The Complexity of Countable Structures

by

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The Complexity of Countable Structures

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Abstract

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We prove various results about the complexity of countable structures, both computable and arbitrary.

We begin by investigating descriptions of countable structures in the infinitary logic $L_{\omega_1\omega}$. Given a countable structure $\mathcal{A}$, we can find a sentence $\varphi$, a Scott sentence for $\mathcal{A}$, which describes $\mathcal{A}$ up to isomorphism in the sense that $\mathcal{A}$ is the unique countable model of $\varphi$. We can assign a complexity, the Scott rank of $\mathcal{A}$, to $\mathcal{A}$; this is the quantifier complexity of the simplest Scott sentence. We investigate the Scott ranks of the models of a theory; we produce several new computable models of high Scott rank; we construct a computable group with no $d$-$\Sigma^1_2$ Scott sentence; and we produce a first-order theory of Ulm type.

Next, we look at structures on a cone. If $\mathcal{A}$ is a natural structure—the informal notion of a structure that might show up in the normal course of mathematics, and which was not constructed explicitly as a computability-theoretic counterexample—then arguments about $\mathcal{A}$ will generally relativize. So we can study natural structures by studying arbitrary structures relativized ‘on a cone’. This gives us a way of making precise statements about the imprecise notion of a natural structure. We give a complete classification of the degrees of categoricity on a cone, and prove a number of results about degree spectra of relation on a cone.

Third, we investigate the deep connections between infinitary interpretations and functors. An interpretation of one structure $\mathcal{A}$ in another structure $\mathcal{B}$ induces a functor which produces copies of $\mathcal{A}$ from copies of $\mathcal{B}$. Moreover, the interpretation induces a homomorphism from the automorphism group of $\mathcal{B}$ to the automorphism group of $\mathcal{A}$. We show that this reverses: given a functor from $\mathcal{B}$ to $\mathcal{A}$, or a homomorphism from the automorphism group of $\mathcal{B}$ to $\mathcal{A}$, we can recover an interpretation. We consider the effective version of this, as well as the corresponding results for bi-interpretations.

Finally, we look at a number of different algebraic structures. We study the effective process of extending automorphisms of fields to their algebraic closure, extending valuations to a field extension, and embedding valued fields into algebraically closed valued fields. We give a metatheorem for finding computable copies of various algebraic structures with an
independence relation, such as differentially closed fields with $\delta$-independence, with a computable basis. We show that there is a computable left-orderable group with no computable copy with a computable ordering.
To Jane, Charlie, and Emma.
Contents

I Descriptions of Structures Using $\mathcal{L}_{\omega_1 \omega}$ Sentences

1 General Introduction

2 Scott Spectra of Theories

3 Some New Computable Structures of High Rank
   with Greg Igusa and Julia Knight

4 Scott Sentences of Finitely Generated Algebraic Structures
   with Meng-Che "Turbo" Ho

5 A First-Order Theory of Ulm Type

II Structures on a Cone

6 Degree Spectra of Relations

7 Computable Categoricity
   with Barbara Csima

III Functors and Interpretations

8 Computable Functors
   with Alexander Melnikov, Russell Miller, and Antonio Montalbán

9 Borel Functors
   with Russell Miller and Antonio Montalbán

IV Computable Algebra

10 Extensions of Embeddings of Fields
   with Alexander Melnikov and Russell Miller

11 Notions of Independence
   with Alexander Melnikov and Antonio Montalbán
12 Computable Valued Fields 333
13 Left-Orderable Computable Groups 355

V Miscellaneous 376
14 The Complexity of Decidable Presentability 377
15 The Gamma Problem for Many-One Degrees 405

VI References 415
Bibliography 416
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Chapter 1

General Introduction

This thesis addresses a number of different topics in computable structure theory. The main objects of study in computable structure theory are countable structures, such as graphs, groups, or fields, consisting of a domain and various functions, relations, and constants. Because we want to use methods from computability theory, the domain of all of our structures will be the natural numbers \( \omega \) (or a subset of \( \omega \)). A structure is computable if its domain is computable, and its functions and relations are computable as functions \( \omega^n \to \omega \) and as subsets of \( \omega^n \) respectively. So, for example, a group with domain \( \omega \) is computable if we have a way of computing the group operation.

Two isomorphic structures are said to be copies or presentations of one another. While generally in mathematics, one should think of two isomorphic structures as being the same, in computable structure theory one should only think of two structures as being the same if there is a computable isomorphism between the two. Two structures which are isomorphic, but not computably isomorphic, may have different computability-theoretic properties—for example, there is a computable vector space with a computable basis, and an isomorphic computable vector space with no computable basis—and often it is the computational properties of the different presentations of the same structure which are important in understanding the complexity of that structure.

We are mainly concerned with the computational properties of structures, and the relationship between these and the structural or algebraic properties of those structures. The computational properties of a structure are the computational properties of its various presentations; for example, the information which is coded in every presentation, the difficulty of computing a presentation, the difficulty of finding an isomorphism between two presentations, or the difficulty of deciding if a given presentation is actually a copy of that structure. By a structural or algebraic property, we mean a property of the isomorphism type of the structure, such as the dimension of a vector space. Indeed, a vector space has a unique computable copy up to computable isomorphism if and only if it has finite dimension. Thus we relate a structural property—the dimension—to a computational property—the number of different computable presentations.

The main results of this thesis are divided into five parts. We begin by proving a number
of different results around Scott ranks and descriptions of structures using sentences of the
infinitary logic $L_{\omega_1 \omega}$. Next, we study “natural” structures by proving results about structures
on a cone. In the third part, we draw connections between interpretations of one structure in
another, and functors between presentations of structures. Next, we consider some algebraic
structures such as various types of fields, rings, and groups. The fifth and final part contains
results which did not fit into the previous parts.

Some chapters contain joint work: 3 with Gregory Igusa and Julia Knight; 4 with Meng-
Che Ho; 7 with Barbara Csima; 8 with Alexander Melnikov, Russell Miller, and Antonio
Montalbán; 9 with Russell Miller and Antonio Montalbán; 10 with Alexander Melnikov and
Russell Miller; and 11 with Alexander Melnikov and Antonio Montalbán. The remaining
chapters—2, 5, 6, 12, 13, 14, and 15—contain work that I have done myself. As the chapters
have been individually submitted for publication, each is written as a self-contained paper.

We will now outline a small amount of background material, following which we give a
brief overview of the main results which appear in this thesis and of the themes which tie
these results together.

1.1 Descriptions of Structures Using $L_{\omega_1 \omega}$ Sentences

In the first part of this thesis, we consider the descriptive complexity of structures. By
a description, we mean a description in the infinitary logic $L_{\omega_1 \omega}$. This is the logic which
allows countably infinite disjunctions and conjunctions, but only finite strings of quantifiers.
Each $L_{\omega_1 \omega}$ formula may be put into normal form, and classified according to the number of
quantifier alternations:

- $\varphi$ is $\Pi_0^{in}$ and $\Sigma_0^{in}$ if it is finitary quantifier-free,
- $\varphi$ is $\Sigma_\alpha^{in}$ if it is a countable disjunction of formulas $(\exists x)\psi$, where $\psi$ is $\Pi_\beta^{in}$ for some $\beta < \alpha$,
- $\varphi$ is $\Pi_\alpha^{in}$ if it is a countable conjunction of formulas $(\forall x)\psi$, where $\psi$ is $\Sigma_\beta^{in}$ for some $\beta < \alpha$,
- $\varphi$ is $d-\Sigma_\alpha^{in}$ if it is a conjunction of a $\Sigma_\alpha^{in}$ formula and a $\Pi_\beta^{in}$ formula.

The beginning of the hierarchy is ordered as follows, from the simplest formulas on the left,
to the most complicated formulas on the right:

$$
\begin{array}{c}
\Sigma_0^0 \rightarrow \Sigma_1^0 \rightarrow \Sigma_2^0 \rightarrow \Sigma_3^0 \\
\Pi_0^0 \rightarrow \Pi_1^0 \rightarrow \Pi_2^0 \rightarrow \Pi_3^0 \\
\end{array}
$$
We can also consider computable $L_{\omega_1\omega}$ formulas, where the conjunctions and disjunctions are taken over computable sets of formulas. We denote these by $\Sigma^c_2$ and $\Pi^c_2$.

Given a countable structure $A$, Scott [Sco65] proved that there is an $L_{\omega_1\omega}$ sentence $\varphi$ such that $A$ is the only countable model of $\varphi$. Such a sentence is called a Scott sentence for $A$. One should think of a Scott sentence for $A$ as a description of $A$ among countable structures. (It is possible for there to also be an uncountable model of the Scott sentence of a computable structure.) For example, we can write down a Scott sentence for the natural numbers as a linear order by saying that, first of all, it is an infinite linear order, and second, that each element has exactly $n$ predecessors for some $n$. This is a $\Pi^c_3$ Scott sentence for the structure $(\omega, \leq)$: the axioms for linear orders are $\Pi^c_1$, saying that it is infinite if $\Pi^c_2$, and the last part of the sentence can be written out as

$$\forall y_0 \bigwedge_{n \in \omega} \exists y_n < \cdots < y_1 < y_0 [\forall z \, (z > y_0) \lor (z = y_0) \lor (z = y_1) \lor \cdots \lor (z = y_n)] .$$

As a second example, one can describe the infinite-dimension $\mathbb{Q}$-vector space using the vector space axioms and the sentence

$$\bigwedge_{n \in \mathbb{N}} (\exists x_1, \ldots, x_n) \text{Indep}(x_1, \ldots, x_n)$$

where $\text{Indep}(x_1, \ldots, x_n)$ is the $\Pi^c_3$ sentence which expresses that $x_1, \ldots, x_n$ are independent. Thus this vector space has a $\Pi^c_3$ Scott sentence.

We can measure the complexity of a structure $A$ by the complexity of an optimal Scott sentence for $A$. We can assign to each structure a rank—its Scott rank—based on the complexity of the simplest description of that structure. We say that $A$ has Scott rank $\alpha$ if $\alpha$ is least such that $A$ has a $\Pi^c_{\alpha+1}$ Scott sentence. This definition is due to Montalbán [Mon15a] but there are many other possible definitions of Scott rank—see Section 2.2.1. Structures with higher Scott rank are more difficult to describe.

### 1.1.1 Scott Ranks of Models of a Theory

In the first chapter of this part—Chapter 2—we consider questions of the following kind: given an $L_{\omega_1\omega}$ sentence $\varphi$, which we think of as a theory defining a class of structures, what might the set of Scott ranks of the models of $\varphi$ be? A first question is, if $\varphi$ is a simple sentence—for example, $\Pi^c_{\alpha+1}$—then must $\varphi$ have a simple model? Surprisingly, the models of $\varphi$ might all be very complicated.

**Theorem 2.1.3** (Harrison-Trainor). Fix $\alpha < \omega_1$. Then there is a $\Pi^c_{\alpha+1}$ sentence $T$ whose models all have Scott rank $\alpha$.

The strategy we use to prove this theorem is to construct a theory of two-sorted structures, the first of which is a linear order, which we view as a (non-standard) ordinal, and the second of which is a tree endowed with non-standard back-and-forth relations indexed by the first sort. The order type of the well-founded part of the first sort determines the Scott rank of
the structure via the non-standard back-and-forth relations. By naming each element of the first sort by a constant, it does not contribute to the Scott rank. The same technique, with some modifications, is used for all of the results in this chapter.

Next we ask what kinds of sets could be the set of Scott ranks of a sentence $\varphi$. We call such a set of Scott ranks the Scott spectrum of $\varphi$:

$$SS(\varphi) = \{\alpha \in \omega_1 : \alpha \text{ is the Scott rank of a countable model of } \varphi\}.$$ 

Under projective determinacy, we get a complete characterization. One can state the characterization in two different ways.

**Theorem 2.1.12** (Harrison-Trainor; in ZFC + PD). The Scott spectra of $\mathcal{L}_{\omega_1 \omega}$-sentences are the sets $\mathcal{C}$ of ordinals in a $\Sigma^1_1$ class of linear orders with the property that, if $\mathcal{C}$ is unbounded below $\omega_1$, then either $\mathcal{C}$ is stationary or $\{\alpha : \alpha + 1 \in \mathcal{C}\}$ is stationary.

**Theorem 2.1.14** (Harrison-Trainor; in ZFC + PD). The Scott spectra of $\mathcal{L}_{\omega_1 \omega}$-sentences are exactly the sets of the following forms, for some $\Sigma^1_1$ class of linear orders $\mathcal{C}$:

1. the well-founded parts of orderings in $\mathcal{C}$,
2. the orderings in $\mathcal{C}$ with the non-well-founded part collapsed to a single element, or
3. the union of (1) and (2).

Given a $\Sigma^1_1$ class $\mathcal{C}$ of linear orders, we can construct an $\mathcal{L}_{\omega_1 \omega}$ sentence with the appropriate Scott spectrum in ZFC without PD. It is only for the other direction that we use PD, and only to differentiate between (1), (2), and (3); in ZFC, we can show that for each Scott spectrum, there is a $\Sigma^1_1$ class $\mathcal{C}$ of linear orders such that:

- each ordinal in the Scott spectrum is either the well-founded part of an order in $\mathcal{C}$ or else an order in $\mathcal{C}$ with the non-well-founded part collapsed to a single element, and
- for each order in $\mathcal{C}$, either the well-founded part of that order, or that order with the non-well-founded part collapsed to a single element, is in the Scott spectrum.

Moreover, we can choose $\mathcal{C}$ so that the only non-well-founded ordinals in $\mathcal{C}$ have, as their well-founded part, some admissible ordinal.

By examining the proof, we also find that each Scott spectrum is the Scott spectrum of a $\Pi^1_n$ sentence, and that every Scott spectrum of an $\mathcal{L}_{\omega_1 \omega}$-pseudo-elementary class is the Scott spectrum of an $\mathcal{L}_{\omega_1 \omega}$-sentence. These observations, combined with results of Sacks [Sac83] and Marker [Mar90], allow us to answer a question of Sacks by showing that the Scott height of the computable $\mathcal{L}_{\omega_1 \omega}$ sentences is $\delta^1_2$, the least non-$\Delta^1_2$-presentable ordinal.

We can also apply the same methods to build new computable structures of high Scott rank. A computable structure must have Scott rank at most $\omega^1_{CK} + 1$, where $\omega^1_{CK}$ is the least non-computable ordinal. There are many well-known examples of computable structures
having each computable rank. The Harrison linear order [Har68], which has order type \( \omega_1^{CK} (1 + \mathbb{Q}) \), has a computable presentation and Scott rank \( \omega_1^{CK} + 1 \). Computable structures of Scott rank \( \omega_1^{CK} \) are the hardest to construct. Makkai [Mak81] constructed an arithmetic structure of Scott rank \( \omega_1^{CK} \), and Knight and Millar [KM10] modified his construction to give a computable structure. Later, Calvert, Knight, and Millar [CKM06] gave a simpler example. These examples were all obtained from a certain kind of tree, and no other examples were known. Each of these examples was computably approximable by computable models of low Scott rank: every computable infinitary sentence true in these examples is also true in some computable structure of low Scott rank. Calvert and Knight [CK06] asked whether every computable structure of high Scott rank is computably approximable. Using the same techniques from the previous results, we construct the first known example of a model of high Scott rank which cannot be approximated in this way.

**Theorem 2.1.6** (Harrison-Trainor). There are computable structures of Scott rank \( \omega_1^{CK} \) and \( \omega_1^{CK} + 1 \) which are not computably approximable.

### 1.1.2 More Computable Models of High Scott Rank

In Chapter 3 we construct more new examples of computable structures of high Scott rank. First we produce an example of a model of Scott rank \( \omega_1^{CK} \) where the computable infinitary theory is not \( \aleph_0 \)-categorical. The computable infinitary theory of a structure is the set of computable \( \mathcal{L}_{\omega_1 \omega} \) sentences true of that structure. Nadel74 [Nad74] showed that two computable structures which satisfy the same computable infinitary sentences must be isomorphic. However, there might be a non-computable structure which shares the same computable infinitary theory as a given computable structure. The computable infinitary theory of a computable structure of computable Scott rank is \( \aleph_0 \)-categorical, because the theory contains a Scott sentence for the structure, while the computable infinitary theory of a computable structure of Scott rank \( \omega_1^{CK} + 1 \) is not \( \aleph_0 \)-categorical, because there is a non-principal type which may be omitted. For computable structures of Scott rank \( \omega_1^{CK} \), the known examples all had an \( \aleph_0 \)-categorical infinitary theory. Millar and Sacks asked whether this was always the case. They were able to find a structure \( A \) whose computable infinitary theory was not \( \aleph_0 \)-categorical, but \( A \) was not computable [MS08]. Instead, \( A \) was “low for \( \omega_1^{CK} \),” i.e., \( \omega_1^A = \omega_1^{CK} \), or equivalently \( A \) could not compute any non-computable ordinals. We resolve this question.

**Theorem 3.2.1** (Harrison-Trainor, Igusa, Knight). There is a computable structure \( M \) with Scott rank \( \omega_1^{CK} \) such that the computable infinitary theory of \( M \) is not \( \aleph_0 \)-categorical.

Thus there are computable structures of Scott rank \( \omega_1^{CK} \) whose computable infinitary theory is \( \aleph_0 \)-categorical, and others whose computable infinitary theory is not.

Second, we consider indiscernible sets. All known examples of computable structures of Scott rank \( \omega_1^{CK} + 1 \) contained an (order) indiscernible set. For example, the Harrison linear order \( \omega_1^{CK} (1 + \mathbb{Q}) \) has an indiscernible set among its non-standard elements; each element...
(a, q) with $a \in \omega^+_{CK}$ and $q \in \mathbb{Q}$ is in the same automorphism orbit as any other element $(a, q')$ for any other $q \in \mathbb{Q}$. We produce two examples of computable structures of Scott rank $\omega^+_{CK}$ with no indiscernible triple. Our first example is the Harrison linear order with each element replaced by an equivalence class consisting of infinitely many elements, and a structure similar to a triangle-free graph placed on the elements of these equivalence classes. However, there is an indiscernible set of equivalence classes in this structure. Our second example does not even have an indiscernible triple of equivalence classes.

**Theorem 3.3.1 & 3.3.9** (Harrison-Trainor, Igusa, Knight). There is a computable structure of Scott rank $\omega^+_{CK}$ with no indiscernible ordered triple.

1.1.3 Scott Sentences for Finitely Generated Structures

In Chapter 4, we move on to look at structures of very low Scott rank. A finitely generated structure always has a description which is $\Sigma^3_{\infty}$ (and hence is of Scott rank at most 3). A finitely generated structure is determined by the atomic type of a generating tuple, and so, fixing the atomic type $p$ of a generating tuple for a finitely generated structure $\mathcal{A}$, we get a Scott sentence for $\mathcal{A}$ by saying that there is a tuple realizing the type $p$ which generates the whole structure; this is $\Sigma^3_{\infty}$. However, it turns out that many groups have a simpler $d$-$\Sigma^2_{\infty}$ Scott sentence—a conjunction of a $\Sigma^2_{\infty}$ and a $\Pi^2_{\infty}$ sentence—and hence have Scott rank at most 2. Knight and Saraph [KS] showed that every abelian group has a $d$-$\Sigma^2_{\infty}$ Scott sentence, and Calvert et al [CHK+12] showed that finitely generated free groups have $d$-$\Sigma^2_{\infty}$ Scott sentences. Ho [Ho] showed that every nilpotent and even polycyclic group has a $d$-$\Sigma^2_{\infty}$ Scott sentence, and also that various other groups of interest in geometric group theory have $d$-$\Sigma^2_{\infty}$ Scott sentences. This led Knight to conjecture that every finitely generated group has a $d$-$\Sigma^2_{\infty}$ Scott sentence. A result of Miller [Mil78] shows that if a structure has a $\Sigma^2_{\infty}$ Scott sentence and also a $\Pi^2_{\infty}$ Scott sentence, then it has a $d$-$\Sigma^2_{\infty}$ Scott sentence; since each finitely generated structure has a $\Sigma^2_{\infty}$ Scott sentence, the only two options for the optimal Scott sentence of a finitely generated group (except for extremely simple groups, such as finite groups) are $\Sigma^3_{\infty}$ and $d$-$\Sigma^2_{\infty}$.

To resolve this conjecture, we first give a general characterization of the finitely generated structure with $d$-$\Sigma^2_{\infty}$ Scott sentences.

**Theorem 4.1.5** (Harrison-Trainor, Ho). Let $\mathcal{M}$ be a finitely generated structure. The following are equivalent:

1. $\mathcal{M}$ has a $d$-$\Sigma^2_{\infty}$ Scott sentence,
2. $\mathcal{M}$ is not self-reflective, i.e., it does not contain a copy of itself as a proper $\Sigma^1_{\infty}$-elementary substructure.

Alvir, Knight, and McCoy [AKM] independently gave another characterization of the finitely generated structures with $d$-$\Sigma^2_{\infty}$ Scott sentences: they are the structures for which the automorphism orbit of some (or equivalently every) generating tuple is $\Pi^2_{\infty}$-definable.
CHAPTER 1. GENERAL INTRODUCTION

Using our characterization, we produce an example of a finitely generated computable group with no d-$\Sigma^0_2$ Scott sentence; in fact, its index set is $\Sigma^0_3$ m-complete, which is as complicated as possible. Thus this group achieves the maximum possible descriptive complexity.

**Theorem 4.1.3** (Harrison-Trainor, Ho). There is a finitely-generated computable group $G$ which has no d-$\Sigma^0_2$ Scott sentence. Its index set is $\Sigma^0_3$ m-complete.

The proof uses HNN extensions and small cancellation theory.

We are also able to show that the group ring $\mathbb{Z}[G]$, where $G$ is as in the previous theorem, has no d-$\Sigma^0_2$ Scott sentence. As a further application of the general characterization, we show that every finitely generated field has a d-$\Sigma^0_2$ Scott sentence. Thus, while a finitely generated group may be difficult to describe, a finitely generated field always has a relatively simple description.

The class of abelian $p$-groups is a well-studied class in computable structure theory. As a result of their classification via Ulm invariants, they have many interesting computability-theoretic properties. The class of abelian $p$-groups is axiomatizable by a simple $L_{\omega_1\omega}$ sentence, but not by an elementary first-order theory. In Chapter 5, we consider the problem of finding a first-order theory which shares many of the same interesting properties as abelian $p$-groups.

Recall that Nadel74 [Nad74] showed that any two computable structures satisfying the same computable infinitary sentences are isomorphic. Because one can express the Ulm invariants using computable infinitary sentences, one can extend this result from computable abelian $p$-groups to all abelian $p$-groups which are low for $\omega^C\omega$, i.e., groups $G$ with $\omega^G_1 = \omega^C\omega$. We say that a class of structures with this property is of Ulm type.

**Definition 5.1.5** (Definition 6 of [FKM+11]). A class of countable structures has **Ulm type** if for any two structures $A$ and $B$ in the class, if $\omega^A_1 = \omega^B_1 = \omega^C\omega$, and if $A$ and $B$ satisfy the same computable infinitary sentences, then $A$ and $B$ are isomorphic.

Knight asked whether there is a first-order theory of Ulm type. This is our first goal.

Our second goal is to find a first-order theory which is Borel equivalent to $p$-groups. In their influential paper [FS89], Friedman and Stanley introduced Borel reductions between invariant Borel classes of structures with universe $\omega$ in a countable language. A class $\mathcal{C}$ Borel reduces to a class $\mathcal{D}$ if there is a Borel operator $\Phi: \mathcal{C} \to \mathcal{D}$ such that for $A,B \in \mathcal{C}$,

$$A \simeq B \iff \Phi(A) \simeq \Phi(B).$$

Thus the isomorphism relation on $\mathcal{C}$ can be reduced, in a Borel way, to the isomorphism relation on $\mathcal{D}$. We say that a class $\mathcal{C}$ of structures is Borel complete if every other class reduces to it; the isomorphism relation on such a class $\mathcal{C}$ cannot be Borel. However, it is possible to have an isomorphism relation which is not Borel, but also not Borel complete. Abelian $p$-groups are the most well-known example of this. Until very recently, they were one of the few such examples, and there were no known examples of elementary first-order theories with similar properties. Recently, Laskowski, Rast, and Ulrich [URL] gave an example of a first-order theory which is not Borel complete, but whose isomorphism relation is not Borel.

In this chapter, we answer both questions simultaneously.
Theorem 5.1.7 & 5.1.6 (Harrison-Trainor). There is a first-order theory of Ulm type which is Borel equivalent to the class of abelian $p$-groups. Thus the isomorphism relation on models of this theory is neither Borel complete nor Borel.

The first-order theory is a theory of abelian $p$-groups with the unary relation for each torsion class. The addition operator is replaced by a stratified sequence of ternary relations representing the addition of elements of particular orders of torsion.

1.2 Structures on a Cone

One of the most important problems from the early days of degree theory, Post’s problem, was to determine whether there is an intermediate c.e. degree, i.e. a c.e. degree strictly between $0$ and $0'$. Post’s problem was resolved in the 1950’s with Friedberg and Muchnik’s construction of a c.e. degree strictly between $0$ and $0'$. However, their construction did not produce a “natural degree”. In computability theory, natural notions tend to relativize nicely. On the other hand, relativizing the Friedberg-Muchnik construction to sets $X$ and $Y$, with $X \equiv_T Y$, produces two different degrees. Noticing this, Sacks asked whether there was a c.e. operator $W$, which maps degrees to degrees, such that for all sets $X$, $X \lt_T W^X \lt_T X'$. Such a c.e. operator would be a natural intermediate degree. This question is still open.

Generalizing this, Martin conjectured that the natural functions on degrees are well-ordered, with the successor being given by the jump. Thus, the first few natural functions on degrees would be the identity, the jump, the second jump, and so on. Formally, under the Axiom of Determinacy, Martin [Mar68] proved that every set of degrees either contains a cone (the set of all Turing degrees above a particular degree) or is disjoint from a cone. “Containing a cone” gives a $\{0, 1\}$-valued countably additive measure on the sets of degrees; a set which contains a cone is “large” and is assigned measure one. Identifying any two functions which are equal on a cone, Martin conjectured that these equivalence classes of functions were well-ordered with the successor being given by the jump. Martin’s conjecture is the origin of studying objects “on a cone” in order to study natural objects. While some special cases of Martin’s conjecture have been solved—see [Ste82, SS88]—the question is far from being solved.

In this thesis, we will mainly consider degrees of categoricity and degree spectra of relations of natural structures. There are also other results about various objects on a cone, some of which were proved before the results of this thesis, and others afterward:

- Martin’s conjecture for uniformly degree-invariant functions from Turing degrees to Turing degrees [SS88],
- Martin’s conjecture for uniformly degree-invariant functions from Turing degrees to many-one degrees [KM],
- non-trivial degree-invariant $\Sigma^0_1$ equivalence relations with $\aleph_1$-many classes [Mon15b],
• $\Delta^0_\alpha$-categoricity and $\Sigma^\inf_{\alpha+2}$ Scott sentences [Mon15a],
• every structure has computable dimension $1$ or $\infty$ [McC02],
• existence of r.i.c.e. structurally complete formulas [Mon12b], and
• Muchnik linear equivalence relations, Vaught’s conjecture, and hyperarithmetic-is-recursive [Mon13b].

Natural structures seem to have have natural degrees of categoricity, such as $0, 0'$, etc., and natural classes of degree spectra, such as the c.e. degrees, the d.c.e. degrees, the $\Delta^0_2$ degrees, etc. On the other hand, many examples have been constructed with other degrees of categoricity and degree spectra. Of course, there is no formal definition of a natural structure; instead, Montalbán has suggested that one can study natural behaviour as follows. Consider some property $P$ of structures and relations. Usually in computability theory, results about natural structures relativize; so if a natural structure has property $P$, then it will have property $P$ relative to any degree $d$. On the other hand, with enough determinacy, for any structure, natural or not, that structure will either have property $P$ relative to all sufficiently high degrees $d$, or not have property $P$ relative to all sufficiently high degrees $d$. (More formally, there will be a degree $c$ relative to which the structure will either have property $P$ (or not have property $P$) relative to all degrees $d \geq c$.) If the structure has property $P$ relative to all degrees $d \geq c$ for some $c$, then we say that the structure has property $P$ on a cone. So by studying the properties of structures on a cone, we can effectively study the properties of natural structures. Results about structures on a cone also shed light on the techniques required to construct various examples: if all structures have property $P$ on a cone, then to construct a computable structure without property $P$, one must use a technique, such as diagonalization, which does not relativize.

1.2.1 Degrees of Categoricity

A computable structure $A$ has degree of categoricity $d$ if for every computable copy $B$ of $A$, $d$ computes an isomorphism between $A$ and $B$, and $d$ is the least degree with this property. Thus the degree of categoricity of $A$ is a measure of the difficulty of computing isomorphisms between $A$ and its computable copies. In the paper where they introduced degrees of categoricity, Fokina, Kalimullin and R. Miller [FKM10] showed that if $d$ is d.c.e. (difference of c.e.) in and above $0^{(n)}$, then $d$ is a degree of categoricity. They also showed that $0^{(\omega)}$ is a degree of categoricity. In [CFS13], Csima, Franklin and Shore extended these results through the hyperarithmetic hierarchy. On the other hand, there have been a number of results about degrees which are not degrees of categoricity. In [CFS13] it was shown that all degrees of categoricity are hyperarithmetic. In [AC16], Anderson and Csima showed that no non-computable hyperimmune-free degree is a degree of categoricity. They also showed that there is a $\Sigma^0_2$ degree that is not a degree of categoricity, and that if $G$ is 2-generic (relative to a perfect tree), then $\text{deg}(G)$ is not a degree of categoricity. For a long time, the question
of whether there are $\Delta_2^0$ degrees that are not degrees of categoricity has remained open. Recently, Csima and Ng have announced that all such degrees are degrees of categoricity.

In the case of degrees of categoricity, we get a complete characterization in Chapter 7. The degrees of categoricity on a cones are exactly the jumps: $0, 0'$, and so on.

**Theorem 7.1.5** (Csima, Harrison-Trainor). Let $A$ be a countable structure. Then, on a cone: $A$ has a strong degree of categoricity, and this degree of categoricity is $\Delta_0^\alpha$-complete.

The construction of a structure with degree of categoricity some d.c.e. (but not c.e.) degree uses a computable approximation to the d.c.e. degree; this requires the choice of a particular index for the approximation, and hence the argument that the resulting structure has degree of categoricity d.c.e. but not c.e. does not relativize. By our theorem, there is no possible construction which does relativize.

### 1.2.2 Degree Spectra of Relations

Given a computable structure $A$ with an additional relation $R$, $R$ may have different Turing degrees in different computable copies of $A$. For example, $A$ might be the linear order $(\mathbb{N}, <)$, and $R$ might be the successor relation. In some computable copies of $A$, $R$ is computable, while in others it is non-computable, but it is always co-c.e. The degree spectrum of $R$ is the set of all Turing degrees of $R$ in different computable copies of $A$. This is a measure of the complexity of that relation. Degree spectra of relations were first introduced by Harizanov [Har87]. The degree spectra of particular relations have been frequently studied, particularly with the goal of finding as many possible different degree spectra as possible. For example, Harizanov [Har93] has shown that there is a $\Delta_0^\alpha$ (but not c.e.) degree $a$ such that $\{0, a\}$ is the degree spectrum of a relation. Hirschfeldt [Hir00] has shown that for any $\alpha$-c.e. degree $b$, with $\alpha \in \omega \cup \{\omega\}$, $\{0, b\}$ is the degree spectrum of a relation. Hirschfeldt has also shown that for any c.e. degree $c$ and any computable ordinal $\alpha$, the set of $\alpha$-c.e. degrees less than or equal to $c$ is a degree spectrum. A number of other papers have been published showing that other degree spectra are possible—see for example Khoussainov and Shore [KS98] and Goncharov and Khoussainov [GK97].

We do not get a complete characterization of the degree spectra on a cone, but we get a number of different results in Chapter 6. Our first observation comes from interpreting a result of Harizanov in this new context. Every degree spectrum on a cone is either just the computable degree, or contains all the c.e. degrees. Thus there are is a minimal degree spectrum, containing just the computable degree, and a second spectrum, containing just the c.e. degrees, which is minimal among the rest. (A strengthening of Harizanov’s theorem appears as Theorem 6.3.1.)

The next natural place to look is at the d.c.e. degrees. We had hoped to show that any degree spectrum which is not contained within the c.e. degrees must contain all of the d.c.e. degrees, and in general to show that the degree spectra were linear ordered. Instead, we found that there are incomparable degree spectra contained within the d.c.e. degrees.
Theorem 6.1.6 (Harrison-Trainor). There are two incomparable degree spectra on a cone contained within the d.c.e. degrees.

This theorem has consequences in pure computability theory: The degree spectra we build form natural classes of Turing degrees which are between the c.e. degrees and the d.c.e. degrees. By natural, we mean that they relativize. These classes of Turing degrees are the beginning of a finer refinement of Ershov's hierarchy.

Our next step is to consider degree spectra which contain non-$\Delta^0_2$ degrees. One might have hoped to show that every degree spectrum on a cone is either contained within the $\Delta^0_2$ degrees, or contains the $\Sigma^0_2$ degrees, but a result of Ash and Knight [AK95, AK97] shows that one must instead work with the 2-CEA degrees.

Theorem 6.1.5 (Harrison-Trainor). Every degree spectrum on a cone is either contained within the $\Delta^0_2$ degrees or contains all of the 2-CEA degrees.

This theorem partially answers a question of Ash and Knight [AK95, AK97]. They had asked for a generalization of Harizanov's result which was described above: Subject to some effectivity conditions, must a degree spectrum which is not contained in the $\Delta^0_\alpha$ degrees contain all of the $\alpha$-CEA degrees? Our result answers this question for $\alpha = 2$.

The work in this part of the thesis forms the first steps of a program to try to understand the degree spectra of natural relations. Our picture so far is as follows:

1. there is a smallest degree spectrum: the computable degree,
2. there is a smallest degree spectrum strictly containing the computable degree: the c.e. degrees,
3. there are two incomparable degree spectra both strictly containing the c.e. degrees and strictly contained in the d.c.e. degrees,
4. any degree spectrum strictly containing the $\Delta^0_2$ degrees must contain all of the 2-CEA degrees.

These results give just the beginning, and there is a lot to know about degree spectra of relations. One hopes to eventually be able to completely classify the degree spectra of relations on a cone.

This chapter of the thesis also contains a discussion of the degree spectra of relations on $(\omega, <)$ in Section 6.5. Surprisingly, while we know more than in the general case, there are still difficult open questions here.

1.3 Functors and Interpretations

Many constructions in mathematics build new structures out of old ones. For example, from an integral domain, we can build its fraction field or polynomial ring. In model theory,
one performs such operations using interpretations, by defining one structure using tuples from the other, with the operations and relation of the new structure defined using the operations and relations of the old structure. The fraction field of an integral domain $\mathcal{R}$ can be defined as the sets of pairs of elements of $\mathcal{R}$, modulo the definable equivalence relation of representing the same fraction. Addition and multiplication of fractions can be defined using the addition and multiplication of $\mathcal{R}$. Traditionally in model theory, the domain consists of tuples all of the same arity, and the definitions are in elementary first-order logic. See, for example, [Mar02, Definition 1.3.9]. Here, we will consider a generalization, introduced in [Mon13a, Definition 1.7], where we use tuples of arbitrary lengths, and our definitions are in the infinitary logic $L_{\omega_1 \omega}$. The construction of the polynomial ring of an integral domain uses tuples of all lengths; one interprets the polynomial $a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n$ as the tuple $(a_0, a_1, \ldots, a_n)$. Thus this is an interpretation in our sense, but not in the traditional sense of an interpretation in model theory.

**Definition 9.1.1.** A structure $\mathcal{A} = (A; P_A^0, P_A^1, \ldots)$ (where $P_A^i \subseteq A^{a(i)}$) is **infinitarily interpretable** in $\mathcal{B}$ if there are relations $\operatorname{Dom}_{\mathcal{B}}^A$, $\sim$, $R_0, R_1, \ldots$, each $L_{\omega_1 \omega}$-definable without parameters in the language of $\mathcal{B}$, such that

1. $\operatorname{Dom}_{\mathcal{B}}^A \subseteq B^{<\omega}$,
2. $\sim$ is an equivalence relation on $\operatorname{Dom}_{\mathcal{B}}^A$,
3. $R_i \subseteq (\operatorname{Dom}_{\mathcal{B}}^A)^{a(i)}$ is closed under $\sim$,

and there exists a function $f_{\mathcal{A}}^B : \operatorname{Dom}_{\mathcal{B}}^A \rightarrow \mathcal{A}$ which induces an isomorphism:

$$f_{\mathcal{A}}^B : (\operatorname{Dom}_{\mathcal{A}}^B/\sim; R_0/\sim, R_1/\sim, \ldots) \cong (A; P_A^0, P_A^1, \ldots),$$

where $R_i/\sim$ stands for the $\sim$-collapse of $R_i$.

If the domain and the relations are uniformly relatively intrinsically computable—that is, they are defined by both a computable $\Sigma^c_1$ formula and a computable $\Pi^c_1$ formula, without parameters—then we say that the interpretation is effective. If a structure $\mathcal{A}$ is effectively interpretable in a structure $\mathcal{B}$, within a copy of $\mathcal{B}$ we can effectively enumerate the elements of the domain of the copy of $\mathcal{A}$ inside of $\mathcal{B}$, and decide whether a relation holds of a particular element. One can think of $\mathcal{A}$ as being reducible to $\mathcal{B}$; $\mathcal{B}$ is at least as complicated as $\mathcal{A}$, because $\mathcal{B}$ contains a copy of $\mathcal{A}$ in a way which is easy to find. Effective interpretability is equivalent to the parameterless version of $\Sigma$-reducibility, introduced by Ershov [En96], which has been studied in Russia over the last 20 years (as in [Puz09, Stu07, Stu08, Stu13, MK08, Kal09]).

Given an interpretation of a structure $\mathcal{A}$ inside of a structure $\mathcal{B}$, we get an induced functor which maps presentations of $\mathcal{B}$ to presentations of $\mathcal{A}$, by mapping a copy of $\mathcal{B}$ to the copy of $\mathcal{A}$ which is interpreted inside of it. (The morphisms of the relevant categories are isomorphisms between presentations.) If the original interpretation was effective, then the induced functor is computable, by which we mean that we can effectively, via a Turing functional, construct
the copy of $A$ from the copy of $B$, and isomorphisms of copies of $A$ from the corresponding isomorphisms of copies of $B$.

**Definition 8.1.4.** A functor from $A$ to $B$ is a map $F$ that assigns to each copy $\tilde{A}$ of $A$ a copy $F(\tilde{A})$ of $B$, and assigns to each isomorphism $f: \tilde{A} \to \tilde{A}$ an isomorphism $F(f): F(\tilde{A}) \to F(\tilde{A})$ so that the two properties hold below:

(N1) $F(\text{id}_{\tilde{A}}) = \text{id}_{F(\tilde{A})}$ for every copy $\tilde{A}$ of $A$, and

(N2) $F(f \circ g) = F(f) \circ F(g)$ for all isomorphisms $f, g$ between copies of $A$.

A functor $F$ from $A$ to $B$ is **computable** if there exist two computable operators $\Phi$ and $\Phi^*$ such that

(C1) for every copy $\tilde{A}$ of $A$, $\Phi^{D(\tilde{A})}$ is the atomic diagram of $F(\tilde{A})$;

(C2) for every isomorphism $f: \tilde{A} \to \tilde{A}$ between copies of $A$, $\Phi^{D(\tilde{A})} \oplus f \oplus D(\tilde{A}) = F(f)$.

Here, $D(\tilde{A})$ denotes the atomic diagram of $\tilde{A}$.

We mentioned that every effective interpretation induces a computable functor; a natural question to ask is whether every computable functor arises from an effective interpretation. It turns out that this is true.

**Theorem 8.1.5 & 8.1.7** (Harrison-Trainor, Melnikov, Miller, Montalbán). Let $A$ and $B$ be countable structures. Every computable functor from $B$ to $A$ is effectively isomorphic to one which is induced by an effective interpretation of $A$ in $B$.

Just as important as notions of reducibility are notions of equivalence. Two structures $A$ and $B$ are bi-interpretable if there is an interpretation of each inside the other, and moreover, if these two interpretations compose in a nice way: the isomorphism between $B$ and the copy of $B$ inside the copy of $A$ inside a copy of $B$ is definable within the ambient copy of $B$, and similarly with $A$ and $B$ reversed. This is much stronger than just asking that $A$ be interpretable in $B$, and $B$ in $A$; in particular, it implies that the automorphism groups of the two structures are isomorphic. Of course, we can also ask that the relevant formulas be computable $\Sigma^c_1$ formulas, in which case we say that the structures are effectively bi-interpretable. Two structures which are effectively bi-interpretable are essentially the same from a computability-theoretic point of view. For example (see [Monb, Lemma 5.3]), two structures which are bi-interpretable have the same degree spectrum, the same computable dimension, and the same Scott rank; their index sets are Turing equivalent; their jumps are effectively bi-interpretable; and so on.

The corresponding functorial notion is an adjoint equivalence of categories: a pair of functors from $A$ to $B$ and from $B$ to $A$, with the compositions of the two being naturally isomorphism to the identity functors. We get a similar reversal.
Theorem 8.1.9 (Harrison-Trainor, Melnikov, Miller, Montalbán). Let $A$ and $B$ be countable structures. Then $A$ and $B$ are effectively bi-interpretable if and only if there is a computable adjoint equivalence of categories between $A$ and $B$.

Computable adjoint equivalences of categories and effective bi-interpretations of this sort have been of much interest recently. It has long been known that, for a number of classes of structures such as graphs, groups, and rings, every countable structure is effectively bi-interpretable with one in that class [HKSS02]. Since two structures which are effectively bi-interpretable have the same computability-theoretic properties, such classes are known as universal classes because any computability-theoretic phenomenon which can be realized by any structure can be realized by a structure in a universal class. However, this result was not originally stated in the language of interpretations or functors; instead, it was proved that given any degree spectrum of a structure, there is a graph, a group, etc. with the same degree spectrum; and similarly for a large list of other computability-theoretic properties. Recently, fields were added to the list of universal classes [MPSS], and it was here that the functorial language was first used. Independently, Montalbán introduced the syntactic perspective of bi-interpretations [Monb]. Our theorem above shows that the two possible notions of universality—one involving interpretations, and the other involving functors—coincide. There have also recently been constructions of an interpretation of graphs into differentially closed fields [MM] and into real closed fields [Oca14].

If we have an interpretation of $A$ inside of $B$, but the formulas are not computable $\Sigma^c_2$, then the resulting functor is Borel. Once again, we get a similar reversal, though the functor may actually be Baire-measurable, as one can show that any Baire-measurable functor can be replaced by a Borel functor. An interpretation also induces a continuous homomorphism on the automorphism groups of the two structures, from $\text{Aut}(B)$ to $\text{Aut}(A)$. The reversal is once again true: every such homomorphism is induced by an infinitary interpretation.

Theorem 9.1.3 & 9.1.9 (Harrison-Trainor, Miller, Montalbán). Let $A$ and $B$ be countable structures. Every Baire-measurable functor from $B$ to $A$, and every continuous homomorphism from $\text{Aut}(B)$ to $\text{Aut}(A)$, is induced by an infinitary interpretation of $A$ in $B$.

Moreover, the corresponding result for bi-interpretations is also true.

Theorem 9.1.6, 9.1.11 & 9.1.12 (Harrison-Trainor, Miller, Montalbán). Let $A$ and $B$ be countable structures. The following are equivalent:

1. $A$ and $B$ are infinitary bi-interpretable.
2. There is a Baire-measurable adjoint equivalence of categories between $A$ and $B$.
3. There is a Borel adjoint equivalence of categories between $A$ and $B$.
4. There is a continuous isomorphism between $\text{Aut}(A)$ and $\text{Aut}(B)$.
Moreover, the functors in (3) and the isomorphism in (4) are those induced by the interpretation.

Both of these theorems are proved using a forcing argument.

As consequences of these theorems we get several characterizations of the structures with particular automorphism groups. For example, the only way for a structure to have an automorphism group with a (continuous) homomorphism onto $S_{\omega}$, the permutation group on countable many elements, is for the structure to contain an $L_{\omega_1\omega}$-definable equivalence relation with infinitely many classes which are indiscernible in the sense that every permutation of the equivalence classes extends to one on the structure.

1.4 Computable Algebra

In computable algebra, we are interested in measuring the effectivity of various construction and theorems from algebra. In this thesis, we will mostly be interested in questions of constructing some set inside of a computable structure, such as finding a basis in a computable vectors space, and on extension problems, such as extending a computable automorphism of a computable field to an automorphism of its algebraic closure. We begin in the first chapter of this section with questions related to this latter question.

1.4.1 Extending Embeddings of Fields

Any embedding of a field $F$ into an algebraically closed field $K$ extends to an embedding of the algebraic closure $\overline{F}$ of $F$ into $K$. In the effective context, every computable field $\mathcal{F}$ embeds, via a computable embedding $\nu: F \to \overline{F}$, within a computable presentation of its algebraic closure. Given a computable embedding $\alpha$ of $F$ into a computable algebraically closed field $K$, we can try to extend $\alpha$ to an embedding $\beta$ of $\overline{F}$ into $K$ (by which we mean that $\alpha = \beta \circ \nu$. If we can decide the irreducibility of polynomials over $F$—in which case, we say that $F$ has a splitting algorithm—then it is not hard to make such an extension effectively. However, we show that if there is no algorithm for deciding the irreducibility of polynomials over $F$, then there is a computable embedding of $F$ into a computable algebraically closed field $K$ which does not extend to the algebraic closure via the fixed embedding.

**Theorem 10.1.1** (Harrison-Trainor, Melnikov, Miller). Let $F$ be a computable field together with a computable embedding $\nu: F \to \overline{F}$ of $F$ into its algebraic closure. Then the following are equivalent:

1. $F$ has a splitting algorithm,

2. every computable automorphism of $F$ can be extended to a computable automorphism of $\overline{F}$ via $\nu$. 
If $F$ is a normal algebraic extension of the prime field, we get a similar theorem for extending automorphisms of $F$ to an automorphism of $\overline{F}$ via the fixed embedding. However, we can modify the question slightly to allow the embedding of $F$ into $\overline{F}$ to vary: for which computable fields $F$ is it true that for every automorphism $\alpha$ of $F$, there is a computable embedding $\iota: F \to \overline{F}$ of $F$ into a computable algebraic closure, and a computable automorphism $\beta$ of $\overline{F}$ which extends $\alpha$? We give a partial answer to this question for field extensions with the “non-covering property” of Definition 10.1.3.

**Theorem 10.1.4** (Harrison-Trainor, Melnikov, Miller). Let $F$ be a computable normal extension of $\mathbb{F}_p$ such that $\text{Gal}(F/\mathbb{F}_p)$ has the non-covering property. The following are equivalent:

1. $F$ has a splitting algorithm,
2. For every computable automorphism $\alpha$ of $F$ an computable embedding $\iota$ of $F$ into $\overline{F}$, there is a computable automorphism of $\overline{F}$ extending $\alpha$.

We give a number of examples of applications of this theorem; in fact, we do not know of a single example where this theorem (at least, in the form of a slight generalization given in Theorem 10.4.6) cannot be applied.

We mentioned above that every computable field embeds into a computable algebraic closure. It is also true that every computable differential field embeds into a computable differential closure. This work on extending automorphisms began with an attempt to determine which computable difference fields embed into a computable difference closed field. A difference field is a field equipped with an automorphism. It turns out that a computable difference field embeds into a computable difference closed field if and only if there is a computable extension of it’s automorphism to an algebraically closed field—this is exactly the property discussed in Theorem 10.1.4.

### 1.4.2 Independence Relations and Bases

In Chapter 11 we turn to the problem of computing a basis for various algebraic structures. Many algebraic structures in mathematics have a notion of independence. Two of the most well-known examples are vector spaces, with linear independence, and fields, with algebraic independence. However, there are many more examples: abelian groups with $\mathbb{Z}$-linear independence, differential fields with $\delta$-independence, and many more. The are all instances of the general notion of a pregeometry (or matroid). Every pregeometry has a basis—a maximal independent set—which, in vector spaces, is just the standard notion of a basis, and in fields is the standard notion of a transcendence base.

We consider algebraic structures equipped with a c.e. pregeometry (which means that we can enumerate the element dependent on a given tuple). It is well known that a computable vector space (of infinite dimension) need not have a computable basis; but it is obvious that every computable vector space is isomorphic (via a $\Delta^0_2$) to a (or more precisely, up to computable isomorphism, the) computable vector space with a computable basis. We name this phenomenon for Mal’cev, who first observed it.
Definition 11.1.1. A class \( K \) has the Mal'cev property if each member \( M \) of \( K \) of infinite dimension has a computable presentation \( G \) with a computable basis and a computable presentation \( B \) with no computable basis such that \( B \cong_{\Delta^0_2} G \).

So though such a structure may not have a computable basis, we can find a computable basis in a different “good” computable presentation. Similarly, if such a structure has a computable basis, then we can find a “bad” computable presentation with no computable basis. In particular, by a result of Goncharov [Gon82], a structure with the Mal'cev property has infinitely many computable copies up to computable isomorphism. We began this work trying to study bases for differentially closed fields, but we quickly realized that we were using only very general properties of differentially closed fields. Abstracting these properties into Conditions \( G \) and \( B \) (see Section 11.1.4), we prove the following metatheorem.

Theorem 11.1.2 (Harrison-Trainor, Melnikov, Montalbán). Let \( K \) be a class of computable structures that admits a r.i.c.e. preregometry cl. If each \( M \) in \( K \) of infinite dimension satisfies Conditions \( G \) and \( B \), then \( K \) has the Mal'cev property.

This metatheorem can be applied to many different examples:

Theorem 11.1.3, 11.1.4, 11.1.5, 11.1.6, 11.1.7, 12.5.6, & 12.5.8. The following algebraic classes have the Mal'cev property:

1. vector spaces with respect to linear independence.
2. algebraically closed fields with respect to algebraic independence.
3. differentially closed fields of characteristic zero with respect to \( \delta \)-independence.
4. difference closed fields with respect to transformal independence.
5. real closed fields with respect to the standard field-theoretic (or, equivalently, model-theoretic) algebraic independence.
6. torsion-free abelian groups with respect to linear independence over \( \mathbb{Z} \).
7. Archimedean ordered abelian groups with respect to linear independence over \( \mathbb{Z} \).
8. algebraically closed valued fields with respect to algebraic independence.
9. \( p \)-adically closed fields with respect to algebraic independence.

(1) and (2) are easy to see without the metatheorem. (3), (4), and (5) are completely new and were proved jointly by the author, Melnikov, and Montalbán, appearing in [HTMM15]. (8) and (9) are due to the author and are also completely new; they appeared in [HTa]. (6) was previously known from results of Nurtazin [Nur74], Dobrica [Dob83], and Goncharov [Gon82], but we give a new perspective on the proof. It was known from work of Goncharov, Lempp, and Solomon [GLS03] that every computable Archimedean ordered
abelian group has a computable copy with a computable base, and that in the case of infinite rank it has infinitely many effectively distinct computable presentations, but the rest of (7) is new, and appeared in [HTMM15].

1.4.3 Computable Valued Fields

In Chapter 12, we consider another extension problem coming from valued fields. Variations of Rabin’s theorem for valued fields were previously studied by Smith [Smi81]; some of our results extend those of that paper. If \((K, v)\) is a computable valued field, then the valuation extends to any field extension of \(K\). We give a characterization of the fields where this can be done computably.

**Theorem 12.1.2** (Harrison-Trainor). Let \((K, v)\) be a computable algebraic valued field. Then the following are equivalent:

1. for every computable embedding \(\iota: K \to L\) of \(K\) into a field \(L\) algebraic over \(K\), there is a computable extension of \(v\) to a computable valuation \(w\) on \(L\),
2. the Hensel irreducibility set

\[ H_K := \{ f = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 \in \mathcal{O}_K[x] : f \text{ is irreducible over } K, \ v(a_{n-1}) = 0, \text{ and } v(a_{n-2}), \ldots, v(a_0) > 0 \} \]

of \((K, v)\) is computable.

Note the relation between (2) and a splitting algorithm for \(K\); in particular, if \(K\) has a splitting algorithm, then (2) holds.

Among the most important examples of valued fields are the \(p\)-adics \(\mathbb{Q}_p\). The theory of \(p\)-adically closed fields is the theory of \(\mathbb{Q}_p\). Just as the theory of real closed fields is the model companion of the formally real fields, the theory of \(p\)-adically closed fields is the model companion of a class of fields called the formally \(p\)-adic fields. Classically, every formally \(p\)-adic embeds into a \(p\)-adic closure. The effective analogue is false:

**Theorem 12.1.3** (Harrison-Trainor). There is a computable formally \(p\)-adic field which does not embed into a computable \(p\)-adic closure.

The issue is that we can construct a formally \(p\)-adic field in which the divisibility relation on the value group is not computable. If we have an algorithm to compute the divisibility relation on the value group of a formally \(p\)-adic field, then we can effectively embed that field into a computable \(p\)-adic closure.

**Theorem 12.1.4** (Harrison-Trainor). Let \((K, v)\) be a computable formally \(p\)-adic valued field with value group \(\Gamma\). Suppose that we can compute, for each \(\gamma \in \Gamma\) and \(k \in \mathbb{N}\), whether \(\gamma\) is divisible by \(k\). Then there is a computable embedding of \(K\) into a computable \(p\)-adic closure \((L, w)\).
1.4.4 Ordered Groups

In the case of abelian groups, bases for \( \mathbb{Z} \)-linear independence are related to orderings. An abelian group is orderable if and only if it is torsion-free. There are computable torsion-free abelian groups which have no computable ordering. However, every computable torsion-free abelian group has a computable presentation with a computable basis with respect to \( \mathbb{Z} \)-linear independence, and Solomon [Sol02] noted that an order on this basis induces an order on the group. So every computable torsion-free abelian group is (classically) isomorphic to a computable group with a computable ordering. Downey and Kurtz [DK86] asked whether every computable orderable non-abelian group is isomorphic to a computable group with a computable ordering. This is the topic of Chapter 13.

For non-abelian groups, there are two different notions of ordering: left-orderings and bi-orderings (right-orderings are essentially the same as left-orderings). For left-orderings, if \( a < b \), then \( ca < cb \); for bi-orderings, it must also be true that if \( a < b \) then \( ac < bc \). We show that, unlike abelian groups, there are computable left-orderable groups which are not isomorphic to a group with a computable left-ordering.

**Theorem 13.1.2** (Harrison-Trainor). There is a computable left-orderable group which has no presentation with a computable left-ordering.

Our strategy is to build a group
\[
G = \mathbb{N} \rtimes \mathcal{H}/\mathcal{R}
\]
and code information into the finite orbits of certain elements of \( \mathbb{N} \) under inner automorphisms given by conjugating by elements of \( \mathcal{H}/\mathcal{R} \). This strategy cannot work to build a bi-orderable group, as in a bi-orderable group there is no generalized torsion—i.e., no product of conjugates of a single element can be equal to the identity—and hence no inner automorphism has a non-trivial finite orbit. It is still an open question whether the same is true for bi-orderable groups.

1.5 Miscellaneous Results

In the fifth and final part of this thesis, we consider some problems which do not fit within the other sections.

1.5.1 Decidably Presentable Structures

In Chapter 14, we consider the problem of characterizing those computable structures which have a decidable presentation. A structure is decidable if its full elementary diagram, rather than just the atomic diagram, is computable. For example, any computable algebraically closed field is decidable as a result of quantifier elimination. There are also examples of structures where some computable presentations are decidable, but others are not. For example, the standard presentation of the linear order \((\mathbb{N},<)\) is decidable. However, there is
also a computable copy of the same structure in which the successor relation is not computable, and hence this copy is not decidable. (Here, $a$ is a successor of $b$ if and only if $(\forall c)[c < b \lor c > a]$.) Though these two computable structures are isomorphic, they are not computably isomorphic.

Goncharov asked whether there is a characterization of the computable structures which have a decidable presentation. We show that there is no such characterization. More formally, we compute the complexity of the index set of the decidably presentable structures.

**Theorem 14.1.1** (Harrison-Trainor). The index set

$$I_{d-pres} = \{i \mid \text{the } i\text{th computable structure is decidably presentable}\}$$

is $\Sigma^1_1$-complete.

As a result, there is no possible reasonable characterization of the computable structures with decidable presentations. What we mean is that there is no simpler way to check whether a computable structure $A$ has a decidable presentation than to ask: *Does there exist a decidable structure $B$ and a classical isomorphism between $A$ and $B$?* This requires searching through all possible isomorphisms, of which there may be continuum-many, between $A$ and $B$. If there were a simpler characterization of the computable structures with decidable presentations, then one would expect that characterization to yield a simpler way of checking whether a computable structure has a decidable presentation.

A similar approach was taken in [DKL+15], where it was shown that there is no reasonable characterization of computable categoricity, and in [DM08], where it was shown that there is no reasonable classification of abelian groups. This approach originated with [GK02]. See also [LS07, Fok07, CFG+07, FGK+15, GBM15a, GBM15b].

We also show that there is also no characterization of the computable structures which have a 1-decidable presentation, i.e., a presentation in which the existential diagram is decidable. Moreover, this generalizes to the $n$-presentable structures for every $n$. In fact,

$$(\Sigma^1_1, \Pi^1_1) \leq_m (I_{d-pres}, I_{2-pres})$$

where $I_{2-pres}$ is the index set of the 2-presentable structures.

### 1.5.2 The Gamma Problem

Chapter 15 is about coarse computability and the Gamma problem. Coarse computation is a generalization of Turing computation where the computation is allowed to make mistakes, but only on a small set, i.e., on one of asymptotic lower density zero. More generally, we can talk about algorithms which are correct half the time, or a third of the time, or almost never. To a set $A \subseteq \omega$, we can assign a real number which measures the highest density to which it can be approximated by a computable set.
CHAPTER 1. GENERAL INTRODUCTION

Definition 1.5.2 ([HJMS16]). A set $A \subseteq \omega$ is coarsely computable at density $r \in [0, 1]$ if there is a computable set $R$ such that $\rho(A \leftrightarrow R) = r$. Here, $A \leftrightarrow R$ is the set on which $A$ and $R$ agree:

$$A \leftrightarrow R := \{x \mid x \in A \iff x \in R\}.$$ 

Definition 1.5.3 ([HJMS16]). The coarse computability bound of a set $A \subseteq \omega$ is

$$\gamma(A) := \sup\{r \mid A \text{ is coarsely computable at density } r\}.$$ 

That is, $\gamma(A)$ is the supremum, over all computable sets $R$, of $\rho(A \leftrightarrow R)$.

It is known that for each $r \in (0, 1]$, there are sets with coarse computability bound $r$ such that the supremum is obtained, and sets where the supremum is not obtained [HJMS16].

Jockusch and Schupp [JS12] have shown that every non-zero Turing degree contains a set which is not coarsely computable. (This follows from the proof of Proposition 15.1.6 below.) Thus, if $\Gamma_T(a) = 1$, then $a = 0$. Andrews, Cai, Diamondstone, Jockusch, and Lempp suggested assigning to each Turing degree a real number which measures the extent to which all sets computable in that degree can be coarsely computed.

Definition 1.5.4 ([ACD+16]). The coarse computability bound of a Turing degree $a$ is

$$\Gamma_T(a) := \inf\{\gamma(A) \mid A \text{ is } a\text{-computable}\}.$$ 

It suffices to take the infimum only over sets in $a$.

Andrews, Cai, Diamondstone, Jockusch, and Lempp showed that $\Gamma_T(a)$ can take on the values $0, 1/2,$ and $1$. Hirschfeldt, Jockusch, McNicholl, and Schupp showed that $\Gamma_T(a)$ cannot take on any values in the open interval $(1/2, 1)$. The Gamma question, from [ACD+16], asks whether the value of $\Gamma_T$ can be strictly between $0$ and $1/2$. Monin [Mona] has recently given a solution to the Gamma question: The only possible values of $\Gamma_T$ are $0, 1/2,$ and $1$.

Our work grew out of an independent attempt to answer the Gamma question. If we replace Turing reducibility by many-one reducibility, we get a Gamma function on many-one degrees:

Definition 1.5.5. The coarse computability bound of an $m$-degree $a$ is

$$\Gamma_m(a) := \inf\{\gamma(A) \mid A \leq_m a\}.$$ 

It suffices to take the infimum only over sets in $a$.

It is still true that $\Gamma_m(a)$ cannot take on any values in the open interval $(1/2, 1)$, and the same examples have $\Gamma_m$ equal to $0, 1/2,$ and $1$. Thus, we can ask the Gamma question for $m$-degrees: Can the value of $\Gamma_m$ be strictly between $0$ and $1/2$? Interestingly, we get the opposite answer from Monin's: Every $p \in [0, 1/2]$ is a possible value of $\Gamma_m$.

Theorem 15.1.8 (Harrison-Trainor). Fix $0 \leq p \leq \frac{1}{2}$. There is an $m$-degree $a$ with $\Gamma_m(a) = p$. 

Versions of the Gamma question for weaker reducibilities have already been asked in the literature: In \[\text{[Hir17]}\], Hirschfeldt asked the Gamma question for truth table degrees. (Monin’s answer to the Gamma question for Turing degrees also yields the same answer for truth table degrees: The value of $\Gamma_{tt}$ cannot be strictly between 0 and 1/2.) An open problem is to determine at which points—i.e., for which reducibility between many-one and Turing reducibility—the behaviour changes.
Part I

Descriptions of Structures Using $\mathcal{L}_{\omega_1 \omega}$ Sentences
Chapter 2
Scott Spectra of Theories

The results presented in this chapter appeared in [HTf].

2.1 Introduction

Scott [Sco65] showed that every countable structure $A$ can be characterized, up to isomorphism, as the unique countable structure satisfying a particular sentence of the infinitary logic $L_{\omega_1\omega}$, called the Scott sentence of $A$. Scott’s proof gives rise to a notion of Scott rank for structures; there are several different definitions, which we will discuss later in Section 2.2.1, but until then we may take the Scott rank of $M$ to be the least ordinal $\alpha$ such that $M$ has a $\Pi^1_{\alpha+1}$ Scott sentence. This paper is concerned with the following general question: given a theory (by which we mean a sentence of $L_{\omega_1\omega}$) for what could the Scott ranks of models of $T$ be? This collection of Scott ranks is called the Scott spectrum of $T$:

**Definition 2.1.1.** Let $T$ be an $L_{\omega_1\omega}$-sentence. The Scott spectrum of $T$ is the set

$$SS(T) = \{ \alpha \in \omega_1 : \alpha \text{ is the Scott rank of a countable model of } T \}.$$  

This is an old definition. For example, in 1981, Makkai [Mak81] defined the Scott spectrum of a theory in this way and showed that there is a sentence of $L_{\omega_1\omega}$ without uncountable models whose Scott spectrum is unbounded below $\omega_1$. In [Vää11, p. 151] a reference is made to gaps in the Scott spectrum—ordinals $\beta$ which are not in the Scott spectrum, but which are bounded above by some other $\alpha$ in the Scott spectrum—but the only results proved about Scott spectra are about bounds below $\omega_1$. This seems to be a general pattern: whenever Scott spectra are mentioned in the literature, it is to say that they are either bounded or unbounded below $\omega_1$. This paper, to the contrary, is about the gaps, and about a classification of the sets of countable ordinals that can be Scott spectra. Our main result is a complete descriptive-set-theoretic classification of the sets of ordinals which are Scott spectra. For this classification, we assume projective determinacy.

This work began with the following question, first asked by Montalbán at the 2013 BIRS Workshop on Computable Model Theory.
Question 2.1.2 (Montalbán). If $T$ is a $\Pi_2^{in}$ sentence, must $T$ have a model of Scott rank two or less?

At the time, we knew very little about how to answer such questions. In this paper, we make a large step forward in our understanding of Scott spectra: not only do we answer the question negatively, but we also answer the generalization to any ordinal $\alpha$ and we apply those techniques to solve other open problems about Scott ranks.

The paper is in two parts. The first part is a general construction in Section 2.3. Given a $\mathcal{L}_{\omega_1\omega}$-pseudo-elementary class of linear orders, we build an $\mathcal{L}_{\omega_1\omega}$-sentence $T$ so that the Scott spectrum of $T$ is related to the set of well-founded parts of linear orders in that class. The construction bears some similarity to work of Marker [Mar90]. In the second part, we apply the general construction to get various results about Scott spectra. We will describe these applications now.

2.1.1 $\Pi_2^{in}$ Theories with No Models of Low Scott Rank

It follows easily from known results that for a given ordinal $\alpha$, there is a theory $T$ all of whose models have Scott rank at least $\alpha$. (We can, for example, take $T$ to be the Scott sentence of a model of Scott rank $\alpha$.) This is not very surprising, as the theory $T$ we get has quantifier complexity about $\alpha$; complicated theories may have only complicated models. The interesting question is whether there is an uncomplicated theory all of whose models are complicated. Such theories exist.

Theorem 2.1.3. Fix $\alpha < \omega_1$. There is a $\Pi_2^{in}$ sentence $T$ whose models all have Scott rank $\alpha$.

In particular, taking $\alpha > 2$ answers the question of Montalbán stated above. In Section 2.4 we will derive Theorem 2.1.3 from the general construction of Section 2.3.

2.1.2 Computable Structures of High Scott Rank

Nadel [Nad74] showed that if $A$ is a computable structure, then its Scott rank is at most $\omega_1^{CK} + 1$. We say that a computable structure with non-computable Scott rank, i.e. with Scott rank $\omega_1^{CK}$ or $\omega_1^{CK} + 1$, has high Scott rank. There are few known examples of computable structures of high Scott rank. Harrison [Har68] gave the first example of a structure of Scott rank $\omega_1^{CK} + 1$: the Harrison linear order $H$, which is a computable linear order of order type $\omega_1^{CK}(1 + \mathbb{Q})$. The Harrison order is the limit of the computable ordinals in the following sense: given $\alpha$ a computable ordinal, there is a computable ordinal $\beta$ such that $H \equiv_{\alpha} \beta$. We say that such a structure is strongly computable approximable:

Definition 2.1.4. A computable structure $A$ of non-computable rank is weakly computably approximable if every computable infinitary sentence $\varphi$ true in $A$ is also true in some computable $B \not\equiv A$. $A$ is strongly computably approximable if we require that $B$ have computable Scott rank.
Makkai [Mak81] gave the first example of an arithmetic structure of Scott rank $\omega_1^{CK}$, and Knight and Millar [KM10] modified the construction to get a computable structure. Calvert, Knight, and Millar [CKM06] showed that this structure is also strongly computably approximable. Calvert and Knight [CK06, Problem 6.2] asked the following question:

**Question 2.1.5** (Calvert and Knight). Is every computable model of high Scott rank strongly (or weakly) computably approximable?

At the time, every known example of a computable structure of high Scott rank was strongly computably approximable. We show here that there are computable structures of Scott rank $\omega_1^{CK}$ and $\omega_1^{CK} + 1$ which are not strongly computably approximable.

**Theorem 2.1.6.** For $\alpha = \omega_1^{CK}$ or $\alpha = \omega_1^{CK} + 1$: There is a computable model $A$ of Scott rank $\alpha$ and a $\Pi^c_2$ sentence $\psi$ such that $A \models \psi$, and whenever $B$ is any structure and $B \models \psi$, $B$ has Scott rank $\alpha$.

We prove Theorem 2.1.6 in Section 2.5. Note that this gives a new type of model of high Scott rank which is qualitatively different from the previously known examples.

### 2.1.3 Bounds on Scott Height

It follows from a general counting argument that there is a least ordinal $\alpha < \omega_1$ such that if $T$ is a computable $L_{\omega_1\omega}$-sentence whose Scott spectrum is bounded below $\omega_1$, then the Scott spectrum of $T$ is bounded below $\alpha$. We call this ordinal the Scott height of $L_{\omega_1\omega}^\alpha$, and we denote it $sh(L_{\omega_1\omega}^\alpha)$.

Sacks [Sac83] and Marker [Mar90] asked:

**Question 2.1.7** (Sacks and Marker). What is the Scott height of $L_{\omega_1\omega}^\omega$?

**Definition 2.1.8.** $\delta_2^1$ is the least ordinal which has no $\Delta^1_2$ presentation.

Sacks [Sac83] showed that $sh(L_{\omega_1\omega}^\omega) \leq \delta_2^1$. Marker [Mar90] was able to resolve this question for pseudo-elementary classes.

**Definition 2.1.9.** A class $\mathcal{C}$ of structures in a language $\mathcal{L}$ is an $L_{\omega_1\omega}$-pseudo-elementary class ($PC_{L_{\omega_1\omega}}$-class) if there is an $L_{\omega_1\omega}$-sentence $T$ in an expanded language $\mathcal{L}' \supseteq \mathcal{L}$ such that the structures in $\mathcal{C}$ are the reducts to $\mathcal{L}$ of the models of $T$. $\mathcal{C}$ is a computable $PC_{L_{\omega_1\omega}}$-class if $T$ is a computable sentence.

We can define the Scott height of $PC_{L_{\omega_1\omega}}$ in a similar way to the Scott height of $L_{\omega_1\omega}$, except that now we consider all $L_{\omega_1\omega}$-pseudo-elementary classes which are the reducts of the models of a computable sentence. Marker [Mar90] showed that $sh(PC_{L_{\omega_1\omega}}) = \delta_2^1$. Using our methods, we can expand this argument to $L_{\omega_1\omega}^\omega$.

**Theorem 2.1.10.** $sh(L_{\omega_1\omega}^\omega) = \delta_2^1$.

We prove this theorem in Section 2.7.
2.1.4 Classifying the Scott Spectra

Assuming projective determinacy, we will define a descriptive set-theoretic class which will give a classification of the Scott spectra.

**Definition 2.1.11.** A set of countable ordinals is a $\Sigma^1_1$ class of ordinals if it consists of the order types in $C \cap On$ for some $\Sigma^1_1$ class $C$ of linear orders on $\omega$.

Note that $C$ and $On$ here are classes of presentations of ordinals as linear orders of $\omega$. Frequently we will pass without comment between viewing a class as a collection of ordinals, i.e., of order types, and as a collection of $\omega$-presentations of linear orders.

**Theorem 2.1.12** (ZF C + PD). The Scott spectra of $L_{\omega_1 \omega}$-sentences are the $\Sigma^1_1$ classes $C$ of ordinals with the property that, if $C$ is unbounded below $\omega_1$, then either $C$ is stationary or $\{\alpha: \alpha + 1 \in C\}$ is stationary.

We can also get an alternate characterization which is more tangible. To state this, we must define two ways to produce an ordinal from an arbitrary linear order.

**Definition 2.1.13.** Let $(L, \leq)$ be a linear order. The well-founded part $\text{wfp}(L)$ of $L$ is the largest initial segment of $L$ which is well-founded. The well-founded collapse of $L$, $\text{wfc}(L)$, is the order type of $L$ after we collapse the non-well-founded part $L \setminus \text{wfp}(L)$ to a single element.

We can identify $\alpha \in \text{wfp}(L)$ with the ordinal which is the order type of $\{\beta \in L: \beta < \alpha\}$. We can also identify $\text{wfp}(L)$ with its order type. If $L$ is well-founded, with order type $\alpha$, then $\text{wfc}(L) = \text{wfp}(L) = \alpha$. If $L$ is not well-founded, $\text{wfc}(L) = \text{wfp}(L) + 1$.

**Theorem 2.1.14** (ZF C + PD). The Scott spectra of $L_{\omega_1 \omega}$-sentences are exactly the sets of the form:

1. $\text{wfp}(C)$,
2. $\text{wfc}(C)$, or
3. $\text{wfp}(C) \cup \text{wfc}(C)$

where $C$ is a $\Sigma^1_1$ class of linear orders of $\omega$.

**Theorem 2.1.15** (ZF C + PD). Each Scott spectrum is the Scott spectrum of a $\Pi^1_n$ sentence.

**Theorem 2.1.16** (ZF C + PD). Every Scott spectrum of a $PC_{L_{\omega_1 \omega}}$-class is the Scott spectrum of an $L_{\omega_1 \omega}$-sentence.

We will prove Theorems 2.1.12, 2.1.14, 2.1.15, and 2.1.16 in Section 2.8. This classification allows us to construct interesting Scott spectra. For example, the successor ordinals and the admissible ordinals are Scott spectra.
2.2 Preliminaries on Back-and-forth Relations and Scott Ranks

All of our structures will be countable structures in a countable language. The infinitary logic $L_{\omega_1\omega}$ consists of formulas which allow countably infinite conjunctions and conjunctions; see [AK00, Sections 6 and 7] for background. We will use $\Sigma_\alpha^a$ for the infinitary $\Sigma_\alpha$ formulas and $\Sigma_a^c$ for the computable infinitary $\Sigma_\alpha$ formulas (and similarly for $\Pi_\alpha^a$ and $\Pi_\alpha^c$).

2.2.1 Scott Rank

Let $A$ be a countable structure. There are a number of ways to define the Scott rank of $A$, not all of which agree. We describe a number of different definitions before fixing one for the rest of the paper. For the most part, it does not matter, modulo some small changes, which definition we choose as our results are quite robust.

The first definition uses the symmetric back-and-forth relations which come from Scott’s proof of his isomorphism theorem [Sco65]. See, for example, [AK00, Sections 6.6 and 6.7].

Definition 2.2.1. The standard symmetric back-and-forth relations $\sim_\alpha$ on $A$, for $\alpha < \omega_1$, are defined by:

1. $\bar{a} \sim_0 \bar{b}$ if $\bar{a}$ and $\bar{b}$ satisfy the same quantifier-free formulas.

2. For $\alpha > 0$, $\bar{a} \sim_\alpha \bar{b}$ if for each $\beta < \alpha$ and $\bar{d}$ there is $\bar{c}$ such that $\bar{a} \bar{c} \sim_\beta \bar{b} \bar{d}$, and for all $\bar{c}$ there is $\bar{d}$ such that $\bar{a} \bar{c} \sim_\beta \bar{b} \bar{d}$.

For each tuple $\bar{a} \in A$, Scott proved that there is a least ordinal $\alpha$, the Scott rank of the tuple, such that if $\bar{a} \sim_\alpha \bar{b}$, then $\bar{a}$ and $\bar{b}$ are in the same automorphism orbit of $A$. Equivalently, $\alpha$ is the least ordinal such that if $\bar{a} \sim_\alpha \bar{b}$, then $\bar{a} \sim_\gamma \bar{b}$ for all ordinals $\gamma < \omega_1$, or such that if $\bar{a} \sim_\alpha \bar{b}$, then $\bar{a}$ and $\bar{b}$ satisfy the same $L_{\omega_1\omega}$-formulas. Then the Scott rank of $A$ is the least ordinal strictly greater than (or, in the definition used by Barwise [Bar75], greater than or equal to) the Scott rank of each tuple of $A$. One can then define a Scott sentence for $A$, that is, a sentence of $L_{\omega_1\omega}$ which characterizes $A$ up to isomorphism among countable structures.

Another definition uses the non-symmetric back-and-forth relations which have been useful in computable structure theory. See [AK00, Section 6.7].

Definition 2.2.2. The standard (non-symmetric) back-and-forth relations $\leq_\alpha$ on $A$, for $\alpha < \omega_1$, are defined by:

1. $\bar{a} \leq_0 \bar{b}$ if for each quantifier-free formula $\psi(\bar{x})$ with Gödel number less than the length of $\bar{a}$, if $A \models \psi(\bar{a})$ then $A \models \psi(\bar{b})$.

2. For $\alpha > 0$, $\bar{a} \leq_\alpha \bar{b}$ if for each $\beta < \alpha$ and $\bar{d}$ there is $\bar{c}$ such that $\bar{b} \bar{d} \leq_\beta \bar{a} \bar{c}$.

Let $\bar{a} \equiv_\alpha \bar{b}$ if $\bar{a} \leq_\alpha \bar{b}$ and $\bar{b} \leq_\alpha \bar{a}$,
For $\alpha \geq 1$, $\bar{a} \leq_\alpha \bar{b}$ if and only if every $\Sigma^1_{\alpha}$ formula true of $\bar{b}$ is true of $\bar{a}$.

Then one can define the Scott rank of a tuple $\bar{a}$ to be the least $\alpha$ such that if $\bar{a} \equiv_\alpha \bar{b}$, then $\bar{a}$ and $\bar{b}$ are in the same automorphism orbit of $\mathcal{A}$. The Scott rank of $\mathcal{A}$ is then least ordinal strictly greater than the Scott rank of each tuple.

A third definition of Scott rank has recently been suggested by Montalbán based on the following theorem:

**Theorem 2.2.3** (Montalbán [Mon15a]). Let $\mathcal{A}$ be a countable structure, and $\alpha$ a countable ordinal. The following are equivalent:

1. $\mathcal{A}$ has a $\Pi^1_{\alpha+1}$ Scott sentence.
2. Every automorphism orbit in $\mathcal{A}$ is $\Sigma^1_\alpha$-definable without parameters.
3. $\mathcal{A}$ is uniformly (boldface) $\Delta^0_\alpha$-categorical without parameters.
4. Every $\Pi^1_\alpha$ type realized in $\mathcal{A}$ is implied by a $\Sigma^1_\alpha$ formula.
5. No tuple in $\mathcal{A}$ is $\alpha$-free.

Montalbán defines the Scott rank of $\mathcal{A}$ to be the least ordinal $\alpha$ such that $\mathcal{A}$ has a $\Pi^1_{\alpha+1}$ Scott sentence. It is this definition which we will take as our definition of Scott rank. We write $SR(\mathcal{A})$ for the Scott rank of the structure $\mathcal{A}$. The $\alpha$-free tuples which appear in the theorem above will also appear later.

**Definition 2.2.4.** Let $\bar{a}$ be a tuple of $\mathcal{A}$. Then $\bar{a}$ is $\alpha$-free if for each $\bar{b}$ and $\beta < \alpha$, there are $\bar{a}'$ and $\bar{b}'$ such that $\bar{a}, \bar{b} \leq_\beta \bar{a}', \bar{b}'$ and $\bar{a}' \not\leq_\alpha \bar{a}$.

Other definitions of Scott rank appear in [Sac07, Section 2] and [Gao07, Section 3].

### 2.2.2 Scott Spectra

Recall that the Scott spectrum of an $\mathcal{L}_{\omega_1}\omega$-sentence $T$ is the set of countable ordinals

$$SS(T) = \{SR(\mathcal{A}) : \mathcal{A} \text{ is a countable model of } T\}.$$ 

More generally, one can define the Scott spectrum $SS(\mathcal{C})$ of a class of countable structures $\mathcal{C}$. For each $\alpha < \omega_1$ there is an $\mathcal{L}_{\omega_1}\omega$-sentence whose Scott spectrum is $\{\alpha\}$. For example, if $\mathcal{A}$ is a structure of Scott rank $\alpha$,\footnote{Such structures exist; for example, the results on linear orders in [AK00, Section 15] can be used to construct examples, or one can use the construction in [CFS13].} then we can take $T$ to be the Scott sentence for $\mathcal{A}$. However, the quantifier complexity of $T$ will be approximately $\alpha$. It is only as a result of our Theorem 2.1.3 that one can obtain such a theory $T$ of low quantifier complexity even when $\alpha$ is very large.

We note some results about producing new Scott spectra by combining existing ones. The proofs are all simple constructions which we omit.


CHAPTER 2. SCOTT SPECTRA OF THEORIES

Proposition 2.2.5. If \((\mathcal{A}_i)_{i \in \omega}\) are the Scott spectra of \(L_{\omega_1 \omega}\)-sentences, then \(\bigcup_{i \in \omega} \mathcal{A}_i\) is also the Scott spectrum of an \(L_{\omega_1 \omega}\)-sentence.

Proposition 2.2.6. If \(\mathcal{A}\) is the Scott spectrum of an \(L_{\omega_1 \omega}\)-sentence and \(\alpha < \omega_1\), then \(\mathcal{B} = \{\beta \in \mathcal{A} : \beta \geq \alpha\}\) is also the Scott spectrum of an \(L_{\omega_1 \omega}\)-sentence.

Proposition 2.2.7. Let \(\mathcal{A}\) and \(\mathcal{B}\) be sets of countable ordinals, and suppose that \(\mathcal{A}\) is the Scott spectrum of an \(L_{\omega_1 \omega}\)-sentence. If there is a countable ordinal \(\alpha < \omega_1\) such that

\[
\mathcal{A} \cap \{\beta : \alpha \leq \beta < \omega_1\} = \mathcal{B} \cap \{\beta : \alpha \leq \beta < \omega_1\}
\]

then \(\mathcal{B}\) is also the Scott spectrum of an \(L_{\omega_1 \omega}\)-sentence.

2.2.3 Non-standard Back-and-Forth Relations

Let \((L, \leq)\) be a linear order. We will consider \((L, \leq)\) to be a non-standard ordinal, i.e., a linear ordering with an initial segment which is an ordinal, but whose tail may not necessarily be well-ordered. Assume that \(L\) has a smallest element 0.

Definition 2.2.8. A sequence of equivalence relations \((\preceq_{\alpha})_{\alpha \in L}\) are non-standard back-and-forth relations on \(A\) if they satisfy the definition of the standard back-and-forth relations (Definition 2.2.2), that is, if:

1. If \(\alpha\) is the smallest element of \(L\), \(\bar{a} \preceq_{\alpha} \bar{b}\) if for each quantifier-free formula \(\psi(\bar{x})\) with Gödel number less than the length of \(\bar{a}\), if \(A \models \psi(\bar{a})\) then \(A \models \psi(\bar{b})\).

2. If \(\alpha\) is not the smallest element of \(L\), \(\bar{a} \preceq_{\alpha} \bar{b}\) if for each \(\beta < \alpha\), for all \(\bar{d}\) there is \(\bar{c}\) such that \(\bar{b}, \bar{d} \preceq_{\beta} \bar{a}, \bar{c}\).

While the standard back-and-forth relations are uniquely defined, this is not the case for non-standard back-and-forth relations. However, they are uniquely determined on the well-founded part of \(L\).

Remark 2.2.9. Let \((L, <)\) be a linear order and \((\preceq_{\alpha})_{\alpha \in L}\) a sequence of non-standard back-and-forth relations on \(A\). The relations \(\preceq_{\alpha}\) for \(\alpha \in \text{wfp}(L)\) are the same as the standard back-and-forth relations \(\preceq_{\alpha}\) on \(A\).

For non-standard \(\alpha \in L\), that is, \(\alpha \in L \setminus \text{wfp}(L)\), the back-and-forth relations hold only between tuples in the same automorphism orbit.

Lemma 2.2.10. Let \((L, <)\) be a linear order and \((\preceq_{\alpha})_{\alpha \in L}\) a sequence of non-standard back-and-forth relations on \(A\). For \(\alpha \in L \setminus \text{wfp}(L)\), if \(\bar{a} \preceq_{\alpha} \bar{b}\), then there is an isomorphism of \(A\) taking \(\bar{a}\) to \(\bar{b}\).
Proof. It is easy to see that \( \{ \bar{a} \mapsto \bar{b} : \bar{a} \preceq_\beta \bar{b} \text{ for some } \beta \in L \setminus \text{wfp}(L) \} \)
is a set of finite maps with the back-and-forth property. If \( \bar{a} \preceq_\beta \bar{b} \) for some \( \beta \in L \setminus \text{wfp}(L) \), then \( \bar{a} \) and \( \bar{b} \) satisfy the same atomic sentences. Thus any such map extends to an automorphism. \( \square \)

2.2.4 Admissible Ordinals and Harrison Linear Orders

Given \( X \subseteq \omega, \omega_1^X \) is the least non-\( X \)-computable ordinal. By a theorem of Sacks [Sac76], the countable admissible ordinals \( \alpha > \omega \) are all of the form \( \omega_1^X \) for some set \( X \). For our purposes, we may take this as the definition of an admissible ordinal.

Harrison [Har68] showed that for each \( X \subseteq \omega \), there is an \( X \)-computable ordering which is not well-ordered, and which has no \( X \)-hyperarithmetic descending sequence. Moreover, any such ordering is of order type \( \omega_1^X \cdot (1 + \mathbb{Q}) + \beta \) for some \( X \)-computable ordinal \( \beta \). We call \( \omega_1^X \cdot (1 + \mathbb{Q}) \) the Harrison linear order relative to \( X \). Note that the property of being the Harrison linear order relative to \( X \) is \( \Sigma_1^1(X) \): a linear order is the Harrison linear order relative to \( X \) if:

1. it is \( X \)-computable,
2. for every \( X \)-computable ordinal \( \alpha \) and element \( x \), there is \( y \) such that the interval \( [x, y) \) has order type \( \alpha \),
3. it has a descending sequence, and
4. for every \( X \)-computable ordinal \( \alpha \) and index \( e \) there is a jump hierarchy on \( \alpha \) which witnesses that \( \varphi_e^{(\alpha)} \) is not a descending sequence.

Later we will use the fact that the set of admissible ordinals contains a club.

Definition 2.2.11. A set \( U \subseteq \omega_1 \) is closed unbounded (club) if it is unbounded below \( \omega_1 \) and is closed in the order topology, i.e., if \( \sup(U \cap \alpha) = \alpha \neq 0 \), then \( \alpha \in U \).

Definition 2.2.12. A set \( U \subseteq \omega_1 \) is stationary if it intersects every club set.

Remark 2.2.13. Given a set \( Y \subseteq \omega \), the set of \( \alpha < \omega_1 \) such that \( L_\alpha[Y] \) is an elementary substructure of \( L_{\omega_1}[Y] \) is a club. Hence the set \( \{ \omega_1^X : X \supseteq Y \} \) contains a club. (Recall also that every club is a stationary set.)
2.3 The Main Construction

In this section we will do the main work of this paper by giving the general construction used in the applications. Given an $\mathcal{L}_{\omega_1\omega}$-pseudo-elementary class $S$ of linear orders, we will build a theory $T$ whose models have Scott ranks in correspondence with the linear orders in $S$.

**Theorem 2.3.1.** Let $S$ be a $PC\mathcal{L}_{\omega_1\omega}$-class of linear orders. Then there is an $\mathcal{L}_{\omega_1\omega}$-sentence $T$ such that

$$\operatorname{SS}(T) = \{\operatorname{wfc}(L) : L \in S\}.$$ 

Moreover, suppose that $S$ is the class of reducts of a sentence $S$. Then:

1. We can choose $T$ to be $\Pi^1_2$ (or $\Pi^1_2$ if $S$ is computable).

2. If $L$ is a computable model of $S$ with a computable successor relation, then there is a computable model $M$ of $T$ with $\operatorname{SR}(M) = \operatorname{wfc}(L)$.

With a little more work, we can replace the well-founded collapse with the well-founded part:

**Theorem 2.3.2.** In Theorem 2.3.1, we can also get

$$\operatorname{SS}(T) = \{\operatorname{wfp}(L) : L \in S\}.$$ 

2.3.1 Overview of the Construction

Our structures will have two sorts, the order sort and the main sort. We will also treat elements of $\omega$ as if they are in the structure (e.g., we will talk about functions with codomain $\omega$). We can identify $S$ with an $\mathcal{L}_{\omega_1\omega}$ sentence $S$ in the language with a symbol $\leq$ for the ordering and possibly further symbols; $S$ is the class of reducts of models of $S$ to the language with just the symbol $\leq$. Let $S^+$ be $S$ together with:

(O1) There are constants $(e_i)_{i \leq \omega}$ such that each element is equal to exactly one constant.

(O2) There is a partial successor function $\alpha \mapsto \alpha + 1$, and each non-maximal element has a successor.

(O3) There is a sequence $(R_n)_{n \leq \omega}$ of subsets satisfying:

- (R1) $R_1$ is not strictly bounded (i.e., there is no $\alpha$ which is strictly greater than each element of $R_1$),
- (R2) $R_n \subseteq R_{n+1}$,
- (R3) $\bigcup_n R_n$ is the whole universe of the order sort.
- (R4) If $\alpha \in R_n$, then $\alpha = \sup(\beta + 1 : \beta \in R_{n+1} \text{ and } \beta < \alpha)$,
(R5) For each \( n \) and \( \beta \), there is a least element \( \gamma \) of \( R_n \) with \( \gamma \geq \beta \).

