots, n \)): This is the case if

\[ \beta \geq \frac{4\pi}{2\pi + \omega_{jj}} \quad (j = 1, \ldots, n) , \]

or

\[ \beta \geq \frac{4\pi}{2\pi + \omega_m} = \beta_m \]  \tag{18} \]

Then (12) becomes:

\[ \|L_{\beta}\| = \max_j \left[ \frac{\beta}{4\pi} \sum_{(i \neq j)} \omega_{ji} \left( 1 - \frac{\beta}{2} \right) + \frac{\beta}{4\pi} \omega_{jj} \right] = \beta - 1 \]  \tag{19} \]

We have \( 0 < \|L_{\beta}\| < 1 \) for \( \beta \in (1,2) \) but (18) implies \( \beta \in \mathcal{I}_c = [\beta_m, 2) \) since \( \omega_m < 2\pi \).

In order to minimize \( \|L_{\beta}\| \) we insert the minimal value \( \beta_m \) of \( \beta \) in (19):

\[ \|L_{\beta_m}\| = \beta_m - 1 = \frac{2\pi - \omega_m}{2\pi + \omega_m} < 1 \]  \tag{20} \]

It can be verified that \( \|L_{\beta_m}\| \leq \|L_{\beta_m}\|_m \). This inequality reduces to the statements

\[ 0 < \omega_m \leq \omega_M. \]

The optimum value \( \beta^* \) of the parameter \( \beta \) is therefore
\[ \beta^* = \beta_m = \frac{4\pi}{2\pi + \omega_m}. \]  

(21)

b3) Some \( c_j \geq 0 \) but \( c_i \leq 0 \) (\( i \neq j \)): These cases reduce to inequalities

\[ \frac{4\pi}{2\pi + \omega_i} \leq \beta \leq \frac{4\pi}{2\pi + \omega_j}, \]  

which make it impossible to define a unique \( \beta \) in general. Thus we accept the value of \( \beta^* \) given in (21).

The iteration (9) converges for \( \beta \in I = I_+ \cup I_- = (0, \beta_U] \cup [\beta_m, 2) \) and optimal convergence rate is realized for \( \beta = \beta^* = \beta_m \).

Observe the gap \( (\beta_U, \beta_m) \) in which convergence is not guaranteed for arbitrary starting vectors \( x^0 \). Nevertheless \( \beta = 1 \in I_+ \) since \( \omega_m < 2\pi \), and this implies the convergence of the ordinary iteration (9) for the regularized systems of the type (4b) or (6).

5. The Optimum B-Extrapolated Method

The iteration (9) has been found to be convergent for \( \beta \in I \) where

\[ I = (0, \beta_U] \cup [\beta_m, 2) \]

the optimum value of the extrapolation parameter \( \beta \) being

\[ \beta^* = \beta_m = \frac{4\pi}{2\pi + \omega_m}. \]  

(21)

The fact that a slightly perturbed value \( \beta < \beta^* \) of the parameter induces a possibly divergent iteration sequence makes the optimum \( \beta \)-extrapolated method unstable.

We restore the stability by eliminating the gap \( (\beta_U, \beta_m) \). This can be achieved by introducing \( n \) parameters \( \beta_j (j = 1, \ldots, n) \) which replace the single parameter \( \beta \), as suggested by the inequalities (22). The corresponding family of methods, the \( B \)-extrapolated methods, have the form:
\[\begin{align*}
x^0 &= \text{given starting vector} \\
x^{k+1} &= x^k + B\delta^k \quad (k = 0, 1, 2, \ldots) \\
B &= (\beta_j \delta_j) = \text{diag} (\beta_1, \ldots, \beta_n). 
\end{align*}\]

The remaining quantities are defined by (6), (7), and (8a). The corresponding iteration function is \(f_B(x) = L_B x + v_B\), where, instead of (9):

\[L_B = I - \frac{1}{2} B - \frac{1}{4\pi} B\Omega, \quad v_B = B b.\]

It is now easy to establish the convergence of the iteration (23) for

\[0 < \beta_j < 2 \quad (j = 1, \ldots, n).\]

and to verify that the optimum values of the \(\beta_j\) are given by

\[\beta_j^* = \frac{4\pi}{2\pi + \omega_{jj}} \quad (j = 1, \ldots, n).\]

Convergence of the resulting optimum \(B\)-extrapolated method \((B^* = \text{diag}(\beta_1^*, \ldots, \beta_n^*))\) is given by:

\[\|L_{B^*}\|_\infty = \max_j \left[ \frac{2\pi - \omega_{jj}}{2\pi + \omega_{jj}} \right] = \frac{2\pi - \omega_m}{2\pi + \omega_m} < 1.\]

while stability follows from (24).

Observe the relationship (25) between the optimum parameters \(\beta_j^*\) and the (solid-) angular defects \(\omega_{jj}\) which are finite contributions of the singularity of the kernel \(K\) to the discretized form (7) of the integrals (2). If we ignore these contributions, i.e., if we set \(\omega_{jj} = 0\) then \(\beta_j = 2\) and the iteration reduces to the divergent ordinary iteration corresponding to (11).

Next we compare the convergence rates of the optimum \(B\)-extrapolated method and the non-extrapolated method. We define here the (asymptotic) convergence rate \(R_{B^*}\) of the iteration (23) with respect to the row-norm \(\|L_B\|_\infty\):

\[R_{B^*} = -\log_{10} \|L_B\|_\infty.\]

For the optimum \(B\)-extrapolated method we have for large \(n\) \((\lim \omega_m = 0)\):
Non-extrapolated iteration $\beta = 1$ of the same equation has the convergence rate (see Eq. (14)):

$$R_{G^*,-} = -\log_{10} \|L_{G^*}\| = -\log_{10} \left( \frac{2\pi - \omega_m}{2\pi + \omega_m} \right)$$

$$= -0.4343 \left[ -2 \left( \frac{\omega_m}{2\pi} \right) - \frac{2}{3} \left( \frac{\omega_m}{2\pi} \right)^3 - \frac{2}{5} \left( \frac{\omega_m}{2\pi} \right)^5 + \cdots \right]$$

$$= 0.4343 \left( \frac{\omega_m}{2\pi} \right)$$

Equations (27) and (28) yield the ratio:

$$\frac{R_{G^*,-}}{R_{1,-}} \approx 2$$

for large $n$. This means that the $B^*$-extrapolated method is about twice as fast as the non-extrapolated method, for practical $n \approx 1000$. Notice that the $\beta^*$-extrapolated method of Section 3 and 4 is only apparently simpler than the $B^*$-extrapolated method: The computational effort in evaluating (25) and $B^* r_k$ in (23) is negligible. In any case, the former method should not be used because of its instability.

6. The Case of the Singular Equations

The $B^*$-extrapolated method defined in Section 5 has a natural analogon for equations of the type (4a), (11):
\[ x^0 = \text{given starting vector}, \]
\[ x^{k+1} = x^k + B^* r^k \quad (k = 0, 1, 2, \ldots), \]
\[ B^* = \text{diag}(\beta_1^*, \ldots, \beta_n^*) \quad \beta_j^* = \frac{2\pi}{2\pi + \omega_{jj}}, \]
\[ r^k = -\frac{1}{2\pi} \Omega x^k + 2b - x^k. \]

The optimum values \( \beta_j^* \) are changed but the convergence rate in the uniform norm is the same as predicted for the regularized equation. At this point, numerical experiments are required in order to validate the argument given in Section 2.

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