For Theorem 2.3.1 (i.e. to get \( \text{SS}(T) = \{ \text{wfc}(L) \mid L \vDash S \} \)) we will add

\[
\text{(O4a)} \quad R_n = L \text{ for all } n.
\]

(O3) is a consequence of (O4a); moreover, (O4a) will make the \( R_n \) trivial (see (Q7) below). (O4a) will only be used for the final computation of the Scott ranks of the models of \( T \), whereas (O3) will be used in the construction itself. For Theorem 2.3.2, we will use a different axiom (O4b) instead of (O4a); (O3) will also be a consequence of (O4b). The general construction will be the same, but (O4b) will give us a different computation of the Scott rank of the resulting models. Thus (O3) is exactly that common part of (O4a) and (O4b) which is required for the construction, and the particulars of (O4a) and (O4b) are what give the Scott ranks. While reading through the construction for the first time, it might be helpful to assume that (O4a) is in effect. Each order type in \( S \) is represented as a model of \( S^+ \).

The order sort will be a model of \( S^+ \). Our next step will be to define, for each model \( L \) of \( S^+ \), an \( \mathcal{L}_{\omega_1 \omega} \)-sentence \( T(L) \). The sentence \( T \) will say that the order sort is a model \( L \) of \( S^+ \) and the main sort is a model of \( T(L) \). In defining \( T(L) \), we will use quantifiers over \( L \), and \( T(L) \) will be uniform in \( L \).

For now, fix a particular model \( L \) of \( S^+ \). As a model of \( S \), \( L \) will be a linear ordering, which we view as a non-standard ordinal. The Scott rank of \( M = T(L) \) will be determined by \( L \); in particular, if \( L \) is actually an ordinal, then the Scott rank of \( M \) will be its order type. If \( (L, M) \) is a model of \( T \), then since by (O1) each element of \( L \) is named by a constant, the Scott rank of \( L \) will be as low as possible, and so the Scott rank will be carried by \( M \). We will have

\[
\text{SS}(T) = \{ \text{SR}(M) \mid M \vDash T(L) \text{ for some } L \vDash S^+ \}.
\]

If \( L \) has a least element and at least two elements, then for \( M \vDash T(L) \), \( \text{SR}(M) \) will be \( \text{wfc}(L) \) (or \( \text{wfp}(L) \) in the case of Theorem 2.3.2). We can then modify \( T \) slightly using Proposition 2.2.7 to get the theorem; we first modify \( S \) so that every \( L \vDash S \) has a least element and at least two elements, and then we use Proposition 2.2.7 to add 0 or 1, if desired, to the Scott spectrum. Since there are structures of Scott rank 0 and 1 which have Scott sentences which are \( \Pi^1_1 \) and \( \Pi^2_2 \) respectively, Proposition 2.2.7 gives the correct quantifier complexity.

\( T(L) \) will be constructed as follows. First, we will let \( \mathbb{K} \) be the class of finite structures satisfying the properties (P1)-(P6) and (Q1)-(Q7) below. We will show that \( \mathbb{K} \) has a Fraïssé limit. This is an ultrahomogeneous structure, and hence has very low Scott rank. We will add to the Fraïssé limit unary relations \( A_i \) indexed by \( i \in \omega \). \( T(L) \) will be a sentence of \( \mathcal{L}_{\omega_1 \omega} \) defining the Fraïssé limit of \( \mathbb{K} \) together with relations \( A_i \) satisfying properties (A1) and (A2).

To see (1) of Theorem 2.3.1, we can take the Morleyization of \( S^+ \). This will be a \( \Pi^2_2 \) sentence which defines the same class of linear orders. The construction of \( T(L) \) relative to \( L \) is \( \Pi^2_2 \), so if we define \( T \) in the same way as above but replacing \( S^+ \) by its Morleyization
CHAPTER 2. SCOTT SPECTRA OF THEORIES

$S_M^+$, $T$ will now be $\Pi_2^\Delta$. Since in each model $(L, M)$ of $T$, each element of $L$ is named by a constant, we still have

$$SS(T) = \{SR(M) : M \models T(L) \text{ for some } L \models S_M^+\}.$$ 

If $S$ is actually a computable formula, then its Morleyization is computable, and this $T$ will be computable.

To see (2), we observe that if $L$ is a computable model of $S$ with a computable successor relation, then it has a computable expansion to a model of (O4a) (and hence of (O3)). Then Lemma 2.3.6 below will show that there is a computable model of $T$ with order sort $L$.

2.3.2 The Definition of $T(L)$

Fix $L \models S^+$. We begin by constructing the age of our Fraïssé limit. Let $\mathbb{K}$ be the class of finite structures $M$ satisfying (P1)-(P6) and (Q1)-(Q7) below. Structures in $\mathbb{K}$ should be viewed as trees.

(P1) $\leq$ a partial tree-ordering, that is, the set of predecessors of any element is linearly ordered.

(P2) $\emptyset$ is the unique $\leq$-smallest element.

(P3) Each element other than $\emptyset$ has a unique predecessor, and $P$ is a unary function $M \to M$ picking out that predecessor.

(P4) Each element has finite length, i.e., there is a finite chain of successors starting at $\emptyset$ and ending at that element.

(P5) $\varrho, \varepsilon : M \setminus \{\emptyset\} \to L$ and $\varepsilon : M \setminus \{\emptyset\} \to \omega$ are unary functions.

(P6) If $x < y$, then $\varrho(x) > \varrho(y)$.

The properties (P1)-(P6) that we have introduced so far already define the age of a Fraïssé limit in the restricted language $\{\emptyset, \leq, P, \varrho, \varepsilon\}$. In reading the properties (Q1)-(Q7) below, it will helpful to have this model in mind.

Lemma 2.3.3. The class of finite structures in the language $\{\emptyset, \leq, P, \varrho, \varepsilon\}$ satisfying (P1)-(P6) has the hereditary property (HP), the amalgamation property (AP), and the joint embedding property (JEP). The Fraïssé limit is (isomorphic to) the following structure $M$.

Fix an infinite set $D$. The domain of $M$ is the set of all finite sequences

$$\sigma = ((\alpha_0, c_0, d_0), \ldots, (\alpha_n, c_n, d_n))$$

with $\alpha_i \in L$, $c_i \in \omega$, and $d_i \in D$, such that $\alpha_0 > \alpha_1 > \cdots > \alpha_n$. We interpret the relations in the natural way: $\leq$ is the standard ordering of extensions of sequences, $P$ is the standard predecessor function, $\varepsilon(\sigma) = c_n$, and $\varrho(\sigma) = \alpha_n$. 

Proof. First, it is easy to see that the age of \( \mathcal{M} \) is the set of finitely generated structures satisfying (P1)-(P6). Then we just have to note that \( \mathcal{M} \) is ultrahomogeneous to see that it is the Fraïssé limit of these structures. 

Given an element 

\[
\sigma = ((\alpha_0, c_0, d_0), \ldots, (\alpha_n, c_n, d_n))
\]

of this structure, write \( \bar{\varrho}(\sigma) \) for \( (\alpha_0, \ldots, \alpha_n) \) and \( \bar{\varepsilon}(x) \) for \( (c_0, \ldots, c_n) \). Write \( |\sigma| = n + 1 \) for the length of \( \sigma \).

We will now add an additional function \( E \) whose properties are axiomatized by (Q1)-(Q7). \( E \) is a function from \( \mathcal{M} \times \{ 0 \} \times \mathcal{M} \times \{ 0 \} \) to \( \{ -\infty \} \cup L \times \omega \). \( E \) is defined only on those pairs \( (x, y) \) with \( |x| = |y| \), \( \bar{\varrho}(x) = \bar{\varrho}(y) \), and \( \bar{\varepsilon}(x) = \bar{\varepsilon}(y) \). Note that the domain of \( E \) is an equivalence relation, for which we write \( \equiv \). For convenience, when we talk about \( E(x, y) \) for some \( x \) and \( y \) we will often implicitly assume that \( x \equiv y \). We view the range of \( E \) as a totally ordered set via the lexicographic ordering on \( L \times \omega \), with \( -\infty \) smaller than every element of \( L \times \omega \). Given \( x, y \in \mathcal{M} \) with \( E(x, y) > -\infty \), let \( E_L(x, y) \) be the first coordinate of \( E(x, y) \), i.e., the coordinate in \( L \), and let \( E_\omega(x, y) \) be the second coordinate. If \( E(x, y) = -\infty \), then we let \( E_L(x, y) = -\infty \).

One can view \( E \) as a nested sequence \( (\sim_{\alpha, n})_{\alpha \in L, n \in \omega} \) of relations on \( M \), defined by \( x \sim_{\alpha, n} y \) if \( E(x, y) \geq \min((\varrho(x), 0), (\alpha, n)) \). If \( E(x, y) = -\infty \), then \( x \) and \( y \) are not at all related. It will follow from (Q1), (Q2), and (Q3) that these are equivalence relations. These equivalence relations are nested and continuous (i.e., \( \sim_{\alpha, 0} \supseteq \bigcap_{\beta < \alpha, n \in \omega} \sim_{\beta, n} \)). The most important relations are the relations \( \sim_{\alpha, 0} \) which we will denote by \( \equiv_{\alpha} \). The relations \( \equiv_{\alpha} \) will be non-standard back-and-forth relations (see Lemma 2.3.9). The definition of the back-and-forth relations is not \( \Pi^1_2 \), so we cannot just ask that \( \equiv_{\alpha} \) satisfy the definition of the back-and-forth relations. This is where we use \( \varepsilon \) and the \( \omega \) in \( L \times \omega \); their role is to convert an existential quantifier into a universal quantifier by acting as a sort of Skolem function.

If \( x \in \mathcal{M} \) is not a dead end, the children of \( x \) are divided into infinitely many subsets indexed by \( \omega \) via the function \( \varepsilon \). If \( E_L(x, y) > \alpha \), then for every child \( x' \) of \( x \), there will be a child \( y' \) of \( y \) with \( E_L(x', y') \geq \alpha \); this is in keeping with the idea of making the equivalence relations \( \sim \) agree with the back-and-forth relations. If \( E_L(x, y) = \alpha \), then this will not be true for all \( x' \). However, it will be true for exactly those \( x' \) with \( \varepsilon(x') < E_\omega(x, y) \). Rather than saying that there is a child \( x' \) of \( x \) such that no child \( y' \) of \( y \) has \( E_L(x', y') = \alpha \), we can say that for all children \( x' \) of \( x \) with \( \varepsilon(x') \geq E_\omega \), there is no child \( y' \) of \( y \) with \( E_L(x', y') = \alpha \). This is of lower quantifier complexity. (Note that we cannot say that for all \( x' \) and \( y' \) children of \( x \) and \( y \), \( E(x', y') < \alpha \). This is for the same reason as the following fact: if \( x \) and \( y \) are such that for all \( \bar{x}' \) and \( \bar{y}' \), \( x \not\equiv_{\alpha} y \), then \( x \not\equiv_{\alpha} y \).

For all \( x, y, \) and \( z \) with \( x \equiv y \equiv z \):

\begin{align*}
(\text{Q1}) & \ E(x, x) = (\varrho(x), 0), \\
(\text{Q2}) & \ E(x, y) = E(y, x), \\
(\text{Q3}) & \ E(x, z) \geq \min(E(x, y), E(y, z)),
\end{align*}
If $E$ divides, then $x$ is a successor of $y$ with $x' \preceq y'$, $E_L(x', y') \leq E_L(x, y)$.

If $E(x, y) > -\infty$, then for every $x'$ a successor of $x$ with $\varepsilon(x') \geq E_\omega(x, y)$, there are no successors $y'$ of $y$ with $E_L(x', y') = E_L(x, y)$.

If $|x| = |y| = n$, then $E_L(x, y) \in R_n \cup \{-\infty\} \cup \{g(x)\}$.

While $\{\}$ was not in the domain of $E$, we will consider $E(\{\}, \{\})$ to be $(L, 0)$, i.e., to be greater than each element of $L$.

(Q1), (Q2), and (Q3) are just saying that the relations $\sim_{\alpha, n}$ defined above are reflexive, symmetric, and transitive respectively (and hence equivalence relations). (Q6) is the axiom which is doing most of the work.

The intuition behind (Q7) will be explained in Subsection 2.3.5. For now, the reader can simply imagine that $R_n = L$ for each $n$ (as it will be for Theorem 2.3.1), so that (Q7) is a vacuous condition.

**Lemma 2.3.4.** The class $\mathbb{K}$ of finite structures satisfying (P1)-(P6) and (Q1)-(Q7) relative to the fixed structure $L$ has the AP, JEP, and HP.

**Proof.** It is easy to see that $\mathbb{K}$ has the hereditary property. Note that every finite structure in $\mathbb{K}$ contains, via an embedding, the structure with one element $\{\}$. So the joint embedding property will follow from the amalgamation property.

For the amalgamation property, let $\mathcal{A}$ be a structure in $\mathbb{K}$ which embeds into $\mathcal{B}$ and $\mathcal{C}$. Identify $\mathcal{A}$ with its images in $\mathcal{B}$ and $\mathcal{C}$, and assume that the only elements common to both $\mathcal{B}$ and $\mathcal{C}$ are the elements of $\mathcal{A}$. By amalgamating $\mathcal{C}$ one element at a time, we may assume that $\mathcal{C}$ contains only a single element $c$ not in $\mathcal{A}$. The element $c$ is the child of some element of $\mathcal{A}$, and has no children in $\mathcal{C}$. We will define a structure $\mathcal{D}$ whose domain is $\mathcal{B} \cup \{c\}$ and then show that $\mathcal{D}$ is in $\mathbb{K}$.

First, we can take the amalgamation of the structures in the language $\{\}, \varepsilon, P, g, \varepsilon\}$ as in Lemma 2.3.3; we just add $c$ to $\mathcal{B}$, setting $P(c), \varepsilon(c)$, and $g(c)$ to be the same as in $\mathcal{C}$. Set $E(c, c) = (g(c), 0)$. We need to define $E(b, c)$ when $b$ is an element from $\mathcal{B}$ with $b \not\preceq c$. Define $E(b, c) = E(c, b)$ to be the maximum of $\min(E(b, a), E(a, c))$ over all $a \in \mathcal{A}$ with $a \not\preceq c$. If there are no such $a \in \mathcal{A}$, set $E(b, c) = -\infty$. By (Q3) this is well-defined, that is, for $a \in \mathcal{A}$,

$E(a, c)$ is the maximum of $\min(E(a, a'), E(a', c))$ over all $a' \in \mathcal{A}$.

We now have to check (Q1)-(Q7). (Q1) and (Q2) are obvious from the definition of the extension of $E$.

For (Q3), we have two new cases to check. For the first case, fix $b, b' \in \mathcal{B}$ with $b \not\preceq b' \not\preceq c$; we will show that $E(b, c) \geq \min(E(b, b'), E(b', c))$. If there is no $a \in \mathcal{A}$ with $a \not\preceq c$, then $E(b, c) = E(b', c) = -\infty$, and so we have $E(b, c) \geq \min(E(b, b'), E(b', c))$. Otherwise, let $a \in \mathcal{A}$
be such that $E(b', c) = \min(E(b', a), E(a, c))$. By definition,

$$
E(b, c) \geq \min(E(b, a), E(a, c))
\geq \min(E(b, b'), E(b', a), E(a, c))
= \min(E(b, b'), E(b', c)).
$$

Now for the second case, again fix $b, b' \in B$ with $b \not\preceq b' \not\preceq c$; now we will show that $E(b, b') \geq \min(E(b, c), E(c, b'))$. If there is no $a \in A$ with $a \not\preceq c$, then $E(b, c) = E(b', c) = -\infty$, and so we have

$$
E(b, b') \geq -\infty = \min(E(b, c), E(c, b')).
$$

Otherwise, let $a \in A$ be such that $E(b, c) = \min(E(b, a), E(a, c))$, and let $a' \in A$ be such that $E(b', c) = \min(E(b', a'), E(a', c))$. Then

$$
E(b, b') \geq \min(E(b, a), E(a, a'), E(a', b'))
\geq \min(E(b, a), E(a, c), E(c, a'), E(a', b'))
= \min(E(b, c), E(c, b')).
$$

For (Q4), suppose that $b \in B$ has $b \not\preceq c$. Then either $E(b, c) = -\infty$, in which case there is nothing to check, or $E(b, c) = \min(E(b, a), E(a, c))$ for some $a \in A$. In the second case, either $E(b, c) = E(b, a) \leq (g(a), 0) = (g(c), 0)$ or $E(b, c) = E(a, c) \leq (g(c), 0)$.

Now we check (Q5). Let $\hat{c} \in A$ be the parent of $c$. Fix $b \in B$ and let $\hat{b}$ be the parent of $b$. We must show that $E_L(b, c) \preceq E_L(\hat{b}, \hat{c})$. Choose $a \in A$ such that $E(b, c) = \min(E(b, a), E(a, c))$. (If there is no such $a$, then we can immediately see that (Q5) holds.) Let $\hat{a}$ be the parent of $a$. Then we have $E_L(\hat{b}, \hat{a}) \geq E_L(b, a)$ and $E_L(\hat{a}, \hat{c}) \geq E_L(a, c)$ so that

$$
E_L(\hat{b}, \hat{c}) \geq \min(E_L(\hat{b}, \hat{a}), E_L(\hat{a}, \hat{c}))
\geq \min(E_L(b, a), E_L(a, c))
= E_L(b, c).
$$

Next we check (Q6). Since $c$ has no children in $C$, the only new case to check is as follows. Let $\hat{c} \in A$ be the parent of $c$, and let $\hat{b} \in B$ be such that $\hat{c} \preceq \hat{b}$. Suppose that $n = \varepsilon(c) \geq E_\omega(\hat{c}, \hat{b})$ and let $\alpha = E_L(\hat{c}, \hat{b}) > -\infty$. Suppose to the contrary that there is $b$ a child of $\hat{b}$ with $E_L(b, c) = \alpha$. Then, by definition there is $a \in A$ such that $E(b, c) = \min(E(b, a), E(a, c))$. Let $\hat{a}$ be the parent of $a$. Since $E_L(b, a) \geq \alpha$ and $E_L(a, c) \geq \alpha$, $E(b, \hat{a}) > (\alpha, n)$ and $E(\hat{a}, \hat{c}) > (\alpha, n)$. Hence $E(b, \hat{c}) > (\alpha, n)$. This is a contradiction.

Finally, for (Q7), if $E(b, c) = -\infty$ we are done. So we may suppose that $E(b, c) = \min(E(b, a), E(a, c))$ for some $a \in A$. Then since $E_L(b, c)$ and $E_L(a, c)$ are both in $R_n \cup \{-\infty\} \cup \{g(c)\}$, the same is true of $E_L(b, c)$.

\begin{lemma}

The reduct of the Fraïssé limit of $\mathbb{K}$ to the language $\{(), \preceq, P, g, \varepsilon\}$ is the structure from Lemma 2.3.3.

\end{lemma}
Lemma 2.3.6. If \( \mathcal{M} \) is a structure in \( \mathbb{K} \), and \( \mathcal{N} \) is a structure in the language \( \mathcal{L}_x = \{ 0, \leq, P, \varphi, \varepsilon \} \) satisfying (P1)-(P6) and with \( \mathcal{M} \subseteq \mathcal{L}_x \mathcal{N} \), then we can expand \( \mathcal{N} \) to a structure \( \mathcal{N}' \) in the language \( \mathcal{L} = \mathcal{L}_x \cup \{ E \} \) with \( \mathcal{M} \subseteq \mathcal{L} \mathcal{N}' \). We can do this simply by setting \( E(x,y) = (\varphi(x),0) \) for \( x \in \mathcal{N}' \setminus \mathcal{M} \), and \( E(x,y) = E(y,x) = -\infty \) for all \( x \in \mathcal{N}' \setminus \mathcal{M} \) and \( y \in \mathcal{N} \). (Q1)-(Q7) are easy to check.

For a fixed \( L \), let \( T(L) \) be the \( \mathcal{L}_{\omega_1 \omega} \)-sentence describing the Fraïssé limit of \( \mathbb{K} \), and to which we add unary relations \( (A_i)_{i \in \omega} \) satisfying (A1) and (A2) below. The relations \( A_i \) will name the equivalence classes \( \sim_0 \), so that while the Fraïssé limit is ultra-homogeneous, the models of \( T(L) \) will not be. The Fraïssé limit is axiomatizable by a \( \Pi^2_1 \) formula. If \( L \) is computable with a computable successor relation, then \( \mathbb{K} \) is a computable age, and hence the Fraïssé limit is axiomatizable by a \( \Pi^2_1 \) formula. Since (P1)-(P6) and (Q1)-(Q7) are all \( \Pi^2_1 \) formulas, \( T(L) \) is \( \Pi^2_1 \) axiomatizable.

(A1) For each \( x \), \( A_i(x) \) for exactly one \( i \).

(A2) For all \( x \) and \( x' \), \( E(x,x') > -\infty \) if and only if for all \( i \), \( A_i(x) \iff A_i(x') \).

Lemma 2.3.6. If \( L \) is computable with a computable successor relation, then \( T(L) \) has a computable model.

Proof. By Theorem 3.9 of [CHMM11], there is a computable Fraïssé limit of \( \mathbb{K} \). Then we can add the relations \( A_i \) in a computable way. \( \square \)

2.3.3 Computation of the Scott Rank

Fix \( L \) a model of \( S^+ \). Let \( \mathcal{M} \) be a countable model of \( T(L) \). The remainder of the proof is a computation of \( SR(\mathcal{M}) \). As remarked earlier, for this section we will assume that \( L \) has a least element 0 and has at least two elements. We will show that \( SR(\mathcal{M}) = \text{wfp}(L) \) for such an \( \mathcal{M} \). Let \( \text{wfp}(L) \) be the well-ordered part of \( L \). Recall that we identify elements of \( \text{wfp}(L) \) with ordinals in the natural way.

Lemma 2.3.7. Fix \( \beta \in \text{wfp}(O) \). Suppose that \( m \in \omega \) and \( u_1, \ldots, u_t, u_1', \ldots, u_t' \), and \( v \) are tuples from \( \mathcal{M} \) such that \( \varepsilon(x) < m \) where \( x \) ranges among all of these elements and their predecessors, and such that:

(i) \( u_1, \ldots, u_t \equiv_a u_1', \ldots, u_t' \),

(ii) for each \( i \), \( E(u_i,u_i') \geq \min((\varphi(u_i),0),(\beta,m)) \).

Suppose moreover that \( u_1, \ldots, u_t \) and \( u_1', \ldots, u_t' \) are closed under the predecessor relation \( P \). Then there is \( v' \) such that \( u_1, \ldots, u_t, v \) and \( u_1', \ldots, u_t', v' \) satisfy (i) and (ii).
CHAPTER 2. SCOTT SPECTRA OF THEORIES

Proof. We may assume that $v$ is not one of $u_1, \ldots, u_t$, as if $v = u_i$ then we could take $v' = u'_i$. Thus for no $u_i$ is $u_i \geq v$. By repeated applications of the claim, we may also assume that $P(v)$ is among the $u_i$.

Let $u$ be the predecessor of $v$, and let $y_1, \ldots, y_k$ be those $u_i$ with $v \not< u_i$. Let $x_1, \ldots, x_k$ be the predecessors of the $y_i$. Let $u', y'_1, \ldots, y'_k$, and $x'_1, \ldots, x'_k$ be the corresponding $u'_i$. We will define a finite structure with domain consisting of $u_1, \ldots, u_t, u'_1, \ldots, u'_t, v$, and a new element $v'$. We will show that this structure is in $\mathbb{K}$, and hence we may take $v'$ to be in $\mathcal{M}$.

Begin by defining $|v'| = |v|$, $\varepsilon(v') = \varepsilon(v)$, $\rho(v') = \rho(v)$, and $u'_i < v'$ if and only if $u_i < v$. Define $E$ as follows:

1. $E(v', v') = (\rho(v'), 0)$,
2. $E(v', y'_i) = E(v, y_i)$.
3. $E(v', y_i)$ is the maximum of $\min(E(v', y'_i), E(y'_j, y_i))$ over all $j$,
4. If $\beta \in R_{|v|}$, then $E(v', v)$ is the maximum of:
   a. $\min(E(v', y'_i), E(y'_j, v))$ over all $i$ and
   b. $\min((\rho(v), 0), (\beta, m))$.

Otherwise, if $\beta \notin R_{|v|}$, then let $\gamma$ be the least element of $R_{|v|}$ with $\gamma > \beta$. ($(R5)$ guarantees that such a $\gamma$ exists.) Then $E(v', v)$ is the maximum of:

a. $\min(E(v', y'_i), E(y'_j, v))$ over all $i$ and
b. $\min((\rho(v), 0), (\gamma, 0))$.

Let $E(\cdot, v') = E(v', \cdot)$ in each of the cases above. Note that by (2), (3) is equivalent to defining $E(v', y_i)$ to be the maximum of $\min(E(v, y_j), E(y'_j, y_i))$ over all $j$, and similarly with (4); so these are all definitions in terms of quantities we are given.

We need to check that this is defines a finite structure in $\mathbb{K}$. It is easy to see that (P1)-(P6) hold. (Q1) and (Q2) are trivial, as we set $E(v', v') = (\rho(v'), 0)$ and $E(\cdot, v') = E(v', \cdot)$ above. (Q4) is easy to see from the definition of $E(v', \cdot)$. We now check (Q3), (Q5), (Q6), and (Q7).

For (Q3), we must show that if $a \not< b \not< c$, then we have $E(a, b) \geq \min(E(a, c), E(c, b))$. We have a number of different cases depending on which values $a$, $b$, and $c$ take. If none of $a$, $b$, or $c$ are $v'$, then it is trivial; also, if there is any duplication, then it is trivial. Unfortunately there are a large number of possible combinations remaining. The reader might find it helpful to draw a picture for each case, using the intuition of (Q3) as corresponding to the transitivity of an equivalence relation. We will frequently use the fact that (Q3) holds in $\mathcal{M}$.
\(a = v', \ b = v, \ c = y_i\): Let \(j\) be such that \(E(v', y_i) = \min(E(v', y_j'), E(y_j', y_i))\). Then
\[
E(v', v) \geq \min(E(v', y_j'), E(y_j', v)) \\
\geq \min(E(v', y_j'), E(y_j', y_i), E(y_i, v)) \\
= \min(E(v', y_i), E(y_i, v)).
\]

\(a = v', \ b = v, \ c = y_i': E(v', v) \geq \min(E(v', y_i'), E(y_i', v))\) by definition.

\(a = v', \ b = y_i, \ c = v\): We have three cases corresponding to (a), (b), and (c) in the definition of \(E(v', v)\).

(a) Suppose that \(E(v', v) = \min(E(v', y_j'), E(y_j', v))\) for some \(j\). Then
\[
E(v', y_i) \geq \min(E(v', y_j'), E(y_j', y_i)) \\
\geq \min(E(v', y_j'), E(y_j', v), E(v, y_i)) \\
= \min(E(v', v), E(v, y_i)).
\]

(b) Suppose that \(E(v', v) = \min((\varrho(v), 0), (\beta, m))\). We know that
\[
E(y_i', y_i) \geq \min((\varrho(v), 0), (\beta, m)),
\]
and so \(E(y_i', y_i) \geq E(v', v)\). Then
\[
E(v', y_i) \geq \min(E(v', y_i'), E(y_i', y_i)) \geq \min(E(v', v), E(v, y_i)).
\]

(c) Suppose that \(E(v', v) = \min((\varrho(v), 0), (\gamma, 0))\). We know that
\[
E(y_i', y_i) \geq \min((\varrho(v), 0), (\beta, m)),
\]
and since \(E(v', y_i) \in R_{[\beta]} \cup \{ -\infty \} \cup \{ \varrho(v) \}, \ E(y_i', y_i) \geq \min((\varrho(v), 0), (\gamma, 0))\). So \(E(y_i', y_i) \geq E(v', v)\). Then
\[
E(v', y_i) \geq \min(E(v', y_i'), E(y_i', y_i)) \geq \min(E(v', v), E(v, y_i)).
\]

\(a = v', \ b = y_i, \ c = y_j\): Let \(k\) be such that \(E(v', y_j) = \min(E(v', y_k'), E(y_k', y_j))\). By definition, we have
\[
E(v', y_i) \geq \min(E(v', y_k'), E(y_k', y_i)) \\
\geq \min(E(v', y_k'), E(y_k', y_j), E(y_j, y_i)) \\
= \min(E(v', y_j), E(y_j, y_i)).
\]

\(a = v', \ b = y_i, \ c = y_i'\): By definition, \(E(v', y_i) \geq \min(E(v', y_i'), E(y_i', y_i))\).
\( a = v', b = y'_i, c = v \): We have three cases corresponding to (a), (b), and (c) in the definition of \( E(v', v) \).

(a) Suppose that \( E(v', v) = \min(E(v', y'_i), E(y'_j, v)) \) for some \( j \). We have

\[
E(v', y'_i) = E(v, y_i) \\
\geq \min(E(v, y_j), E(y_j, y_i)) \\
= \min(E(v', y'_j), E(y'_j, y_i)) \\
\geq \min(E(v', y'_j), E(y'_j, v), E(v, y'_i)) \\
= \min(E(v', v), E(v, y'_i)).
\]

(b) Suppose that \( E(v', v) = \min((\rho(v), 0), (\beta, m)) \). Then

\[
E(v', y'_i) = E(v, y_i) \\
\geq \min(E(v, y'_i), E(y'_i, y_i)) \\
\geq \min(E(v, y'_i), (\rho(y_i), 0), (\beta, m)) \\
= \min(E(v, y'_i), E(v', v)).
\]

(c) Suppose that \( E(v', v) = \min((\rho(v), 0), (\gamma, 0)) \). Then, as before, we have that \( E(y'_i, y_i) \geq \min((\rho(y_i), 0), (\gamma, 0)) \). So

\[
E(v', y'_i) = E(v, y_i) \\
\geq \min(E(v, y'_i), E(y'_i, y_i)) \\
\geq \min(E(v, y'_i), (\rho(y_i), 0), (\gamma, 0)) \\
= \min(E(v, y'_i), E(v', v)).
\]

\( a = v', b = y'_i, c = y_j \): Let \( k \) be such that \( E(v', y_j) = \min(E(v', y'_k), E(y'_k, y_j)) \). We have

\[
E(v', y'_i) = E(v, y_i) \\
\geq \min(E(v, y_k), E(y_k, y_i)) \\
= \min(E(v', y'_k), E(y'_k, y_i)) \\
\geq \min(E(v', y'_k), E(y'_k, y_j), E(y_j, y'_i)) \\
= \min(E(v', y_j), E(y_j, y'_i)).
\]

\( a = v', b = y'_j, c = y'_j \): We have

\[
E(v', y'_i) = E(v, y_i) \\
\geq \min(E(v, y_j), E(y_j, y_i)) \\
\geq \min(E(v', y'_j), E(y'_j, y'_i)).
\]
\[ a = y_i, \ b = y_j, \ c = v': \] Let \( k \) and \( \ell \) be such that \( E(v', y_i) = \min(E(v', y_k'), E(y_k', y_i)) \) and also \( E(v', y_j) = \min(E(v', y_k'), E(y_k', y_j)) \). We have

\[
E(y_i, y_j) \geq \min(E(y_i, y_k'), E(y_k', y_j)) \\
= \min(E(y_i, y_k'), E(y_k', y_j)) \\
\geq \min(E(y_i, y_k'), E(y_k', v), E(v, y_j)) \\
= \min(E(y_i, y_k'), E(y_k', v'), E(v', y_j)) \\
= \min(E(y_i, v'), E(v', y_j)).
\]

\[ a = y_i, \ b = y_j, \ c = v': \] Let \( k \) be such that \( E(v', y_i) = \min(E(v', y_k'), E(y_k', y_i)) \). Then

\[
E(y_i, y_j') \geq \min(E(y_i, y_k'), E(y_k', y_j')) \\
= \min(E(y_i, y_k'), E(y_k', y_j')) \\
\geq \min(E(y_i, y_k'), E(y_k', v), E(v, y_j')) \\
= \min(E(y_i, y_k'), E(y_k', v'), E(v', y_j')) \\
= \min(E(y_i, v'), E(v', y_j')).
\]

\[ a = y_i', \ b = y_j', \ c = v': \] We have

\[
E(y_i', y_j') = E(y_i, y_j) \\
= \min(E(y_i, v), E(v, y_j)) \\
= \min(E(y_i', v'), E(v', y_j')).
\]

\[ a = v, \ b = y_i, \ c = v': \] Let \( j \) be such that \( E(v', y_i) = \min(E(v', y_j'), E(y_j', y_i)) \). We have three cases corresponding to (a), (b), and (c) in the definition of \( E(v', v) \).

(a) Suppose that \( E(v', v) = \min(E(v', y_j'), E(y_j', v)) \) for some \( j \) and that \( E(v', y_i) = \min(E(v', y_k'), E(y_k', y_i)) \) for some \( k \). Then

\[
E(v, y_i) \geq \min(E(v, y_k'), E(y_k', y_j), E(y_j', y_i)) \\
= \min(E(v, y_k'), E(y_k', v), E(v, y_j), E(y_j', y_i)) \\
\geq \min(E(v, y_k'), E(y_k', v'), E(v', y_j'), E(y_j', y_i)) \\
= \min(E(v, v'), E(v', y_i)).
\]

(b) Suppose that \( E(v', v) = \min((\rho(v), 0), (\beta, m)) \). We have

\[
E(v, y_i) \geq \min(E(v, y_i), E(y_j, y_i)) \\
= \min(E(v', y_j'), E(y_j', y_i)) \\
\geq \min(E(v', y_j), E(y_j', y_i), E(y_i, y_i)) \\
= \min(E(v', y_i), E(y_i, y_i)) \\
\geq \min(E(v', y_i), (\rho(y_i), 0), (\beta, m)) \\
= \min(E(v', y_i), E(v', v)).
\]
(c) Suppose that $E(v', v) = \min((\varrho(v), 0), (\gamma, 0))$. As before, we have that $E(y_i, y'_i) \geq \min((\varrho(v), 0), (\gamma, 0))$. We have

$$E(v, y_i) \geq \min(E(v, y_j), E(y_j, y_i))$$

$$= \min(E(v', y'_j), E(y'_j, y'_i))$$

$$\geq \min(E(v', y'_j), E(y'_j, y_i), E(y_i, y'_i))$$

$$= \min(E(v', y_i), E(y_i, y'_i))$$

$$\geq \min(E(v', y_i), (\varrho(y_i), 0), (\gamma, 0))$$

$$= \min(E(v', y_i), E(v', v)).$$

$a = v, b = y'_i, c = v'$: We have three cases corresponding to (a), (b), and (c) in the definition of $E(v', v)$.

(a) Suppose that $E(v', v) = \min(E(v, y'_j), E(v, y_j))$ for some $j$. Then

$$E(v, y'_j) \geq \min(E(v, y'_j), E(y'_j, y'_i))$$

$$= \min(E(v, y'_j), E(y_j, y_i))$$

$$\geq \min(E(v, y'_j), E(y_j, v), E(v, y_i))$$

$$\geq \min(E(v, y'_j), E(y'_j, v'), E(v', y'_i))$$

$$= \min(E(v, v'), E(v', y'_i)).$$

(b) Suppose that $E(v', v) = \min((\varrho(v), 0), (\beta, m))$. Then

$$E(v, y'_i) \geq \min(E(v, y_i), E(y_i, y'_i))$$

$$\geq \min(E(v', y'_i), (\varrho(v), 0), (\beta, m))$$

$$= \min(E(v', y'_i), E(v', v)).$$

(c) Suppose that $E(v', v) = \min((\varrho(v), 0), (\gamma, 0))$. As before, we have that $E(y_i, y'_i) \geq \min((\varrho(v), 0), (\gamma, 0))$. Then

$$E(v, y'_i) \geq \min(E(v, y_i), E(y_i, y'_i))$$

$$\geq \min(E(v', y'_i), (\varrho(v), 0), (\gamma, 0))$$

$$= \min(E(v', y'_i), E(v', v)).$$

That completes the last case in the verification of (Q3).

For (Q5), we have three cases to check.

(1) We will show that $E_L(v', v) \leq E_L(u', u)$. We have three subcases.
(a) \(E(v', v) = \min(E(v', y_i), E(y'_i, v))\) for some \(i\). Then
\[
E_L(u', u) \geq \min(E_L(u', x'_i), E_L(x'_i, u)) \\
= \min(E_L(u, x_i), E_L(x'_i, u)) \\
\geq \min(E_L(v, y_i), E_L(y'_i, v)) \\
= \min(E_L(v', y'_i), E_L(y'_i, v)) \\
= E_L(v', v).
\]

(b) \(E_L(v', v) = \min(\rho(v), \beta)\). Then \(E_L(u, u') \geq \min(\rho(u), \beta) \geq E_L(v', v)\).

(c) \(E_L(v', v) = \min(\rho(v), \gamma)\). So \(\beta \not\in R_{[v]}\). Then, since \(R_{[v]} \subseteq R_{[u]}\), \(\beta \not\in R_{[u]}\) and the least element of \(R_{[u]}\) which is greater than \(\beta\) is at least \(\gamma\). Since \(E_L(u, u') \geq \min(\rho(u), \beta), \ E_L(u, u') \geq \min(\rho(u), \gamma) \geq E_L(v', v)\).

(2) We will show that \(E_L(u', x_i) \geq E_L(v', v)\). Then we can pick \(j\) be such that \(E(v', y_i) = \min(E(v', y'_j), E(y'_j, y_i))\). Then
\[
E_L(u', x_i) \geq \min(E_L(u', x'_j), E_L(x'_j, x_i)) \\\n= \min(E_L(u, x_j), E_L(x'_j, x_i)) \\\n\geq \min(E_L(v, y_j), E_L(y'_j, y_i)) \\\n= \min(E_L(v', y'_j), E_L(y'_j, y_i)) \\\n= E_L(v', v).
\]

(3) \(E_L(v', y'_i) = E_L(v, y_i) \leq E_L(u, x_i) = E_L(u', x'_i)\).

For (Q6), we again have three cases to check.

(1) Suppose that \(\alpha = E_L(v', y_i) = E_L(u', x_i)\) and \(n = \varepsilon(v') \geq E_\omega(u', x_i)\). Let \(j\) be such that
\[
E(v', y_i) = \min(E(v', y'_j), E(y'_j, y_i)) = \min(E(v, y_j), E(y'_j, y_i)).
\]

Thus \(E(v, y_j) \geq \alpha\) and \(E(y'_j, y_i) \geq \alpha\). Either \(E_\omega(x'_j, x_i) > n\) or
\[
E_L(x'_j, x_i) > E_L(y'_j, y_i) \geq \alpha.
\]

We always have \(E_L(x'_j, x_i) \geq \alpha\), and hence \(E_L(x'_j, x_i) > (\alpha, n)\). Similarly, either \(E_\omega(u, x_j) > n\) or
\[
E_L(u, x_j) > E_L(v, y_j) \geq \alpha
\]
and so \(E_L(u, x_j) > (\alpha, n)\). Thus
\[
E(u', x_i) \geq \min(E(u', x'_j), E(x'_j, x_i)) = \min(E(u, x_j), E(x'_j, x_i)) > (\alpha, n).
\]

This is a contradiction.
(2) Suppose that $E_L(v', y_i') = E_L(u', x_i')$ and $\varepsilon(v') \geq E_\omega(u', x_i')$. We have $E(v', y_i') = E(v, y_i)$, $\varepsilon(v') = \varepsilon(v)$, and $E(u', x_i') = E(u, x_i)$. This contradicts $(Q6)$.

(3) Suppose that $E_L(v', v) = E_L(u', u)$ and $\varepsilon(v') \geq E_\omega(u, u')$. We have three subcases, depending on how $E(v', v)$ gets its value.

(a) Suppose that for some $i$,

$$E(v', v) = \min(E(v', y_i'), E(y_i', v)) = \min(E(v, y_i), E(y_i', v)).$$

We have

$$E(u', u) \geq \min(E(u', x_i'), E(x_i', u)) = \min(E(u, x_i), E(x_i', u)).$$

Either $E_L(v, y_i) < E_L(u, x_i)$ or $\varepsilon(v) < E_\omega(u, x_i)$, and either $E_L(v, y_i') < E_L(u, x_i')$ or $\varepsilon(v) < E_\omega(u, x_i')$.

Suppose that $E(u', u) = E(u, x_i) \leq E(x_i', u)$. The other case is similar. Then, since $E_L(u', u) = E_L(v', v)$ we must have $E_L(v, y_i) = E_L(u, x_i)$; otherwise, we would have

$$E_L(v', v) \leq E_L(v, y_i) < E_L(u, x_i) = E_L(u', u).$$

Hence $\varepsilon(v) < E_\omega(u, x_i)$. Thus $E_\omega(u', u) > \varepsilon(v)$, a contradiction.

(b) Suppose that $E_L(v', v) = \min((\varrho(v), 0), (\beta, m))$. Then since $\varrho(v) < \varrho(u)$ and $E(u, u') \geq \min((\varrho(u), 0), (\beta, m))$, if $E_L(v', v) = E_L(u, u')$ then they are both equal to $\beta$, and $E(u, u') \geq (\beta, m)$. But $\varepsilon(v') < m$, which contradicts $E_L(v', v) \geq E_\omega(u, u') \geq m$.

(c) Suppose that $E_L(v', v) = \min((\varrho(v), 0), (\gamma, 0))$. Then, by choice of $\gamma$ and using the fact that $R_{[\varrho]} \subseteq R_{[\varrho]}$ and since $E(u, u') \geq \min((\varrho(u), 0), (\beta, m))$, $E(u, u') \geq \min((\varrho(u), 0), (\gamma, 0))$. If $E_L(v', v) = E_L(u, u')$ then they are both equal to $\gamma$. But then $\gamma \in R_{[\varrho]}$, and so by $(R3)$, in $R_{[\varrho]}$ there is some $\gamma'$ with $\beta \leq \gamma' < \gamma$. This contradicts the choice of $\gamma$.

For $(Q7)$, we once more have three cases to check.

(1) We will show that $E_L(v', v) \in R_{[\varrho]} \cup \{-\infty, \varrho(v)\}$. As usual, we have three subcases.

(a) $E_L(v', v) = \min(E_L(v', y_i'), E_L(y_i', v))$ for some $i$. Then each of $E_L(v', y_i') = E_L(v, y_i)$ and $E_L(y_i', v) = \min(E(v, y_i), E(y_i', v))$ is in $R_{[\varrho]} \cup \{-\infty, \varrho(v)\}$, the same is true of $E_L(v', v)$.

(b) $E_L(v', v) = \min((\varrho(v), 0), (\beta, m))$ and $\beta \in R_{[\varrho]}$. This is immediate.

(c) $E_L(v', v) = \min((\varrho(v), 0), (\gamma, 0))$ and $\gamma \in R_{[\varrho]}$. This is also immediate.

(2) Let $j$ be such that $E(v', y_j) = \min(E(v', y_j'), E(y_j', y_i))$. Then each of $E_L(v', y_j') = E_L(v, y_j)$ and $E_L(y_j', y_i)$ is in $R_{[\varrho]} \cup \{-\infty, \varrho(v)\}$, so the same is true of $E_L(v', y_i)$. 

(3) $E_L(v', y'_i) = E_L(v, y_i)$ which is in $R_{[v]} \cup \{-\infty, \rho(v)\}$.

We have now finished showing that the finite structure we defined above is in the class $\mathbb{K}$. So we can assume that $v'$ is in $\mathcal{M}$. Note that $E(v', v) > -\infty$, so by (A2), $A_i(v') \iff A_i(v)$. Thus

\[ u_1, \ldots, u_t, v \equiv_{at} u'_1, \ldots, u'_t, v'. \]

By definition, we have $E(v', v) \geq \min((\rho(v), 0), (\beta, m))$. This completes the proof of the lemma.

Recall that we defined an equivalence relation $\sim_{\alpha}$ by $x \sim_{\alpha} y$ if $E_L(x, y) \geq \min(\rho(x), \alpha)$. We can expand this to an equivalence relation on tuples as follows.

**Definition 2.3.8.** Given $\alpha \in L$ and $x_1, \ldots, x_r$ and $x'_1, \ldots, x'_r$ from $\mathcal{M}$ both closed under the predecessor relation $P$, define:

\[ x_1, \ldots, x_r \sim_{\alpha} x'_1, \ldots, x'_r \]

if and only if

\[ x_1, \ldots, x_r \equiv_{at} x'_1, \ldots, x'_r \text{ and for all } i, x_i \sim_{\alpha} x'_i. \]

If $x_1, \ldots, x_r$ and $x'_1, \ldots, x'_r$ are not closed under predecessors, we can close them under predecessors in the natural way to extend $\sim_{\alpha}$ to a relation on all pairs of tuples.

Note that $\bar{x} \sim_0 \bar{y}$ asks that $\bar{x}$ and $\bar{y}$ satisfy the same atomic formulas, whereas $\bar{x} \preceq_0 \bar{y}$ asks that they satisfy the same atomic formulas with bounded Gödel numbers. However, if we replace $\sim_0$ by $\preceq_0$, these relation $\sim_{\alpha}$ are non-standard back-and-forth relations. Note that the relations $\sim_{\alpha}$ are symmetric, whereas back-and-forth relations are, a priori, not necessarily symmetric.

**Lemma 2.3.9.** $\preceq_0$ and $(\sim_{\alpha})_{0 \alpha \in L}$ are non-standard back-and-forth relations in the sense of Definition 2.2.8.

**Proof.** Suppose that $\alpha > 0$ and

\[ x_1, \ldots, x_r \sim_{\alpha} x'_1, \ldots, x'_r. \]

Suppose that we are given $y_1, \ldots, y_s$ and $\beta < \alpha$. We will find $y'_1, \ldots, y'_s$ such that

\[ x_1, \ldots, x_r, y_1, \ldots, y_s \sim_{\beta} x'_1, \ldots, x'_r, y'_1, \ldots, y'_s. \]

We already know that

\[ x_1, \ldots, x_r \equiv_{at} x'_1, \ldots, x'_r. \]

We may assume that the $y_i$ are closed under predecessors, and that the predecessor of each $y_i$ appears earlier in the list (or in $x_i$). Let $m \in \omega$ be large enough that for any element $z$ which we have mentioned so far (the $x_i, x'_i$ and $y_i$) we have $\bar{z}(z) \in \{0, \ldots, m\}^\omega$. Note that,
by choice of $m, x_1, \ldots, x_r$ and $x_1', \ldots, x_r'$ satisfy (i) and (ii) of Lemma 2.3.7 (for this $\beta$ and $m$). So using the lemma we get a $y_i'$ such that $x_1, \ldots, x_r, y_i$ and $x_1', \ldots, x_r', y_i'$ also satisfy (i) and (ii). But then we can use the lemma to get a $y_i'$, and so on, until we have $y_1', \ldots, y_s'$ as desired.

On the other hand, suppose that $\alpha > 0$ and

$$x_1, \ldots, x_r \not\equiv_{\alpha} x_1', \ldots, x_r'.$$

We need to show that there are $y_1, \ldots, y_s$ and $\beta < \alpha$ such that for all $y_1', \ldots, y_s'$,

$$x_1, \ldots, x_r, y_1, \ldots, y_s \not\equiv_{\beta} x_1', \ldots, x_r', y_1', \ldots, y_s'.$$

There are three cases.

**Case 1.** $x_1, \ldots, x_r \not\equiv_{\alpha} x_1', \ldots, x_r'$.

*Proof.* There are only finitely many constant symbols from $L$ required to determine the values of all of the functions in the language on $x_1, \ldots, x_r$, and by (A1), finitely many indices $j$ for relations $A_j$ are required to determine which of the $A_j$ hold of $x_1, \ldots, x_r$. In particular, a finite set of formulas from the language determines the entire atomic diagram of $x_1, \ldots, x_r$. Hence, for any arbitrary choice of $y_1, \ldots, y_s$ with $s$ an upper bound on the Gödel numbers of those finitely many formulas, for all $y_1', \ldots, y_s'$,

$$x_1, \ldots, x_r, y_1, \ldots, y_s \not\equiv_{0} x_1', \ldots, x_r', y_1', \ldots, y_s'.$$  

In the other two cases, we have

$$x_1, \ldots, x_r \equiv_{\alpha} x_1', \ldots, x_r'.$$  

It follows (by (A2)) that $E(x_i, x_i') > -\infty$ for each $i$. Also, since

$$x_1, \ldots, x_r \not\equiv_{\alpha} x_1', \ldots, x_r'$$

there is some $i$ such that $E_L(x_i, x_i') < \min(\varrho(x_i), \alpha)$.

**Case 2.** $x_1, \ldots, x_r \equiv_{\alpha} x_1', \ldots, x_r'$ and for some $i$, $E_L(x_i, x_i') < \varrho(x_i) < \alpha$.

*Proof.* We have

$$x_1, \ldots, x_r \not\equiv_{\varrho(x_i)} x_1', \ldots, x_r'.$$  

Since $\varrho(x_i) < \alpha$, we are done in this case.

**Case 3.** $x_1, \ldots, x_r \equiv_{\alpha} x_1', \ldots, x_r'$ and for some $i$, $E_L(x_i, x_i') < \alpha \leq \varrho(x_i)$.  


**Proof.** Recall that \( E(x_i, x'_i) > -\infty \). Let \( \beta = E_L(x_i, x'_i) \) and \( \ell = E_{\omega}(x_i, x'_i) \). There is a successor \( y \) of \( x_i \) with \( \rho(y) = \beta < \rho(x_i) \) and \( \varepsilon(y) \geq \ell \). By (Q5) and (Q6), for all \( y' \) successors of \( x'_i \), \( E_L(y, y') < \beta \). Then

\[
x_1, \ldots, x_r, y \not<_\beta x'_1, \ldots, x'_r, y'
\]

for all \( y' \). If \( \beta > 0 \), then we are already done. Otherwise, if \( \beta = 0 \) is the least element of \( L \), then by (A2) for all \( y' \) we have

\[
x_1, \ldots, x_r, y \not<_{\alpha} x'_1, \ldots, x'_r, y'.
\]

As in Case 1 for any arbitrary choice of \( y_1, \ldots, y_s \) with \( s \) sufficiently large, for all \( y', y'_1, \ldots, y'_s \), \( x_1, \ldots, x_r, y, y_1, \ldots, y_s \not< \alpha x'_1, \ldots, x'_r, y'_1, \ldots, y'_s \). \( \square \)

These three cases end the proof of the lemma. \( \square \)

**Corollary 2.3.10.** For \( \alpha \in \text{wfp}(O) \), \( \alpha > 0 \), and \( \bar{a}, \bar{b} \in M \), the following are equivalent:

1. \( \bar{a} <_{\alpha} \bar{b} \),
2. \( \bar{a} \geq_{\alpha} \bar{b} \),
3. \( \bar{a} \leq_{\alpha} \bar{b} \).

**Lemma 2.3.11.** Let \( M \models T(L) \). Then there is an automorphism of \( M \) taking \( \bar{a} \) to \( \bar{a}' \) if and only if \( \bar{a} <_{\alpha} \bar{a}' \) for all \( \alpha \in \text{wfp}(L) \).

**Proof.** If, for some \( \alpha \in \text{wfp}(L) \), \( \bar{a} \not<_{\alpha} \bar{a}' \), then by Lemma 2.3.10 \( \bar{a} \not<_{\alpha} \bar{a}' \). So there is no automorphism taking \( \bar{a} \) to \( \bar{a}' \).

On the other hand, suppose that for all \( \alpha \in \text{wfp}(L) \), \( \bar{a} \sim_{\alpha} \bar{a}' \). We have three cases.

**Case 1.** \( L \) is well-founded and has a maximal element.

Let \( \alpha \) be the maximal element of \( L \). We claim that the set of finite partial maps

\[
\{ \bar{a} \mapsto \bar{a}' : \bar{a} \sim_{\alpha} \bar{a}' \}
\]

has the back-and-forth property. It suffices to assume that \( \bar{a} \) and \( \bar{a}' \) are closed under predecessors. It also suffices to check the back-and-froth property for adding an element \( b \) which is a child of one of the \( a_i \). Since \( \alpha \) is the maximal element of \( L \), \( x \sim_{\alpha} y \) if and only if \( E(x, y) = \rho(x) \) for each \( i \).

Let \( \bar{a} = (a_1, \ldots, a_t) \) and \( \bar{a}' = (a'_1, \ldots, a'_t) \) be such that \( \bar{a} \sim_{\alpha} \bar{a}' \). Note that \( \bar{a} \equiv_{\alpha} \bar{a}' \) and for each \( i \), \( E(a_i, a'_i) = (\rho(a_i), 0) \). Given \( b \) a child of \( a_i \), by Lemma 2.3.7 (with \( \beta = \alpha \) and \( m = 0 \)) there is \( b' \) such that \( \bar{a}, b \equiv_{\alpha} \bar{a}', b' \) and \( E(b, b') = (\rho(b), 0) \). Hence \( \bar{a}, b \sim_{\alpha} \bar{a}', b' \).

**Case 2.** \( L \) is well-founded and has no maximal element.
We claim that the set of finite partial maps
\[ \{ \bar{a} \mapsto \bar{a}' : \bar{a} \sim_{\alpha} \bar{a}' \text{ for all } \alpha \in L \} \]
has the back-and-forth property. It suffices to assume that \( \bar{a} \) and \( \bar{a}' \) are closed under predecessors. It also suffices to check the back-and-forth property for adding an element \( b \) which is a child of one of the \( a_i \).

Let \( \bar{a} = (a_1, \ldots, a_t) \) and \( \bar{a}' = (a'_1, \ldots, a'_t) \) be such that \( \bar{a} \sim_{\alpha} \bar{a}' \) for all \( \alpha \in L \). Note that \( \bar{a} \equiv_{\text{at}} \bar{a}' \) and for each \( i \), \( E(a_i, a'_i) = (\varphi(a_i), 0) \). Given \( b \) a child of \( a_i \), let \( \beta \) be such that \( \beta > \varphi(b), \varphi(a_1), \ldots, \varphi(a_t) \). Then by Lemma 2.3.7 (with this \( \beta \) and \( m = 0 \)), there is \( b' \) such that \( \bar{a}, b \equiv_{\text{at}} \bar{a}', b' \) and \( E(b, b') = (\varphi(b), 0) \), and hence \( \bar{a}, b \sim_{\alpha} \bar{a}', b' \) for all \( \alpha \in L \).

**Case 3.** \( L \) is not well-founded.

Let \( \bar{a} = (a_1, \ldots, a_t) \) and \( \bar{a}' = (a'_1, \ldots, a'_t) \) be such that \( \bar{a} \sim_{\alpha} \bar{a}' \) for all \( \alpha \in L \). We claim that there is \( \alpha \notin \text{wfp}(L) \) such that \( \bar{a} \sim_{\alpha} \bar{a}' \). Then, by Lemmas 2.3.9 and 2.2.10, we would get an automorphism of \( \mathcal{M} \) taking \( \bar{a} \) to \( \bar{a}' \).

We claim that for each \( i \), either \( \varphi(a_i) \in \text{wfp}(L) \) or \( E(a_i, a'_i) \notin \text{wfp}(L) \). This is enough to get \( \bar{a} \sim_{\alpha} \bar{a}' \) for some \( \alpha \notin \text{wfp}(L) \). If \( \varphi(a_i) \notin \text{wfp}(L) \), then since \( a_i \sim_{\alpha} a'_i \) for all \( \alpha \in \text{wfp}(L) \), \( E(a_i, a'_i) \geq \alpha \) for all \( \alpha \in \text{wfp}(L) \). By (O2), \( E(a_i, a'_i) \notin \text{wfp}(L) \).

**Lemma 2.3.12.** Given \( \bar{x} \) a tuple in \( \mathcal{M} \) and \( \alpha \in \text{wfp}(L) \), there is a \( \Pi^{\text{in}}_{\alpha} \) formula which defines the set of \( \bar{y} \) with \( \bar{x} \sim_{\alpha} \bar{y} \).

**Proof.** Let \( \bar{y} \) be such that \( \bar{x} \not\equiv_{\alpha} \bar{y} \). By Lemma 2.3.10, \( \bar{x} \preceq_{\alpha} \bar{y} \). Proposition 15.1 of [AK00] says that \( \bar{x} \preceq_{\alpha} \bar{y} \) if and only if every \( \Sigma^{\text{in}}_{\alpha} \) formula true of \( \bar{y} \) is true of \( \bar{x} \). So there is a \( \Sigma^{\text{in}}_{\alpha} \) formula \( \varphi_{\bar{y}} \) true of \( \bar{y} \) but not of \( \bar{x} \). Let
\[ \psi = \bigwedge_{\bar{x} \preceq_{\alpha} \bar{y}} \neg \varphi_{\bar{y}}. \]
Note that \( \psi \) is a \( \Pi^{\text{in}}_{\alpha} \) formula. If \( \mathcal{M} \models \neg \psi(\bar{z}) \) then there is \( \bar{y} \) such that \( \mathcal{M} \models \varphi_{\bar{y}}(\bar{z}) \) and so \( \bar{x} \not\equiv_{\alpha} \bar{z} \) (and hence \( \bar{x} \not\equiv_{\alpha} \bar{z} \)). On the other hand, if \( \bar{x} \not\equiv_{\alpha} \bar{z} \), then \( \mathcal{M} \models \varphi_{\bar{z}}(\bar{z}) \) and so \( \mathcal{M} \models \neg \psi(\bar{z}) \).

**2.3.4 Computation of Scott Rank for Theorem 2.3.1**

Recall that for Theorem 2.3.1, we add to \( S^+ \):

(O4a) \( R_n = L \) for all \( n \).

So (Q7) is a vacuous condition. The following lemma completes the proof of Theorem 2.3.1:

**Lemma 2.3.13.** Let \( \mathcal{M} \models T(L) \). Then \( \text{SR}(\mathcal{M}) = \text{wfc}(L) \).

**Proof.** Recall Theorem 2.2.3, which says that the Scott rank of \( \mathcal{M} \) is the least \( \alpha \) such that every automorphism orbit is \( \Sigma^{\text{in}}_{\alpha} \)-definable without parameters. We have two cases.
Case 1. \( L \) is well-founded.

Let \( \bar{x} = (x_1, \ldots, x_n) \) be a tuple in \( M \). Let \( \alpha \in L \) be such that \( \alpha = \varphi(x_1), \ldots, \varphi(x_n) \). Then for \( \gamma \geq \alpha \) and \( y \in M \), \( \bar{x} \sim_\gamma y \) if and only if \( \bar{x} \sim_\alpha y \). So, by Lemma 2.3.11, \( \bar{x} \sim_\alpha y \) if and only if \( \bar{x} \) and \( y \) are in the same automorphism orbit. By Lemma 2.3.12, the orbit of \( \bar{x} \) is \( \Pi^{in}_\alpha \)-definable. Thus the orbits of all of the tuples \( \bar{x} \) from \( M \) are \( \Sigma^{in}_{\text{wfp}(L)} \)-definable.

Let \( \alpha \in L \), \( \alpha > 0 \). By Lemma 2.3.5 there is \( x \in M \) a successor of \( \emptyset \) with \( \varphi(x) = \alpha \). We claim that the automorphism orbit of \( x \) is not definable by a \( \Sigma^{in}_\alpha \) formula. Suppose to the contrary that it was, say by a formula \( \varphi \). Let \( \bar{y} = (y_1, \ldots, y_s) \) be a tuple in \( M \) and \( \psi \) a \( \Pi^{in}_\beta \) formula for some \( \beta < \alpha \) which witness that \( M \models \varphi(x) \). Let \( m \in \omega \) be such that \( m > \varepsilon(y_1), \ldots, \varepsilon(y_s) \). Using the construction of \( M \) as a Fraïssé limit, there is \( x' \in M \) such that \( (\alpha,0) > E(x,x') > (\beta,m) \). So \( x \triangleright_\alpha x' \). By Lemma 2.3.7 with these values of \( \beta \) and \( m \), there is a tuple \( \bar{y}' = (y'_1, \ldots, y'_s) \) such that \( x, \bar{y} \sim_\beta x', \bar{y}' \). By Lemma 2.3.10, \( x, \bar{y} \preceq_\beta x', \bar{y}' \). Hence \( M \models \varphi(x') \). So \( x \) and \( x' \) are in the same automorphism orbit; but then Lemma 2.3.11 contradicts the fact that \( x \triangleright_\alpha x' \).

So the automorphism orbits of \( M \) are definable by \( \Sigma^{in}_{\text{wfp}(L)} \) formulas, but there is no \( \alpha < \text{wfp}(L) \) such that all of the automorphism orbits are definable by \( \Sigma^{in}_\alpha \) formulas. So \( SR(M) = \text{wfp}(L) = \text{wfc}(L) \) since \( L \) is well-founded.

Case 2. \( L \) is not well-founded.

Fix a tuple \( \bar{x} \). By Lemma 2.3.11, \( \bar{y} \) is in the automorphism orbit of \( \bar{x} \) if and only if \( \bar{x} \sim_\alpha \bar{y} \) for each \( \alpha \in \text{wfp}(L) \). By Lemma 2.3.12, the set of such \( \bar{y} \) is \( \Pi^{in}_\alpha \)-definable for each fixed \( \alpha \). So the set of \( \bar{y} \) for which \( \bar{x} \sim_\alpha \bar{y} \) for all \( \alpha \in \text{wfp}(L) \) is \( \Pi^{in}_{\text{wfp}(L)} \)-definable, and therefore \( \Sigma^{in}_{\text{wfp}(L)+1} \)-definable.

By Lemma 2.3.5 there is \( x \in M \) a successor of \( \emptyset \) with \( \varphi(x) = \alpha \notin \text{wfp}(L) \). The argument from the previous case shows that the automorphism orbit of \( x \) is not definable by a \( \Sigma^{in}_\beta \) formula for any \( \beta \in \text{wfp}(L) \). Hence it is not definable by a \( \Sigma^{in}_{\text{wfp}(L)} \) formula since \( \text{wfp}(L) \) is a limit ordinal by (O2).

We have shown that the automorphism orbits of \( M \) are all definable by \( \Sigma^{in}_{\text{wfp}(L)+1} \) formulas, but that not all of the automorphism orbits are definable by \( \Sigma^{in}_{\text{wfp}(L)} \) formulas. Since \( L \) is not well-founded, \( \text{wfp}(L) + 1 = \text{wfc}(L) \). So \( SR(M) = \text{wfc}(L) \).

\[ \square \]

### 2.3.5 Computation of Scott Rank for Theorem 2.3.2

For Theorem 2.3.2, we want to have: If \( M \models T(L) \), then \( SR(M) = \text{wfp}(L) \) (rather than \( \text{wfc}(L) \)). We accomplish this by adding to \( S^+ \):

(O4b) There is a function \( G: L \to \mathcal{P}(L) \) such that for each \( \alpha \in L \), \( G(\alpha) \) is an increasing sequence of order type \( \omega \) whose limit is \( \alpha \) if \( \alpha \) is a limit ordinal, or a finite set containing \( \alpha - 1 \) if \( \alpha \) is a successor ordinal. \( R_1 \) is an increasing sequence with order type \( \omega \) which
is unbounded in $L$, or $R_1$ is $\{\gamma\}$ if $\gamma$ is the maximal element of $L$. For each $n$,

$$R_{n+1} = \{\beta_n\} \cup R_n \cup \bigcup_{\alpha \in R_n} G(\alpha)$$

where $\beta_n$ is an element of $L$. If $\gamma < \alpha < \beta$, and $\gamma \in G(\beta)$, then $\gamma \in G(\alpha)$.

In this case, (Q7) is no longer a vacuous condition. Recall that while (O4b) is not $\Pi^1_2$, this does not matter as we can take its Morleyization.

(O4b) plays a similar role to the “thin trees” in Knight and Millar’s [KM10] construction of a computable structure of Scott rank $\omega_1^{CK}$; the ordinals in $R_n$ put a bound, at level $n$ of the tree, on the Scott ranks of the elements at that level. In Lemma 2.3.14 below, we will see that for each $n$ there is a bound, below $\text{wfp}(L)$, on the ordinals in $R_n \cap \text{wfp}(L)$.

We begin by showing that (O4b) implies (O3). (R1), (R2), (R3), and (R4) follow immediately from (O4b). To see (R5), we show in the following lemma that the $R_n$ are well-founded.

**Lemma 2.3.14.** Suppose that $G$ and $R_n$ satisfy (O4b) and that $L$ is not well-founded. For each $n$, $R_n$ is well-founded and there is a bound $\alpha_n \in \text{wfp}(L)$ on $R_n \cap \text{wfp}(L)$.

**Proof.** We argue by simultaneous induction that:

1. for each $n$, $R_n$ is well-founded,
2. for each $n$ there is a bound $\alpha_n \in \text{wfp}(L)$ on $R_n \cap \text{wfp}(L)$.

For $n = 1$, $R_1$ has order type at most $\omega$ and is unbounded in $L$. Thus $R_1$ is well-founded, and has only finitely many elements in $\text{wfp}(L)$.

For the case $n + 1$, we have that

$$R_{n+1} = \{\beta_n\} \cup R_n \cup \bigcup_{\alpha \in R_n} G(\alpha).$$

We claim that $R_{n+1}$ is well-ordered. It suffices to show that $\bigcup_{\alpha \in R_n} G(\alpha)$ is well-ordered. Suppose that $(\beta_n)_{n \in \omega}$ is a decreasing sequence in $\bigcup_{\alpha \in R_n} G(\alpha)$. For each $n$, let $\alpha_n$ be the least $\alpha \in R_n$ such that $\beta_n \in G(\alpha_n)$. We claim that $(\alpha_n)_{n \in \omega}$ is a non-increasing sequence. If not, then for some $n$, $\alpha_{n+1} > \alpha_n$. We have $\alpha_{n+1} > \alpha_n \geq \beta_n \geq \beta_{n+1}$, so by (O4b) we have $\beta_{n+1} \in G(\alpha_n)$. This contradicts the choice of $\alpha_{n+1}$. Since $R_n$ is well-founded, there is $\alpha$ such that for all sufficiently large $n$, $\alpha_n = \alpha$. Thus, throwing away some initial part of the sequence $(\beta_n)_{n \in \omega}$, we may assume that each $\beta_n \in G(\alpha)$. This is a contradiction, since $G(\alpha)$ has order type at most $\omega$.

Now we claim that there is a bound $\alpha_{n+1} \in \text{wfp}(L)$ on $R_{n+1} \cap \text{wfp}(L)$. Let $\gamma$ be the least element of $R_n$ which is not in $\text{wfp}(L)$. Then we claim that

$$R_{n+1} \cap \text{wfp}(L) = \{\beta_{n+1}\} \cup [R_n \cap \text{wfp}(L)] \cup [G(\gamma) \cap \text{wfp}(L)] \cup \bigcup_{\alpha \in R_n \cap \text{wfp}(L)} G(\alpha).$$
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CHAPTER 2. SCOTT SPECTRA OF THEORIES

52

(If \( \beta_{n+1} \notin \text{wfc}(L) \), then we omit it.) Clearly the right hand side is contained in the left hand side. To see that we have equality, suppose that \( \alpha \in \text{wfp}(L) \cap G(\delta) \) for some \( \delta \in R_n \setminus \text{wfp}(L) \). Then \( \alpha < \gamma \leq \delta \), and so \( \alpha \in G(\gamma) \) by (O4b).

Then since \( G(\gamma) \) has order type at most \( \omega \) and \( \gamma \notin \text{wfp}(L) \), \( G(\gamma) \cap \text{wfp}(L) \) is finite and hence bounded. Also, each element of \( G(\alpha) \) for \( \alpha \in R_n \cap L \) is bounded above by \( \alpha \), and since \( R_n \cap L \) is bounded above by \( \alpha_n \), \( \cup_{\alpha \in R_n \cap \text{wfp}(L)} G(\alpha) \) is also bounded above by \( \alpha_n \). Thus \( R_{n+1} \cap \text{wfp}(L) \) is bounded above by some \( \alpha_{n+1} \in \text{wfp}(L) \).

We also need to know that for each \( L \in S \), there are \( G \) and \( R_n \) satisfying (O4b). Moreover, if \( L \) is computable with a computable successor relation, then we need to find \( G \) and \( R_n \) computable in order to have a computable model of \( T(L) \).

**Lemma 2.3.15.** There are \( G \) and \( R_1 \) satisfying (O4b) such that \( L = \bigcup_n R_n \). If \( L \) is computable with a computable successor relation, then we can pick \( G, (\beta_n)_{n \in \omega} \), and \( R_1 \) such that the \( R_n \) are uniformly computable.

**Proof.** Let \( \alpha_0, \alpha_1, \alpha_2, \ldots \) be a listing of \( L \). To define \( R_1 \), greedily pick an increasing subsequence of \( \alpha_0, \alpha_1, \alpha_2, \ldots \) (or, if there is a maximal element \( \gamma \) of \( L \), let \( R_1 = \{ \gamma \} \)).

We begin by defining \( G(\alpha_0) \), after which we define \( G(\alpha_1) \), and so on. To define \( G(\alpha_0) \), greedily pick an increasing subsequence of \( \alpha_1, \alpha_2, \ldots \) each element of which is less than \( \alpha_0 \) (stopping if we ever find a predecessor of \( \alpha_0 \)). Now suppose that we have defined \( G(\alpha_0), \ldots, G(\alpha_n) \). Suppose that \( \beta_1, \ldots, \beta_t \) are those \( \alpha_i, 0 \leq i \leq n \), with \( \alpha_i < \alpha_{n+1} \). Let \( \beta = \max(\beta_1, \ldots, \beta_t) \). Let \( \gamma_1, \ldots, \gamma_m \) be those \( \alpha_i, 0 < i \leq n \), with \( \alpha_i > \alpha_{n+1} \). Begin by putting into \( G(\alpha_{n+1}) \) the finitely many elements of \( G(\gamma_1), \ldots, G(\gamma_m) \) which are less than \( \alpha_{n+1} \). If \( \alpha_{n+1} \) is the successor of one of these elements, then we are done; if \( \alpha_{n+1} \) is the successor of one of \( \alpha_1, \ldots, \alpha_n \), then add that element to \( G(\alpha_{n+1}) \). Otherwise, greedily pick an increasing subsequence of \( \alpha_{n+2}, \alpha_{n+3}, \ldots \), each element of which is at least \( \beta \) and less than \( \alpha_{n+1} \). If we ever find a predecessor of \( \alpha_{n+1} \), then we can stop there. It is easy to check that \( G \) is as required by (O4b). In particular, if \( \gamma < \alpha < \beta \), and \( \gamma \in G(\beta) \), then \( \gamma \in G(\alpha) \).

Now for each \( n \), let

\[
R_{n+1} = \{ \alpha_n \} \cup R_n \cup \bigcup_{\alpha \in R_n} R(\alpha).
\]

Note that \( L = \bigcup_n R_n \).

We claim that if \( \alpha_0, \alpha_1, \ldots \) was an effective listing, then each \( G(\alpha_i) \) and each \( R_n \) is computable. First, note that \( R_1 \) is computable. It is also easy to see that each \( G(\alpha_n) \) is computable. Given a way of computing \( R_n \), we will show that to compute \( R_{n+1} \). To check whether \( \alpha_i \in R_{n+1} \), first check whether \( i = n \) (as we know that \( \alpha_n \in R_{n+1} \)). Second, check whether \( \alpha_i \in R_n \). Third, check whether \( \alpha_i + 1 \) is in \( R_n \); if it is, then \( \alpha_i \in G(\alpha_i + 1) \cap R_{n+1} \). Fourth, check whether \( \alpha_i \) is in one of \( G(\alpha_0), \ldots, G(\alpha_{i-1}) \). Note from the construction above that if \( \alpha_i \) is not in one of these sets (and \( \alpha_i \) is not in \( R_{n+1} \) for one of the first three reasons), then it is not in \( G(\alpha_j) \) for any \( j \) with \( \alpha_j \in R_n \). Finally, we show that models of \( T(L) \) have the correct Scott rank.
Lemma 2.3.16. Let $M \models T(L)$. Then $SR(M) = \text{wfc}(L)$.

Proof. Recall Theorem 2.2.3, which says that the Scott rank of $M$ is the least $\alpha$ such that every automorphism orbit is $\Sigma^0_\alpha$-definable without parameters. We have two cases.

Case 1. $L$ is well-founded.

This case is the same as the corresponding case of Lemma 2.3.13.

Case 2. $L$ is not well-founded.

Fix a tuple $\bar{x}$. By Lemma 2.3.11, $\bar{y}$ is in the automorphism orbit of $\bar{x}$ if and only if $\bar{x} \sim_\alpha \bar{y}$ for each $\alpha \in \text{wfp}(L)$. By Lemma 2.3.14, there is a bound $\gamma \in \text{wfp}(L)$ such that $\bar{y}$ is in the same orbit as $\bar{x}$ if and only if $\bar{x} \sim_\alpha \bar{y}$ for each $\alpha \leq \gamma$. By Lemma 2.3.12, the set of such $\bar{y}$ is $\Pi^0_\alpha$-definable for each fixed $\alpha$. So the set of $\bar{y}$ for which $\bar{x} \sim_\alpha \bar{y}$ for all $\alpha \leq \gamma$ is $\Pi^0_\gamma$-definable.

Let $\alpha \in \text{wfp}(L)$ and by (O1) let $n$ be such that $\alpha \in R_n$. By Lemma 2.3.5 there is $x \in M$, $|x| = n$, with $\varrho(x) = \alpha \notin \text{wfp}(L)$. The argument from the previous case shows that the automorphism orbit of $x$ is not definable by a $\Sigma^0_\beta$ formula for any $\beta \leq \alpha$.

We have shown that the automorphism orbits of $M$ are all definable by $\Sigma^0_{\text{wfp}(L)}$ formulas, but that there is no $\alpha \in \text{wfp}(L)$ such that all of the automorphism orbits are definable by $\Sigma^0_\alpha$ formulas. So $SR(M) = \text{wfp}(L)$.

\[ \square \]

2.4 $\Pi^0_2$ Theories

Recall that Theorem 2.1.3 stated that given $\alpha < \omega_1$, there is a $\Pi^0_2$ sentence $T$ all of whose models have Scott rank $\alpha$. This theorem is a simple application of the main construction.

Proof of Theorem 2.1.3. Let $A = (A, <_A)$ be a presentation of $(\alpha, <)$ as a structure with domain $A = (a_i)_{i \in \omega}$. Consider the atomic diagram of $A$ in the language with constant symbols $(a_i)_{i \in \omega}$ for the elements of $A$. Let $S$ be the conjunction of all of the sentences in the atomic diagram of $A$, together with the sentence $(\forall x) \forall i(x = a_i)$. Let $T$ be the $\Pi^0_2$ sentence obtained from Theorem 2.3.1. Then

\[ SS(T) = \{ \alpha \} . \]

\[ \square \]

In the main construction, the sentence we built had uncountably many existential types. This was necessary: an omitting types argument shows that if a $\Pi^0_2$ sentence has only countably many existential types, then it must have a model of Scott rank 1.

2.5 Computable Models of High Scott Rank

In this section, we will prove Theorem 2.1.6 by producing a $\Pi^0_2$ sentence $T$ all of whose models have Scott rank $\omega_1^{CK} + 1$ (and a $\Pi^0_2$ sentence whose models all have Scott rank $\omega_1^{CK}$). Moreover $T$ will have a computable model. If $A$ is a model of this sentence, then whenever
B is another structure with $A \equiv B$, $B$ will also be a model of $T$ and hence will also have non-computable Scott rank. Thus it is not the case that every computable structure $A$ of high Scott rank is approximated by models of lower Scott rank in the sense that for each $\alpha < \omega_1^{CK}$, there is a structure $B_\alpha$ with $\text{SR}(B_\alpha) < \omega_1^{CK}$ such that $A \equiv_\alpha B_\alpha$.

**Proof of Theorem 2.1.6.** Let $\mathcal{H} = (H, \prec_H)$ be a computable presentation of the Harrison linear ordering of order type $\omega_1^{CK}(1 + \mathbb{Q})$ as a structure with domain $H = (h_i)_{i \in \omega}$. We may assume that the successor relation is computable by replacing each element of $\mathcal{H}$ by $\omega$ (this does not change the order type). Let $S$ be the conjunction of the sentences of the atomic diagram of $\mathcal{H}$ (in the language with constants $h_i$) together with the sentence $(\forall x) \bigwedge_i (x = h_i)$. Let $T$ be the $\Pi_2$ sentence obtained from Theorem 2.3.1 applied to $S$. Then

$$\{\text{SR}(M) : M \models T\} = \{\text{wfc}(L) : L \models S\} = \{\omega_1^{CK} + 1\}.$$  

To get Scott rank $\omega_1^{CK}$, we simply use Theorem 2.3.2 instead of Theorem 2.3.1. □

A similar argument also gives the following:

**Theorem 2.5.1.** Let $\alpha$ be a computable ordinal. There is a $\Pi_2$ sentence with a computable model whose computable models all have Scott rank $\alpha$.

**Proof.** The proof is the same as that of the previous theorem, using the fact that every computable ordinal has a presentation where the successor relation is computable [Ash86a, Ash87, AK00]. □

### 2.6 A Technical Lemma

The general construction of Section 2.3 references $PC_{\mathcal{L}_{\omega_1^\omega}}$-classes of linear orders, whereas our classification in Theorems 2.1.12 and 2.1.14 references $\Sigma_1^1$ classes of linear orders. The following technical lemma shows that, if we are only interested in the order types represented in the class, the two are equivalent.

**Lemma 2.6.1.** Let $C$ be a class of linear orders (i.e., of order types). The following are equivalent:

1. $C$ is the set of order types of a $\Sigma_1^1$ class of linear orders.

2. $C$ is the set of order types of a $PC_{\mathcal{L}_{\omega_1^\omega}}$-class of linear orders.

Moreover, the lightface notions are also equivalent:

1. $C$ is the set of order types of a (lightface) $\Sigma_1^1$ class of linear orders.

2. $C$ is the set of order types of a computable $PC_{\mathcal{L}_{\omega_1^\omega}}$-class of linear orders.
CHAPTER 2. SCOTT SPECTRA OF THEORIES

Proof. (2) ⇒ (1) is clear. For (1) ⇒ (2), suppose that \( C \) is a class of linear orders defined by \( \exists X \varphi(X, \leq) \) where \( \varphi \) has only quantifiers over \( \omega \). Consider the class \( C^+ \) of pairs \((\leq, X) \in \omega^2 \times \omega\) with \( \varphi(X, \leq) \). Then \( C^+ \) is a Borel class.

Let \( D \) be the class of models in the language \( \{\leq, Y\} \cup \{a_i : i \in \omega\} \) such that each element of the domain is named by a unique constant \( a_i \), and such that with \( \leq \subseteq \omega^2 \) defined by \( i \leq j \iff a_i \leq a_j \) and \( X \subseteq \omega \) defined by \( i \in X \iff a_i \in Y \), \((\leq, X) \in C^+\). This gives a Borel reduction from \( D \) to \( C^+ \), and hence \( D \) is Borel. By a theorem of Lopez-Escobar [LE65], \( D \) is \( \mathcal{L}_{\omega_1 \omega} \)-axiomatizable since it is closed under isomorphism. Moreover, the order types of models in \( D \) are the same as the order types of the linear orders in \( C^+ \) and hence the same as those in \( C \).

For the lightface notions, the proof is the same except that we use Vanden Boom’s [VB07] lightface analogue of the Lopez-Escobar theorem. \( \square \)

2.7 Bounds on Scott Height

Recall that Sacks [Sac83] showed that \( \text{sh}(\mathcal{L}_{\omega_1 \omega}) \leq \delta_1^2 \) and that Marker [Mar90] showed that \( \text{sh}(\mathcal{P}
C_{\omega_1 \omega}) = \delta_1^2 \). We now show that \( \text{sh}(\mathcal{L}_{\omega_1 \omega}^c) = \delta_1^2 \).

Proof of Theorem 2.1.10. Fix \( \alpha < \delta_1^2 \). We may assume that \( \alpha \geq \omega_1^{CK} \). Let \( S \) be a computable \( \mathcal{P}
C_{\omega_1 \omega} \)-class in a language \( L \) with \( \alpha < \text{sh}(S) < \delta_1^2 \). When we say that \( S \) is a computable class, we mean that there is a computable \( \mathcal{L}_{\omega_1 \omega} \)-sentence \( T \) in a language \( L' \supseteq L \) such that \( S \) is the class of reducts of models of \( T \) to \( L \).

We define a (lightface) \( \Sigma_1^1 \) class \( \mathcal{C} \) of linear orders as follows. \((L, \leq) \) is in \( \mathcal{C} \) if and only if there is:

1. a model \( A \) of \( T \),
2. a set \( X \subseteq \omega \),
3. a Harrison linear order \( \mathcal{H} \) relative to \( X \),
4. an embedding \( f : L \to \mathcal{H} \) such that \( f(L) \) is an initial segment of \( \mathcal{H} \), and
5. non-standard back-and-forth relations \( \preceq_\alpha \) on \( A \) indexed by \( L \) (in the language \( \mathcal{L} \) of \( S \), not of \( T \)), such that:

   a) for all \( \alpha \in L \), there is \( \bar{x} \) which is \( \alpha \)-free, i.e. for all \( \bar{y} \) and \( \beta < \alpha \), there are \( \bar{x}' \) and \( \bar{y}' \) such that \( \bar{x}, \bar{y} \preceq_\beta \bar{x}', \bar{y}' \) and \( \bar{x}' \not\preceq_\alpha \bar{x} \),
   b) the set of partial maps

   \[ \{\bar{a} \mapsto \bar{b} : \bar{a} \preceq_\alpha \bar{b} \text{ for all } \alpha\} \]

   has the back-and-forth property.
If \( A \models T \) has Scott rank \( \alpha \) (where we compute Scott rank in the language \( \mathcal{L} \) of \( S \)), then take \( X \) such that \( \omega^X_1 \succ \alpha \). Let \( \mathcal{H} \) be a Harrison linear order relative to \( X \). Then \( \alpha \) embeds into an initial segment of \( \mathcal{H} \), and we can take the standard back-and-forth relations on \( A \), indexed by elements of \( \alpha \). By Theorem 2.2.3, (5a) and (5b) are satisfied. Thus \( \alpha \in \mathcal{C} \).

On the other hand, if \((\mathcal{L}, \leq)\) is well-founded, and there is a model \( A \) of \( T \) with back-and-forth relations indexed by \( \mathcal{L} \) satisfying (5a) and (5b), then \( L \) is the Scott rank of \( A \). Thus \( \mathcal{C} \cap On = SS(T) \).

We claim that if \( L \in \mathcal{C} \), then \( wfp(L) \leq \sup(SS(S)) \). If \( L \) is well-founded, then this is clear, so assume that \( L \) is not well-founded. Thus, for some set \( X \), \( L \) embeds into a Harrison linear order \( \mathcal{H} \) relative to \( X \) as an initial segment (and there is a model \( A \) of \( T \) and non-standard back-and-forth relations as above). Since \( L \) is not well-founded, its image in \( \mathcal{H} \) includes the well-founded part \( \omega^X_1 \). Now, for each \( \alpha \in wfp(L) \), by (5a) there is \( \bar{x} \) which is \( \alpha \)-free. Thus the Scott rank of \( A \) is at least \( \omega^X_1 \). Hence \( wfp(L) \leq SR(A) \), and since \( L \) was arbitrary, \( wfp(L) \leq \sup(SS(S)) \).

By the lightface version of Theorem 2.3.2, there is a computable \( \mathcal{L}_{\omega_1 \omega} \)-sentence \( T' \) such that

\[
SS(T') = \{ wfp(L) : L \in \mathcal{C} \supseteq \mathcal{C} \cap On = SS(S) \} \ni \alpha.
\]

Since if \( L \in \mathcal{C} \) then \( wfp(L) \leq \sup(SS(S)) \), we have \( sh(T) \leq sh(S) \). So \( \alpha \leq sh(T') < \omega_1 \). Since \( \alpha \) was arbitrary, \( sh(\mathcal{L}_{\omega_1 \omega}) \geq \delta^1_2 \). This proves the theorem.

### 2.8 Possible Scott Spectra of Theories

In this section, we will prove Theorems 2.1.12 and 2.1.14 which completely classify the possible Scott spectra under the assumption of Projective Determinacy. We begin by going as far as we can without any assumptions beyond ZFC, and then we assume Projective Determinacy in order to get a cone of sets \( X \) where the Scott spectrum contains either only \( \omega^X_1 \) for all \( X \) on the cone, or only \( \omega^X_1 + 1 \) for all \( X \) on the cone, or both for all \( X \) on the cone.

The following result is well-known.

**Lemma 2.8.1.** Let \( T \) be an \( \mathcal{L}_{\omega_1 \omega} \)-sentence. If the Scott spectrum of \( T \) is unbounded below \( \omega_1 \), then there is a set \( Y \) such that for every \( X \geq_T Y \), there is a model \( A \) of \( T \) with \( SR(A) \geq \omega^A_1 \omega = \omega^X_1 \). In particular, \( SR(A) \geq \omega^X_1 \) or \( SR(A) = \omega^X_1 + 1 \).

**Proof.** Choose \( Y \) such that \( T \) is \( Y \)-computable. This is a well-known application of Gandy’s basis theorem; see [Mon13b, Lemma 3.4].

In the next lemma, we consider the linear orders which support back-and-forth relations on models of an \( \mathcal{L}_{\omega_1 \omega} \)-sentence \( T \). This will give a \( \Sigma^1_1 \) class of linear orders, which if it is unbounded will contain non-well-founded members (supporting non-standard back-and-forth relations).
Theorem 2.8.2. Let $T$ be an $\mathcal{L}_{\omega_1\omega}$-sentence. There is a $\Sigma_1$ class of linear orders $\mathcal{C}$ such that

$$SS(T) = \mathcal{C} \cap On.$$ 

If $SS(T)$ is bounded below $\omega_1$, then $\mathcal{C} \subseteq On$. Otherwise, if $SS(T)$ is unbounded below $\omega_1$, then there is a set $Y$ such that

1. for all $X \geq_T Y$, at least one of $\omega_1^X$ or $\omega_1^X + 1$ is in $\mathcal{C}$, and
2. the only non-well-founded members of $\mathcal{C}$ are Harrison linear orders relative to some $X \geq_T Y$, i.e. $\omega_1^Y \cdot (1 + \mathbb{Q}) + \beta$ for some $Y$-computable $\beta$.

Proof. If $SS(T)$ is bounded below $\omega_1$, then we can just take $\mathcal{C} = SS(T)$; this is a $\Sigma_1$ class. So suppose that $SS(T)$ is unbounded below $\omega_1$.

By Lemma 2.8.1, there is a set $Y$ such that for every $X \geq_T Y$, there is a model $A$ of $T$ with $SR(A) = \omega_1^X$ or $SR(A) = \omega_1^X + 1$.

The proof of the theorem is similar to the proof of Theorem 2.1.10. Let $\mathcal{C}$ be the $\Sigma_1$ class of linear orders defined as follows. $(L, \leq)$ is in $\mathcal{C}$ if and only if there are:

1. a model $A$ of $T$,
2. a set $X \geq_T Y$,
3. a Harrison linear order $\mathcal{H}$ relative to $X$,
4. an embedding $f: L \rightarrow \mathcal{H}$ such that $f(L)$ is an initial segment of $\mathcal{H}$, and
5. non-standard back-and-forth relations $\preceq_\alpha$ on $A$ indexed by $L$ (in the language $\mathcal{L}$ of $S$, not of $T$), such that:
   a) for all $\alpha \in L$, there is $\bar{x}$ which is $\alpha$-free, i.e. for all $\bar{y}$ and $\beta < \alpha$, there are $\bar{x}'$ and $\bar{y}'$ such that $\bar{x}, \bar{y} \preceq_\beta \bar{x}', \bar{y}'$ and $\bar{x}' \preceq_\alpha \bar{x}$,
   b) the set of partial maps
   $$\{ a \mapsto \bar{b} : a \preceq_\alpha \bar{b} \text{ for all } \alpha \}$$
   has the back-and-forth property.

Note that while in Theorem 2.1.10 $\mathcal{C}$ was a (lightface) $\Sigma_1$ class, now $\mathcal{C}$ is $\Sigma_1^n(Y,T)$. We still prove, in the same way, that $\mathcal{C} \cap On = SS(T)$.

Now if $(L, \leq)$ is not well-founded, then for some $X \geq_T Y$, $L$ embeds into the Harrison linear order $\mathcal{H}$ relative to $X$. Since $L$ is not well-founded, it is itself isomorphic to $\omega_1^X \cdot (1 + \mathbb{Q}) + \beta$ for some $X$-computable ordinal $\beta$. By choice of $Y$, either $\omega_1^X$ or $\omega_1^X + 1$ is the Scott spectrum of a model of $T$. \qed
Our use of projective determinacy will be to have a uniform choice of whether \( \omega_1^X \) or \( \omega_1^X + 1 \) (or both) is in the Scott spectrum; that is, we will be able to choose \( Y \) such that for all \( X \geq_T Y \), \( \omega_1^X \) is in the Scott spectrum, or such that for all \( X \geq_T Y \), \( \omega_1^X + 1 \) is in the Scott spectrum, or both. We now prove Theorem 2.1.14, which said that each Scott spectrum is built from a \( \Sigma_1^1 \) class by either taking the well-founded part, the well-founded collapse, or both.

**Proof of Theorem 2.1.14.** Theorems 2.3.1 and 2.3.2 show that each of these is a Scott spectrum. In the other direction, Theorem 2.8.2 says that each Scott spectrum is \( \mathcal{C} \cap \text{On} \) for some \( \Sigma_1^1 \) class \( \mathcal{C} \). Either \( \mathcal{C} \subseteq \text{On} \) (in which case \( \mathcal{C} \) is bounded below \( \omega_1 \), and we are done) or there is a set \( Y \) such that

1. for all \( X \geq_T Y \), \( \omega_1^X \) or \( \omega_1^X + 1 \) is in \( \mathcal{C} \), and
2. the only non-well-founded members of \( \mathcal{C} \) are Harrison linear orders relative to some \( X \geq_T Y \).

By projective determinacy, there is \( Z \geq_T Y \) such that one of the following is true for all \( X \geq_T Z \):

1. \( \omega_1^X \in \mathcal{C} \) and \( \omega_1^X + 1 \notin \mathcal{C} \),
2. \( \omega_1^X \notin \mathcal{C} \) and \( \omega_1^X + 1 \in \mathcal{C} \), or
3. \( \omega_1^X \in \mathcal{C} \) and \( \omega_1^X + 1 \in \mathcal{C} \).

Let \( \mathcal{C}' \) be the set of linear orders in \( \mathcal{C} \) which embed into an initial segment of \( \mathcal{H}^X \) for some \( X \geq_T Z \). Then \( \mathcal{C} \cap \text{On} = \mathcal{C}' \cap \text{On} \), and depending on which case we were in above, we have:

1. \( \mathcal{C} \cap \text{On} = \mathcal{C}' \cap \text{On} = \text{wfp}(\mathcal{C}') \),
2. \( \mathcal{C} \cap \text{On} = \mathcal{C}' \cap \text{On} = \text{wfc}(\mathcal{C}') \), or
3. \( \mathcal{C} \cap \text{On} = \mathcal{C}' \cap \text{On} = \text{wfp}(\mathcal{C}') \cup \text{wfc}(\mathcal{C}') \).

We now give the proof of our alternate characterization.

**Proof of Theorem 2.1.12.** The Scott spectra which are bounded below \( \omega_1 \) clearly correspond to \( \Sigma_1^1 \) sets of ordinals which are bounded below \( \omega_1 \). So it remains only to deal with the unbounded case.

First we show that if \( \mathcal{C} \) is a \( \Sigma_1^1 \) set of linear orders with either \( \mathcal{C} \cap \text{On} \) or \( \{ \alpha : \alpha + 1 \in \mathcal{C} \cap \text{On} \} \) stationary, then \( \mathcal{C} \cap \text{On} \) is the Scott spectrum of a theory. Note that for each set \( Y \), \( \{ \omega_1^X : X \geq_T Y \} \) contains a club (see Remark 2.2.13), and hence intersects either \( \mathcal{C} \cap \text{On} \) or \( \{ \alpha : \alpha + 1 \in \mathcal{C} \cap \text{On} \} \). Thus exactly one of the following is true for cofinally many \( X \) in the Turing degrees: \( \omega_1^X \) is in \( \mathcal{C} \cap \text{On} \) (but \( \omega_1^X + 1 \) is not), or \( \omega_1^X + 1 \) is in \( \mathcal{C} \cap \text{On} \) (but \( \omega_1^X \) is not), or both are in \( \mathcal{C} \cap \text{On} \). By Projective Determinacy, one of these is true on a cone, say the cone above a set \( Y \).
Now let $C'$ be the $\Sigma^1_1$ set of linear orders $(L, \leq)$ in $C$ such that for some $X \geq_T Y$, $L$ embeds into an initial segment of $H^X$. Then $C' \cap On = C \cap On$, and either:

(1) whenever $(L, \leq) \in C'$ is not well-founded, $wfp(L)$ is in $C' \cap On$,

(2) whenever $(L, \leq) \in C'$ is not well-founded, $wfp(L) + 1$ is in $C' \cap On$, or

(3) whenever $(L, \leq) \in C'$ is not well-founded, $wfp(L)$ and $wfp(L) + 1$ are in $C' \cap On$.

By the proof of Theorem 2.1.14, $C' \cap On$ is a Scott spectrum.

On the other hand, if $T$ is an $L_{\omega_1 \omega}$-sentence whose Scott spectrum is unbounded below $\omega_1$, then by Theorem 2.1.14 there is a set $Y$ such that for all $X \geq_T Y$, either $\omega_1^X$ or $\omega_1^X + 1$ is in $SS(T)$. By projective determinacy, there is $Z \geq_T Y$ such that either for all $X \geq_T Z$, $\omega_1^X \in SS(T)$, or for all $X \geq_T Z$, $\omega_1^X + 1 \in SS(T)$. In Remark 2.2.13, we noted that $\{\omega_1^X : X \geq_T Z\}$ is stationary. This completes the proof.

**Remark 2.8.3.** In Theorems 2.3.1 and 2.3.2 we can get $T$ to be $\Pi^0_2$. Thus Theorem 2.1.15 follows from Theorem 2.1.14.

**Remark 2.8.4.** Note that the proofs of Lemma 2.8.1 and Theorems 2.8.2, 2.1.12, and 2.1.14 go through if we replace $T$ by a $PC_{L_{\omega_1 \omega}}$ class of structures. Thus, under projective determinacy, the Scott spectra of $PC_{L_{\omega_1 \omega}}$-classes are the same as the Scott spectra of $L_{\omega_1 \omega}$ sentences. Thus we have proved Theorem 2.1.16.

We can use the classification to find some interesting examples of Scott spectra.

**Proposition 2.8.5.** The following are all Scott spectra of $L_{\omega_1 \omega}$-sentences:

1. $\{\alpha + 1 : \alpha < \omega_1\}$.
2. $\{\alpha : \alpha < \omega_1$ is an admissible ordinal\}.
3. $\{\alpha + 1 : \alpha < \omega_1$ is an admissible ordinal\}.

**Proof.** Note that if $(L, \leq)$ is not well-founded, then $wfc(L)$ is a successor ordinal. If $C$ is the $L_{\omega_1 \omega}$-definable class of all linear orders with an initial element and containing a dense interval (with endpoints), $C$ contains no well-founded orders. Moreover, for each ordinal $\alpha < \omega_1$, $\alpha + Q \in C$ and $wfc(\alpha + Q) = \alpha + 1$. Thus

$$\{wfc(L) : L \in C\} = \{\alpha : \alpha < \omega \text{ is a successor ordinal}\}$$

is a Scott spectrum.

In Subsection 2.2.4, we remarked that class of Harrison linear orders is a $\Sigma^1_1$ class. Thus

$$\{wfp(L) : L \in C\} = \{\omega_1^X : X \subseteq \omega\}$$

and

$$\{wfc(L) : L \in C\} = \{\omega_1^X + 1 : X \subseteq \omega\}$$

are Scott spectra. (Note that $\omega$ is an admissible which is not in the first spectrum, but we can easily add it in via Proposition 2.2.5.)
2.9 Open Questions

We begin by asking whether one can remove the assumption of Projective Determinacy in the classification of Scott spectra.

**Open Question.** Classify the Scott spectra of $\mathcal{L}_{\omega_1\omega}$-sentences in ZFC.

We would also like to know a lightface classification. The proofs of Theorems 2.1.12 and 2.1.14 do not go through for computable sentences because of the use of Projective Determinacy.

**Open Question.** Classify the Scott spectra of computable $\mathcal{L}_{\omega_1\omega}$-sentences.

Finally, our construction relied upon being able to take infinite disjunctions, such as when we named each element of the order sort by a constant. A first-order theory cannot name each element of an infinite sort by a constant. Can our results be expanded to first-order theories?

**Open Question.** Classify the Scott spectra of first-order theories.
Chapter 3

Some New Computable Structures of High Rank

The results presented in this chapter appeared in [HTIK]. They are joint work with Greg Igusa and Julia Knight and appear here with their permission.

3.1 Introduction

Our main result answers an open problem posed by Millar and Sacks [MS08]. They asked whether every computable structure of Scott rank \( \omega_{CK}^{1} \) is completely determined by the computable sentences it satisfies. We give a negative answer by building a computable structure of Scott rank \( \omega_{CK}^{1} \) whose computable infinitary theory is not \( \aleph_0 \)-categorical. This is a new model of high Scott rank which is fundamentally different from all previously constructed models.

The Scott rank of a structure measures the internal complexity. We give one definition below. There are other definitions, which assign slightly different ordinals, but the important distinctions are the same for all definitions in current use. In particular, if one definition assigns Scott rank \( \omega_{CK}^{1} + 1 \), or \( \omega_{CK}^{1} \), to a particular computable structure, then the other definitions do the same.

Let \( \mathcal{A} \) be a countable structure for a computable language. Our definition of Scott rank is based on a family of equivalence relations \( \sim^\alpha \), for countable ordinals \( \alpha \). Scott’s original definition [Sco65] was based on a slightly different family of equivalence relations.

Definition 3.1.1. Let \( \bar{a} \) and \( \bar{b} \) be tuples in \( \mathcal{A} \) of the same finite length. Then

1. \( \bar{a} \sim^0 \bar{b} \) if \( \bar{a} \) and \( \bar{b} \) satisfy the same atomic formulas

2. For \( \alpha > 0 \), \( \bar{a} \sim^\alpha \bar{b} \) if for each \( \beta < \alpha \), for each \( \bar{c} \), there exists \( \bar{d} \), and for each \( \bar{d} \), there exists \( \bar{c} \), such that \( \bar{a}, \bar{c} \sim^\beta \bar{b}, \bar{d} \).

For later use, we extend the definition \( \sim^\alpha \) to allow tuples from different structures.
Definition 3.1.2. Let $A$ and $B$ be structures for the same language, and let $\bar{a}$ and $\bar{b}$ be tuples of the same length in $A$, $B$, respectively.

1. $(A, \bar{a}) \sim_0^0 (B, \bar{b})$ if $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas in their respective structures.

2. for $\alpha > 0$, $(A, \bar{a}) \sim_{\alpha} (B, \bar{b})$ if for $\beta < \alpha$, for each $\bar{c}$ in $A$, there exists $\bar{d}$ in $B$, and for each $\bar{d}$ in $B$, there exists $\bar{c}$ in $A$, such that $(A, \bar{a}, \bar{c}) \sim_{\beta} (B, \bar{b}, \bar{d})$.

Remark: If $(A, \bar{a}) \sim_{\alpha} (B, \bar{b})$, then for any $\Sigma_\alpha$ formula $\varphi(\bar{x})$ of $L_{\omega_1 \omega}$, $A \models \varphi(\bar{a})$ iff $B \models \varphi(\bar{b})$.

We define Scott rank, first for a tuple in a structure $A$, and then for the structure itself.

Definition 3.1.3. 

1. The Scott rank of a tuple $\bar{a}$ is the least $\alpha$ such that for all $\bar{b}$, if $\bar{a} \sim_{\alpha} \bar{b}$, then for all $\gamma > \alpha$, $\bar{a} \sim_{\gamma} \bar{b}$.

2. The Scott rank of the structure $A$ is the least ordinal greater than the Scott ranks of all tuples in $A$.

Nadel [Nad74] observed that for a computable structure $A$, two tuples are automorphic just in case they satisfy the same computable infinitary formulas. This implies that the Scott rank of $A$ is at most $\omega_1^{CK} + 1$. The following is well-known.

Fact. Let $A$ be a computable structure.

1. $A$ has computable Scott rank iff there is a computable ordinal $\alpha$ such that for all tuples $\bar{a}$ in $A$, the orbit of $\bar{a}$ is defined by a computable $\Sigma_\alpha$ formula.

2. $A$ has Scott rank $\omega_1^{CK}$ iff for each tuple $\bar{a}$, the orbit is defined by a computable infinitary formula, but for each computable ordinal $\alpha$, there is a tuple $\bar{a}$ whose orbit is not defined by a computable $\Sigma_\alpha$ formula.

3. $A$ has Scott rank $\omega_1^{CK} + 1$ iff there is a tuple $\bar{a}$ whose orbit is not defined by a computable infinitary formula.

There are familiar examples of computable structures having various computable ordinal ranks. The canonical example of a computable structure of Scott rank $\omega_1^{CK} + 1$ is the Harrison ordering, a linear order with order type $\omega_1^{CK} (1 + \eta)$ (see [Har68]). Producing a computable structure of Scott rank $\omega_1^{CK}$ took longer. Makkai gave an example of an arithmetical structure of Scott rank $\omega_1^{CK}$ [Mak81], a “group-tree”. In [KM10], Makkai’s construction is re-worked to give a computable structure. In [CKM06], there is a simpler example, a computable tree of Scott rank $\omega_1^{CK}$. In [CGKM09], the tree is used to produce further structures in familiar classes—a field, a group, etc.\footnote{Although [KM10] was not published until 2011, it was written before [CKM06], which was published in 2006, and [CGKM09], which was published in 2009.}
These examples of computable structures of Scott rank $\omega_1^{CK}$ all have the feature that the computable infinitary theory is $\aleph_0$-categorical. The conjunction of the computable infinitary theory forms a Scott sentence. In [MS08], Millar and Sacks produced a structure $A$ of Scott rank $\omega_1^{CK}$ such that the computable infinitary theory of $A$ is not $\aleph_0$-categorical. The structure $A$ is not computable; it is not even hyperarithmetical, but it has the feature that $\omega_1^A = \omega_1^{CK}$. This means that $A$ lives in a fattening of the admissible set $L_{\omega_1^{CK}}$.

In [MS08], Millar and Sacks asked whether there is a computable structure of Scott rank $\omega_1^{CK}$ whose computable infinitary theory is not $\aleph_0$-categorical. The question is asked again in [CGKM09]. Millar and Sacks also asked whether there are similar examples for other countable admissible ordinals. In [Fre08], Freer proved the analog of the result of Millar and Sacks, producing, for an arbitrary countable admissible ordinal $\alpha$, a structure $A$ with $\omega_1^A = \alpha$, such that the theory of $A$ in the admissible fragment $L_\alpha$ is not $\aleph_0$-categorical. Freer’s structure is not in $L_\alpha$, but in a fattening of $L_\alpha$. The main result of the present paper says that there is a computable structure of Scott rank $\omega_1^{CK}$ for which the conjunction of the computable infinitary theory is not a Scott sentence. This answers positively the question of Millar and Sacks mentioned above. The construction appears in Section 2.

By an “indiscernible sequence”, we mean a infinite sequence that is indiscernible for $L_{\omega_1^\omega}$ formulas.

**Definition 3.1.4.** Fix a structure $A$. An *indiscernible sequence* in $A$ is a sequence $(a_i)_{i<\omega}$ of elements of $A$ such that for any two finite subsequences $a_{i_1}, \ldots, a_{i_n}$ and $b_{j_1}, \ldots, b_{j_n}$ (with $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_n$) satisfy the same $L_{\omega_1^\omega}$ formulas.

The Harrison ordering obviously has an infinite indiscernible sequence. Other examples of computable structures of Scott rank $\omega_1^{CK} + 1$ share this feature. Goncharov and Knight (unpublished) asked whether every computable structure of Scott rank $\omega_1^{CK} + 1$ has an infinite indiscernible sequence. At the same time, they also noticed that the structures of Scott rank $\omega_1^{CK}$ constructed by Makkai [Mak81], and Knight and Millar [KM10], did not have an infinite indiscernible sequence.

In Section 3, we describe two constructions producing computable structures of Scott rank $\omega_1^{CK} + 1$ with not even an indiscernible ordered triple. The first is produced by taking a Fraïssé limit with infinitely many infinite equivalence classes, and putting the structure of the Harrison ordering on the equivalence classes. Although this structure has no indiscernible triples, it is effectively bi-interpretable [HTMMM] with the Harrison ordering, and hence it has an infinite indiscernible sequence of imaginaries.

The second example is a modified version of Makkai’s construction [Mak81]. Makkai [Mak81] gave a “computable operator” taking an input tree $T$ to a group-tree $A(T)$—the group-tree $A(T)$ is computable uniformly in the input tree $T$. The structure $A(T)$ is built by putting a group structure on each level of the tree (the language of the structure $A(T)$ does not include the group operation, but, instead, has a collection of unary functions). Makkai constructed a $\Delta^0_0$ “thin” tree $T$ such that $A(T)$ had Scott rank $\omega_1^{CK}$. Knight and Millar [KM10] modified the construction to make the input tree (and hence the output group-tree)
computable. All of the elements of \( A(T) \) are definable from a collection of parameters \( g_0 \), one at each level. Hence, \( A(T) \) does not have an indiscernible ordered triple. We will show that if our input tree \( T \) is the sequence of descending sequences in the Harrison ordering, then the resulting group-tree \( A(T) \) has Scott rank \( \omega_1^{CK} + 1 \), but it still does not have an indiscernible triple (or even an indiscernible triple of imaginaries).

### 3.2 Scott Rank \( \omega_1^{CK} \)

In this section, our goal is to prove the following.

**Theorem 3.2.1.** There is a computable structure \( M \) with Scott rank \( \omega_1^{CK} \) such that the computable infinitary theory of \( M \) is not \( \aleph_0 \)-categorical.

We will use some material on trees from [CKM06]. The trees are isomorphic to subtrees of \( \omega^\omega \). We use a language with a successor relation. Here are the facts that we need. There is a computable tree \( T^* \) of Scott rank \( \omega_1^{CK} \). In addition, for a fixed \( \Pi^1_1 \) path \( P \) through \( \mathcal{O} \), there is a family of approximating trees \( (T^a)_{a \in P} \) of computable Scott rank. For \( a \in P \) such that \( |a| = \alpha \), \( T^a \) has tree rank at most \( \omega(\alpha + 1) \), and \( T^* \sim^\alpha T^0 \). The family \( (T^a)_{a \in P} \) is computable uniformly in \( a \), and the tree ranks of the nodes of \( T^a \) are also computable, uniformly in \( a \), in the sense that we can effectively label the nodes of \( T^a \) by pairs \( (b, n) \), where \( \sigma \in T^a \) has label \( (b, n) \) for \( b \in P \) just in case \( \sigma \) has tree rank \( \omega \cdot \beta + n \) for \( |b| = \beta \). The tree \( T^* \) and the approximations \( T^a \) are all rank-homogeneous.

**Definition 3.2.2.** \( T \) is rank-homogeneous provided that for each node \( x \) at level \( n \), if \( x \) has tree rank \( \alpha \) and there is a node at level \( n + 1 \) (not necessarily a successor of \( x \)) of tree rank \( \beta < \alpha \), then \( x \) has infinitely many successors of tree rank \( \beta \). Also, for each node \( x \) at level \( n \), if \( x \) has infinite rank, then it has infinitely many successors of infinite rank, in addition to infinitely many of each ordinal rank \( \beta \) that occurs at level \( n + 1 \).

**Definition 3.2.3.** For \( A \) and \( B \) rank-homogeneous trees, with \( \bar{a} \) in \( A \) and \( \bar{b} \) in \( B \), we write \((A, \bar{a}) \equiv^\alpha (B, \bar{b})\) provided that

1. for all \( n \), the tree ranks less than \( \omega \alpha \) of nodes at level \( n \) are the same in \( A \) and \( B \),

2. the subtree of \( A \) “generated” by \( \bar{a} \) (by closing under predecessors) is isomorphic to the subtree of \( B \) generated by \( \bar{b} \), with an isomorphism taking \( \bar{a} \) to \( \bar{b} \),

3. for corresponding elements \( x \) in the subtree of \( A \) generated by \( \bar{a} \) and \( x' \) in the subtree of \( B \) generated by \( \bar{b} \), either the tree ranks of \( x \) and \( x' \) match, or else both are at least \( \omega \cdot \alpha \).

**Lemma 3.2.4.** Let \( A \) and \( B \) each be one of our trees \( T^* \) or \( T^0 \). If \((A, \bar{a}) \equiv^\alpha (B, \bar{b})\), then \((A, \bar{a}) \sim^\alpha (B, \bar{b})\).
Proof. The statement is clear for $\alpha = 0$. Also, if $\alpha$ is a limit ordinal, and the statement holds for all $\beta < \alpha$, then it holds for $\alpha$. Supposing that it holds for $\alpha$, we prove it for $\alpha + 1$. For simplicity, suppose that $\bar{a}$ and $\bar{b}$ are subtrees. Let $a$ be an element of $\bar{a}$ that has a new successor $c$ at the top of a finite subtree $\bar{c}$. Let $b$ be the element corresponding to $a$. We need $d$ and $\bar{d}$ matching $c$ and $\bar{c}$. If the tree ranks of $a$ and $b$ match, and are less than $\omega(\alpha + 1)$, then we can choose $d$ and $\bar{d}$ with tree ranks matching the corresponding elements of $c$ and $\bar{c}$. If the tree ranks of $a$ and $b$ are at least $\omega(\alpha + 1)$, and the tree rank of $c$ and the elements of $\bar{c}$ are at least $\omega \cdot \alpha$, then we choose $d$ and $\bar{d}$ also with tree ranks at least $\omega \cdot \alpha$. We can choose $d$ of rank $\omega \cdot \alpha + n$ for $n$ sufficiently large to leave room for choosing the rest of $\bar{d}$. If some of the elements of $\bar{c}$ have tree ranks less than $\omega \cdot \alpha$, then we choose the corresponding elements of $\bar{d}$ with matching tree ranks.

Given $a$, we can effectively find a Scott sentence for $T^a$. We give the tree rank of the top node, and for each $n$, we say what are the tree ranks of nodes at level $n$. Finally, we say that the tree is “rank-homogeneous”; i.e., for all $x$ at level $n$ of tree rank $\beta$, and all $\gamma < \beta$ such that there is a node of tree rank $\gamma$ at level $n + 1$, $x$ has infinitely many successors of tree rank $\gamma$. This is effective since we have, uniformly in $a$, a function giving the ranks of the nodes of $T^a$.

We want a computable copy $(U, <_U, S_U)$ of the Harrison ordering with the successor relation, with a family of trees $(T_u)_{u \in U}$, uniformly computable in $u$, such that if $\text{pred}(u)$ has order type $\alpha$ with notation $a \in P$, then $T_u \cong T^a$, and if $\text{pred}(u)$ is not well-ordered, then $T_u \not\cong T^a$.

Lemma 3.2.5. There is a computable structure $A$ with universe the union of disjoint sets $U$ and $V$, with an ordering $<$ of type $\omega_1^{CK}(1 + \eta)$ and successor relation $S$ on $U$, and with a function $Q$ from $V$ to $U$ such that for each $u \in U$, $Q^{-1}(u)$ is an infinite set, with a tree structure $T_u$. If $\text{pred}(u)$ has order type $\alpha$ with notation $a \in P$, then $T_u \cong T^a$. If $\text{pred}(u)$ is not well ordered, then $T_u \not\cong T^a$.

Proof. To prove the lemma, we use Barwise-Kreisel Compactness. Let $\Gamma$ be a $\Pi^1_1$ set of computable infinitary sentences in the language of $A$ saying that $U$ and $V$ are disjoint, $U$ is linearly ordered by $<_U$, $S_U$ is the successor relation on the ordering, $Q$ maps $V$ onto $U$ such that for each $u \in U$, $Q^{-1}(u)$ is infinite, $S_T$ is the union of successor relations putting a tree structure $T_u$ on the set $Q^{-1}(u)$, with further axioms guaranteeing the following:

1. for each computable ordinal $\alpha$, the ordering $(U, <_U)$ has an initial segment of type $\omega^\alpha$,
2. for each computable ordinal $\alpha$, each $u \in U$ is the left endpoint of an interval of type $\omega^\alpha$,
3. the ordering $(U, <_U)$ has no infinite hyperarithmetic decreasing sequence,
4. for each $u \in U$, if $\text{pred}(u)$ has order type $\alpha$, where $a \in P$ is the notation for $\alpha$, then $T_u \cong T^a$.
(5) For a computable ordinal \( \alpha \), if \( u <_U v \), where \( \text{pred}(u) \) has order type \( \alpha \), then \( T_u \) and \( T_v \) satisfy the same computable \( \Sigma_\alpha \) sentences.

For a hyperarithmetic set \( \Gamma' \subseteq \Gamma \), there is a computable ordinal \( \gamma \) bounding the ordinals \( \alpha \) corresponding to sentences in \( \Gamma' \) of types (1), (2), (4), and (5). Then we get a model of \( \Gamma' \) as follows. We fix computable sets \( U \) and \( V \) in advance. Let \( c \) be the notation for \( \gamma \) in \( P \). Since \( \prec_\text{pred}(c) \) is c.e., we have a computable function \( f \) from \( U \to 1 - 1 \) onto \( \prec_\text{pred}(c) \). For \( x, y \in U \), \( x <_U y \) iff \( f(x) <_O f(y) \). Let \( S_U(x, y) \) iff \( f(y) = 2f(x) \). If \( f(x) = a \), then \( T_x \cong T^a \).

Since every hyperarithmetical subset of \( \Gamma \) has a model, the whole set does. In this model, \( (U,<_U) \) has order type \( \omega^{\text{CK}}_1(1 + \eta) \). For \( u \in U \) such that \( \text{pred}(u) \) is not well-ordered, \( T_u \) satisfies the computable infinitary sentences true in \( T^* \). Since \( T_u \) is computable, it must be isomorphic to \( T^* \). \( \square \)

**Lemma 3.2.6.** Let \( I \) be the well-ordered initial segment of \( U \) of order type \( \omega^{\text{CK}}_1 \). There is a uniformly computable sequence \( (R_n)_{n \in \omega} \) of infinite subsets of \( U \) with the following properties:

1. \( R_0 \) contains some element of \( I \),
2. for each \( n \), there exists \( u \in I \) that is an upper bound on \( R_n \cap I \),
3. for each \( u \in R_n \), there exists \( v \in R_{n+1} \) such that \( u < v \),
4. for each \( u \in R_n \cap I \), there exists \( v \in R_{n+1} \cap I \) such that \( u < v \),
5. \( \cup_n R_n \) is unbounded in the well-ordered initial segment of \( U \).

**Proof.** Fix \( u_0 \notin I \), \( u_1 \in I \). Let \( R_0 \) consist of all elements of \( u \geq u_0 \) plus \( u_1 \). Given \( R_m \) for \( m \leq n \), let \( R_{n+1} \) consist of the successors in \( U \) of each \( u \in R_n \), plus the \( \omega \)-first element of \( U \) not in \( \cup_{m \leq n} R_m \). \( \square \)

Let \( T \) be the tree of finite sequences \((u_1, \ldots, u_n)\) that are increasing in \((U,<_U)\), with \( u_n \in R_n \). We define a function \( H \) that takes each non-empty sequence \( \sigma \in T \) to its last term \( u_n \). This function is obviously computable. We are about to describe the structure \( \mathcal{M} \). The language includes the following:

1. unary predicates \( A \) and \( B \)—these will be disjoint,
2. for each \( \tau \in T \), a binary relation \( C_\tau \) that associates to each \( x \in A \) a subset \( T_{(\tau,x)} \) of \( B \), where for distinct pairs \((\tau, x)\) and \((\tau', x')\), the sets \( T_{(\tau,x)} \) and \( T_{(\tau', x')} \) are disjoint,
3. a binary successor relation \( S \) that puts a tree structure on each set \( T_{(\tau,x)} \).

For the structure \( \mathcal{M} \), we let \( A^\mathcal{M} \) consist of (codes for) the elements of \( T \). For each \( \sigma \in A \) and \( \tau \in T \), we define \( S^\mathcal{M} \) so that

\[
T_{(\tau, \sigma)} = \begin{cases} 
T_u & \text{if } \tau \leq \sigma \land H(\tau) = u \\
T^* & \text{if } \tau \notin \sigma 
\end{cases}
\]
The structure $\mathcal{M}$ is computable. We note that for $u \in T$, if $\text{pred}(u)$ is well ordered, then $T_u$ is isomorphic to the appropriate $T^u$, while if $\text{pred}(u)$ is not well ordered, then $T_u$ is isomorphic to $T^*$. 

To show that the computable infinitary theory of $\mathcal{M}$ is not $\aleph_0$-categorical, we produce a second model $\mathcal{N}$, not isomorphic to $\mathcal{M}$. We want a path through $T$ with special features.

**Lemma 3.2.7.** There is a path $\pi$ through $T$ such that for all $n$, $\pi(n) \in R_n$, and $\text{ran}(\pi)$ is co-final in $I$.

*Proof.* Let $(u_n)_{n \in \omega}$ be a list of the elements of $I$. Let $\pi(0) \in R_0 \cap I$. Given $\pi(n)$, take the first $k$ such that $u_k > \pi(n)$. We choose $\pi(n + 1)$ to be some $v > \pi(n)$ in $R_{n+1} \cap I$. If possible, we take $v \geq u_k$. Since $I$ is co-final in $\bigcup_n R_n$, there is some $m$ such that $R_m \cap I$ has an element $v \geq u_k$, and then the same is true for all $m' \geq m$. So, for each $k$, we will come to $m$ such that we can choose $\pi(m) \geq u_k$. \hfill $\Box$

Let $\mathcal{N}$ be the extension of $\mathcal{M}$ with an additional element of $\mathcal{A}^\mathcal{N}$ representing the path $\pi$. We define $S^\mathcal{N}$ on $T_{(\tau, \pi)}$ so that

$$T_{(\tau, \pi)} \cong \begin{cases} T_u & \text{if } H(\tau) = u \text{ and } \tau \leq \pi \\ T^* & \text{otherwise} \end{cases}$$

**Lemma 3.2.8.** $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic.

*Proof.* For a fixed $\sigma \in T$, there are only finitely many $\tau$ such that $\tau \leq \sigma$. In $\mathcal{M}$, for a fixed $\sigma$, all but finitely many of the trees $T_{(\tau, \sigma)}$ are isomorphic to $T^*$. On the other hand, in $\mathcal{N}$, there are infinitely many initial segments $\tau$ of $\pi$ with $T_{(\tau, \pi)}$ isomorphic to some $T_u \neq T^*$. Thus, no element of $\mathcal{M}$ can be mapped isomorphically to $\pi \in \mathcal{N}$.

**Definition 3.2.9.** We write $\mathcal{A} \leq_\infty \mathcal{B}$ if for any computable infinitary formula $\varphi(\bar{x})$ and any $\bar{a} \in \mathcal{A}$, $\mathcal{A} \models \varphi(\bar{a})$ iff $\mathcal{B} \models \varphi(\bar{a})$.

To show that $\mathcal{N}$ satisfies the computable infinitary theory of $\mathcal{M}$, we show that $\mathcal{M} \leq_\infty \mathcal{N}$. For this, it is enough to show that for any computable ordinal $\alpha$ and any tuple $\bar{a} \in \mathcal{M}$, $(\mathcal{M}, \bar{a}) \sim_\alpha (\mathcal{N}, \bar{a})$.

**Lemma 3.2.10.** Let $\mathcal{A}$ and $\mathcal{B}$ be structures, each isomorphic to one of $\mathcal{M}$ or $\mathcal{N}$. Let $\bar{a} = (a_1, \ldots, a_n)$ be a tuple in $\mathcal{A}$, and let $\bar{b} = (b_1, \ldots, b_n)$ be a tuple in $\mathcal{B}$ of the same length. Suppose that:

1. $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas,
2. for each $a_i$ and the corresponding $b_i$ in the predicate $A$ (and with $u$ being the element of $U$ with $\text{pred}(u)$ having order type $\alpha$), for each $n$, if one of $a_i(n)$ or $b_i(n)$ is defined and $\leq_U u$, then $a_i(n) = b_i(n)$, and
(3) for each \( a_i \) and corresponding \( b_i \), both in \( A \), and for each \( \tau \in T \), we have \( (T_{(\tau,a_i)}, c) \sim_{\alpha} (T_{(\tau,b_i)}, \bar{d}) \), where \( c \) consists of the elements from \( \bar{a} \) that are in \( T_{(\tau,a_i)} \), and \( \bar{d} \) consists of the corresponding elements from \( \bar{b} \).

(We assume that for each element \( a \) of the tuple \( \bar{a} \) that is in the predicate \( B \), the corresponding element \( a' \) of the predicate \( A \) with \( a \in T_{(\tau,a')} \) is also present in the tuple \( \bar{a} \). We make a similar assumption about \( b \).) Then \( (A, \bar{a}) \sim_{\alpha} (B, \bar{b}) \).

Note that if \( \bar{a} \) and \( \bar{b} \) both consist solely of elements from the predicate \( A \), then (1) and (2) imply (3).

**Proof.** We argue by induction on \( \alpha \). Suppose that \( \bar{a} \) and \( \bar{b} \) satisfy the conditions above for \( \alpha \). Given \( \beta < \alpha \) and \( \bar{a}' \) a tuple in \( A \), we will find a tuple \( \bar{b}' \) in \( B \) such that \( \bar{a}, \bar{a}' \) and \( \bar{b}, \bar{b}' \) satisfy the conditions above for \( \beta \). It suffices to assume about \( \bar{a}' \) that for each element \( a \) of the tuple \( \bar{a}' \) that is in the predicate \( B \), the corresponding element \( a' \) of the predicate \( A \) with \( a \in T_{(\tau,a')} \) is present in the tuple \( \bar{a}, \bar{a}' \).

First, for each \( a'_i \in A \), we choose \( b'_i \) such that for all \( \tau \in T \), \( T_{(\tau,a'_i)} \sim_{\alpha} T_{(\tau,b'_i)} \). If \( a'_i \in T \), then we choose \( b'_i = a'_i \), if possible. However, it may be that \( a'_i \) is already in \( \bar{b} \). So, instead, let \( n \) be the length of \( a'_i \), and choose \( b'_i = a'_i \cdot (v) \) for some sufficiently large \( v \in R_n \) with \( \text{pred}(v) \) ill-founded. For each \( \tau \), if \( \tau \leq a'_i \), then \( \tau \leq b'_i \), so that \( T_{(\tau,a'_i)} \equiv T_u \equiv T_{(\tau,b'_i)} \) for some \( u \). If \( \tau \nleq a'_i \), then either \( \tau \nleq b'_i \) or \( \tau = b'_i \) (whence \( H(\tau) = v \)). Either way, \( T_{(\tau,a'_i)} \equiv T^* \equiv T_{(\tau,b'_i)} \) by choice of \( u \).

If, instead, \( a'_i = \pi \), let \( u \in U \) be such that \( \text{pred}(u) \) is well-founded with order type \( \alpha \). Let \( \sigma \) be the initial segment of \( \pi \) consisting of all of the entries \( v \) of \( \pi \) with \( v \leq_U u \). Let \( b'_i \) be a code for \( \sigma \cdot (v) \) for some \( v \in R_n \) with \( \text{pred}(v) \) ill-founded, which is sufficiently large that \( b'_i \) codes a new element. Then for all \( \tau \leq \sigma \), we have \( \tau \leq \pi \) and so \( T_{(\tau,a'_i)} \equiv T_{(\tau,b'_i)} \). For all \( \tau \nleq \sigma \), either \( \tau \nleq \sigma \), in which case \( T_{(\tau,a'_i)} \equiv T^* \equiv T_{(\tau,b'_i)} \), or \( \tau \leq \pi \), in which case, \( T_{(\tau,a'_i)} \equiv T_v \sim_{\alpha} T^* \equiv T_{(\tau,b'_i)} \) for some \( v \geq_U u \). In this manner, we may reduce to the case where \( \bar{a}'_i \) contains only elements of the predicate \( B \).

For each element \( a \) from \( \bar{a} \) in the predicate \( A \), let \( b \) be the corresponding element from \( \bar{b} \). Fix \( \tau \in T \). Let \( \bar{c} \) consist of the elements from \( \bar{a} \), and let \( \bar{c}' \) consist of the elements from \( \bar{a}' \) that are in \( T_{(\tau,a)} \). Similarly, let \( \bar{d} \) consist of the elements from \( \bar{b} \) that are in \( T_{(\tau,b)} \). By assumption, \( (T_{(\tau,a)}, \bar{c}) \sim_{\alpha} (T_{(\tau,b)}, \bar{d}) \). Thus, there is an \( \bar{d}' \) such that \( (T_{(\tau,a)}, \bar{c}, \bar{c}') \sim_{\beta} (T_{(\tau,b)}, \bar{d}, \bar{d}') \). The tuple \( \bar{b}' \) consists of the elements of the tuples \( \bar{d}' \) for each \( a \) and \( \tau \).

**Lemma 3.2.11.** \( \mathcal{M} \preceq_{\infty} \mathcal{N} \)

**Proof.** Let \( \bar{a} \) be a tuple in \( \mathcal{M} \). Then by the previous lemma, \( (\mathcal{M}, \bar{a}) \sim_{\alpha} (\mathcal{N}, \bar{a}) \). It follows that for any \( \Sigma_\alpha \) formula \( \varphi(\bar{x}) \), if \( \mathcal{M} \models \varphi(\bar{a}) \), then \( \mathcal{N} \models \varphi(\bar{a}) \). This proves the lemma.

**Lemma 3.2.12.** \( \mathcal{M} \) has Scott rank \( \omega^{CK}_1 \).
Proof. First, note that there is an automorphism of \( M \) taking \( \sigma_1 \in A \) to \( \sigma_2 \in A \) if and only if for each \( \tau \in T \), \( T(\tau,\sigma_1) \) is isomorphic to \( T(\tau,\sigma_2) \). This is the case if and only if for each \( n \), if \( \text{pred}(\sigma_1(n)) \) or \( \text{pred}(\sigma_2(n)) \) is well-founded, then \( \sigma_1(n) = \sigma_2(n) \).

Fix \( \sigma \in A \). We define the orbit of \( \sigma \) by saying, for the finitely many \( \tau \leq \sigma \) with \( H(\tau) = u \) and \( \text{pred}(u) \) well-founded, that \( T(\tau,\sigma) \) is isomorphic to \( T_u \), and for each other \( \tau \), that \( T(\tau,\sigma) \cong T^* \). We can express the former by a computable formula using the Scott sentences of the \( T_u \). For the latter, note that it suffices to say that \( T(\tau,\sigma) \) is isomorphic to \( T^* \) for only those \( \tau \) of length at most \( n + 1 \), where \( n \) is the length of \( \sigma \). We cannot express this directly by a computable infinitary formula, but there is a computable infinitary formula that is satisfied in \( M \) exactly by such elements of \( A \). Let \( \alpha < \omega_1^{CK} \) be large enough that for all elements \( v \) of \( R_0,\ldots,R_n \) with \( \text{pred}(v) \) well-founded (recalling that such \( v \) are not cofinal in the initial segment of \( U \) of order type \( \omega_1^{CK} \)), \( T_v \not\equiv^\alpha T^* \). Then, for \( \tau \) of length at most \( n + 1 \), \( T(\tau,\sigma) \) is isomorphic to \( T^* \) if and only if \( T(\tau,\sigma) \not\equiv^\alpha T^* \). This can be expressed by a computable infinitary formula.

The Scott rank of a tuple \( \bar{b} \) in \( T(\tau,\sigma) \) is not greater than the Scott rank of \( \bar{b} \) in \( M \). Therefore, the Scott rank of \( M \) is at least \( \omega_1^{CK} \), since there are many \( \tau \) and \( \sigma \) such that \( T(\tau,\sigma) \) is isomorphic to \( T^* \), which has Scott rank \( \omega_1^{CK} \). The Scott rank of \( M \) is at most \( \omega_1^{CK} \), since we can define the orbit of any tuple by a computable infinitary formula. For a tuple \( \bar{u}, \bar{v} \) in \( M \), where \( \bar{u} \in A \) and \( \bar{v} \in B \), we can define the orbit as follows: for each element \( \sigma \in A \) in the tuple \( \bar{u} \), we give a definition as above, and if \( \bar{b} \) is the part of the tuple \( \bar{v} \) in a particular tree \( T(\tau,\sigma) \), then we say what is the orbit of \( \bar{b} \) in \( T(\tau,\sigma) \). Here we use the fact that each of the trees \( T(\tau,\sigma) \) itself has Scott rank at most \( \omega_1^{CK} \). 

\[ \Box \]

### 3.3 Scott Rank \( \omega_1^{CK} + 1 \)

We begin this section by proving the following:

**Theorem 3.3.1.** There is a computable structure of Scott rank \( \omega_1^{CK} + 1 \) with no indiscernible ordered triple.

Our structure will be a Fraïssé limit, obtained from a class \( K \) of finite structures satisfying the hereditary, amalgamation, and joint embedding properties, abbreviated \( HP \), \( AP \), and \( JEP \). For a discussion of Fraïssé limits from the point of view of computability, see [CHMM11]. We note that Henson [Hen71] gave an example of a homogeneous triangle-free graph.

**Proof.** We define a class \( K \) of finite structures with signature consisting of binary relations \( E \) and \( (C_i)_{i \in \omega} \). We view the relations \( C_i \) as “colors” (in the sense of Ramsey theory) with which we color (unordered) pairs of vertices. A finite structure \( A \) will be in \( K \) if \( E \) is an equivalence relation, the \( C_i \) color the unordered pairs of vertices (i.e., \( xC_iy \) if and only if \( yC_ix \)) with exactly one color per edge, and there are no monochromatic triangles; i.e., there is no \( i \in \omega \) and \( x, y, z \in A \) with \( xC_iyC_izC_ix \).
Claim 1. \( K \) satisfies the hereditary property (HP), amalgamation property (AP), and the joint embedding property (JEP).

Proof of Claim 1. The HP is clear. To see that \( K \) has the AP, suppose that \( A \subseteq \mathcal{B}, \mathcal{C} \) are structures in \( K \). We define a structure \( \mathcal{D} \subseteq K \) extending \( \mathcal{B} \) and \( \mathcal{C} \). We can extend the equivalence relation to \( \mathcal{D} \). Since there are only finitely many elements of \( A \) and \( \mathcal{B} \), only finitely many colors have been used so far. To color edges \((x, y)\), where \( x \in \mathcal{B} - A \) and \( y \in \mathcal{C} - A \), simply choose \( i \) that has not colored any edge yet, and color \((x, y)\) with \( i \). This cannot introduce any monochromatic triangles. A similar argument, omitting \( A \), shows that \( K \) has the JEP. \( \square \)

Note that we can effectively list the structures of \( K \). Thus, \( K \) has a computable Fraïssé limit \( \mathcal{M} \), and \( \mathcal{M} \) has, as its Scott sentence, a computable infinitary sentence \( \varphi \). Models of \( \varphi \) have infinitely many equivalence classes, all of which have infinitely many elements. Now, let \( \mathcal{N} \) be an expansion of \( \mathcal{M} \) with a linear order \( \leq \) of order type \( \omega_1^{CK}(1 + \eta) \) on the equivalence classes. The structure \( \mathcal{N} \) has a computable copy, since we can find an effective labeling of the equivalence classes by elements of \( \omega \), and use the Harrison ordering. Let \( R : A \to \omega_1^{CK}(1 + \eta) \) be the resulting effective order-preserving map, which respects equivalence classes, and which induces a bijection between the equivalence classes and elements of \( \omega_1^{CK}(1 + \eta) \).

Claim 2. For \( \alpha \geq 1 \) and tuples \( \bar{x}, \bar{y} \) in \( \mathcal{N} \), \( \bar{x} \sim^\alpha \bar{y} \) in \( \mathcal{N} \) if and only if \( R(\bar{x}) \sim^\alpha R(\bar{y}) \) in \( \omega_1^{CK}(1 + \eta) \) and \( \bar{x} \equiv_{at} \bar{y} \), where \( \bar{x} \equiv_{at} \bar{y} \) if \( \bar{x} \) and \( \bar{y} \) satisfy the same atomic formulas in \( \mathcal{N} \).

Proof of Claim 2. Suppose that \( R(\bar{x}) \sim^\alpha R(\bar{y}) \) and \( \bar{x} \equiv_{at} \bar{y} \). Then we will show that \( \bar{x} \sim^\alpha \bar{y} \). Take \( \beta < \alpha \) and \( \bar{x}' \) a new tuple of elements. Then there is \( \bar{a} \) in \( \omega_1^{CK}(1 + \eta) \) such that \( R(\bar{x})R(\bar{x}') \sim^\beta R(\bar{y})\bar{a} \). Choose \( \bar{z} \) such that \( R(\bar{z}) = \bar{a} \). Let \( \bar{y}' \) be a tuple of new symbols of the same length as \( \bar{z} \). Consider the finite structure, in the signature of \( K \), defined on the elements \( \bar{y}, \bar{y}', \bar{z} \) as follows. The relations \( E \) and \( C_i \) are defined on \( \bar{y} \) and \( \bar{z} \) as in \( \mathcal{N} \). We set \( y'_i Ez_i \) and the equivalence classes are completely determined by this. Define \( y_iC_ky'_j \) if and only if \( x_iC_kx'_j \) (and \( y'_jC_ky_i \) if and only if \( x'_jC_kx_i \)). There are no monochromatic triangles among \( \bar{y}, \bar{y}', \) or among \( \bar{y}, \bar{z} \). Since we have only used finitely many colors so far, we can color the remaining pairs so that there are no monochromatic triangles.

The finite structure we have defined is in the class \( K \), so we can find a realization of \( \bar{y}' \) in \( \mathcal{N} \). Then \( R(\bar{y}') = \bar{a} \), so that \( R(\bar{x}), R(\bar{x}') \sim^\beta R(\bar{y}), R(\bar{y}') \). Also, \( \bar{x}, \bar{x}' \equiv_{at} \bar{y}, \bar{y}' \). Thus, \( \bar{x}, \bar{x}' \sim^\beta \bar{y}, \bar{y}' \) by the inductive hypothesis (or, for \( \beta = 0 \), because \( \bar{x}, \bar{x}' \sim^\beta \bar{y}, \bar{y}' \)). So, we have shown that \( \bar{x} \sim^\alpha \bar{y} \). On the other hand, if \( \bar{x} \not\equiv_{at} \bar{y} \), then it is immediate that \( \bar{x} \not\sim^\alpha \bar{y} \). If \( R(\bar{x}) \not\sim^\alpha R(\bar{y}) \), then it is not hard to see that \( \bar{x} \not\sim^\alpha \bar{y} \). \( \square \)

Claim 3. \( SR(\mathcal{N}) = \omega_1^{CK} + 1 \).

Proof of Claim 3. Let \( x \in \mathcal{N} \) be such that \( R(x) \) is in the ill-founded part of \( \omega_1^{CK}(1 + \eta) \). We claim that \( SR(x) = \omega_1^{CK} \). Fix \( \alpha < \omega_1^{CK} \). Let \( y \) be such that \( \text{pred}(R(y)) \) is well-founded and \( R(x) \sim^\alpha R(y) \). Now, there is no automorphism of the Harrison ordering taking \( R(y) \) to \( R(x) \), so there is no automorphism of \( \mathcal{N} \) taking \( y \) to \( x \). Thus, \( x \) and \( y \) are in different
automorphism orbits. Since \(x\) and \(y\) are singletons, \(x \equiv_y y\). Thus, by the previous claim, \(x \sim^{\alpha} y\). Since \(\alpha\) was arbitrary, \(SR(x) = \omega_1^{CK}\), completing the proof of the claim.

**Claim 4.** \(N\) has no indiscernible ordered triple.

**Proof of Claim 4.** It suffices to show that no three singleton elements are order indiscernible. Given \(x, y,\) and \(z,\) let \(i\) be such that \(xC_i y\). Since \(N\) has no monochromatic triangles, it cannot be the case that \(yC_i z\) and \(xC_i z\). Thus \(x, y,\) and \(z\) are not indiscernible.

This completes the proof of Theorem 3.3.1.

Note that this construction is, in some sense, cheating. The structure \(N\) is effectively bi-interpretable (see [HTMMM]) with the Harrison ordering: the Harrison ordering lives inside \(N\) as the definable quotient modulo the definable equivalence relation \(E\). The indiscernible sequence of the Harrison ordering becomes an indiscernible sequence of imaginaries in \(N\).

**Definition 3.3.2.** Fix a structure \(A\). An indiscernible sequence of imaginaries of \(A\) is a sequence \((E_i)_{i \in \omega}\) of equivalence classes of \(A\), modulo some \(L_{\omega_1\omega}\)-definable equivalence relation, such that for any two finite subsequences \(E_{i_1}, \ldots, E_{i_n}\) and \(E_{j_1}, \ldots, E_{j_n}\) (with \(i_1 < i_2 < \cdots < i_n\) and \(j_1 < j_2 < \cdots < j_n\)) there is an automorphism of \(A\) mapping \(E_{i_k}\) to \(E_{j_k}\).

**Proposition 3.3.3.** Let \(N\) be the structure from Theorem 3.3.1. Then \(N\) has an indiscernible sequence of imaginaries.

**Proof.** The map \(R\) from above induced a bijection between the \(E\)-equivalence classes and the elements of \(\omega_1^{CK}(1 + \eta)\). We claim that each automorphism of \(\omega_1^{CK}(1 + \eta)\) induces an automorphism of \(N\). Then, since \(\omega_1^{CK}(1 + \eta)\) has an indiscernible sequence, \(N\) will have an indiscernible sequence of imaginaries, namely the \(E\)-equivalence classes in bijection to the indiscernible sequence of \(\omega_1^{CK}(1 + \eta)\).

It suffices to see that in the Fraïssé limit \(M\), any permutation \(\pi\) of the \(E\)-equivalence classes extends to an automorphism of \(M\). Let \(\bar{a}\) and \(\bar{b}\) be tuples of elements of \(M\) of the same length, satisfying the same atomic formulas, and such that if \(a_i\) is in the \(j\)th equivalence class then \(b_i\) is in the \(\pi(j)\)th equivalence class. Let \(c\) be an additional element of \(M\). We can find an element \(d\) of \(M\) such that \(d\) is in the \(\pi(j)\)th equivalence class (if \(c\) was in the \(j\)th equivalence class) and such that \(\bar{b}, d\) is colored in the same way as \(\bar{a}, c\). We can do this since \(\bar{a}, c\) has no monochromatic triangles. This lets us construct the desired automorphism using a back-and-forth construction.

A construction similar to that of Theorem 3.3.1 allows us to turn any structure \(M\) into a structure \(M^*\) that is effectively bi-interpretable with \(M\), but has no indiscernible triples. Two structures which are effectively bi-interpretable have many of the same computability-theoretic properties; for example, they have the same computable dimension (see [HKSS02]). In light of this, we want not just a structure \(M\) with Scott rank \(\omega_1^{CK} + 1\) and no indiscernible sequence, but a structure \(M\) with Scott rank \(\omega_1^{CK} + 1\) and no indiscernible sequence of imaginaries. To produce such a structure, we use a construction originally due to Makkai
There is a computable tree \( T \subseteq \omega^\omega \) be a tree. We will define a new structure \( \mathcal{A}(T) \). Let \( T_n \) be the set of nodes at the \( n \)-th level of \( T \). For each \( n \), we define \( G_n = \mathcal{P}_\omega(T_n) \) to be the collection of finite subsets of \( T_n \). Now, \( G_n \) forms an abelian group under symmetric difference \( \Delta \). The identity element of \( G_n \) is the empty set, which we denote by \( \text{id}_n \). Let \( G = \bigcup_n G_n \). The tree structure on \( T \) induces a tree structure on \( G \), which we will define using a predecessor relation \( p \).

Given \( a \in G_{n+1} \), write \( a = \{t_1, \ldots, t_n\} \). Then set \( p(a) \) to be the sum of the predecessors of \( t_1, \ldots, t_n \). An element \( t^* \) is in \( p(a) \) if and only if the number of successors of \( t^* \) in \( a \) is odd. We have \( p(\text{id}_{n+1}) = \text{id}_n \). Note that \( p \) is a homomorphism from \( G_{n+1} \) to \( G_n \).

Lemma 3.3.4 (Lemma 3.3 of \([\text{KM10}]\)). Let \( a \in G_n \), with \( a \neq \text{id}_n \). Then the tree rank of \( a \) is the minimum of the tree ranks of \( t \) for \( t \in a \).

Lemma 3.3.5 (Lemma 3.6 of \([\text{KM10}]\)). For \( a \in G_n \), \( a \equiv^\beta \text{id}_n \) if and only if the tree rank of \( a \) is at least \( \omega \cdot \beta \).

Theorem 3.3.6 (Theorem 3.7 and Lemma 4.3 of \([\text{KM10}]\)). There is a computable tree \( T \) such that \( \text{SR}(\mathcal{A}(T)) = \omega_1^{CK} \).

Lemma 3.3.7. Let \( T \) be a tree. Then \( \mathcal{A}(T) \) does not have an indiscernible ordered triple of imaginaries.

Proof. Suppose to the contrary that there is a definable equivalence relation with three equivalence classes \( E_1, E_2, \) and \( E_3 \) that form an indiscernible triple. Fix \( a \in E_1 \). Let \( n \) be such that \( a \in G_n \). There are automorphisms of \( \mathcal{A}(T) \), one taking \( E_1 \) to \( E_2 \), and one taking \( E_1 \) to \( E_3 \). So, \( E_2 \) and \( E_3 \) both contain elements of \( G_n \). Let \( g \) be an automorphism of \( \mathcal{A}(T) \) fixing \( E_1 \). Then for each \( b \in G_n \), \( b = f_b(a) \), and so \( g(b) = f_b(g(a)) \). Hence, the action of \( g \) on \( G_n \) is entirely determined by where \( g \) sends \( a \). Thus, there cannot be two automorphisms of \( \mathcal{A}(T) \), one fixing \( E_1, E_2, \) and \( E_3 \) and the other fixing \( E_1 \) and mapping \( E_2 \) to \( E_3 \). This contradicts the order-indiscernibility of \( E_1, E_2, \) and \( E_3 \).

Theorem 3.3.8. There is a computable tree \( T \) such that \( \mathcal{A}(T) \) has Scott rank \( \omega_1^{CK} + 1 \).
Proof. Let $T$ be the tree of finite decreasing sequences in the Harrison ordering. We claim that $A(T)$ has Scott rank $\omega^\CK_1 + 1$. Note that at each level of $T$, there are elements of every computable tree rank, and there are elements with infinite tree rank. Let $G$ be the tree defined from $T$ as above. Then by Lemma 3.3.4, at each level of $G$, there are elements of every computable tree rank, and there are elements with infinite tree rank. Fix $n$. Given $\beta < \omega^\CK_1$, there is $a \in G_n$ with computable tree rank at least $\omega \cdot \beta$. By Lemma 3.3.5, $a \equiv^\beta \id_n$, but $a \not\equiv^\gamma \id_n$ for some $\gamma > \beta$. Hence $SR(\id_n) = \omega^\CK_1$. It follows that $SR(A(T)) = \omega^\CK_1 + 1$. 

**Corollary 3.3.9.** There are computable structures $M$ of Scott rank $\omega^\CK_1$, and of $\omega^\CK_1 + 1$, that have no indiscernible ordered triples of imaginaries.

**Proof.** This follows from Lemma 3.3.7 and Theorems 3.3.6 and 3.3.8.

We end with an open question.

**Question.** Is there a structure of Scott rank $\omega^\CK_1$ that is computably approximable and has no indiscernible sequences of imaginaries?
Chapter 4

Scott Sentences of Finitely Generated Algebraic Structures

The results presented in this chapter appeared in [HTH]. They are joint work with Meng-Che Ho and appear here with his permission.

4.1 Introduction

Given a countable structure $\mathcal{M}$, we can describe $\mathcal{M}$, up to isomorphism, by a sentence of the infinitary logic $\mathcal{L}_{\omega_1\omega}$ which allows countable conjunctions and disjunctions. (See Section 4.1.1 for the formal description of this logic; in this brief introduction, we will write down sentences in an informal way.) To measure the complexity of $\mathcal{M}$, we want to write down the simplest possible description of $\mathcal{M}$. For example, one can describe the countably infinite-dimensional $\mathbb{Q}$-vector space by the vector space axioms together with the sentence

for all $n$, there are $x_1, \ldots, x_n$ such that for all $r_1, \ldots, r_n \in \mathbb{Q}$, if $r_1 x_1 + \cdots + r_n x_n = 0$ then some $r_i = 0$.

This sentence has a universal quantifier, followed by an existential quantifier, followed by a universal quantifier. There is a hierarchy of sentences depending on the number of quantifier alternations. The $\Sigma^0_n$ sentences have $n$ alternations of quantifiers, beginning with existential quantifiers; the $\Pi^0_n$ sentences have $n$ alternations of quantifiers, beginning with a universal quantifier; and the $d-\Sigma^0_n$ sentences are the conjunction of a $\Sigma^0_n$ and a $\Pi^0_n$ sentence. The hierarchy is ordered as follows, from the simplest formulas on the left, to the most complicated formulas on the right:

$$
\begin{align*}
\Sigma^0_1 & \rightarrow \Sigma^0_1 \land \Pi^0_1 \\
\Pi^0_1 & \rightarrow \Sigma^0_2 \\
\Sigma^0_2 & \rightarrow \Sigma^0_1 \land \Pi^0_2 \\
\Pi^0_2 & \rightarrow \Sigma^0_2 \land \Pi^0_2 \\
\Sigma^0_3 & \rightarrow \Sigma^0_2 \land \Pi^0_3 \\
\Pi^0_3 & \rightarrow \Sigma^0_3 \land \Pi^0_3 \\
d-\Sigma^0_1 & \rightarrow \cdots 
\end{align*}
$$
We use this hierarchy to measure the complexity of a sentence. The sentence given above describing the infinite-dimensional \( \mathbb{Q} \)-vector space is a \( \Pi^0_3 \) sentence, and it turns out that this is the best possible; there is no \( d-\Sigma^0_2 \) description of this vector space. There is a \( d-\Sigma^0_2 \) description of any finite-dimensional \( \mathbb{Q} \)-vector space, and so these structures are “simpler” than the infinite-dimensional vectors space.

In this paper, we consider descriptions of finitely generated structures, and particularly of finitely generated groups. Any finitely generated structure \( \mathcal{M} \), with generating tuple \( \bar{a} \), has a \( \Sigma^0_3 \) description of the form:

there is a tuple \( \bar{x} \), satisfying the same atomic formulas as \( \bar{a} \) (i.e., for all atomic formulas true of \( \bar{a} \), the formula is true of \( \bar{x} \) (i.e., for all \( y \), there is a term \( t \) in the language such that \( y = t(\bar{x}) \)).

However, many finitely generated groups have a simpler description which is \( d-\Sigma^0_2 \). For the group \( \mathbb{Z} \), for example, the \( \Pi^0_2 \) axioms of torsion-free abelian groups, together with the following two sentences, which are \( \Pi^0_2 \) and \( \Sigma^0_2 \) respectively, form a \( d-\Sigma^0_2 \) description:

for all \( x \) and \( y \), there are \( n, m \in \mathbb{Z} \), not both zero, such that \( nx = my \)

and

there is \( x \neq 0 \) which has no proper divisors.

Indeed, all previously known examples of finitely generated groups had a \( d-\Sigma^0_2 \) Scott sentences, including all polycyclic (including nilpotent) groups and many finitely-generated solvable groups [Ho]. The main result of this paper is an example of a computable group which has no \( d-\Sigma^0_2 \) Scott sentence. Our group has \( \Sigma^0_3 \) \( m \)-complete index set.

This paper is divided into two main sets of results. The first is a general investigation of conditions for a finitely-generated structure to have (or not have) a \( d-\Sigma^0_2 \) Scott sentence. The second is an application of these general results to constructing the group mentioned above. We also include some results on finitely generated fields and rings.

\subsection*{4.1.1 Scott Sentences}

The infinitary logic \( \mathcal{L}_{\omega_1 \omega} \) is the logic which allows countably infinite conjunctions and disjunctions but only finite quantification. If the conjunctions and disjunctions of a formula \( \varphi \) are all over computable sets of indices for formulas, then we say that \( \varphi \) is computable.

We use the following recursive definition to define the complexity of classes:

- An \( \mathcal{L}_{\omega_1 \omega} \) formula is both \( \Sigma^0_0 \) and \( \Pi^0_0 \) if it is quantifier free and does not contain any infinite disjunction or conjunction.
- An \( \mathcal{L}_{\omega_1 \omega} \) formula is \( \Sigma^0_\alpha \) if it is a countable disjunction of formulas of the form \( \exists x \phi \) where each \( \phi \) is \( \Pi^0_\beta \) for some \( \beta < \alpha \).
- An \( \mathcal{L}_{\omega_1 \omega} \) formula is \( \Pi^0_\alpha \) if it is a countable disjunction of formulas of the form \( \forall x \phi \) where each \( \phi \) is \( \Sigma^0_\beta \) for some \( \beta < \alpha \).
We say a formula is d-$\Sigma^0_{\alpha}$ if it is a conjunction of a $\Sigma^0_{\alpha}$ formula and a $\Pi^0_{\alpha}$ formula.

Scott [Sco65] showed that if $\mathcal{A}$ is a countable structure in a countable language, then there is a sentence $\varphi$ of $L_{\omega_1\omega}$ whose countable models are exactly the isomorphic copies of $\mathcal{A}$. Such a sentence is called a Scott sentence for $\mathcal{A}$. We remark that because $\alpha \land \neg(\beta \land \neg\gamma)$ is equivalent to $(\alpha \land \neg\beta) \lor (\alpha \land \gamma)$, the complexity classes $n\Sigma^0_{\alpha}$ of Scott sentences collapse for $n \geq 2$.

We can measure the complexity of a countable structure by looking for a Scott sentence of minimal complexity, as measured by the quantifier complexity hierarchy of computable formulas described above. [Mil78] showed that if $\mathcal{A}$ has a $\Pi^0_{\alpha}$ Scott sentence and a $\Sigma^0_{\alpha}$ Scott sentence, then it must have a d-$\Sigma^0_{\beta}$ Scott sentence for some $\beta < \alpha$. So for a given structure, the optimal Scott sentence is $\Sigma^0_{\alpha}$, $\Pi^0_{\alpha}$, or d-$\Sigma^0_{\alpha}$ for some $\alpha$.

We refer the interested readers to Chapter 6 of [AK00] for a more complete description of $L_{\omega_1\omega}$ formulas and Scott sentences.

4.1.2 Index Set Complexity

Given a structure $\mathcal{A}$ and a Scott sentence $\varphi$ for $\mathcal{A}$, we want to determine whether $\varphi$ is an optimal Scott sentence for $\mathcal{A}$, or whether there is a simpler Scott sentence which we have not yet found. We can use index set calculations to resolve this problem.

**Definition 4.1.1.** Let $\mathcal{A}$ be a structure. The index set $I(\mathcal{A})$ is the set of all indices $e$ such that the $e$th Turing machine $\Phi_e$ gives the atomic diagram of a structure $\mathcal{B}$ isomorphic to $\mathcal{A}$. We can also relativize this to any set $X$: $I^X(\mathcal{A})$ is the set of all indices $e$ such that the $e$th Turing machine $\Phi_e^X$ with oracle $X$ gives the atomic diagram of a structure $\mathcal{B}$ isomorphic to $\mathcal{A}$.

There is a connection between index sets and Scott sentences:

**Proposition 4.1.2.** If a countable structure $\mathcal{A}$ has an $X$-computable $\Sigma^0_{\alpha}$ (respectively $\Pi^0_{\alpha}$ or d-$\Sigma^0_{\alpha}$) Scott sentence, then the index set $I^X(\mathcal{A})$ is in $\Sigma^0_{\alpha}(X)$ (respectively $\Pi^0_{\alpha}(X)$ or d-$\Sigma^0_{\alpha}(X)$).

So if, for example, we have a computable $\Sigma^0_{\alpha}$ Scott sentence for a structure $\mathcal{A}$, we will try to show that the index set $I(\mathcal{A})$ is $\Sigma^0_{\alpha}$ $m$-complete. If we can do this, then we know that our Scott sentence is optimal. In general, any $L_{\omega_1\omega}$ sentence is $X$-computable for some $X$.

4.1.3 Summary of Prior Results

There are many results using the strategy above to find the complexities of optimal Scott sentences of structures. For example, Knight et al. [CHKM06], [CHK+12] determined the complexities of optimal Scott sentences for finitely generated free abelian groups, reduced abelian groups, free groups, and many other structures.
However, this strategy does not work when the complexity of the optimal Scott sentence is strictly higher than the complexity of the index set. Indeed, Knight and McCoy gave the first such example in [KM14], showing there is a subgroup $G$ of $\mathbb{Q}$ such that $I(G)$ is $\Sigma^0_2$-d, but it has no computable $d$-$\Sigma^0_2$ Scott sentence.

It was observed in [KS] that any computable finitely generated group, and indeed any computable finitely generated structure, has a computable $\Sigma^0_3$ Scott sentence. In [Ho], it was shown that many classes of “nice” groups in the sense of geometric group theory, including polycyclic groups (which includes nilpotent groups and abelian groups), and certain solvable groups all have computable $d$-$\Sigma^0_2$ Scott sentence. However, none of these examples achieves the $\Sigma^0_2$ bound that was given in [KS].

4.1.4 New Results

In this paper, we give an example of a finitely-generated group which has no $d$-$\Sigma^0_2$ Scott sentence. As mentioned above, we do this by showing that the index set is $\Sigma^0_3$ m-complete.

**Theorem 4.1.3.** There is a finitely-generated computable group $G$ which has no $d$-$\Sigma^0_2$ Scott sentence. The index set of $G$ is $\Sigma^0_3$ m-complete, relative to any set.

The proof is in two parts. First, in Section 4.2, we develop some general results on when a finitely generated structure of any kind has a $d$-$\Sigma^0_2$ Scott sentence. These results are of interest independent of their application to groups.

**Definition 4.1.4.** Let $A$ be a finitely generated structure. Then $A$ is self-reflective if it contains a proper $\Sigma^0_1$-elementary substructure isomorphic to itself. ($B$ is a $\Sigma^0_1$-elementary substructure of $A$, and we write $B \preceq_1 A$, if, for each existential formula $\varphi(\bar{x})$ and $\bar{b} \in B$, $A \vDash \varphi(\bar{b})$ if and only if $B \vDash \varphi(\bar{b})$).

We prove, using an index-set calculation, the equivalence of (1) and (2) in the following characterization of finitely-generated structures with no $d$-$\Sigma^0_2$ Scott sentence.

**Theorem 4.1.5.** Let $\mathcal{M}$ be a finitely generated structure. The following are equivalent:

1. $\mathcal{M}$ has a $d$-$\Sigma^0_2$ Scott sentence,
2. $\mathcal{M}$ is not self-reflective,
3. for all (or some) generating tuples of $\mathcal{M}$, the orbit is defined by a $\Pi^0_1$ formula.

The equivalence of (3) to (1) has been proved by Alvir, Knight, and McCoy [AKM].

Second, in Section 4.4, we apply this characterization to finitely generated groups. Using small cancellation theory and HNN extensions, we produce a computable group $G$ which is self-reflective. Thus—using Theorem 4.1.5—this group has no $d$-$\Sigma^0_2$ Scott sentence. Using the group ring construction, we generalize this in Section 4.5 to produce a ring which is self-reflective.
We also apply our results to finitely generated fields in Section 4.3. A simple argument shows that no finitely generated field is self-reflective. Thus:

\textbf{Theorem 4.1.6.} Every finitely generated field has a $d$-$\Sigma_2^0$ Scott sentence.

\subsection*{4.1.5 Open Questions}

We leave here several open questions. First, a special class of finitely generated groups are the finitely presented groups. Is there a (computable) finitely presented group with no $d$-$\Sigma_2^0$ Scott sentence?

\textbf{Question 4.1.7.} Does every finitely presented group with solvable word problem have a $d$-$\Sigma_2^0$ Scott sentence?

Second, one can consider structures other than fields and groups. A natural class to consider is rings. Using the group ring construction, we get a self-reflective ring. However, if we insist that the ring be commutative, then such a construction no longer works.

\textbf{Question 4.1.8.} Does every commutative ring have a $d$-$\Sigma_2^0$ Scott sentence?

One can also place further restrictions on the ring. A natural restriction is that there be no zero-divisors.

\textbf{Question 4.1.9.} Does every integral domain have a $d$-$\Sigma_2^0$ Scott sentence?

We expect the answer to be yes, as integral domains have a good dimension theory.

\section*{4.2 General Theory}

Our goal in this section is to prove Theorem 4.1.5. The proof is in two parts. First we will show that if $\mathcal{A}$ is not self-reflective, then it has a $d$-$\Sigma_2^0$ Scott sentence. Second, we will show that if $\mathcal{A}$ is self-reflective, then its index set is as complicated as possible.

\textbf{Theorem 4.2.1.} Let $\mathcal{A}$ be a finitely generated structure. If $\mathcal{A}$ is not self-reflective, then $\mathcal{A}$ has a $d$-$\Sigma_2^0$ Scott sentence.

\textit{Proof.} Let $\vec{g}$ be a generating tuple for $\mathcal{A}$. Let $p$ be the atomic type of $\vec{g}$. For any tuple $\vec{g}'$ satisfying $p$, the substructure generated by $\vec{g}$ is isomorphic to $\mathcal{A}$. Since $\mathcal{A}$ is not self-reflective, if $\vec{g}'$ does not generate $\mathcal{A}$, then there is a tuple $\vec{a}$ and a quantifier-free formula $\psi(\vec{x}, \vec{y})$ with $\mathcal{A} \models \psi(\vec{g}', \vec{a})$, such that there is no $\vec{b} \in \mathcal{A}$ such that $\mathcal{A} \models \psi(\vec{g}, \vec{b})$. Let $S$ be the set of formulas $\psi(\vec{x}, \vec{y})$ such that for some tuple $\vec{g}'$ satisfying the atomic type $p$ but not generating $\mathcal{A}$, and some $\vec{a}$, $\mathcal{A} \models \psi(\vec{g}', \vec{a})$, but there is no $\vec{b} \in \mathcal{A}$ such that $\mathcal{A} \models \psi(\vec{g}, \vec{b})$.

Using the set $S$, we can now define the Scott sentence for $\mathcal{A}$. The Scott sentence for $\mathcal{A}$ is the conjunction of the $\Sigma_2^0$ sentence which says:
there exists a tuple \( \bar{x} \) satisfying \( p \) and such that for all \( \bar{z} \) and \( \psi \in S \), \( \bar{x} \bar{z} \) does not satisfy \( \psi \), and the \( \Pi^0_2 \) sentence which says:

for all tuples \( \bar{x} \) which satisfy \( p \), either for all \( y, y \in (\bar{x}) \), or there is a formula \( \psi \in S \) and a tuple \( \bar{z} \) such that \( \bar{x}, \bar{z} \) satisfies \( \psi \).

This latter sentence is of the form \( (\forall \bar{x}) [\theta \rightarrow (\alpha \lor \beta)] \) where \( \theta \) is \( \Pi^0_1 \), \( \alpha \) is \( \Pi^0_2 \), and \( \beta \) is \( \Sigma^0_3 \).

It is easy to see that \( \mathcal{A} \) models this sentence. Now suppose that \( \mathcal{M} \) is any structure which satisfies this sentence. Since \( \mathcal{M} \) satisfies the \( \Sigma^0_2 \) part of the sentence, there is a tuple \( \bar{h} \in \mathcal{M} \) which satisfies the atomic type \( p \), and such that for all \( \bar{c} \in \mathcal{M} \) and \( \psi \in S \), \( \mathcal{M} \not\models \psi(\bar{h}, \bar{c}) \). We claim that \( \bar{h} \) generates \( \mathcal{M} \); since \( \bar{h} \) satisfies the atomic type \( p \), this would imply that \( \mathcal{M} \) is isomorphic to \( \mathcal{A} \). Indeed, by the \( \Pi^0_2 \) part of the sentence, either \( \bar{h} \) generates \( \mathcal{M} \) or there is a formula \( \psi \in S \) and a tuple \( \bar{c} \) such that \( \mathcal{M} \models \psi(\bar{h}, \bar{c}) \). The latter cannot happen, and so \( \bar{h} \) generates \( \mathcal{M} \). \( \square \)

We will now show that if \( \mathcal{A} \) is self-reflective, then (relativizing everything to \( \mathcal{A} \)) its index set is \( \Sigma^0_3 \) m-complete. We will use the following remark in the proof.

**Theorem 4.2.2.** Let \( \mathcal{A} \) be \( X \)-computable and self-reflective. Then \( I^X(\mathcal{A}) \) is \( \Sigma^0_3(X) \) m-complete (relative to \( X \)).

**Proof.** We will assume that \( \mathcal{A} \) is computable; the general result can be obtained by relativizing. Fix a \( \Sigma^0_3 \) set \( S \). We may assume that \( S \) is of the form

\[
n \in S \iff (\exists e) W_f(e, n) \text{ is infinite}
\]

for some computable function \( f \). We will define, uniformly in \( n \), a computable structure \( \mathcal{B}_n \) such that if \( n \in S \), then \( \mathcal{B}_n \simeq \mathcal{A} \), and if \( n \notin S \), then \( \mathcal{B}_n \) is not finitely generated. We may assume that at each stage \( s \), there is at most one \( e \) for which an element is enumerated into \( W_f(e, n) \).

For convenience, we will suppress \( n \), writing \( B \) for \( B_n \) and \( f(e) \) for \( f(e, n) \). We will build \( B \) with domain \( \omega \) as a union of finite substructures (in a finite sublanguage) \( B[s] \), viewing the language as a relation language as is usual for this kind of construction.

Since \( \mathcal{A} \) sits properly inside itself as a \( \Sigma^0_1 \)-elementary substructure, we can create an infinite chain

\[
\mathcal{A}_0 \prec \mathcal{A}_1 \prec \mathcal{A}_2 \prec \cdots \prec \mathcal{A}^\ast
\]

where each \( \mathcal{A}_i \) is (effectively) isomorphic to \( \mathcal{A} \) and \( \mathcal{A}_i \) is a c.e. (but not necessarily computable) subset of \( \mathcal{A}_{i+1} \). The structure \( \mathcal{A}^\ast \) is the union of all of the \( \mathcal{A}_i \)'s, and is not finitely generated (and hence not isomorphic to \( \mathcal{A} \)).

At each stage \( s \), the domain of \( B[s] \) will be the union of finitely many unary relations \( R_0[s] \subseteq \cdots \subseteq R_k[s] \). We will also have computable partial embeddings \( j[s] : B[s] \to \mathcal{A}^\ast \) such that \( j[s](R_k[s]) \subseteq \mathcal{A}_k \).

We will build \( R_0 \) isomorphic to \( \mathcal{A}_0 \), \( R_1 \) isomorphic to \( \mathcal{A}_1 \), and so on, via \( j \). While \( W_f(e) \) does not have any elements enumerated into it, we will keep building \( R_e \) to copy \( \mathcal{A}_e \). However,
when an element is enumerated into $W_{f(e)}$ we will collapse each $R_j$, $j > e$ into $R_e$. If $e$ is least such that $W_{f(e)}$ is infinite, then $B$ will consist just of the domain $R_e$, as each $R_j$, $j > e$, will be collapsed infinitely many times, and $B$ will be isomorphic to $A$. On the other hand, if each $W_{f(e)}$ is finite, then $B$ will be isomorphic to $A^*$, and hence $B$ will not be isomorphic to $A$.

**Construction.** Begin at stage 0 with $B[0]$ empty and $k_0 = 0$, with $R_0[0]$ empty.

**Action at stage** $s + 1 = 3t + 1$. Set $k = k_s$. We will have $k_{s+1} = k$. For each $n = 0,\ldots,k$, let $a_n$ be the first element of $A_n$ not in $j[s](R_n[s])$. Define $B[s + 1] \supseteq B[s]$ so that $j[s + 1] : B[s + 1] \to A^*$ is a partial embedding, extending $j[s]$, whose range also contains $a_0,\ldots,a_k$. Given $x \in B[s + 1]$, set $R_n[s + 1]$ to be $R_n[s]$ plus the elements $x$ such that $j(x)$ is among the first $s$ elements of $A_n$.

**Action at stage** $s + 1 = 3t + 2$. Set $k_{s+1} = k_s + 1$ and $j[s + 1] = j[s]$. Let $R_{k_{s+1}}$ be empty. For each $n = 0,\ldots,k_s$, let $R_n[s + 1] = R_n[s]$.

**Action at stage** $s + 1 = 3t + 3$. If for some $e < k_s$, an element entered $W_{f(e)}$ at stage $t$, do the following. Otherwise, do nothing. Let $k_{s+1} = e$. Let $\bar{u}$ be the elements of $R_e[s]$ and let $\bar{v}$ be the other elements of $B[s]$ which are not in $R_e[s]$. Let $\psi(\bar{x},\bar{y})$ be the conjunction of the atomic diagram of $B[s]$, so that $B[s] \models \psi(\bar{u},\bar{v})$. Then $A_{k_e} \models \psi(j[s](\bar{u}),j[s](\bar{v}))$. Since $j[s](\bar{u}) \in A_e \prec_1 A_{k_e}$, there is a tuple $\bar{a} \in A_e$ such that $A_e \models \psi(j[s](\bar{u}),\bar{a})$. Then define $R_e[s + 1] = R_e[s] \cup \{\bar{v}\}$ and define $j[s + 1] = j[s] |_{R_e[s]}$ to map $\bar{v}$ to $\bar{a}$. For $n < e$, define $R_n[s + 1] = R_n[s]$.

Note that at every stage $s$, $j[s](R_n) \subseteq A_n$.

**End construction.**

Let $k = \liminf_s k_s$. If $n \in S$, then $k$ is the least $e$ such that $W_{f(e)}$ is infinite. Otherwise, if $n \notin S$, then $k = \infty$.

**Claim 4.2.3.** Fix $n \leq k$. Let $s$ be a stage such that $k_s \geq n$ and after which no element is ever enumerated into $W_{f(e)}$ for any $e < n$. Then:

1. For all $t_2 > t_1 \geq s$, $R_n[t_1] \subseteq R_n[t_2]$ and $j[t_1] |_{R_n[t_1]} \subseteq j[t_2]$.
2. $R_n = \bigcup_{t \geq s} R_n[t]$ is a substructure (in the relational language) of $B$.
3. $j_n = \bigcup_{t \geq s} j[t] |_{R_n[t]}$ is an isomorphism between $R_n$ and $A_n$.

Given $m \leq n \leq k$, $R_m \subseteq R_n$.

**Proof.** (1) is easy to see from the construction. (2) is also clear. For (3) it remains to see that $j_n$ is surjective onto $A_n$. If $a \in A_n$ is the least element which is not in the image of $j$, then there is some stage $t \geq s$ at which each lesser element of $A_n$ is already in the image of $j[t]$, and $a$ is among the first $t$ elements of $A_n$. For each lesser element $a'$ of $A_n$, $a' = j[3t+1](b')$
for some \( b' \), and \( b' \in R_n[3t+1] \); hence \( j[t'](b') = a' \) at each later stage \( t' \geq 3t+1 \). Then at some stage, say, \( 3t+4 \), we put \( a \) into the image of \( j \), say with \( j(b) = a \), and we have \( b \in R_n[3t+4] \), so that \( j[t'](b) = a \) at each later stage \( t' \geq 3t+4 \). This is a contradiction; thus \( j \) contains all of \( A_n \) in its image. \( \Box \)

**Claim 4.2.4.** \( B = \bigcup_{n \leq k} R_n \).

*Proof.* If an element enters \( W_f(e) \) at stage \( t \), and no element ever enters \( W_f(e') \), for \( e' < e \), after stage \( t \), then \( B[3t+3] = R_e[3t+3] \subseteq R_e \). If \( k < \infty \), then there are infinitely many stages \( 3t+3 \) at which \( B[3t+3] = R_k[3t+3] \), and so \( B = R_k \). If \( k = \infty \), then there is a sequence \( (e_1,t_1),(e_2,t_2),(e_3,t_3), \ldots \), with \( e_1 < e_2 < e_3 < \ldots \) and \( t_1 < t_2 < t_3 < \ldots \), at which \( B[3t_i+3] \subseteq R_{e_i}[3t_i+3] \subseteq R_{e_i} \). Then \( B = \bigcup_{n \leq k} R_n \). \( \Box \)

**Claim 4.2.5.** If \( m \in S \), then \( B_m \cong A \).

*Proof.* We have \( k < \infty \). Then \( B_m = \bigcup_{n \leq k} R_n = R_k \), and \( R_k \) is isomorphic to \( A \) via \( j_k \). \( \Box \)

**Claim 4.2.6.** If \( m \notin S \), then \( B_m \) is not finitely generated.

*Proof.* Fix a tuple \( \bar{g} \in B_m \). Then \( \bar{g} \in R_n \) for some \( n \). Pick \( a \in A_{n+1} \setminus A_n \). Since \( a \notin A_n \), \( a \notin j(R_n) \). Thus there is \( h \in R_{n+1} \setminus R_n \) with \( j(h) = a \). Thus \( R_n \) is a proper substructure of \( B \). Since \( \bar{g} \in R_n \), \( \bar{g} \) cannot generate \( B \).

This completes the proof of the theorem. \( \Box \)

*Proof of (1)⇒(2) in Theorem 4.1.5.* Let \( A \) be a finitely generated self-reflective structure which has a \( d\Sigma_2^0 \) Scott sentence. Let \( X \geq_T A \) be such that this Scott sentence is \( X \)-computable. Then by Theorem 4.2.2, the index set \( I^X(A) \) is \( \Sigma^0_3(X) \) m-complete relative to \( X \), contradicting that \( I^X(A) \) is in \( d\Sigma_2^0(X) \).

**4.3 Finitely Generated Fields**

It is not hard to show that every finitely-generated field is self-reflective, and hence has a \( d\Sigma_2^0 \) Scott sentence.

*Proof of Theorem 4.1.6.* Let \( F \) be a finitely generated field of characteristic \( p \) which is possibly zero. We claim that \( F \) is not self-reflective, and hence by Theorem 4.2.1, \( F \) has a \( d\Sigma_2^0 \) Scott sentence.

Let \( \mathbb{F}_p \) be the prime field of characteristic \( p \). Write \( F = \mathbb{F}_p(a_1,\ldots,a_m,b_1,\ldots,b_n) \), with \( a_1,\ldots,a_m \) a transcendence basis for \( F \) over \( \mathbb{F}_p \), and let \( \varphi:F \to E \subseteq F \) be an isomorphism between \( F \) and a proper subfield \( E \) of \( F \). We claim that \( E \) is not a \( \Sigma_1^0 \)-elementary substructure of \( F \).

Let \( a'_1,\ldots,a'_m \) be the images of \( a_1,\ldots,a_m \) under \( \varphi \), and let \( b'_1,\ldots,b'_n \) be the images of \( b_1,\ldots,b_n \) under \( \varphi \). Since \( F \) and \( E = \mathbb{F}_p(a'_1,b'_1) \) are isomorphic, \( a'_1,\ldots,a'_m \) are a transcendence base for \( E \), and so \( \bar{a},\bar{b} \) are algebraic over \( \mathbb{F}_p(a',b') \). Thus the atomic type \( tp_{\Sigma_1}(\bar{a},\bar{b}/\mathbb{F}_p(a',b')) \)
is isolated by a formula \( \psi(\bar{a}', \bar{b}', \bar{x}, \bar{y}) \). We claim that there is no tuple \( \bar{c}, \bar{d} \in E \) with \( E \models \psi(\bar{a}', \bar{b}', \bar{c}, \bar{d}) \). Suppose to the contrary that there was such a tuple \( \bar{c}, \bar{d} \); then \( E(c, d) \) would be isomorphic to \( F \) over \( E \); but since \( \bar{c}, \bar{d} \in E \), \( E(c, d) = E \), and so \( E \) is isomorphic to \( F \) over \( E \). This cannot happen as \( E \) is a proper subfield of \( F \). This is a contradiction; thus \( E \) is not a \( \Sigma^0_1 \)-elementary substructure of \( F \), proving the theorem. \( \square \)

4.4 Finitely Generated Groups

In this section, we first introduce some group theory background on HNN extensions (Section 4.4.1) and small cancellation theory (Section 4.4.2). Then we will use this machinery to construct a self-reflective group in Section 4.4.3. We refer the interested reader to [LS01, §IV, §V] for more details on the group theoretic tools we are using here.

4.4.1 HNN Extensions

**Definition 4.4.1.** For a group \( G \) with presentation \( G = \langle S \mid R \rangle \) and an isomorphism \( \alpha : H \to K \) between two subgroups \( H, K \subseteq G \), we define the **HNN extension** of \( G \) by \( \alpha \) to be

\[
G*_{\alpha} = \langle S, t \mid R, \{th^{-1} = \alpha(h)\}_{h \in H} \rangle.
\]

The key lemma about HNN extensions we will need is the following, which says every trivial word in the HNN extension is either already trivial in \( G \), or “reducible” by a conjugation of \( t \) or \( t^{-1} \).

**Lemma 4.4.2 (Britton’s Lemma).** With the notation above, let

\[
w = g_0 t^{e_1} g_1 t^{e_2} \cdots t^{e_n} g_n \in G*_{\alpha}
\]

with \( g_i \in G \), and \( e_i = \pm 1 \). Suppose \( w = 1 \), then one of the following is true:

1. \( n = 0 \) and \( g_0 = 1 \) in \( G \),
2. there is \( k \) such that \( e_k = 1 \), \( e_{k+1} = -1 \), and \( g_k \in H \), or
3. there is \( k \) such that \( e_k = -1 \), \( e_{k+1} = 1 \), and \( g_k \in K \).

One can show using Britton’s Lemma that the natural homomorphism from \( G \) to \( G*_{\alpha} \) is an embedding, so that we can think of \( G \) as a subgroup of \( G*_{\alpha} \).

4.4.2 Small Cancellation

**Definition 4.4.3.** We say a presentation \( \langle S \mid R \rangle \) is **symmetrized** if every relation is cyclically reduced and the relation set \( R \) is closed under inverse and cyclic permutation.
Let $\langle S \mid R \rangle$ be a symmetrized presentation. We say a word $u \in F(S)$ is a piece if there are two $r_1 \neq r_2 \in R$ such that $u$ is an initial subword of both $r_1$ and $r_2$. We also say the presentation satisfies the $C'(\lambda)$ small cancellation hypothesis if for every relation $r$ and every piece $u$ with $r = uv$, we have $|u| < \lambda|r|$. Furthermore, we shall say a non-symmetrized presentation satisfies the small cancellation hypothesis if it does once we replace the relation set with its symmetrized closure. We shall also say a group is a small cancellation when it is clear which presentation we are using.

The key lemma we will need for small cancellation groups is the following, which says that every presentation of the trivial word must contain a long common subword with a relator.

**Lemma 4.4.4** (Greendlinger’s Lemma). Let $G = \langle S \mid R \rangle$ be a $C'(\lambda)$ small cancellation group with $0 \leq \lambda \leq \frac{1}{6}$. Let $w$ be a non-trivial freely reduced word representing the trivial element of $G$. Then there is a cyclic permutation $r$ of a relation in $R$ or its inverse with $r = uv$ such that $u$ is a subword of $w$, and $|u| > (1 - 3\lambda)|r|$. We say that a word $w$ is Dehn-minimal if it does not contain any subword $v$ that is also a subword of a relator $r = vu$ such that $|v| > |r|/2$. Greendlinger’s lemma implies that, given a $C'(1/6)$ presentation of a group, we can solve the word problem using the following observation: a Dehn-minimal word is equivalent to the identity if and only if it is the trivial word. Given a word $w$, we replace $w$ by equivalent words of shorter length until we have replaced $w$ by a Dehn-minimal word $w'$. Then $w$ is equivalent to the identity if and only if $w'$ is the trivial word. This is the Dehn’s algorithm.

### 4.4.3 A Self-Reflective Group

Let $T$ be the tree (directed acyclic graph) with vertex set $V(T) = \{(n, \tau) \mid n \in \omega \text{ and } \tau \in \mathbb{Z}^\omega\}$. The parent $(n, \tau)^{-}$ of $(n, \tau)$ is $(n, \tau^{-})$ if $\tau \neq (\cdot)$, and $(n+1,(\cdot))$ otherwise. See Figure 4.1.

Let $u(x, y) = xyx^2y\cdots x^{100}y$ be a word in $F(x, y)$. Let $K$ be the group on generators $V(T) \cup \{a\} \cup B$ (where $B = \{b_i \mid i \in \mathbb{Z}\}$) with relations:

- $u((n, \tau), a) = (n, \tau^{-})$ for every $(n, \tau) \in T$.
- $u((n, \tau), b_i) = (n, \tau^{\cdot}(i))$ for every $(n, \tau) \in T$ and $i \in \mathbb{Z}$.

Note that $K$ is generated by $(0,(\cdot)), a,$ and $B$: we can generate any vertex $(n,(\cdot))$ by $(1,(\cdot)) = u((0,(\cdot)), a), (2,(\cdot)) = u((1,(\cdot)), a),$ and so on, and then we can generate, for example, $(2,(5,3))$, as $(2,(5,3)) = u(u((2,(\cdot)), b_3), b_3)$. Also note that $K$ is a $C'(\frac{1}{10})$ small cancellation group. Noting that any reduced word in $B$ is Dehn-minimal, we see that $B$ freely generates a free subgroup of $K$.

**Claim 4.4.5.** Let $v$ be a word in $V(T), a, B$, such that $v$ is Dehn-minimal. Then $v$ is in the subgroup $F(B)$ of $K$ generated by $B$ if and only if $v$ is a word in $B$. 
Figure 4.1: The tree $T$. 
Proof. The if direction is obvious. For the only if direction, assume we have a word \( v \), in \( V(T) \), \( a \), and \( B \), which is equal to a reduced word \( v' \) in \( B \). If \( v' \) was the trivial word, then since \( v \) is Dehn-minimal, \( v \) would also be the trivial word. So we may assume that \( v' \) is not the trivial world. Also, we may assume without loss of generality that \( v \) and \( v' \) have no common prefix, so that \( v^{-1}v' \) is a reduced word. Then, by applying Greendlinger’s lemma to \( v^{-1}v' \), we get a subword \( u \) of \( v^{-1}v' \) which is also a subword of a relator \( r \), with \( |u| > (\frac{r}{10})|r| \).

Noting that none of the relators of \( K \) has two consecutive \( b \)'s, we see that the subword \( u \) of \( v^{-1}v' \) given by Greendlinger’s lemma has to be contained in \( v^{-1} \) except possibly the last letter of \( u \). If \( u' \) is the part of \( u \) which is contained in \( v^{-1} \), we have \( |u'| \geq |u| - 1 \geq \frac{1}{2} |r| \) as \( |r| > 100 \). This contradicts the Dehn-minimality of \( v \). □

Now let \( G \) be the HNN extension \( \langle K, t \mid tb_it^{-1} = \alpha(b_i) = b_{i+1} \rangle \) of \( K \) by the partial isomorphism \( \alpha(b_i) = b_{i+1} \). \( G \) is then finitely-generated by \( (0,()) \), \( a, b_0 \), and \( t \).

Lemma 4.4.6. \( G \) is self-reflective.

Proof. Let \( H \subseteq G \) be the subgroup generated by \( (1,()) \), \( a, b_0 \), and \( t \). We claim that \( H \) is a proper \( \Sigma_2 \)-elementary subgroup of \( G \) which is isomorphic to \( G \).

Claim 4.4.7. \( H \) is isomorphic to \( G \).

Proof. Define the homomorphism \( \iota : G \to H \subseteq G \) given by sending \( (n, \tau) \) to \( (n + 1, \tau) \) and fixing \( a, b_i \), and \( t \). We must check that this does indeed define a homomorphism:

- \( u(\iota(n, \tau), \iota(a)) = u((n + 1, \tau), a) = (n + 1, \tau)^{-1} = \iota((n, \tau)^{-1}) \) for every \( (n, \tau) \in T \).
- \( u(\iota(n, \tau), \iota(b_i)) = u((n + 1, \tau), b_i) = (n + 1, \tau^{-1}(i)) = \iota((n, \tau^{-1}(i))) \)
- \( \iota(t)\iota(b_i)\iota(t)^{-1} = tb_it^{-1} = b_{i+1} = \iota(b_{i+1}) \)

Since \( \iota \) maps relators of \( G \) to relators of \( G \), it defines a homomorphism.

Now we will check that \( \iota \) is an embedding. Suppose \( \iota(v) = 1 \) for some word \( v \) in \( V(T), a, B, t \). Without loss, we may assume \( v \neq 1 \) is a word of minimum length among the words representing the same group element. By abusing notation, we will use \( \iota(v) \) to denote the word obtained by replacing each \( (n, \sigma) \) in \( v \) by \( (n + 1, \sigma) \); this is a word that represents the group element \( \iota(v) \).

Now since \( \iota(v) = 1 \), by Britton’s lemma, either \( \iota(v) \) does not contain \( t, t^{-1} \), or it contains a subword \( tut^{-1} \) or \( t^{-1}ut \) with \( u \in F(B) \). We claim that we must be in the first case, where \( \iota(v) \) (and hence \( v \)) does not contain \( t \) or \( t^{-1} \). In the second case, if \( \iota(v) \) does contain a subword \( tut^{-1} \) or \( t^{-1}ut \) with \( u \in F(B) \), we can write \( u = \iota(w') \), where \( tu't^{-1} \) or \( t^{-1}u't \) appears as a subword of \( v \) as \( \iota \) leaves \( t \) unchanged. If we can show that \( u \) is Dehn-minimal and so, by Lemma 4.4.5, \( u \) is actually a word in \( B \), then, as \( \iota \) leaves \( B \) unchanged, \( u' \) would also be a word in \( B \). By conjugating each \( b_i \) in \( u' \) by \( t \) (or \( t^{-1} \)) to get \( b_{i+1} \) (or \( b_{i-1} \)), we get a shorter word representing the same element, contradicting the minimality of \( v \). We will now argue that \( u \) is Dehn-minimal. If \( u \) was not Dehn-minimal, this would be witnessed by a subword
w of a relator r, with $|w| > \frac{1}{2}|v|$. Then looking at all of relators of K, we see that $w = \iota(w')$ and $r = \iota(r')$ where $w'$ is a subword of a relator $r'$ of K, and also a subword of $u'$. Thus $v$ is not of minimal length, a contradiction. So $u$ is Dehn-minimal, and so $\iota(v)$ does not contain $t,t^{-1}$.

Since $\iota(v) = 1$ and contains only $V(T)$, $a$, and $B$, by Greendlinger’s lemma, $\iota(v)$ is not Dehn-minimal. However, since any relator $r$ that holds on $\iota(V(T))$, $a$, and $B$ is the image, under $\iota$, of a relator that holds on $V(T)$, $a$, and $B$, this shows that $v$ is also not Dehn-minimal, a contradiction. \hfill \Box

Claim 4.4.8. $H$ is a proper subgroup of $G$.

Proof. We will show that $(0,\epsilon) \notin H$. Suppose $(0,\epsilon) \in H$. Choose a shortest spelling $v$ of $(0,\epsilon)$ in $\iota(V(T)),a,B,t$. By applying Britton’s lemma to $(0,\epsilon)^{-1}v$ and using the same argument as above, we see that $v$ does not contain $t$. Thus, we may apply Greendlinger’s lemma on $(0,\epsilon)^{-1}v$ to get a subword that is also a subword of some relator $r$ with length more than half of the length of $r$. However, this subword can not contain $(0,\epsilon)^{-1}$, as the only relation containing $(0,\epsilon)^{-1}$ but not $(0,\sigma)$ for any $\sigma \neq \epsilon$ is $u((0,\epsilon),a) = (1,\epsilon)$, but any long subword of it will contain more than one instance of $(0,\epsilon)$. Thus, the subword must be strictly in $v$, and contradicts the minimality of $v$. \hfill \Box

Claim 4.4.9. $H$ is a $\Sigma^0$-elementary subgroup of $G$.

Proof. We only need to show that for every tuple $\overline{g} \in G$, and every quantifier free formula $\psi(\overline{x},\overline{g})$ such that $G \models \psi((1,\epsilon),a,b_0,t,\overline{g})$, there is a tuple $\overline{h} \in H$ such that $H \models \psi((1,\epsilon),a,b_0,t,\overline{h})$. It suffices to show the (stronger) statement that for every finite subset $1 \notin S \subset G$, there is a (retraction) $\kappa : G \to G$ such that $\kappa|_H = \id_H$, $\kappa(G) = H$, and $1 \notin \kappa(S)$. Fixing a shortest spelling in $V(T),a,B,t$ for each element in $S$, we define $\kappa$ by fixing the generators of $H$ and sending $(0,\epsilon)$ to $(1,\sigma)$ for $n$ sufficiently large relative to the (length and subscripts of) spelling of elements of $S$.

Suppose there is $s \in S$ with $\kappa(s) = 1$. Write $s$ in the shortest spelling fixed above. We spell $\kappa(s)$ by replacing every $(0,\tau)$ in the shortest spelling of $s$ by $(1,n^{-1}\tau)$. By Britton’s lemma, either there is no $t$ in $\kappa(s)$, or there is a subword $tvt^{-1}$ or $t^{-1}vt$ with $v \in F(B)$. In the second case, by minimality of $s$ and Claim 4.4.5, we see that $v$ only contains letters $b_i$’s, and thus we can reduce the length of $s$ by replacing each $b_i$ by $b_{i+1}$ (or $b_{i-1}$) to get a shorter spelling of $s$, a contradiction. Thus, $s$ does not have any $t$ in it.

Now, applying Greendlinger’s lemma to $\kappa(s)$, we get a subword of $\kappa(s)$ that can be replaced by a shorter string. We will argue that a corresponding replacement can also be carried out for $s$, possibly with a different relator, contradicting the minimality of $s$. We divide into three cases, depending on which relator is used. First, note that the replacement cannot be given by any relator involving $b_m$ for $|m| \geq n$ since $n \gg 1$ implies $s$ does not contain the letter $b_m$ in it; thus the following three cases exhaust the possibilities.

Case 1. The relator is $u((1,\epsilon)^{-1}\sigma),a) = (1,\epsilon)^{-1}\sigma)$ for $|\epsilon| \geq n$. 
Since \( n \gg 1 \), each instance of \((1, (i)^{-} \sigma)\) in \( \kappa(s) \) comes from an instance of \((0, \sigma)\) in \( s \), and each instance of \((1, (i)^{-} \sigma)\) comes from an instance of \((0, \sigma)\) in \( s \). (It is important here that \((0,(i))^{-} = (1,(i))^{-}\).) Thus we can perform a replacement in \( s \) using the relator \( u((0, \sigma), a) = (0, \sigma)^{-} \).

**Case 2.** The relator is \( w((1, (i)^{-} \sigma), b_{k}) = (1, (i)^{-} \sigma \{k\}) \) for \( |i| \geq n \).

Since \( i \gg 1 \), each instance of \((1, (i)^{-} \sigma)\) in \( \kappa(s) \) comes from an instance of \((0, \sigma)\) in \( s \), and each instance of \((1, (i)^{-} \sigma \{k\})\) in \( \kappa(s) \) comes form an instance of \((0, \sigma \{k\})\) in \( s \). Thus we can perform a replacement in \( s \) using \( w((0, \sigma), b_{k}) = (0, \sigma \{k\}) \).

**Case 3.** The relator does not involve any letters \((1, (i)^{-} \sigma)\) with \(|i| \geq n\).

In this case, we can apply exactly the same relator to \( s \).

Thus we have shown that \( G \) contains a copy \( H \) of itself as a \( \Sigma_{1}^{0} \)-elementary subgroup, and hence is self-reflective.

**Proposition 4.4.10.** \( G \) is computable.

**Proof.** We use the following algorithm to solve the word problem in \( G \): for any string in \( V(T), a, B, t \), we search and replace the following three types of subwords:

1. \( tvt^{-1} \) with \( v \) containing only \( b_{i}'s \). Replace such subwords by deleting \( t \) and \( t^{-1} \) and replacing each \( b_{i} \) by \( b_{i-1} \).

2. \( t^{-1}vt \) with \( v \) containing only \( b_{i}'s \). Replace such subwords by deleting \( t^{-1} \) and \( t \) and replacing each \( b_{i} \) by \( b_{i+1} \).

3. Subword \( v \) such that \( v \) is also a subword of a relator \( r \) and \(|v| > \frac{1}{2}|r|\). Replace such subwords by the rest of the relator \( r \) after deleting \( v \).

Since any word can only mention finitely many letters, there are only finitely many possible relators for case (3). Thus, even though we have infinitely many relators, the search in (3) is still finite. Since these replacements shorten the length of the word, for any input word, sequences of such replacements terminate. If the resulting word is trivial, we output “The input word is equal to the identity.”, otherwise output “The input word does not equal the identity.”.

To verify this algorithm is valid, consider a word that represents the identity, on which the algorithms terminates with a non-trivial word \( v \). Since we can not perform any more replacement of the third kind, \( v \) is Dehn-minimal. Thus, by Lemma 4.4.5 and Britton’s lemma, we should be able to do a replacement of either the first or the second kind, a contradiction.
4.5 Finitely Generated Rings

In this section, we use the group ring construction to produce a ring that is self-reflective. Notice that the group ring $R[G]$ is computable if both $G$ and $R$ are.

**Theorem 4.5.1.** Let $G$ be the self-reflective group defined in Section 4.4. Then the group ring $R[G] = \{ f : G \to R \mid |\text{supp}(f)| < \infty \}$ over any finitely generated ring $R$ is also self-reflective.

*Proof.* Note that any endomorphism $\alpha$ of $G$ induces an endomorphism $\alpha^*$ of $R[G]$ by pre-composition and fixing $R$. Furthermore, if the endomorphism on $G$ is injective, then the induced endomorphism on $R[G]$ is also injective.

Let $\iota$ be as defined in Lemma 4.4.6. Then $\iota^*$, the induced endomorphism of $R[G]$, is also injective and not surjective. Call $B = \iota^*(R[G])$. Note that $B$ is just $R[H]$, where $H = \iota(G)$.

Now, as in Lemma 4.4.6, it suffices to show that for every finite subset $0 \notin T \subset R[G]$, there is a retraction $\beta : R[G] \to R[G]$ with $\beta(R[G]) = B$, $\beta|_B = \text{id}|_B$, and $0 \notin \beta(T)$. Let $U$ be all the group elements that appear in some members of $T$, and $S = \{ u_1 u_2^{-1} \mid u_1 \neq u_2 \in U \}$. Since $1 \notin S$, the proof of Lemma 4.4.6 gives a retraction $\kappa : G \to G$ such that $1 \notin \kappa(S)$. Now the induced endomorphism $\beta = \kappa^*$ is also a retraction. Furthermore, if $\kappa^*(t) = 0$ for some $t \in T$, since $1 \notin \kappa(S)$, $\kappa^*$ is injective on the support of $t$, thus we must have $t = 0$, a contradiction. Thus $R[G]$ is also self-reflective. \(\square\)
Chapter 5

A First-Order Theory of Ulm Type

The results presented in this chapter appeared in [HTc].

5.1 Introduction

The class of abelian $p$-groups is a well-studied example in computable structure theory. A simple compactness argument shows that abelian $p$-groups are not axiomatizable by an elementary first-order theory, but they are definable by the conjunction of the axioms for abelian $p$-groups (which are first-order $\forall \exists$ sentences) and the infinitary $\Pi^0_2$ sentence which says that every element is torsion of order some power of $p$.

Abelian $p$-groups are classifiable by their Ulm sequences [Ulm33]. Due to this classification, abelian $p$-groups are examples of some very interesting phenomena in computable structure theory and descriptive set theory. We will define a theory $T_p$ whose models behave like the class of abelian $p$-groups, giving a first-order example of these phenomena. In particular, Theorem 5.1.6 below answers a question of Knight.

5.1.1 Infinitary Formulas

The infinitary logic $\mathcal{L}_{\omega_1\omega}$ is the logic which allows countably infinite conjunctions and disjunctions but only finite quantification. If the conjunctions and disjunctions of a formula $\varphi$ are all over computable sets of indices for formulas, then we say that $\varphi$ is computable. We use $\Sigma^1_\alpha$ and $\Pi^1_\alpha$ to denote the classes of all infinitary $\Sigma_\alpha$ and $\Pi_\alpha$ formulas respectively. We will also use $\Sigma^c_\alpha$ and $\Pi^c_\alpha$ to denote the classes of computable $\Sigma_\alpha$ and $\Pi_\alpha$ formulas, where $\alpha < \omega_1^{CK}$ the least non-computable ordinal. See Chapter 6 of [AK00] for a more complete description of computable formulas.

5.1.2 Bi-Interpretability

One way in which we will see that the models of $T_p$ are essentially the same as abelian $p$-group is using bi-interpretations using infinitary formulas [Monb, HTMMM, HTMB].
structure $A$ is infinitary interpretable in a structure $B$ if there is an interpretation of $A$ in $B$ where the domain of the interpretation is allowed to be a subset of $B^{<\omega}$ and where all of the sets in the interpretation are definable using infinitary formulas. This differs from the classical notion of interpretation, as in model theory [Mar02, Definition 1.3.9], where the domain is required to be a subset of $B^n$ for some $n$, and the sets in the interpretation are first-order definable.

**Definition 5.1.1.** We say that a structure $A = (A; P_0^A, P_1^A, \ldots)$ (where $P_i^A \subseteq A^{a(i)}$) is **infinitary interpretable** in $B$ if there exists a sequence of relations $(\text{Dom}_A^B, \sim, R_0, R_1, \ldots)$, definable using infinitary formulas (in the language of $B$, without parameters), such that

1. $\text{Dom}_A^B \subseteq B^{<\omega}$,
2. $\sim$ is an equivalence relation on $\text{Dom}_A^B$,
3. $R_i \subseteq (B^{<\omega})^{a(i)}$ is closed under $\sim$ within $\text{Dom}_A^B$,

and there exists a function $f_B^A: \text{Dom}_A^B \rightarrow A$ which induces an isomorphism:

$$(\text{Dom}_A^B/\sim; R_0/\sim, R_1/\sim, \ldots) \cong (A; P_0^A, P_1^A, \ldots),$$

where $R_i/\sim$ stands for the $\sim$-collapse of $R_i$.

Two structures $A$ and $B$ are infinitary bi-interpretable if they are each effectively interpretable in the other, and moreover, the composition of the interpretations—i.e., the isomorphisms which map $A$ to the copy of $A$ inside the copy of $B$ inside $A$, and $B$ to the copy of $B$ inside the copy of $A$ inside $B$—are definable.

**Definition 5.1.2.** Two structures $A$ and $B$ are infinitary bi-interpretable if there are infinitary interpretations of each structure in the other as in Definition 5.1.1 such that the compositions

$$f_B^A \circ \tilde{f}_B^A: \text{Dom}_B^{(\text{Dom}_A^B)} \rightarrow B \quad \text{and} \quad f_B^A \circ \tilde{f}_B^A: \text{Dom}_B^{(\text{Dom}_A^B)} \rightarrow A$$

are definable in $B$ and $A$ respectively. (Here, $\text{Dom}_B^{(\text{Dom}_A^B)} \subseteq (\text{Dom}_B^B)^{<\omega}$ and $\tilde{f}_B^A: (\text{Dom}_A^B)^{<\omega} \rightarrow A^{<\omega}$ is the obvious extension of $f_B^A: \text{Dom}_A^B \rightarrow A$ mapping $\text{Dom}_B^{(\text{Dom}_A^B)}$ to $\text{Dom}_B^B$.)

If we ask that the sets and relations in the interpretation (or bi-interpretation) be (uniformly) relatively intrinsically computable, i.e., definable by both a $\Sigma_i^c$ formula and a $\Pi_i^c$ formula, then we say that the interpretation (or bi-interpretation) is effective. Any two structures which are effectively bi-interpretable have all of the same computability-theoretic properties; for example, they have the same degree spectra and the same Scott rank. See [Monb, Lemma 5.3].

Here, we will use interpretations which use (lightface) $\Delta^c_i$ formulas. It is no longer true that any two structures which are $\Delta^c_i$-bi-interpretable have all of the same computability-theoretic properties, but it is true, for example, that any two such structures either both have computable, or both have non-computable, Scott rank.
Theorem 5.1.3. Each abelian $p$-group is effectively bi-interpretable with a model of $T_p$. Each model of $T_p$ is $\Delta^2_2$-bi-interpretable with the disjoint union of an abelian $p$-group and a pure set.

This theorem will follow from the constructions in Sections 5.3 and 5.4. Given a model $M$ of $T_p$, $M$ is bi-interpretable with an abelian $p$-group $G$ and a pure set. The domain of the copy of $G$ inside of $M$ is definable by a $\Sigma^c_1$ formula but not by a $\Pi^c_1$ formula. This is the only part of the bi-interpretation which is not effective.

5.1.3 Classification via Ulm Sequences

Let $G$ be an abelian group. For any ordinal $\alpha$, we can define $p^\alpha G$ by transfinite induction:

- $p^0 G = G$;
- $p^{\alpha+1} G = p(p^\alpha G)$;
- $p^\beta G = \bigcap_{\alpha < \beta} p^\alpha G$ if $\beta$ is a limit ordinal.

These subgroups $p^\alpha G$ form a filtration of $G$. This filtration stabilizes, and we call the smallest ordinal $\alpha$ such that $p^\alpha G = p^{\alpha+1} G$ the length of $G$. We call the intersection $p^\infty G$ of these subgroups, which is a $p$-divisible group, the $p$-divisible part of $G$. Any countable $p$-divisible group is isomorphic to some direct product of the Prüfer group

$$\mathbb{Z}(p^\infty) = \mathbb{Z}[1/p, 1/p^2, 1/p^3, \ldots]/\mathbb{Z}.$$

Denote by $G[p]$ the subgroup of $G$ consisting of the $p$-torsion elements. The $\alpha$th Ulm invariant $u_\alpha(G)$ of $G$ is the dimension of the quotient

$$\left( p^\alpha G[p] / (p^{\alpha+1} G[p] \right)$$

as a vector space over $\mathbb{Z}/p\mathbb{Z}$.

Theorem 5.1.4 (Ulm’s Theorem, see [Fuc70]). Let $G$ and $H$ be countable abelian $p$-groups such that for every ordinal $\alpha$ their $\alpha$th Ulm invariants are equal, and the $p$-divisible parts of $G$ and $H$ are isomorphic. Then $G$ and $H$ are isomorphic.

5.1.4 Scott Rank and Computable Infinitary Theories

Scott [Sco65] showed that if $\mathcal{M}$ is a countable structure, then there is a sentence $\varphi$ of $\mathcal{L}_{\omega_1 \omega}$ such that $\mathcal{M}$ is, up to isomorphism, the only countable model of $\varphi$. We call such a sentence a Scott sentence for $\mathcal{M}$. There are many different definitions [AK00, Sections 6.6 and 6.7] of the Scott rank of $\mathcal{M}$, which differ only slightly in the ranks they assign. The one we will use, which comes from [Mon15b], defines the Scott rank of $\mathcal{A}$ to be the least ordinal $\alpha$ such that $\mathcal{A}$ has a $\Pi^\infty_\alpha$ Scott sentence. We denote the Scott rank of a structure $\mathcal{A}$ by $\text{SR}(\mathcal{A})$. It is always the case that $\text{SR}(\mathcal{A}) \leq \omega^4_1 + 1$ [Nad74]. We could just as easily use any of the other definitions of Scott rank; for all of these definitions, given a computable structure $\mathcal{A}$:
(1) \( \mathcal{A} \) has computable Scott rank if and only if there is a computable ordinal \( \alpha \) such that for all tuples \( \bar{a} \) in \( \mathcal{A} \), the orbit of \( \bar{a} \) is defined by a computable \( \Sigma_\alpha \) formula.

(2) \( \mathcal{A} \) has Scott rank \( \omega_1^{CK} \) if and only if for each tuple \( \bar{a} \), the orbit is defined by a computable infinitary formula, but for each computable ordinal \( \alpha \), there is a tuple \( \bar{a} \) whose orbit is not defined by a computable \( \Sigma_\alpha \) formula.

(3) \( \mathcal{A} \) has Scott rank \( \omega_1^{CK} + 1 \) if and only if there is a tuple \( \bar{a} \) whose orbit is not defined by a computable infinitary formula.

Given a structure \( \mathcal{M} \), define the computable infinitary theory of \( \mathcal{M} \), \( \text{Th}_\infty(\mathcal{M}) \), to be collection of computable \( L_{\omega_1\omega} \) sentences true of \( \mathcal{M} \). We can ask, for a given structure \( \mathcal{M} \), whether \( \text{Th}_\infty(\mathcal{M}) \) is \( \aleph_0 \)-categorical, or whether there are other countable models of \( \text{Th}_\infty(\mathcal{M}) \).

For \( \mathcal{M} \) a hyperarithmetic structure:

(1) If \( \text{SR}(\mathcal{M}) < \omega_1^{CK} \), then \( \text{Th}_\infty(\mathcal{M}) \) is \( \aleph_0 \)-categorical. Indeed, \( \mathcal{M} \) has a computable Scott sentence [Nad74].

(2) If \( \text{SR}(\mathcal{M}) = \omega_1^{CK} \), then \( \text{Th}_\infty(\mathcal{M}) \) may or may not be \( \aleph_0 \)-categorical [HTIK].

(3) If \( \text{SR}(\mathcal{M}) = \omega_1^{CK} + 1 \), then \( \text{Th}_\infty(\mathcal{M}) \) is not \( \aleph_0 \)-categorical as \( \mathcal{M} \) has a non-principal type which may be omitted.

In the case of abelian \( p \)-groups, we can say something even when we replace the assumption that \( \mathcal{M} \) is hyperarithmetic with the assumption that \( \omega_1^G = \omega_1^{CK} \).

**Definition 5.1.5** (Definition 6 of [FKM+11]). A class of countable structures has **Ulm type** if for any two structures \( \mathcal{A} \) and \( \mathcal{B} \) in the class, if \( \omega_1^A = \omega_1^B = \omega_1^{CK} \) and \( \text{Th}_\infty(\mathcal{A}) = \text{Th}_\infty(\mathcal{B}) \), then \( \mathcal{A} \) and \( \mathcal{B} \) are isomorphic.

It is well-known that abelian \( p \)-groups are of Ulm type; however, we do not know of a good reference with a complete proof, so we will give one in Section 5.2. We also note that there are indeed non-hyperarithmetic abelian \( p \)-groups \( G \) with \( \text{SR}(G) < \omega_1^{CK} \).

Knight asked whether there was a (non-trivial) first-order theory of Ulm type. By a non-trivial example, we mean that the elementary first-order theory should have non-hyperarithmetic models which are low for \( \omega_1^{CK} \). Our theory \( T_p \) is such an example.

**Theorem 5.1.6.** The models of \( T_p \) are of Ulm type. Moreover, given \( \mathcal{M} \models T_p \) with \( \omega_1^{CK} = \omega_1^M \) and \( \text{SR}(\mathcal{M}) < \omega_1^{CK} = \omega_1^M \), \( \text{Th}_\infty(\mathcal{M}) \) is \( \aleph_0 \)-categorical.

**Proof.** Let \( \mathcal{M} \) be a model of \( T_p \). Now \( \mathcal{M} \) is bi-interpretable, using computable infinitary formulas, with the disjoint union of an abelian \( p \)-group \( G \) and a pure set. Thus \( \mathcal{M} \) inherits these properties from \( G \) (see Theorem 5.2.1). \( \square \)

Of course, there will be non-hyperarithmetic models of \( T_p \) with Scott rank below \( \omega_1^{CK} \).
5.1.5 Borel Incompleteness

In their influential paper [FS89], Friedman and Stanley introduced Borel reductions between invariant Borel classes of structures with universe $\omega$ in a countable language. Such classes are of the form $\text{Mod}(\varphi)$, the set of models of $\varphi$ with universe $\omega$, for some $\varphi \in \mathcal{L}_{\omega_1\omega}$. A Borel reduction from $\text{Mod}(\varphi)$ to $\text{Mod}(\psi)$ is a Borel map $\Phi: \text{Mod}(\varphi) \to \text{Mod}(\psi)$ such that

$$\mathcal{M} \cong \mathcal{N} \iff \Phi(\mathcal{M}) \cong \Phi(\mathcal{N}).$$

If such a Borel reduction exists, we say that $\text{Mod}(\varphi)$ is Borel reducible to $\text{Mod}(\psi)$ and write $\varphi \leq_B \psi$. If $\varphi \leq_B \psi$ and $\psi \leq_B \varphi$, then we say that $\text{Mod}(\varphi)$ and $\text{Mod}(\psi)$ are Borel equivalent and write $\varphi \equiv_B \psi$. Friedman and Stanley showed that graphs, fields, linear orders, trees, and groups are all Borel equivalent, and form a maximal class under Borel reduction.

If $\text{Mod}(\varphi)$ is Borel complete, then the isomorphism relation on $\text{Mod}(\varphi) \times \text{Mod}(\varphi)$ is $\Sigma_1$-complete. The converse is not true, and the most well-known example is abelian $p$-groups, whose isomorphism relation is $\Sigma_1$-complete but not Borel complete. Until very recently, they were one of the few such examples, and there were no known examples of elementary first-order theories with similar properties. Recently, Laskowski, Rast, and Ulrich [URL] gave an example of a first-order theory which is not Borel complete, but whose isomorphism relation is not Borel. Our theory $T_p$ is another such example.

**Theorem 5.1.7.** The class of models of $T_p$ is Borel equivalent to abelian $p$-groups.

Because abelian $p$-groups are not Borel complete, but their isomorphism relation is $\Sigma_1$-complete, we get:

**Corollary 5.1.8.** The class of models of $T_p$ is not Borel complete but the isomorphism relation is $\Sigma_1$-complete.

Theorem 5.1.7 is a specific instance of the following general question asked by Friedman:

**Question 5.1.9.** Is it true that for every $\mathcal{L}_{\omega_1\omega}$ sentence there is a Borel equivalent first-order theory?

5.2 Proof of Theorem 5.2.1

In this section we will describe a proof of the following well-known theorem, which shows that abelian $p$-groups are of Ulm type.

**Theorem 5.2.1.** Let $G$ be an abelian $p$-group with $\omega_1^{CK} = \omega_1^G$. Then:

1. $G$ is the only countable model of $\text{Th}_\infty(G)$ with $\omega_1^G = \omega_1^{CK}$, and
2. if $\text{SR}(G) < \omega_1^{CK} = \omega_1^G$, then $\text{Th}_\infty(G)$ is $\aleph_0$-categorical.
CHAPTER 5. A FIRST-ORDER THEORY OF ULM TYPE

The proof of Theorem 5.2.1 consists essentially of expressing the Ulm invariants via computable infinitary formulas.

**Definition 5.2.2.** Let $G$ be an abelian $p$-group. For each ordinal $\alpha < \omega_1^{CK}$, there is a computable infinitary sentence $\psi_\alpha(x)$ which defines $p^\alpha G$ inside of $G$:

- $\psi_0(x) = x$;
- $\psi_{\alpha + 1}(x) = (\exists y)[\psi_\alpha(y) \land py = x]$;
- $\psi_\beta(x) = \bigwedge_{\alpha < \beta} \psi_\alpha(x)$ for limit ordinals $\beta$.

**Definition 5.2.3.** For each ordinal $\alpha < \omega_1^{CK}$ and $n \in \omega \cup \{\omega\}$, there is a computable infinitary sentence $\varphi_{\alpha,n}$ such that, for $G$ an abelian $p$-group,

$$G \vDash \varphi_{\alpha,n} \iff u_\alpha(G) = n.$$

For $n \in \omega$, define $\varphi_{\alpha,2n}$ to say that there are $x_1, \ldots, x_n$ such that:

- $\psi_\alpha(x_1) \land \cdots \land \psi_\alpha(x_n)$,
- $px_1 = \cdots = px_n = 0$, and
- for all $c_1, \ldots, c_n \in \mathbb{Z}/p\mathbb{Z}$ not all zero, $\neg \psi_{\alpha + 1}(c_1x_1 + \cdots + c_nx_n)$.

Then for $n \in \omega$, $\varphi_{\alpha,n}$ is $\varphi_{\alpha,2n} \land \neg \varphi_{\alpha,2n+1}$, and $\varphi_{\alpha,\omega}$ is $\bigwedge_{n \in \omega} \varphi_{\alpha,2n}$.

**Lemma 5.2.4** (Theorem 8.17 of [AK00]). Let $G$ be an abelian $p$-group. Then:

1. the length of $G$ is at most $\omega_1^G$, and
2. if $G$ has length $\omega_1^G$ then $G$ is not reduced (in fact, its $p$-divisible part has infinite rank) and $\text{SR}(G) = \omega_1^G + 1$.

We are now ready to give the proof of Theorem 5.2.1.

**Proof of Theorem 5.2.1.** Since $\omega_1^{CK} = \omega_1^G$, $G$ has length at most $\omega_1^{CK}$. Note that $\text{Th}_\infty(G)$ contains the sentences $\varphi_{\alpha,u_\alpha(G)}$ for $\alpha < \omega_1^{CK}$. Thus any model of $\text{Th}_\infty(G)$ has the same Ulm invariants as $G$, for ordinals below $\omega_1^{CK}$.

If $\text{SR}(G) < \omega_1^{CK}$, let $\lambda$ be the length of $G$. Then $\text{Th}_\infty(G)$ includes the computable formula $(\forall x)[\psi_\lambda(x) \leftrightarrow \psi_{\lambda+1}(x)]$, so that any countable model of $\text{Th}_\infty(G)$ has length at most $\lambda$. Note that in such a model, $\psi_\lambda$ defines the $p$-divisible part. Let $n \in \omega \cup \{\omega\}$ be such that $p^n G$ is isomorphic to $\mathbb{Z}(p^n \mathbb{Z})^n$. Then, if $n \in \omega$, $\text{Th}_\infty(G)$ contains the formula which says that there are $x_1, \ldots, x_n$ such that:

- $\psi_\lambda(x_1) \land \cdots \land \psi_\lambda(x_n)$,
• for all $c_1, \ldots, c_n < p$ not all zero and $k_1, \ldots, k_n \in \omega$, 
\[ \frac{c_1}{p^{k_1}} x_1 + \cdots + \frac{c_n}{p^{k_n}} x_n \neq 0, \]

• for all $y$ with $\psi_\lambda(y)$, there are $c_1, \ldots, c_n < p$ and $k_1, \ldots, k_n \in \omega$ such that 
\[ y = \frac{c_1}{p^{k_1}} x_1 + \cdots + \frac{c_n}{p^{k_n}} x_n. \]

If $n = \omega$, then $\text{Th}_\infty(G)$ contains the formula which says that for each $m \in \omega$, there are $x_1, \ldots, x_m$ such that

• $\psi_\lambda(x_1) \land \cdots \land \psi_\lambda(x_m)$, and

• for all $c_1, \ldots, c_m < p$ not all zero and $k_1, \ldots, k_m \in \omega$, 
\[ \frac{c_1}{p^{k_1}} x_1 + \cdots + \frac{c_m}{p^{k_m}} x_m \neq 0. \]

Any countable model of $\text{Th}_\infty(G)$ has $p$-divisible part isomorphic to $\mathbb{Z}(p^\infty)^n$. So any countable model of $\text{Th}_\infty(G)$ has the same Ulm invariants and $p$-divisible part as $G$, and hence is isomorphic to $\text{Th}_\infty(G)$. Hence $\text{Th}_\infty(G)$ is $\aleph_0$-categorical. This gives (2), and (1) for the case where $\text{SR}(G) < \omega_1^{CK}$.

If $\text{SR}(G) = \omega_1^{CK} + 1$, let $H$ be any other countable model of $\text{Th}_\infty(G)$ with $\omega_1^H = \omega_1^G = \omega_1^{CK}$. Thus $G$ and $H$ both have length $\omega_1^{CK}$ and their $p$-divisible parts have infinite rank. As remarked before, they have the same Ulm invariants, and so they must be isomorphic. This completes the proof of (1).

\[ \square \]

### 5.3 The Theory $T_p$

Fix a prime $p$. The language $L_p$ of $T_p$ will consist of a constant $0$, unary relations $R_n$ for $n \in \omega$, and ternary relations $P_{\ell,m}^n$ for $\ell, m \in \omega$ and $n \leq \max(\ell, m)$. The following transformation of an abelian $p$-group into an $L_p$-structure will illustrate the intended meaning of the symbols.

**Definition 5.3.1.** Let $G$ be an abelian $p$-group. Define $M(G)$ to be $L_p$-structure obtained as follows, with the same domain as $G$, and the symbols of $L_p$ interpreted as follows:

• Set $0_{M(G)}$ to be the identity element of $G$.

• For each $n$, let $R_n^{M(G)}$ be the elements which are torsion of order $p^n$.

• For each $\ell, m \in \omega$ and $n \leq \max(\ell, m)$, and $x, y, z \in G$, set $P_{\ell,m}^n(x,y,z)$ if and only if $x + y = z$, $x \in R_{\ell}^{M(G)}$, $y \in R_{m}^{M(G)}$, and $z \in R_n^{M(G)}$. 

**CHAPTER 5. A FIRST-ORDER THEORY OF ULM TYPE**
One should think of such $\mathcal{L}_p$-structures as the canonical models of $T_p$. The theory $T_p$ will consist of following axiom schemata:

(A1) For all $\ell, m, n \in \omega$:
\[
(\forall x \forall y \forall z) \left[ P^n_{\ell,m}(x, y, z) \rightarrow (R_\ell(x) \land R_m(x) \land R_n(z)) \right].
\]

(A2) $(R_n$ contains the elements which are torsion of order $p^n$.)
\[
(\forall x)[R_0(x) \leftrightarrow x = 0].
\]

and, for all $n \geq 1$:
\[
(\forall x)\left[ x \in R_n \leftrightarrow (\exists x_2 \cdots \exists x_{p-1})\left[ P^n_{n,n}(x, x_2, x_3) \land \cdots \land P^n_{n,n}(x, x_{p-1}, x_p) \right] \right].
\]

(A3) $(P$ defines a partial function.) For all $\ell, m, n, n' \in \omega$:
\[
(\forall x\forall y\forall z\forall z')\left[ (P^n_{\ell,m}(x, y, z) \land P^{n'}_{\ell,m}(x, y, z')) \rightarrow z = z' \right].
\]

(A4) $(P$ is total.) For all $\ell, m \in \omega$:
\[
(\forall x\forall y)\left[ (R_\ell(x) \land R_m(y)) \rightarrow \bigvee_{n \leq \max(\ell, m)} (\exists z) P^n_{\ell,m}(x, y, z) \right].
\]

(A5) $(Identity.)$ For all $\ell \in \omega$:
\[
(\forall x)[R_\ell(x) \rightarrow \left[ P^\ell_{0,\ell}(0, x, x) \land P^\ell_{0,0}(x, 0, x) \right]].
\]

(A6) $(Inverses.)$ For all $\ell \in \omega$:
\[
(\forall x)(\exists y)\left[ R_\ell(x) \rightarrow \left[ P^0_{\ell,\ell}(x, y, 0) \land P^0_{\ell,0}(y, x, 0) \right] \right].
\]

(A7) $(Associativity.)$ For all $\ell, m, n \in \omega$:
\[
(\forall x\forall y\forall z)\left[ \left( R_\ell(x) \land R_m(y) \land R_n(z) \right) \rightarrow \bigvee_{r \leq \max(\ell, m)} \left( \exists u \exists v \exists w \right) \left[ P^r_{\ell,m}(x, y, u) \land P^r_{\ell,n}(u, z, w) \land P^s_{m,n}(y, z, v) \land P^t_{l,s}(x, v, w) \right] \right].
\]

(A8) $(Abelian.)$ For all $\ell, m \in \omega$ and $n \leq \max(\ell, m)$:
\[
(\forall x\forall y\forall z)\left[ \left( R_\ell(x) \land R_m(y) \land R_n(z) \right) \land P^n_{\ell,m}(x, y, z) \right] \rightarrow P^m_{\ell,\ell}(y, x, z)].
CHAPTER 5. A FIRST-ORDER THEORY OF ULM TYPE

We must now check that the definition of $T_p$ works as desired, that is, that if $G$ is an abelian $p$-group, then $\mathfrak{M}(G)$ is a model of $T_p$.

**Lemma 5.3.2.** If $G$ is an abelian $p$-group, then $\mathfrak{M}(G) \models T_p$.

**Proof.** We must check that each instance of the axiom schemata of $T_p$ holds in $\mathfrak{M}(G)$.

(A1) Suppose that $x$, $y$, and $z$ are elements of $G$ with $P_{m,\ell}^{n,\mathfrak{M}(G)}(x, y, z)$. Then, by definition, $x + y = z$, $x \in R_{\ell}^{m,\mathfrak{M}(G)}(y) \in R_{m}^{n,\mathfrak{M}(G)}$, and $z \in R_{n}^{\mathfrak{M}(G)}$.

(A2) $R_{0}^{\mathfrak{M}(G)}$ contains the elements of $G$ which are torsion of order $p^0 = 1$, so $R_0$ contains just the identity. For each $n > 0$, $R_{n}^{\mathfrak{M}(G)}$ contains the elements of order $p^n$. An element $x$ has order $p^n$ if and only if $px$ has order $p^{n-1}$. It remains only to note that if $x$ has order $p^n$, then $x, 2x, 3x, \ldots, (p-1)x$ all have order $p^n$ as well. The existential quantifier is witnessed by $x_2 = 2x$, $x_3 = 3x$, and so on.

(A3) If, for some $x$, $y$, $z$, and $z'$, $P_{\ell,\mathfrak{M}(G)}^{n,\mathfrak{M}(G)}(x, y, z)$ and $P_{\ell,\mathfrak{M}(G)}^{n',\mathfrak{M}(G)}(x, y, z')$, then $x + y = z$ and $x + y = z'$, so that $z = z'$.

(A4) Given $x$ and $y$ in $G$ which are of order $p^m$ and $p^\ell$ respectively, $x + y$ is of order $p^n$ for some $n \leq \max(m, \ell)$, and so we have $P_{m,\ell}^{n,\mathfrak{M}(G)}(x, y, x + y)$.

(A5) If $x \in G$ is of order $p^\ell$, then $x + 0 = 0 + x = x$ and so we have $P_{\ell,0}^{\ell,\mathfrak{M}(G)}(x, 0, x)$.

(A6) If $x \in G$ is of order $p^\ell$, then $-x$ is also of order $p^\ell$, and $x + (-x) = 0 = (-x) + x$. So we have $P_{\ell,0}^{\ell,\mathfrak{M}(G)}(x, -x, 0)$.

(A7) Given $x, y, z \in G$ of order $p^\ell$, $p^m$, and $p^n$ respectively, there are $r \leq \max(\ell, m)$ and $s \leq \max(m, n)$ such that $x + y$ and $y + z$ are of order $p^r$ and $p^s$ respectively. Then there is $t$ such that $x + y + z$ is of order $p^t$; $t \leq \max(r, n)$ and $t \leq \max(\ell, s)$.

(A8) Given $x, y, z \in G$ of order $p^\ell$, $p^m$, and $p^n$ respectively, $n \leq \max(\ell, m)$, and with $x + y = z$, we have $y + x = z$ as $G$ is abelian.

Thus we have shown that $\mathfrak{M}(G)$ is a model of $T_p$. □

Note that $G$ and $\mathfrak{M}(G)$ are effectively bi-interpretable, proving one half of Theorem 5.1.3.

5.4 From a Model of $T_p$ to an Abelian $p$-Group

Given an abelian $p$-group $G$, we have already described how to turn $G$ into a model of $T_p$. In this section we will do the reverse by turning a model of $T_p$ into an abelian $p$-group.

**Definition 5.4.1.** Let $\mathcal{M}$ be a model of $T_p$. Define $\mathfrak{S}(\mathcal{M})$ to be the group obtained as follows.
CHAPTER 5. A FIRST-ORDER THEORY OF ULM TYPE

- The domain of \( \mathcal{G}(\mathcal{M}) \) will be the subset of the domain of \( \mathcal{M} \) given by \( \bigcup_{n \in \omega} R^\mathcal{M}_n \).
- The identity element of \( \mathcal{G}(\mathcal{M}) \) will be 0\( \mathcal{M} \).
- We will have \( x + y = z \) in \( \mathcal{G}(\mathcal{M}) \) if and only if, for some \( \ell, m, \) and \( n \), \( P^n_{\ell,m}(x, y, z) \).

We will now check that \( \mathcal{G}(\mathcal{M}) \) is always an abelian \( p \)-group.

**Lemma 5.4.2.** If \( \mathcal{M} \) is a model of \( T_p \), then \( \mathcal{G}(\mathcal{M}) \) is an abelian \( p \)-group.

**Proof.** First we check that the operation \( + \) on \( \mathcal{G}(\mathcal{M}) \) defines a total function. Given \( x, y \in \mathcal{G}(\mathcal{M}) \), choose \( \ell \) and \( m \) such that \( x \in R^\mathcal{M}_\ell \) and \( y \in R^\mathcal{M}_m \). Then by (A3) and (A4), there is a unique \( n \leq \max(\ell, m) \) and a unique \( z \) such that \( P^n_{\ell,m}(x, y, z) \). Thus \( x + y = z \), and \( z \) is unique.

Second, we check that \( \mathcal{G}(\mathcal{M}) \) is in fact a group. To see that 0\( \mathcal{M} \) is the identity, given \( x \in \mathcal{G}(\mathcal{M}) \), there is \( \ell \) such that \( x \in R^\mathcal{M}_\ell \). By (A5), \( P^\ell_{0,0}(x, 0^\mathcal{M}, x) \) and \( P^\ell_{0,0}(0^\mathcal{M}, x, 0^\mathcal{M}) \). Thus \( x + 0^\mathcal{M} = 0^\mathcal{M} + x = x \), and \( 0^\mathcal{M} \) is the identity of \( \mathcal{G}(\mathcal{M}) \). To see that \( \mathcal{G}(\mathcal{M}) \) has inverses, given \( x \in \mathcal{G}(\mathcal{M}) \), there is \( \ell \) such that \( x \in R^\mathcal{M}_\ell \), and by (A6) there is \( y \in R^\mathcal{M}_\ell \) such that \( P^\ell_{0,0}(y, x, 0^\mathcal{M}) \). Thus \( x + y = y + x = 0^\mathcal{M} \), and so \( y \) is the inverse of \( x \).

Finally, we see that \( \mathcal{G}(\mathcal{M}) \) is associative, given \( x, y, z \in \mathcal{G}(\mathcal{M}) \), there are \( \ell, m, \) and \( n \) such that \( x \in R^\mathcal{M}_\ell, y \in R^\mathcal{M}_m, \) and \( z \in R^\mathcal{M}_n \). Then by (A7) there are \( r, s, t, \) and \( u, v, \) and \( w, \) such that \( P^\ell_{r,s}(x, y, u), P^m_{t,u}(u, z, w), P^n_{v,w}(y, z, v), \) and \( P^n_{w,v}(x, v, w) \). Thus \( x + y = u, u + z = w, y + z = v, \) and \( x + v = w \). So \( (x + y) + z = x + (y + z) \). Thus \( \mathcal{G}(\mathcal{M}) \) is associative.

Third, to see that \( \mathcal{G}(\mathcal{M}) \) is abelian, let \( x, y \in \mathcal{G}(\mathcal{M}) \). There are \( \ell \) and \( m \) such that \( x \in R^\mathcal{M}_\ell \) and \( y \in R^\mathcal{M}_m \). Let \( n \leq \max(\ell, m) \) be such that \( z = x + y \in R^\mathcal{M}_n \). (Such an \( n \) and \( z \) exist by the arguments above that \( + \) is total, via (A3) and (A4).) Then \( P^n_{\ell,m}(x, y, z) \), and so by (A8), \( P^n_{\ell,m}(y, x, z) \). Thus \( y + x = z \) and so \( \mathcal{G}(\mathcal{M}) \) is abelian.

Finally, we need to see that \( \mathcal{G}(\mathcal{M}) \) is a \( p \)-group. We claim, by induction on \( n \geq 0 \), that \( R^\mathcal{M}_n \) consists of the elements of \( \mathcal{G}(\mathcal{M}) \) which are of order \( p^n \). From this claim, it follows that \( \mathcal{G}(\mathcal{M}) \) is a \( p \)-group. For \( n = 0 \), the claim follows directly from (A2). Given \( n > 0 \), suppose that \( x \in R^\mathcal{M}_n \). Then the witnesses \( x_2, x_3, \ldots, x_p \) to (A2) must be \( 2x, 3x, \ldots, px \). Note that since \( P^{n-1}_{n,n}(x, (p-1)x, px) \), \( px \in R^\mathcal{M}_{n-1} \). Thus \( px \) is of order \( p^{n-1} \), and so \( x \) is of order \( p^n \). On the other hand, if \( x \) is of order \( p^n \), then \( px \) is of order \( p^{n-1} \) and so \( px \in R^\mathcal{M}_{n-1} \). Moreover, \( x_2 = 2x, x_3 = 3x, \ldots, x_{p-1} = (p-1)x \) are all of order \( p^n \). So we have \( P^n_{n,n}(x, x, x_2), P^n_{n,n}(x, x_2, x_3), \ldots, P^{n-1}_{n,n}(x, x_{p-1}, x_p) \). By (A2), \( x \in R^\mathcal{M}_n \). This completes the inductive proof. 

We now have two operations, one which turns an abelian \( p \)-group into a model of \( T_p \), and another which turns a model of \( T_p \) into an abelian \( p \)-group. These two operations are almost inverses to each other. If we begin with an abelian \( p \)-group, turn it into a model of \( T_p \), and then that model into an abelian \( p \)-group, we will obtain the original group. However, if we start with a \( \mathcal{M} \) model of \( T_p \), turn it into an abelian \( p \)-group, and then turn that abelian \( p \)-group into a model of \( T_p \), we may obtain a different model of \( T_p \). The problem is that the
of elements of $\mathcal{M}$ which are not in any of the sets $R^M_n$ are discarded when we transform $\mathcal{M}$ into an abelian $p$-group. However, these elements form a pure set, and so the only pertinent information is their size.

**Definition 5.4.3.** Given a model $\mathcal{M}$ of $T_p$, the size of $\mathcal{M}$, $\#\mathcal{M} \in \omega \cup \{\infty\}$, is the number of elements of $\mathcal{M}$ not in any relation $R_n$.

**Lemma 5.4.4.** Given an abelian $p$-group $G$, $\mathcal{G}(\mathcal{M}(G)) = G$.

*Proof.* Since $\#\mathcal{M}(G) = 0$, we see that $G$, $\mathcal{M}(G)$, and $\mathcal{G}(\mathcal{M}(G))$ all have the same domain. The identity of $\mathcal{G}(\mathcal{M}(G))$ is $0^{\mathcal{M}(G)}$ which is the identity of $G$. If $x + y = z$ in $G$, then, for some $\ell, m, n \in \omega$, we have $P_{\ell,m}^{n,\mathcal{M}(G)}(x, y, z)$. Thus, in $\mathcal{G}(\mathcal{M}(G))$, we have $x + y = z$. So $\mathcal{G}(\mathcal{M}(G)) = G$. \qed

We make a simple extension to $\mathcal{M}$ as follows.

**Definition 5.4.5.** Let $G$ be an abelian $p$-group and $m \in \omega \cup \{\infty\}$. Define $\mathcal{M}(G, m)$ to be $\mathcal{L}_p$-structure with domain $G \cup \{a_1, \ldots, a_m\}$ with the relations interpreted as in $\mathcal{M}(G)$. Thus, no relations hold of any of the elements $a_1, \ldots, a_m$.

**Lemma 5.4.6.** Given a model $\mathcal{M}$ of $T_p$, $\mathcal{M}(G(M), \#\mathcal{M}) \cong \mathcal{M}$.

*Proof.* We will show that if $\#\mathcal{M} = 0$, then $\mathcal{M}(\mathcal{G}(\mathcal{M})) = \mathcal{M}$. From this one can easily see that $\mathcal{M}(\mathcal{G}(\mathcal{M}), \#\mathcal{M}) \cong \mathcal{M}$ in general.

If $\#\mathcal{M} = 0$, then $\mathcal{M}$, $\mathcal{G}(\mathcal{M})$, and $\mathcal{M}(\mathcal{G}(\mathcal{M}))$ all share the same domain. It is clear that $0^\mathcal{M} = 0^{\mathcal{G}(\mathcal{M})} = 0^{\mathcal{M}(\mathcal{G}(\mathcal{M}))}$. From the proof of Lemma 5.4.2, we see that for each $n$, $R^\mathcal{M}_n$ defines the set of elements of $\mathcal{G}(\mathcal{M})$ which are torsion of order $p^n$, and so $R^\mathcal{M}_n = R^\mathcal{G}(\mathcal{M})_n$. Given $\ell, m \in \omega$ and $n \leq \max(\ell, m)$, and $x$, $y$, and $z$ elements of the shared domain, we have $P_{\ell,m}^{n,\mathcal{M}}(x, y, z)$ if and only if $x + y = z$ in $\mathcal{G}(\mathcal{M})$ and $x \in R^\mathcal{M}_\ell$, $y \in R^\mathcal{M}_m$, and $z \in R^\mathcal{M}_n$. Since $R^\mathcal{M}_i = R^\mathcal{G}(\mathcal{M})_i$ for each $i$, this is the case if and only if $P_{\ell,m}^{n,\mathcal{M}(\mathcal{G}(\mathcal{M}))}(x, y, z)$. Thus we have shown that $\mathcal{M}(\mathcal{G}(\mathcal{M})) = \mathcal{M}$. \qed

Note that $\mathcal{M}$ and the disjoint union of $\mathcal{G}(\mathcal{M})$ with a pure set of size $\#\mathcal{M}$ are bi-interpretable, using computable infinitary formulas, completing the proof of Theorem 5.1.3.

### 5.5 Borel Equivalence

In this section we will prove Theorem 5.1.7 by showing that the class of models of $T_p$ and the class of abelian $p$-groups are Borel equivalent. $G \mapsto \mathcal{G}(\mathcal{M}(G)) = \mathcal{G}(\mathcal{M}(G,0))$ is a Borel reduction from isomorphism on abelian $p$-groups to isomorphism on models of $T_p$. However, $\mathcal{M} \mapsto \mathcal{G}(\mathcal{M})$ is not a Borel reduction in the other direction, because two non-isomorphic
models of $T_p$ might be mapped to isomorphic groups. We need to find a way to turn $\mathfrak{G}(M)$ and $\#M$ into an abelian $p$-group $\mathcal{H}(\mathfrak{G}(M), \#M)$, so that $M$ and $\#M$ can be recovered from $\mathcal{H}(\mathfrak{G}(M), \#M)$.

We will define $\mathcal{H}(G, m)$ for any abelian $p$-group $H$ and $m \in \omega \cup \{\infty\}$. It is helpful to think about what this reduction will do to the Ulm invariants: The first Ulm invariant of $\mathcal{H}(G, m)$ will be $m$, and for each $\alpha$, then $1 + \alpha$th Ulm invariant of $\mathcal{H}(G, m)$ will be the same as the $\alpha$th Ulm invariant of $G$.

**Definition 5.5.1.** Given an abelian $p$-group $G$, and $m \in \omega \cup \{\infty\}$, define an abelian $p$-group $\mathcal{H}(G, m)$ as follows. Let $\hat{B}$ be a basis for the $\mathbb{Z}_p$-vector space $G/pG$. Let $B \subseteq G$ be a set of representatives for $\hat{B}$. Let $G^*$ be the abelian group $(G, a_b : b \in B \mid pa_b = b)$. Then define $\mathcal{H}(G, m) = G^* \oplus (\mathbb{Z}_p)^m$.

To make this Borel, we can take $B$ to be the lexicographically first set of representatives for a basis. It will follow from Lemma 5.5.4 that the isomorphism type of $\mathcal{H}(G, m)$ does not depend on these choices. First, we require a couple of lemmas.

**Lemma 5.5.2.** Each element of $G$ can be written uniquely as a (finite) linear combination $h + \sum_{b \in B} x_b b$ where $h \in pG$ and each $x_b < p$.

*Proof.* Given $g \in G$, let $\hat{g}$ be the image of $g$ in $G/pG$. Then, since $\hat{B}$ is a basis for $G/pG$, we can write

$$\hat{g} = \sum_{b \in \hat{B}} x_b \hat{b}$$

with $x_b < p$, where $\hat{b}$ is the image of $b$ in $G/pG$. Thus setting

$$h = g - \sum_{b \in B} x_b b \in pG$$

we get a representation of $g$ as in the statement of the theorem.

To see that this representation is unique, suppose that

$$h + \sum_{b \in B} x_b b = h' + \sum_{b \in B} y_b b.$$  

Then, modulo $pG$,

$$\sum_{b \in B} x_b \hat{b} = \sum_{b \in B} y_b \hat{b}.$$  

Since $\hat{B}$ is a basis, $x_b = y_b$ for each $b \in B$. Then we get that $h = h'$ and the two representations are the same. \hfill $\Box$

**Lemma 5.5.3.** Each element of $G^*$ can be written uniquely in the form $h + \sum_{b \in B} x_b a_b$ where $h \in G$ and each $x_b < p$. 


Proof. It is clear that each element of \(G^*\) can be written in such a way. If
\[
  h + \sum_{b \in B} x_b a_b = h' + \sum_{b \in B} y_b a_b
\]
then, in \(G\),
\[
  ph + \sum_{b \in B} x_b b = ph' + \sum_{b \in B} y_b b.
\]
This representation is unique, so \(x_b = y_b\) for each \(b \in B\), and so \(h = h'\).

\(\square\)

**Lemma 5.5.4.** The isomorphism type of \(\mathcal{F}(G, m)\) depends only on the isomorphism type of \(G\), and not on the choice of \(B\).

**Proof.** It suffices to show that if \(C\) is another choice of representatives for a basis of \(G/pG\), then \(G^*_B = G^*_C\), where the former is constructed using \(B\), and the latter is constructed using \(C\). Let \(f : B \to C\) be an bijection.

Given \(g \in G^*_B\), write \(g = g' + \sum_{b \in B} x_b a_b\) with \(g' \in G\) and \(0 \leq x_b < p\). This representation of \(g\) is unique by Lemma 5.5.3. Define \(\varphi(g) = g' + \sum_{b \in B} x_b a_f(b)\). It is not hard to check that \(\varphi\) is a homomorphism. The inverse of \(\varphi\) is the map \(\psi\) which is defined by \(\psi(h) = h' + \sum_{c \in C} y_c a_{f^{-1}(c)}\), where \(h = h' + \sum_{c \in C} y_c a_c\).

The next two lemmas will be used to show that if \(G\) is not isomorphic to \(G'\), or if \(m\) is not equal to \(m'\), then \(\mathcal{F}(G, m)\) will not be isomorphic to \(\mathcal{F}(G', m')\).

**Lemma 5.5.5.** \(G = pG^*\).

**Proof.** Each element of \(G\) can be written as \(g + \sum_{b \in B} x_b b\) with \(g \in pG\). Let \(g' \in G\) be such that \(pg' = g\). Then
\[
  p(g' + \sum_{b \in B} x_b a_b) = g + \sum_{b \in B} x_b b.
\]
Hence \(G \subseteq pG^*\). Given \(h \in G^*\), write \(h = g + \sum_{b \in B} x_b a_b\). Then \(ph = pg + \sum_{b \in B} x_b b \in G\). So \(pG^* \subseteq G\), and so \(G = pG^*\).

If \(G\) is a group, recall that we denote by \(G[p]\) the elements of \(G\) which are torsion of order \(p\).

**Lemma 5.5.6.** \(\mathcal{F}(G, m)[p] / (p\mathcal{F}(G, m))[p] \cong (\mathbb{Z}_p)^m\).

**Proof.** Note that
\[
  \mathcal{F}(G, m)[p] / (p\mathcal{F}(G, m))[p] \cong (G^*[p] / (pG^*)[p]) \oplus ((\mathbb{Z}_p)^m[p] / (p(\mathbb{Z}_p)^m[p])
\]
\[
\cong (G^*[p] / G[p]) \oplus (\mathbb{Z}_p)^m.
\]
We will show that \((G^*[p] / G[p])\) is the trivial group by showing that if \(g \in G^*\), \(pg = 0\), then \(g \in G\). Indeed, write \(g = g' + \sum_{b \in B} y_b a_b\) with \(g' \in G\). Then
\[
0 = pg = pg' + \sum_{b \in B} py_b a_b = pg' + \sum_{b \in B} y_b b.
\]
Since $0 \in pG$ has a unique representation (by Lemma 5.5.2) $0 = 0 + \sum_{b \in B} 0b$, we get that $y_b = 0$ for each $b \in B$, and so $g = g' \in G$.

By the previous lemma, we can recover $m$ from $\mathcal{H}(G, m)$. We have

$$p\mathcal{H}(G, m) = pG^* \oplus p(\mathbb{Z}_p)^m \cong pG^* = G$$

so that we can also recover $G$.

Thus, using Lemma 5.4.6, $\mathcal{M} \mapsto \mathcal{H}(\mathcal{G}(\mathcal{M}), \#\mathcal{M})$ gives a Borel reduction from $T_p$ to abelian $p$-groups. This completes the proof of Theorem 5.1.7.
Part II

Structures on a Cone
Chapter 6

Degree Spectra of Relations

The results presented in this chapter appeared in [HTb].

6.1 Introduction

The aim of this monograph is to introduce the study of “nice” or “natural” relations on a computable structure via the technical device of relativizing to a cone of Turing degrees.

Let \( A \) be a mathematical structure, such as a graph, poset, or vector space, and \( R \subseteq A^n \) an additional relation on that structure (i.e., not in the diagram). The relation \( R \) might be the set of nodes of degree three in a graph or the set of linearly independent pairs in a vector space. The basic question we ask in the computability-theoretic study of such relations is: how do we measure the complexity of the relation \( R \)? One way to measure the complexity of \( R \) is the degree spectrum of \( R \). As is often the case in computability theory, many examples of relations with pathological degree spectra have been constructed in the literature but these tend to require very specific constructions. In this work, we restrict our attention to natural relations to capture those structures and relations which tend to show up in normal mathematical practice. We find that the degree spectra of natural relations are much better behaved than those of arbitrary relations, but not as well-behaved as one might hope.

The study of relations on a structure began with Ash and Nerode [AN81] who showed that, given certain assumptions about \( A \) and \( R \), the complexity of the formal definition of \( R \) in the logic \( L_{\omega_1\omega} \) is related to its intrinsic computability. For example, \( R \) has a computable \( \Sigma_n \) definition if and only if \( R \) is intrinsically \( \Sigma_n \), that is, for any computable copy \( B \) of \( A \), the copy of \( R \) in \( B \) is \( \Sigma_n \).

Harizanov [Har87] introduced the degree spectrum of \( R \) to capture a finer picture of the relation’s complexity. The degree spectrum of \( R \) is the collection of all Turing degrees of copies of the relation \( R \) inside computable copies \( B \) of \( A \). The degree spectra of particular relations have been frequently studied, particularly with the goal of finding as many possible different degree spectra as possible. For example, Harizanov [Har93] has shown that there is a \( \Delta^0_2 \) (but not c.e.) degree \( a \) such that \( \{0, a\} \) is the degree spectrum of a relation. Hirschfeldt
[Hir00] has shown that for any $\alpha$-c.e. degree $b$, with $\alpha \in \omega \cup \{\omega\}$, $\{0, b\}$ is the degree spectrum of a relation. Hirschfeldt has also shown that for any $\alpha$-c.e. degree $c$ and any computable ordinal $\alpha$, the set of $\alpha$-c.e. degrees less than or equal to $c$ is a degree spectrum. A number of other papers have been published showing that other degree spectra are possible—see for example Khoussainov and Shore [KS98] and Goncharov and Khoussainov [GK97].

These results require complicated constructions and one would not expect relations which one finds in nature to have such degree spectra. Instead, we expect to find simpler degree spectra such as the set of all c.e. degrees, the set of all d.c.e. degrees, or the set of all $\Delta^0_2$ degrees. The goal of this paper is to begin to answer the question of what sorts of degree spectra we should expect to find in nature. Since we cannot formally describe what we mean by a relation found in nature, we will prove our results relative to a cone, that is, relativized to any sufficiently high degree. One expects a result which holds on a cone to hold for any “nice” or “natural” relations and structures because natural properties tend to relativize. Such structures include vector spaces and algebraically closed fields, but not first-order arithmetic. We hope to be able to convince the reader that the study of relations relative to a cone is an interesting and useful way of approaching the study of relations that one might come across in nature. Our results are the beginning and there is a large amount of work still to be done. An interesting picture is already starting to emerge.

Before proceeding further, we will try to give an intuitive idea of what “on a cone” means in relation to computable structures. A property holds of a structure $A$ on a cone if it holds (relative to $X$) of all $X$-computable copies of $A$, for all $X$ on a cone (i.e., for all $X \geq_T Y$ for some set $Y$). One can view complexity in computable structure theory as coming from two sources: complexity coming from coding subsets of $\omega$ and structural complexity. For an example of what we mean, first consider a computable graph $G$ with a loop of length $n$ for each $n$. Let $S \subseteq \omega$ be any set, computable or non-computable. Let $R_S$ be the set of elements of $G$ that are contained in a loop of length $n$ for some $n \in S$. If $S$ is computable, then $R_S$ is computable; but if $S$ is non-computable, then $R_S$ is non-computable. The same is true in any computable copy of $G$. Thus $R_S$ codes $S$. And yet, if we relativize everything to the set $S$ (including looking at $S$-computable copies of $G$), then $R_S$ becomes computable in every copy. So $R_S$ is complicated only insofar as it codes $S$. On the other hand, consider the graph $H$ which has infinitely many vertices of degree zero, and infinitely many cycles of length three. Let $R$ be the relation consisting of all points in a cycle of length three. Then in some computable presentations of $H$, $R$ is computable; in others, $R$ is non-computable because we cannot decide that a point has degree zero just because we have not yet seen any edges from it. Even relativizing to any set $S$, there are still $S$-computable presentations of $H$ in which the relation $R$ is non-computable. $R$ does not code any non-computable subset of $\omega$; its complexity is all structural complexity. In general, a relation can have complexity of both types. By relativizing to a cone, we can ignore complexity which comes from coding sets, and focus on structural complexity. On a cone, we can have access to any fixed set of information, but we must use the same information to view all copies of the structure.

We will introduce the definition, suggested by Montalbán, of a degree spectrum of a relation on a cone. The results in this paper can be viewed as studying the partial ordering
of degree spectra on a cone. The following is a simplification of the full definition which will come later in Section 6.2.

**Definition 6.1.1** (Montalbán). Let $A$ and $B$ be structures with relations $R$ and $S$ respectively. We say that $R$ and $S$ have the same degree spectrum on a cone if there is a degree $d$ such that for all degrees $c \geq d$ (i.e., for all $c$ on a cone),

\[
\{d(R^A) \oplus c : \hat{A} \cong A \text{ and } \hat{A} \leq_T C\} = \{d(S^B) \oplus c : \hat{B} \cong B \text{ and } \hat{B} \leq_T C\}
\]

where $d(D)$ is the Turing degree of the set $D$.

We are particularly interested in whether or not there are “fullness” results for particular types of degree spectra, by which we mean results which say that degree spectra on a cone must contain many degrees. This is in opposition to the pathological examples of many small (even two-element) degree spectra that can be constructed when not working on a cone. There are a small number of previous “fullness” results, though not in the language which we use here, starting with Harizanov [Har91] who proved that (assuming $(\ast)$ below), as soon as a degree spectrum contains more than the computable degree, it must contain all c.e. degrees:

**Theorem 6.1.2** (Harizanov [Har91, Theorem 2.5]). Let $A$ be a computable structure and $R$ a computable relation on $A$ which is not relatively intrinsically computable. Suppose moreover that the effectiveness condition $(\ast)$ holds of $A$ and $R$. Then for every c.e. set $C$, there is a computable copy $B$ of $A$ such that $R^B \equiv_T C$.

$(\ast)$ For every $\bar{c}$, we can computably find $\bar{a} \in R$ such that for all $\bar{b}$ and quantifier-free formulas $\theta(\bar{z}, \bar{x}, \bar{y})$ such that $A \models \theta(\bar{c}, \bar{a}, \bar{b})$, there are $\bar{a}' \notin R$ and $\bar{b}'$ such that $A \models \theta(\bar{c}, \bar{a}', \bar{b}')$

The result is stated using the effectiveness condition $(\ast)$ which says that $R$ must be a nice relation in some particular way. When we relativize to a cone, the effectiveness condition becomes trivial, as we can relativize to a degree which can compute what we require in $(\ast)$ (about the fixed computable copy $A$). We are left with the statement:

**Corollary 6.1.3** (Harizanov). Relative to a cone, every degree spectrum either is the computable degree, or contains all c.e. degrees.

This result stands in contrast to the state of our knowledge of degree spectra when not working on a cone, where we know almost no restrictions on what sets of degrees may be degree spectra.

Ash and Knight tried to generalize Harizanov’s result in the papers [AK95] and [AK97]. They wanted to replace “c.e.” by “$\Sigma_\alpha$”. In our language of degree spectra on a cone, they wanted to show that every degree spectrum is either contained in the $\Delta_\alpha$ degrees or contains all of the $\Sigma_\alpha$ degrees. However, they discovered that this was false: there is a computable structure $A$ with a computable relation $R$ where $R$ is intrinsically $\Sigma_\alpha$, not intrinsically $\Delta_\alpha$, and for any computable copy $B$, $R^B$ is $\alpha$-CEA. Moreover, the proof of this relativizes.
So instead of asking whether “c.e.” can be replaced by \( \Sigma^0_\alpha \), Ash and Knight asked whether “c.e.” can be replaced by \( \alpha \text{-CEA} \). A set \( S \) is \( \alpha \text{-CEA} \) if there are sets \( S_0, S_1, S_2, \ldots, S_n = S \) such that \( S_0 \) is c.e., \( S_1 \) is c.e. in and above \( S_0, S_2 \) is c.e. in and above \( S_1, \) and so on. For now, the reader can ignore what this means for an infinite ordinal \( \alpha \) (the definition in general will follow in Section 6.2). Ash and Knight were able to show that Harizanov’s result can be extended in this manner when the coding is done relative to a \( \Delta^0_\alpha \)-complete set (note that every \( \Sigma^0_\alpha \) set, when joined with a \( \Delta^0_\alpha \)-complete set, becomes \( \alpha \text{-CEA} \); each \( \alpha \text{-CEA} \) set is already \( \Sigma^0_\alpha \)).

**Theorem 6.1.4** (Ash-Knight [AK97, Theorem 2.1]). Let \( A \) be a computable structure and \( R \) a computable relation on \( A \) which is not relatively intrinsically \( \Delta^0_\alpha \). Suppose moreover that the effectiveness condition \((***)\) holds of \( A \) and \( R \). Then for any \( \Sigma^0_\alpha \) set \( C \), there is a computable copy \( B \) of \( A \) such that

\[
R^B \oplus \Delta^0_\alpha \equiv_T C \oplus \Delta^0_\alpha
\]

where \( \Delta^0_\alpha \) is a \( \Delta^0_\alpha \)-complete set.

\((***)\) \( A \) is \( \alpha \)-friendly\(^1\) and that for all \( \bar{c} \), we can find \( \bar{a} \notin R \) which is effectively \( \alpha \)-free\(^2\) over \( \bar{c} \).

The reader need not worry about the effectiveness condition, which is somewhat technical; as before, after relativizing to a cone, the effectiveness condition becomes trivial. On a cone, this theorem says that either \( R \) is intrinsically \( \Delta^0_\alpha \) or for every \( \Sigma^0_\alpha \) set \( C \), there is a computable copy \( B \) of \( A \) such that

\[
R^B \oplus \Delta^0_\alpha \equiv_T C \oplus \Delta^0_\alpha.
\]

This is not enough to show that, on a cone, every degree spectrum is either contained in the \( \Delta_\alpha \) degrees or contains all of the \( \alpha \text{-CEA} \) degrees. A much better result, and one which would be sufficient, would be to show that \( R^B \equiv_T C \) rather than \( R^B \oplus \Delta^0_\alpha \equiv_T C \oplus \Delta^0_\alpha \). This was the goal of Knight in [Kni98] where she showed that it could be done with strong assumptions on the relation \( R \). The general question, without these strong assumptions, was left unresolved.

One of our main results in this paper in Section 6.6 is a positive answer to this question in the case \( \alpha = 2 \).

**Theorem 6.1.5.** Let \( A \) be a structure, and let \( R \) be an additional relation. Suppose that \( R \) is not intrinsically \( \Delta^0_2 \) on any cone. Then, on a cone, the degree spectrum of \( R \) contains the \( 2 \text{-CEA} \) sets.

\(^1\)A computable structure is \( \alpha \)-friendly if for \( \beta < \alpha \), the back-and-forth relations \( \leq_\beta \) are c.e. uniformly in \( \beta \). See [AK00, Section 15.2].

\(^2\)By a theorem of Barker, under certain effectiveness conditions such \( \bar{a} \) exist because \( R \) is not relatively intrinsically \( \Delta^0_\alpha \). See [AK00, page254] for the definition of \( \alpha \)-free. Note that, as an unfortunate consequence of the standard terminology, this notion of \( \alpha \)-free tuples is different from that of Section 6.5.
The proof uses an interesting method which we have not seen before and which we think is of independent interest. We will describe the method briefly here. During the construction, we are presented with two possible choices of how to continue, but it is not clear which will work. We are able to show that one of the two choices must work, but in order to find out which choice it is we must consider a game in which we play out the rest of the construction against an opponent who attempts to make the construction fail. By finding a winning strategy for this game, we are able to decide which choice to make.

Up to this point, degree spectra on a cone are looking very well-behaved, and in fact one might start to hope that they are linearly ordered. However, this is not the case as we see by considering Ershov’s hierarchy. Suppose (once again working on a cone) that there is a computable copy $B$ such that $R^B$ is not of c.e. degree. Is it necessarily the case that for every d.c.e. set $W$, there is a computable copy $C$ such that $R^C \equiv_T W$? We will show in Section 6.4 that this is not the case. Moreover, we will show that there is a computable structure $A$ with relatively intrinsically d.c.e. relations $R$ and $S$ which have incomparable degree spectra relative to every oracle.

**Theorem 6.1.6.** There is a computable structure $A$ and relatively intrinsically d.c.e. relations $R$ and $S$ such that neither $R$ nor $S$ are intrinsically of c.e. degree, even relative to any cone, and the degree spectra of $R$ and $S$ are incomparable relative to any cone (i.e., the degree spectrum of $R$ is not contained in that of $S$, and vice versa).

In proving this, we will also give a structural condition equivalent to being intrinsically of c.e. degree (which, as far as we are aware, is a new definition; we mean that in any computable copy, the relation has c.e. degree). The structural condition works for relations which are intrinsically d.c.e., and it does not seem difficult to extend it to work in more general cases.

The following is a summary of all that we know about the possible degree spectra of relations on a cone:

1. there is a smallest degree spectrum: the computable degree,
2. there is a smallest degree spectrum strictly containing the computable degree: the c.e. degrees (Corollary 6.1.3, see [Har91]),
3. there are two incomparable degree spectra both strictly containing the c.e. degrees and strictly contained in the d.c.e. degrees (Theorem 6.1.6),
4. any degree spectrum strictly containing the $\Delta^0_2$ degrees must contain all of the 2-CEA degrees (Theorem 6.1.5).

Figure 6.1 gives a graphical representation of all that is known. There are many more questions to be asked about degree spectra on a cone. In general, at least at the lower levels, there seem to be far fewer degree spectra on a cone than there are degrees (or degree spectra not on a cone). We expect this pattern to continue. However, an interesting phenomenon is
that not all degree spectra are “named” (e.g., as the $\Sigma^0_1$ or $\Delta^0_2$ degrees are), though perhaps this is just because we do not understand enough about them to name them. The hope would be to classify and name all of the possible degree spectra. The degree spectra from (3) give rise to new “natural” classes of degrees, where by natural we mean that they relativize.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{degree_spectra.png}
\caption{A visual summary of everything we know so far (including results from this paper) about the possible degree spectra of relations on a cone. The possible degree spectra are labeled or contained within one of the two enclosed areas (i.e., between $\Sigma^0_1$ and $\Delta^0_2$, or above 2-CEA). The two strictly d.c.e. degrees shown are incomparable.}
\end{figure}

In this paper, we will also consider the special case of the structure $(\omega, \lt)$. This special case has been studied previously by Downey, Khoussainov, Miller, and Yu [DKMY09], Knoll [Kno09], and Wright [Wri13]. Knoll showed that for the standard copy of $(\omega, \lt)$, the degree spectrum of any unary relation which is not intrinsically computable consists of exactly the $\Delta^0_2$ degrees. So while the counterexamples of Theorem 6.1.6 exist in general, such counterexamples may not exist for particular structures. Wright [Wri13] later independently found this same result about unary relations, and also showed that for $n$-ary relations on $(\omega, \lt)$, any degree spectrum which contains a non-computable set contains all c.e. sets. In Section 6.5, we begin an investigation of what the partial order of degree spectra on a cone looks like when we restrict ourselves to relations on $(\omega, \lt)$. We show that every relation which is intrinsically $\alpha$-c.e. is intrinsically of c.e. degree. We also introduce the notion of what it means for a relation to uniformly have its degree spectrum, and show that either

(1) there is a computable relation $R$ on $(\omega, \lt)$ whose degree spectrum strictly contains the c.e. degrees but does not contain all of the d.c.e. degrees, or
(2) there is a computable relation \( R \) on \((\omega, <)\) whose degree spectrum is all of the \( \Delta^0_2 \) degrees but does not have this degree spectrum uniformly.

Both (1) and (2) are interesting situations, and one or the other must occur. It seems likely, but very difficult to show, that (1) occurs.

In this monograph, we would like to advocate for this program of studying degree spectra on a cone. We hope that some of the results in this paper will support the position that this is a fruitful view. We believe that the study of degree spectra on a cone is interesting beyond the characterization of the degree spectra of naturally occurring structures. Even when considering structures in general, knowing which results can and cannot be proven on a cone is still illuminating. If a result holds in the computable case, but not on a cone, then that means that the result relies in some way on the computable presentation of the original structure (for example, diagonalization arguments can often be used to produce pathological examples; such arguments tend not to relativize). Understanding why certain a result holds for computable structures but fails on a cone is a way of understanding the essential character of the proof of that result.

We will begin by making some preliminary definitions in Section 6.2. In Section 6.3 we include a discussion of the technical details around the definition of degree spectra on a cone, in particular in relation to relativizing Harizanov’s results on c.e. degrees. In Section 6.4, we will consider d.c.e. degrees. We begin by introducing the notion of being intrinsically of c.e. degree and give a characterization of such relations. We then prove Theorem 6.1.6. In Section 6.5, we apply some of the results from Section 6.4 to the structure \((\omega, <)\) and study the degree spectra of such relations. Section 6.6 is devoted to the proof of Theorem 6.1.5 on 2-CEA degrees. Finally, in Section 6.7 we will state some interesting open questions and describe what we see as the future of this program.

6.2 Preliminaries

In this section, we will introduce some background from computability theory and computable structure theory before formally defining what we mean by “on a cone.”

6.2.1 Computability Theory

We will assume that the reader has a general knowledge of computability theory. For the most part, the reader will not need to know about computable ordinals. Occasionally we will state general questions involving a computable ordinal \( \alpha \), but the reader should feel free to assume that \( \alpha \) is a finite number \( n \). See Chapter 4 of [AK00] for a reference on computable ordinals.

There are two classes of sets which will come up frequently which we will define here. Ershov’s hierarchy \([Er\check{s}68a, Er\check{s}68b, Er\check{s}70]\) generalizes the c.e. and d.c.e. sets to classify sets by how many times their computable approximation is allowed to change.
**Definition 6.2.1** (Ershov’s hierarchy). A set $X$ is $\alpha$-c.e. if there are computable functions $g: \omega \times \omega \to \{0, 1\}$ and $n: \omega \times \omega \to \{\beta : \beta \leq \alpha\}$ such that for all $x$ and $s$,

1. $g(x, 0) = 0$,
2. $n(x, 0) = \alpha$,
3. $n(x, s + 1) \leq n(x, s)$,
4. if $g(x, s + 1) \neq g(x, s)$ then $n(x, s + 1) < n(x, s)$, and
5. $\lim_{s \to \infty} g(x, s) = X(x)$.

The function $g$ guesses at whether $x \in X$, with $n$ counting the number of changes. We could instead have made the following equivalent definition:

**Definition 6.2.2** (Ershov’s hierarchy, alternate definition). $X$ is $\alpha$-c.e. if there are uniformly c.e. families $(A_\beta)_{\beta < \alpha}$ and $(B_\beta)_{\beta < \alpha}$ such that

$$X = \bigcup_{\beta < \alpha} (A_\beta - \bigcup_{\gamma < \beta} B_\gamma)$$

and if $x \in A_\beta \cap B_\beta$, then $x \in A_\gamma \cup B_\gamma$ for some $\gamma < \beta$.

See Chapter 5 of [AK00] for more on $\alpha$-c.e. sets.

A set $X$ is c.e. in and above (CEA) a set $Y$ if $X$ is c.e. in $Y$ and $X \geq_T Y$. We can easily generalize this to any finite $n$ by iterating the definition: $X$ is $n$-CEA in $Y$ if there are $X_0 = Y, X_1, \ldots, X_n = X$ such that $X_i$ is CEA in $X_{i-1}$ for each $i = 1, \ldots, n$. We can even generalize this to arbitrary $\alpha$:

**Definition 6.2.3** (Ash-Knight [AK95]). A set $X$ is $\alpha$-CEA in a set $Y$ if there is a sequence $(X_\beta)_{\beta \leq \alpha}$ such that

1. $X_0$ is recursive,
2. $X_{\beta+1}$ is CEA in $X_\beta$ uniformly in $\beta$,
3. $X_\delta$ is CEA in $\bigoplus_{\beta < \delta} X_\beta$ uniformly in $\delta$ when $\delta$ is a limit ordinal, and
4. $X_\alpha = X$.

Ash and Knight [AK95] note that a set which is $\alpha$-CEA is $\Sigma_0^\alpha$, but that the converse is not necessarily true (one can see that this follows from the existence of a minimal $\Delta_2^0$ set; a minimal $\Delta_2^0$ set is not c.e. and hence not 2-CEA).
6.2.2 Computable Structure Theory

We will consider only countable structures, so we will say “structure” when we mean “countable structure.” For an introduction to computable structures as well as much of the other background, see [AK00]. We view the atomic diagram \( D(A) \) of a structure \( A \) as a subset of \( \omega \), and usually we will identify \( A \) with its diagram. A computable presentation (or computable copy) \( B \) of a structure \( A \) is another structure \( B \) with domain \( \omega \) such that \( A \cong B \) and the atomic diagram of \( B \) is computable.

The infinitary logic \( L^{\omega_1\omega} \) is the logic which allows countably infinite conjunctions and disjunctions but only finite quantification. If the conjunctions and disjunctions of a formula \( \varphi \) are all over computable sets of indices for formulas, then we say that \( \varphi \) is computable. We use \( \Sigma^n_\alpha \) and \( \Pi^n_\alpha \) to denote the classes of all infinitary \( \Sigma_\alpha \) and \( \Pi_\alpha \) formulas respectively. We will also use \( \Sigma^c_\alpha \) and \( \Pi^c_\alpha \) to denote the classes of computable \( \Sigma_\alpha \) and \( \Pi_\alpha \) formulas. These formulas will often involve finitely many constant symbols from the structure. See Chapter 6 of [AK00] for a more complete description of computable formulas.

By a relation \( R \) on a structure \( A \), we mean a subset of \( A^n \) for some \( n \). We say that \( R \) is invariant if it is fixed by all automorphisms of \( R \). It is a theorem, following from the Scott Isomorphism Theorem [Sco65], that a relation is invariant if and only if it is defined in \( A \) by a formula of \( L^{\omega_1\omega} \). All of the relations that we will consider will be invariant relations. If \( B \) is a computable copy of \( A \), then there is a unique interpretation \( R^B \) of \( R \) in \( B \), either by using the \( L^{\omega_1\omega} \)-definition of \( R \), or using the invariance of \( R \) under automorphisms (so that if \( f : A \to B \) is an isomorphism, \( f(R) \) is a relation on \( B \) which does not depend on the choice of the automorphism \( f \)).

The study of invariant relations began with Ash and Nerode [AN81]. They made the following definition: if \( \Gamma \) is some property of sets, then \( R \) is intrinsically \( \Gamma \) if among all of the computable copies \( B \) of \( A \), \( R^B \) is \( \Gamma \). Usually we will talk about relations which are intrinsically computable (or more generally \( \Delta_\alpha \)), intrinsically c.e. (or more generally \( \Sigma_\alpha \) or \( \Pi_\alpha \)), intrinsically \( \alpha \)-CEA, or intrinsically \( \alpha \)-c.e. Ash and Nerode showed that (making some assumptions on the structure and on the relation) a relation is intrinsically c.e. if and only if is defined by a \( \Sigma^c_1 \) formula:

**Theorem 6.2.4** (Ash-Nerode [AN81, Theorem 2.2]). Let \( A \) be a computable structure and \( R \) a relation on \( A \). Suppose that for any tuple \( \bar{c} \in A \) and any finitary existential formula \( \varphi(\bar{c}, \bar{x}) \), we can decide whether or not there is \( \bar{a} \notin R \) such that \( A \models \varphi(\bar{c}, \bar{a}) \). Then the following are equivalent:

1. \( R \) is intrinsically c.e.
2. \( R \) is defined by a \( \Sigma^c_1 \) formula with finitely many parameters from \( A \) (we say that \( R \) is formally \( \Sigma_1 \) or formally c.e.).

In practice, most naturally occurring structures and relations satisfy the effectiveness condition from this theorem. However, there are structures which do not have the effective-
ness condition, and some of these structures are counterexamples to the conclusion of the theorem.

Barker [Bar88] later generalized this to a theorem about intrinsically $\Sigma_\alpha$ relations. Ash and Knight [AK96] also proved a result for intrinsically $\alpha$-c.e. relations (with the formal definition being of the form of Definition 6.2.2 above).

Ash, Knight, Manasse, and Slaman [AKMS89] and independently Chisholm [Chi90] considered a relativized notion of intrinsic computability. We say that $R$ is relatively intrinsically $\Sigma_\alpha$ (or $\Pi_\alpha$, etc.) if, in every copy $B$ of $A$, $R^B$ is $\Sigma^0_\alpha(B)$ ($\Pi^0_\alpha(B)$, etc.). Then they were able to prove a theorem similar to Theorem 6.2.4 above but without an effectiveness condition:

**Theorem 6.2.5** (Ash-Knight-Manasse-Slaman [AKMS89], Chisholm [Chi90]). Let $A$ be a computable structure and $R$ a relation on $A$. The following are equivalent:

1. $R$ is relatively intrinsically $\Sigma_\alpha$,
2. $R$ is defined by a $\Sigma^0_\alpha$ formula with finitely many parameters from $A$.

These theorems say that the computational complexity of a relation is strongly tied to its logical complexity.

In order to give a finer measure of the complexity of a relation, Harizanov [Har87] introduced the degree spectrum.

**Definition 6.2.6** (Degree Spectrum of a Relation). The degree spectrum of a relation $R$ on a computable structure $A$ is the set

$$\text{dgSp}(R) = \{d(R^B) : B \text{ is a computable copy of } A\}.$$ 

### 6.2.3 Relativizing to a Cone

In this section, we will formally describe what we mean by working on a cone, and by the degree spectrum of a relation on a cone. Consider the degree spectrum of a relation. For many natural structures and relations, the degree spectrum of a relation is highly related to the model-theoretic properties of the relation $R$. However, for more pathological structures (and first-order arithmetic), the degree spectra of relations can often be badly behaved. Some examples of such relations were given in the introduction. On the other hand, Theorem 6.1.2 says that many relations have degree spectra which are nicely behaved (of course, there are relations which do not satisfy the effectivity condition from this theorem and which do not satisfy the conclusion—see [Har91]).

It is a common phenomenon in computable structure theory that there are unnatural structures which are counterexamples to theorems which would otherwise hold for natural structures. This unnatural behaviour tends to disappear when the theorem is relativized to a sufficiently high cone; in the case of Harizanov’s result, relativizing the conclusion to any degree above $0''$ allows the theorem to be stated without the effectivity condition (since $0''$ can compute what is required by the effectivity condition).
A Turing cone is a collection of sets of the form \( \{ X : X \geq_T A \} \) for some fixed set \( A \). A collection of sets is Turing invariant if whenever \( X \) is in the collection, and \( X \equiv_T Y \), then \( Y \) is in the collection (i.e., the collection is a set of Turing degrees). Martin [Mar68] noticed that any Turing invariant collection \( A \) which is determined\(^3\) either contains a cone, or contains a cone in its complement. Note that only one of these can happen for a given set \( A \), as any two cones intersect non-trivially and contain a cone in their intersection. Moreover, by Borel determinacy (see [Mar75]) every Borel invariant set is determined. Thus we can form a \( \{0,1\} \)-valued measure on the Borel sets of Turing degrees, selecting as “large” those sets of Turing degrees which contain a cone.

Given a statement \( P \) which relativizes to any degree \( d \), we say that \( P \) holds on a cone if the set of degrees \( d \) for which the relativization of \( P \) to \( d \) holds contains a cone. If \( P \) defines a Borel set of degrees in this way, then either \( P \) holds on a cone or \( \neg P \) holds on a cone. If \( P \) holds on a cone, then \( P \) holds for most degrees, or for sufficiently high degrees.

We say that \( R \) is intrinsically \( \Sigma_\alpha \) on a cone if for all degrees \( d \) on a cone, and all copies \( B \) of \( A \) with \( B \) computable in \( d \), \( R^B \) is \( \Sigma_\alpha^0(d) \). Then by relativizing previous results (Theorem 6.2.4 or Theorem 6.2.5), we see that \( R \) is intrinsically \( \Sigma_\alpha \) on a cone if and only if it is defined by a \( \Sigma_\alpha^\text{up} \) formula, without any computability-theoretic assumptions on either \( A \) and \( R \) or the \( \Sigma_\alpha^\text{up} \) formula. Note that a relation is intrinsically \( \Sigma_0^0 \) on a cone if and only if it is relatively intrinsically \( \Sigma_0^0 \) on a cone; so after relativizing to a cone, both notions coincide. We make similar definition for \( \Pi_\alpha \) and other classes of degrees.

Note that when we work on a cone, we do not need to assume that the structure \( A \) or the relation \( R \) is computable, because we can consider only cones with bases above \( A \oplus R \). Also, for the same reason, we can work with arbitrary ordinals, even those which are not computable, by working on a cone on which that ordinal is computable. Thus, for example, we can say that a relation is intrinsically \( \Sigma_\alpha \) on a cone even when \( \alpha \) is not a computable ordinal. What we mean is that there is a cone above which \( \alpha \) is computable, and the relation is intrinsically \( \Sigma_\alpha \) relative to all degrees on that cone.

Now we will define what we mean by the degree spectrum of a relation on a cone. First, there is a natural relativisation of the degree spectrum of a relation to a degree \( d \). The degree spectrum of \( R \) below the degree \( d \) is

\[
\text{dgSp}(A, R)_{\leq d} = \{ d(R^B) \oplus d : (B, R^B) \text{ is an isomorphic copy of } (A, R) \text{ with } B \leq_T d \}.
\]

An alternate definition would require the isomorphic copy \( B \) to be Turing equivalent to \( d \), rather than just computable in \( d \):

\[
\text{dgSp}(A, R)_{\leq d} = \{ d(R^B) \oplus d : (B, R^B) \text{ is an isomorphic copy of } (A, R) \text{ with } B \equiv_T d \}.
\]

If \( B \leq_T d \), then by Knight’s theorem on the upwards closure of the degree spectrum of structures (see [Kni86]), there is an isomorphic copy \( C \) of \( B \) with \( C \equiv_T d \) and a \( d \)-computable isomorphism \( f : B \to C \). Then \( R^C \oplus d \equiv_T R^B \oplus d \). So these two definitions are equivalent.

\(^{3}\)A set \( A \) (viewed as set of reals in Cantor space \( 2^\omega \)) is determined if one of the two players has a winning strategy in the Gale-Stewart game \( G_A \), where players I and II alternate playing either 0 or 1; I wins if the combined sequence of plays is in \( A \), and otherwise II wins. See [GS53] and [Jec03].
The proof of Theorem 6.1.2 relativizes to show that for any degree $d \geq 0''$, if $\text{dgSp}(A, R)_{\preceq d}$ contains a degree which is not computable in $d$, then it contains every degree CEA in $d$. One could also have defined the degree spectrum of a relation to be the set

$$\text{dgSp}^*(A, R)_{\preceq d} = \{ d(R^B) : (B, R^B) \text{ is an isomorphic copy of } (A, R) \text{ with } B \preceq_T d \}.$$ 

In this case, Harizanov's proof of Theorem 6.1.2 does not relativize. In Appendix 6.3, we will consider a new proof of Harizanov's result which relativizes in the correct way for this definition of the degree spectrum. However, our proof becomes much more complicated than Harizanov's original proof. For the other results in the paper, we will not consider $\text{dgSp}^*(A, R)_{\preceq d}$. Though it is quite possible that there are similar ways to extend our proofs, it would distract from main content of those results. We are interested in whether there is any real difference between $\text{dgSp}(A, R)_{\preceq d}$ and $\text{dgSp}^*(A, R)_{\preceq d}$, or whether any result provable about one transfers in a natural manner to the other. For example, is it always the case that restricting $\text{dgSp}^*(A, R)_{\preceq d}$ to the degrees above $d$ gives $\text{dgSp}(A, R)_{\preceq d}$ for sufficiently high $d$?

Now we want to make our relativisation of the degree spectrum independent of the degree $d$. Thus we turn to Definition 6.1.1 due to Montalbán, which we will now develop more thoroughly. To each structure $A$ and relation $R$, we can assign the map $f_R : d \mapsto \text{dgSp}(A, R)_{\preceq d}$. Given two pairs $(A, R)$ and $(B, S)$, for any degree $d$, either $\text{dgSp}(A, R)_{\preceq d}$ is equal to $\text{dgSp}(B, S)_{\preceq d}$, one is strictly contained in the other, or they are incomparable. By Borel determinacy, there is a cone on which exactly one of these happens. Thus we get a pre-order on these functions $f_R$, and taking the quotient by equivalence, we get a partial order on degree spectrum. Denote the elements of the quotient by $\text{dgSp}_{rel}(A, R)$. We call $\text{dgSp}_{rel}(A, R)$ the degree spectrum of $R$ on a cone.

For many classes $\Gamma$ of degrees which relativize, for example the $\Sigma_\alpha$ degrees, there is a natural way of viewing them in this partial ordering by considering the map $\Gamma : d \mapsto \Gamma(d)$. By an abuse of notation, we will talk about such a class $\Gamma$ as a degree spectrum (in fact, it is easy to see for many simple classes of degrees that they are in fact the degree spectrum of some relation on some structure). Thus we can say, for example, that the degree spectrum, on a cone, of some relation contains the $\Sigma_\alpha$ degrees, or is equal to the d.c.e. degrees, and so on.

In particular, using this notation, we see that Theorem 6.1.2 yields:

**Corollary 6.2.7** (Harizanov). Let $A$ be a structure and $R$ a relation on $A$. Then either:

1. $\text{dgSp}_{rel}(A, R) = \Delta^0_1$ or
2. $\text{dgSp}_{rel}(A, R) \supseteq \Sigma^0_1$.

The cone on which this theorem holds is $(A \oplus R)''$—i.e., one could replace $\text{dgSp}_{rel}$ with $\text{dgSp}_{\preceq d}$ for any degree $d \geq_T (A \oplus R)''$.

We also get the following restatements of Theorems 6.1.5 and 6.1.6:

**Corollary 6.2.8**. Let $A$ be a structure and $R$ a relation on $A$. Then either:
Corollary 6.2.9. There is a structure $A$ and relations $R$ and $S$ on $A$ such that $\text{dgSp}_\text{rel}(A, R)$ and $\text{dgSp}_\text{rel}(A, S)$ contains the c.e. degrees and are contained within the d.c.e. degrees, but neither $\text{dgSp}_\text{rel}(A, R) \subseteq \text{dgSp}_\text{rel}(A, S)$ nor $\text{dgSp}_\text{rel}(A, S) \subseteq \text{dgSp}_\text{rel}(A, R)$.

Note that these two concepts that we have just introduced—intrinsic computability on a cone and degree spectra on a cone—are completely independent of the presentations of $A$ and $R$. Moreover, the intrinsic computability of a relation $R$ is completely dependent on its model-theoretic properties. So by looking on a cone, we are able to look at more model-theoretic properties of relations while using tools of computability theory.

The reader should always keep in mind the motivation behind this work. The theorems we prove are intended to be applied to naturally occurring structures. For well-behaved structures, “property $P$” and “property $P$ on a cone” should be viewed as interchangeable.

This work was originally motivated by a question of Montalbán first stated in [Wri13]. There is no known case of a structure $(A, R)$ where $\text{dgSp}(A, R)_{\text{sd}}$ does not have a maximum degree for $d$ sufficiently large. When the degree spectrum does contain a maximum degree, define the function $f_{A,R}$ which maps a degree $d$ to the maximum element of $\text{dgSp}(A, R)_{\text{sd}}$. This is a degree-invariant function, and hence the subject of Martin’s conjecture. Montalbán has asked whether Martin’s conjecture is true of this function $f_{A,R}$; that is, is it true that for every structure $A$ and relation $R$, there is an ordinal $\alpha < \omega_1$ such that for all $d$ on a cone, $d^{(\alpha)}$ is the maximal element of $\text{dgSp}(A, R)_{\text{sd}}$?

Recall the question of Ash and Knight from the introduction, which we stated as: is it true that every degree spectrum is either contained in the $\Delta_\alpha$ degrees or contains all of the $\alpha$-CEA degrees? If this were true, then Montalbán’s question would (almost) be answered positively, as every relation $R$ has a definition which is $\Sigma_\alpha^0$ and $\Pi_\alpha^0$ for some $\alpha < \omega_1$; choosing $\alpha$ to be minimal, if $\alpha$ is a successor ordinal $\alpha = \beta + 1$, then for all degrees $d$ on a cone, there is a complete $\Delta_\alpha^0(d)$ degree which is $\beta$-CEA above $d$ and hence is the maximal element of $\text{dgSp}(A, R)_{\text{sd}}$. The relativized version of Harizanov’s Theorem 6.1.2 answers Ash and Knight’s question (and hence Montalbán’s question) for relations which are defined by a $\Sigma_1^0$ formula, and our Theorem 6.1.5 answers these questions for relations which are $\Sigma_2^0$-definable.\(^5\)

\(^4\)Technically, the function we are considering maps a set $C$ to some set $D$ which is of maximum degree in $\text{dgSp}(A, R)_{\text{sd}(C)}$. For now, we can ignore which sets we choose.

\(^5\)Note that the proofs of these results also show that $f_{A,R}$ is uniformly degree-invariant for these relations, which also implies that Martin’s conjecture holds for these $f_{A,R}$—see [Ste82] and [SS88]. What we mean is that, given sets $C$ and $D$ of degree $d$ (with $C \equiv_T D$), Theorem 6.1.2 (respectively Theorem 6.1.5) provide isomorphic copies $C$ and $D$ of $A$ with $C \leq_T C$ and $D \leq_T D$ with $R^C \equiv_T C'$ and $R^D \equiv_T D'$ (resp. $R^C \equiv_T C''$ and $R^D \equiv_T D''$) and these Turing equivalences are uniform in $C'$ and $D'$ (resp. $C''$ and $D''$). So given an index for the equivalence $C \equiv_T D$, we can effectively find an index for the equivalence $R^C \equiv_T R^D$.\(^6\)
6.3 Relativizing Harizano’s Theorem on C.E. Degrees

Theorem 6.1.2 had, as a consequence, Corollary 6.2.7 which said that for any structure \( A \) and relation \( R \), either \( \text{dgSp}_{rel}(A, R) = \Delta^0_1 \) or \( \text{dgSp}_{rel}(A, R) \supseteq \Sigma^0_1 \). Now \( \text{dgSp}_{rel}(A, R) \) was defined using the behaviour, on a cone, of the degree spectrum relativized to a degree \( d \):

\[
\text{dgSp}(A, R)_{sd} = \{ d(R^B) \oplus d : (B, R^B) \text{ is an isomorphic copy of } (A, R) \text{ with } B \leq_T d \}.
\]

In Chapter 6.2, we briefly considered the alternate definition

\[
\text{dgSp}^*(A, R)_{sd} = \{ d(R^B) : (B, R^B) \text{ is an isomorphic copy of } (A, R) \text{ with } B \leq_T d \}
\]

from which one could define an alternate degree spectrum on a cone, \( \text{dgSp}^*_{rel}(A, R) \). The relativization of Theorem 6.1.2 does not suffice to prove that either \( \text{dgSp}^*_{rel}(A, R) = \Delta^0_1 \) or \( \text{dgSp}^*_{rel}(A, R) \supseteq \Sigma^0_1 \). This is because, as we will see later, the relativization to a degree \( d \) only shows that \( R^B \oplus d \equiv_T C \oplus d \) (for \( C \) c.e. in \( d \)).

Our goal in this section is to give a new proof of Theorem 6.1.2 whose revitalization is strong enough to apply to the alternative degree spectrum \( \text{dgSp}^*_{rel}(A, R) \). We will show that it is possible, though the proof is significantly more complicated. The reader may skip this section without any impact on their understanding of the rest of this work. The results in this section are in a similar style, but much easier, than Theorem 6.6.1 and hence may be read as an introduction to the proof of that result.

Theorem 6.3.1. Suppose that \( A \) is a computable structure and \( R \) is a computable relation which is not intrinsically computable. Suppose that \( A \) satisfies the following effectiveness condition: the \( \exists_1 \)-diagram of \((A, R)\) is computable, and given a finitary existential formula \( \varphi(c, \bar{x}) \), we can decide whether there are finitely many or infinitely many solutions. Then for any sets \( X \leq_T Y \) with \( Y \) c.e. in \( X \), there is an \( X \)-computable copy \( B \) of \( A \) with

\[
R^B \equiv_T Y.
\]

Moreover, \( Y \) can compute the isomorphism between \( A \) and \( B \).

From this theorem, we get the following corollary:

Corollary 6.3.2. Suppose that \( A \) is a structure and \( R \) is a relation on \( A \) which is not intrinsically computable on a cone. Then \( \text{dgSp}^*_{rel}(A, R) \) contains the c.e. degrees.

Our goal in proving Theorem 6.3.1 is to give evidence towards two ideas. First, we want to give evidence that results that can be proved for \( \text{dgSp}_{rel} \) can also be proved for \( \text{dgSp}^*_{rel} \). Second, we will see that if we try to prove results for \( \text{dgSp}^*_{rel} \), we have to deal with many complications which distract from the heart of the proof.

We will begin by describing Harizano’s proof of Theorem 6.1.2. This will both show us why it does not relativize in the way we desire, and also guide us as to what we need to do. Let \( A \) be a structure and \( R \) a computable relation which is not intrinsically computable, say \( R \) is not intrinsically \( \Pi^0_1 \). Harizano’s construction uses the following definition from Ash-Nerode [AN81].
Definition 6.3.3. Let $\bar{c}$ be a tuple from $A$. We say that $\bar{a} \notin R$ is free over $\bar{c}$ if for any finitary existential formula $\psi(\bar{c}, \bar{x})$ true of $\bar{a}$ in $A$, there is $\bar{a}' \in R$ which also satisfies $\psi(\bar{c}, \bar{x})$.

If $A$ is assumed to have an effectiveness condition—namely that for each $\bar{c}$ in $A$ and finitary existential formula $\varphi(\bar{x}, \bar{c})$, we can decide whether there is $\bar{a} \notin R$ such that $A \models \varphi(\bar{a}, \bar{c})$—then for any tuple $\bar{c}$, we can effectively find a tuple $\bar{a} \notin R$ which is free over $\bar{c}$. Harizanov uses these free elements to code a c.e. set $C$ into a computable copy $B$ of $A$. Building $B$ via a $\Delta^0_2$ isomorphism, for each $x \in \omega$, there is a tuple from $B$ which codes, by being in $R$ or not in $R$, whether or not $x$ is in $C$. For a given $x$, she fixes a tuple $\bar{b}_x$ from $B$ and maps it to a tuple $\bar{a}$ from $A$ which is free; the fact that $\bar{a} \notin R$ codes that $x \notin C$. If, at a later stage, $x \in C$, then using the fact that $\bar{a}$ is free, she modifies the $\Delta^0_2$ isomorphism to instead map $\bar{b}$ to a tuple $\bar{a}' \in R$, coding that $x \in C$. The argument involves finite injury. Given $R^B$, one can compute $\bar{b}_0$ and decide whether or not $0 \in C$; then, knowing this, we can wait until a stage at which $0$ enters $C$ (if necessary) and can compute $\bar{b}_1$. We need to wait, since the choice of $\bar{b}_1$ may be injured before this stage. Once we know $\bar{b}_1$, we can use $R^B$ to decide whether or not $1 \in C$, and so on. On the other hand, given $C$, one understands the injury from the construction and can compute the isomorphism between $A$ and $B$.

Now consider the relativisation to a degree $d$. Let $C$ be CEA in $d$ and try to use the same construction (building $B$ computable in $d$). Given $C$, we can once again run the construction and compute the isomorphism between $A$ and $B$. However, given $R^B$, we can not necessarily compute $d$, and hence do not necessarily have access to the enumeration of $C$. Without this, we cannot run through the construction and compute the coding locations $\bar{b}_i$. We do, however, get that $R^B \oplus d \equiv_T C$.

To prove Theorem 6.3.1, we need a strategy to divorce the coding locations from the construction of $B$. The trick we will use is as follows. Fix beforehand tuples from $B$ to act as coding locations, and number them in increasing order using $\omega$. Choose a computable infinite-to-one bijection $g: \omega \rightarrow 2^{<\omega}$. A coding location may either be “on” or “off” depending on whether or not it is in $R^B$ (though whether “on” means in $R^B$ and “off” means out of $R^B$, or vice versa, will depend on the particular structure and relation). We will show that we make two choices for a coding location: we can either choose for them to be permanently off no matter what happens with the other coding locations, or we can choose to have a coding location start on and later turn off (after which we no longer have control of the coding location—if some earlier location turns from on to off, the later coding location may turn back on again).

We will ensure that there is a unique increasing sequence $k_0 < k_1 < k_2 < \cdots$ of coding locations which are “on” such that $g(k_0)$ has length one, and $g(k_{i+1})$ extends $g(k_i)$ by a single element. Thus $\bigcup_{i \in \omega} g(k_i)$ will be a real, and we will ensure that it is the set $C$ which we are trying to code. We call such a sequence an active sequence. Since this sequence is unique, we can compute it using $R^B$ by looking for the first coding location $k_0$ with $g(k_0)$ of length one which is on, then looking for the next coding location $k_1$ with $g(k_1)$ an extension of $g(k_0)$ of length two and which is on, and so on.

We will illustrate how we build the sequence using the following example where we code
whether two elements 0 and 1 are in C. To code that 0 \notin C, start by choosing a coding location \( k_0 \) with \( g(k_0) = 0 \) and have \( k_0 \) be on. Set every smaller coding location to be permanently off. Then, to code that 1 \notin C, find a coding location \( k_1 > k_0 \) with \( g(k_1) = 00 \) and have \( k_1 \) be on, while every coding location between \( k_0 \) and \( k_1 \) is off. Now if 0 enters \( C \), switch \( k_0 \) off; \( k_1 \) might be on, but every other coding location less than \( k_1 \) is permanently off. Because there is no coding location \( i < k_1 \) with \( g(i) \) of length one and \( g(i) < g(k_1) \), \( k_1 \) can never be part of an active sequence even if it is on. Find some \( k'_0 \) which has no appeared yet with \( g(k'_0) = 1 \) and set \( k'_0 \) to be on, while every coding location between \( k_1 \) and \( k'_0 \) is permanently off. Thus \( k'_0 \) will be the first coding location in the active sequence.

In the remainder of this chapter, we give the proof of Theorem 6.3.1.

### 6.3.1 Framework of the Proof

Let \( A, R, X, \) and \( Y \) be as in the theorem. Note that the effectiveness condition is robust in the following sense: if \( Q \) is definable from \( R \) via both a finitary existential formula and a finitary universal formula, then the effectiveness conditions holds for \( Q \) as well.

The proof of the theorem is by induction on the arity \( r \) of \( R \). We will argue that we can make three assumptions, (I), (II), and (III). Let \( (A)^r \) denote the tuples in \( A^r \) with no duplicate entries and let \( A^r_{i=j} \) denote the set of tuples from \( A^r \) with \( i \)th entry equal to the \( j \)th entry. Note that \( (A)^r \) and \( A^r_{i=j} \) are defined by both finitary existential and universal formulas. If the restriction \( R \cap A^r_{i=j} \) of \( R \) to some \( A^r_{i=j} \) is not intrinsically computable, then as the restriction is essentially an \((r-1)\)-ary relation, by the induction hypothesis, there is an \( X \)-computable copy \( B \) of \( A \) with \( R^B \cap B^r_{i=j} \equiv_T Y \). Now the set \( Y \) computes the isomorphism between \( A \) and \( B \), and since \( R \) is computable in \( A, Y \) computes \( R^B \). Also, \( R^B \cap B^r_{i=j} \subseteq_T R^B \) and hence \( R^B \equiv_T Y \) and we are done. So we may assume that the restrictions \( R \cap A^r_{i=j} \) are intrinsically computable and hence are defined by finitary existential and universal formulas. Since

\[
R = (R \cap (A)^r) \cup \bigcup_{i=j} (R \cap A^r_{i=j})
\]

and \( R \cap (A)^r \) is disjoint from \( \bigcup_{i=j} (R \cap A^r_{i=j}) \), we must have that \( R \cap (A)^r \) is not intrinsically computable. So we may replace \( R \) by \( R \cap (A)^r \). This is assumption (I): that \( R \subseteq (A)^r \). When we say \( \bar{a} \in R \) or \( \bar{a} \notin R \), we really mean that \( \bar{a} \in (A)^r \) as well.

An important aspect of the proof will be whether we can find “large” formally \( \Sigma^0_1 \) sets contained in \( R \) or its complement. Let us formally define what we mean by large. We say that two tuples \( \bar{a} = (a_1, \ldots, a_n) \) and \( \bar{b} = (b_1, \ldots, b_n) \) are disjoint if they do not share any entries, that is, \( a_i \neq b_j \) for each \( i \) and \( j \). We say that a set \( S \subseteq A^n \) is thick if it contains an infinite set of pairwise disjoint tuples; otherwise, we say that \( S \) is thin.

**Lemma 6.3.4.** If \( S \subseteq A^k \) is a thin set, there is a bound on the size of sets of pairwise disjoint tuples from \( S \).

**Proof.** Suppose that there is no such bound. We claim that \( S \) is thick. Let \( \bar{a}_1, \ldots, \bar{a}_n \) be a maximal pairwise disjoint subset of \( S \). Since there is no bound on the size of sets of pairwise
disjoint tuples form $S$, we can pick $\bar{b}_1, \ldots, \bar{b}_{n+k+1}$ all pairwise disjoint. There are $k \cdot n$ distinct entries that appear in $\bar{a}_1, \ldots, \bar{a}_n$; since each entry can only appear in a single $\bar{b}_i$, some $\bar{b}_i$ must be disjoint from each $\bar{a}_j$. This contradicts the maximality of $\bar{a}_1, \ldots, \bar{a}_n$. 

We’ll argue that we may assume that $R$ is thick. Suppose that $R$ is not thick. Then there are finitely many tuples $\bar{a}_1, \ldots, \bar{a}_n$ such that no more elements of $R$ are disjoint from $\bar{a}_1, \ldots, \bar{a}_n$. Write $\bar{a}_i = (a^1_i, \ldots, a^r_i)$. Let $R_{i,j,k}$ be the set of tuples $\bar{b} = (b^1, \ldots, b^n)$ in $R$ with $b^j = a^k_i$. Then

$$R = \bigcup_{i,j,k} R_{i,j,k}. $$

Then one of the $R_{i,j,k}$ is not intrinsically computable, or else $R$ would be intrinsically computable. But $R_{i,j,k}$ is essentially an $(r-1)$-ary relation and $R_{i,j,k} \leq_T R$. So we can use the induction hypothesis as above to reduce to the case where $R$ and $\neg R$ are both thick. This is assumption (II).

Let $\bar{c}$ be any tuple. By a similar argument as above, we may assume that $R$, when restricted to tuples not disjoint from $\bar{c}$, is intrinsically computable, and that $R$ restricted to tuples disjoint from $\bar{c}$ is not intrinsically computable. This is assumption (III).

Now we have three cases to consider.

1. $R$ is not intrinsically $\Pi^0_1$, but $R$ contains a thick set defined by a $\Sigma^c_1$ formula.

2. $R$ is not intrinsically $\Sigma^0_1$, but $\neg R$ contains a thick defined by a $\Sigma^c_1$ formula.

3. Neither $R$ nor $\neg R$ contains a thick set defined by a $\Sigma^c_1$ formula.

These exhaust all of the possibilities. If we are not in the third case, then either $R$ or $\neg R$ must contain a thick set defined by a $\Sigma^c_1$ formula. If neither $R$ nor $\neg R$ are definable by a $\Sigma^c_1$ formula, then we must be in one of the first two cases. Finally, it is possible that $R$ is definable by a $\Pi^c_1$ or $\Sigma^c_1$ formula. In the first case, $R$ is not definable by a $\Sigma^c_1$ formula, but $\neg R$ is (and, as $\neg R$ is thick, (2) holds). In the second case, $R$ is definable by a $\Sigma^c_1$ formula, but not by any $\Pi^c_1$ formula, and so (1) holds.

Note that cases one and two include the particular cases where $R$ is intrinsically $\Sigma^0_1$ but not intrinsically $\Pi^0_1$ and intrinsically $\Pi^0_1$ but not intrinsically $\Sigma^0_1$ respectively.

### 6.3.2 The First Two Cases

The second case is similar to the first, but with $R$ replaced by $\neg R$. We will just consider the first case.

We assume that $R$ satisfies (I), (II), and (III). Now we have the following lemma which applies to $R$ (since our effectiveness condition is strong enough to imply the condition in the lemma, as well as to find the free elements from the lemma).

**Lemma 6.3.5** (Ash-Nerode [AN81]). Suppose that $R$ is not defined in $A$ by $\Pi^c_1$ formula. Furthermore, suppose that for each tuple $\bar{c}$ in $A$ and finitary existential formula $\varphi(\bar{c}, x)$, we
can decide whether there exists \( \bar{a} \notin R \) such that \( A \models \varphi(\bar{a}, \bar{c}) \). Then for each tuple \( \bar{c} \in A \), there is \( \bar{a} \in R \) disjoint from \( \bar{c} \) and free over \( \bar{c} \).

Let \( \bar{c} \) be a tuple from \( A \). We say that \( \bar{a} \in (A)^r \) is \textit{constrained over} \( \bar{c} \) if there is a finitary existential formula \( \psi(\bar{c}, \bar{x}) \) true of \( \bar{a} \) in \( A \) such that for any \( \bar{a}' \in (A)^r \) which also satisfies \( \psi(\bar{c}, \bar{x}) \), \( \bar{a}' \in R \) if and only if \( \bar{a} \in R \).

Let \( \bar{d} \) be a tuple in \( A \) and \( \varphi(\bar{d}, \bar{x}) \) a \( \Sigma^e_1 \) formula defining a thick subset of \( R \). Now there are infinitely many disjoint tuples \( \bar{b} \in (A)^r \) satisfying \( \varphi(\bar{d}, \bar{x}) \) and hence in \( R \). Each of these tuples satisfies a finitary existential formula which is a disjunct in \( \varphi(\bar{d}, \bar{x}) \), and hence is constrained over \( \bar{d} \). We may find such tuples computably.

The following proposition completes the theorem in the first case.

**Proposition 6.3.6.** Suppose that there is a tuple \( \bar{d} \) such that for any \( \bar{c} \supseteq \bar{d} \), we can compute new elements \( \bar{a} \notin R \) and \( \bar{b} \in R \) disjoint from \( \bar{c} \) such that \( \bar{a} \) is free over \( \bar{c} \) and \( \bar{b} \) is constrained over \( \bar{d} \). Then for any sets \( X \preceq_T Y \) with \( Y \) c.e. in \( X \) there is an \( X \)-computable copy \( B \) of \( A \) with \( R^B \equiv_T Y \). Moreover, \( Y \) can compute the isomorphism between \( A \) and \( B \).

**Proof.** We may assume that the constants \( \bar{d} \) are part of the language, and hence ignore them.

Let \( \bar{c} \) be a tuple. Suppose that \( \bar{a} \notin R \) is free over \( \bar{c} \), and \( \bar{b}_1, \ldots, \bar{b}_m \in R \) are constrained, and \( \bar{e} \) any elements. Let \( \varphi(\bar{c}, \bar{u}, \bar{v}_1, \ldots, \bar{v}_m, \bar{w}) \) be a quantifier-free formula true of \( \bar{a}, \bar{b}_1, \ldots, \bar{b}_m, \bar{e} \). We claim that there is \( \bar{a}' \in R \), \( \bar{b}'_1, \ldots, \bar{b}'_m \in R \) constrained, and \( \bar{e} \) such that \( A \models \varphi(\bar{c}, \bar{a}', \bar{b}'_1, \ldots, \bar{b}'_m, \bar{e}') \). For \( i = 1, \ldots, m \) choose \( \psi_i(\bar{d}, \bar{v}_i) \) an existential formula true of \( \bar{b}_i \) in \( A \) such that for any \( \bar{b}'_i \in A \) which also satisfies \( \psi_i, \bar{b}'_i \in R \); note that any such \( \bar{b}'_i \) is also constrained over \( \bar{d} \) since it satisfies \( \psi_i \). Then consider the existential formula

\[
\phi(\bar{c}, \bar{u}) = \exists \bar{v}_1, \ldots, \bar{v}_m, \bar{w}(\varphi(\bar{c}, \bar{u}, \bar{v}_1, \ldots, \bar{v}_m, \bar{w}) \land \bigwedge_{i=1}^{m} \psi_i(\bar{d}, \bar{v}_i))
\]

Since \( \bar{a} \) is free over \( \bar{c} \), there is \( \bar{a}' \in R \) with \( A \models \phi(\bar{c}, \bar{a}') \). Then let \( \bar{b}'_1, \ldots, \bar{b}'_m \in R \) and \( \bar{e} \) be the witnesses to the existential quantifier. These are the desired elements.

Let \( Y \) be the enumeration of \( Y \) relative to \( X \). Using \( X \) we will construct by stages a copy \( B \) of \( A \) with \( R^B \equiv_T Y \). Let \( B \) be an infinite set of constants disjoint from \( A \). We will construct a bijection \( F : B \to A \) and use \( F^{-1} \) to define the structure \( B \) on \( B \). At each stage, we will give a tentative finite part of the isomorphism \( F \). It will be convenient to view \( B \) as a list of \( r \)-tuples \( B = \{ \bar{b}_0, \bar{b}_1, \ldots \} \); that is, if \( B = \omega \), then \( \bar{b}_0 = (0, \ldots, r-1) \), \( \bar{b}_1 = (r, \ldots, 2r-1) \), and so on.

Fix a computable infinite-to-one bijection \( g : \omega \to 2^{\omega} \cup \{ \emptyset \} \). We will code \( Y \) into \( R^B \) in the following way. We will ensure that if \( k_1 < \cdots < k_n \) are such that \( \bar{b}_{k_1}, \ldots, \bar{b}_{k_n} \in R^B \) and \( |g(k_i)| = i \) and \( g(k_1) < g(k_2) < \cdots < g(k_n) \) then \( g(k_n) < Y \). Moreover, we will ensure that there is an infinite sequence \( k_1, k_2, \ldots \) with this property. Thus \( R^B \) will be able to compute \( Y \) by reconstructing such a sequence. On the other hand, since \( Y \) will be able to compute the isomorphism, it will be able to compute \( R^B \). Note that the empty string \( \emptyset \) has length zero, so it can never appear in such a sequence, and thus marks a position that never does any coding.
We will promise that if at some stage we are mapping some tuple \( \bar{b} \in B \) to a constrained tuple in \( R \), \( \bar{b} \) will be mapped to a constrained tuple of \( R \) at every later stage.

At a stage \( s + 1 \), let \( F_s : \{ \bar{b}_0, \ldots, \bar{b}_s \} \to A \) be the partial isomorphism determined in the previous stage, and let \( B_s \) be the finite part of the diagram of \( B \) which has been determined so far. We will also have numbers \( i_{1,s} < \cdots < i_{n,s} \) with \( i_{1,s} < \cdots < i_{n,s} \) which indicate tuples \( \bar{b}_{k,s} \).

For each \( k \), \( g(i_{k,s}) \) will code a string of length \( k \). We are trying to ensure that if \( g(i_{k,s}) < Y_t \) for every stage \( t \geq s \), then we keep \( \bar{b}_{k,s} \notin R^B \), and otherwise we put \( \bar{b}_{k,s} \in R^B \). Once the first \( k \) entries of \( Y \) have stabilized, \( i_{k,s} \) will stabilize.

We define a partial isomorphism \( G \) extending \( F_s \) which, potentially with some corrections, will be \( F_{s+1} \). Let \( G(\bar{b}) = F_s(\bar{b}) \) for \( 0 \leq i \leq \ell \). We may assume, by extending \( G \) by adding constrained tuples, that \( g(\ell + 1) = \emptyset \). Let \( \bar{a}_{\ell+1} \) be the first \( r \) new elements not yet in the image of \( F_s \). Now let \( k > \ell \) be first such that \( g(k) = Y_{s+1} \upharpoonright_{n_{s+1}} \). Find new tuples \( \bar{a}_{\ell+2}, \ldots, \bar{a}_k \in R \) which are constrained. Also find \( \bar{a}_{k+1} \in \neg R \), and free over \( G(\bar{b}_0), \ldots, G(\bar{b}_\ell), \bar{a}_{\ell+1}, \ldots, \bar{a}_k \).

Set \( G(\bar{b}_i) = \bar{a}_i \) for \( \ell + 1 \leq i \leq k \).

Let \( B_{s+1} \supseteq B_s \) be the atomic formulas of Gödel number at most \( s \) which are true of the images of \( \bar{b}_0, \ldots, \bar{b}_k \) in \( A \).

Now we will act to ensure that for each \( m \), \( g(i_{m,s+1}) < Y_{s+1} \). Find the first \( m \), if any exists, such that \( g(i_{m,s}) \notin Y_{s+1} \). If such an \( m \) exists, using the fact that \( g(i_{m,s}) \) is free over the previous elements, choose \( \bar{a}_{m'}, \ldots, \bar{a}_k' \) such that:

1. \( G(\bar{b}_0), \ldots, G(\bar{b}_{m-1}), \bar{a}_m', \ldots, \bar{a}_k' \) satisfy the same existential formula that \( \bar{b}_0, \ldots, \bar{b}_k \) does in \( B_{s+1} \),

2. for any \( m' > m \) such that \( \bar{a}_m' \in R \) is constrained, so is \( \bar{a}_{m'} \in R \), and

3. \( \bar{a}_{i_{m,s}}' \in R \).

Set \( F_{s+1}(\bar{b}_j) = G(\bar{b}_j) \) for \( j < m \) and \( F_{s+1}(\bar{b}_j) = \bar{a}_j \) for \( m + 1 \leq j \leq k \). Also set \( n_{s+1} = m - 1 \) and \( i_{0,s+1} = i_{0,s}, \ldots, i_{m-1,s+1} = i_{m-1,s} \).

On the other hand, if no such \( m \) exists, set \( F_{s+1} = G \), set \( n_{s+1} = n_s + 1 \), \( i_{n_{s+1},s+1} = k \), and \( i_{j,s+1} = i_{j,s} \) for \( 0 \leq j \leq n_s \).

This completes the construction. It is a standard finite-injury construction and it is easy to verify that the construction works as desired.

\[ \square \]

### 6.3.3 The Third Case

We may suppose that, for each \( n \) and restricting to tuples in \( (A)^n \), no thick subset of \( R^1 \times \cdots \times R^n \) (where \( i_1, \ldots, i_n \in \{-1,1\} \)) is definable by a \( \Sigma^1_1 \) formula or a \( \Pi^1_1 \) formula. Moreover, we may assume that the same is true for particular fibers of such a set; we may assume that for any \( \bar{c}_2, \ldots, \bar{c}_n \), the fiber

\[ S = \{ (\bar{x}, \bar{y}_2, \ldots, \bar{y}_n) | \bar{x} \in R^1, \bar{y}_j \bar{c}_j \in R^j \} \]
Lemma 6.3.8. If many solutions.
over $\bar{\phi}$ tuple existential formula The proof is by induction on Lemma 6.3.7. Since we considered only tuples of $A^n$ with no repeated entries, $S$ satisfies assumption (I). As $R$ and $\neg R$ are thick by assumption (II), $S$ and $\neg S$ are also thick. Moreover, the restriction of $S$ to those tuples which are disjoint from some particular tuple $\bar{c}$ is not intrinsically computable as the restriction of $R$ is not. So $S$ satisfies assumptions (I), (II), and (III), and falls under either case one or case two. Note that we are not using the induction hypothesis here (and indeed it does not apply since $S$ is possibly of higher arity than $R$) because $S$ is already understood as a relation on tuples with no repeated entries satisfying the assumptions, so we can appeal directly to Proposition 6.3.6. Thus we may make the assumption that each such set $S$ has no thick subset.

Now the remainder of the proof is an analysis of the definable sets in order to run the construction in the previous case even without constrained elements.

**Lemma 6.3.7.** Let $\bar{c}$ be a tuple. Suppose that every tuple $\bar{a} \in (A)^r$ satisfies some finitary existential formula $\varphi(\bar{c}, \bar{a})$ with $\varphi(\bar{c}, (A)^r) = \{b \in (A)^r : A \models \varphi(\bar{c}, b)\}$ thin. Then there is a tuple $\bar{d}$ over which every $\bar{a} \in (A)^r$ satisfies a finitary existential formula with only finitely many solutions.

First, we need another lemma.

**Lemma 6.3.8.** If $\varphi(\bar{c}, v)$ is a finitary existential formula such that $S = \varphi(\bar{c}, \langle A \rangle^n)$ is a thin set, and $\bar{a} \in S$ where $\bar{a} = (a_1, \ldots, a_n)$, then for some $i$, $a_i$ satisfies a finitary existential formula over $\bar{c}$ with finitely many solutions.

**Proof.** The proof is by induction on $n$. For $n = 1$, a thin set is just a finite set and the result is clear. Now suppose that we know the result for $n$. Let $\varphi(\bar{c}, \bar{a})$ be a finitary existential formula with $|\bar{a}| = n + 1$, and suppose that $\varphi(\bar{c}, \langle A \rangle^{n+1})$ is thin. Let $k$ be maximal such that there are $k$ pairwise disjoint tuples satisfying $\varphi(\bar{c}, \bar{a})$. Write $\bar{a} = \bar{v}, w$ with $|\bar{v}| = n$.

If $\exists \bar{v} \varphi(\bar{c}, \bar{v}, A)$ is finite (including the case where $\varphi(\bar{c}, \langle A \rangle^{n+1})$ is finite) then we are done. If the set of solutions of $\exists \bar{w} \varphi(\bar{c}, \bar{v}, \bar{w})$ is thin, then we are done by the induction hypothesis.

We claim that there are only finitely many (in fact at most $k$) elements $\bar{d}$ with $\varphi(\bar{c}, \langle A \rangle^n, \bar{d})$ containing $n \cdot k + 1$ or more disjoint tuples. If not, then choose $d_1, \ldots, d_{k+1}$ distinct. Then, for $d_1$, choose $\bar{e}_1$ satisfying $\varphi(\bar{c}, \bar{v}, d_1)$. Now $d_2$ has at least $n + 1$ disjoint tuples satisfying $\varphi(\bar{c}, \bar{v}, d_2)$, so it must have some tuple disjoint from $\bar{e}_1$. Continuing in this way, we contradict the choice of $k$ by constructing $k + 1$ disjoint solutions $d_i \bar{e}_i$ of $\varphi(\bar{c}, \bar{a}, \bar{v})$. So there are finitely many $d$ such that $\varphi(\bar{c}, \langle A \rangle^n, d)$ contains $n \cdot k + 1$ or more disjoint tuples, and the set of such $d$ is definable by an existential formula. If for our given tuple $\bar{a}$, $a_{n+1}$ is one of these $d$, then we are done.

Otherwise, since the set is finite, say there are exactly $m$ such $d$, the set of $\bar{v}, w$ with $\varphi(\bar{c}, \bar{v}, \bar{w})$ and $w$ not one of these $d$ is also existentially definable and thin. Add to $\varphi(\bar{c}, \bar{v}, \bar{w})$ the existential formula which says that there are $d_1, \ldots, d_m$ with $\varphi(\bar{c}, \langle A \rangle^n, d_i)$ containing
\(n \cdot k + 1\) or more disjoint tuples, and that \(w\) is not one of the \(d_i\). So we may assume that for all \(d\), \(\varphi(\bar{c}, (A)^m, d)\) contains at most \(n \cdot k\) many disjoint tuples and that the set \(\exists w \varphi(\bar{c}, \bar{v}, w)\) is not thin. Choose \(\bar{e}_1, d_1\) satisfying \(\varphi(\bar{c}, \bar{v}, w)\). Choose \(\bar{f}_1, \ldots, \bar{f}_{n_k}\) pairwise disjoint from each other and also disjoint from \(\bar{e}_1\), and \(g_1, \ldots, g_{n_k}\) with \(\bar{f}_i, g_i\) satisfying \(\varphi\). Then we cannot have \(g_1 = \cdots = g_{n_k} = d_1\), so we can choose \(\bar{e}_2, d_2\) pairwise disjoint and disjoint from from \(\bar{e}_1, d_1\). Now choose \(\bar{f}_1, \ldots, \bar{f}_{2n_k}\) disjoint from \(\bar{e}_1, \bar{e}_2\), and \(g_1, \ldots, g_{2n_k}\). Then some \(g_i\) must be distinct from \(d_1\) and \(d_2\). Continue in this way; we contradict the fact that \(\varphi(\bar{c}, (A)^{n+1})\) is thin. This exhausts the possibilities.

We return to the proof of Lemma 6.3.7.

**Proof of Lemma 6.3.7.** There must be fewer than \(r\) elements of \(A\) which do not satisfy some existential formula over \(\bar{c}\) with finitely many solutions. If not, then there are \(a_1, \ldots, a_r \in A\) that are not contained in any such existential formula. Let \(\varphi(\bar{c}, \bar{u})\) define a thin set containing \((a_1, \ldots, a_r)\); then by the previous lemma one of \(a_1, \ldots, a_r\) must be contained in some existential formula over \(\bar{c}\) with finitely many solutions.

Let \(\bar{d}\) be \(\bar{c}\) together with these finitely many exceptions. Then every tuple \(\bar{a} \in (A)^r\) satisfies some existential formula \(\varphi(\bar{d}, \bar{u})\) with \(\varphi(\bar{d}, (A))\) finite.

**Lemma 6.3.9.** For each tuple \(\bar{c}\), there is a tuple \(\bar{a}\) such that the set of solutions of each existential formula over \(\bar{c}\) satisfied by \(\bar{a}\) is thick.

**Proof.** Suppose not. Then Lemma 6.3.7 applies. For each \(\bar{a} \in (A)^m\), let \(\varphi_{\bar{a}}\) be such that \(\varphi_{\bar{a}}(\bar{d}, (A)^m)\) is finite and as small as possible and contains \(a\). Suppose that \(\bar{b} \in (A)^m\) and \(\varphi_{\bar{a}}(\bar{d}, \bar{b})\). Also suppose that \(A \models \psi(\bar{d}, \bar{b})\) but \(A \not\models \psi(\bar{d}, \bar{a})\) where \(\psi\) is existential. Let \(n = |\varphi_{\bar{a}}(\bar{d}, (A)^m) \cap \psi(\bar{d}, (A)^m)|\). Then

\[
\varphi_{\bar{a}}(\bar{d}, \bar{a}) \land \exists_{\Sigma^2_n} \bar{u}(\varphi_{\bar{a}}(\bar{d}, \bar{u}) \land \psi(\bar{d}, \bar{u}) \land \bar{u} \neq \bar{a})
\]

but this formula has fewer solutions than \(\varphi(\bar{d}, \bar{x})\) since \(\bar{b}\) is not a solution. So every tuple \(\bar{a}\) in \((A)^m\) has some existential formula \(\varphi_{\bar{a}}\) it satisfies over \(\bar{d}\), with the property that each pair of tuples satisfying \(\varphi_{\bar{a}}\) satisfy all the same existential formulas. By Proposition 6.10 of [AK00], \(\Phi = \{\varphi_{\bar{a}} : \bar{a} \in (A)^m, m \in \omega\}\) is a Scott family. In particular, \(\varphi_{\bar{a}}(\bar{d}, (A)^r) \subseteq R\) or \(\varphi_{\bar{a}}(\bar{d}, (A)^r) \subseteq \neg R\) for each \(\bar{a} \in (A)^r\), and so \(R\) is defined by both a \(\Sigma^2_1\) formula and a \(\Pi^1_1\) formula. This is a contradiction.

**Corollary 6.3.10.** For each tuple \(\bar{c}\), there is a tuple \(\bar{a} \in (A)^r\) such that the set of solutions of each existential formula over \(\bar{c}\) satisfied by \(\bar{a}\) is thick. Also, \(\bar{a}\) is free over \(\bar{c}\).

**Proof.** Let \(\bar{c}\) be a tuple. We know that there is some \(\bar{a}_1\) such that every existential formula over \(\bar{c}\) satisfied by \(\bar{a}_1\) is thick. If \(|\bar{a}_1| \geq r\), then we can just truncate \(\bar{a}_1\). Otherwise, if \(|\bar{a}_1| < r\), we can find \(\bar{a}_2\) such that every existential formula over \(\bar{c}\bar{a}_1\) satisfied by \(\bar{a}_2\) is thick. Let \(\varphi(\bar{c}, \bar{x}, \bar{y})\) be an existential formula satisfied by \(\bar{a}_1, \bar{a}_2\). We claim that the set of solutions of \(\varphi\) is thick. Suppose not, and say that there are at most \(k\) disjoint solutions. Now the solution
set of $\varphi(c, \bar{a}_1, \bar{y})$ is thick, so there is an existential formula $\psi(c, \bar{x})$ true of $\bar{a}_1$ over $\bar{c}$ which says that there are at least $(k+1) \cdot (|\bar{a}_1| + |\bar{a}_2|) + 1$ disjoint solutions $\bar{y}$ to $\varphi(c, \bar{x}, \bar{y})$. Then the solution set of $\psi(c, \bar{x})$ is thick, so we can choose $k+1$ disjoint solutions $\bar{b}_1, \ldots, \bar{b}_{k+1}$. Then choose $\bar{d}_1$ a solution to $\varphi(c, \bar{b}_1, \bar{y})$. Now there are $2|\bar{a}_1| + |\bar{a}_2|$ entries in $\bar{b}_1 \bar{d}_1$, so one of the $(k+1) \cdot (|\bar{a}_1| + |\bar{a}_2|) + 1$ solutions to $\varphi(c, \bar{b}_2, \bar{y})$ is disjoint from $\bar{b}_1, \bar{d}_1$, and $\bar{b}_2$. We can pick some such solution $\bar{d}_2$. Continuing in this way, we get $k+1$ disjoint solutions $\bar{b}_1 \bar{d}_1, \ldots, \bar{b}_{k+1} \bar{d}_{k+1}$ to $\varphi(c, \bar{x}, \bar{y})$, a contradiction.

Now let $\bar{a} \in (A)^r$ be such that every existential formula over $\bar{c}$ satisfied by $\bar{a}$ is thick. If $\bar{a}$ is not free over $\bar{c}$, then there is some existential formula $\varphi_{\bar{a}}(\bar{c}, \bar{x})$ true of $\bar{a}$ and not true of any $\bar{a}' \in R$. Then $\varphi(c, (A)^r)$ is a $\Sigma^0_1$-definable subset of $R$, and hence thin, a contradiction.

We also want another pair of corollaries of the above lemma.

**Corollary 6.3.11.** Let $S$ be any existentially defined set, and $\bar{a} \in S$. Then there is an existentially defined $S' \subseteq S$ containing $\bar{a}$ and $\varphi(c, \bar{u}, \bar{v})$ defining $S'$ such that $\exists \bar{u} \varphi(c, \bar{u}, \bar{v})$ is thick and $\exists \bar{u} \varphi(c, \bar{u}, \bar{v})$ is finite.

**Proof.** Let $\bar{a} = \bar{a}' \bar{a}''$ where $\bar{a}'$ is contained in some finite existentially definable set over $\bar{c}$, and no entry of $\bar{a}''$ is. Let $\psi(c, \bar{u})$ be a defining formula of this finite set, and let $\chi(c, \bar{u}, \bar{v})$ define $S$. Let $S'$ be the set of solutions to $\chi(c, \bar{u}, \bar{v}) \land \psi(c, \bar{u})$. Then by Lemma 6.3.8 since no entry of $\bar{a}''$ is in a finite existentially definable set over $\bar{c}$, $\exists \bar{u} (\chi(c, \bar{u}, \bar{v}) \land \psi(c, \bar{u}))$ is thick. \qed

**Corollary 6.3.12.** Let $S$ be any existentially defined set, and $\bar{a} \in S$. Then we can write $\bar{a} = \bar{a}' \bar{a}''$ where $\bar{a}'$ is in an existential formula over $\bar{c}$ with finitely many solutions and there is an existential formula $\varphi(c, \bar{a}', \bar{u})$ which is thick and contains $\bar{a}''$.

**Proof.** Let $\bar{a} = \bar{a}' \bar{a}''$ be as in the previous corollary, and let $\chi(c, \bar{u}, \bar{v})$ define $S$. Now, let $\psi(c, \bar{u})$ be the formula defining the finite set containing $\bar{a}'$; we may choose $\psi$ so that every solution has the same existential type over $\bar{c}$ by the argument on the previous page. Since $\exists \bar{u} (\chi(c, \bar{u}, \bar{v}) \land \psi(c, \bar{u}))$ is thick, and is the union of $\chi(c, \bar{b}, \bar{v})$ for each $\bar{b}$ satisfying $\psi(c, \bar{u})$, $\chi(c, \bar{b}, \bar{v})$ is thick for some $\bar{b}$. This is witnessed by (a family of) existential formulas about $\bar{b}$ saying that there are arbitrarily many disjoint solutions to $\chi(c, \bar{b}, \bar{v})$. But all such formulas are true of each other solution of $\psi(c, \bar{u})$, and in particular of $\bar{a}'$. So $\chi(c, \bar{a}', \bar{v})$ is thick. \qed

**Lemma 6.3.13.** Let $\bar{c}$ be a tuple, and $\bar{a}, \bar{b}_1, \ldots, \bar{b}_n \in (A)$, and $\bar{a}$ is contained in no thin set over $\bar{c}$. Let $\varphi(c, \bar{u}, \bar{v}_1, \ldots, \bar{v}_n)$ be an existential formula true of $\bar{a}, \bar{b}_1, \ldots, \bar{b}_n$. Then there are $\bar{a}', \bar{d}_1, \ldots, \bar{d}_n$ satisfying $\varphi$ with $\bar{a}' \in R \iff \bar{a} \notin R$ and $\bar{d}_i \in R \iff \bar{b}_i \in R$.

**Proof.** Using the above lemma, for each $i$ write $\bar{b}_i = \bar{b}_i' \bar{b}_i''$ be contained in finite definable sets over $\bar{c}$ with $\varphi(c, \bar{u}, \bar{b}_i', \bar{v}_1, \ldots, \bar{b}_i'', \bar{v}_n)$ defining a thick set. Note that $\bar{a}$ is not contained in any thin set over $\bar{c}$, and in particular, none of its entries are in any existentially definable sets over $\bar{c}$. Using the fact that $R \times S$ (where $S$ is a fiber over some tuple of a product of $R$ and $\neg R$) has no thick subset, we can choose $\bar{d}_i = \bar{d}_i'$ for $1 \leq i \leq m$ and choose $\bar{a}'$ and $\bar{d} = \bar{d}_1' \bar{d}_n''$ to satisfy $\bar{a}' \in R \iff \bar{a} \notin R$ and $\bar{d}_i \in R \iff \bar{b}_i \in R$. \qed
These lemmas are exactly what is required for the construction in Proposition 6.3.6. Instead of choosing elements which are free, we choose elements which are not contained in any thin set, and use the above lemma to move them into \( R \) while keeping later elements from \( R \) in \( R \). We need to know that we can effectively find such elements. We can do this using the effectiveness condition and Lemma 6.3.8.

6.4 Degree Spectra Between the C.E. Degrees and the D.C.E. Degrees

We know that every degree spectrum (on a cone) which contains a non-computable degree contains all of the c.e. degrees. In this section, we will consider relations whose degree spectra strictly contain the c.e. degrees. The motivating question is whether any degree spectrum on a cone which strictly contains the c.e. degrees contains all of the d.c.e. degrees. We will show that this is false by proving Theorem 6.1.6 which says that there are two incomparable degree spectra which contain only d.c.e. degrees. In the process, we will define what it means to be \textit{intrinsically of c.e. degree} (as opposed to simply being c.e.) and give a characterization of the relatively intrinsically d.c.e. relations which are intrinsically of c.e. degree, and at the same time a sufficient (but not necessary) condition for a relation to not be intrinsically of c.e. degree.

6.4.1 Necessary and Sufficient Conditions to be Intrinsically of C.E. Degree

We begin by defining what it means to be intrinsically of c.e. degree.

Definition 6.4.1. \( R \) on \( A \) is \textit{intrinsically of c.e. degree} if in every computable copy \( B \) of \( A \), \( R^B \) is of c.e. degree.

We can make similar definitions for \textit{relatively intrinsically of c.e. degree} and \textit{intrinsically of c.e. degree on a cone}. As far as we are aware, these are new definitions.

Any relation which is intrinsically c.e. is intrinsically of c.e. degree, but the following example shows that the converse implication does not hold (even on a cone).

Example 6.4.2. Let \( A \) be two-sorted with sorts \( B \) and \( C \). There is a relation \( S \) in the signature of \( A \) of type \( B \times C \). The sort \( B \) is a directed graph, each connected component of which consists of two elements and one directed edge. Each element of \( B \) is related via \( S \) to zero, one, or two elements of \( C \), and each element of \( C \) is related to exactly one element of \( A \). \( A \) consists of infinitely many disjoint copies of each of the following three connected components and nothing else, with the edge adjacency relation and \( S \):

\[
0 \rightarrow 1 \quad 1 \rightarrow 2 \quad 2 \rightarrow 2.
\]
The numbers show how many elements of $C$ a particular element of $B$ is related to. For example, $0 \rightarrow 1$ is a two element connected component with a single directed edge, and the first element is not related to any elements of $C$, while the second element is related to a single element of $C$. The additional relation $R$ (not in the signature of $A$) consists of those elements of $B$ which are related to exactly one element of $C$.

In any copy $B$ of $A$, the set $T^B$ of elements related to exactly two elements of $C$ is c.e. in $B$. We claim that this set has the same Turing degree as $R^B$. Let $a$ and $b$ be elements in $A$, with a directed edge from $a$ to $b$. Then there are three possibilities:

1. $a \notin R^B$, $b \in R^B$ and $a \notin T^B$, $b \notin T^B$,
2. $a \in R^B$, $b \notin R^B$ and $a \notin T^B$, $b \in T^B$, or
3. $a \notin R^B$, $b \notin R^B$ and $a \in T^B$, $b \in T^B$.

Each of these three possibilities is distinct from the others both in terms of $R$ and also in terms of $T^B$. So knowing whether $a \in R^B$ and $b \in R^B$ determines whether $a \in T^B$ and $b \in T^B$, and vice versa. Hence $R^B \oplus B \equiv T^B \oplus B$. Since $T^B$ is c.e., $R^B$ is of c.e. degree in $D(B)$. Note that $R^B$ is always d.c.e. in $D(B)$, but one can show using a standard argument that $R^B$ is not always c.e. in $D(B)$.

We will begin by finding a necessary and sufficient condition for a relation to be intrinsically of c.e. degree. We will assume, for one of the directions, that the relation is relatively intrinsically d.c.e. A relation which is not intrinsically $\Delta_2$ cannot be intrinsically of c.e. degree (and, assuming sufficient effectiveness conditions, the same is true for the relative notions). We leave the question open for relations which are relatively intrinsically $\Delta_2$ but not relatively intrinsically d.c.e.

An important idea in most of the results in this work are the free tuples from the theorem of Ash and Nerode on intrinsically computable relations [AN81], and other variations.

**Definition 6.4.3.** Let $\bar{c}$ be a tuple from $A$. We say that $\bar{a} \notin R$ is free (or 1-free) over $\bar{c}$ if for any finitary existential formula $\psi(\bar{c}, \bar{x})$ true of $\bar{a}$ in $A$, there is $\bar{a}' \in R$ which also satisfies $\psi(\bar{c}, \bar{x})$.

Such free elements, and many variations, have been used throughout the literature, including in many of the results we referenced in the previous sections. We will only use 1-free elements in Section 6.3, but we will use other variants in Sections 6.4, 6.5, and 6.6.

In the spirit of the definitions made just before Propositions 2.2 and 2.3 of [AK96], we will make the following definition of what it means to be difference-free, or d-free. Let $A$ be a computable structure and $R$ a computable relation on $A$. We begin with the case where $R$ is unary, where the condition is simpler to state. We say that $a \notin R$ is d-free over $\bar{c}$ if for every $b_1, \ldots, b_n$ and existential formula $\varphi(\bar{c}, u, v_1, \ldots, v_n)$ true of $a, b_1, \ldots, b_n$, there are $a' \in R$ and $b'_1, \ldots, b'_n$ which satisfy $\varphi(\bar{c}, u, v_1, \ldots, v_n)$ such that for every existential
formula \( \psi(\bar{c}, u, v_1, \ldots, v_n) \) true of them, there are \( a'', b''_1, \ldots, b''_n \) satisfying \( \psi \) with \( a'' \notin R \) and \( b_i \in R \iff b''_i \in R \).

Note that this is different from the 2-free elements which are defined just before Propositions 2.2 and 2.3 in [AK96]. The definitions are the same, except that for \( a \) to be 2-free over \( \bar{c} \), there is no requirement on the \( b_i \) and \( b''_i \). Note that an element \( a \) may be 2-free over \( \bar{c} \), but not d-free over \( \bar{c} \) (but if \( a \) is d-free over \( \bar{c} \), then it is 2-free over \( \bar{c} \)).

Now suppose that \( R \) is an \( m \)-ary relation. We say that \( \bar{a} \) is d-free over \( \bar{c} \) if for every \( \bar{b} \) and existential formula \( \varphi(\bar{c}, \bar{u}, \bar{v}) \) true of \( \bar{a}, \bar{b} \), there are \( \bar{a}' \) and \( \bar{b}' \) which satisfy \( \varphi(\bar{c}, \bar{u}, \bar{v}) \) such that \( R \) restricted to tuples from \( \bar{c}\bar{a}' \) is not the same as \( R \) restricted to tuples from \( \bar{c}\bar{a} \) and also such that for every existential formula \( \psi(\bar{c}, \bar{u}, \bar{v}) \) true of them, there are \( \bar{a}'' \), \( \bar{b}'' \) satisfying \( \psi \) and such that \( R \) restricted to \( \bar{c}\bar{a}''\bar{b}'' \) is the same as \( R \) restricted to \( \bar{c}\bar{a} \). If \( R \) is unary, a tuple \( \bar{a} \) is d-free over \( \bar{c} \) if and only if one of its entries \( a_i \) is.

Under sufficient effectiveness conditions we will show—for a formally d.c.e. relation \( R \) on a structure \( A \)—that \( R \) is not intrinsically of c.e. degree if and only if for each tuple \( \bar{c} \) there is some \( \bar{a} \) which is d-free over \( \bar{c} \) (note that under the effectiveness conditions of Proposition 2.2 of Ash and Knight [AK96], a relation is formally d.c.e. if and only if it is intrinsically d.c.e.). In fact, the existence of a tuple \( \bar{c} \) over which no tuple \( \bar{a} \) is d-free will imply that \( R \) is not intrinsically of c.e. degree even if \( R \) is not formally d.c.e. We will use this in Theorem 6.5.19 of Section 6.5.

When stated in terms of degree spectra on a cone, our result is:

**Proposition 6.4.4.** Let \( A \) be a structure and \( R \) a relation on \( A \). Then if, for each tuple \( \bar{c} \), there is \( \bar{a} \) which is d-free over \( \bar{c} \), then the degree spectrum \( \text{dgSp}_{rel}(A, R) \) on a cone strictly contains the c.e. degrees. Moreover, if \( R \) is formally d.c.e., then this is a necessary condition.

The (relativizations of) the next two propositions prove the two directions of this using the appropriate effectiveness conditions.

**Proposition 6.4.5.** Let \( R \) be a formally d.c.e. relation on a computable structure \( A \). Suppose that there is a tuple \( \bar{c} \) over which no \( \bar{a} \) is d-free. Assume that given tuples \( \bar{a} \) and \( \bar{c} \), we can find witnesses \( \bar{b} \) and \( \varphi(\bar{c}, \bar{u}, \bar{v}) \) to the fact that \( \bar{a} \) is not d-free over \( \bar{c} \) (and furthermore, given \( \bar{a}' \) and \( \bar{b}' \) satisfying \( \varphi \), find \( \psi(\bar{c}, \bar{u}, \bar{v}) \)) as in the definition of d-freeness. Then for every computable copy \( B \) of \( A \), \( R^B \) is of c.e. degree.

**Proof.** Let \( A \) and \( R \) be as in the statement of the proposition. We will assume that \( R \) is unary. The proof when \( R \) is not unary uses exactly the same ideas, but is a little more complicated as we cannot ask whether individual elements are or are not in \( R^B \), but instead we must ask about tuples (including tuples which may include elements of \( \bar{c} \)). The translation of the proof to the case when \( R \) is not unary requires no new ideas, and considering only unary relations will make the proof much easier to understand.

Let \( \bar{c} \in A \) be such that no \( a \notin R \) is d-free over \( c \). We may omit any reference to \( \bar{c} \) by assuming that it is in our language. Let \( B \) be a computable copy of \( A \). We must show that
$R^B$ is of c.e. degree. We will use $a, a', \bar{b}, \bar{b}'$, etc. for elements of $A$, and $d, e$, etc. for elements of $B$.

We will begin by making some definitions, following which we will explain the intuitive idea behind the proof. Finally, we will define two c.e. sets $A$ and $B$ such that $R^B = A \ominus B \equiv_T A \oplus B$.

Since there are no $d$-free elements, for each $a \notin R$ there is a tuple $\bar{b}^a = b_1^a, \ldots, b_{n_a}^a$ and an existential formula $\varphi_a(u, v_1, \ldots, v_{n_a})$ such that

$$A \models \varphi_a(a, \bar{b}^a)$$

which witness the fact that $a$ is not $d$-free.

Now let $a \notin R$, $a' \in R$, and $\bar{b}'$ be such that

$$A \models \varphi_a(a', \bar{b}') = \varphi_{\bar{a}''}(a, \bar{b}^a)$$

By choice of $\varphi_a$ and $\bar{b}^a$, there is an existential formula $\psi(u, \bar{v})$ true of $a', \bar{b}'$ and extending $\varphi_a(u, \bar{v})$ such that for all $a''$ and tuples $\bar{b}'' = (b_1'', \ldots, b_{n_a}''\ldots)$ with

$$A \models \psi(a'', \bar{b}''),$$

if $b_k \in R \iff b_k'' \in R$ for all $k$ then $\bar{a}'' \in R$. Note that $\psi$ depends only on $a$, $a'$, and $\bar{b}'$ (as $\bar{b}^a$ depends on $a$). Let $\psi_{a, a', \bar{b}'}$ be this formula $\psi$. We can find $\bar{b}^a, \varphi_a$, and $\psi_{a, a', \bar{b}'}$ effectively using the hypothesis of the theorem.

Let $R$ be defined by $\alpha(u) \land \neg \beta(u)$ with $\alpha(u)$ and $\beta(u)$ being $\Sigma_1^e$ formulas with finitely many parameters (which we may assume are included in our language). We may assume that every solution of $\exists \bar{v} \psi_{a, a', \bar{b}'}(u, \bar{v})$ for every $a$, $a'$, and $\bar{b}'$ is a solution of $\alpha(u)$ by replacing each formula by its conjunction with $\alpha(u)$. Let $\alpha_s(u)$ and $\beta_s(u)$ be the finitary existential formulas which are the disjunctions of the disjuncts enumerated in $\alpha(u)$ and $\beta(u)$ respectively by stage $s$.

We can effectively enumerate the $\Sigma_1^e$ formulas which are true of elements of $B$. At each stage $s$, we have a list of formulas which we have found to be true so far. This is the partial existential diagram of $B$ at stage $s$, which we denote by $D_{3,s}(B)$. We say that an element $d \in B$ appears to be in $R^B$ at stage $s$ if one of the disjuncts of $\alpha_s(d)$ is in $D_{3,s}(B)$, and no disjunct of $\beta_s(d)$ is in $D_{3,s}(B)$. Otherwise, we say that $d$ appears to be in $\neg R^B$ at stage $s$.

Now note that $\alpha$ defines a c.e. set $\tilde{A}$ in $B$, and $\beta$ defines a c.e. set $\tilde{B}$, and $R^B = \tilde{A} \ominus \tilde{B}$. Then $\tilde{A} \oplus \tilde{B} \geq_T R^B$. If in fact we had $\tilde{A} \oplus \tilde{B} \equiv_T R^B$, then $R^B$ would be of c.e. degree. However, $R^B$ may not compute $\tilde{A} \oplus \tilde{B}$ because it may not be able to tell the difference between an element of $\tilde{B}$ and an element not in $\tilde{A}$. So we will come up with appropriate sets $A$ and $B$ where $R^B$ can tell the difference, i.e. $R^B \geq_T A$ and $R^B \geq_T B$. We can always assume that $B \subseteq A$ and hence $B = A \ominus R^B$, so it suffices to show that $R^B \geq_T A$.

The set $A$ will consist of the elements $d \in B$ with the following property: for some stages $t > s$,

1. $d$ appears to be in $R^B$ at stage $s$ and at stage $t$, and
(2) for every \( a \in A \) with \( a \notin R \) and \( \bar{e} = (e_1, \ldots, e_n) \in B \) we have found at stage \( s \) with the property that \( \varphi_a(d, \bar{e}) \) is in \( D_{3,s}(B) \), by stage \( t \) we have found an \( a' \in A \) and \( \bar{b}' \in A \) with \( \psi_{a,a',\bar{b}'}(d, \bar{e}) \) in \( D_{3,t}(B) \).

When we say that at stage \( s \) we have found \( a \in A \) and \( \bar{e} \in B \), we mean that \( a \) comes from the first \( s \) elements of \( A \) and \( \bar{e} \) from the first \( s \) elements of \( B \).

**Claim 6.4.6.** \( R^B \subseteq A \) and thus \( R^B = A - (A - R^B) \).

**Proof.** Let \( d \) be an element of \( R^B \). There is some stage \( s \) at which \( d \) appears to be in \( R^B \) and moreover, at any stage \( t > s \), \( d \) still appears to be in \( R^B \). For any \( a \in A \) and \( \bar{e} \in B \) with

\[
 B = \varphi_a(d, \bar{e}),
\]

there are \( a' \in A \) and \( \bar{b}' \in A \) with

\[
 B = \psi_{a,a',\bar{b}'}(d, \bar{e}).
\]

If \( f : A \to B \) is any isomorphism, then \( a' = f^{-1}(d) \) and \( \bar{b}' = f^{-1}(\bar{e}) \) are one possible choice. This suffices to show that \( d \in A \), and so \( R^B \subseteq A \).

The second part of the claim follows immediately. \( \square \)

**Claim 6.4.7.** \( A \) and \( A - R^B \) are c.e.

**Proof.** \( A \) is c.e. because to check whether \( d \in A \), we search for \( s \) and \( t \) satisfying the two conditions in the definition of \( A \). For a given \( s \) and \( t \), these two conditions are computable to check.

\( A - R^B \) is c.e. because it is just equal to the elements of \( A \) which satisfy the formula \( \beta \), as every element of \( A \) satisfies \( \alpha \). \( \square \)

**Claim 6.4.8.** \( R^B \geq_T A \).

**Proof.** Given \( d \), we want to check (using \( R^B \) as an oracle) whether \( d \in A \). First ask \( R^B \) whether \( d \in R^B \). If the answer is yes, then we must have \( d \in A \).

Otherwise, \( d \notin R^B \). Now, since \( A \) is c.e., it suffices to show that checking whether \( d \) is in its complement is also c.e. in \( R^B \). Suppose that at some stage \( s \), \( d \) has not yet been enumerated into \( A \), and we find \( a \in A \) and \( \bar{e} \in B \) such that:

1. \( d \) does not appear to be in \( R^B \) at stage \( s \),
2. \( a \) and \( \bar{e} \) are from among the first \( s \) elements of \( A \) and \( B \) respectively,
3. \( \varphi_a(d, \bar{e}) \) is in \( D_{3,s}(B) \), and
4. the oracle \( R^B \) tells us that \( e_j \in R^B \) if and only if \( b_j \in R \).
We claim that $d \notin A$. Suppose to the contrary that $d \in A$. Then there is a stage $s'$ at which $d$ appears to be in $R^B$, and a stage $t'$ at which $d$ enters $A$ (i.e., $s'$ and $t'$ are the $s$ and $t$ from the definition of $A$). Now at these two stages $s'$ and $t'$, $d$ appears to be in $R^B$. Moreover, we know that at stage $s$, $d$ has not yet entered $A$, and so $s < t'$. Then we must have $s < s'$ since between stages $s'$ and $t'$, $d$ appears to be in $R^B$, and this is not the case at stage $s$. Then there must be elements $a', b' \in A$ with $\psi_{a', b'}(d, \bar{e})$ in $D_{a', b'}(B)$. This is a contradiction, since by choice of $\psi_{a', b'}$ we cannot have $d \notin R^B$ and $e_j \in R^B \iff b_j \in R$.

Now, since $d \notin R^B$, there exists some $a$ and $\bar{e}$ such that

$$B \models \varphi_a(d, \bar{e})$$

and $e_j \in R^B$ if and only if $b_j \in R$. If $f : A \to B$ is an isomorphism, $a = f^{-1}(d)$ and $\bar{e} = f(\bar{b'})$ are one possible choice for $a$ and $\bar{e}$. Then one of two things must happen first—either $d$ is enumerated into $A$, or we find $a$ and $\bar{e}$ as above at some stage $s$ when $a$ does not appear to be in $R^B$ and hence $d$ is not in $A$. Since finding such $a$ and $\bar{e}$ is a computable search, the complement of $A$ is c.e. in $R^B$.

Now since $R^B \geq_T A$, $R^B \geq_T A - R^B$. Thus $R^B \geq_T A \oplus (A - R^B)$. It is trivial to see that $A \oplus (A - R^B) \geq R^B$. Finally, $A$ and $A - R^B$ are c.e. sets, and so their join is of c.e. degree. Thus $R^B$ is of c.e. degree.

For the second direction, we do not have to assume that $R$ is formally d.c.e. We will use this direction in Section 4.2 and in Section 5, and the style of the proof will be a model for Propositions 6.4.15 and 6.4.17, and Theorem 6.5.19 later on.

**Proposition 6.4.9.** Let $A$ be a computable structure and $R$ a computable relation on $A$. Suppose that for each $\bar{c}$, there is $a \in R$ d-free over $\bar{c}$. Also, suppose that for each tuple $\bar{c}$, we can effectively find a tuple $\bar{a}$ which is d-free over $\bar{c}$, and moreover we can find $\bar{a}', \bar{b}'$, and $\bar{a}''$ as in the definition of d-freeness. Then there is a computable copy $B$ of $A$ such that $R^B$ is not of c.e. degree.

**Proof.** We will assume that $R$ is a unary relation, and hence that for each $\bar{c}$, there is $a \notin R$ d-free over $\bar{c}$. We begin by describing a few conventions. We denote by $X[0, \ldots, n]$ the initial segment of $X$ (written as a binary string) of $X$ of length $n + 1$; $X[0, \ldots, n]$ can be identified with the finite set of elements of $X$ up to and including $n$. We say that a computation $\Phi^X = Y$ has use $u$ if it takes fewer than $u$ steps and it only uses the oracle $X[0, \ldots, u]$.

The proof will be to construct a computable copy $B$ of $A$ which diagonalizes against every possible Turing equivalence with a c.e. set. The construction will be a finite-injury priority construction. We use the d-free elements to essentially run a standard proof that there are d.c.e. degrees which are not c.e. degrees.

We will construct $B$ with domain $\omega$ by giving at each stage $s$ a tentative finite isomorphism $F_s : \omega \to A$. In the limit, we get $F : \omega \to A$ a bijection, and $B$ is the pullback along $F$ of the structure on $A$. We will maintain values $a_{e, i, j}[s]$, $u_{e, i, j}[s]$, $t_{e, i, j}[s]$, and $v_{e, i, j}[s]$ which
reference computations that have converged. See Figure 6.2 for a visual representation of what these values mean.

We will meet the following requirements:

\( R_{e,i,j} \): If \( \Phi_i \) and \( \Phi_j \) are total, then either \( R^B \neq \Phi^W_i \) or \( W_e \neq \Phi^R_j \).

\( S_i \): The \( i \)th element of \( A \) is in the image of \( F \).

Put a priority ordering on these requirements.

At each stage, each requirement \( R_{e,i,j} \) will be in one of four states: initialized, waiting-for-computation, waiting-for-change, or diagonalized. A requirement will move through these four stages in order, and be satisfied when it enters the state diagonalized. If it is injured, a requirement will return to the state initialized.

![Figure 6.2: The values associated to a requirement for Proposition 6.4.9. An arrow shows a computation converging. The computations use an oracle and compute some initial segment of their target. The tail of the arrow shows the use of the computation, and the head shows the length. So, for example, we will have \( R^B[0, \ldots, a_{e,i,j}] = \Phi^W_i[0, \ldots, a_{e,i,j}] \) with use \( u_{e,i,j} \).](image)

We are now ready to describe the construction.

**Construction.**

At stage 0, let \( F_s = \emptyset \) and for each \( e,i,j \), let \( a_{e,i,j}[0] \), \( u_{e,i,j}[0] \), \( t_{e,i,j}[0] \), and \( v_{e,i,j}[0] \) be 0 (i.e., undefined).

At a stage \( s + 1 \), let \( F_s : \{ 0, \ldots, \xi_s \} \to A \) be the partial isomorphism determined in the previous stage, and let \( B_s \) be the finite part of the diagram of \( B \) which has been determined so far. We have an approximation \( R^{B_s} \) to \( R^B \) which we get by taking \( k \in R^{B_s} \) if \( F_s(k) \in R \).

We will deal with a single requirement—the highest priority requirement which requires attention at stage \( s + 1 \). We say that a requirement \( S_i \) requires attention at stage \( s + 1 \) if the \( i \)th element of \( A \) is not in the image of \( F_s \). If \( S_i \) is the highest priority requirement which requires attention, then let \( a \) be the \( i \)th element of \( A \). Let \( F_{s+1} \) extend \( F_s \) with \( F_{s+1}(\xi_s+1) = a \). Set \( \xi_{s+1} = \xi_s + 1 \). Injure each requirement of lower priority.
CHAPTER 6. DEGREE SPECTRA OF RELATIONS

The conditions for a requirement \( R_{e,i,j} \) to require attention at stage \( s+1 \) depend on the state of the requirement. Below, we will list for each possible state of \( R_{e,i,j} \), the conditions for \( R_{e,i,j} \) to require attention, and the action that the requirement takes if it is the highest priority requirement that requires attention.

**Initialized:** The requirement has been initialized, so \( a_{e,i,j}[0] \), \( u_{e,i,j}[0] \), \( t_{e,i,j}[0] \), and \( v_{e,i,j}[0] \) are all 0.

**Requires attention:** The requirement always requires attention.

**Action:** Choose a new element \( a \) of \( \mathcal{A} \) which is d-free over the image of \( F_s \). Note that \( \dagger \). Set \( F_{s+1}(\xi_s + 1) = a \), \( a_{e,i,j}[s + 1] = \xi_s \), and \( \xi_{s+1} = \xi_s + 1 \).

**Waiting-for-computation:** We have set \( a_{e,i,j} \), so we need to wait for the computations (6.1) and (6.2) below to converge. Once they do, we can use the fact that \( F(a_{e,i,j}) = a \notin R \) was chosen to be d-free to modify \( F \) so that \( F(a_{e,i,j}) \in R \) to break the computation (6.1).

**Requires attention:** The requirement requires attention if there is a computation

\[
R^{B_s}\left[0, \ldots, a_{e,i,j}[s]\right] = \Phi^{W_{e,s}}_{i,s}\left[0, \ldots, a_{e,i,j}[s]\right]
\]  

(6.1)

with use \( u \), and a computation

\[
W_{e,s}\left[0, \ldots, u\right] = \Phi^{B_{s+1}}_{j,s}\left[0, \ldots, u\right]
\]  

(6.2)

with use \( v \).

**Action:** Set \( u_{e,i,j}[s + 1] = u \), \( t_{e,i,j}[s + 1] = s \), and \( v_{e,i,j}[s + 1] = v \). Let

\[
\bar{c} = (F_s(0), \ldots, F_s(a_{e,i,j}[s] - 1))
\]
\[
a = F_s(a_{e,i,j}[s])
\]
\[
\bar{b} = (F_s(a_{e,i,j}[s] + 1), \ldots, F_s(\xi_s)).
\]

Write \( \bar{c} = (c_0, \ldots, c_{a_{e,i,j}[s]}) \) and \( \bar{b} = (b_{a_{e,i,j}[s]}, \ldots, b_{\xi_s}) \). We will have ensured during the construction that \( a \) is d-free over \( \bar{c} \). So we can find \( a' \in R \) and \( \bar{b}' \) such that \( \bar{c}, a', \bar{b}' \) satisfies the same quantifier-free formulas determined so far in \( B_s \) as \( \bar{c}, a, \bar{b} \), and so that for any further existential formula \( \psi(\bar{c}, u, \bar{v}) \) true of \( \bar{c}, a', \bar{b}' \), there are \( a'' \notin R \) and \( \bar{b}'' = (b_{a_{e,i,j}[s]}, \ldots, b_{\xi_s}) \) satisfying \( \psi \) and with \( b_k'' \in R \) if and only if \( b_k \in R \). Define

\[
F_{s+1}(k) = c_k \quad \text{for } 0 \leq k < a_{e,i,j}[s]
\]
\[
F_{s+1}(a_{e,i,j}[s]) = a'
\]
\[
F_{s+1}(k) = b_k' \quad \text{for } a_{e,i,j}[s] < k \leq \xi_s.
\]

Set the state of this requirement to \textit{waiting-for-change}. Each requirement of lower priority has been injured. Reset the state of all such requirements to \textit{initialized} and set all of the corresponding values to 0.
**Waiting-for-change:** In the previous state, \( F \) was modified to break the computation (6.1). If we are to have \( R^B = \Phi_{t}^W \), then \( W_e \) must change below the use of that computation. In this state, we wait for this to happen, and then use the fact that \( a \) was chosen to be d-free in state INITIALIZED to return \( R^B \) to the way it was for the computation (6.2).

**Requires attention:** Let \( u = u_{e,i,j}[s] \), \( v = v_{e,i,j}[s] \) and \( t = t_{e,i,j}[s] \). The requirement requires attention if
\[
W_{e,s}[0,\ldots,u] \neq W_{e,t}[0,\ldots,u].
\]

**Action:** Let
\[
\begin{align*}
\bar{c} &= (F_{s}(0), \ldots, F_{s}(a_{e,i,j}[s] - 1)) \\
\bar{a}' &= F_{s}(a_{e,i,j}[s]) \\
\bar{b}' &= (F_{s}(a_{e,i,j}[s] + 1), \ldots, F_{s}(v)) \\
\bar{d}' &= (F_{s}(v + 1), \ldots, F_{s}(\xi_s))
\end{align*}
\]

As before, write \( \bar{c} = (c_0, \ldots, c_{a_{e,i,j}[s]-1}) \), \( \bar{b}' = (b'_{a_{e,i,j}[s]+1}, \ldots, b'_{e,s}[v]) \), and \( \bar{d}' = (d'_{v+1}, \ldots, d'_{\xi_s}) \). Now \( a' \) was chosen in state WAITING-FOR-COMPUTATION. So we can choose \( a'' \notin R \), \( b'' = (b''_{a''_{e,i,j}[s]+1}, \ldots, b''_{e,s}[v]) \), and \( d'' = (d''_{v+1}, \ldots, d''_{\xi_s}) \) such that \( c\bar{a}'\bar{b}'\bar{d}' \) satisfies any formula determined by \( B_s \), to be satisfied by \( \bar{c}a''\bar{b}''\bar{d}'' \), and moreover \( b''_{e,s}[k] \in R \) if and only if \( F_t(k) \in R \) (note that \( F_t(k) \) is the value \( b_k \) from state WAITING-FOR-COMPUTATION). Define
\[
\begin{align*}
F_{s+1}(k) &= c_k & \text{for } 0 \leq k < a_{e,i,j}[s] \\
F_{s+1}(a_{e,i,j}[s]) &= a'' \\
F_{s+1}(k) &= b''_k & \text{for } a_{e,i,j}[s] < k \leq v \\
F_{s+1}(k) &= d''_k & \text{for } v < k \leq \xi_s.
\end{align*}
\]

Then we will have
\[
R_{s+1}^B[0,\ldots,v] = R_s^B[0,\ldots,v].
\]

So
\[
\Phi_{j}^{R_s^B}[0,\ldots,u] = \Phi_{j}^{R_s^B}[0,\ldots,u] = W_{e,t}[0,\ldots,u] \neq W_{e,s+1}[0,\ldots,u]
\]
since the use of this computation at stage \( t \) was \( v \). Set the state of this requirement to DIAGONALIZED. Each requirement of lower priority has been injured. Reset the state of all such requirements to INITIALIZED and set all of the corresponding values to \( \emptyset \).

**Diagonalized:** In this state, \( R^B \) is the same as it was under the use \( v \) in the computation (6.2) from state WAITING-FOR-COMPUTATION. By (6.2), if we are to have \( W_e = \Phi_{j}^{R_s^B} \), then \( W_e \) restricted to the elements \( 0,\ldots,u \) must be the same as it was then. But this cannot happen, because some such element has entered \( W_e \) since then. So we have satisfied the requirement \( R_{e,i,j} \).
Requires attention: The requirement never requires attention.

Action: None.

Set $B_{s+1}$ to be the atomic and negated atomic formulas true of $0, \ldots, \xi_{s+1}$ with Gödel number at most $s$.

End construction.

Note that at any stage $s$, the $a_{e,i,j}$ are ordered by the priority of the corresponding requirements. This is because if a requirement is injured, each lower priority requirement is injured at the same time, and then new values of $a_{e,i,j}$ are defined in order of priority. Moreover, if $R_e, i, j$ is of higher priority than $R_{e', i', j'}$ and $v_{e,i,j}$ is defined, then $a_{e,i,j} < v_{e,i,j} < a_{e', i', j'}$.

If $R_{e,i,j}$ is never injured after some stage $s$, then it only acts at most three times—once in each of the stages INITIALIZED, WAITING-FOR-COMPUTATION, and WAITING-FOR-CHANGE, in that order— and it never moves backwards through the states. A requirement $S_i$ acts only once if it is not injured. So every requirement is injured only finitely many times.

It remains to show that every requirement is eventually satisfied. Suppose to the contrary that some requirement is not satisfied. There must be some least such requirement. First, suppose that it is a requirement $S_i$. Then there is a stage $s$ after which each higher priority requirement never acts. Then at the next stage, $S_i$ acts, and is never again injured. So $S_i$ is satisfied.

Now suppose that $R_{e,i,j}$ is the least requirement which is not satisfied, and let $s$ be a stage after which each higher priority requirement never acts. So $R_{e,i,j}$ is never injured after the stage $s$. Also, since $R_{e,i,j}$ is not satisfied, we have

$$W_e = \Phi_j^{R^B_e}$$

and

$$R^B_e = \Phi_i^{W^B_e}.$$ 

If $R_{e,i,j}$ was in state INITIALIZED at stage $s$, then at a later stage, $a_{e,i,j}$ is defined and the requirement enters stage WAITING-FOR-COMPUTATION. Eventually, at a stage $t$, the following computations must converge:

$$R^{B_t^e}[0, \ldots, a_{e,i,j}] = \Phi_{t}^{W_{e,t}}[0, \ldots, a_{e,i,j}]$$

with use $u$ \hspace{1cm} (6.3)

$$W_{e,t}[0, \ldots, u] = \Phi_{j,t}^{R^{B_t^e}}[0, \ldots, u]$$

with use $v$. \hspace{1cm} (6.4)

Then $R_{e,i,j}$ requires attention at stage $t + 1$. We modify $F$ to have $F(a_{e,i,j}) \in R$, breaking computation (6.3). Requirement $R_{e,i,j}$ also moves to state WAITING-FOR-CHANGE.

Since $R^B_e = \Phi_i^{W_e}$, eventually at some stage $t'$, $W_e$ must change below the use $u$ of the computation (6.3). Then $R_{e,i,j}$ requires attention at stage $t' + 1$. We modify $F$ by moving $F(a_{e,i,j})$ back to $-R$ and ensuring that

$$R^{B_{t'+1}}[0, \ldots, v] = R^{B_t}[0, \ldots, v].$$
But for every stage $t'' > t'$ we have $W_{e,t''}[0, \ldots, u] \neq W_{e,t}[0, \ldots, u]$ since $W_e$ is a c.e. set and $W_{e,t'}[0, \ldots, u] \neq W_{e,t}[0, \ldots, u]$.

This, together with computation (6.4) contradicts the assumption that $R^B = \Phi^W_e$. So every requirement is satisfied. \qed

In Proposition 6.4.5, we showed that a condition about the non-existence of free elements is equivalent to a condition on the possible degrees of the relation $R^B$ in computable copies $B$ of $A$. In particular, we showed that the relation $R^B$ is Turing equivalent to the join of c.e. sets. In, for example, the theorems of Ash and Nerode [AN81] and Barker [Bar88] there are two parts: first, that a condition on the existence of free tuples is equivalent to a condition on the possible computability-theoretic properties of $R^B$; and second, that a condition on the existence of free tuples is equivalent to a syntactic condition on the relation $R$. In Propositions 6.4.5 and 6.4.9, we are missing this second syntactic part.

We might hope that there is a syntactic condition which is equivalent (under some effectiveness conditions) to being intrinsically of c.e. degree. For example, one candidate (and certainly a sufficient condition) would be that there are formally $\Sigma^0_1$ sets $A \supseteq R$ and $B = A - R$ such that $A$ is formally $\Sigma^0_1$ and $\Pi^0_1$ relative to $R$.

In the proof of Proposition 6.4.5, we found c.e. sets $A$ and $B$ in our particular copy $B$ of $A$ such that $R^B = A - B$ and $R^B \geq_T A$. These c.e. sets were not necessarily definable by $\Sigma^0_1$ formulas, but instead depended on the enumeration of $B$. When $R$ was defined by $\alpha(x) \land \neg \beta(x)$, whether or not an element $a \notin R^B$ which satisfied both $\alpha(x)$ and $\beta(x)$ was in $A$ depended on the order in which we discovered certain facts in $B$.

The following example should be taken as (very strong) evidence that we cannot find an appropriate syntactic condition.

**Example 6.4.10.** Consider a structure as in Example 6.4.2, except that the connected components are different. There are infinitely many copies of each of the following five connected components, and no others:

```
    0 → 1 → 1 → 0 → 1 → 0 → 1 → 1 → 0 → 1 → 1 → 0 → 1 → 1 → 0 → 1 → 1 
```

The formally d.c.e. relation $R$ consists of the nodes which are labeled 1. Note that the elements at the center of their connected component are definable in both a $\Sigma^0_1$ and a $\Pi^0_1$ way (in a $\Sigma^0_1$ way as they are the only nodes of degree at least three, and in a $\Pi^0_1$ way because they are the unique such nodes in their connected components, and so they are the only node...
which are not connected to some other node of degree at least three). In particular, given a connected component, we can compute the center.

**Claim 6.4.11.** $R$ is relatively intrinsically of c.e. degree.

**Proof.** We will use Proposition 6.4.5.\(^6\) First, we claim that no tuple $\bar{a}$ is d-free over $\emptyset$. Since $R$ is unary, it suffices to show that no single element $a$ is d-free over $\emptyset$. If $a$ is not in the center of its connected component, then there is an existential formula $\varphi(u)$ which witnesses this. If $a' \in R$ also satisfies $\varphi(u)$, then $\varphi(u) \land \psi(u)$ is true of $a'$ where $\psi(u)$ is the existential formula which says that $u$ is labeled “1” or “2”. Every solution of $\varphi(u) \land \psi(u)$ is in $R$. So $a$ is not d-free over $\emptyset$. Now suppose that $a \notin R$ is the center element of its connected component. If $a$ is labeled “2”, then it is not d-free over $\emptyset$. So suppose that $a$ is labeled “0”. There is a $b \notin R$ which is connected to $a$, and an existential formula $\varphi(u,v)$ which says that $u$ is the center element of its connected component and $v$ is connected to $u$. Now if $a' \in R$ satisfies $(\exists v) \varphi(u,v)$, then $a'$ also satisfies $\psi(u)$ which says that there is a chain of length two leading off of $a'$. Now suppose that $a'' \notin R$ satisfies $\psi(u) \land (\exists v) \varphi(u,v)$. Let $b''$ be such that we have $\psi(a'',b'')$. Now, since $a''$ satisfies $\psi(u)$, any such $b''$ must be labeled “1” and hence be in $R$. But we had $b \notin R$. Thus $a$ cannot have been d-free. So there is no tuple $\bar{a}$ which is d-free over $\emptyset$. The effectiveness conditions of Proposition 6.4.5 are immediate, because everything we did above is computable. Moreover, this relativizes. Thus $R$ is relatively intrinsically of c.e. degree. \(\square\)

Now we argue that there is no syntactic fact about $R$ which explains this. Such a syntactical fact should say that $R$ is intercomputable with a join of formally $\Sigma_1$ sets. So it should say something like: there is a join $S$ of formally $\Sigma_1$ sets which is formally $\Pi_1(R)$, and $R$ is formally $\Sigma_1(S)$ and $\Pi_1(S)$. We will show that there are no non-trivial sets which are formally $\Sigma^0_1$ and $\Pi^0_1(R)$.

Let $A$, $B$, $C$, $D$, and $E$ be the sets of points which are at the center of connected components of the first, second, third, fourth and fifth types respectively. It is not hard to check that the formally $\Sigma^0_1$ sets are:

1. $A \cup B \cup C \cup D \cup E$,
2. $B \cup C \cup D \cup E$,
3. $B \cup C$,
4. $C \cup D \cup E$, and
5. $C$.

The formally $\Sigma^0_1(R)$ sets are those above, and in addition:

1. $B$,

\(^6\)This will also give us our first non-trivial application of Proposition 6.4.5.
CHAPTER 6. DEGREE SPECTRA OF RELATIONS

(2) $A \cup D$,
(3) $A \cup B \cup D$,
(4) $A \cup C \cup D \cup E$.

Every $\Sigma_1(R)$ set containing $E$ also contains $C$ and $D$. If a set $X$ and its complement $\bar{X}$ are both $\Sigma_1(R)$ and one of them is $\Sigma_1$, then one of them must contain $E$, and hence $C$ and $D$. Then the other must be contained within $A \cup B$. The only possibilities for $X$ and $\bar{X}$ are $B$ and $A \cup C \cup D \cup E$, but neither of these are formally $\Sigma_1$.

So there are no formally $\Sigma_1$ sets which are also formally $\Pi_1(R)$. Thus it seems impossible to have a syntactic condition which is necessary and sufficient for a relation to be intrinsically of c.e. degree.

6.4.2 Incomparable Degree Spectra of D.C.E. Degrees

In this section our goal is to prove the following theorem, which will yield Theorem 6.1.6:

**Theorem 6.4.12.** There are structures $\mathcal{A}$ and $\mathcal{M}$ with relations $R$ and $S$ respectively which are relatively intrinsically d.c.e. such that the degree spectra of $R$ and of $S$ are incomparable even relative to every cone.

This is a surprising result which is interesting because it says that there is no “fullness” result for d.c.e. degrees over the c.e. degrees, that is, no result that says that any degree spectrum on a cone which strictly contains the c.e. degrees contains all of the d.c.e. degrees. Also, it implies that the partial ordering of degree spectra on a cone is not a linear order. The two structures $\mathcal{A}$ and $\mathcal{M}$, and the relations $R$ and $S$, are as follows.

**Example 6.4.13.** Let $\mathcal{A}$ be two-sorted with sorts $A_1$ and $A_2$. The first sort $A_1$ will be the tree $\omega^{<\omega}$ with the relation “is a child of” and the root node distinguished. The second sort $A_2$ will be an infinite set with no relations. There will be a single binary relation $U$ of type $A_1 \times A_2$. Every element of $A_2$ will be related by $U$ to exactly one element of $A_1$, and each element of $A_1$ will be related to zero, one, or two elements of $A_2$. The only elements of $A_1$ related to no elements of $A_2$ are those of the form $0^n = 0\ldots0$. Any other element of the form $\sigma.a$ is related to one element of $A_2$ if $a$ is odd, and to two if $a$ is even. The structure $\mathcal{A}$ consists of these two sorts $A_1$ and $A_2$, the “is a child of” relation, the root of the tree, and $U$.

We say that an element of $A_1$ is of type $(n)$ (so possibly of type $(0)$, type $(1)$, or type $(2)$) if it is related by $U$ to exactly $n$ elements of $A_2$. The relation $R$ on $\mathcal{A}$ is the set of elements of $A_1$ which are related by $U$ to exactly one element of $A_2$, that is, the elements of type $(1)$.

**Example 6.4.14.** Let $\mathcal{M}$ be a three-sorted model with sorts $M_1$, $M_2$, and $M_3$. The sort $M_1$ will be the tree $\omega^{<\omega}$, however, instead of defining the tree with the relation “is a child of,”
the tree will be given as a partial order. We will have a relation $V$ of type $M_1 \times M_2$ which is defined in the same way as $U$, except with $M_1$ replacing $A_1$ and $M_2$ replacing $A_2$. There will be another relation $W$ on $M_1 \times M_3$ such that each element of $M_3$ is related by $W$ to exactly one element of $M_1$, and each element of $M_1$ is related to either no elements or one element of $M_3$. An element of $M_1$ will be related (via $W$) to an element of $M_3$ exactly if its last entry is odd, but it is not of the form $0^n 1$.

Once again, we give elements of $M_1$ a “type”. We say that an element of $M_1$ is of type $\langle n, m \rangle$ if it is related by $V$ to exactly $n$ elements of $M_2$ and by $W$ to exactly $m$ elements of $M_3$. The possible types of elements are $\langle 0, 0 \rangle$, $\langle 1, 0 \rangle$, $\langle 1, 1 \rangle$, and $\langle 2, 0 \rangle$. The relation $S$ on $M$ will be defined in the same way as $R$, but again $A_1$ is replaced by $M_1$ and $A_2$ by $M_2$; so $S$ consists of the elements of types $\langle 1, 0 \rangle$ and $\langle 1, 1 \rangle$. Note that every element of $B$ which is in $S$, except for those of the form $0^n 1$, is of type $\langle 1, 1 \rangle$, and hence satisfies an existential formula which is only satisfied by elements of $S$.

Both examples have d-free elements over any tuple $\bar{c}$; these elements are of the form $0^n$ for $n$ large enough that no children of $0^n$ appear in $\bar{c}$ (i.e., the elements of type $\langle 0 \rangle$ in $A$ or type $\langle 0, 0 \rangle$ in $M$). In either structure, any existential formula (over $\bar{c}$) satisfied by $0^n$ is also satisfied by $0^{n-1} 1$, and any existential formula satisfied by $0^{n-1} 1$ is also satisfied by $0^{n-1} b$ for $b$ even. Moreover, the relation $R$ (or $S$) on the subtrees of $0^n$ and $0^{n-1} b$ is the same under the natural identification. Both structures satisfy the effectiveness condition from Proposition 6.4.9, so for all degrees $d$, $\text{dgSp}(A, R)_{sd}$ strictly contains $\Sigma^0_1(d)$.

Note that in $A$, there is an existential formula $\varphi(u)$ which says that $u$ is of type $\langle 2 \rangle$, and an existential formula $\psi(u)$ which says that $u$ is of type $\langle 1 \rangle$ or of type $\langle 2 \rangle$ (i.e., of type $\langle n \rangle$ with $n \geq 1$). Similarly, in $M$, for each $n_0$ and $m_0$ there are existential formulas which say that an element $u$ is of type $\langle n, m \rangle$ with $n \geq n_0$ and $m \geq m_0$.

We begin by proving in Proposition 6.4.15 that there is a Turing degree in the (unrelativized) degree spectrum of $S$ which is not in the degree spectrum of $R$ before proving in Proposition 6.4.17 that there is a Turing degree in the degree spectrum of $R$ which is not in the degree spectrum of $S$. After each proposition, we give the relativized version.

**Proposition 6.4.15.** There is a computable copy $N$ of $M$ such that no computable copy $B$ of $A$ has $S_N \equiv_T R_B$.

**Proof.** The proof will be very similar to that of Proposition 6.4.9, though it will not be as easy to diagonalize as both structures have d-free elements. We will construct $N$ with domain $\omega$ by giving at each stage a tentative finite isomorphism $F_s: \omega \to M$. $F = \lim F_s$ will be a bijection, giving $N$ as an isomorphic copy.

We need to diagonalize against computable copies of $A$. Given a computable function $\Phi_e$, we can try to interpret $\Phi_e$ as giving the diagram of a computable structure $B_e$ isomorphic to $A$. At each stage, we get a finite substructure $B_{e,s}$ which is isomorphic to a finite substructure of $A$ by running $\Phi_e$ up to stage $s$, and letting $B_{e,s}$ be the greatest initial segment on which all of the relations are completely determined and which is isomorphic to a finite substructure of $A$ (because $A$ is relatively simple, this can be checked computably). Let $\mathcal{B}_e$ be the union
of the \( B_{e,s} \). If \( \Phi_e \) is total and gives the diagram of a structure isomorphic to \( A \), then \( B_e \) is that structure. Otherwise, \( B_e \) will be some other, possibly finite, structure. For elements of \( B_{e,s} \), we also have an approximation of their type in \( B_{e,s} \) by looking at how many elements of the second sort they are connected to in \( B_{e,s} \). By further reducing the domain of \( B_{e,s} \), we may assume that \( B_{e,s} \) has the following property since \( A \) does: all of the elements of \( B_{e,s} \) which are of type \((0)\) in \( B_{e,s} \) are linearly ordered.

We will meet the following requirements:

\[
R_{e,i,j} : \text{If } B_e \text{ is isomorphic to } A, \text{ and } \Phi_i^{R_e} \text{ and } \Phi_j^{S^N} \text{ are total, then either } S^N \neq \Phi_i^{R_e} \text{ or } R^B \neq \Phi_j^{S^N}.
\]

\( S_i \): The \( i \)th element of \( M \) is in the image of \( F \).

Note that the \( d \)-free elements of both structures are linearly ordered. Suppose that in \( A \), \( p \notin R \) is \( d \)-free (so \( p = 0^\ell \) for some \( \ell \) ), and \( q \notin R \) is \( d \)-free over \( p \) and in the subtree below \( p \). Then, using the fact that \( p \) is \( d \)-free, we can replace it by \( p' \in R \) (replacing \( q \) by \( q' \)) while maintaining any existential formula, and then we can replace \( p' \) by \( p'' \notin R \) (and \( q' \) by \( q'' \)). However, \( q'' \) will no longer be \( d \)-free, because it will not be of the form \( 0^k \). The same is true in \( M \). This is what we will exploit for both this proof and the proof of the next proposition.

Figure 6.3: The values associated to a requirement for Proposition 6.4.15. An arrow shows a computation converging. The computations use an oracle and compute some initial segment of their target. The tail of the arrow shows the use of the computation, and the head shows the length.

The way we will meet \( R_{e,i,j} \) will be to put a \( d \)-free element \( x \notin S \) into \( N \). If there is no \( p \) in \( B_{e,s} \) which is \( d \)-free, we would be able to diagonalize by moving \( x \) to \( x' \in S \), and
then later to $x'' \notin S$ and using appropriate computations as in Proposition 6.4.9. So we may assume that there is $p$ in $B_{e,s}$ which is d-free. Now if $B_e$ is an isomorphic copy of $A$, we will eventually find a chain $p_0, p_1, \ldots, p_n = p$ from the root node $p_0$ to $p$, where $p_{i+1}$ is a child of $p_i$. Then in $A$ every d-free element aside from $p_0, \ldots, p_n = p$ is in the subtree below $p$.

In $N$, we just have to respect the tree-order, so no matter how much of the diagram of $N$ we have built so far, we can always add a new d-free element $y$ such that $x$ is in the subtree below $y$. Then we will use the fact described above about d-free elements which are in the subtree below another d-free element. By moving $x$ to $x' \in S$ and then to $x'' \notin S$, we can force $p$ to move to $p' \in R$ and then $p'' \notin R$. Then, by moving $y$ to $y' \in S$ and then $y'' \notin S$, we can diagonalize as $B_e$ will have no d-free elements which it can use: all of the d-free elements which were below $p$ have now been moved to be below $p''$ are are no longer d-free. We could still move $y$ to $y'$ and then $y''$ as $x$ was in the subtree below $y$ rather than vice versa. As in Proposition 6.4.9, we will use various computations to force $B_e$ to follow $N_e$.

The requirement $R_{e,i,j}$ will have associated to it at each stage $s$ values $a_{e,i,j}[s]$, $u_{e,i,j}[s]$, $v_{e,i,j}[s]$, $m_{e,i,j}[s]$, $t_{e,i,j}[s]$, and $b_{e,i,j}[s]$, $\mu_{e,i,j}[s]$, $\nu_{e,i,j}[s]$, and $\tau_{e,i,j}[s]$. These values will never be redefined, but may be canceled. When a requirement is injured, its corresponding values will be canceled. Figure 6.3 shows how these values are related.

At each stage, each of the requirements $R_{e,i,j}$ will be in one of the following states:

- INITIALIZED,
- WAITING-FOR-FIRST-COMPUTATION,
- WAITING-FOR-SECOND-COMPUTATION,
- WAITING-FOR-FIRST-CHANGE,
- WAITING-FOR-SECOND-CHANGE,
- WAITING-FOR-THIRD-CHANGE, or
- DIAGONALIZED.

Every requirement will move through these linearly in that order.

We are now ready to describe the construction.

**Construction.**

At stage $0$, let $F_s = \emptyset$ and for each $e$, $i$, and $j$ let $a_{e,i,j}[0]$, $u_{e,i,j}[0]$, and so on be 0 (i.e., undefined).

At a stage $s + 1$, let $F_s : \{0, \ldots, \xi_s\} \rightarrow M$ be the partial isomorphism determined in the previous stage, and let $D(N_s)$ be the finite part of the diagram of $N$ which has been determined so far. We have an approximation $S^N_s$ to $S^N$ which we get by taking $k \in S^N_s$ if $F_s(k) \in S$. For each $e$, we have a guess $R^S_{e,s}$ at $R^S_e$ using the diagram of the finite structure
$B_{e,s}$, given by $x \in R_{e,s}^B$ if and only if in $B_{e,s}$, $x$ is of type (1) (i.e., related by $U_{B_{e,s}}$ to exactly one element of the second sort).

We will deal with a single requirement—the highest priority requirement which requires attention at stage $s + 1$. A requirement $S_i$ requires attention at stage $s + 1$ if the $i$th element of $A$ is not in the image of $F_s$. If $S_i$ is the highest priority requirement which requires attention, then let $c$ be the $i$th element of $A$. Let $F_{s+1}$ extend $F_s$ with $c$ in its image. Injure each requirement of lower priority.

The conditions for a requirement $R_{e,i,j}$ to require attention at stage $s + 1$ depend on the state of the requirement. Below, we will list for each possible state of $R_{e,i,j}$, the conditions for $R_{e,i,j}$ to require attention, and the action that the requirement takes if it is the highest priority requirement that requires attention. We will also loosely describe what is happening in the construction, but a more rigorous verification will follow.

**Initialized:** The requirement has been initialized, so $a_{e,i,j}[0]$, $u_{e,i,j}[0]$, and so on are all 0.

**Requires attention:** The requirement always requires attention.

**Action:** Let $F_{s+1}$ extend $F_s$ by adding to its image the element $0^\ell$, where $\ell$ is large enough that $0^\ell$ has no children in $\text{ran}(F_s)$. Then $0^\ell$ is d-free over $\text{ran}(F_s)$. Let $a_{e,i,j}[s+1]$ be such that $F_{s+1}(a_{e,i,j}[s+1]) = 0^\ell$. Change the state to WAITING-FOR-FIRST-COMPUTATION.

**Waiting-for-first-computation:** We have set $F(a_{e,i,j}) = 0^\ell \notin R$ a d-free element.

We wait for the computations (6.5) and (6.6) below. Then, we use the fact that $M$ is given using the tree-order to insert an element $b_{e,i,j}$ in $\mathcal{N}$ above $a_{e,i,j}$ (so that now the image of $b_{e,i,j}$ under $F$ is $0^\ell$ and the image of $a_{e,i,j}$ under $F$ is $0^{\ell+1}$).

**Requires attention:** The requirement requires attention if:

1. there is a computation

$$S_{s}^N[0, \ldots, a_{e,i,j}[s]] = \Phi_{i,s}^{R^B_{e}}[0, \ldots, a_{e,i,j}[s]]$$

with use $u < s$,

2. each element $p$ of the first sort in $B_{e,s}$, with $p \leq u$, is part of a chain $p_0, p_1, \ldots, p_n = p$ in $B_{e,s}$ where $p_0$ is the root node and $p_{i+1}$ is a child of $p_i$, and

3. there is a computation

$$R_{s}^B[0, \ldots, m] = \Phi_{j,s}^{S^N}[0, \ldots, m]$$

with use $v < s$ where $m \geq u$ is larger than each $p_i$ above.
**Action:** Set $u_{e,i,j}[s+1] = u$, $v_{e,i,j}[s+1] = v$, $m_{e,i,j}[s+1] = m$, and $t_{e,i,j}[s+1] = s$. Let $a = a_{e,i,j}[s]$. We have $F_s(a) = 0^\ell$, where $\ell$ is large enough that no child of $0^\ell$ appears earlier in the image of $F_s$. Set

$$F_{s+1}(w) = \begin{cases} 0^{\ell+1}\sigma & F_s(w) = 0^\ell\sigma \\ F_s(w) & \text{otherwise} \end{cases}.$$  

What we have done is taken every element of $N$ which was mapped to the subtree below $0^\ell$, and moved it to the subtree below $0^{\ell+1}$. Now let $b = b_{e,i,j}[s+1]$ be the first element on which $F_{s+1}$ is not yet defined and set $F_{s+1}(b) = 0^\ell$. So $F_{s+1}(b)$ is the parent of $F_{s+1}(a)$. Any existential formula which was true of the tree below $0^\ell$ is also true of the tree below $0^{\ell+1}$. Also, for $w \in \text{dom}(F_s)$, $F_s(w) \in R$ if and only if $F_{s+1}(w) \in R$. Change the state to WAITING-FOR-SECOND-COMPUTATION.

**WAITING-FOR-SECOND-COMPUTATION:** In the previous state we defined $b_{e,i,j}$, so now we have to wait for the computations (6.7) and (6.8) below involving it. Then we modify $F$ so that it now looks like $a_{e,i,j} \in S$, breaking the computation (6.5) above.

**Requires attention:** The requirement requires attention if there are computations

$$S^N_s[0, \ldots, b_{e,i,j}[s]] = \Phi^{R_b}_{i,s}[0, \ldots, b_{e,i,j}[s]]$$  

with use $\mu < s$, and

$$R^b_s[0, \ldots, \mu] = \Phi^{S^N}_{j,s}[0, \ldots, \mu]$$  

with use $\nu < s$.

**Action:** Set $\mu_{e,i,j}[s+1] = \mu$, $\nu_{e,i,j}[s+1] = \nu$, and $t_{e,i,j}[s+1] = s$. Let $a = a_{e,i,j}[s]$ and $b = b_{e,i,j}[s]$. We have $F_s(a) = 0^{\ell+1}$ and $F_s(b) = 0^\ell$, where $\ell$ is large enough that no child of $0^\ell$ appears before $a$ in the image of $F_s$. Choose $x$ odd and larger than any number we have encountered so far, and define $F_{s+1}$ by

$$F_{s+1}(w) = \begin{cases} 0^\ell x^{-}\sigma & F_s(w) = 0^{\ell+1}\sigma \\ F_s(w) & \text{otherwise} \end{cases}.$$  

What we have done is taken every element of $N$ which was mapped to the subtree below $0^{\ell+1}$, and moved it to the subtree below $0^\ell x$. Note that $F_{s+1}(b) = F_s(b)$ and for $w < a$, $F_s(w) = F_{s+1}(w)$. Any existential formula which was true of the tree below $0^\ell$ is also true of the tree below $0^\ell x$ (but not vice versa, since $0^\ell x$ is of type (1) but $0^{\ell+1}$ is of type (0)). Also, for $w \in \text{dom}(F_s)$, $F_s(w) \in S$ if and only if $F_{s+1}(w) \in S$ with the single exception of $w = a$. In that case, $F_s(a) \notin S$ and $F_{s+1}(a) \in S$. Change to state WAITING-FOR-FIRST-CHANGE.
 CHAPTER 6. DEGREE SPECTRA OF RELATIONS

Waiting-for-first-change: In the previous state, we modified $F$ to break the computation (6.5). If we are to have $S^N = \Phi_i^{R_{e\ell}}$, then $R_{e\ell}$ must change below its use $u_{e,i,j}$. So some element of $B_e$ which was previously of type (0) becomes of type (1), or some element which was previously of type (1) becomes of type (2). When this happens, we modify $F$ (by changing the image of $a_{e,i,j}$ again) so that $S^N$ becomes the same as it was originally (below $\nu$).

Requires attention: The requirement requires attention if

$$R^B_x[0, \ldots, u_{e,i,j}[s]] \neq R^B_{\ell e,i,j}[s][0, \ldots, u_{e,i,j}[s]].$$

Action: Let $a = a_{e,i,j}[s]$ and $b = b_{e,i,j}[s]$. We have $F_s(a) = 0^\ell x$, where $x$ is odd. Choose $y > 0$ even and larger than any number we have encountered so far, and define $F_{s+1}$ by

$$F_{s+1}(w) = \begin{cases} 
0^\ell y \cdot \sigma & F_s(w) = 0^\ell x \cdot \sigma \\
F_s(w) & otherwise
\end{cases}.$$

This is moving the subtree below $0^\ell x$ to the subtree below $0^\ell y$. Once again, $F_{s+1}(b) = F_s(b) = 0^\ell$. For $w \in \text{dom}(F_s), w \neq a$, we have $F_s(w) \in S$ if and only if $F_{s+1}(w) \in S$. For $w = a$, we have $F_s(a) \in S$ and $F_{s+1}(a) \notin S$. Change the state to waiting-for-second-change.

Waiting-for-second-change: In the previous state, we modified $F$ so that $S^N$ is the same as it was previously in state waiting-for-first-computation. By the computation (6.6), $R_{e\ell}$ must return, below the use $u_{e,i,j}$, to the way it was previously (i.e., as it was when the computation (6.6) was found). It must be that the element from state waiting-for-first-change which changed its type then (from type (0) to type (1) or from type (1) to type (2)) must now change its type again, and so it must have gone from type (0) to type (1) and now changes from type (1) to type (2). Call this element $p$. When this happens, we modify $F$ so that $b_{e,i,j}$ looks like it is in $S^N$. We can do this because so far we have only modified the image of $a_{e,i,j}$, and $a_{e,i,j}$ was in the subtree below $b_{e,i,j}$. This breaks the computation (6.7).

Requires attention: The requirement requires attention if

$$R^B_x[0, \ldots, \mu_{e,i,j}[s]] = R^B_{\ell e,i,j}[s][0, \ldots, \mu_{e,i,j}[s]].$$

and also, each of the elements $m_{e,i,j}[s] + 1, \ldots, \mu_{e,i,j}[s]$ is of type (1) or type (2).

Action: Let $b = b_{e,i,j}[s]$. We have $F_s(b) = 0^\ell$. Choose $x > 0$ odd and larger than any number we have encountered so far, and define $F_{s+1}$ by

$$F_{s+1}(w) = \begin{cases} 
0^{\ell-1} x \cdot \sigma & F_s(w) = 0^\ell \cdot \sigma \\
F_s(w) & otherwise
\end{cases}.$$
This is moving the subtree below $0^\ell$ to the subtree below $0^{\ell-1}$x. For $w \in \text{dom}(F_s)$, $w \neq b$, we have $F_s(w) \in S$ if and only if $F_{s+1}(w) \in S$. For $w = b$, we have $F_s(b) \notin S$ and $F_{s+1}(b) \in S$. Change the state to \textsc{waiting-for-third-change}.

\textbf{Waiting-for-third-change:} In the previous state, we broke the computation (6.7). If we are to have $S^N = \Phi_i^{R^B_e}$, then $R^B_e$ must change below the use $\mu$ of this computation. But since $S^N[0,\ldots,v]$ is the same as it was before, by the computation (6.6), $R^B_e[0,\ldots,u]$ cannot change. So $R^B_e$ must change on one of the elements $u + 1, \ldots, \mu$. Let $p$ be the element we described in the previous state. By (2) from state \textsc{waiting-for-first-computation}, the only elements from among $u + 1, \ldots, \mu$ in $B_e$ which can be of type (0) are in the subtree below $p$. So when, in state \textsc{waiting-for-first-change}, $p$ becomes of type (1), each of the elements in the subtree below $p$ becomes of type (1) or type (2). So now, when $R^B_e$ changes on one of the elements $u + 1, \ldots, \mu$, it does so by some such element which was of type (1) becoming of type (2). Now modify $F$ so that $S^N$ looks the same as it did originally (below $\nu$).

\textbf{Requires attention:} The requirement requires attention if

$$R^B_e[0,\ldots,\mu_{e,i,j}[s]] \neq R^B_{\tau_{e,i,j}[s]}[0,\ldots,\mu_{e,i,j}[s]].$$

\textbf{Action:} Let $b = b_{e,i,j}[s]$. We have $F_s(b) = 0^{\ell-1}x$. Choose $y > 0$ even and larger than any number we have encountered so far, and define $F_{s+1}$ by

$$F_{s+1}(w) = \begin{cases} 0^{\ell-1}y^{-}\sigma & F_s(w) = 0^{\ell-1}x^{-}\sigma \\ F_s(w) & \text{otherwise} \end{cases}.$$

This is moving the subtree below $0^{\ell-1}x$ to the subtree below $0^{\ell-1}y$. For $w \in \text{dom}(F_s)$, $w \neq b$, we have $F_s(w) \in S$ if and only if $F_{s+1}(w) \in S$. For $w = b$, we have $F_s(b) \notin S$ and $F_{s+1}(b) \notin S$. Change the state to \textsc{diagonalized}.

\textbf{Diagonalized:} In the previous state, we made sure that one of the elements $u + 1, \ldots, \mu$ of $B_e$ which previously looked like it was in $R^B_e$ is now not in $R^B_e$, so that that element is now of type (2) and hence must be in $R^B_e$. We also modified $F$ so that $S^N$ is the same as it was in state \textsc{initialized} (below $\nu$). Then, by computation (6.8), we cannot have $R^B_e = \Phi^S_j$. So we have satisfied $R_{e,i,j}$.

\textbf{Requires attention:} The requirement never requires attention.

\textbf{Action:} None.

When a requirement of higher priority than $R_{e,i,j}$ acts, $R_{e,i,j}$ is injured. When this happens, $R_{e,i,j}$ is returned to state \textsc{initialized} and its values $a_{e,i,j}$, $u_{e,i,j}$, etc. are set to 0. Now injure all requirements of lower priority than the one that acted.
Set \( D(N_{s+1}) \) to be the pullback along \( F_{s+1} \) of the atomic and negated atomic formulas true of \( \text{ran}(F_{s+1}) \) with Gödel number at most \( s \).

End construction.

If \( R_{e,i,j} \) is never injured after some stage \( s \), then it acts at most once at each stage, and it never moves backwards through the states. A requirement \( S_i \) only acts once if it is not injured. So every requirement is injured only finitely many times.

Now we will show that each requirement is satisfied. Each requirement \( S_i \) is satisfied, because there is a stage \( s \) after which each higher priority requirement never acts, and then at the next stage, \( S_i \) acts if it is not already satisfied, and is never again injured.

Now suppose that \( R_{e,i,j} \) is the least requirement which is not satisfied, and let \( s \) be the last stage at which it is injured (or \( s = 0 \) if it is never injured). So \( R_{e,i,j} \) is never injured after the stage \( s \). Also, since \( R_{e,i,j} \) is not satisfied, \( B_e \) is isomorphic to \( A_s \), \( S^N = \Phi_{i}^{R^B_e} \), and \( R^B_e = \Phi_{j}^{S^N} \).

Since \( R_{e,i,j} \) was just injured at stage \( s \), it is in state \textsc{initialized}, and so it requires attention at stage \( s_1 = s + 1 \). Then we define \( a \) such that \( F_{s_1+1}(a) \notin S \) and change to state \textsc{waiting-for-first-computation}.

Since \( S^N = \Phi_{i}^{R^B_e}, R^B_e = \Phi_{j}^{S^N} \), and \( B_e \) is isomorphic to \( A_s \), at some stage \( t > s_1 \), \( R_{e,i,j} \) will require attention. Thus we get \( u, m \), and \( v \) such that

\[
S^N_t[0, \ldots, u] = \Phi_{i,t}^{R^B_e}[0, \ldots, a]
\]  
(6.9)

with use \( u < m \),

\[
R^B_e[i,0, \ldots, m] = \Phi_{j,i}^{S^N}[0, \ldots, m]
\]  
(6.10)

with use \( v \), each element \( p \) from among \( 0, \ldots, u \) in \( B_e \) has a chain in \( B_{e,t} \) from the root node to itself, and these chains are in \( \{0, \ldots, m\} \). We define \( b > m \) and \( F_{t+1}(b) \notin S \).

Then we change to state \textsc{waiting-for-second-computation}.

Once again, using the fact that \( R^B_e = \Phi_{i}^{S^N} \) and \( S^N = \Phi_{j}^{R^B_e} \), at some stage \( \tau > t \), we will have

\[
S^N_\tau[0, \ldots, b] = \Phi_{i,\tau}^{R^B_e}[0, \ldots, b]
\]  
(6.11)

with use \( \mu \) and

\[
R^B_e[0, \ldots, \mu] = \Phi_{j,\tau}^{S^N}[0, \ldots, \mu]
\]  
(6.12)

with use \( \nu \). Note that (6.11) implies that

\[
R^B_e[0, \ldots, \mu] = R^B_e[0, \ldots, u].
\]  
(6.13)

Then \( R_{e,i,j} \) requires attention at stage \( \tau + 1 \). We define \( F_{\tau+1} \) so that \( F_{\tau+1}(a) \notin S \). Then we have

\[
S^N_{\tau+1}[0, \ldots, u] \neq S^N_\tau[0, \ldots, u]
\]  
(6.14)

The state is changed to \textsc{waiting-for-first-change}.
CHAPTER 6. DEGREE SPECTRA OF RELATIONS

Now by (6.11), (6.14), and the fact that \( S^N = \Phi_j^{R_{\mathcal{B}_e}} \), at some stage \( s_1 > \tau \), we have

\[
R_{s_1}^{B_e}[0, \ldots, u] \neq R_{\tau}^{B_e}[0, \ldots, u] = R_{\tau}^{B_e}[0, \ldots, u]
\]

(6.15)

and so \( R_{e,i,j} \) requires attention at stage \( s_1 + 1 \). \( F_{s_1+1} \) is defined such that

\[
S_{s_1+1}^N[0, \ldots, \nu] = S_{\tau}^N[0, \ldots, \nu].
\]

(6.16)

Then the state is changed to waiting-for-second-change.

Since \( R_{8}^{B_e} = \Phi_j^{S_N} \) and using (6.12) and (6.16), at some stage \( s > s_1 \), we have

\[
R_{s}^{B_e}[0, \ldots, \mu] = R_{\tau}^{B_e}[0, \ldots, \mu].
\]

(6.17)

Now by (6.15) and (6.17), there must be some \( p \in \{0, \ldots, u\} \) such that \( p \notin R_{\tau}^{B_e} \), \( p \in R_{s_1}^{B_e} \), and \( p \notin R_{s}^{B_e} \). Then in \( B_{e,s} \), \( p \) must be of type \( (2) \). All of the elements of \( B_e \) from \( \{m + 1, \ldots, \mu\} \) which looked like they were of type \( (0) \) at stage \( t \) were in the subtree below \( p \), so there is a stage \( s' > s \) at which each of them is of type \( (1) \) or type \( (2) \). Then, at some stage \( s_2 > s' \), we still have

\[
R_{s_2}^{B_e}[0, \ldots, \mu] = R_{\tau}^{B_e}[0, \ldots, \mu].
\]

So \( R_{e,i,j} \) requires attention at stage \( s_2 + 1 \). \( F_{s_2+1} \) is defined so that \( F_{s_2+1}(b) \in S \). The state is changed to waiting-for-third-change.

Since we had \( F_{s_2}(b) \notin S \),

\[
S_{s_2+1}^N[0, \ldots, \nu] = S_{\tau}^N[0, \ldots, \nu] = S_{\tau}^N[0, \ldots, \nu].
\]

So by (6.10) and (6.11) and since \( R_{8}^{B_e} = \Phi_j^{S_N} \), at some stage \( s_3 > s_2 \), we have

\[
R_{s_3}^{B_e}[0, \ldots, \mu] \neq R_{\tau}^{B_e}[0, \ldots, \mu]
\]

but

\[
R_{s_3}^{B_e}[0, \ldots, \mu] = R_{\tau}^{B_e}[0, \ldots, \mu]
\]

so that

\[
R_{s_3}^{B_e}[0, \ldots, \mu] = R_{\tau}^{B_e}[0, \ldots, \mu]
\]

(6.18)

So \( R_{e,i,j} \) requires attention. \( F_{s_3+1} \) is defined so that

\[
S_{s_3+1}^N[0, \ldots, \mu] = S_{\tau}^N[0, \ldots, \mu].
\]

The state is changed to diagonalized.

Since \( R_{8}^{B_e} = \Phi_j^{S_N} \), at some stage \( s_4 > s_3 \), by (6.12), we have

\[
R_{s_4}^{B_e}[0, \ldots, \mu] = R_{\tau}^{B_e}[0, \ldots, \mu].
\]

Then in \( B_e \), there must be \( q \in \{m + 1, \ldots, \mu\} \) such that \( F_{s_2}(q) \notin R \), \( F_{s_3}(q) \in R \), and \( F_{s_4}(q) \notin R \). Then in \( B_{e,s_2} \) at stage \( s_2 \), \( q \) must have looked like it was of type \( (0) \). We have already established that all of the elements in \( \{m + 1, \ldots, \mu\} \) were of type \( (1) \) or of type \( (2) \). This is a contradiction. Hence all of the requirements are satisfied. \( \square \)
The proposition relativizes as follows:

**Corollary 6.4.16.** For every degree \( d \), there is a copy \( N \) of \( M \) with \( N \leq_T d \) such that no copy \( B \) of \( A \) with \( B \leq_T d \) has \( S^N \oplus d \equiv_T R^B \oplus d \).

Now we have the proposition in the other direction, in the unrelativized form:

**Proposition 6.4.17.** There is a computable copy \( B \) of \( A \) such that no computable copy \( N \) of \( M \) has \( R^B \equiv_T S^N \).

*Proof.* We will construct a computable copy \( B \) of \( A \) with domain \( \omega \). We will diagonalize against every possible Turing equivalence with a computable copy \( N \) of \( M \). We will build \( B \) with an infinite injury construction using subrequirements, where each subrequirement is injured only finitely many times.

We will construct \( B \) by giving at each stage a tentative finite isomorphism \( F_s : \omega \to A \). \( F = \lim F_s \) will be a bijection, giving \( B \) as an isomorphic copy. The proof will be very similar in style to the proofs of Propositions 6.4.9 and 6.4.15, but there are some significant complications. In particular, we have to introduce subrequirements.

We need to diagonalize against computable copies of \( M \). As in the previous proposition, given a computable function \( \Phi_e \), we can try to interpret \( \Phi_e \) as giving the diagram of a computable structure \( N_e \) isomorphic to \( M \). At each stage, we get a finite substructure \( N_{e,s} \) isomorphic to a substructure of \( M \). If \( \Phi_e \) is total and gives the diagram of a structure isomorphic to \( M \), then \( N_e = \bigcup N_{e,s} \) is that structure. Otherwise, \( N_e \) will be some structure which may be finite and may or may not be isomorphic to \( M \). Also, recall that elements of \( N_{e,s} \) have a type which approximates their type in \( N_e \), and that we can assume that our approximation \( N_{e,s} \) has the the following property: all of the elements of \( N_{e,s} \) which are of type \((0,0)\) in \( N_{e,s} \) are linearly ordered.

We will meet the following requirements:

**\( R_{e,i,j} \):** If \( N_e \) is isomorphic to \( M \), and \( \Phi_i^{SNe} \) and \( \Phi_j^{R^B} \) are total, then either \( R^B \neq \Phi_i^{SNe} \) or \( S^N = \Phi_j^{R^B} \).

**\( S_i \):** The \( i \)th element of \( A \) is in the image of \( F \).

The strategy for satisfying a requirement \( R_{e,i,j} \) is as follows. The requirement \( R_{e,i,j} \) will have, associated to it at each stage \( s \), values \( a_{e,i,j}[s] \), \( u_{e,i,j}[s] \), \( v_{e,i,j}[s] \), and \( \tau_{e,i,j}[s] \). Also, for each \( n \), there will be values \( b^n_{e,i,j}[s] \), \( \mu^n_{e,i,j}[s] \), \( \nu^n_{e,i,j}[s] \), and \( \tau^n_{e,i,j}[s] \). See Figure 6.4 for a depiction of what these values mean. If \( R_{e,i,j} \) is not satisfied, then \( N_e \) will be isomorphic to \( M \), and we will have

\[
R^B = \Phi_i^{SNe} \text{ and } S^N = \Phi_j^{R^B}.
\]

Thus given, for example, a value for \( a_{e,i,j} \), there will be some \( u_{e,i,j} \) such that

\[
R^B[0, \ldots, a_{e,i,j}] = \Phi_j^{SNe}[0, \ldots, a_{e,i,j}]
\]
with use $u_{e,i,j}$. In the overview of the proof which follows below, we will just assume that these values (and the corresponding computations) always exist, since otherwise $R_{e,i,j}$ is trivially satisfied. We also write $a$ for $a_{e,i,j}$ etc.

![Diagram](image)

Figure 6.4: The values associated to a requirement for Proposition 6.4.17. An arrow shows a computation converging. The computations use an oracle and compute some initial segment of their target. The tail of the arrow shows the use of the computation, and the head shows the length of the output. For example, $R^S[0,\ldots,a_{e,i,j}] = \Phi^{S_{N_e}}$ with use $u_{e,i,j}$.

We begin by mapping $a_{e,i,j}$ in $B$ to the d-free element $0^\ell$ in $A$. Because of the computation

$$R^B[0,\ldots,a_{e,i,j}] = \Phi^{S_{N_e}}_j,$$

there must be at least one element in $N_e$ below the use $u_{e,i,j}$ of this computation which still looks like it could be d-free in $N_e$, i.e. of type $(0,0)$. If not, we could modify $F$ to map $a_{e,i,j}$ to $0^{\ell-1}x$ where $x$ is odd (so that $F(a_{e,i,j}) \in R$) and then later modify it again to map $a_{e,i,j}$ to $0^{\ell-1}y$ where $y$ is even (and so $F(a_{e,i,j}) \notin R$) to immediately diagonalize against the computation above (while maintaining, as usual, any existential formulas). So one of the elements $0,\ldots,u_{e,i,j}$ of $N_e$ must look like an element of the form $0^m$, i.e., be of type $(0,0)$. All of the elements in $M$ of the form $0^m$ (i.e., of type $(0,0)$) are linearly ordered; if $N_e$ is isomorphic to $M$, the same is true in $N_e$. Let $p$ be the element from $0,\ldots,u_{e,i,j}$ which is of type $(0,0)$ and which is furthest from the root node of all such elements.
Now map \( b^1_{e,i,j} \in B \), a new element of the domain, to an element of the form \( 0^{\ell_1}1 \in A \) where \( \ell_1 \geq \ell \). We claim that one of the elements \( u_{e,i,j} + 1, \ldots, \mu^1_{e,i,j} \) of \( N_e \) has to be:

(i) not in the subtree below \( p \) and

(ii) of type \( (0,0) \) or type \( (1,0) \).

Otherwise, we will be able to diagonalize to satisfy the requirement \( R_{e,i,j} \) in the following way. First, modify \( F \) to map \( a_{e,i,j} \) to \( 0^{\ell-1}x \) where \( x \) is odd and then to \( 0^{\ell-1}y \) where \( y \) is even. Then \( b^1_{e,i,j} \) is now mapped to an element of the form \( \rho1 \) for some \( \rho \). This will force \( p \), or some other element between \( p \) in the root node, to enter \( S^{N_e} \) and then leave \( S^{N_e} \). Then \( p \) must be look like an element of the form \( 0^mz \) for \( z \) even, or \( \tau z \) where \( \tau \) is not of the form \( 0^m \). In either case, by assumption each of the elements \( u_{e,i,j} + 1, \ldots, \mu^1_{e,i,j} \) of \( N_e \) must now either be of type \( (1,1) \) or type \( (2,0) \) (if one of these elements was not before, then it is now as it was in the subtree below \( p \) and \( p \) is not of type \( (0,0) \)). So all of these elements satisfy some existential formula which forces them to be in \( S^{N_e} \) (if they are of type \( (1,1) \)) or which forces them out of \( S^{N_e} \) (if they are of type \( (2,0) \)). Recall that \( F \) is now mapping \( b^1_{e,i,j} \) to \( \rho1 \) for some \( \rho \). We also have the computations

\[
R^B[0, \ldots, b^1_{e,i,j}] = \Phi^{S^{N_e}}_j[0, \ldots, b^1_{e,i,j}]
\]

with use \( \mu^1_{e,i,j} \) and

\[
S^{N_e}[0, \ldots, u_{e,i,j}] = \Phi^R_i[0, \ldots, u_{e,i,j}]
\]

with use \( v_{e,i,j} \). So by now modifying \( F \) to map \( b^1_{e,i,j} \) to an element of the form \( \rho z \) for \( z \) even, we now have \( b^1_{e,i,j} \notin R^B \). We break the first computation, causing \( S^{N_e} \) to change below the use \( \mu^1_{e,i,j} \). Because the use \( v_{e,i,j} \) of the second computation is less than \( b^1_{e,i,j} \), \( R^B \) has not changed on the use of the second computation. So \( S^{N_e} \) must stay the same on the elements \( 0, \ldots, u_{e,i,j} \). Thus \( S^{N_e} \) must change on the elements \( u_{e,i,j} + 1, \ldots, \mu_{e,i,j} \). But this cannot happen as remarked before. Thus we have diagonalized and satisfied \( R_{e,i,j} \).

Thus, if we cannot diagonalize to satisfy \( R_{e,i,j} \) in this way, one of the elements \( u_{e,i,j} + 1, \ldots, \mu^1_{e,i,j} \) of \( N_e \) satisfies (i) and (ii) above. Choose one such element, and call it \( q^1 \). Defining \( b^2_{e,i,j}, b^3_{e,i,j}, \ldots \) and so on in the same way, we get \( q^2, q^3, \ldots \). Thus we find, in \( N_e \), infinitely many elements satisfying (i) and (ii). But if \( N_e \) is isomorphic to \( M \), there can only be finitely many such elements: if \( p \) is the isomorphic image of \( 0^m \), then the only elements satisfying (i) and (ii) are the isomorphic images of \( 0^m1 \) and \( 0^{m'}1 \) for \( m' < m \). Thus we force \( N_e \) to be non-isomorphic to \( M \) and satisfy \( R_{e,i,j} \) in that way.

The requirement \( R_{e,i,j} \) will have subrequirements \( R_{e,i,j}^n \) for \( n \geq 1 \). The main requirement will choose \( a \), while each subrequirement will choose \( b^n \). A subrequirement will act only when all of the previous requirements have chosen their values \( b^n \). The main requirement \( R_{e,i,j} \) will monitor \( N_e \) to see whether it has given us elements \( p \) and \( q^n \), and if not, it can attempt to diagonalize. Either the parent requirement \( R_{e,i,j} \) will at some point diagonalize and be satisfied, or each subrequirement will be satisfied guaranteeing that \( N_e \) is not isomorphic to \( M \).
Because the subrequirements use infinitely many values \( b^n \), we need to assign the subrequirements a lower priority than the main requirement in order to give the other requirements a chance to act. \( R_{e,i,j} \) will be of higher priority than its subrequirements \( R^n_{e,i,j} \), and the subrequirements will be of decreasing priority as \( n \) increases. The subrequirements will be interleaved in this ordering, so that, for example, the ordering might begin (from highest priority to lowest priority):

\[
R_{e_1,i_1,j_1} > R^1_{e_1,i_1,j_1} > R_{e_2,i_2,j_2} > R^2_{e_1,i_1,j_1} > R^1_{e_2,i_2,j_2} > R^3_{e_1,i_1,j_1} > \ldots
\]

The requirement \( R_{e,i,j} \) will have, associated to it at each stage \( s \), the values \( a_{e,i,j}[s] \), \( u_{e,i,j}[s] \), \( v_{e,i,j}[s] \), and \( t_{e,i,j}[s] \). A subrequirement \( R^n_{e,i,j} \) will be associated with the values \( b^n_{e,i,j}[s] \), \( \mu^n_{e,i,j}[s] \), \( \nu^n_{e,i,j}[s] \), and \( \tau^n_{e,i,j}[s] \). These values will never be redefined, but may be canceled. When a requirement is injured, its corresponding values will be canceled, with one exception. If \( R_{e,i,j} \) finds an opportunity to diagonalize using \( b^n_{e,i,j}[s] \), then it will protect the values \( b^n_{e,i,j}[s] \), \( \mu^n_{e,i,j}[s] \), \( \nu^n_{e,i,j}[s] \), and \( \tau^n_{e,i,j}[s] \) using its own priority.

At each stage, each requirement \( R_{e,i,j} \) will be in one of the following states:

- **INITIALIZED**,
- **WAITING-FOR-COMPUTATION**,  
- **NEXT-SUBREQUIREMENT**,  
- **WAITING-FOR-CHANGE**,  
- **DIAGONALIZED**,  
- **WAITING-FOR-FIRST-CHANGE-\( n \)**,  
- **WAITING-FOR-SECOND-CHANGE-\( n \)**, or  
- **DIAGONALIZED-\( n \)**.

The requirement will move through these in the following order (where, at the branch, the requirement will move along either the left branch or the right branch, and if it moves along the right branch it does so for some specific value of \( n \)):

```
INITIALIZED  ↓  WAITING-FOR-COMPUTATION  ↓  NEXT-SUBREQUIREMENT
            ↓                       ↓
WAITING-FOR-CHANGE   DIAGONALIZED   WAITING-FOR-FIRST-CHANGE-\( n \)  ↓  DIAGONALIZED-\( n \)
            ↓                       ↓                       ↓
WAITING-FOR-CHANGE
            ↓                       ↓
DIAGONALIZED   WAITING-FOR-SECOND-CHANGE-\( n \)
            ↓                       ↓
DIAGONALIZED-\( n \)
```
When the requirement is in state NEXT-SUBREQUIREMENT, the subrequirements will begin acting. The requirement will then monitor them for a chance to diagonalize. There are two ways in which the requirement can diagonalize, either by going to state WAITING-FOR-CHANGE if the element \( p \) described above does not exist, or WAITING-FOR-FIRST-CHANGE-\( n \) if some element \( q^n \) described above does not exist. Recall that in the second case, the values \( b^n_{e,i,j}, \mu^n_{e,i,j}, \tau^n_{e,i,j} \), and \( \tau^n_{e,i,j} \) are protected by the requirement \( R_{e,i,j} \) at its priority.

Each subrequirement \( R^n_{e,i,j} \) will be in one of three states: INITIALIZED, WAITING-FOR-COMPUTATION, or NEXT-SUBREQUIREMENT. The subrequirement will move through these states in order. When one subrequirement is in state NEXT-SUBREQUIREMENT, it has finished acting and the next one can begin.

We are now ready to describe the construction.

**Construction.**

At stage 0, let \( F_s = \varnothing \) and for each \( e, i \), and \( j \) let \( a_{e,i,j}[0], u_{e,i,j}[0], v_{e,i,j}[0], \) and \( t_{e,i,j}[0] \) be \( \varnothing \). For each \( n \), let \( b^n_{e,i,j}[0], \mu^n_{e,i,j}[0], \nu^n_{e,i,j}[0], \) and \( \tau^n_{e,i,j}[0] \) be 0 as well.

At a stage \( s + 1 \), let \( F_s : \{0, \ldots, \xi_s\} \to A \) be the partial isomorphism determined in the previous stage, and let \( D(\mathcal{B}_s) \) be the finite part of the diagram of \( \mathcal{B} \) which has been determined so far. We have an approximation \( R^S_0 \to R^S \) which we get by taking \( k \in R^S_0 \) if \( F_s(k) \in R \). For each \( e \), we have a guess \( S^n_{e,s} \) at \( S^n_{e} \) using the diagram of the finite structure \( \mathcal{N}_{e,s} \) given by \( x \in S^n_{e,s} \) if and only if in \( \mathcal{N}_{e,s} \), \( x \) is in the first sort and is related by \( V^{\mathcal{N}_{e,s}} \) to exactly one element of the second sort.

We will deal with a single requirement—the highest priority requirement which requires attention at stage \( s + 1 \). A requirement \( S_i \) requires attention at stage \( s + 1 \) if the \( i \)th element of \( A \) is not in the image of \( F_s \). If \( S_i \) is the highest priority requirement which requires attention, then let \( c \) be the \( i \)th element of \( A \). Let \( F_{s+1} \) extend \( F_s \) with \( c \) in its image. Injure each requirement of lower priority.

The conditions for a requirement \( R_{e,i,j} \) or a subrequirement \( R^n_{e,i,j} \) to require attention at stage \( s + 1 \) depend on the state of the requirement. Below, we will list for each possible state of \( R_{e,i,j} \), the conditions for \( R_{e,i,j} \) to require attention, and the action that the requirement takes if it is the highest priority requirement that requires attention. The subrequirements will follow afterward.

**Initialized:** The requirement has been initialized, so \( a_{e,i,j}[0], u_{e,i,j}[0], \) and so on are all 0.

**Requires attention:** The requirement always requires attention.

**Action:** Let \( F_{s+1} \) extend \( F_s \) by adding to its image the element \( 0^\ell \), where \( \ell \) is large enough that \( 0^\ell \) has no children in \( \operatorname{ran}(F_s) \). Let \( a_{e,i,j}[s+1] \) be such that \( F_{s+1}(a_{e,i,j})[s+1] = 0^\ell \). Change the state to WAITING-FOR-COMPUTATION.
Waiting-for-computation: We have set \( F(a_{e,i,j}) = 0 \notin R \) a d-free element. We wait for the computations (6.18) and (6.19) below. Then we can begin to satisfy the subrequirements.

Requires attention: The requirement requires attention if there is a computation
\[
R^B_s[0, \ldots, a_{e,i,j}[s]] = \Phi_i^{S^N_e}[0, \ldots, a_{e,i,j}[s]]
\]
with use \( u < s \), and
\[
S^N_e[0, \ldots, u] = \Phi_j^{R^B}[0, \ldots, u]
\]
with use \( v < s \).

Action: Let \( u \) and \( v \) be the uses of the computations which witness that this requirement requires attention. Set \( u_{e,i,j}[s+1] = u \), \( v_{e,i,j}[s+1] = v \), and \( t_{e,i,j}[s+1] = s \). We have \( F_{s+1} = F_s \). Change the state to next-subrequirement.

Next-subrequirement: While in this state, we begin trying to satisfy the subrequirements, building elements \( b^1_{e,i,j}, b^2_{e,i,j} \), and so on. At the same time, we look for a way to immediately satisfy \( R_{e,i,j} \). The requirement requires attention during this state if we see such a way to satisfy \( R_{e,i,j} \). There are two possible ways that we might immediately diagonalize. The first is that we can diagonalize using only \( a_{e,i,j} \) and the computations (6.18) and (6.19), because none of the elements of \( N_e \) below the use \( u_{e,i,j} \) of (6.18) are d-free. The second is that we can diagonalize using \( a_{e,i,j} \) and some \( b^n_{e,i,j} \), because we use \( a_{e,i,j} \) to force \( S^N_e \) to change below the use \( u_{e,i,j} \) of (6.18), and this will mean that we can diagonalize by changing \( b^n_{e,i,j} \) from being in \( R^B \) to being out of \( R^B \). If we see a chance to diagonalize, we modify \( F \) to put \( a_{e,i,j} \) into \( R^B \), breaking the computation (6.18).

Requires attention: There are two possible ways that this requirement might require attention. The requirement requires attention of the first kind if in \( N_{e,s} \):

1. each of the elements \( 0, \ldots, u_{e,i,j}[s] \) of \( N_e \) which is in the first sort is of type \((1,0)\), type \((1,1)\), or type \((2,0)\),
2. we still have
\[
S^N_e[0, \ldots, u_{e,i,j}[s]] = S^N_{S_{e,i,j}[s]}[0, \ldots, u_{e,i,j}[s]].
\]

The requirement requires attention of the second kind if for some \( n \):

1. each of the subrequirements \( R^m_{e,i,j} \) is in state next-requirement for all \( m < n \),
2. each of the elements \( \mu^{n-1}_{e,i,j}[s] + 1, \ldots, \mu^n_{e,i,j}[s] \) of \( N_{e,s} \) (with \( \mu^{n-1}_{e,i,j}[s] \) replaced by \( u_{e,i,j}[s] \) if \( n = 0 \)) which is in the first sort and is either of type \((0,0)\) or type \((1,0)\) in \( N_{e,s} \) is in the subtree below those elements from among \( 0, \ldots, u_{e,i,j}[s] \) which are not related to any elements of the second sort,
Waiting-for-change: In this state, we are trying to diagonalize against $R_{e,i,j}$ in the first way described above. The computation (6.18) was broken, and so as usual, $S^{N_e}$ must change below the use $u_{e,i,j}$. When we first entered this state, all of the elements $0,\ldots,u_{e,i,j}$ of $N_e$ were of types $(1,0)$, $(1,1)$, or $(2,0)$ (i.e., were not d-free). So in order for $S^{N_e}$ to change below the use $u_{e,i,j}$, one of these elements (call it $p$) which was connected to one element of the second sort must become connected to two elements of the second sort. We then modify $F$ to make $R^S$ the same as it was originally (below $u_{e,i,j}$). This will successfully satisfy the requirement, because $S^{N_e}$ cannot return to the way it was originally because $p$ cannot return to being in $S^{N_e}$, and so the computation (6.19) from the state WAITING-FOR-COMPUTATION means that we cannot have $S^{N_e} = \Phi^R_{e,j}$. 

Requires attention: This requirement requires attention if

$$S^{N_e}_{a_{e,i,j}}[0,\ldots,u_{e,i,j}[s]] = S^{N_e}_{a_{e,i,j}}[0,\ldots,u_{e,i,j}[s]].$$

Action: Let $a = a_{e,i,j}[s]$. We have $F_s(a) = 0^{\ell-1}x$, where $x$ is odd and no child of $0^{\ell-1}x$ appears earlier in the image of $F_s$. Choose $y > 0$ even and larger than any number we have encountered so far, and define $F_{s+1}$ by

$$F_{s+1}(w) = \begin{cases} 0^{\ell-1}y \cdot \sigma & F_s(w) = 0^{\ell-1}x \cdot \sigma \\ F_s(w) & \text{otherwise} \end{cases}.$$
This is moving the subtree below $0^\ell x$ to the subtree below $0^\ell y$. For $w \in \text{dom}(F_s)$, $w \neq a$, we have $F_s(w) \in R$ if and only if $F_{s+1}(w) \in R$. For $w = a$, we have $F_s(a) \in R$ and $F_{s+1}(a) \notin R$. Change the state to DIAGONALIZED.

**Diagonalized:** In this state, we have successfully satisfied $\mathcal{R}_{e,i,j}$ in the first way described above.

**Requires attention:** The requirement never requires attention.

**Action:** None.

**Waiting-for-first-change-n:** In this state, we are trying to diagonalize against $\mathcal{R}_{e,i,j}$ in the second way described above. The computation (6.18) was broken, and so as usual, $S^{N_e}$ must change below the use $u_{e,i,j}$. So some such element (which we call $p$) which was connected to no elements of the second sort (i.e., of type $(0,0)$) must become connected to one element of the second sort\(^7\) (i.e., it is now some other type). We then modify $F$ to make $R^B$ the same as it was originally (below $v_{e,i,j}$).

**Requires attention:** This requirement requires attention if

$$S^{N_e}_s[0,\ldots,u_{e,i,j}[s]] \neq S^{N_e}_{t_{e,i,j}}[s][0,\ldots,u_{e,i,j}[s]].$$

**Action:** Do the same thing as in state waiting-for-change, except that instead of moving to state diagonalized, change to state waiting-for-second-change-n.

**Waiting-for-second-change-n:** In the previous state, $F$ was modified so that $R^B$ is the same as it was originally below $v_{e,i,j}$. By the computation (6.19) from state waiting-for-computation, $S^{N_e}$ must return to the same as it was originally below $u_{e,i,j}$, i.e., the element $p$ from the previous state must become connected to two elements of the second sort. Now, in state next-subrequirement, we had the condition (2). This condition implied that each of the elements $\mu^{n-1}_{e,i,j} + 1, \ldots, \mu^n_{e,i,j}$ of $N_e$ which is not either forced to be in $S^{N_e}$ (by being of type $(1,1)$) or forced to be not in $S^{N_e}$ (by being of type $(2,0)$) was in the subtree below $p$. But now $p$ is of type $(2,0)$, and so there are no such elements below $p$. So each of $\mu^{n-1}_{e,i,j} + 1, \ldots, \mu^n_{e,i,j}$ is either forced to be in $S^{N_e}$ or forced to not be in $S^{N_e}$. Now $b^n_{e,i,j}$ in $B$ is currently in $R^B$, and we can modify $F$ so that $b^n_{e,i,j}$ is not in $R^B$. By the computation (6.20) below from the $n$th subrequirement, we cannot have $R^B = \Phi^s_i S^{N_e}$.

**Requires attention:** This requirement requires attention if

$$S^{N_e}_2[0,\ldots,\mu^n_{e,i,j}[s]] = S^{N_e}_{t_{e,i,j}}[s][0,\ldots,\mu^n_{e,i,j}[s]].$$

\(^7\)It is possible for some element which was connected to one element of the second sort to become connected to two elements, but in this case we will successfully satisfy $\mathcal{R}_{e,i,j}$ in much the same way as above as a byproduct of our general construction.
and also in $N_{e,s}$, each of the elements $\mu_{e,i,j}[s] + 1, \ldots, \mu_{e,i,j}[s]$ of $N_{e,s}$ is either of type $(1, 1)$ or of type $(2, 0)$ in $N_{e,s}$.

**Action:** Let $a = a_{e,i,j}[s]$ and $b = b_{e,i,j}[s]$. We have $F_s(a) = 0^{\ell_1-1} y$ for some even $y$, and $F_s(b) = 0^{\ell_1-1} y 0^{\ell_2} 1$. Let $\rho = 0^{\ell_1-1} y 0^{\ell_2}$ so that $F_s(b) = \rho 1$. Choose $z > 0$ even and larger than any number we have encountered so far, and define $F_{s+1}$ by

$$F_{s+1}(w) = \begin{cases} \rho \cdot z \cdot \sigma & F_s(w) = \rho \cdot 1 \cdot \sigma \\ F_s(w) & \text{otherwise} \end{cases}$$

This is moving the subtree below $\rho \cdot 1$ to the subtree below $\rho \cdot z$. For $w \in \text{dom}(F_s)$, $w \neq b$, we have $F_s(w) \in R$ if and only if $F_{s+1}(w) \in R$. For $w = b$, we have $F_s(b) \in R$ and $F_{s+1}(b) \notin R$. Change the state to **Diagonalized-**$n$.

**Diagonalized-**$n$: In this state, we have successfully satisfied $R_{e,i,j}$ in the second way using $b_{e,i,j}^n$.

**Requires attention:** The requirement never requires attention.

**Action:** None.

In order for a subrequirement $R_{e,i,j}^n$ to require attention (in any state), there is a necessary (but not sufficient) condition: the parent requirement $R_{e,i,j}$ must be in state **Next-Subrequirement**. If this condition is satisfied, then the whether the requirement requires attention depends on its state:

**Initialized:** The subrequirement has been initialized, so $b_{e,i,j}^n$, $\mu_{e,i,j}^n$, and so on are all 0. We define $b_{e,i,j}^n[s + 1]$.

**Requires attention:** The subrequirement always requires attention.

**Action:** Let $F_{s+1}$ extend $F_s$ by adding to its image the element $0^\ell 1$, where $\ell$ is large enough that $0^\ell$ has no children in $\text{ran}(F_s)$. Let $b_{e,i,j}^n[s + 1]$ be such that $F_{s+1}(b_{e,i,j}^n[s + 1]) = 0^\ell 1$. Change the state to **Waiting-for-computation**.

**Waiting-for-computation:** In the previous state, we defined $b_{e,i,j}^n$. We now wait for the computations below.

**Requires attention:** This subrequirement requires attention if there are computations

$$R_s^{R_i}[0, \ldots, b_{e,i,j}^n[s]] = \Phi_{i}^{S_{N_e}}[0, \ldots, b_{e,i,j}^n[s]]$$

(6.20)

with use $\mu < s$, and

$$S_{e}^{N_e}[0, \ldots, \mu] = \Phi_{j}^{R_s}[0, \ldots, \mu]$$

(6.21)

with use $\nu < s$. 


**CHAPTER 6. DEGREE SPECTRA OF RELATIONS**

**Action:** Set $\mu_{e,i,j}^n[s+1] = \mu$, $\nu_{e,i,j}^n[s+1] = \nu$, and $\tau_{e,i,j}^n[s+1] = s$. We have $F_{s+1} = F_s$. Change the state to NEXT-SUBREQUIREMENT.

**Next-subrequirement:** In the previous state, we found the computations (6.20) and (6.21). This subrequirement is done acting, and the next subrequirement can begin.

**Requires attention:** The subrequirement never requires attention.

**Action:** None.

Now we will say what happens when we say that we injure a requirement or subrequirement. When a requirement $R_{e,i,j}$ is injured, it is returned to state INITIALIZED and its values $a_{e,i,j}$, $u_{e,i,j}$, $v_{e,i,j}$, and $t_{e,i,j}$ are set to 0. Moreover, if it is in one of the states WAITING-FOR-FIRST-CHANGE-$n$, WAITING-FOR-SECOND-CHANGE-$n$, or DIAGONALIZED-$n$, then for $m \leq n$ set $b_{e,i,j}^m$, $\mu_{e,i,j}^m$, $\nu_{e,i,j}^m$, and $\tau_{e,i,j}^m$ to 0.

When a subrequirement $R_{e,i,j}^n$ is injured, it is returned to state INITIALIZED. Unless its parent requirement $R_{e,i,j}$ is in one of the states WAITING-FOR-FIRST-CHANGE-$m$, WAITING-FOR-SECOND-CHANGE-$m$, or DIAGONALIZED-$m$ for $m \geq n$, set $b_{e,i,j}^m$, $\mu_{e,i,j}^m$, $\nu_{e,i,j}^m$, and $\tau_{e,i,j}^m$ to 0. In this way, by being in one of these three states WAITING-FOR-FIRST-CHANGE-$n$, WAITING-FOR-SECOND-CHANGE-$n$, or DIAGONALIZED-$n$ the parent requirement can take over control of the values associated to the subrequirements $R_{e,i,j}^m$ for $m \leq n$ and protect them with its own priority level.

Set $D(B_{s+1})$ to be the pullback along $F_{s+1}$ of the atomic and negated atomic formulas true of $\text{ran}(F_{s+1})$ with Gödel number at most $s$.

End construction.

Each requirement and subrequirement, if it is not injured, only acts finitely many times. We must show that each requirement is satisfied. Suppose not. Then there is a least requirement which is not satisfied. It is easy to see that a requirement $S_i$ is always eventually satisfied, so let $R_{e,i,j}$ be the least requirement which is not satisfied. Then $N_e$ is a computable structure isomorphic to $B$, $R^B = \Phi_i^{S_i^{N_e}}$, and $S_i^{N_e} = \Phi_j^{R^B}$.

There is some last stage at which $R_{e,i,j}$ is injured. At this stage $R_{e,i,j}$ and its subrequirements are in state INITIALIZED.

We will use the following fact implicitly throughout the rest of the proof. It is easy to prove.

**Lemma 6.4.18.** If a requirement $R_{e,i,j}$ is never injured after the stage $s$, then after the stage $s$, $F$ is only changed on the domain $[0, \ldots, v_{e,i,j}]$ by $R_{e,i,j}$. If a subrequirement $R_{e,i,j}^n$ is never injured after the stage $s$, then after the stage $s$, $F$ is only changed on the domain $[0, \ldots, v_{e,i,j}^n]$ by $R_{e,i,j}$. Also, if a requirement $R_{e,i,j}$ is in one of the states WAITING-FOR-FIRST-CHANGE-$n$, WAITING-FOR-SECOND-CHANGE-$n$, or DIAGONALIZED-$n$ and is never injured after the stage $s$, then $F$ is only changed on $[0, \ldots, v_{e,i,j}^n]$ by $R_{e,i,j}$.
We will show that eventually $R_{e,i,j}$ enters state diagonalized or diagonalized-$n$ and diagonalizes against the two computations above, a contradiction. We will write $a$ for $a_{e,i,j}[s]$ since the $e$, $i$, and $j$ are fixed, and the value is never redefined since $R_{e,i,j}$ is never injured. Similarly, we write $u$ for $u_{e,i,j}[s]$, $\mu^n$ for $\mu^n_{e,i,j}[s]$, and so on.

In state INITIALIZED, $R_{e,i,j}$ always requires attention, so we will always define $a$ such that $F_s(a) = 0^\ell$ for some $\ell$ and move on to state WAITING-FOR-COMPUTATION.

Now because $R^B = \Phi_i^{sN_e}$ and $SN_e = \Phi_j^{R^B}$, at some later stage $t$ we will have computations

$$R^B_t[0, \ldots, a_{e,i,j}[t]] = \Phi_i^{sN_e}_{t} [0, \ldots, a_{e,i,j}[t]] \quad (6.22)$$

with use $u < t$, and

$$S^N_t[0, \ldots, u] = \Phi_j^{R^B}_{t} [0, \ldots, u] \quad (6.23)$$

with use $v < t$. Then $u$, $v$, and $t$ will be defined to be these values and the requirement will move to state NEXT-SUBREQUIREMENT.

Now we have three cases. First, it might be that at some later stage, $R_{e,i,j}$ leaves state NEXT-SUBREQUIREMENT and enters state WAITING-FOR-CHANGE. Second, it might be that it enters the state WAITING-FOR-FIRST-CHANGE-$n$. Third, the requirement might never leave state NEXT-SUBREQUIREMENT. We have to find a contradiction in each case.

**Case 1.** $R_{e,i,j}$ leaves state NEXT-SUBREQUIREMENT and enters WAITING-FOR-CHANGE.

At some stage $s_1 + 1 > t$, $R_{e,i,j}$ requires attention of the first kind. We change $F$ so that $F_{s_1+1}(a) \in R$ and change to state WAITING-FOR-CHANGE.

Now since $R^B = \Phi_i^{sN_e}$, at some stage $s_2 > s_1$, we have

$$R^B_{s_2}[0, \ldots, a] = \Phi_i^{sN_e}_{s_2} [0, \ldots, a]$$

and hence, by (6.22) and the fact that $F_{s_2}(a) = F_{s_1+1}(a) \neq F_t(a)$, we have

$$S^N_{s_2}[0, \ldots, u] \neq S^N_t[0, \ldots, u]. \quad (6.24)$$

So the requirement requires attention at stage $s_2 + 1$.

We change $F$ so that $F_{s_2+1}(a) \notin R$ and change to state DIAGONALIZED. We make sure that

$$R^B_{s_2+1}[0, \ldots, v] = R^B_t[0, \ldots, v] \quad (6.25)$$

Since $SN_e = \Phi_j^{R^B}$, at some stage $s_3 > s_2$, we have

$$S^N_{s_3}[0, \ldots, u] = \Phi_j^{R^B}_{s_3} [0, \ldots, u].$$

Then by (6.25) and (6.23) we have

$$S^N_{s_3}[0, \ldots, u] = S^N_t[0, \ldots, u] \quad (6.26)$$
Combining this with (6.24) we see that there is some \( p \) in \([0, \ldots, u]\) with \( p \notin S_1^{N_e} \), \( p \in S_{s_2}^{N_e} \), and \( p \notin S_{s_3}^{N_e} \).

Now \( s_1 + 1 \) was the stage at which \( R_{e,i,j} \) required attention of the first kind while in state \textsc{next-subrequirement}. First of all, this means that

\[
S_{s_1}^{N_e}[0, \ldots, u] = S_t^{N_e}[0, \ldots, u]
\]

and so \( p \notin S_{s_1}^{N_e} \).

Also, in \( N_{e,s_1} \), \( p \) must be related by \( V_{N_e} \) to at least one element of the second sort (i.e., \( p \) is of one of the types \((1, 0)\), \((1, 1)\), or \((2, 0)\)). Since \( p \notin S_{s_1}^{N_e} \), \( p \) must be related to two elements of the second sort, so of type \((2, 0)\). But then the same is true at stage \( s_2 \), which contradicts the fact that \( p \notin S_{s_1}^{N_e} \).

**Case 2.** \( R_{e,i,j} \) leaves state \textsc{next-subrequirement} and enters state \textsc{waiting-for-first-change-}n for some \( n \).

The beginning of the proof of this case is the same as the beginning of the last case (with the states \textsc{waiting-for-change} and \textsc{diagonalized} replaced by \textsc{waiting-for-first-change-}n and by \textsc{waiting-for-second-change-}n respectively). Only the part of the proof after we conclude that \( p \notin S_{s_1}^{N_e} \), \( p \in S_{s_2}^{N_e} \), and \( p \notin S_{s_3}^{N_e} \) is different—this no longer leads to a contradiction. The requirement is in state \textsc{waiting-for-second-change-}n.

Since \( R_{e,i,j} \) required attention of the second kind at stage \( s_1 + 1 \), the subrequirement \( R_{e,i,j}^m \) must have been in state \textsc{next-subrequirement}. It will have defined, for \( m \leq n \), \( b^m \), \( \mu^m \), and \( \tau^m \) with

\[
R_{\tau^m}[0, \ldots, b^m] = \Phi_{i,\tau}^{S_{e,s}^{N_e}}[0, \ldots, b^m] \tag{6.27}
\]

with use \( \mu^m \), and

\[
S_{\tau^m}[0, \ldots, \mu^m] = \Phi_{j,\tau}^{R_{\tau^m}}[0, \ldots, \mu^m] \tag{6.28}
\]

with use \( \nu^m \).

Now if \( q \) is an element of \( N_e \) from among \( \mu^{n-1} + 1, \ldots, \mu^n \), and in \( N_{e,s_1} \) it looks like \( q \) is either of type \((1, 1)\) or type \((2, 0)\), then \( q \) has the same type in the diagram at stage \( s_3 \). Now since \( R_{e,i,j} \) required attention of the second kind while in state \textsc{next-subrequirement} at stage \( s_1 + 1 \), any other element \( q \) from among \( \mu^{n-1} + 1, \ldots, \mu^n \) not satisfying either of the above conditions was in the subtree below \( p \). At stage \( s_3 \), \( p \) is of type \((2, 0)\), and so we can see from the definition of \( \mathcal{M} \) that any \( q \) in the subtree below \( p \) must be of type \((1, 1)\) or type \((2, 0)\).

Thus at some stage \( s_4 > s_3 \), the requirement requires attention. Each element \( q \) from among \( \mu^{n-1} + 1, \ldots, \mu^n \) has been determined to either not be in \( S_{N_e} \) if it is of type \((2, 0)\), or to be in \( S_{N_e} \) if it is of type \((1, 1)\). So for all stages \( s > s_4 \), we have

\[
S_{s}^{N_e}[\mu^{n-1} + 1, \ldots, \mu^n] = S_{s_4}^{N_e}[\mu^{n-1} + 1, \ldots, \mu^n] = S_{\tau}^{N_e}[\mu^{n-1} + 1, \ldots, \mu^n] \tag{6.29}
\]

We change \( F \) so that \( F_{s_4 + 1}^{n}(b^n) \notin R \) (while \( F_{\tau}^{n}(b^n) \) was in \( R \)) and

\[
R_{s_4 + 1}[0, \ldots, \nu^{n-1}] = R_{\tau}^{n}[0, \ldots, \nu^{n-1}].
\]
This is also true with \( s_4 \) replaced by any \( s \geq s_4 \). By (6.28), for sufficiently large \( s > s_4 \) we have
\[
S^{Ne}_s[0, \ldots, \mu^{n-1}] = S^{Ne}_{\tau^n}[0, \ldots, \mu^{n-1}].
\] (6.30)
Then since for all \( s > s_4 \) we have
\[
R^B_s[0, \ldots, b^n] \neq R^B_{\tau^n}[0, \ldots, b^n]
\]
by (6.27) for sufficiently large \( s > s_4 \) we have
\[
S_s[0, \ldots, \mu^n] \neq S_{\tau^n}[0, \ldots, \mu^n].
\]
From this and (6.30), we see that
\[
S_s[\mu^{n-1}+1, \ldots, \mu^n] \neq S_{\tau^n}[\mu^{n-1}+1, \ldots, \mu^n].
\]
This contradicts (6.29).

**Case 3.** \( \mathcal{R}_{e,i,j} \) never leaves state **NEXT-SUBREQUIREMENT**.

Suppose to the contrary that while in state **NEXT-SUBREQUIREMENT** the requirement \( \mathcal{R}_{e,i,j} \) never requires attention. Then for all stages \( s > t \), we have
\[
R^B_s[0, \ldots, v] = R^B_t[0, \ldots, v].
\]
Since \( S^{Ne}_e = \Phi^R_s \) and using (6.23), for sufficiently large stages \( s \) we have
\[
S^{Ne}_s[0, \ldots, u] = S^{Ne}_t[0, \ldots, u].
\] (6.31)
So for sufficiently large stages \( s \), the subrequirement \( \mathcal{R}_{e,i,j}^1 \) always requires attention in state **INITIALIZED**. At some stage, each requirement of higher priority than \( \mathcal{R}_{e,i,j}^1 \) has acted, and so \( \mathcal{R}_{e,i,j}^1 \) is never injured after this point.

Then the subrequirement \( \mathcal{R}_{e,i,j}^1 \) will require attention and we will define \( b^1 \) such that
\[
F_s(b^1) = 0^\ell 1 \text{ for some } \ell \text{ and move on to state **WAITING-FOR-COMPUTATION**.}
\]
Because \( R^B = \Phi^{SNe}_s \) and \( S^{Ne}_s = \Phi^{R^B}_s \), at some later stage \( \tau^1 \) we will have computations
\[
R^B_{\tau^1}[0, \ldots, b^1] = \Phi^{SNe}_{e,i,j,\tau^1}[0, \ldots, b^1]
\]
with use \( \mu^1 < \tau^1 \), and
\[
S^{Ne}_{\tau^1}[0, \ldots, \mu^1] = \Phi^{R^B}_{e,i,j,\tau^1}[0, \ldots, \nu^1]
\]
with use \( \nu^1 < \tau^1 \). Then \( \mu^1, \nu^1, \) and \( \tau^1 \) will be defined to be these values and the requirement will move to state **NEXT-SUBREQUIREMENT**.

Continuing a similar argument, each of the subrequirements defines \( b^n, \mu^n, \nu^n, \) and \( \tau^n \) such that
\[
S^{Ne}_{\tau^n}[0, \ldots, \mu^n] = \Phi^{R^B}_{e,i,j,\tau^n}[0, \ldots, \mu^n]
\] (6.32)
with use $\nu^n$.

Now we claim that $N_e$ is not isomorphic to $M$. First, $R_{e,i,j}$ never requires attention of the first kind. Because of (6.31), the only way this is possible is if there is an element $p$ from among $0, \ldots, u$ which is of type $(0,0)$. That is, $p$ is the isomorphic image of $0^\ell$ in $M$ for some $\ell$.

Let $n$ be arbitrary. By (6.32) and the fact that $S^N_e = \Phi^j_N$, for sufficiently large stages $s$, we have
\[ S^N_e[0, \ldots, \nu^n] = S^N_e[0, \ldots, \nu^n]. \]

Then since $R_{e,i,j}$ does not ever require attention of the second kind, there is $q_n$ from $\nu^{n-1} + 1, \ldots, \nu^n$ which is of type $(0,0)$ or type $(1,0)$ and not in the subtree below $p$.

But one can easily see from the definition of $M$ that there cannot be infinitely many such elements $q_n$, a contradiction. \qed

This proposition relativizes as follows:

**Corollary 6.4.19.** For every degree $d$, there is a copy $B$ of $A$ with $B \leq_T d$ such that no copy $N'$ of $M$ with $N' \leq_T d$ has $R^B \oplus d \equiv_T S^N \oplus d$.

From the relativized versions of these two propositions (Corollaries 6.4.16 and 6.4.19), we get Theorem 6.4.12.

### 6.5 Degree Spectra of Relations on the Naturals

In this chapter, we will consider the special case of the structure $(\omega, <)$. We will generally be working with relations on the standard computable copy of this structure. Downey, Khoussainov, Miller, and Yu [DKMY09] studied relations on $\omega$ and though they were mostly interested in the degree spectra of non-computable relations, they showed that any computable unary relation $R$ on $(\omega, <)$ has a maximal degree in its degree spectrum, and this degree is either $0$ or $0'$. Knoll [Kno09] (and later independently Wright [Wri13]) extended this to show:

**Theorem 6.5.1** (Knoll [Kno09, Theorem 2.2], Wright [Wri13, Theorem 1.2]). Let $R$ be a computable unary relation on $(\omega, <)$. Then either $R$ is intrinsically computable, or its degree spectrum consists of all $\Delta^0_2$ degrees.

For relations which are not unary, Wright was able to show:

**Theorem 6.5.2** (Wright [Wri13, Theorem 1.3]). Let $R$ be a computable $n$-ary relation on $(\omega, <)$ which is not intrinsically computable. Then the degree spectrum of $R$ contains all of the c.e. degrees.
Note that this is the same as the conclusion of Harizanov’s Theorem 6.1.2 for this particular structure. One could adapt Wright’s proof to check Harizanov’s effectiveness condition. All of these results relativize.

In the case of unary relations on the standard copy of \((\omega, <)\), Knoll was able to classify the possible degree spectra completely—they are either just the computable degree, or all \(\Delta^0_2\) degrees. This suggests the following idea: study the partial order of degree spectra (on a cone) for relations on a single fixed structure (or class of structures). While in general, we know from §6.4 that there are incomparable d.c.e. degree spectra, this is not the case for unary relations on \((\omega, <)\); in fact, there are only two possible degree spectra for such relations. We know that there are at least three possible degree spectra for arbitrary relations on \((\omega, <)\): the computable degree, the c.e. degrees, and the \(\Delta^0_2\) degrees. Are there more, and if so, is there a nice classification of the possible degree spectra?

To begin, we will study the intrinsically \(\alpha\)-c.e. relations on \(\omega\). We need some definitions and a lemma about \(L_{\omega_1, \omega}\)-definable sets which is implicit in Wright’s work. A significant portion of the lemma coincides with Lemma 2.1 of [Mon09].

A partial order is a well-quasi-order if it is well-founded and has no infinite anti-chains (see, for example, [Kru72] where Kruskal first noticed that the same notion of a well-quasi-order had been used in many different places under different names). Note that a total ordering which is well-ordered is a well-quasi-order. There are two simple constructions that we will use which produce a new well-quasi-ordering from an existing one. First, if \((A_i, \leq_i)\) for \(i = 1, \ldots, n\) are partial orders, then their product \(A_1 \times \cdots \times A_n\) is partially ordered by \(\bar{a} \leq \bar{b}\) if and only if \(a_i \leq b_i\) for each \(i\). If each \((A_i, \leq_i)\) is a well-quasi-order, then so is the product. Second, if \((A, \preceq)\) is a partial order, then let \(P_{\omega}(A)\) be the set of finite subsets of \(A\). Define \(\preceq_P\) on this set by \(U \preceq_P V\) if and only if for all \(a \in U\), there is \(b \geq a\) in \(V\). If \((A, \preceq)\) is a well-quasi-order, then \((P_{\omega}(A), \preceq_P)\) is a well-quasi-order.

**Lemma 6.5.3.** The set of \(\Sigma^1_n\)-definable \(n\)-ary relations on \((\omega, <)\) over a fixed set of parameters \(\bar{c}\) is a well-quasi-order under (reverse) inclusion. That is, there is no infinite strictly increasing sequence, and there are no infinite anti-chains.

**Proof.** We will begin by considering the relations defined without parameters. Let \(R\) be a \(\Sigma^1_n\) \(n\)-ary relation. There are various possible orderings of an \(n\)-tuple, for example the entries may be increasing, decreasing, some entries may be equal to other entries, or many other possible orderings. But however many ways an \(n\)-tuple may be ordered, there are only finitely many possibilities. We can write \(R\) as the disjoint union of its restrictions to each of these orderings. Each of these restrictions is also \(\Sigma^1_n\)-definable. For any two such relations, we have \(S \subseteq R\) if and only if each of these restrictions of \(S\) is contained in the corresponding restriction of \(R\). Hence inclusion on \(n\)-ary relations is the product order of the inclusion order on each of these restrictions, and so it suffices to show that each of these restrictions is a well-quasi-order. Without loss of generality, it suffices to show this for increasing tuples.

Let \(R\) be \(\Sigma^1_n\)-definable relation on increasing \(n\)-tuples. Then \(R\) is defined by a \(\Sigma^1_n\) formula
We may assume that $\varphi(\bar{x})$ can be written in the form
\[ \varphi(\bar{x}) = (x_1 < x_2 < \cdots < x_n) \land \bigvee_i (\exists y) \psi_i(\bar{x}, y). \]

Now each of these disjuncts can be written in turn as the disjunct of finitely many formulas $\chi_\bar{p}(\bar{x})$ where $\bar{p} = (p_1, \ldots, p_n) \in \omega^n$ and $\chi_\bar{p}(\bar{x})$ is the formula which say that $x_1 < x_2 < \cdots < x_n$ and that there are at least $p_1$ elements less than $x_1$, $p_2$ elements between $x_1$ and $x_2$, and so on. So we can write $\varphi(\bar{x})$ as the disjunction of such formulas $\chi_\bar{p}$. For each $\bar{p} \in \omega^n$, let $D_\bar{p}$ be the set of solutions to $\chi_\bar{p}(\bar{x})$ in $\omega^n$. Then $R$ is the union of some of these sets $D_\bar{p}$.

Then note that $\bar{p} \mapsto D_\bar{p}$ is an order-maintaining bijection between the product order $\omega^n$ and the relations $D_\bar{p}$ ordered by reverse inclusion (that is, $\bar{p} \leq \bar{q}$ in the product order if and only if $D_\bar{p} \supseteq D_\bar{q}$). Since $\omega^n$ is well-quasi-ordered, the set of relations $D_\bar{p}$ is also well-quasi-ordered. Thus any set of such relations $D_\bar{p}$ contains finitely many maximal elements ordered under inclusion (or minimal elements ordered under reverse inclusion), and each other relation is contained in one of those maximal elements. In particular, $R$ is the union of finitely many sets of the form $D_\bar{p}$. What we have done so far is the main content of Montalbán’s work in [Mon09, Lemma 2.1].

Let $\bar{p} = (p_1, \ldots, p_n)$. Now note that the element $\hat{\bar{p}} = (p_1, p_1 + p_2 + 1, \ldots, p_1 + \cdots + p_n + n - 1)$ is in $D_i$ if and only if $p_i \geq r_i$ for each $i$, in which case $D_\bar{p} \subseteq D_\hat{\bar{p}}$. Now suppose that $D_\bar{p} \subseteq D_{\bar{q}_1} \cup \cdots \cup D_{\bar{q}_k}$. Then $\hat{\bar{p}} \in D_\bar{p}$ and so $\hat{\bar{p}} \in D_{\bar{q}_k}$ for some $k$. Hence for this $k$, $D_\bar{p} \subseteq D_{\bar{q}_k}$. Thus $D_\bar{p} \subseteq D_{\bar{q}_i} \cup \cdots \cup D_{\bar{q}_k}$ if and only if, for some $k$, $D_\bar{p} \subseteq D_{\bar{q}_k}$.

From this it follows that the partial order on $\Sigma_1^{\text{def}}$-definable relations on increasing $n$-tuples is isomorphic to the finite powerset order on $\omega^n$. This is a well-quasi-order.

Now we must consider formulas over a fixed tuple of parameters $\bar{c}$. Note that if $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{y})$ are $\Sigma_1^{\text{def}}$ formulas with no parameters, and every solution of $\varphi(\bar{x}, \bar{y})$ is a solution of $\psi(\bar{x}, \bar{y})$, then every solution of $\varphi(\bar{c}, \bar{y})$ is a solution of $\psi(\bar{c}, \bar{y})$. Thus there is no infinite anti-chain of sets $\Sigma_1^{\text{def}}$-definable over $\bar{c}$ (as any such anti-chain would yield an anti-chain of sets definable without parameters).

Now suppose that there is a strictly increasing chain of sets definable over $\bar{c}$, $A_1 \subsetneq A_2 \subsetneq \ldots$, which are definable by $\Sigma_1^{\text{def}}$ formulas $\varphi(\bar{c}, \bar{y})$. Let $B_1, B_2, \ldots$ be the corresponding sets definable by the formulas $\varphi(\bar{x}, \bar{y})$. Then $B_1, B_2, \ldots$ cannot be a strictly increasing sequence, nor can it be an anti-chain. Thus for some $i < j$, $B_i \supseteq B_j$. But then $A_i \supseteq A_j$, and so $A_i = A_j$. This is a contradiction. Hence there is no strictly increasing chain of sets $\Sigma_1^{\text{def}}$-definable over $\bar{c}$. This completes the proof.

From now on, by $\Sigma_1^{\text{def}}$-definable we mean definable with finitely many parameters. Then:

**Corollary 6.5.4.** Let $R$ be a $\Sigma_1^{\text{def}}$-definable relation on $(\omega, <)$. Then $R$ is defined by a finitary existential formula and $R$ is computable (in the standard copy of $(\omega, <)$). Moreover, $R$ is computable uniformly in the finitary existential formula and the tuple $\bar{c}$ over which it is defined.

Note that while $R$ is computable as a relation on $(\omega, <)$, it may not be intrinsically computable.
Proof. Suppose that $R$ is definable by a $\Sigma^1_{\omega}$ formula $\varphi(c, y)$. By the proof of the previous lemma, $\varphi(\bar{x}, \bar{y})$ is a disjunction of finitely many formulas of the form $\chi_{\beta}(\bar{x}, \bar{y})$. Since in $(\omega, \lessdot)$ we can compute the number of elements between any two particular elements, the solution set in $\omega$ of each of these formulas $\chi_{\beta}(\bar{x}, \bar{y})$ is a computable set. Then $\varphi(\bar{x}, \bar{y})$ is equivalent to a finitary existential formula and its solutions are computable; thus the same is true of $\varphi(\bar{c}, \bar{y})$.

Recall that $R$ is said to be intrinsically $\alpha$-c.e. if in all computable copies $\mathcal{B}$ of $\mathcal{A}$, $R^\mathcal{B}$ is $\alpha$-c.e. There is a theorem due to Ash and Knight (Propositions 3.2 and 3.3 of [AK96]), like that of Ash and Nerode [AN81], which relates the notion of intrinsically $\alpha$-c.e. to formally $\alpha$-c.e. definitions. The theorem uses the following notion of $\alpha$-free for a particular (computable presentation of) an ordinal $\alpha$. Given tuples $\bar{c}$ and $\bar{a}$ in a structure, we say that $\bar{a}$ is $\alpha$-free over $\bar{c}$ if for any finitary existential formula $\varphi(\bar{c}, \bar{x})$ true of $\bar{a}$, and any $\beta < \alpha$, there is a $\bar{a}'$ satisfying $\varphi(\bar{c}, \bar{x})$ which is $\beta$-free over $\bar{c}$ and such that $\bar{a}' \in R$ if and only if $a' \in R$. Then there are two theorems which together describe the intrinsically $\alpha$-c.e. relations:

**Theorem 6.5.5** (Ash-Knight [AK96, Proposition 3.3]). Let $\alpha$ be a computable ordinal. Let $\mathcal{A}$ be a computable structure and $R$ a computable relation on $\mathcal{A}$. Let $\bar{c}$ be a tuple. Suppose that no $\bar{a} \in R$ is $\alpha$-free over $\bar{c}$, and for each tuple $\bar{a}$ we can find a formula $\varphi(\bar{c}, \bar{a})$ which witnesses this. Also, suppose that for any $\beta < \alpha$, we can effectively decide whether a tuple $\bar{a}$ is $\beta$-free over $\bar{c}$, and if not then we can find the witnessing formula $\varphi(\bar{c}, \bar{a})$. Then $R$ is formally $\alpha$-c.e., that is, there are computable sequences $(\varphi_\beta(\bar{c}, \bar{x}))_{\beta < \alpha}$ and $(\psi_\beta(\bar{c}, \bar{x}))_{\beta < \alpha}$ such that

1. For all $\beta \leq \alpha$ and tuples $\bar{a}$, if

\[ \mathcal{A} \models \varphi_\beta(\bar{c}, \bar{a}) \land \psi_\beta(\bar{c}, \bar{a}) \]

then for some $\gamma < \beta$, $\mathcal{A} \models \varphi_\gamma(\bar{c}, \bar{a}) \lor \psi_\gamma(\bar{c}, \bar{a})$, and

2. $R$ is defined by

\[ \bigvee_{\beta < \alpha} (\varphi_\beta(\bar{c}, \bar{x}) \land \neg \bigvee_{\gamma < \beta} \psi_\gamma(\bar{c}, \bar{x})) \]

**Theorem 6.5.6** (Ash-Knight [AK96, Proposition 3.2]). Let $\alpha$ be a computable ordinal, $\mathcal{A}$ be a computable structure, and $R$ a computable relation on $\mathcal{A}$. Suppose that for each tuple $\bar{c}$, we can find a tuple $\bar{a} \in R$ which is $\alpha$-free over $\bar{c}$. Suppose if $\beta < \alpha$, $\bar{a}$ is $\beta$-free, and given an existential formula $\varphi$, we can find the witness $\bar{a}'$ to $\beta$-freeness. Then $R$ is not intrinsically $\alpha$-c.e.

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8Note that this is different from the notion of $\alpha$-free in Theorem 6.1.4.
Recall that when working on a cone, we can work with non-computable ordinals by relativizing to a cone on which they are computable (see page 114). For any ordinal \( \alpha \), even those which are not computable, a relation \( R \) is intrinsically \( \alpha \)-c.e. on a cone if and only if it has a formally \( \alpha \)-c.e. definition in the sense of Theorem 6.5.5, where the sequences \( \varphi_\beta \) and \( \psi_\beta \) are not necessarily computable. We see this by working on a cone above which \( \alpha \) is computable and above which the effectiveness conditions of the above theorems hold; then the equivalence follows from the relativized versions of those theorems.

There is also the usual relationship between relationship between relativel intrinsically \( \alpha \)-c.e. and formally \( \alpha \)-c.e. This was shown in the d.c.e. case by McCoy \[McC02\], and also independently McNicholl \[McN00\] who proved the result for all \( n \)-c.e. degrees. The general theorem for any ordinal \( \alpha \) was shown by Ash and Knight in \[AK00\].

**Theorem 6.5.7** (Ash-Knight \[AK00, Theorem 10.11\]). Let \( \alpha \) be a computable ordinal, \( \mathcal{A} \) be a computable structure, and \( R \) a computable relation on \( \mathcal{A} \). Then \( R \) is relatively intrinsically \( \alpha \)-c.e. if and only if it is formally \( \alpha \)-c.e. in the sense of Theorem 6.5.5.

We will show that for the structure \((\omega, <)\), all of these notions coincide. We begin by showing that intrinsically \( \alpha \)-c.e. implies relatively intrinsically \( \alpha \)-c.e. We do this by checking in the next two lemmas that in \((\omega, <)\), the effectiveness conditions from Theorems 6.5.6 and 6.5.5 are always satisfied.

**Lemma 6.5.8.** The structure \((\omega, <)\) satisfies the effectiveness conditions of Theorem 6.5.6 for any computable relation \( R \) and computable ordinal \( \alpha \) (i.e., if for each tuple \( \bar{c} \) there is \( \bar{a} \) \( \alpha \)-free over \( \bar{c} \), then we can find such an \( \bar{a} \), etc.).

**Proof.** Our argument will be very similar to the proof of Theorem 1.3 of \[Wri13\]. Using similar arguments, we may assume that \( R \) is a relation on increasing \( n \)-tuples. Suppose that for each tuple \( \bar{c} \), there is a tuple \( \bar{a} \) which is \( \alpha \)-free over \( \bar{c} \).

We will show that if \( \bar{a} \) is \( \alpha \)-free over the empty tuple, and each element of \( \bar{a} \) is greater than each element of a tuple \( \bar{c} \), then \( \bar{a} \) is \( \alpha \)-free over \( \bar{c} \). Let \( \bar{c} = (c_1, \ldots, c_n) \) and let \( m = \max(c_i) \). Then for any existential formula \( \varphi(\bar{c}, \bar{x}) \) true of \( \bar{a} \), there is a corresponding formula \( \psi(\bar{x}) \) which says that there are elements \( y_0 < \cdots < y_m \) smaller than each element of \( \bar{x} \), and that \( y_{c_1}, \ldots, y_{c_n}, \bar{x} \) satisfy \( \varphi \). Any solution of \( \varphi(\bar{c}, \bar{x}) \) is also a solution of \( \psi(\bar{x}) \), and vice versa (note that if \( \psi(\bar{b}) \) holds for some \( \bar{b} \), then \( y_0 = 0, \ldots, y_m = m \) witness the existential quantifier, since \( \varphi \) is existential).

Consider for each \( \beta \leq \alpha \) the following sets:

\[ C_\beta = \{ \bar{a} : \bar{a} \text{ is not } \beta \text{-free over } \varnothing \} \]

and its complement

\[ F_\beta = \{ \bar{a} : \bar{a} \text{ is } \beta \text{-free over } \varnothing \}. \]

Now suppose that the set

\[ L_\alpha = \{ \min(\bar{a}) : \bar{a} \in F_\alpha \} \]
is not unbounded, say its maximum is $m$. (Here, $\min(\bar{a})$ is the least entry of $\bar{a}$.) Then consider the $(n - 1)$-ary relations $R(0, x_1, \ldots, x_{n-1})$, $R(1, x_1, \ldots, x_{n-1})$, and so on up to $R(m, x_1, \ldots, x_{n-1})$. One of these relations must have $\alpha$-free tuples over any tuple $\bar{c}$. We may replace $R$ with this new relation. Continuing in this way, eventually we may assume that $L_\alpha$ is unbounded.

Now if $\bar{a}$ is not $\beta$-free, this is because there is a finitary existential formula $\varphi(\bar{x})$ true of $\bar{a}$ which witnesses that $\bar{a}$ is not $\beta$-free. Thus $C_\beta$ can be written in the form

$$C_\beta = (D_\beta \cap R) \cup (E_\beta \cap \neg R)$$

where $D_\beta$ and $E_\beta$ are $\Sigma^0_1$-definable (and hence computable and definable by a finitary existential formula). Since $R$ is computable, $C_\beta$ is computable, and hence $F_\beta$ is computable as well for each $\beta$. Moreover, these are uniformly computable, because the sets $C_\beta$ are increasing and the $\Sigma^0_1$-definable sets are well-quasi-ordered. So there are $\beta_1, \ldots, \beta_m \leq \alpha$ such that $C_0 = \cdots = C_{\beta_1}$, $C_{\beta_1+1} = \cdots = C_{\beta_2}$, and so on until $C_{\beta_m+1} = \cdots = C_\alpha$, and each of these sets are computable.

Thus, for any $\bar{c}$, we can find some tuple $\bar{a}$ which is $\alpha$-free over $\bar{c}$ (by searching through $F_\alpha$ for a tuple $\bar{a}$ all of whose elements are greater than each element of $\bar{c}$).

Now suppose that $\bar{a}$ is $\beta$-free over a tuple $\bar{c}$ for some $\beta \leq \alpha$. Then for any $\gamma < \beta$ and existential formula $\varphi(\bar{c}, \bar{x})$ true of $\bar{a}$, there is $\bar{b}$ satisfying $\varphi(\bar{c}, \bar{x})$ and $\gamma$-free over $\bar{c}$. Note that there must be such a $\bar{b}$ all of whose elements are greater than each element of $\bar{c}$, since this true of $\bar{a}$, and that any such element of $F_\gamma$ is $\gamma$-free over $\bar{c}$. So we can compute such a $\bar{b}$. \qed

**Lemma 6.5.9.** $(\omega, <)$ satisfies the effectiveness condition of Theorem 6.5.5 for any computable relation $R$ and ordinal $\alpha$.

**Proof.** Fix $\bar{c}$. Suppose that there are no $\alpha$-free tuples over $\bar{c}$. For each $\beta \leq \alpha$, let

$$C_\beta = \{ \bar{a} : \bar{a} \text{ is not } \beta\text{-free over } \bar{c} \}.$$

Once again, $C_\beta \cap R$ and $C_\beta \cap \neg R$ are $\Sigma^0_1$-definable over $\bar{c}$, and so by the well-quasi-ordering of such sets, they are uniformly computable and the finitary existential definitions can be uniformly determined. This is enough to have the effectiveness condition of Theorem 6.5.5. \qed

Now as a corollary of the previous two lemmas, we can prove the following fact.

**Corollary 6.5.10.** Suppose that $R$ is a computable relation on $(\omega, <)$ which is intrinsically $\alpha$-c.e. Then $R$ is relatively intrinsically $\alpha$-c.e.

**Proof.** Suppose that $R$ is computable and intrinsically $\alpha$-c.e. for some computable ordinal $\alpha$. We must show that $R$ is relatively intrinsically $\alpha$-c.e., that is, that $R$ is formally $\alpha$-c.e.

**Claim 6.5.11.** There is a tuple $\bar{c}$ such that no $\bar{a} \in R$ is $\alpha$-free over $\bar{c}$.
Proof. Suppose for a contradiction that for each tuple \( \bar{c} \), there is \( \bar{a} \in R \) which is \( \alpha \)-free over \( \bar{c} \). By Theorem 6.5.6 and Lemma 6.5.8, \( R \) is not intrinsically \( \alpha \)-c.e. This contradicts the fact that \( R \) is intrinsically \( \alpha \)-c.e. \( \square \)

Now let \( \bar{c} \) be as in the claim. By Theorem 6.5.5 and Lemma 6.5.9, \( R \) is formally \( \alpha \)-c.e., and hence relatively intrinsically \( \alpha \)-c.e. \( \square \)

Now we use this to show that the notions of intrinsically \( \alpha \)-c.e., relatively intrinsically \( \alpha \)-c.e., and intrinsically \( \alpha \)-c.e. on a cone all coincide for \( (\omega, \prec) \). One can view this as saying that \( (\omega, \prec) \) and such relations on it are “natural.”

**Proposition 6.5.12.** If \( R \) is a relation on \( (\omega, \prec) \) and \( \alpha \) is any (possibly non-computable) ordinal, then if \( R \) is intrinsically \( \alpha \)-c.e. on a cone then \( R \) is computable and intrinsically \( m \)-c.e. for some finite \( m \).

Proof. Now suppose that \( R \) is a possibly non-computable relation on \( (\omega, \prec) \), \( \alpha \) is a possibly non-computable countable ordinal, and \( R \) is intrinsically \( \alpha \)-c.e. on a cone. As remarked after Theorems 6.5.5 and 6.5.6, \( R \) has a formally \( \alpha \)-c.e. definition in the sense of Theorem 6.5.5, but where the sequences \( \varphi_\beta \) and \( \psi_\beta \) are not necessarily computable. Then there are sets \( A_\beta \) and \( B_\beta \) for \( \beta < \alpha \) which are \( \Sigma^1_\infty \)-definable over a tuple \( \bar{c} \) such that

\[
R = \bigcup_{\beta < \alpha} (A_\beta - \bigcup_{\gamma < \beta} B_\gamma)
\]

and if \( x \in A_\beta \cap B_\beta \) then for some \( \gamma < \beta \), \( x \in A_\gamma \cup B_\gamma \). We may replace \( A_\beta \) by \( \bigcup_{\gamma < \beta} A_\gamma \) for each \( \beta \), and similarly for \( B_\beta \). Then the sequences \( A_\beta \) and \( B_\beta \) are increasing in \( \beta \). Since the \( \Sigma^1_\infty \)-definable relations form a well-quasi-order under inclusion, there is some sequence \( 0 = \beta_1, \ldots, \beta_m \leq \alpha \) such that \( A_\gamma \) and \( B_\gamma \) are constant on the intervals \( [\beta_1, 0), [\beta_2, \beta_1), \ldots, [\beta_m, \beta_{m-1}) \), and so on up to \( [\beta_m, \alpha] \). Otherwise, we could construct an infinite strictly increasing chain.

So

\[
R = (A_{\beta_m} - B_{\beta_{m-1}}) \cup (A_{\beta_{m-1}} - B_{\beta_{m-2}}) \cup \cdots \cup A_{\beta_1}
\]

Suppose that \( x \in A_{\beta_i} \cap B_{\beta_i} \). Then for some least \( \gamma < \beta_i \), \( x \in A_\gamma \cup B_\gamma \). By the minimality of \( \gamma \), \( \gamma \leq \beta_{i-1} \). Thus, for some \( j < i \), \( x \in A_{\beta_j} \cup B_{\beta_j} \).

Since each of these sets \( A_{\beta_i} \) and \( B_{\beta_i} \) is \( \Sigma^1_\infty \)-definable, they are all computable subsets of \( \omega \) which are definable by a finitary existential (and hence \( \Sigma^1_1 \)) formula. Thus \( R \) is intrinsically \( m \)-c.e. for some \( m \) and \( R \) is computable by Corollary 6.5.4. \( \square \)

**Proposition 6.5.13.** Let \( R \) be a relation on \( (\omega, \prec) \). Then the following are equivalent for any computable ordinal \( \alpha \):

1. \( R \) is intrinsically \( \alpha \)-c.e. and computable in \( (\omega, \prec) \),
2. \( R \) is relatively intrinsically \( \alpha \)-c.e.,
3. \( R \) is intrinsically \( \alpha \)-c.e. on a cone.
In this case, $R$ is intrinsically $m$-c.e. for some $m$.

**Proof.** Corollary 6.5.10 shows that if $R$ is intrinsically $\alpha$-c.e. and computable in $(\omega,<)$, then $R$ is relatively intrinsically $\alpha$-c.e.

If $R$ is relatively intrinsically $\alpha$-c.e., then $R$ is formally $\alpha$-c.e., and hence intrinsically $\alpha$-c.e. on a cone.

The previous proposition shows that if $R$ is intrinsically $\alpha$-c.e. on a cone, then it is intrinsically $\alpha$-c.e. and computable in $(\omega,<)$. \qed

Now we will show that any intrinsically $\alpha$-c.e. relation on $(\omega,<)$ (which must be $m$-c.e. for some $m$) is intrinsically of c.e. degree. One example of such a relation is the intrinsically d.c.e. relation $S$ consisting of pairs $(a,b)$ which are separated by exactly one element. In any computable copy, $S$ computes the adjacency relation (which is always co-c.e.). Two elements $a$ and $b$ are adjacent if and only if there is some $c > b$ such that $c$ and $a$ are separated by a single element (which is $b$), and so the adjacency relation is c.e. in $S$ in any computable copy of $(\omega,<)$. Since it is always co-c.e., it is always computable in $S$. On the other hand, the adjacency relation in any computable copy computes an isomorphism between that copy and $(\omega,<)$, and hence computes $S$ in that copy. Thus we see that $S$ is intrinsically of c.e. degree. The proof of the following proposition is just a generalization of this idea.

Note that it is possible to have a relation which is formally $\Delta^0_2$ but not formally $\alpha$-c.e. for any computable ordinal $\alpha$. Indeed, there is a formally $\Delta^0_2$ relation $R$ on a structure $A$ whose degree spectrum, relativized to any degree $d$, consists of all of the $\Delta^0_2(d)$ degrees. (One can take $A$ to be an equivalence structure with infinitely many equivalence classes, each of which is ordered as either $\omega$ or $\omega^*$; the relation $R$ picks out those equivalence classes which are ordered as $\omega$.) The degree spectrum of any formally $\alpha$-c.e. relation does not consist, relative to any degree $d$, of all of the $\Delta^0_2(d)$ degrees; each degree in the degree spectrum is $\alpha$-c.e. for the particular computable presentation of $\alpha$ used in the formal $\alpha$-c.e. definition. So such a relation $R$ is formally $\Delta^0_2$ but not formally $\alpha$-c.e. for any computable ordinal $\alpha$. This is in contrast to the $\Delta^0_2$ degrees, all of which are $\omega^2$-c.e. for some computable presentation of $\omega^2$ (see Theorem 8 of [EHK81]), though no single presentation of $\omega^2$ suffices.

**Proposition 6.5.14.** Let $R$ be an intrinsically $m$-c.e. $n$-ary relation on $(\omega,<)$ for some finite $m$. Then $R$ is intrinsically of c.e. degree.

**Proof.** Let $A_1 \supseteq \cdots \supseteq A_m$ be intrinsically $\Sigma^0_1$ sets such that (depending on whether $m$ is odd or even) either

$$R = (A_1 - A_2) \cup (A_3 - A_4) \cup \cdots \cup (A_{m-1} - A_m)$$

or

$$R = (A_1 - A_2) \cup (A_3 - A_4) \cup \cdots \cup (A_{m-2} - A_{m-1}) \cup A_m.$$  

We claim that in any computable copy $B \cong (\omega,<)$, from $R^B$ we can compute the successor relation on $B$. The successor relation computes the isomorphism between $B$ and $(\omega,<)$ and hence computes $R^B$. The successor relation is intrinsically $\Pi^0_1$ and hence of c.e. degree, so this will suffice to complete the proof.
We will begin with a simple case to which we will later reduce the general case. For \( \bar{p} \in \omega^{n-1} \), let \( E_{\bar{p}} = D_{\bar{p}} \), that is, \( E_{\bar{p}} \) consists of those \( \bar{x} = (x_1, \ldots, x_n) \) such that there are \( p_1 \) points between \( x_1 \) and \( x_2 \), \( p_2 \) point between \( x_2 \) and \( x_3 \), and so on (but no restriction on the number of points before \( x_1 \)).

**Claim 6.5.15.** Suppose that \( B \subseteq A \) are intrinsically \( \Sigma^0_1 \) sets of the form

\[
A = E_{\bar{p}_1} \cup \cdots \cup E_{\bar{p}_{t_1}}
\]

and

\[
B = E_{\bar{q}_1} \cup \cdots \cup E_{\bar{q}_{t_2}}.
\]

Furthermore, suppose that \( A \neq B \) and \( B \neq \varnothing \), \( A - B \subseteq R \), and \( B \subseteq \neg R \). Then \( R \) computes the successor relation.

**Proof.** Since \( A - B \) is non-empty, we may write \( A \) and \( B \) as in the statement of the claim with the additional property that there is some \( j \) such that \( E_{\bar{q}_j} \) is not contained in any of the others, and \( E_{\bar{q}_j} \) is not equal to \( E_{\bar{p}_i} \) for any \( i \).

Now, in the proof of Lemma 6.5.3 we showed that if

\[
E_{\bar{q}_j} \subseteq E_{\bar{p}_1} \cup \cdots \cup E_{\bar{p}_{t_2}}
\]

then \( E_{\bar{q}_j} \subseteq E_{\bar{p}_i} \) for some \( i \) (though we showed this with \( E \) replaced by \( D \), the same result still applies here). Fix such an \( i \). There must be some index \( t \in \{1, \ldots, n-1\} \) such that \( \bar{p}_i(t) < \bar{q}_j(t) \).

Then let

\[
\bar{u} = (u_1, \ldots, u_{n-1}) = (\bar{q}_j(1), \bar{q}_j(2), \ldots, \bar{q}_j(t) - 1, \ldots, \bar{q}_j(n-1))
\]

and

\[
\bar{v} = (v_1, \ldots, v_{n-1}) = \bar{q}_j.
\]

Thus \( u_i = v_i \) except for \( i = t \), in which case \( u_t = u_i + 1 \). So \( E_{\bar{v}} \subseteq E_{\bar{u}} \).

Now \( E_{\bar{v}} = E_{\bar{q}_j} \subseteq B \) and \( E_{\bar{u}} \subseteq E_{\bar{p}_i} \subseteq A \). Moreover, if, for any \( \bar{w} \), \( E_{\bar{v}} \subseteq E_{\bar{w}} \subseteq E_{\bar{u}} \), then either \( E_{\bar{u}} = E_{\bar{w}} \) or \( E_{\bar{w}} = E_{\bar{v}} \).

So, by the choice of \( j \), we have

\[
E_{\bar{u}} - E_{\bar{v}} \subseteq A - B \subseteq R
\]

and

\[
E_{\bar{v}} \subseteq B \subseteq \neg R.
\]

Let \( B \) be a computable copy of \( (\omega, <) \). Let \( S \) be the successor function on \( B \). We claim that, using \( R^B \), we can compute \( S \). Suppose that we wish to compute whether an element \( y \) is the successor of \( x \). We can, in a c.e. way, find out if \( y \) is not the successor of \( x \), so we just need to show how to recognize that \( y \) is the successor of \( x \) if this is the case. We can
non-uniformly assume that we know some initial segment of $B$, say the first $t + u_1 + \cdots + u_t + 1$ elements.

First, we must have $x < y$. If $x \leq t + u_1 + \cdots + u_t$, then we can non-uniformly decide whether $y$ is the successor of $x$. Otherwise, search for $z_1 < z_2 < \cdots < z_{t-1} < x < y = z_t < \cdots < z_n$ such that

1. there are $u_1$ elements between $z_1$ and $z_2$, $u_2$ elements between $z_2$ and $z_3$, and so on,
2. there are $u_t - 1$ elements between $z_{t-1}$ and $x$, and
3. $(z_1, \ldots, z_n) \in R^B$.

Then since $(z_1, \ldots, z_n) \in E_{\bar{u}}$, and $E_{\bar{u}} \subseteq \neg R$,

$$(z_1, \ldots, z_n) \in E_{\bar{u}} - E_{\bar{u}}.$$ 

In particular, there cannot be more than $u_t$ elements between $z_{t-1}$ and $z_t$. As $z_{t-1} < x < y = z_t$, and there are $u_t = v_t - 1$ elements between $z_{t-1}$ and $x$, $y$ is the successor of $x$. If $y$ is the successor of $x$ and $x \geq t + u_1 + \cdots + u_t$, then it is possible to find such elements $z_1, \ldots, z_n$. \hfill \Box

Now we will finish the general case. Let $B$ be a computable copy of $(\omega, <)$ and let $S$ be the successor relation on $B$. Suppose that

$$R = (A_1 - A_2) \cup (A_3 - A_4) \cup \cdots \cup (A_{m-1} - A_m)$$

or

$$R = (A_1 - A_2) \cup (A_3 - A_4) \cup \cdots \cup (A_{m-2} - A_{m-1}) \cup A_m$$

where each of these sets is an intrinsically $\Sigma^0_1$ set definable over constants $\bar{c}$.

Let $M$ be a constant greater than each entry of $\bar{c}$. Then we can non-uniformly know the which elements of $B$ correspond to $\{0, \ldots, M\} \subseteq \omega$. It remains to compute the successor relation of $B$ on the rest of the elements of $B$. Consider the restriction $\hat{R}$ of $R$ to $\{M, M + 1, \ldots\}^n$; $R^B$ computes $\hat{R}^B$. By restricting $B$ to the elements which correspond to $\{M, M + 1, \ldots\} \subseteq \omega$, and making the natural identification $M \mapsto 0$, $M + 1 \mapsto 1$, and so on, we have reduced to the case where (for some possibly smaller $m$) we have

$$R = (A_1 - A_2) \cup (A_3 - A_4) \cup \cdots \cup (A_{m-1} - A_m)$$

or

$$R = (A_1 - A_2) \cup (A_3 - A_4) \cup \cdots \cup (A_{m-2} - A_{m-1}) \cup A_m$$

and each of these sets is intrinsically $\Sigma^0_1$ and definable without parameters. Note that $\hat{R}^B$ may not compute $R^B$. However, if we can show that $\hat{R}^B$ computes the successor relation $S$, then since $S$ computes $R^B$, all three of the relations $R^B$, $\hat{R}^B$, and $S$ will be intercomputable.

Now by the arguments of Lemma 6.5.3 each $A_i$ is a finite union of sets of the form $D_{\bar{p}}$ for $\bar{p} \in \omega^n$. Let $N$ be larger than the first entry $p_0$ of each of these tuples $\bar{p}$. Then make the same reduction as before to reduce to the case where each $A_i$ is a union of sets $E_{\bar{q}}$. 

\hfill \Box
CHAPTER 6. DEGREE SPECTRA OF RELATIONS

Now if
\[ R = (A_1 - A_2) \cup (A_3 - A_4) \cup \cdots \cup (A_{m-1} - A_m) \]
than \( A = A_{m-1} \) and \( B = A_m \) are both as in the claim above. If
\[ R = (A_1 - A_2) \cup (A_3 - A_4) \cup \cdots \cup (A_{m-2} - A_{m-1}) \cup A_m \]
than \( A = A_{m-1} \) and \( B = A_m \) are as in the claim above, except with \( R \) and \( \neg R \) interchanged.

So far, we still only know of three possible degree spectra for a relation on \((\omega, <)\): the computable degree, the c.e. degrees, and the \(\Delta^0_2\) degrees. It is possible that there is another degree spectrum in between the c.e. degree and the \(\Delta^0_2\) degrees, but we do not know whether such a degree spectrum exists. This is the main open question of this section:

**Question 6.5.16.** Is there a relation on \((\omega, <)\) whose degree spectrum is strictly contained between the c.e. degrees and the \(\Delta^0_2\) degrees on a cone?

This question appears to be a difficult one. If the answer to the question is no, that such a degree spectrum cannot exist, and the proof was not too hard, then it would probably be of the following form. Let \( R \) be a relation on \((\omega, <)\). Working on a cone, there is a computable function \( f \) such that given an index \( e \) for a computable function \( \varphi_e(x, s) \) of two variables which is total and gives a \(\Delta^0_2\) approximation of a set \( C \), \( f(e) \) is an index for a structure \( A \) isomorphic to \((\omega, <)\) with \( R^A \equiv_T C \). If \( \varphi_e(x, s) \) does not give a \(\Delta^0_2\) approximation, then we place no requirements on \( f(e) \). Moreover, if the coding is simple, then we will have indices \( g_1(e) \) and \( g_2(e) \) for the computations \( R^A \leq_T C \) and \( C \leq_T R^A \) respectively. We capture this situation with the following definition:

**Definition 6.5.17.** Let \( A \) be a structure and \( R \) a relation on \( A \). Let \( \Gamma \) be a class of degrees indexed by some subset of \( \omega \) which relativizes. Then the degree spectrum of \( R \) is uniformly equal to \( \Gamma \) on a cone if, on a cone, there are computable functions \( f \), \( g_1 \), and \( g_2 \) such that given an index \( e \) for \( C \in \Gamma \), the computable structure \( A \) with index \( f(e) \) has
\[ R^A = \Phi^{C}_{g_1(e)} \text{ and } C = \Phi^{R^A}_{g_2(e)}. \]
We do not know of any relations which are not uniformly equal to their degree spectrum.

**Question 6.5.18.** Must every relation obtain its degree spectrum uniformly on a cone?

This question is not totally precise, because we have not classified all possible degree spectra on a cone and so we do not have an indexing for each of them. But this question is precise for relations with particular degree spectra, such as the c.e. degrees or the \(\Delta^0_2\) degrees.

Theorem 6.5.19 below says that one of the two questions we have just introduced is resolved in a surprising way: either there is a relation \( R \) on \((\omega, <)\) with degree spectrum all of the \(\Delta^0_2\) degrees, but not uniformly, or the relation \( R \) has a degree spectrum contained strictly in between the c.e. degrees and the \(\Delta^0_2\) degrees.
If the answer to Question 6.5.16 is no, that there are only three possible degree spectra for a relation on $(\omega, <)$, then by Theorem 6.5.19 the answer to Question 6.5.18 must also be no. If the answer to Question 6.5.16 is yes, then the answer to Question 6.5.18 might be either yes or no.

If the answer to Question 6.5.16 is yes, there are more than three possible degree spectra, then we can ask what sort of degree spectra are possible. We know that any relation which is intrinsically $\alpha$-c.e. has degree spectra consisting only of the c.e. degrees, but can there be a relation which is intrinsically of $\alpha$-c.e. degree, while not being intrinsically of c.e. degree?

We are now ready to prove Theorem 6.5.19.

**Theorem 6.5.19.** There is a relation $R$ on $(\omega, <)$ whose degree spectrum on a cone is contained in, but not uniformly equal to, the $\Delta^0_2$ degrees, but strictly contains the c.e. degrees.

**Proof.** We will begin by working computably, and everything will relativize. We will exhibit the relation $R$ and show that there is a computable function $h$ which, given indices $e, i, j$, produces a $\Delta^0_2$ set $C$ with index $h(e, i, j)$ such that if $e$ is the index for a computable structure $A$ isomorphic to $(\omega, <)$, then either $R^A \neq \Phi^A_i$ or $C \neq \Phi^A_j$. Note that $h(e, i, j)$ will be total and will always give an index for a $\Delta^0_2$ set, even if $e$ is not an index of the desired type.

We will also show that $R$ is not intrinsically of c.e. degree. The construction of $R$ and $h$ will follow later, but first we will show that such an $R$ has a degree spectrum which is not uniformly equal to the $\Delta^0_2$ degrees.

Suppose to the contrary that there are computable functions $f$, $g_1$, and $g_2$ such that, given an index $e$ for a function $\varphi_e(x, s)$ which is total and gives a $\Delta^0_2$ approximation of a set $C$, $f(e)$ is the index of a computable copy $A$ of $(\omega, <)$ with

$$R^A = \Phi^C_{g_1(e)} \text{ and } C = \Phi^A_{g_2(e)}.$$

Then we would like to take the composition $\theta(e) = h(f(e), g_1(e), g_2(e))$, except that this may not be total when applied to indices $e$ where $\varphi_e(x, s)$ is either not total or not a $\Delta^0_2$ approximation. So instead define $h$ as follows. Given an input $e$, we will define a $\Delta^0_2$ set uniformly by giving an approximation by stages. Try to compute $f(e), g_1(e)$, and $g_2(e)$, and while these do not converge, make the $\Delta^0_2$ approximation equal to zero. When they do converge, compute $h(f(e), g_1(e), g_2(e))$ (since $h$ is total, this will always converge and give an index for a $\Delta^0_2$ set) and have our approximation instead follow the $\Delta^0_2$ set with index $h(f(e), g_1(e), g_2(e))$. Thus $\theta(e) = h(f(e), g_1(e), g_2(e))$ whenever the right hand side is defined, and is an index for the empty set otherwise.

Now, by the recursion theorem, there is a fixed point $e$ of $\theta$, so that

$$\varphi_{\theta(e)}(x, s) = \varphi_e(x, s).$$

Now the left hand side is always total and is a $\Delta^0_2$ approximation, so the same is true of the right hand side. Thus $\theta(e) = h(f(e), g_1(e), g_2(e))$. Let $C$ be the $\Delta^0_2$ set with this
approximation, so that $C$ has indices $e$ and $\theta(e)$. Thus $f(e)$ is the index of a computable copy $A$ of $(\omega, <)$ with

$$R^A = \Phi^C_{g_1(e)} \text{ and } C = \Phi^R_{g_2(e)}.$$ 

But then by definition of $h$,

$$R^A \neq \Phi^C_{g_1(e)} \text{ or } C \neq \Phi^R_{g_2(e)}.$$ 

This is a contradiction. Hence no such functions $f$, $g_1$, and $g_2$ can exist, and the degree spectrum of $R$ is not uniformly equal to the $\Delta^0_2$ degrees on a cone.

**Construction of $R$.**

We will begin by defining $a_n$, $b_n$, and $c_n$ with $a_1 < b_1 = c_1 < a_2 < b_2 < c_2 < a_3 < \ldots$. Begin with $a_1 = 0$. Let $b_n = a_n + n + 1$, $c_n = b_n + a_n$, and $a_{n+1} = c_n + n + 1$. This divides $\omega$ up into disjoint intervals $[a_n, b_n)$, $[b_n, c_n)$, and $[c_n, a_{n+1})$ as $n$ varies. Note that $b_1 = 2 = c_1$, and so the interval $[b_1, c_1)$ is empty. Every other interval is non-empty. Also, the length of the interval $[b_n, c_n)$ is the same as the length of the interval $[0, a_n)$.

![Figure 6.5: The relation $R$ on the first sixteen elements of $(\omega, <)$](image)

Figure 6.5: The relation $R$ on the first sixteen elements of $(\omega, <)$. The arrows $\leftrightarrow$ are those from the cycles $a_n \leftrightarrow a_n + 1 \leftrightarrow a_n + 2 \leftrightarrow \cdots \leftrightarrow a_n + n \leftrightarrow a_n$ and $c_n \leftrightarrow c_n + 1 \leftrightarrow c_n + 2 \leftrightarrow \cdots \leftrightarrow c_n + n \leftrightarrow c_n$. The arrows $\rightarrow$ (which curve above) are those between $a_n$ and $c_n$ for some $n$. The arrows $\leftarrow$ (which curve below) are those from $y$ to $c_n + n$ for $y \geq a_{n+1}$.

The relation $R$ will be a binary relation which we can interpret as a directed graph. For each $n$, we will have cycles of edges

$$a_n \leftrightarrow a_n + 1 \leftrightarrow a_n + 2 \leftrightarrow \cdots \leftrightarrow a_n + n \leftrightarrow a_n$$

and

$$c_n \leftrightarrow c_n + 1 \leftrightarrow c_n + 2 \leftrightarrow \cdots \leftrightarrow c_n + n \leftrightarrow c_n.$$
These edges all go in both directions; i.e., there is an edge from \( a_n \) to \( a_n + 1 \) and from \( a_n + 1 \) to \( a_n \).

Now add an edge from \( a_n \) to \( c_n \). These edges are directed, and go from the smaller element to the larger element. Also, add edges from \( y \) to \( c_n + n \) for all \( y \geq a_{n+1} \). These edges are also directed, but go from the larger element to the smaller element. By looking at whether an edge goes in both directions, in the increasing direction, or in the decreasing direction, we can decide what type of edge it is (i.e., is it from a cycle, from \( a_n \) to \( c_n \) for some \( n \), or from \( y \) to \( c_n + n \) for some \( n \) and some \( y \geq a_{n+1} \)).

For \( x, y \in [b_n, c_n] \), put an edge from \( x \) to \( y \) if and only if there is an edge from \( (x - b_n) \) to \( (y - b_n) \). So the relation \( R \) on the interval \([b_n, c_n]\) looks the same as it does on the interval \([0, a_n]\). This completes the definition of the relation \( R \). Note that \( R \) is computable. Figure 6.5 shows the relation \( R \) on an initial segment of \( \omega \).

We say that elements \( \bar{y} = (y_0, y_1, \ldots, y_m) \) of \( (\omega, <) \) form an \( m + 1 \)-cycle (for \( m \geq 1 \)) if \( y_0 < y_1 < \cdots < y_m \) and there is a cycle

\[
y_0 \leftrightarrow y_1 \leftrightarrow \cdots \leftrightarrow y_m \leftrightarrow y_0
\]

and there are no other edges between any of the \( y_i \). An \( m + 1 \)-cycle is just a bi-directional \( m + 1 \)-cycle of the graph. Note that every \( m + 1 \)-cycle is either

\[
a_m \leftrightarrow a_m + 1 \leftrightarrow \cdots \leftrightarrow a_m + m \leftrightarrow a_m,
\]

or

\[
c_m \leftrightarrow c_m + 1 \leftrightarrow \cdots \leftrightarrow c_m + m \leftrightarrow c_m,
\]

or contained in \([b_n, c_n]\) for some \( n > m \).

**Remark 6.5.20.** Each element \( x \in \omega \) is part of exactly one cycle.

**Proof.** This can easily be seen by an induction argument on the \( n \) such that \( x \in [a_n, c_n] \). If \( x \) is in \([a_n, b_n]\) or \([c_n, a_{n+1}]\) then this is obvious, and if \( x \) is in \([b_n, c_n]\) then this follows by the induction hypothesis. \( \square \)

If \( \bar{x} = (x_0, \ldots, x_m) \) and \( \bar{y} = (y_0, \ldots, y_m) \) are \( m + 1 \)-cycles, listed in increasing order, and there is an edge from \( x_0 \) to \( y_0 \) (so that \( x_0 < y_0 \)), then we say that \( \bar{x} \) and \( \bar{y} \) are a matching pair of \( m + 1 \)-cycles. Note that, by convention, we list the two tuples of a matching pair in increasing order: thus, if \( \bar{x} \) and \( \bar{y} \) are a matching pair, then \( x_0 < x_1 < \cdots < x_m < y_0 < y_1 < \cdots < y_m \). Each \( m + 1 \)-cycle is part of a matching pair of \( m + 1 \)-cycles.

**Remark 6.5.21.** If \( \bar{x} = (x_0, \ldots, x_m) \) and \( \bar{y} = (y_0, \ldots, y_m) \) are a matching pair of \( m + 1 \)-cycles, then there is some \( n \) such that both \( \bar{x} \) and \( \bar{y} \) are contained in \([a_n, a_{n+1}]\).

**Proof.** Since \( \bar{x} \) and \( \bar{y} \) form a matching pair, there is an edge from \( x_0 \) to \( y_0 \) but not vice versa. We can see by the definition of \( R \) that either \( \bar{x} = (a_m, \ldots, a_m + m) \) and \( \bar{y} = (c_m, \ldots, c_m + m) \), or \( \bar{x} \) and \( \bar{y} \) are both contained in \([b_n, c_n]\) for some \( n \). \( \square \)
Remark 6.5.22. If \( \bar{u} = (u_0, \ldots, u_m) \) and \( \bar{v} = (v_0, \ldots, v_m) \) are a matching pair of \( m + 1 \)-cycles, and \( \bar{x} = (x_0, \ldots, x_m) \) and \( \bar{y} = (y_0, \ldots, y_m) \) are another matching pair of \( m + 1 \)-cycles (for the same \( m \)), then the relation \( R \) restricted to the interval \([u_0, v_m]\) is the same as the relation \( R \) restricted to the interval \([x_0, y_m]\). In particular, the lengths of these intervals are the same: \( v_m - u_0 = y_m - x_0 \).

Proof. We may assume that \( \bar{u} = (a_m, \ldots, a_m + m) \) and \( \bar{v} = (c_m, \ldots, c_m + m) \). Suppose to the contrary that there are \( \bar{x} = (x_0, \ldots, x_m) \) and \( \bar{y} = (y_0, \ldots, y_m) \) a matching pair of \( m + 1 \)-cycles such that the relation \( R \) restricted to the interval \([u_0, v_m]\) is not the same as the relation \( R \) restricted to the interval \([x_0, y_m]\). Assume that \( x_0 \) is least with this property.

Now by the previous fact, \( \bar{x} \) and \( \bar{y} \) are contained in \([a_n, a_{n+1}]\) for some \( n \). We must have \( x_0 > c_m + m \) and so \( n > m \). Then \( \bar{x} \) and \( \bar{y} \) are contained in \([b_n, c_n]\). Then by definition of \( R \), \( \bar{x}' = (x_0 - b_n, \ldots, x_m - b_n) \) and \( \bar{y}' = (y_0 - b_n, \ldots, y_m - b_n) \) are a matching pair of \( m + 1 \)-cycles and \( R \) restricted to the interval \([x_0, y_m]\) is the same as \( R \) restricted to the interval \([x_0 - b_n, y_m - b_n]\). But by the induction hypothesis, \( R \) restricted to the interval \([x_0 - b_n, y_m - b_n]\) is the same as \( R \) restricted to the interval \([u_0, v_m]\). This contradiction finishes the proof.

In any computable copy \( \mathcal{A} \) of \((\omega, <)\), using \( R^A \) as an oracle we can compute for each \( x \in \mathcal{A} \) the unique \( m + 1 \)-cycle in which \( x \) is contained, \( x \)'s position in that cycle, and we can compute the other \( m + 1 \)-cycle with which this \( m + 1 \)-cycle forms a matching pair.

\( R \) is not intrinsically of \( c.e. \) degree.

Let \( \bar{c} \) be a tuple and \( a_n \) be such that \( \bar{c} < a_n \). We will show that the tuple \( \bar{a} = (a_n, \ldots, a_n + n) \) is \( d \)-free over \( \bar{c} \). First, we will introduce some notation. Given a tuple \( \bar{x} = (x_0, \ldots, x_n) \), and \( r \in \omega \), write \( \bar{x} + r \) for \((x_0 + r, \ldots, x_n + r)\).

Recall what it means for \( \bar{a} \) to be \( d \)-free over \( \bar{c} \): for every \( \bar{b} \) and existential formula \( \phi(\bar{c}, \bar{u}, \bar{v}) \) true of \( \bar{a}, \bar{b} \), there are \( \bar{a}' \) and \( \bar{b}' \) which satisfy \( \phi(\bar{c}, \bar{u}, \bar{v}) \) such that \( R \) restricted to tuples from \( \bar{c}\bar{a}' \) is not the same as \( R \) restricted to tuples from \( \bar{c}\bar{a} \) and also such that for every existential formula \( \psi(\bar{c}, \bar{u}, \bar{v}) \) true of them, there are \( \bar{a}'' \), \( \bar{b}'' \) satisfying \( \psi \) and such that \( R \) restricted to \( \bar{c}\bar{a}''\bar{b}'' \) is the same as \( R \) restricted to \( \bar{c}\bar{a}\).

Figure 6.6 shows the choices of \( \bar{a}', \bar{b}' \), etc. in one particular example. Given \( \bar{b} \) and some existential formula \( \psi(\bar{c}, \bar{u}, \bar{v}) \) true of \( \bar{a}, \bar{b} \), we may assume that \( \phi \) is quantifier-free by expanding \( \bar{b} \). Then \( \psi \) just says that \( \bar{c}\bar{a}\bar{b} \) are ordered in some particular way. Now, some of the entries of \( \bar{b} \) are less than \( a_n \), and the rest are greater than or equal to \( a_n \). Rearranging \( \bar{b} \) as necessary, write \( \bar{b} = \bar{b}_1\bar{b}_2 \) where each entry of \( \bar{b}_1 \) is less than \( a_n \), and each entry of \( \bar{b}_2 \) is greater than or equal to \( a_n \). Let \( \bar{a}' = \bar{a} + 1 \) (recall that this is shifting each entry by one) and \( \bar{b}' = \bar{b}_1\bar{b}_2' \) where \( b_2' = b_2 + 1 \). Then \( \bar{c}\bar{a}'\bar{b}' \) are ordered in the same way as \( \bar{c}\bar{a}\bar{b} \) and hence still satisfy \( \phi \). Also, the valuation of \( R \) on \( \bar{a} \) is different from that of \( R \) on \( \bar{a}' \), since there are edges between \( a_n \) and \( a_n + n \), but not between \( a_n + 1 \) and \( a_n + n + 1 = b_n \).

Now suppose that \( \psi(\bar{c}, \bar{u}, \bar{v}) \) is some further existential formula true of \( \bar{c}\bar{a}'\bar{b}' \). Let \( \bar{e} \) be the witnesses to the existential quantifiers, and \( \chi(\bar{c}, \bar{u}, \bar{v}, \bar{w}) \) be the quantifier-free formula which holds of \( \bar{c}\bar{a}'\bar{b}'\bar{e} \). Write \( \bar{e} = \bar{e}_1\bar{e}_2 \) with each entry of \( \bar{e}_1 \) less than \( a_n \), and each entry of \( \bar{e}_2 \)
greater than or equal to $a_n$. Let $k$ be such that $b_k$ is larger than all of the entries of $\bar{a}$ and $\bar{b}$. Let $\bar{a}'' = \bar{a} + b_k$, $\bar{b}'' = \bar{b}_1\bar{b}_2''$ where $\bar{b}_2'' = b_2 + b_k$, and $\bar{e}' = \bar{e}_1\bar{e}_2'$ where $\bar{e}_2' = \bar{e} + b_k$. Then $\bar{c}\bar{a}''\bar{b}''\bar{e}'$ is ordered in the same way as $\bar{c}\bar{a}'\bar{b}'\bar{e}$, and so $\bar{c}\bar{a}''\bar{b}''\bar{e}'$ satisfies $\chi$. Thus $\bar{c}\bar{a}''\bar{b}''$ satisfies $\psi$. We need to show that the relation $R$ restricted to $\bar{c}\bar{a}''\bar{b}''$ is the same as $R$ restricted to $\bar{c}\bar{a}\bar{b}$.

Note that $\bar{a}''$ and $\bar{b}_2''$ are contained in the interval $[b_k, c_k)$. By definition of $R$, there is an edge between $x$ and $y$ in $[b_k, c_k)$ if and only if there is an edge between $x - b_k$ and $y - b_k$. Since $\bar{a}'' = \bar{a} + b_k$ and $\bar{b}_2'' = b_2 + b_k$, $R$ restricted to $\bar{a}''$ and $\bar{b}_2''$ is the same as $R$ restricted to $\bar{a}$ and $b_2$. Now $\bar{c}$ and $\bar{b}_1$ are contained in the interval $[0, a_n)$. There are no edges from some $x < a_n$ to some $y \geq a_n$. Note from the construction of $R$ that there is an edge from $x \geq a_n$ to $y < a_n$ if and only if there is an edge from every $z \geq a_n$ to $y$. This completes the proof that $R$ restricted to $\bar{c}\bar{a}\bar{b}$ is the same as $R$ restricted to $\bar{c}\bar{a}''\bar{b}''$.

So for any tuple $\bar{c}$, there is a tuple $\bar{a}$ which is d-free over $\bar{c}$. Moreover, everything was effective. Thus, by Proposition 6.4.9, the degree spectrum of $R$ contains a non-c.e. degree.

Construction of $h$.

Let $e$ be an index for a computable function $\varphi_e$, which we attempt to interpret as the diagram of a computable structure $A$. Let $i$ and $j$ be indices for the Turing functionals $\Phi_i$ and $\Phi_j$. We will build a $\Delta^0_2$ set $C$ such that if $A$ is a computable copy of $(\omega, <)$, then either

$$C \neq \Phi_i^R A \text{ or } R^A \neq \Phi_i^C.$$

By constructing $C$ via a $\Delta^0_2$ approximation uniformly from $e$, $i$, and $j$ we will obtain the required function $h$. 

Figure 6.6: The choice of $\bar{a}'$, $\bar{b}'$, $\bar{a}''$, $\bar{b}''$, and $\bar{e}'$. The figure shows the case when $\bar{c} = \emptyset$, we choose $\bar{a} = (0, 1)$, and $\bar{b} = \bar{b}_2 = (2, 3)$. We choose $k = 2$ so that $b_k = b_2 = 7$. 

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Let $e$ be an index for a computable function $\varphi_e$, which we attempt to interpret as the diagram of a computable structure $A$. Let $i$ and $j$ be indices for the Turing functionals $\Phi_i$ and $\Phi_j$. We will build a $\Delta^0_2$ set $C$ such that if $A$ is a computable copy of $(\omega, <)$, then either

$$C \neq \Phi_i^R A \text{ or } R^A \neq \Phi_i^C.$$
At each stage $s$, we get a finite linear order $\mathcal{A}_s$ which approximates $\mathcal{A}$. Now, the domain of $\mathcal{A}_s$ is contained in $\omega$, but $\mathcal{A}$ may also be isomorphic to $\omega$. To avoid confusion, when we say $\omega$, we refer to the underlying domain of $\mathcal{A}_s$, and by $(\omega, <)$ we mean the structure as a linear order. We may assume that the elements of $\mathcal{A}_s$ form a finite initial segment of the domain $\omega$. To differentiate between the ordering (as part of the language of the structure) on $\mathcal{A}_s$ and the underlying order on the domain as a subset of $\omega$, we will use $\leq_\mathcal{A}$ for the former and $\leq$ for the latter. We can write the elements of $\mathcal{A}_s$ as $x_1^s <_\mathcal{A} x_2^s <_\mathcal{A} \cdots <_\mathcal{A} x_n^s$, and given $z \in \omega$, we write $N_z(z) = k$ if $z = x_k^s$. So $N_z(z)$ is a guess at which element of $(\omega, <)$ the element $z \in \mathcal{A}_s$ represents. For $z \in \mathcal{A}$, let $N(z) \in (\omega, <)$ be the element which $z$ is isomorphic to (if $\mathcal{A}$ is isomorphic to $(\omega, <)$). We also get an approximation $R^A_s$ of $R^A$ by setting $z \in R^A_s$ if and only if $N_z(z) \in R$. Then $R^A_s$ is a $\Delta^0_2$ approximation of $R^A$ in the case that $\mathcal{A}$ is isomorphic to $(\omega, <)$.

The general idea of the construction is as follows. At each stage $s$, we will have finite set $C_s$ such that $C(n) = \lim_{s \to \infty} C_s(n)$. If we do not explicitly say that we change $C$ from stage $s$ to stage $s + 1$, then we will have $C_{s+1} = C_s$. Suppose that at stage $s$ we have $C_s(0) = 0$ and we have computations $C_s(0) = 0 = \Phi_{i,s}^{R^A_s}(0)$ with use $u$ and $R^A_s[0, \ldots, u] = \Phi_{j,s}^{C_z}[0, \ldots, u]$ with use $v$. By putting 0 into $C$, we destroy the first computation, forcing $R^A_s$ to change below the use $u$ (or else we are done); then, by removing 0 from $C$, because of the second computation we force $R^A_s$, at some later stage $t$, to change back to the way it was before (i.e., $R^A_s[0, \ldots, u] = R^A_t[0, \ldots, u]$). This means that some $q \leq u$ in $\mathcal{A}$ must have had some element enumerated $\leq_\mathcal{A}$-below it, so that $N_q(q) > N_z(q)$. By moving 0 in and out of $C$ in this way, we can force arbitrarily many elements to be enumerated $\leq_\mathcal{A}$-below one of $[0, \ldots, u]$. If we could enumerate infinitely many such elements, then we would have prevented $\mathcal{A}$ from being isomorphic to $(\omega, <)$. However, this would require moving 0 in and out of $C$ infinitely many times, which would make $C$ not $\Delta^0_2$. We must be more clever.

Let $p_0 = 0$. We will wait for computations as above (with uses $u_0$ and $v_0$), and then choose $p_1 > v$. We wait for computations $C_s[0, \ldots, p_1] = 0 = \Phi_{i,s}^{R^A_s}[0, \ldots, p_1]$ with use $u_1$ and $R^A_s[0, \ldots, u_1] = \Phi_{j,s}^{C_z}[0, \ldots, u_1]$ with use $v_1$. We will move $p_0 = 0$ into and then out of $C$ as above to enumerate an element in $\mathcal{A} \leq_\mathcal{A}$-below one of $[0, \ldots, u]$. At the same time, we will create a “link” between $[0, \ldots, u_0]$ and $[u_0 + 1, \ldots, u_1]$ so that if some element gets enumerated $\leq_\mathcal{A}$-below one of $[u_0 + 1, \ldots, u_1]$, then some element will also get enumerated $\leq_\mathcal{A}$-below one of $[0, \ldots, u_0]$. (Exactly how these links work will be explained later.) We have moved $p_0$ into and then out of $C$, but from now on it will stay out of $C$. We will find a $p_2$, and move $p_1$ into and out of $C$. On the
one hand, this will cause an element to be enumerated \( \leq \mathcal{A} \)-below one of \([u_0 + 1, \ldots, u_1]\), and hence \( \leq \mathcal{A} \)-below one of \([0, \ldots, u_0]\). On the other hand, we will create a “link” between \([u_0 + 1, \ldots, u_1]\) and \([u_1 + 1, \ldots, u_2]\). Now when an element gets enumerated \( \leq \mathcal{A} \)-below one of \([u_1 + 1, \ldots, u_2]\), an element gets enumerated \( \leq \mathcal{A} \)-below one of \([u_0 + 1, \ldots, u_1]\), and thus some element gets enumerated \( \leq \mathcal{A} \)-below one of \([0, \ldots, u_0]\). Continuing in this way, defining \( p_3, p_4, \) and so on, and maintaining these links, we force infinitely many elements to be enumerated \( \leq \mathcal{A} \)-below some element of \([0, \ldots, u_0]\). We will describe exactly how these links work in the construction. Figure 6.7 shows the computations which we use during the construction.

![Diagram](image)

Figure 6.7: The values associated to a requirement for Proposition 6.5.19. An arrow shows a computation converging. The computations use an oracle and compute some initial segment of their target. The tail of the arrow shows the use of the computation, and the head shows the length.

The construction will consist of three steps defined below, which are repeated for each of \( n = 0, 1, \ldots \). The proof of the following lemma will be interspersed with the construction below:
Lemma 6.5.23. If $A$ is an isomorphic copy of $(\omega, \prec)$, and

$$C = \Phi_i^{R^A} \text{ and } R^A = \Phi_j^C$$

then the construction does not get stuck in any step.

After describing the action taking place at each step, we will prove that if $A$ is an isomorphic copy of $(\omega, \prec)$, and

$$C = \Phi_i^{R^A} \text{ and } R^A = \Phi_j^C$$

then the construction eventually finishes that step.

Begin the construction with $p_0 = 0$ and $C_0 = \varnothing$. Before beginning to repeat the three steps, wait for a stage $s$ where we have computations

$$C_s[p_0] = 0 = \Phi_i^{R^A_s}[p_0] \quad (6.33)$$

with use $u_0$ and

$$R^A_s[0, \ldots, u_0] = \Phi_j^{C_s}[0, \ldots, u_0] \quad (6.34)$$

with use $v_0$. Let $s_0 = s$. If $C = \Phi_i^{R^A}$ and $R^A = \Phi_j^C$, we eventually find computations as in (6.33) and (6.34).

Now repeat, in order, the following steps. We call each repetition of these steps a Rep. Begin at Rep $0$ with $n = 0$. At the beginning of Rep $n$, we will have defined values $p_i$, $s_i$, $u_i$, and $v_i$ for $0 \leq i \leq n$, $m_i$, $w_i$, and $q_i$ for $0 \leq i < n$, and $\rho_i$ and $\nu_i$ for $1 \leq i \leq n$ (note that we never define $\rho_0$ or $\nu_0$). At the beginning of each repetition, we will have $C = \varnothing$; in Step One, we will add an element to $C$, and in Step Two we will remove that element from $C$ returning it to the way it was before.

**Step One.** Wait for a stage $s$ and an $m > \rho_n$ (or $m > u_0$ if $n = 0$) such that

1. at this stage $s$ we still have the computation

   $$R^A_s[0, \ldots, \rho_n] = \Phi_j^{C_s}[0, \ldots, \rho_n] \quad (6.35)$$

   with use $\nu_n$ from Step 3 of the previous step (see equation (6.44)). If $n = 0$, then instead we ask that

   $$R^A_s[0, \ldots, u_0] = \Phi_j^{C_s}[0, \ldots, u_0], \quad (6.36)$$

2. all of the elements $x$ of $A_s$ with $x \preceq_A p_n$ come from among $[0, \ldots, m]$,

3. there is a computation

   $$R^A_s[0, \ldots, m] = \Phi_j^{C_s}[0, \ldots, m] \quad (6.37)$$

   with use $w$,
there are computations

\[ C_{s}[0, \ldots, w + 1] = \Phi_{i,s}^{R_{s}^{A}}[0, \ldots, w + 1] \]  \hspace{1cm} (6.38)

with use \( u \) and

\[ R_{s}^{A}[0, \ldots, u] = \Phi_{j,s}^{C_{s}}[0, \ldots, u] \]  \hspace{1cm} (6.39)

with use \( v \),

for each \( z \in [0, \ldots, u] \), among the elements \( [0, \ldots, m] \) of \( A_{s} \), there is a matching pair of \( k + 1 \)-cycles (for some \( k \in \omega \)) \( \bar{x} = (x_{0}, \ldots, x_{k}) \) and \( \bar{y} = (y_{0}, \ldots, y_{k}) \) in \( A_{s} \) with \( x_{0} \leq_{A} z \leq_{A} y_{k} \) and there are edges from each element of \( [m + 1, \ldots, u] \) to \( y_{k} \) (and an edge from \( x_{0} \) to \( y_{0} \) witnessing that the \( k + 1 \)-cycles form a matching pair).

Set \( m_{n} = m, w_{n} = w, p_{n+1} = w + 1, u_{n+1} = u, v_{n+1} = v, \) and \( s_{n+1} = s \). Set \( C_{s+1}(p_{n}) = 1 \) to break the computation 6.38 the previous Rep.

The idea at this step is to find, for each \( z \in [0, \ldots, u] \), a matching pair of \( k + 1 \)-cycles \( \bar{x} \) and \( \bar{y} \) which contain \( z \) between them. These \( k \)-cycles are all contained in the elements \( [0, \ldots, m_{n}] \). Moreover, \( \bar{x} \) and \( \bar{y} \) look like they correspond to the \( k \)-cycles \( (a_{k}, \ldots, a_{k} + k) \) and \( (c_{k}, \ldots, c_{k} + k) \) respectively from the standard copy of \( (\omega, \rho) \) in the sense that there is an edge from each element of \( [m_{n} + 1, \ldots, u] \) to \( y_{k} \). We also define the next value \( p_{n+1} \) during this step. All of this is to set up the “link” between \( p_{n+1} \) to \( p_{n} \) (but the link will not be completed until Step Three).

Note that, since we set \( p_{n+1} = w + 1, u_{n+1} = u, \) and \( s_{n+1} = s \), the computation (6.38) is really

\[ C_{s_{n+1}}[0, \ldots, p_{n+1}] = \Phi_{i,s_{n+1}}^{R_{s}^{A}}[0, \ldots, p_{n+1}] \]  \hspace{1cm} (6.40)

with use \( u_{n+1} \). At the end of this step, we set \( C_{s+1}(p_{n}) = 1 \) to break the computation

\[ C_{s}[0, \ldots, p_{n}] = \Phi_{i,s}^{R_{s}^{A}}[0, \ldots, p_{n}] \]

with use \( u_{n} \) from (6.40) of Rep \( n - 1 \).

*Proof of Lemma 6.5.23 for Step One.* Suppose that we never leave *Step One*. Let \( t \) be the stage at which we entered *Step One*. Then \( C = C_{t} \). For sufficiently large \( s > t \), we have the true computation

\[ R_{s}^{A}[0, \ldots, \rho_{n}] = \Phi_{j,s}^{C_{s}}[0, \ldots, \rho_{n}] \]

as in (6.35). Thus we satisfy (1). Recall that \( N \) is the isomorphism \( \mathcal{A} \to (\omega, \rho) \). Since \( \mathcal{A} \) is an isomorphic copy of \( (\omega, \rho) \), there are only finitely many elements \( x \in \mathcal{A} \) with \( x \leq_{A} a_{n} \) and so for sufficiently large \( m \), they all come from \( [0, \ldots, m] \). So (2) is satisfied as well. For each \( z \in [0, \ldots, u_{n}] \), there is \( k \) such that \( N(z) \) is in the interval \( [a_{k}, a_{k+1}) \). Let \( \bar{x} \) and \( \bar{y} \) be the matching pair of \( k + 1 \)-cycles in \( \mathcal{A} \) which are the pre-images, under the isomorphism \( N \), of \( (a_{k}, \ldots, a_{k} + k) \) and \( (c_{k}, \ldots, c_{k} + k) \). Increasing \( m \) if necessary, we may assume that these
Let \( w, u, v \). So (3) and (4) are satisfied.

Finally, for each \( z \in [0, \ldots, u_n] \) the \( k \)-cycles \( \bar{x} \) and \( \bar{y} \) chosen above satisfy \( x_0 \leq_A z \leq_A y_k \) and there is an edge from \( x_0 \) to \( y_0 \). Since all of the elements of \( A \) which are \( \leq_A \)-less than any entry of \( \bar{x} \) and \( \bar{y} \) are contained in \([0, \ldots, m]\), each of \( m + 1, \ldots, u \) are \( \leq_A \)-greater than \( y_k \). So there is an edge from each of these to \( y_k \). Thus (5) is satisfied. This contradicts our assumption that we never leave Step One.

**Step Two.** Wait for a stage \( s \) such that

\[
R_s^A[m_{n-1} + 1, \ldots, u_n] \neq R_{s_n}^A[m_{n-1} + 1, \ldots, u_n] = R_{s_{n+1}}^A[m_{n-1} + 1, \ldots, u_n]. \tag{6.41}
\]

Let \( q_n \) be the \( \leq_A \)-greatest element in \([m_{n-1} + 1, \ldots, u_n]\); note that \( N_q(q_n) > N_{s_{n+1}}(q_n) \). Set \( C_{s+1}(p_n) = 0 \).

In the previous step, we broke the computation

\[
C_{s_n}[0, \ldots, p_n] = \Phi_{i,s_n}^R[0, \ldots, p_n].
\]

In order for this computation to again hold at some \( s \), \( R^A \) must change below the use \( u_n \) of that computation. That means that some element must be enumerated in \( A \leq_A \)-below one of \( 0, \ldots, u_n \) (in fact, it will have to be enumerated below one of \( m_{n-1} + 1, \ldots, u_n \)). We let \( q \) be the \( \leq_A \)-greatest such element, so we know that some an element has been enumerated below \( q_n \). At the end of this stage, we set \( C_{s+1}(p_n) = 0 \) (so that \( C_{s+1} = C_{s_n} \)) to restore the computation above and force \( R^A \) to return to the way it was before.

**Proof of Lemma 6.5.23 for Step Two.** Suppose that the construction does not leave Step Two. Then for all \( t > s_{n+1} \), \( C_t(a_n) = C_{s_{n+1}+1}(a_n) \neq C_{s_{n+1}}(a_n) = C_{s_n}(a_n) \). Since \( C = \Phi_i^R \), at some stage \( t > s_{n+1} \), we have a true computation

\[
C_t[0, \ldots, a_n] = \Phi_i^R[0, \ldots, a_n].
\]

By (6.40) of the previous repetition, and since \( C_t(a_n) \neq C_{s_n}(a_n) \), we have

\[
R_t^A[0, \ldots, u_n] \neq R_{s_n}^A[0, \ldots, u_n]. \tag{6.42}
\]

Since \( C_t[0, \ldots, w_{n-1}] = C_{s_n}[0, \ldots, w_{n-1}] \), by (6.37) we have

\[
R_t^A[0, \ldots, m_{n-1}] = R_{s_n}^A[0, \ldots, m_{n-1}].
\]

Thus

\[
R_t^A[m_{n-1} + 1, \ldots, u_n] \neq R_{s_n}^A[m_{n-1} + 1, \ldots, u_n].
\]

This contradicts our assumption that we never leave Step Two. \( \square \)
Step Three. Wait for a stage \( s \) and \( \rho > u_{n+1} \) such that

1. we have
   \[
   R^A_s[0, \ldots, u_{n+1}] = R^A_{s_{n+1}}[0, \ldots, u_{n+1}],
   \]  

2. among the elements \([0, \ldots, \rho]\) of \( A_{s} \), there is a matching pair of \( \ell + 1 \)-cycles (for some \( \ell \)) \( \bar{\sigma} = (\sigma_0, \ldots, \sigma_\ell) \) and \( \bar{\tau} = (\tau_0, \ldots, \tau_\ell) \) in \( A_{s} \) with \( \sigma_\ell < A_{s} q_n \) and \( z < A_{s} \tau_0 \) for each \( z \in [m_n + 1, \ldots, u_{n+1}] \) (and an edge from \( \sigma_0 \) to \( \tau_0 \) but not vice versa witnessing that the \( \ell + 1 \)-cycles are matching),

3. we have the computation
   \[
   R^A_s[0, \ldots, \rho] = \Phi^{C_s}_{j,s}[0, \ldots, \rho],
   \]  

with use \( \nu \).

Set \( \rho_{n+1} = \rho \) and \( \nu_{n+1} = \nu \). Return to Step One for \( Rep n + 1 \).

In this step, we wait for the computation (6.43) to be restored. Now, this forces \( R^A \) to be the same as it was during Step One:

\[
R^A_s[0, \ldots, u_{n+1}] = R^A_{s_n}[0, \ldots, u_{n+1}].
\]

At Step One, there was a matching pair of \( k + 1 \)-cycles \( \bar{x} \) and \( \bar{y} \) in \([0, \ldots, m_n]\) which contained \( q_n \leq_A \)-between them. Since \( R^A[0, \ldots, m_n] \) is the same at this stage as it was at that stage, \( \bar{x} \) and \( \bar{y} \) are still a matching pair of \( k + 1 \)-cycles. But some element has been enumerated \( \leq_A \)-below \( q_n \) since then, and by Remark 6.5.22, it must be enumerated below \( \bar{x} \). So \( \bar{x} \) and \( \bar{y} \) are not the \( \leq_A \)-least matching pair of \( k + 1 \)-cycles. Thus, they correspond to some elements in an interval \([b_\ell, c_\ell]\) from \((\omega, <)\) for some \( \ell \). The \( \bar{\sigma} \) and \( \bar{\tau} \) in (2) above are intended to be \((a_\ell, \ldots, a_\ell + \ell)\) and \((c_\ell, \ldots, c_\ell + \ell)\) respectively. Since, in Step One, there was an edge from each \( z \in [m_n + 1, \ldots, u_{n+1}] \) to \( y_k \), each such \( z \) must also correspond to some element from the interval \([b_\ell, c_\ell]\) (one can see from the definition of the relation \( R \) that there are no such edges from an element \( z \geq c_\ell \) to some \( y \in [b_\ell, c_\ell] \)).

This establishes the desired “link”. By Remark 6.5.22, since the interval \([\sigma_0, \tau_\ell]\) is of a fixed length determined by \( \ell \), no new elements can be enumerated between \( \sigma_0 \) and \( \tau_\ell \). So if some new element is enumerated below one of \([m_n + 1, \ldots, u_{n+1}]\), it must be enumerated below \( \sigma_0 \) and hence below \( q_n \).

Proof of Lemma 6.5.23 for Step Three. Let us suppose that the construction never leaves Step Three. Suppose that the construction entered Step Three at stage \( s \). For \( t \geq s \), we have \( C_t = C_{s_{n+1}} = \emptyset \). Since \( R^A = \Phi^{C_t}_j \), and using (6.39), for sufficiently large \( t > s \) we have

\[
R^A[0, \ldots, u_{n+1}] = R^A_t[0, \ldots, u_{n+1}] = R^A_{s_{n+1}}[0, \ldots, u_{n+1}].
\]
Since \( q_n \in [m_{n-1} + 1, \ldots, u_n] \), at stage \( s_{n+1} \), there was a matching pair of \( k + 1 \)-cycles \( \bar{x} = (x_0, \ldots, x_k) \) and \( \bar{y} = (y_0, \ldots, y_k) \) among the elements \([0, \ldots, m_n]\) as in (5) of \textit{Step One} with \( x_0 \preceq_A q_n \preceq_A y_k \). Since
\[
R^A[0, \ldots, u_{n+1}] = R^A_{s_{n+1}}[0, \ldots, u_{n+1}],
\]
\( \bar{x} \) and \( \bar{y} \) are actually \( k + 1 \)-cycles in \( A \), and there is actually an edge from \( x_0 \) to \( y_0 \). Also, \( N_i(q_n) > N_{s_{n+1}}(q_n) \). By Remark 6.5.22, \( N_i(x_0) > N_{s_{n+1}}(x_0) \). Let \( \bar{x}' \in (\omega, <) \) be the tuple \((N_{s_{n+1}}(x_0), \ldots, N_{s_{n+1}}(x_k))\) and similarly for \( \bar{y}' \). Let \( \bar{x}'' \in (\omega, <) \) be \((N_i(x_0), \ldots, N_i(x_k))\) and similarly for \( \bar{y}'' \). Then in \((\omega, <)\) we know that \( \bar{x}' \) and \( \bar{y}' \) are a matching pair of \( k + 1 \)-cycles, and so are \( \bar{x}'' \) and \( \bar{y}'' \). So \( \bar{x}'' \) and \( \bar{y}'' \) are not the first matching pair of \( k + 1 \)-cycles, and so they are contained in some interval \([b_\ell, c_\ell]\) for some \( \ell \). Moreover, in \( R^A \), each element \( z \) of \([m_n, \ldots, u_{n+1}]\) has an edge from it to \( x_0 \). This is only possible if \( N_i(z) \in [b_\ell, c_\ell] \) for each such \( z \). Let \( \bar{\sigma} \) and \( \bar{\tau} \) be the matching pair of \( \ell + 1 \)-cycles in \( A \) whose images in \((\omega, <)\) under the isomorphism \( N \) are \((a_\ell, \ldots, a_\ell + \ell)\) and \((c_\ell, \ldots, c_\ell + \ell)\) respectively. Then \( \bar{\sigma} \) and \( \bar{\tau} \) the required \( \ell + 1 \)-cycles in \textit{Step Three}. We get the computation for (6.44) because \( R^A = \Phi_j^C \). This contradicts our assumption that we never leave \textit{Step Three}.

This ends the construction. In the process, we have proved Lemma 6.5.23. The next two lemmas complete the proof that the construction works as desired.

**Lemma 6.5.24.** \( C \) is a d.c.e. set and the approximation \( C_s \) is a d.c.e. approximation

**Proof.** We change the approximation \( C_s(x) \) at most twice for each \( x \)—since \( x = p_n \) for at most one \( n \), we change \( C_s(x) \) at most once in \textit{Step One} and once in \textit{Step Two}.

**Lemma 6.5.25.** If the construction does not get stuck in any step, then \( A \) is not isomorphic to \((\omega, <)\).

**Proof.** In the construction above we remarked that \( N_{s_{n+2}}(q_n) > N_{s_{n+1}}(q_n) \). We claim that for all \( n \geq 1 \), \( N_{s_{n+1}}(q_0) > N_n(q_0) \). This would imply that \( A \) is not isomorphic to \((\omega, <)\), as \( q_n \in A \) would have infinitely many predecessors.

The key to the proof will be to use the “links” that we created during the construction. We will show that if, for \( n' > n + 1 \), \( N_{s_{n'+1}}(q_{n+1}) > N_{s_{n'}}(q_{n+1}) \), then \( N_{s_{n'+1}}(q_n) > N_{s_{n'}}(q_n) \). This will suffice to prove the lemma.

During \textit{Step Three} of the \( n \)th repetition, we saw that among the elements \([0, \ldots, p_{n+1}]\), there is a matching pair of \( \ell + 1 \)-cycles \( \bar{\sigma} = (\sigma_0, \ldots, \sigma_\ell) \) and \( \bar{\tau} = (\tau_0, \ldots, \tau_\ell) \) in \( A_s \) with \( \sigma_\ell <_A q_n \) and \( z <_A \tau_0 \) for each \( z \in [m + 1, \ldots, v] \). Moreover, by (6.35), at every stage \( s_{n'} \) for \( n' > n + 1 \), \( \bar{\sigma} \) and \( \bar{\tau} \) are \( \ell + 1 \)-cycles in \( A_{s_{n'}} \).

Then \( q_{n+1} \in [m + 1, \ldots, v] \), so \( q_{n+1} <_A \tau_0 \). Thus if \( N_{s_{n'+1}}(q_{n+1}) > N_{s_{n'}}(q_{n+1}) \), then \( N_{s_{n'+1}}(\tau_0) > N_{s_{n'}}(\tau_0) \). Since \( \bar{\sigma} \) and \( \bar{\tau} \) are a matching pair of \( \ell + 1 \)-cycles at stages \( s_{n'} \) and \( s_{n'+1} \), by Remark 6.5.22 no new elements are added between \( \bar{\sigma} \) and \( \bar{\tau} \) in between these stages. So \( N_{s_{n'+1}}(\sigma_0) > N_{s_{n'}}(\sigma_0) \), and since \( \sigma_0 <_A q_n \), \( N_{s_{n'+1}}(q_n) > N_{s_{n'}}(q_n) \).
If \( A \) is an isomorphic copy of \((\omega, <)\), then Lemma 6.5.23 and Lemma 6.5.25 combine to show that
\[
C \neq \Phi_i^{R_A} \text{ or } R_A \neq \Phi_j^C.
\]
This completes the proof of Theorem 6.5.19.

One can view the proof as a strategy for satisfying a single requirement \( r_{e,i,j} \). For a fixed \( e_0 \), it does not add too much difficulty to satisfy multiple requirements of the form \( r_{e_0,i,j} \) at the same time—since these requirement are all working with the same structure \( A_{e_0} \), only one requirement has to force \( A_e \) to not be isomorphic to \((\omega, <)\). However, if one tries to satisfy every requirement \( r_{e,i,j} \) for different \( e \)'s at the same time, one runs into a problem. Each requirement tries to restrain infinitely much of \( \omega \), and in order to build \( p_{n+1} \), the requirement must move \( p_n \). Thus if \( p_{n+1} \) is injured, \( p_n \) may injure other requirements.

### 6.6 A “Fullness” Theorem for 2-CEA Degrees

In this section, we will prove Theorem 6.1.5. Recall that a set \( A \) is 2-CEA in a set \( B \) if there is \( C \) such that \( A \) is c.e. in and above \( C \) and \( C \) is c.e. in and above \( B \).

We will prove the theorem in the following form:

**Theorem 6.6.1.** Let \( C \) be a computable structure, and let \( R \) be an additional computable relation on \( C \). Suppose that \( R \) is not formally \( \Delta^0_2(0'^{\omega+1}) \). Then for all degrees \( d \geq 0^{(\omega+1)} \) and sets \( A \) 2-CEA in \( d \) there is an isomorphic copy \( D \) of \( C \) with \( D \equiv_T d \) and
\[
R^D \oplus D \equiv_T A.
\]

Our construction will actually build \( D \leq_T d \). We can use Knight’s theorem on the upwards closure of the degree spectrum of a structure (see [Kni86]) to obtain \( D \equiv_T d \) as follows. Suppose that we have built \( D \leq_T d \) as in the theorem. There is an isomorphic copy \( D^* \equiv_T d \) of \( D \). Moreover, \( D^* \) is obtained by applying a permutation \( f \equiv_T d \) to \( D \). Then \( f \oplus D \equiv_T f \oplus D^* \equiv_T D^* \) and \( f \oplus R^D \equiv_T f \oplus R^{D^*} \). Hence
\[
A \equiv_T D^* \oplus R^{D^*}.
\]

Theorem 6.1.5 is obtained from Theorem 6.6.1 by relativizing the proof. Given any structure \( C \) and relation \( R \) on \( C \), we can build a \( d \)-computable copy \( D \equiv_T d \) as in Theorem 6.6.1 for any \( d \) in the cone above \((C \oplus R)^{(\omega+1)} \). We could also give effectiveness conditions on computable \( C \) and \( R \) which would suffice to take \( d = 0 \), but these would be quite complicated.

Finally, the simplest way to state the theorem is as follows:

\[\text{Note that formally } \Delta^0_2(0'^{\omega+1}) \text{ does not mean the same thing as formally } \Delta^0_5; \text{ } R \text{ is formally } \Delta^0_2(0'^{\omega+1}) \text{ means that } R \text{ can be defined by } 0'^{\omega+1}\text{-computable } \Sigma^1_2 \text{ and } \Pi^1_2 \text{ formulas, whereas } R \text{ is formally } \Delta^0_5 \text{ means that } R \text{ can be defined by computable } \Sigma^1_5 \text{ and } \Pi^1_5 \text{ formulas.}\]
Corollary 6.6.2. Let $C$ be a structure and $R$ a relation on $C$. Then either

$$\text{dgSp}_{rel}(C, R) \subseteq \Delta^0_2$$

or

$$2\text{-CEA} \subseteq \text{dgSp}_{rel}(C, R).$$

The proof of the theorem will use free elements as in Barker’s proof that formally $\Sigma_\alpha$ is the same as intrinsically $\Sigma_\alpha$ [Bar88]. It will probably be helpful to understand the proof of that result at least for the case $\alpha = 2$.

Definition 6.6.3. We say that $\bar{a} \notin R$ is 2-free over $\bar{c}$ if for all $\bar{a}_1$, there are $\bar{a}' \in R$ and $\bar{a}'_1$ such that

$$\bar{c}, \bar{a}, \bar{a}_1 \leq \bar{c}, \bar{a}', \bar{a}'_1.$$  

Recall that $\leq_0$ and $\leq_1$ are the first two back-and-forth relations; $\bar{a} \leq_0 \bar{b}$ if all of the quantifier-free formulas with Gödel number less than $|\bar{a}|$ which are true of $\bar{a}$ are true of $\bar{b}$, while $\bar{a} \leq_1 \bar{b}$ if every $\Sigma^0_1$ formula true of $\bar{b}$ is true of $\bar{a}$ (see Chapter 15 of [AK00]). If $F : \{0, \ldots, m\} \to C$ and $G : \{0, \ldots, n\} \to C$ are functions with $n > m$, then $F \leq_i G$ means that

$$F(0), \ldots, F(m) \leq_i G(0), \ldots, G(m).$$

If $R$ is not formally $\Delta^0_2(0''')$, then either $R$ is not defined by a $0'''$-computable $\Pi^0_2$ formula or $\lnot R$ is not defined by a $0'''$-computable $\Pi^0_2$ formula. We may suppose without loss of generality that it is $R$ which is not defined by a $0'''$-computable $\Pi^0_2$ formula. We will relativize Proposition 16.1 of [AK00] to show that for any tuple $\bar{c}$, there is a tuple $\bar{a}$ which is 2-free over $\bar{c}$. Moreover, using $0(4)$ we can check whether a tuple is 2-free, and hence find these 2-free tuples.

Proposition 6.6.4. Let $C$ be a computable structure, and let $R$ be a further computable relation on $C$. Suppose that $\bar{c}$ is a tuple over which no $\bar{a} \notin R$ is 2-free. Then there is a $0'''$-computable $\Sigma_2$ formula $\varphi (\bar{c}, \bar{x})$ defining $\lnot R$.

Proof. We have $\bar{x} \leq_1 \bar{y}$ exactly when all $\exists_1$ formulas true of $\bar{y}$ are true of $\bar{x}$; so $\bar{x} \leq_1 \bar{y}$ if and only if

$$\bigwedge_{\varphi \text{ a } \exists_1 \text{ formula}} \left[ \varphi (\bar{y}) \Rightarrow \varphi (\bar{x}) \right].$$

This is a computable $\Pi_2$ formula. In particular, $\bar{C}$ is 2-friendly relative to $0''$. By Theorem 15.2 of Ash-Knight, for each $\bar{a} \notin R$ and $\bar{a}_1$ there is (uniformly in $\bar{a}$ and $\bar{a}_1$) a $0'''$-computable $\Pi_1$ formula $\varphi_{\bar{c}, \bar{a}, \bar{a}_1}(\bar{c}, \bar{x}, \bar{u})$ which says that $\bar{c}, \bar{a}, \bar{a}_1 \leq_1 \bar{c}, \bar{x}, \bar{u}$.

Since there are no tuples in $\lnot \bar{R}$ which are 2-free over $\bar{c}$, for each $\bar{a} \notin R$ there is $\bar{a}_1$ such that for every $\bar{a}' \in R$ and $\bar{a}'_1$, $\bar{c}, \bar{a}, \bar{a}_1 \notin \bar{c}, \bar{a}', \bar{a}'_1$. Since $\leq_1$ is computable in $0''$, we can find such an $\bar{a}_1$ for each $\bar{a} \notin R$ using $0'''$. For each $\bar{a}$, using this $\bar{a}_1$, define

$$\psi_{\bar{a}} (\bar{c}, \bar{x}) = (\exists \bar{u}) \varphi_{\bar{c}, \bar{a}, \bar{a}_1}(\bar{c}, \bar{x}, \bar{u}).$$
This formula is true of \( \bar{a} \), but it is not true of any element of \( R \). Thus \( \neg R \) is defined by

\[
\forall \bar{a} \in R \psi_\bar{a}(\bar{c}, \bar{x}).
\]

This is a \( 0'' \)-computable \( \Sigma_2 \) formula.

The proof of Theorem 6.6.1 is quite complicated and will take the rest of this section.

### 6.6.1 Approximating a 2-CEA Set

Let \( B \) be c.e. and let \( A \) be c.e. in and above \( B \). As \( A \) is \( \Sigma^0_2 \), there is a computable approximation \( f(x,s) \) for \( A \) such that \( x \in A \) if and only if \( f(x,s) = 1 \) for sufficiently large \( s \), and \( x \notin A \) if and only if \( f(x,s) = 0 \) for infinitely many \( s \). However, if \( A \) is an arbitrary \( \Sigma^0_2 \) set, and \( x \in A \), then \( A \) cannot necessarily compute a stage \( s \) after which \( f(x,t) = 1 \). We will begin this section by showing that \( A \) in fact has such an approximation by virtue of computing \( B \).

#### Lemma 6.6.5

Let \( B \) be c.e. and let \( A \) be c.e. in and above \( B \). There is a computable approximation \( f : \omega^2 \to \{0,1\} \) such that \( x \in A \) if and only if \( f(x,s) = 1 \) for sufficiently large \( s \), and \( x \notin A \) if and only if \( f(x,s) = 0 \) for infinitely many \( s \). Moreover, \( A \) can compute uniformly whether for all \( t \geq s \), \( f(x,t) = 1 \).

**Proof.** As \( B \) is c.e., it has a computable approximation \( B_s \). Let \( e \) be such that \( A = W^B_e \). Set \( f(x,0) = 0 \). Suppose that we have defined \( f(x,s) \). If \( x \notin W^B_{e,s} \), then let \( f(x,s+1) = 0 \). If \( x \in W^B_{e,s} \) and \( f(x,s) = 0 \), set \( f(x,s+1) = 1 \). If \( x \in W^B_{e,s} \) and \( f(x,s) = 1 \), then \( x \in W^B_{e,s+1} \) with some use \( u \) (where by the use of the computation, we mean the length of the initial segment of the oracle which the computation looked at before it halted). We set \( f(x,s+1) = 1 \) if the computations \( x \in W^B_{e,s} \) and \( x \in W^B_{e,s+1} \) are the same (that is, if \( B_s \) and \( B_{s+1} \) agree up to the use \( u \)). Otherwise, set \( f(x,s+1) = 0 \). It is easy to see that this construction works as desired.

We want to put an approximation like the one in the previous lemma on a tree so that the true path is the leftmost path visited infinitely often, while still maintaining the ability of \( A \) to compute stages at which it stabilizes. We will also keep track of how many separate times a particular node in \( 2^\omega \) is visited. Let \( \omega^+ = \{\omega\} \cup \omega \) (with \( \omega \) viewed as being to the left of each element of \( \omega \)). Let \( T \) be the tree \((\omega^+)^\omega \). If \( \sigma \in T \), we write \( v(\sigma) \) for the string in \( 2^\omega \) of the same length as \( \sigma \) which replaces each \( \omega \) in \( \sigma \) with 0, and each other entry with 1. For \( f \in [T] \), \( v(f) \) is defined similarly. We denote by \( T_\infty \) the set of nodes which end in \( \omega \), and by \( T_\omega \) the set of nodes which end in an element of \( \omega \). For \( \sigma \in T \), we denote by \( \sigma^- \) the proper initial segment of \( \sigma \) of length one less. We denote by \( \ell(\sigma) \) the last entry of \( \sigma \).

#### Lemma 6.6.6

Let \( B \) be c.e. and \( A \) be c.e. in and above \( B \). There is a computable approximation \((\sigma_s)_{s<\omega} \in T^\omega \) such that there is a unique \( g \in [T] \) with \( \rho \in g \) if and only if \( \rho \in \sigma_s \) for infinitely many \( s \). From \( A \) we can compute \( g \), and \( v(g) = A \). Moreover,
(i) for each $\tau \in T$ of length $n$, if $s_0, s_1, \ldots$ are the stages $s$ at which $\tau \in \sigma_s$, then $\sigma_{s_0} = \tau$ and the sequence $a_0 = \sigma_{s_1}(n), a_1 = \sigma_{s_2}(n), \ldots$ has the property that $a_0 = \infty$ and if $a_i \in \omega$, then $a_i$ counts the number of $j < i - 1$ such that $a_j = \infty$ and $a_{j+1} \in \omega$, and

(ii) if $s < t$ are such that $\sigma_s$ and $\sigma_t$ are compatible, then $\sigma_t$ is a strict extension of $\sigma_s$, and if $\sigma_s$ is the largest initial segment of $\sigma_t$ which has appeared before stage $t$, then $\sigma_t$ extends $\sigma_s$ by a single element.

It is a consequence of (i) that if we visit some node $\sigma$, and then move further left in the tree than $\sigma$, we will never again return to $\sigma$. We may however return to a node further to the right of $\sigma$. For example, if $f(0) = 0, f(1) = 1, f(2) = 0$, and $f(3) = 1$ then we might have $s_0 = \infty, s_1 = 0, s_2 = \infty \in \omega$, and $s_3 = 1$. Since $s_2$ is to the left of $s_1$, we can no longer visit $s_1$; instead, we visit $s_3$ which is further to the right.

\textit{Proof sketch.} Begin with the approximation $f$ from the previous lemma. The approximation $(\sigma_s)_{s \in \omega}$ is built in a similar way as the outcomes (on a tree) of an infinite injury priority argument. The true path $g$ is the leftmost path visited infinitely often.

We will give a quick sketch of how to go about defining the $\sigma_s$. The first entry $\sigma_s(0)$ of each $\sigma_s$ is defined following $f$: $\sigma_s(0) = \infty$ if $f(0, s) = 0$, and if $f(0, s) = 1$ then $\sigma_s(0)$ counts the number of $t < s - 1$ such that $\sigma_t(0) = \infty$ and $\sigma_{t+1}(0) \in \omega$. Then, to define the second entry $\sigma_s(1)$ of $\sigma_s$ for some $s$, we find the previous stages $s_0 < \cdots < s_n < s$ which agree with stage $s$ about the first entry (i.e., with $\sigma_t(0) = \sigma_s(0)$) and follow the approximation $f$ along those stages in the sense that $\sigma_s(1)$ corresponds to $f(1, i)$. We can continue in this way, with (ii) determining the length of each $\sigma_s$.

The lemma relativizes as follows.

\textbf{Corollary 6.6.7.} Let $A$ be 2-CEA relative to $d$. There is a $d$-computable approximation $(\sigma_s)_{s \in \omega} \in T^\omega$ and a unique $g \in [T]$ as in the lemma with $A \geq_T g$ and $v(g) = A$.

\subsection{6.6.2 Basic Framework of the Construction}

In this section, we will describe what we are building during the construction and eight properties that we will maintain at each stage (the ninth and final property will come later). Let $\mathcal{C}, \mathbf{d}, R$, and $A$ be as in Theorem 6.6.1: $\mathcal{C}$ is a computable structure, $R$ is an additional computable relation on $\mathcal{C}$, $\mathbf{d}$ is a degree above $0^{(\omega+1)}$, and $A$ is 2-CEA in $\mathbf{d}$. For simplicity, assume that $R$ is unary.

We will build a model $\mathcal{D} \leq \mathbf{d}$ with domain $\omega$. The construction will be by stages, at each of which we define increasing finite portions $D_s$ of the diagram. At each stage we will define a finite partial isomorphism $F_s$. This partial isomorphism will map $\{0, \ldots, n\}$ into $\mathcal{C}$ for some $n$. These partial isomorphisms will not necessarily extend each other; however, they will all be consistent with the partial diagram $D_s$ we are building. Moreover, there will be a bijective limit $F : \omega \to \mathcal{C}$ along the true stages of the construction. Then $\mathcal{D}$ will be given by
the pullback, along $F$, of the diagram of $C$. $D$ will be computable because its diagram will be $\bigcup D_s$. To simplify the notation, denote by $F_s(a_1, \ldots, a_t)$ the tuple $F_s(a_1), \ldots, F_s(a_t)$.

Here is the basic idea. To code that some element $x$ is not in $A$, we will put an element $a \notin R$ which is 2-free into the image of $F$ (there may already be some elements in the range of $F$; $a$ should be free over them as well) At following stages, we may add more elements $\bar{a}_1$ into the image of $F$. If, at some later stage, we think that $x \in A$ then we will replace $a, \bar{a}_1$ with $a', \bar{a}'_1$ where $a' \in R$ and so that every existential formula true of $a', \bar{a}'_1$ is true of $a, \bar{a}_1$ (thus the finite partial diagram of $D$ at this stage is maintained even though we changed $F$). Then it is possible that at some later stage we again think that $x \notin A$. We can replace $a', \bar{a}'_1$ with $a, \bar{a}_1$, returning to the previous stage in the construction at which we thought that $x \notin A$ while still maintaining the partial diagram. We have probably added some more elements to the image of $F$ while we thought that $x \in A$, but the fact that $a, \bar{a}_1 \not\leq_1 a', \bar{a}'_1$ means that we can find corresponding witnesses over $a, \bar{a}_1$. Using $R^D$, we can figure out what happened during the construction, and hence whether or not $x \in A$. Now we know how to code a single fact.

Now we will describe how to code two elements $x < y$. To code the fact that $x \notin A$, add to the range of $F$ an element $a \notin R$ which is 2-free. To code that $y \notin A$, add to the range of $F$ another element $b \notin R$ which is 2-free over $a$. Now if at some later stage, we think that $y \in A$, we can act as above. But if we think that $x \in A$, then we replace $a, b$ with some $a', b'$ with $a' \in R$ and $ab \not\leq_1 a'b'$. Now $b'$ is not necessarily 2-free; so to code that $y \notin A$ while we think that $x \in A$, we need to add a new element $c$ which is 2-free over $a'b'$. At some later stage, we might think again that $x \notin A$, so we need to return to $ab$; we can do this because $ab \not\leq_1 a'b'$ (and $c$ is replaced by some $c'$ which is not necessarily 2-free over $ab$, but that does not matter because now $b$ is doing the coding again). So far, this is essentially describing the argument for Theorem 2.1 of [AK97]. This succeeds in coding $A$ into $R^D$, but we require $d'$ in order to decode it. The problem is the following situation. Suppose that we have done the construction as described above, coding whether $x$ or $y$ is in $A$, and we currently think that neither is. Our function $F$ looks like $abc'$ so far. But then at some later stage we might think for a second time that $x \in A$. We replace $abc'$ by $a''b''c''$, with $a'' \in R$ and $abc' \leq_1 a''b''c''$. Now $c''$ may not be 2-free over $a''b''$, so we need to add a new element $d$ which is 2-free over $a''b''c''$, and use $d$ to code whether $y \in A$. Using just $R^D$ and $d$, we cannot distinguish between the case $a'b'c$ where $y$ is being coded by the third element, or $a''b''c''d$ where it is being coded by the fourth element ($a'$ and $a''$ are both in $R$, and we cannot control whether or not $b', b'', c''$ are in $R$). We can differentiate between the two cases using $d'$, which is the basis of the proof by Ash and Knight.

What we do to solve this is the first time we think that $x \in A$, after choosing $a'$ and $b'$, we add a new element $a_0$ before adding $c$ which is 2-free over $a'b'a_0$. When we later believe that $x \notin A$ again, we return to $aba'_0c'$ for some $a'_0$; and then later when we believe that $x \in A$ once more, we choose $a'_0$ as well as $a''$, $b''$, and $c''$. The trick will be to ensure that $a_0 \in R$ if and only if $a'_0 \notin R$. Thus $R$ can differentiate between the two cases. The argument as to why we can choose an appropriate $a_0$ is complicated, and is the main difficulty in this proof. The element $a_0$ will actually need to be a finite list of elements. For now, we do not need to
CHAPTER 6. DEGREE SPECTRA OF RELATIONS

189

worry about how we choose \(a_0\)—this will be done in the following sections.

In the previous section we had a computable approximation for \(A\) which kept track of how many labels we switched between believing an element was in \(A\) and not in \(A\). We will assign labels from the tree \(T\) to positions in the partial isomorphism to say what those positions are coding. In the situation above, we will assign to the position of \(a\) is label \(\infty\); it codes whether or not \(\infty\) (i.e. \(x \notin A\)) is the correct approximation to \(A\). The position of \(b\) would code \(\infty\infty\) since it codes, if the correct approximation for whether or not \(x \in A\) is \(\infty\) (or \(x \notin A\)). The position of \(a_0\) codes \(0\), since it represents the first time that we think that \(x \in A\). The position of \(c\) codes \(0\infty\) since, if \(0\) is the correct approximation for \(x\), it tries to code \(y \notin A\). When we add in \(d\) after \(a^nb^ma_0^n\), we will actually first add another element \(a_1\), playing a similar role as \(a_0\); if, at a later stage, we think that \(x \notin A\), and then later think that \(x \in A\) for a third time, \(a_1\) will signal that \(d\) is no longer actively coding, just as \(a_0\) did for \(c\). Thus \(a_1\) will code \(1\), and \(d\) will code \(1\infty\).

To keep track of these labels, we will define a partial injection \(L_s : T \to \text{dom}(F_s)\). While \(L_s\) is dependent on the stage \(s\), once we set \(L_s(\sigma)\) at some stage \(s\), we will never change the image of \(\sigma\). So if \(s < t\), we will have \(L_s \subseteq L_t\).

Let \(\sigma_s\) and \(f\) be as in Corollary 6.6.7 with \((\sigma_s)_{s \in \omega} \leq d\) and \(A \equiv_T f\). At each stage, we will increase the domain of \(L_s\) by a finite amount to add coding locations for \(\sigma_s\), and so the domain of each \(L_s\) will be finite (in fact, the domain of \(L_s\) will be \(\{\sigma_i : 0 \leq i \leq s\}\)). Newer coding locations will always come after older ones: if \(s < t\), then we will have \(L_s(\sigma_s) = L_t(\sigma_s) < L_t(\sigma_t)\). Also, because the domain of \(L_s\) is \(\{\sigma_i : 0 \leq i \leq s\}\), the domain will satisfy two closure properties. First, if \(L_s\) is defined at some string, then it is defined at every initial segment; and second, if \(L_s\) is defined at some string \(\sigma^+b\) ending in \(b\), and \(a < b\) in \(\omega^+\), then \(L_s\) is defined at \(\sigma^+a\).

Each \(\sigma \in T_\omega\) will label not just \(L_s(\sigma)\), but a whole tuple \(L_s(\sigma), \ldots, L_s(\sigma) + k - 1\) where \(k = k_s(\sigma) > 0\) is a value defined at stage \(s\). For each \(\sigma \in \text{dom}(L_s) \cap T_\omega\), we will also maintain a valuation \(m_s(\sigma) \in \{-1, 1\}^k\) which represents a choice of \(R\) or \(-R\) for the \(k\) elements labeled by \(\sigma\). We will write \(\bar{a} \in R_{m_s(\sigma)}\) if \(a_i \in R\) whenever \(m_s(\sigma)(i) = 1\) and \(a_i \notin R\) whenever \(m_s(\sigma)(i) = -1\). Like \(L_s\), once we set \(k_s(\sigma)\) or \(m_s(\sigma)\), the value will be fixed.

We will take a moment to show how we will compute \(f\) from \(R^D\) and \(d\). Let \(L\) be the union of all the \(L_s\), and similarly for \(m\) and \(k\). The domain of \(L\) contains each initial segment of \(f\). These are partial \(d\)-computable functions. We will build \(D\) in such a way that

(C1\(^+\)): if \(\sigma \in T_\infty\) and \(\sigma \in f\) then \(L(\sigma) \notin R^D\);

(C2\(^+\)): if \(\sigma \in T_\infty\) and \(\sigma \notin f\) but \(\sigma^- \subset f\), then \(L(\sigma) \in R^D\);

(C3\(^+\)): if \(\sigma \in T_\omega\) and \(\sigma \in f\) then \((L(\sigma), \ldots, L(\sigma) + k(\sigma) - 1) \in (R^D)^m(\sigma)\); and

(C4\(^+\)): if \(\sigma \in T_\omega\) and \(\sigma \notin f\) but \(\sigma^- \subset f\) then \((L(\sigma), \ldots, L(\sigma) + k(\sigma) - 1) \notin (R^D)^m(\sigma)\).

In the cases of \((C1^+)\) and \((C2^+)\), i.e. when \(\sigma \in T_\infty\), we will always have \(k(\sigma) = 1\) and \(m(\sigma) = 1\) (and hence we write \((C1^+)\) and \((C2^+)\) without reference to \(k\) or \(m\)).
We will use $R^D$ to recursively compute longer and longer initial segments of $f$. Suppose that we have computed $\tau \in f$. First, check whether $L(\tau \cdot x) \notin R^D$; if it is not in $R^D$, then by (C2*) we must have $\tau \cdot x \in f$. Otherwise, by (C1*) there is some $x \in \omega$ such that $\tau \cdot x \in f$.

For each of $x = 0, 1, \ldots$, let $a_x = L(\tau \cdot x)$ and check whether $(a_x, a_x + 1, \ldots, a_x + k(\tau \cdot x)) \in (R^D)^m(\tau \cdot x)$; if so for some $x$, then by (C4*), $\tau \cdot x \in f$; otherwise, continue searching. By (C3*), we will eventually find the correct initial segment of $f$. Thus we will have $f \leq_T R^D \oplus L \leq_T R^D \oplus d$.

Now we will describe some properties which the partial isomorphisms $F_s$ will have. We required above that $\text{dom}(L)$ consisted of $\sigma_0, \sigma_1, \sigma_2, \ldots$. So at each stage $s$, we must make sure that $\sigma_s$ is assigned a coding location:

(CLoc): $\text{dom}(L_s)$ contains $\sigma_s$.

We also need to make sure the four properties (C1*), (C2*), (C3*), and (C4*) of $L$ and $R^D$ are true by doing the correct coding during the construction. At each stage $s$ and for each $\sigma \in \text{dom}(L_s)$, we will ensure that:

(C1): if $\sigma \in T_0$ and $\sigma \in \sigma_s$ then $F_s(L_s(\sigma)) \notin R$;

(C2): if $\sigma \in T_0$ and $\sigma \notin \sigma_s$ but $\sigma \in \sigma_s$ then $F_s(L_s(\sigma)) \in R$;

(C3): if $\sigma \in T_\omega$ and $\sigma \in \sigma_s$ then $F_s(L_s(\sigma), \ldots, L_s(\sigma) + k_s(\sigma) - 1) \in R^{m_s(\sigma)}$; and

(C4): if $\sigma \in T_\omega$ and $\sigma \notin \sigma_s$ but $\sigma \in \sigma_s$ then $F_s(L_s(\sigma), \ldots, L_s(\sigma) + k_s(\sigma) - 1) \notin R^{m_s(\sigma)}$.

The $F_s$ will maintain the same atomic diagram, even if they do not agree on particular elements. If $s < t$ then

(At): $F_s \leq_0 F_t$.

However, if two stages $s$ and $t$ agree on some part of the approximation, then they will also agree on how that part of the approximation is being coded. This will ensure that we can construct a limit $F$ by looking at the values of the $F_s$ at stages where the approximation is correct.

(Ext): If $\alpha \in \sigma_s$ and $\alpha \in \sigma_t$, then for all $x \leq L(\alpha) + k(\alpha) - 1$, $F_s(x) = F_t(x)$; if $\sigma_s \in \sigma_t$, then in fact $F_s \subset F_t$.

We want the final isomorphism $F$ to be surjective. To do this, we need to ensure that we continue to add new elements into its image.

(Surj): The first $|\sigma_s| - 1$ elements of $C$ appear in $\text{ran}(F_s \upharpoonright L_s(\sigma_s))$.

It may be helpful to remember what the labels stand for: (CLoc) for coding location; (C1), (C2), (C3), and (C4) for coding 1, coding 2, coding 3, and coding 4 respectively; (At) for atomic agreement; (Ext) for extension; and (Surj) for surjective.
CHAPTER 6. DEGREE SPECTRA OF RELATIONS

There is one last condition which ensures that we can complete the construction while still satisfying the above conditions. Before stating it in the next section, we have already done enough to describe the $A$-computable isomorphism $F : D \rightarrow C$ and see that $A \geq_{T} R^D$. Recall that $f$ is the path approximated by the $\sigma_s$ as in Corollary 1.3. Let $s_1, s_2, \ldots$ be the list of stages $s_n$ such that $\sigma_{s_n} < f$. Then $\sigma_s, \not\in_{\sigma_s} \not\in_{\sigma_s} \ldots$ is a proper chain and $f = \bigcup \sigma_{s_n}$. As before, let $L = \bigcup L_s$ and similarly for $k$ and $m$. Now $L(\alpha_{s_1}), L(\alpha_{s_2}), \ldots$ is a strictly increasing sequence in $\omega$, and for each $i < j$, $F_{s_i} \subseteq F_{s_j}$ by (Ext). Define $F = \bigcup F_{s_n}$; this is a total function because the $F_{s_i}$ are defined on increasingly large initial segments of $\omega$. By (Surj), $F$ is onto as $|\sigma_{s_n}| \geq i$ and so the first $i - 1$ elements of $C$ appear in the range of $F_{s_i}$ below $G(\sigma_{s_n})$, and hence appear in the range of $F$. $F$ is injective since each $F_{s_i}$ is; and the pullback along $F$ gives an isomorphic structure $D$ whose diagram is the union of the diagrams of the $F_{s_i}$, and these diagrams agree with each other because of (At). So the atomic diagram of $D$ is computable in $d$. The sequence $s_1, s_2, \ldots$ can be computed by $A$ because $A$ can compute $f$; and, knowing the sequence $s_1, s_2, \ldots$, we can compute $F$. Hence $A$ can compute $F$ and so $A \geq_{T} R^D$ (recall that $C$ and $R$ were computable). Also, (C1), (C2), (C3), and (C4) imply respectively (C1*), (C2*), (C3*), and (C4*). Earlier we argued that as a consequence, $f \leq_{T} R^D \oplus d$. Hence $A \equiv_{T} R^D \oplus d$ as required.

6.6.3 An Informal Description of the Construction

Recall the intuitive picture of how the coding is done in the previous section, but now using some of the more precise notation just developed. We will begin by looking at coding a single element, but now choosing the element $a_0$ which we had to add when we were coding two or more elements. Things will start to get more complicated than they were before, so Figure 6.8 shows the isomorphism $F_s$ as it changes. We began by choosing an element $a \notin R$ which is 2-free. We labeled $a$ with $\infty$ coding that $0 \notin A$. Now, at some stage, we might think that $0 \in A$, so we replace $a$ by $a'$ with $a' \in R$. At this point, we must choose some $a_0$ to code 0. We are now concerned with the issue, which we ignored before, of how to choose $a_0$. What properties does $a_0$ need to have? If at some later stage after we have added $b$ to the image of $F$, we think that it is actually the case that $0 \notin A$, we replace $a' a_0 b$ by $a a_0' b' c$. Then, if at some further later stage after we have added more elements $\bar{c}$ to the image of $F$, we once more think that $0 \in A$, we replace $aaa_0' b' \bar{c}$ by $aa'a_0'' b'' \bar{c} \bar{d}$ where

$$aaa_0' b' \bar{c} \leq_1 a'' a_0'' b'' \bar{c}$$

and $a'' \in R$. We need to have $a_0 \in R$ if and only if $a_0'' \notin R$. So what we need to know is that, no matter what elements $b$ are added to the image of $F$, we can choose $a_0'$ so that no matter which elements $\bar{c}$ are then added to the image of $F$, we can choose $a''$, $a_0''$ and so on such that $a_0 \in R$ if and only if $a_0'' \notin R$. This is a sort of game where each player gets two moves—we are choosing $a_0$, $a''$, $a_0''$, etc. satisfying the required properties while both we and our opponent together choose the tuples $b$ and $\bar{c}$. By this, we mean that we choose a tuple, and then the tuple $\bar{b}$ (or $\bar{c}$) that our opponent plays must extend the tuple which we chose.
We can do this because at any point, we can add any elements we want to the isomorphism. We want to know that we have a winning move in this game.

<table>
<thead>
<tr>
<th>Coding locations</th>
<th>( \sigma_s )</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \infty )</td>
<td>([a])</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>(a'_{\bar{x}})</td>
<td>([a_0])</td>
<td>(\bar{b})</td>
</tr>
<tr>
<td>(0\ldots)</td>
<td>(a''_{\bar{x}})</td>
<td>(a'_0)</td>
<td>(\bar{b}')</td>
</tr>
<tr>
<td>(\infty\ldots)</td>
<td>(a''_{\bar{x}})</td>
<td>(a_0'')</td>
<td>(\bar{b}'')</td>
</tr>
</tbody>
</table>

Figure 6.8: The values of the isomorphism \( F_s \) when coding a single element of \( A \). The column \( \sigma_s \) shows the approximation at a particular stage, and the coding location shows an indexing via \( L \). An entry is surrounded by brackets “[ ]” to show that it is coding “yes” (i.e., if it is at coding location \( \tau \), then it is coding \( \tau \)). If an entry is “active” in the sense that it is of the form \( \tau \cdot x \) and \( \tau \) is being coded as “yes”, but \( \tau \cdot x \) is coding “no” then it is marked as “..".

Any other entries which are not active are unmarked.

Now let \( y \notin R \) be 2-free over \( a' \). So there is \( x \in R \) with \( a'x \leq a'y \). Now we can try choosing \( a_0 = y \). If this works (in the sense that for \( a_0 = y \), no matter which \( \bar{b} \) and then \( \bar{c} \) our opponent chooses we can choose \( a'_0 \) and \( a''_0 \) as required), then we can just choose \( a_0 = y \) and we are done. If this choice does not work, then we want to argue that choosing \( a_0 = x \) does work. If \( y \) does not work, that means that for some \( \bar{b} \) which our opponent plays, every \( a'_0 \) we choose puts us in a losing position. This means that there is some existential formula \( \varphi(u, v) \) (which is witnessed by the elements \( \bar{b} \)) so that \( y \) satisfies \( \varphi(a', v) \) but that every \( a'_0 \) which satisfies \( \varphi(a, v) \) puts us in a losing position. This means that for every such \( a'_0 \) there is a tuple \( \bar{c} \) which our opponent can play so that any \( a'', a''' \), etc. we choose with

\[
\alpha \bar{a}_0 \bar{b}' \bar{c} \leq_1 \alpha'' \alpha_0' \bar{b}'' \bar{c}'
\]

has \( a''_0 \in R \) since \( y \) was also in \( R \). Now if we instead started with \( a_0 = x \), then that same existential formula \( \varphi(u, v) \) which was true of \( a' \) and so is also true of \( a', x \) since \( a'x \leq a'y \). We can add to the isomorphism a tuple witnessing that \( \varphi(u, v) \) holds of \( a', x \). So no matter which tuple \( \bar{b} \) our opponent actually plays, the existential formula \( \varphi(a', x) \) is witnessed by elements from \( \bar{b} \). Then, since \( a \leq_1 a' \), there are \( a_0' = x \) and \( \bar{b}' \) such that \( a'x \bar{b} \leq_0 ax' \bar{b}' \); thus \( \varphi(a, a_0) \) holds. But then this \( a_0' \) is one which did not work for the choice \( a_0 = y \). Now we add to the isomorphism the tuple which our opponent used to beat us at this point when we chose \( a_0 = y \) and then also chose this value of \( a_0' \). So no matter which tuple \( \bar{c} \) our opponent actually plays, it contains the tuple which they used to win when \( a_0 \) was \( y \). Thus, for every \( a'' \), \( a''' \), etc. we choose with

\[
\alpha \bar{a}_0 \bar{b} \bar{c} \leq_1 \alpha'' \alpha_0' \bar{b}'' \bar{c}'
\]
we have \( a''_0 \in R \). This did not work for \( a_0 = y \), but since \( x \notin R \), it does work for \( a_0 = x \).

What we have done is taken our opponent’s strategy from the case \( a_0 = y \), and forced them to use it when \( a_0 = x \). Their strategy consists only of choosing the tuples \( \bar{c} \) and \( \bar{d} \), and these tuples are chosen by us together with our opponent (because we can add them to the isomorphism before our opponent does). So when \( a_0 = x \) we can force our opponent to play tuples \( \bar{c} \) and \( \bar{d} \) which extend the tuples they played when \( a_0 = y \). But because \( y \in R \) and \( x \notin R \), the winning conditions are different for the different choices of \( a_0 \), so what won our opponent the game for \( a_0 = y \) now loses him the game for \( a_0 = x \).

Choosing \( a_0 \) can begin to get more complicated when we are coding two elements. Figure 6.9 shows the isomorphism when coding two elements. For example, suppose that we choose \( a \) and \( b \), both not in \( R \), with \( a \) 2-free and \( b \) 2-free over \( a \). We label \( a \) with \( \infty \), coding \( 0 \notin A \), and \( b \) with \( \infty \infty \), coding \( 1 \notin A \). We think that \( 1 \in A \), and replace \( b \) by some \( b' \) with \( ab \leq_1 ab' \). Then we add a new element \( \bar{b}_0 \) which is labeled by \( \infty \infty \). At some later stage, we think that \( 0 \in A \), and replace \( ab'b_0 \) with \( ab''b_0' \) with \( ab'b_0 \leq_1 ab''b_0' \). Now we need to add a tuple \( \bar{a}_0 \) which is labeled by \( 0 \), followed by an element \( c \) which is 2-free over \( ab''b_0'\bar{a}_0 \).

**Coding locations**

<table>
<thead>
<tr>
<th>( \sigma_s )</th>
<th>( \infty )</th>
<th>( \infty \infty )</th>
<th>( 0 )</th>
<th>( 0 \infty )</th>
<th>( 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \infty )</td>
<td>( [a] )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \infty \infty )</td>
<td>( [a] )</td>
<td>( [b] )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \infty 0 )</td>
<td>( [a] )</td>
<td>( [b'] )</td>
<td>( [b_0] )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 0 )</td>
<td>( [a] )</td>
<td>( b'' )</td>
<td>( b'_0 )</td>
<td>( [\bar{a}_0] )</td>
<td></td>
</tr>
<tr>
<td>( 0 \infty )</td>
<td>( [a'] )</td>
<td>( b'' )</td>
<td>( b'_0 )</td>
<td>( [\bar{a}_0] )</td>
<td>( [c] )</td>
</tr>
</tbody>
</table>
| \( \infty 0 \) | \( [a] \) | \( [b'] \) | \( [b_0] \) | \( \bar{a}_0' \) | \( c' \) | \( \bar{d}' \) | \( \bar{e} \)

**First case:** opponent plays 1 immediately

\[
1 \mid \xi a''_0 \ y'' \ y''_0 \ a''_0 \ c'' \ \bar{d}'' \ \bar{e}' \ [\bar{a}_1]
\]

**Second case:** opponent plays \( \infty \infty \) then 1

\[
\infty \infty \ | \ [a] \ | [b] \ | \bar{b}'_0'' \ | \ a''_0 \ | \ c'' \ | \bar{d}'' \ | \bar{e}' \ | \bar{f}' \ | [\bar{a}_1]
\]

Figure 6.9: The values of the isomorphism \( F_s \) when coding two elements of \( A \). Two possibilities are shown, depending on what the opponent in the game described plays as the approximation—1, or \( \infty \infty \) followed by 1.

Now, as before, we need to see what properties we want to be true of \( \bar{a}_0 \). The tuple \( \bar{a}_0 \) will have two entries. Suppose that we have added some tuple \( \bar{d} \) to the isomorphism, and then we believe that \( 0 \notin A \) and \( 1 \in A \), so now we want to code \( \infty 0 \). We make our isomorphism \( ab\bar{b}_0\bar{a}_0 c'd' \) for some \( \bar{a}_0' \), \( c' \), and \( d' \), and then an additional tuple \( \bar{e} \) gets added to the isomorphism. Now at some later stage, we believe that \( 0 \in A \) once again, so we find
CHAPTER 6. DEGREE SPECTRA OF RELATIONS

194

in R, \( b'' \), \( \bar{a}'_0'' \), \( c'' \), \( d'' \), and \( \bar{e}' \) such that

\[
ab'\bar{b}_0\bar{a}_0''c'd\bar{e} \leq_1 a''b''\bar{b}_0''\bar{a}_0''c''d''\bar{e}'.
\]

In order to do our coding, for any \( \bar{d} \), we must be able to choose our elements so that for any \( \bar{e} \), we can find such \( a'' \in R \), \( b'' \), etc. with the first entry of \( \bar{a}_0'' \) in \( R \) if and only if the first coordinate of \( \bar{a} \) is not in \( R \).

On the other hand, the approximation might turn out to be different. After adding \( \bar{d} \) to the isomorphism, and again believing that \( 0 \notin A \) and \( 1 \in A \), and making our isomorphism \( ab'b_0\bar{a}_0'c'd' \), we add a new tuple \( \bar{e} \) to the isomorphism. In the previous case, the approximation next told us that \( 0 \notin A \) once again; it might instead be the case that first, the approximation says that \( 0 \notin A \) and \( 1 \notin A \). Then we must change the isomorphism to \( ab'b_0\bar{a}_0'c'd'\bar{e}' \). After we add some tuple \( \bar{f} \) to the isomorphism, then we later believe that \( 0 \in A \), and so we must code 1. Now we need to find \( a'' \in R \), \( b'' \), \( \bar{a}'_0'' \), \( c'' \), \( d'' \), \( \bar{e}' \), and \( \bar{f}' \) such that

\[
ab'b_0\bar{a}_0''c''d''\bar{e}' \leq_1 a''b''\bar{b}_0''\bar{a}_0''c''d''\bar{e}'\bar{f}'.
\]

To do our coding, we must have the second element of \( \bar{a}_0'' \) in \( R \) if and only if the second element of \( a_0 \) is not in \( R \).

When we choose the tuple \( a_0 \), we do not know which case we will be in, and so we must be able to handle both cases. There are actually more possibilities than this (for example, we could think that \( 1 \notin A \), then \( 1 \in A \), then \( 1 \notin A \), then \( 1 \in A \), and so on), but it will turn out that we get these possibilities for free, and so for now we will just consider the two possibilities outlined above.

Looking at this as a game again, in addition to adding tuples to the range of the isomorphism, our opponent can now choose whether the first possibility for the approximation of \( A \) described above will happen, or whether the second possibility will happen (he chooses the approximation stage by stage—so he chooses, at the same time as choosing the tuple \( \bar{e} \), which possibility we must respond to). We will choose a pair \( \bar{a}_0 = (a_0^1, a_0^2) \) to defeat our opponent: \( a_0^1 \) to defeat our opponent when he chooses to first possibility for the approximation, and \( a_0^2 \) for the second. We can choose \( a_0^1 \) exactly as before when we were only coding a single element, in order to defeat our opponent if he uses the first possibility; that is, if he chooses \( \bar{d} \) and \( \bar{e} \), and the approximation says that \( 0 \notin A \), and then \( 0 \in A \) (while saying that \( 1 \in A \) the whole time), we can choose \( a_0'' \) which is in \( R \) if and only if \( a_0^1 \) is not. Now we have to argue that we can choose \( a_0^2 \) so that not only do we defeat our opponent if he uses the second possibility for the approximation of \( A \), but that we still beat our opponent if he uses the first possibility. Choose \( y \notin R \) which is 2-free over \( a'b''\bar{b}_0'a_0'' \), and \( x \in R \) such that \( a'b''\bar{b}_0'a_0''x \leq_1 a'b''\bar{b}_0'a_0''y \). Suppose that we cannot beat our opponent if we choose \( a_0^2 = y \), so that he has some winning strategy for this game. Call our opponent’s winning strategy for \( a_0^2 = y \) their \( y \)-strategy.

Then we will choose \( a_0^2 = x \). We will use our winning strategy for the first approximation to ensure that the only way in which we can lose is if our opponent uses the second approximation (\( \infty \infty \) followed by 1) and forces us to have \( a_0'''' \in R \) (recall that \( a_0^2 = x \in R \)). Now, other
than choosing the approximation, the only plays our opponent can make are to choose $\bar{d}, \bar{c}$, and $\bar{f}$. Now these tuples are played by us together with our opponent, so we can force our opponent to use their $y$-strategy by forcing them to play tuples extending those they used in the $y$-strategy (note that if our opponent plays a larger tuple, it puts more of a restriction on what we can play, and so even though our opponent is not, strictly speaking, using his $y$-strategy, he is using a strategy which is even stronger). We had a winning strategy when our opponent could only choose the first approximation. We will still follow this winning strategy, playing against the tuples our opponent plays which extend the tuples which we add to the isomorphism.

If our opponent uses the first approximation, then we will win because we used the strategy that we already had to beat them in this case. If they use the second approximation, then we force them to play their winning strategy from the case $a_0 = y$. So no matter what we play, we are forced to choose $a''_0 \notin R$ because this is the only way that our winning strategy from before with just the first approximation could lose in this new game. But now $x \in R$, so we win.

Now we said above that we do not have to worry about more complicated possibilities for the approximation, like if we think that the approximation is $\infty 0$, then 1, then $\infty 0$, then 2, then $\infty 0$, and so on. This is because every time that we think that the approximation is $\infty 0$, our partial isomorphism looks like $ab'b_0a'_0c'd'e$ for some additional tuple $\bar{e}$. Because our opponent can play any tuple they like, and also we can respond in the same way whether the approximation becomes 1, 2, and so on, these are all essentially the same position in the game—we can play from any of these positions in the same way that we would play from any other. The values 1, 2, etc. are effectively the same for our purposes in the games above because we have not added a coding location for 1, 2, etc. and so they all put only the requirement that 0 not be coded as "yes." If, for example, 1, 2, and 3 all had coding locations as well, then it would be 4, 5, and so on that were all equivalent. We play the game starting at a stage $s$ only for those coding locations that exist at the stage $s$. If we are only worrying about coding finitely many elements of $A$, then after some bounded number of steps, the approximation our opponent plays will have to repeat in this way. So there is some bound $N$ such that if we can beat our opponent when he plays only $N$ stages of the approximation, then we can beat him when he plays any number of stages.

The process starts to get more complicated when we are coding more than two elements. There become even more possibilities for what could happen with the coding which our opponent could play. If we are only coding finitely many elements of $A$, the tuple $\bar{a}_0$ will be exactly as long as the number of possibilities which we have to consider. Viewing the requirements on $\bar{a}_0$ as a game means that we can ignore the exact details of all of the possibilities for the approximation, while keeping the important properties, like the fact that there are finitely many possibilities. In the next section, we will formally define the game which we have used informally throughout this section.

Now in general, we are trying to code all of the elements of $A$. At each stage $s$, we code only finitely many facts, each labeled by the function $L_s$, and we add only finitely many new coding locations at each stage. We will maintain the property, at each stage $s$, that we
can win the game described above for those coding locations (by which we mean that we must maintain properties (C1), (C2), (C3), and (C4) for those coding locations). Then whenever we add new coding locations, we must show that our winning strategy for the game at the previous stage gives rise to a winning strategy which includes these new coding locations. In this way, even though there are infinitely many coding locations which we will have to deal with eventually, at each point we only consider a game where we deal with finitely many of them. The choice of the new coding locations will have to take into account the winning strategy for the game at the previous stage.

6.6.4 The Game $G_s$ and the Final Condition

We return to giving the last condition (WS). So far, we have put no requirement on the construction to reflect the fact that some of the elements have to be chosen to be 2-free, or that we can mark how many separate occasions we have believed that some $x$ is in $A$. At each stage $s$ of the construction, we will associate a game $G_s$ with two players, I and II. Condition (WS) will simply be

(WS): I has an arithmetic winning strategy for $G_s$.

This stands for winning strategy.

In the game $G_s$, I goes first. On their turns, I plays a partial injection $G : \omega \rightarrow C$, interpreted as a partial isomorphism $D \rightarrow C$; on the first turn, I is required to play a partial isomorphism extending $F_s$. In response to $G$, II plays some elements $\bar{c} \in C$ which are not in the range of $G$, viewed as elements in the range of an extension of $G$, and a string $\alpha$ which is a possible value for $\sigma_t$ for $t > s$. The string $\alpha$ must either be a string in $\text{dom}(L)$ which has no proper extensions in $\text{dom}(L)$, or $\sigma \hat{\cdot} \eta$ where $\sigma \hat{\cdot} x \in \text{dom}(L)$ for some $x \in \omega^+$. Let $\text{dom}(L)^+$ be the set of these strings. Here, $\eta$ is a symbol which we can think of as representing some $k \in \omega$ for which $\sigma^k$ has not yet been visited, but we do not want to differentiate between different values of $k$. Thus $\sigma \hat{\cdot} \eta$ should be viewed as being to the right of every extension of $\sigma$ in $\text{dom}(L)$. By convention, $\sigma \hat{\cdot} \eta$ is not in $T$. The approximation $(\sigma_t)_{t \in \omega}$ must satisfy the properties from Lemma 6.6.6. So we also restrict II and force him to play strings which form subsequences of sequences with the properties from the lemma. So if II plays some string to the left of $\tau$ after playing $\tau$, then II can never play any string extending $\tau$ again. Also, for any string $\tau$ and $x \in \omega$, if II plays a string $\tau \hat{\cdot} y$ for $y > x$ (or $y = \eta$), then II can never again play a string extending $\tau \hat{\cdot} x$. Note that this does not apply to strings which end with $\eta$ (conceptually, II should be thought of as playing $\sigma \hat{\cdot} k$ for increasingly large values of $k$).

At stage $s$, we do not yet know the actual values of the approximation $(\sigma_t)_{t \in \omega}$ after stage $s$. We could compute finitely many future stages, but not all of them. In playing the $\alpha \in \text{dom}(L)^+$, II plays a possible future value of the approximation which we have to be able to handle. When I plays a partial isomorphism $G$ in response, it is an attempt to continue the construction assuming that the approximation is as II has played it. But I only has to continue the construction in a limited manner: they must maintain the coding given by $L_s$,
but they do not need to add more coding locations to \(L_s\). Since we will not be adding new elements to \(L_s\), we will let \(L = L_s\), \(k = k_s\), and \(m = m_s\) for the rest of this section and the next.

Now during the construction there will be certain elements which we will have to add to our partial isomorphism. For example, condition (Surj) requires us to add elements in order to make the isomorphism bijective. We will also have to add free elements in order to code new strings; we have some control over these in that we can choose which free element we choose, but not total control in that we are restricted to the free elements. This is the role of the tuples \(c\) which are played by \(\Pi\) in response to a play \(G\) by \(I\): they are elements which \(I\) is required to add to \(G\) before continuing the construction. They will also be useful for more technical reasons in the next section.

Now we will state the ways in which \(I\) can lose. If \(I\) does not lose at any finite stage, then they win (thus it is a closed game). First, there are some conditions on the coding by \(G\). If \(I\) plays \(G\) in response to \(\alpha\), then I must ensure that for each \(\sigma \in \text{dom}(L)\):

\((C1^1)\): if \(\sigma \in T_\infty\) and \(\sigma \subset \alpha\) then \(G(L(\sigma)) \notin R;\)

\((C2^1)\): if \(\sigma \in T_\infty\) and \(\sigma \not\subset \alpha\) but \(\sigma^- \subset \alpha\), then \(G(L(\sigma)) \in R;\)

\((C3^1)\): if \(\sigma \in T_\omega\) and \(\alpha \subset \sigma\) then \(G(L(\sigma), \ldots, L(\sigma) + k(\sigma) - 1)) \in R^m(\sigma);\) and

\((C4^1)\): if \(\sigma \in T_\omega\) and \(\sigma \not\subset \alpha\) but \(\sigma^- \subset \alpha\) then \(G(L(\sigma), \ldots, L(\sigma) + k(\sigma) - 1) \notin R^m(\sigma).\)

These are conditions which ensure that \(G\) codes \(\alpha\) using the coding locations given by \(L\); they are the equivalents of conditions (C1), (C2), (C3), and (C4) respectively for the \(F_s\).

Now we also have global agreement conditions which are the equivalents of (At) and (Ext). There is a slight modification to (At) and (Ext), which is that if \(I\) plays \(G\), and \(\Pi\) responds by playing \(\bar{c}\) and \(\alpha\), then we use \(G^\nu\bar{c}\) rather than \(G\) because the \(\bar{c}\) are elements that must be added to the image of \(G\) before the next move. Suppose that so far, \(I\) has played \(G_0 \supseteq F_s, G_1, \ldots, G_n\) and \(\Pi\) has played \((\bar{c}_0, \alpha_1), \ldots, (\bar{c}_n, \alpha_{n+1})\). Note that the two indices of a move by \(\Pi\) differ by one, so that \(\Pi\) plays \((\bar{c}_i, \alpha_{i+1})\) rather than \((\bar{c}_i, \alpha_i)\). This will turn out to be more convenient later. By convention, we let \(\alpha_0 = \sigma_s\) and \(\bar{c}_{n+1}\) the empty tuple (or, if \(\Pi\) plays the strings \(\beta_i\), then \(\beta_0 = \sigma_s\), and so on). Now \(I\) must play a partial isomorphism \(G_{n+1}\) which codes \(\alpha_{n+1}\). In addition to the four requirements above, \(I\) must also ensure that:

\((At^1)\): \(G_i \preceq \bar{c}_i \leq G_{i+1},\)

\((Ext1^1)\): for each \(0 \leq i < n + 1\), if \(\sigma \in T\) (so \(\sigma\) does not end in \(\eta\)), \(\sigma \subset \alpha_i\), and \(\sigma \subset \alpha_{n+1}\), then for all \(x \leq L(\sigma) + k(\sigma) - 1\), \(G_i(x) = G_{n+1}(x);\) if \(\alpha_i = \alpha_{n+1}\) and they do not end in \(\eta\), then in fact \(G_i \preceq \bar{c}_i \subset G_{n+1}\), and

\((Ext2^1)\): for each \(t \leq s\) and \(0 \leq i < n + 1\), if \(\sigma \in T\) (so \(\sigma\) does not end in \(\eta\)), \(\sigma \subset \sigma_t\), and \(\sigma \subset \alpha_{n+1}\), then for all \(x \leq L(\sigma)\), \(F_t(x) = G_{n+1}(x);\) if \(\sigma_t = \alpha_{n+1}\) and they do not end in \(\eta\), then in fact \(F_t \subset G_{n+1}\.\)
A winning strategy for I is a just way to continue the construction, but without having to add any new strings to $L_s$. Because II can play any appropriate string $\alpha$, the strategy is independent of the future values $\sigma_{s+1}, \sigma_{s+2}, \ldots$ of the approximation.

### 6.6.5 Basic Plays and the Basic Game $G_s^b$

The game $G_s$ requires I to play infinitely many moves in order to win. However, II has only finitely many different strings in $\text{dom}(L)^*$ which they can play, and so if they extend the approximation for infinitely many stages, they must repeat some strings infinitely many times. In this section, we will define a game which is like $G_s$, except that II is not allowed to have the approximation loop more than once. The main lemma here will be that if I can beat II when II is restricted to only playing one loop, then I can win in general. The idea is that at the end of a loop in the approximation, the game ends up in essentially the same place it was before the loop. So if I does not lose to any single loop, they do not lose to any sequence of loops. These plays with only one loop will be called the basic plays, and the game with no loops the basic game $G_s^b$.

Whether or not a play by II is a basic play depends only on the strings $\alpha$ in the play, and is independent of the tuples $\bar{c}$. We say that a play $(\bar{c}_0, \alpha_1), \ldots, (\bar{c}_{\ell-1}, \alpha_{\ell})$ by II is based on the list $\alpha_1, \ldots, \alpha_{\ell}$. A play based on $\alpha_1, \ldots, \alpha_{\ell}$ is a basic play (and the list of strings it is based on is a basic list) if it satisfies:

**(B1):** for $i < \ell - 1$, $\alpha_i \neq \alpha_{i+1}$ (we allow $\alpha_{\ell-1}$ to be equal to $\alpha_\ell$), and

**(B2):** if for some $i < j$ there is some $\tau \in T$ such that $\tau \omega \subset \alpha_i$, $\tau \omega \subset \alpha_j$, and for all $k$ with $i < k < j$, $\alpha_k = \tau \omega \eta$, then $j = \ell$ and $\alpha_i = \alpha_j$.

These conditions include $\alpha_0 = \sigma_s$; so, for example, if $\alpha_1 = \sigma_s$, then $\ell = 1$ by (B1). Note that all of these definitions are dependent on the stage $s$. So really, we are defining what it means to be a basic play at stage $s$.

The first of two important facts about the basic plays is the following lemma.

**Lemma 6.6.8.** There are finitely many basic lists.

**Proof.** The domain of $L$ is finite, and hence $\text{dom}(L)^*$ is finite. Since II can only play strings from $\text{dom}(L)^*$, there are only finitely many different $\sigma$ which can appear as one of the $\alpha_i$ in a basic list. Let $\alpha_1, \ldots, \alpha_\ell$ be a basic list. We will show that $\ell$ is bounded, and hence there are only finitely many basic lists. Let $M$ be the size of $\text{dom}(L)^*$. In any sufficiently long basic list, say $\alpha_1, \ldots, \alpha_N$ of length at least $N$ (depending on $M$), there must be three indices $i_1 < i_2 < i_3 < N$ such that $\alpha_{i_1} = \alpha_{i_2} = \alpha_{i_3}$. Since $i_3 < N$, by (B1) there must be $j_1$ and $j_2$ with $i_1 < j_1 < i_2$, $i_2 < j_2 < i_3$, $\alpha_{j_1} \neq \alpha_{i_1}$, and $\alpha_{j_2} \neq \alpha_{i_2}$. Let $\tau_1$ and $\tau_2$ be the greatest common initial segments of the $j$ with $i_1 \leq j \leq i_2$ and $i_2 \leq j \leq i_3$ respectively. Let $x$ be such that $\tau_1 \cdot x \notin \alpha_{i_1} = \alpha_{i_2}$. Since there is $j_1$ with $i_1 < j_1 < i_2$ and $\tau_1 \cdot x \notin \alpha_{j_1}$, we cannot have $x \in \omega$. So $x = \infty$ or $x = \eta$. First, suppose that $x = \infty$. Then for all $j_1$ with $i_1 < j_1 < i_2$, we cannot have $\tau_1 \cdot z \notin \alpha_{j_1}$ for some $z \in \omega$, since for any such $z$ with $\tau_1 \cdot z \in L$, some string $\tau' \supseteq \tau_1 \cdot z$ appeared
before \(i_1\) and hence \(\tau_1 \preceq z\) can never appear again. So, by decreasing \(i_2\) and increasing \(i_1\), we may assume that \(\tau_1 \preceq \alpha_{i_1}, \tau_1 \preceq \alpha_{i_2}\), and for all \(j_1\) with \(i_1 < j_1 < i_2, \tau_1 \preceq \eta \subseteq \alpha_{j_1}\) (though we may no longer have \(\alpha_{i_1} = \alpha_{i_2}\)). This contradicts \((B2)\). So we must have \(x = \eta\), and so \(\tau_1 = \alpha_{i_1} = \alpha_{i_2}\). Similarly, we must have \(\tau_2 = \alpha_{i_3} = \alpha_{i_4}\). Thus \(\tau_1 = \tau_2\)—call this \(\tau\). Then there are \(j_1\) and \(j_2\) with \(i_1 < j_1 < i_2 < j_2 < i_3\) and \(\tau \preceq \alpha_{j_1}\) and \(\tau = \alpha_{j_2}\). For all \(i\) with \(j_1 < i < j_2\), we have \(\tau \subseteq \alpha_i\). Thus, increasing \(j_1\) and decreasing \(j_2\), we may assume that for all such \(i\), \(\tau \preceq \alpha_i\). This contradicts \((B2)\). So there is no basic list of length greater than \(N\). \qed

The basic game \(G^b_s\) is the same as the game \(G_s\), except that we add a new requirement for \(\Pi\): any play by \(\Pi\) must be a basic play. If, at any point in the game, \(\Pi\) has violated one of the conditions of the basic plays, then \(\Pi\) loses. The next lemma is the second important fact about the basic plays; it says that they are the only plays which \(I\) has to know how to win against.

**Lemma 6.6.9.** If \(I\) has a winning strategy for the basic game \(G^b_s\), then \(I\) has a winning strategy for the game \(G_s\). Moreover, if \(I\) has an arithmetic winning strategy for \(G^b_s\), then they have an arithmetic winning strategy for \(G_s\).

**Proof.** Let \(S^b\) be a winning strategy for \(I\) in the basic game \(G^b_s\). We must give a winning strategy \(S\) for \(I\) in the game \(G_s\). To each play \(P\) by \(\Pi\) in the game \(G_s\), \(S\) must give \(I\)'s response \(S(P)\). To each play \(P\), we will associate a basic play \(P^*\). I will respond to \(P\) in the same way that they responded to the corresponding basic play \(P^*\); that is, \(S(P)\) will be \(S^b(P^*)\). If \(P\) is \((c_0, \alpha_1), \ldots, (c_{m-1}, \alpha_m)\) and \(P^*\) is \((d_0, \beta_1), \ldots, (d_{n-1}, \beta_n)\) then we will have \(\alpha_m = \beta_n\). Thus since \((C1^t),(C2^t),(C3^t),(C4^t)\), and \((\text{Ext}2^t)\) hold for \(I\) playing \(S^b(P^*)\) in response to \(P^*\), they will also hold for \(I\) playing \(S(P) = S^b(P^*)\) in response to \(P\).

The general strategy to construct \(P^*\) from \(P\) will be to build \(P^*\) up inductively. If \(P\) is not a basic play, it is because it fails to satisfy \((B1)\) and \((B2)\). In the first case, this means that for some \(i\), \(\alpha_i = \alpha_{i+1}\), and so in \(P^*\) we will omit \(\alpha_i\). In the second case, we will be able to omit everything between the \(i\) and \(j\) witnessing the failure of \((B2)\). The difficulty is to do this in a well-defined way, so that we have a nice relationship between \(Q^*\) and \(P^*\) when \(P\) is a longer play which includes \(Q\). This is necessary to ensure that \(S^b(Q)\) and \(S^b(P)\) are related in the correct way, e.g. by \((\text{Ext}1^t)\) and \((\text{Ext}2^t)\). It will be sufficient to build up \(P^*\) inductively from the definition of \(Q^*\) in a natural way, but there are a number of things to check.

There is a condition \((*)\) relating \(P\) and \(P^*\) which, intuitively, says that \(P^*\) captures the essence of \(P\) (i.e., \(P^*\) omits only non-essential plays from \(P\)). Let \(G_0 \supseteq F_s, G_1, \ldots, G_m\) be \(I\)'s response to \(P\) and \(H_0 \supseteq F_s, H_1, \ldots, H_n\) be \(I\)'s response to \(P^*\). Let \(\bar{F} : \omega \rightarrow C\) be a partial isomorphism and \(\gamma \in \text{dom}(L)^*\). Then denote by \(\Phi(P^*, \bar{F}, \gamma, i)\) the statement:

- for all \(\sigma \in T\), if \(\sigma \in \beta_i\) and \(\sigma \in \gamma\), then for all \(x \leq L(\sigma)\), \(H_i(x) = \bar{F}(x)\). If \(\beta_i = \gamma\) and they do not end in \(\eta\), then in fact \(H_i \bar{d}_i \subset \bar{F}\),

and by \(\Psi(P, \bar{F}, \gamma, j)\) the statement:
• for all $\sigma \in T$, if $\sigma \preceq \alpha_j$ and $\sigma \preceq \gamma$, then for all $x \leq \ell(\sigma)$, $G_j(x) = \bar{F}(x)$. If $\alpha_j = \gamma$ and they do not end in $\eta$, then in fact $G_j \circ \bar{c}_j \subseteq \bar{F}$.

Then (\*) says that for any $\bar{F}$ and $\gamma$ such that for all $i$ with $0 \leq i \leq n$ the statement $\Phi(P^*, \bar{F}, \gamma, i)$ holds, then for the same $\bar{F}$ and $\gamma$ and for all $j$ with $0 \leq j \leq m$ the statement $\Psi(P, \bar{F}, \gamma, j)$ holds.

Now suppose that (\*) holds for $P$ and $P^*$, and that we want to check $\text{(Ext1)}^\ast$ and $\text{(Ext2)}^\ast$ for the response $H_n = G_m = S(P) = S^b(P^*)$ to $P$. Choose $\bar{F} = H_n = G_m$ and $\gamma = \alpha_m = \beta_n$. We know that $S^b$ is a winning strategy in the basic game, and so I does not lose by playing $H_n$. Thus $\text{(Ext1)}^\ast$ and $\text{(Ext2)}^\ast$ must hold in the basic game. We immediately get $\text{(Ext1)}^\ast$ for the full game. Also, $\text{(Ext2)}^\ast$ for the basic game implies, for all $i \leq n$, the statement $\Phi(P^*, H_n, \gamma, i)$. Then by (\*), we know that for all $i \leq m$, we have $\Psi(P, G_m, \gamma, i)$. This implies $\text{(Ext2)}^\ast$ for the full game at $P$. Thus, instead of checking $\text{(Ext1)}^\ast$ and $\text{(Ext2)}^\ast$, it suffices to check property (\*).

We will define the operation $P \rightarrow P^*$ by induction on the length of $P$. At the same time, we will show that the strategy $S$ for I does not lose at any finite point in the game (and hence, it must win), and also that $P$ and $P^*$ have (\*). If we show (\*), then the only thing remaining to see that I does not lose is to check $\text{(At)}^\ast$. Let $P$ be $(\bar{c}_0, \alpha_1), \ldots, (\bar{c}_\ell, \alpha_{\ell+1})$ and suppose that the operation $Q \rightarrow Q^*$ is defined for plays of length up to $\ell$. Let $Q$ be first $\ell$ plays in $P$, that is, $(\bar{c}_0, \alpha_1), \ldots, (\bar{c}_\ell, \alpha_\ell)$, so that $Q^*$ has already been defined. We have three possibilities.

**Case 1.** $Q^*$ followed by $(\bar{c}_\ell, \alpha_{\ell+1})$ is a basic play.

Let $P^*$ be $Q^*$ followed by $(\bar{c}_\ell, \alpha_{\ell+1})$; this is already a basic play.

First we check (\*). Fix $\gamma$ and $\bar{F}$. Let $H_0, H_1, \ldots, H_n$ be I’s response to $Q^*$ and let $G_0, G_1, \ldots, G_\ell$ be I’s response to $Q$. Let $H_{n+1} = G_{\ell+1} = S^b(P^*) = S(P)$. Suppose that, for each $i \leq n + 1$, we have $\Phi(P^*, \bar{F}, \gamma, i)$, then, from $\Phi(P^*, \bar{F}, \gamma, i)$ for $i \leq n$ and (\*) for $Q$ and $Q^*$, we get $\Psi(P, \bar{F}, \gamma, j)$ for $i \leq \ell$. Note that $\Phi(P^*, \bar{F}, \gamma, \ell + 1)$ is the same as $\Psi(P, \bar{F}, \gamma, n + 1)$ since $H_{n+1} = G_{\ell+1}$. So, for all $j \leq \ell + 1$, we have $\Psi(P, \bar{F}, \gamma, j)$. Thus we have (\*).

Now we need to check $\text{(At)}^\ast$ to see that I does not lose $G^b_s$ by responding to $P$ with $S^b(P^*) = H$. So we need to show that $S(Q) \circ \bar{c}_\ell \subseteq H_{n+1}$. Since I does not lose $G^b_s$ when responding to $P^*$ with $H_{n+1}$, we have $S^b(Q^*) \circ \bar{c}_\ell \subseteq H_{n+1}$. But $S^b(Q^*) = S(Q)$, so $(\text{At})^\ast$ holds for $S$.

**Case 2.** $Q^*$ followed by $(\bar{c}_\ell, \alpha_{\ell+1})$ does not satisfy $\text{(B1)}$.

Let $Q^*$ be $(\bar{d}_0, \beta_1), \ldots, (\bar{d}_{n-1}, \beta_n)$. It must be the case that $\beta_{n-1} = \beta_n$. Let $H_0, H_1, \ldots, H_n$ be I’s response to $Q^*$; then $H_{n-1} \circ \bar{d}_{n-1} \subseteq H_n$. Let $\bar{e}$ be a tuple of elements from $C$ such that $H_{n-1} \circ \bar{d}_{n-1} \circ \bar{e} = H_n$ and let $P^*$ be the play $(\bar{d}_0, \beta_1), \ldots, (\bar{d}_{n-2}, \beta_{n-1}), (\bar{d}_{n-1}, \bar{e} \circ \bar{c}_\ell, \alpha_{\ell+1})$. Since $Q^*$ satisfies $(\text{B2})$ and $\beta_{n-1} = \beta_n$, $P^*$ satisfies $(\text{B2})$ and hence is a basic play.

Now we will check (\*). Fix $\gamma$ and $\bar{F}$. Recall that $H_0, H_1, \ldots, H_n$ is I’s response to $Q^*$ in the basic game, and let $G_0, G_1, \ldots, G_\ell$ be I’s response to $Q$; let $G_{\ell+1} = H_n$ be $S^b(P^*) = S(P)$. Then $H_0, H_1, \ldots, H_{n-1}, H_n'$ is I’s response to $P^*$. Suppose that for each $i \leq n$ we have
CHAPTER 6. DEGREE SPECTRA OF RELATIONS

Φ(P*, F, γ, i). Then since G_{i+1} = H'_n and α_{i+1} is the last play by II in both P and P*, we immediately have Ψ(P, F, γ, i). To show Ψ(P, F, γ, j) for j ≤ ℓ, it suffices to show Ψ(Q, F, γ, j) for j ≤ ℓ, and hence (by (⋆) for Q and Q*) to show Φ(Q*, F, γ, i) for i ≤ n.

Now P* and Q* agree on (d_0, β_1), ..., (d_{n-2}, β_{n-1}), so we have Φ(Q*, F, γ, i) for i ≤ n - 2. Now Φ(P*, F, γ, n - 1) says that: for all σ ∈ T, if σ ⊂ β_{n-1} and σ ⊂ γ, then for all x ≤ L(σ), H_n(x) = F(x); and moreover, if β_{n-1} = γ and they do not end in η, then in fact H_{n-1}^\top \tilde{d}_{n-1} ^\top \tilde{e}^\top \tilde{c}_n = H_n^\top \tilde{e}_n \subset F. This implies both Φ(Q*, F, γ, n - 1) and Φ(Q*, F, γ, n) since H_{n-1} \subset H_n. Thus, for all i ≤ n, we have Φ(Q*, F, γ, i). This completes the proof of (⋆).

Now we need to check (Ati). Since I does not lose \mathcal{G}^b_n when responding to P* with H'_n = G_{i+1}, by (Ati) we have that H_{n-1}^\top \tilde{d}_{n-1} ^\top \tilde{e}^\top \tilde{c}_n ≥ 0 H'_n. But H_{n-1}^\top \tilde{d}_{n-1} ^\top \tilde{e} = S^b(Q^*) = S(Q), so S(Q) ≤ S(P). So (Ati) holds for S.

Case 3. Q* followed by (\tilde{e}, α_{i+1}) satisfies (B1) but does not satisfy (B2).

Let Q* be (d_0, β_1), ..., (d_{n-1}, β_n). Now Q* satisfies (B2), so there are two possible reasons that Q* followed by (\tilde{e}, α_{i+1}) might fail to satisfy (B2).

Subcase 1. \tau^- ∞ ⊂ β_m = β_n but for each k with m < k < n, β_k = \tau^- η

Choose m to be least with the above property. For any k with m < k < n, β_k = \tau^- η. Let H_0, H_1, ..., H_n be I's play in response to Q*. We have H_m \subset H_n. Let \tilde{e} be a tuple of elements of \mathcal{C} such that H_m^\top \tilde{e} = H_n. Then let P* be (d_0, β_1), ..., (d_{m-1}, β_m), (\tilde{e}^\top \tilde{c}_m, α_{m+1}). This is a basic play since Q* was. Note that I's play in response to P* is H_0, H_1, ..., H_m, H' for some partial isomorphism H'.

Now we will check (⋆). Let G_0, G_1, ..., G_ℓ be I's response to Q, so that I's response to P is G_0, G_1, ..., G_ℓ, G_{i+1} = H'. Fix γ and F. Suppose that for each i ≤ n we have Φ(P*, F, γ, i). Since both P and P* end in α_{i+1}, Φ(P*, F, γ, i) implies Ψ(P, F, γ, ℓ + 1). Now for j ≤ ℓ, Ψ(Q, F, γ, j) implies Ψ(P, F, γ, j), and so by (⋆) for Q and Q*, it suffices to show that for all i ≤ n, we have Φ(Q*, F, γ, i).

For the first m turns, Q* and P* agree, so for i ≤ m, Φ(P*, F, γ, i) implies Φ(Q*, F, γ, i). So we have established Φ(Q*, F, γ, i) for i ≤ m.

For m < i < n, we have β_i = \tau^- η. To show Φ(Q*, F, γ, i), it suffices to check that if σ ∈ T has σ ⊂ τ and σ ⊂ γ, then for all x ≤ L(σ), G_i(x) = F(x). Now σ ⊂ τ ⊂ α_i, so G_i(x) = F(x) for all x ≤ G(σ). By (Ext1), for all x ≤ L(σ), G_i(x) = G_j(x), and hence G_j(x) = F(x).

Finally, we have the case i = n. We have β_n = β_i. If σ ∈ T has σ ⊂ β_n and σ ⊂ γ, then σ ⊂ β_i and so for all x ≤ L(σ), G_n(x) = G_i(x) = F(x). We also need to consider the case where γ = β_n. By Φ(P*, F, γ, i), we have that G_i^\top \tilde{e} = F; but G_i^\top \tilde{e} = G_n and so G_n \subset F as desired. Thus we have Ψ(Q*, F, γ, n). This completes the proof of (⋆).

So I responds to P with S^h(P*) = H. To see that this does not lose the game for I, we just need to check (At). Since I does not lose \mathcal{G}^b_n when responding to P* with H, by (At) we have G_i^\top \tilde{e} \tilde{c}_{i+1} ≤_0 H. But G_i^\top \tilde{e} = G_n, so G_n \tilde{c}_{i+1} ≤_0 H as required.

Subcase 2. There is i ≤ n and τ ∈ T such that \tau^- ∞ ⊂ β_i and \tau^- ∞ ⊂ α_{i+1} but for each k with i < k ≤ n, \tau^- η = β_k. Also, β_i = α_{i+1}. 
Let \( \hat{Q} = (\bar{d}_0, \beta_1), \ldots, (\bar{d}_{n-1}, \beta_n), (\bar{c}_\ell, \beta_\ell) \). We can now use the same argument as in the previous subcase, with \( \hat{Q} \) being extended by \((\emptyset, \alpha_{\ell+1})\).

The whole construction of \( S \) from \( S^b \) by transforming plays \( P \) into basic plays \( P^* \) is arithmetic.

Now any winning strategy for I in \( \mathcal{G}_s \) is also a winning strategy in \( \mathcal{G}_b^s \); in particular, if I has an arithmetic winning strategy for \( \mathcal{G}_s \), then they have an arithmetic winning strategy in \( \mathcal{G}_b^s \). However, it would be nice if we did not have to worry about the computability-theoretic properties of the winning strategy during our arguments. The following lemma lets us do exactly that.

**Corollary 6.6.10.** The following are equivalent:

1. I has a winning strategy for \( \mathcal{G}_s \).
2. I has an arithmetic winning strategy for \( \mathcal{G}_s \).
3. I has a winning strategy for \( \mathcal{G}_b^s \).
4. I has an arithmetic winning strategy for \( \mathcal{G}_b^s \).

**Proof.** We have \((4) \Rightarrow (2)\) from the previous lemma. \((2) \Rightarrow (1)\) is immediate. \((1) \Rightarrow (3)\) is because any winning strategy for I in \( \mathcal{G}_s \) is a winning strategy for \( \mathcal{G}_b^s \). So it remains to prove \((3) \Rightarrow (4)\). In \( \mathcal{G}_b^s \), each player makes only finitely many moves in every play of the game, and there is a computable bound on the number of moves. Any such game with a winning strategy for I has an arithmetic winning strategy for I. \(\square\)

So in order to check \((\text{WS})\), it suffices to show that I has a winning strategy without worrying about whether it is arithmetic.

### 6.6.6 The Construction

Begin at stage \( s = -1 \) with \( F_{-1} = L_{-1} = \emptyset \).

At each subsequent stage \( s + 1 \), we will first update the partial isomorphism from the previous stage according to the new approximation. To do this, we will use the winning strategy from the game \( \mathcal{G}_s \) which we had at the previous stage. Then we will add new elements to the image of the isomorphism, and add a new coding location. We must add these new elements so that all of the properties from Section 6.6.2 are satisfied.

At stage \( s + 1 \), \( \sigma_{s+1} \notin \text{dom}(L_s) \) but \( \sigma^-_{s+1} \in \text{dom}(L_s) \). (Recall that \( \sigma^-_{s+1} = \sigma_{s+1} \) with the last entry removed.) We begin by using our winning strategy for \( \mathcal{G}_s \) to code \( \sigma^-_{s+1} \) correctly. If there is some \( x \in \omega^* \) such that \( \sigma^-_{s+1} \dashv x \in \text{dom}(L_s) \), then \( \ell(\sigma_{s+1}) \in \omega; \) let \( \tau = \sigma^-_{s+1} \dashv \eta \). Otherwise, let \( \tau = \sigma^-_{s+1} \). I has an arithmetic winning strategy \( S \) for the game \( \mathcal{G}_s \) from the previous stage. Consider these first couple moves of the game where I uses the strategy \( S \): I plays \( G_0 \supset F_s \), II plays \( (\emptyset, \tau) \), and I responds with \( G \) according to their winning strategy \( S \). \( F_{s+1} \) will be
an extension of $G$. Since $G$ was part of the winning strategy for $I$, it did not lose the game. So, automatically, $F_{s+1}$ will satisfy (C1), (C2), (C3), and (C4) for $\sigma \in \text{dom}(L_s)$, (At), and (Ext) since $G$ already satisfies these.

Suppose that the domain of $G$ is $\{0, \ldots, n\}$. Let $t < s + 1$ be the previous stage at which $\sigma_t = \sigma_{s+1}$. Then by (Surj) at stage $t$, the first $|\sigma_t| - 1$ elements of $\mathcal{C}$ appear in $\text{ran}(F_{t} \upharpoonright L_t(\sigma_t))$. By (Ext), $F_t \subset G$. If the first $|\sigma_{s+1}| - 1$ elements of $\mathcal{C}$ do not appear in $\text{ran}(G)$, then it is because the $\lfloor |\sigma_{s+1}| - 1 \rfloor$th element is not in $\text{ran}(G)$; let this element be $a$, and define $G'$ extending $G$ by $G'(n + 1) = a$. Set $L_{s+1}$ to be the extension of $L_s$ with $L_{s+1}(\sigma_{s+1}) = n + 2$. Then any $F_{s+1}$ extending $G'$ will satisfy (Surj). Also, since $\sigma_{s+1} \in \text{dom}(L_{s+1})$, we satisfy (CLoc).

While we have set $L_{s+1}(\sigma_{s+1}) = n + 2$, we have not yet added an element to the range of $G$ in that position. What remains to be done is to add some element in the position $n + 2$ (or a tuple in the positions $n + 2, n + 3, \ldots$ if $\sigma_{s+1} \in T_n$) to satisfy (C1), (C2), (C3), and (C4) for $\sigma = \sigma_{s+1}$ and also to give a winning strategy witnessing (WS). There are two cases depending on the last entry of $\sigma_{s+1}$. The first is relatively easy while the second is much harder.

### 6.6.7 The Case $\sigma_{s+1} \in T_{\infty}$

Let $b \notin R$ be an element which is 2-free over $\text{ran}(G')$. Let $F_{s+1}$ extend $G'$ with $F_{s+1}(n + 2) = b$. Define $m_{s+1}(\sigma_{s+1}) = -1$ and $k_{s+1}(\sigma_{s+1}) = 1$. Then $F_{s+1}$ satisfies (C1), (C2), (C3), and (C4) for $\sigma = \sigma_{s+1}$, which was the last remaining case of those properties.

We must show that $I$ has a winning strategy in the game $G^b_{s+1}$. The only difference between $G^b_{s+1}$ and $G^b_s$ is that in $G^b_{s+1}$ we have added a new coding location $\sigma_{s+1}$ to $L$, and $G^b_s$ starts a turn earlier than $G^b_{s+1}$. We can accommodate the latter by considering plays in $G^b_s$ which begin with $(\varnothing, \sigma_{s+1})$, thus making $I$ play $G$ as their first play. So every part of the approximation that our opponent plays in $G^b_{s+1}$ is an approximation that our opponent could have played in $G^b_s$, except that in $G^b_{s+1}$ II can also play $\sigma_{s+1}$ and $\sigma_{s+1}^\eta$.

As a first approximation, we could use the strategy $S$ from the previous stage (and when our opponent plays $\sigma_{s+1}$ or $\sigma_{s+1}^\eta$, we respond instead to $\sigma_{s+1}$). This works except for one thing, which is that when our opponent plays either $\sigma_{s+1}$ or $\sigma_{s+1}^\eta$, we are not guaranteed to code $\sigma_{s+1}$ correctly. Now, when our opponent plays $\sigma_{s+1}$, our response using $S$ will extend $F_{s+1} = Gab$ with $b \notin R$ coding $\sigma_{s+1}$ (this is because by (Ext1) and (Ext2)) our response must extend our response at the previous stage where the approximation was $\sigma_{s+1}$. The only problem is that when our opponent plays $\sigma_{s+1}^\eta$, $S$ will also have us respond with an extension of $F_{s+1} = Gab$, but now $b$ is coding the wrong thing. This is relatively easy to fix. Say $S$ has us respond with $Gab\breve{c}$. Since $b$ is 2-free over $Ga$, we can find $b' \in R$ and $c'$ with $Gab\preceq_1 Gab/b'$. Then we will play $Gab/c'$.

In order to continue to follow the strategy $S$ at later stages, we cannot tell $S$ that our opponent plays $\sigma_{s+1}$ or $\sigma_{s+1}^\eta$ because this is an illegal play in $G^b_s$. However, we can tell $S$ that our opponent played $\sigma_{s+1}^\eta$ and pretend that we responded with $Gab\breve{c}$. We will keep track of these corresponding plays for the purposes of using $S$. 

CHAPTER 6. DEGREE SPECTRA OF RELATIONS
So along with defining the strategy \( T \) for \( G_s^b \), we will describe a for each play by II in \( G_{s+1}^b \) a corresponding play in \( G_s \). (Note the corresponding play is in the full game, rather than the basic game). If II plays \((\bar{c}_0, \alpha_1), \ldots, (\bar{c}_{n-1}, \alpha_n)\) in \( G_{s+1}^b \), the corresponding play in \( G_s^b \) will be of the form \((\varnothing, \sigma_{s+1}^-), (ab\bar{d}_0, \beta_1), \ldots, (d_{n-1}, \beta_n)\) in \( G_s \). Note that the length of the play in \( G_s \) is one more than that in \( G_{s+1}^b \); this is because \( G_s \) begins at the previous stage, and so we need to begin by playing \( G \) and adding \( a \) and \( b \) to the image of the isomorphism. Thus the first play using the strategy \( S \) will be \( G \), and before the second play I will be forced to add \( ab \) to \( G \), and thus I is essentially playing \( F_{s+1} \). We can already define \( \beta_1, \ldots, \beta_{n-1} \), but the \( \bar{d}_i \) will be defined at the same time as we define \( T \). If \( \alpha_i \) is in \( \text{dom}(L_s)^* \), then \( \beta_i \) and \( \alpha_i \) will be equal. Otherwise, \( \alpha_i \) is either \( \sigma_{s+1}^- \) or \( \sigma_{s+1}^- \), and \( \beta_i \) will be the longest initial segment which is in \( \text{dom}(L_s)^* \), which is \( \beta_i = \sigma_{s+1}^- \).

Now we will define \( T \) by an alternating inductive definition. Suppose that so far we have defined I’s response \( G_0 \supset F_{s+1}, G_1, \ldots, G_{n-1} \) when II plays \((\bar{c}_0, \alpha_1), \ldots, (\bar{c}_{n-2}, \alpha_{n-1})\). We will also have defined a corresponding play \((\varnothing, \sigma_{s+1}^-), (ab\bar{d}_0, \beta_1), \ldots, (d_{n-2}, \beta_{n-1})\) by II in \( G_s^b \). Let \( F' \supset F_s, G, H_1, \ldots, H_{n-1} \) be II’s response to this using \( S \). We will have ensured that if \( \alpha_i = \sigma_{s+1}^- \) then \( H_i \triangleq 1 G_i \), and otherwise that \( H_i = G_i \) and for each \( i \) and \( \bar{d}_i = \bar{c}_i \). Recall that \( G \cdot ab = F_{s+1} \).

It is now II’s turn, and suppose that II plays \((\bar{c}_{n-1}, \alpha_n)\), and that this is a basic play by II. We must define \( \bar{d}_{n-1} \) and then define I’s response \( G_n \). Note that if \( \alpha_i = \alpha_{i+1} = \sigma_{s+1}^- \), then \( i + 1 = n \) by \((B1)\).

There are four cases depending on the values of \( \alpha_{n-1} \) and \( \alpha_n \). When neither of \( \alpha_{n-1} \) nor \( \alpha_n \) are \( \sigma_{s+1}^- \), then we can just follow \( S \). Then we have three more cases depending on whether one (or both) of \( \alpha_{n-1} \) and \( \alpha_n \) are \( \sigma_{s+1}^- \).

Case 1: \( \alpha_{n-1}, \alpha_n \not= \sigma_{s+1}^- \). In this case we can simply follow \( S \). Let \( \bar{d}_{n-1} = \bar{c}_{n-1} \). Let \( G_n = H_n \) be I’s response, using \( S \), to

\[(\varnothing, \sigma_{s+1}^-), (ab \cdot \bar{d}_0, \beta_1), \ldots, (d_{n-1}, \beta_n).\]

Case 2: \( \alpha_{n-1} = \sigma_{s+1}^- \) and \( \alpha_n \not= \sigma_{s+1}^- \). We have \( H_{n-1} \triangleq 1 G_{n-1} \). Let \( \bar{d}_{n-1} \) be such that \( G_{n-1} \bar{c}_{n-1} \triangleq 0 H_{n-1} \bar{d}_{n-1} \). Now let \( G_n = H_n \) be I’s response, using \( S \), to

\[(\varnothing, \sigma_{s+1}^-), (ab \cdot \bar{d}_0, \beta_1), \ldots, (d_{n-1}, \beta_n).\]

Case 3: \( \alpha_{n-1} \not= \sigma_{s+1}^- \) and \( \alpha_n = \sigma_{s+1}^- \). Let \( \bar{d}_{n-1} = \bar{c}_{n-1} \). Now let \( H_n \) be I’s response, using \( S \), to

\[(\varnothing, \sigma_{s+1}^-), (ab \cdot \bar{d}_0, \beta_1), \ldots, (d_{n-1}, \beta_n).\]

Note that since \( \alpha_n = \sigma_{s+1}^- \), \( \beta_n = \sigma_{s+1}^- \). Then \( H_n \supset F_{s+1} = Gab \), say \( H_n = Gab \bar{e} \). Then using the fact that \( b \) is free over \( Ga \), choose \( b' \in R \) and \( \bar{e}' \) such that \( H_n \triangleq 1 Gab \bar{e}' \). Then set \( G_n = Gab \bar{e}' \).

Case 4: \( \alpha_{n-1} = \alpha_n = \sigma_{s+1}^- \). Set \( G_n = G_{n-1} \bar{c}_{n-1} \). Since \( \alpha_{n-1} = \alpha_n \), there are no longer basic plays than this, and so we do not need to define \( \bar{d}_{n-1} \).
It is tedious but easy to see that none of these plays by I is a losing play. Hence I has a winning strategy \( T \) in \( G_{s+1}^b \).

### 6.6.8 The Case \( \sigma_{s+1} \in T_\omega \)

By Lemma 6.6.8, there are only finitely many basic lists on which II’s basic plays are based. Let \( b^1, \ldots, b^m \) be these basic lists, where \( b^i \) is the list \( \beta^i_1, \beta^i_2, \ldots, \beta^i_{\ell^i} \).

We must add a tuple \( \bar{b} \) to the image of our partial isomorphism, setting \( F_{s+1} = G\bar{b} \). The tuple \( \bar{b} \) will be made up of tuples \( \bar{b}_1, \ldots, \bar{b}_m \) from \( C \). Let \( \bar{\epsilon}_1, \ldots, \bar{\epsilon}_m \) be tuples of elements in \{−1, 1\} be such that \( \bar{b}_j \in R^{\bar{\epsilon}_j} \). If we set \( m_{s+1}(\sigma) = \bar{\epsilon}_1 \cdots \bar{\epsilon}_m \) and \( k_{s+1}(\sigma_{s+1}) = |\bar{b}_1| + \cdots + |\bar{b}_m| \), then (C1)-(C4) will be satisfied. So we just have to make sure that \( \bar{b}_1, \ldots, \bar{b}_m \) are chosen such that (WS) is satisfied, that is, I has a winning strategy for the game \( G_{s+1} \) (or, equivalently, \( G_{s+1}^b \)).

To choose the tuples \( \bar{b}_1, \ldots, \bar{b}_m \), we will define a new class of games. For each \( r \leq m \), \( \bar{b}_1, \ldots, \bar{b}_r \) tuples of elements of \( C \), and tuples \( \bar{\epsilon}_1, \ldots, \bar{\epsilon}_r \) which are tuples of 1s and −1s (with \( \epsilon_i \) the same length as \( b_i \)), we have a game \( H(\bar{b}_1, \bar{\epsilon}_1; \ldots; \bar{b}_r, \bar{\epsilon}_r) \). We do not require, for the definition of the game, that \( \bar{b}_j \in R^{\bar{\epsilon}_j} \). We allow the case \( r = 0 \); that is, \( H(\varnothing) \) is a game. \( H(\varnothing) \) will be essentially the game \( G_s^b \), and we will be able to easily turn a winning strategy for I in \( G_s^b \) into a winning strategy for I in \( H(\varnothing) \). Then we will use the winning strategy for \( H(\varnothing) \) to show that we can choose \( \bar{b}_1 \) and \( \bar{\epsilon}_1 \) so that we have a winning strategy for I in \( H(\bar{b}_1, \bar{\epsilon}_1) \), and we will use that winning strategy to show that we can choose \( \bar{b}_2 \) and \( \bar{\epsilon}_2 \) so that we have a winning strategy for I in \( H(\bar{b}_1, \bar{\epsilon}_1; \bar{b}_2, \bar{\epsilon}_2) \), and so on. Eventually we will be able to choose \( \bar{b}_1, \ldots, \bar{b}_m \) and \( \bar{\epsilon}_1, \ldots, \bar{\epsilon}_m \) so that we have a winning strategy for \( H(\bar{b}_1, \bar{\epsilon}_1; \ldots; \bar{b}_m, \bar{\epsilon}_m) \). We will choose the tuples so that if we make the definition of \( F_{s+1}, m_{s+1}, \) and \( k_{s+1} \) above, this winning strategy will immediately yield a winning strategy for I in \( G_{s+1}^b \).

Here are the rules for the game \( H(\bar{b}_1, \bar{\epsilon}_1; \ldots; \bar{b}_r, \bar{\epsilon}_r) \). I begins by playing a partial isomorphism \( G_0 \) which extends \( G\bar{b}_1 \cdots \bar{b}_r \). Then, II and I alternate, with II playing a tuple of elements from \( C \) and a string in \( \text{dom}(L_1)^* \), and I playing a partial isomorphism. As in \( G_{s+1}^b \), II must make a play which is based on one of the basic lists. I can lose by violating one of (C1′)-(C4′) for \( \sigma \in \text{dom}(L_1)^* \) (not \( \text{dom}(L_{s+1}^\ast) \), since we have not yet defined \( m_{s+1}(\sigma_{s+1}) \) and \( k_{s+1}(\sigma_{s+1}) \)). I can also lose by violating (At′)-(Ext2′) using \( L = L_{s+1} \). Finally, I must ensure that

(CB1′): whenever I is responding to a basic play \((\bar{c}_0, \beta^1_1), \ldots, (\bar{c}_{\ell-1}, \beta^1_{\ell-1})\) based on \( b^i \) for \( i \leq r \), and \( \beta^i_{\ell_i} = \sigma_{s+1}^\ast \delta \), using a partial isomorphism \( H \), then

\[
H(n + |\bar{b}_1| + \cdots + |\bar{b}_{i-1}| + 1), \ldots, H(n + |\bar{a}_1| + \cdots + |\bar{a}_{i-1}| + |\bar{a}_i|) \notin R^{\bar{\epsilon}_i}.
\]

This condition (CB1′) will be what is required to ensure (C4′) for \( \sigma = \sigma_{s+1} \).

The choice of \( \bar{b}_1, \ldots, \bar{b}_m \) involves three lemmas. We will begin by stating the first two lemmas and using them to prove the induction step of the third lemma before returning to the proofs of the first two.
**Lemma 6.6.11.** Suppose that I wins \( \mathcal{H}(\bar{b}_1, \bar{e}_1; \ldots; \bar{b}_r, \bar{e}_r) \). Let \( \bar{c} \) be such that I's first play using their winning strategy for this game is \( \text{Gab}_1 \cdots \bar{b}_r \bar{c} \). Let \( \bar{\nu} \) be such that \( \bar{c} \in R^\nu \). Let \( x \in \mathcal{C} \). Then for one of \( \bar{c} \) using their winning strategy for this game is \( \text{Gab}_1 \cdots \bar{b}_r \bar{c} \).

**Lemma 6.6.12.** Let \( \bar{c} \) and \( \bar{\nu} \) be as above. Let \( \nu = 1 \) or \( \nu = -1 \). If I wins the game \( \mathcal{H}(\bar{b}_1, \bar{e}_1; \ldots; \bar{b}_r, \bar{e}_r; \bar{c}x, \bar{\nu}i) \) and

\[
\text{Gab}_1 \cdots \bar{b}_r \bar{c}x \leq_1 \text{Gab}_1 \cdots \bar{b}_r \bar{c}y,
\]

then I wins \( \mathcal{H}(\bar{b}_1, \bar{e}_1; \ldots; \bar{b}_r, \bar{e}_r; \bar{c}y, \bar{\nu}i) \).

**Lemma 6.6.13.** There are \( \bar{b}_1, \ldots, \bar{b}_m \in \mathcal{C} \) and \( \bar{e}_1, \ldots, \bar{e}_m \) with \( (\bar{b}_1, \ldots, \bar{b}_m) \in R(\bar{e}_1, \ldots, \bar{e}_m) \) such that I wins the game \( \mathcal{H}(\bar{b}_1, \bar{e}_1; \ldots; \bar{b}_m, \bar{e}_m) \).

*Proof.* The proof is by induction on \( r \). The first step is to show that I has a winning strategy for the game \( \mathcal{H}(\varnothing) \). Recall that I has a winning strategy \( S \) for \( \mathcal{G}_s \), and \( S \) responds to \( (\varnothing, \sigma_{s+1} \hat{\eta}) \) with \( G \). To any play \( P = (\bar{d}_0, \alpha_1), \ldots, (\bar{d}_{\ell-1}, \alpha_\ell) \) by II in \( \mathcal{H}(\varnothing) \), associate the play \( P' = (\varnothing, \sigma_{s+1} \hat{\eta}), (\bar{a}\bar{d}_0, \alpha_1), \ldots, (\bar{d}_{\ell-1}, \alpha_\ell) \) in \( \mathcal{G}_s \). A response \( F_s, G, H_1, \ldots, H_\ell \) to \( P' \) according to \( S \) which does not lose \( \mathcal{G}_s \), gives rise to a response \( Ga, H_1, \ldots, H_\ell \) which does not lose \( \mathcal{H}(\varnothing) \) (the conditions for II losing in each game are essentially the same; the only difference is that \( \mathcal{G}_s \) involves an additional turn at the beginning). So the winning strategy \( S \) for \( \mathcal{G}_s \) gives rise to a winning strategy for \( \mathcal{H}(\varnothing) \).

Now for the induction step, suppose that we have tuples \( \bar{b}_1, \ldots, \bar{b}_r \) and \( \bar{e}_1, \ldots, \bar{e}_r \) such that I wins \( \mathcal{H}(\bar{b}_1, \bar{e}_1; \ldots; \bar{b}_r, \bar{e}_r) \) and \( \bar{b}_r \in R^\nu \). Let \( \bar{c} \) and \( \bar{\nu} \) be as in the previous lemmas, that is, I's winning strategy begins by playing \( \text{Gab}_1 \cdots \bar{b}_r \bar{c} \). Now choose an element \( x \in R \) which is 2-free over \( \text{Gab}_1 \cdots \bar{b}_r \bar{c} \). Then choose \( y \in R \) such that

\[
\text{Gab}_1 \cdots \bar{b}_r \bar{c}x \leq_1 \text{Gab}_1 \cdots \bar{b}_r \bar{c}y
\]

Now by Lemma 6.6.11, I wins \( \mathcal{H}(\bar{b}_1, \bar{e}_1; \ldots; \bar{b}_r, \bar{e}_r; \bar{c}x, \bar{\nu}i) \) for either \( \nu = 1 \) or \( \nu = -1 \). If \( x \in R^\nu \) (i.e., \( \nu = -1 \)), then we are done. Otherwise, if \( \nu = 1 \), then by Lemma 6.6.12, I also wins \( \mathcal{H}(\bar{b}_1, \bar{e}_1; \ldots; \bar{b}_r, \bar{e}_r; \bar{c}y, \bar{\nu}i) \). But \( y \in R \) and \( \nu = 1 \), so this completes the induction step.

The winning strategy for \( \mathcal{H}(\bar{b}_1, \bar{e}_1; \ldots; \bar{b}_m, \bar{e}_m) \) is a winning strategy for \( \mathcal{G}_s^b \). We need to check that this strategy satisfies \( \text{(C3)}^* \) and \( \text{(C4)}^* \) for \( \sigma_{s+1} \). The first follows from the choice of \( m_{s+1} \), and the fact that once II plays some string other than \( \sigma_{s+1} \), they can never again play \( \sigma_{s+1} \). The second is because the winning strategy for \( \mathcal{H}(\bar{b}_1, \bar{e}_1; \ldots; \bar{b}_m, \bar{e}_m) \) satisfies \( \text{(CB)}^* \) for each basic list.

Now we return to the omitted proofs.

*Proof of Lemma 6.6.11.* In order to simplify the notation, we will denote by \( \mathcal{G} \) the game \( \mathcal{H}(\bar{b}_1, \bar{e}_1; \ldots; \bar{b}_r, \bar{e}_r) \) and by \( \mathcal{G}(i) \) the game \( \mathcal{H}(\bar{b}_1, \bar{e}_1; \ldots; \bar{b}_r, \bar{e}_r; \bar{c}x, \bar{\nu}i) \) for \( \nu = 1 \) and \( \nu = -1 \). We must show that I has a winning strategy for either \( \mathcal{G}(-1) \) or \( \mathcal{G}(1) \).
Suppose that I does not have a winning strategy for $G(1)$. Then II has a winning strategy for $G(1)$. We also know that I has a winning strategy in $G$. We will show that I has a winning strategy for $G(-1)$.

The strategy for I will try to do two things. First, it will try to be the same as I’s winning strategy for $G$. Then the only way for II to lose while using such a strategy will be for II to use a basic play based on $b^{r+1}$, the $r+1$ st basic list, and to have I fail to satisfy $(\text{CB}^\dagger)$ for this basic list. The second thing that I will try to do is, if II follows the basic list $b^{r+1}$, to try and force II to use their winning strategy from $G(1)$. Since this is a winning strategy for II, I will fail to satisfy $(\text{CB}^\dagger)$ in $G(1)$ for $b^{r+1}$. But failing to satisfy $(\text{CB}^\dagger)$ in $G(1)$ for some basic list is the same as satisfying $(\text{CB}^\dagger)$ in $G(-1)$ for that basic list. So I will win $G(-1)$.

Let $S$ be I’s winning strategy for $G$, and $T$ be II’s winning strategy for $G(1)$. We will define $S'$, a winning strategy for I in $G(-1)$.

I must play first. The first move in $G$ according to $S$ is $H_0 = G\bar{a}_1\cdots\bar{a}_r\bar{c}$. Let $(\bar{d}_0, \beta_1)$ be II’s response to $H_0x$ according to their winning strategy $T$ for $G(1)$. Then the strategy $S'$ for $G(-1)$ will play $G_0 = H_0\bar{b}\bar{d}_0$ as I’s initial play. Note that II has not actually played $(\bar{d}_0, \beta_1)$; I has just looked ahead at what II would play if they were following the strategy $T$.

Now II must actually respond to $G_0$. Suppose that they respond with $(\bar{e}_0, \alpha_1)$. If $\alpha_1 \neq \beta_1^{r+1}$, then since $\alpha_1$ is not part of the basic list $\beta^{r+1}$, the winning and losing conditions are the same as in $G$; have I respond to $(\bar{e}_0, \alpha_1)$ as they would, using $S$, to $(\bar{d}_0\bar{e}_0, \alpha_1)$ in $G$. After this, I just continues to use $S$ to win.

If instead $\alpha_1 = \beta_1^{r+1}$, then let $H_1$ be I’s response, using $S$, to $(\bar{d}_0\bar{e}_0, \beta_1^{r+1})$. Let $(\bar{d}_1, \beta_2)$ be II’s response to $H_1$ using the strategy $T$. Then once again the strategy $S'$ will tell I to play $G_1 = H_1\bar{d}_1$.

If I ever plays an $\alpha_i \neq \beta_i^{r+1}$, then II can win by following the strategy $S$. Otherwise, $S'$ will have I play $G_0, G_1, \ldots$ in response to II playing $(\bar{e}_0, \beta_1), (\bar{e}_1, \beta_2), \ldots$. There will be $H_0, H_1, \ldots$ which are plays according to $S$ in response to II playing $(\bar{d}_0\bar{e}_0, \beta_1), (\bar{d}_1\bar{e}_1, \beta_2), \ldots$. Moreover, we will have $H_i \in G_i$.

Since $H_0, H_1, \ldots$ is a winning play against $(\bar{d}_0\bar{e}_0, \beta_1), (\bar{d}_1\bar{e}_1, \beta_2), \ldots$ in $G$, we can see that because all for the conditions $(\text{C}1^\dagger)$-$(\text{Ext}2^\dagger)$ are satisfied for $H_0, H_1, \ldots$, they are also satisfied for $G_0, G_1, \ldots$ (the extra elements $\bar{d}_i$ in $G_i$ but not $H_i$ are included in $(\text{At}^\dagger)$, $(\text{Ext}1^\dagger)$, and $(\text{Ext}2^\dagger)$ as the tuples which II plays). Also, $(\text{CB}^\dagger)$ is satisfied for the basic lists $b^1, \ldots, b^r$.

So the only way $G_0, G_1, \ldots$ could be a losing play against $(\bar{e}_0, \beta_1), (\bar{e}_1, \beta_2), \ldots$ is if $\beta_1, \beta_2, \ldots$ is the basic play $b^{r+1}$ and $(\text{CB}^\dagger)$ fails for this basic play. $H_0, H_1, \ldots$ is a play by I against which II wins in $G(1)$ using the strategy $T$. Then $(\text{CB}^\dagger)$ fails in $G(1)$ for the basic list $b^{r+1}$, and so $(\text{CB}^\dagger)$ must be satisfied in $G(-1)$ for this basic list.

So I wins this play of $G(-1)$, and $S'$ is a winning strategy.

To prove Lemma 6.6.12, we first need a quick technical lemma.

**Lemma 6.6.14.** Suppose that $\bar{x} \leq_1 \bar{x}'$. Let $\bar{y}$ be a tuple so that for no $y \in \bar{y}$ is there a $y'$ with $\bar{xy} \leq_1 \bar{x}'y'$. Then for each tuple $\bar{z}'$, there is $\bar{z}$ such that $\bar{x}'\bar{z}' \leq_1 \bar{x}\bar{z}$ and moreover $\bar{z}$ is disjoint from $\bar{y}$.
\textbf{Proof.} For each \( y \in \bar{y} \), and each \( y' \in C \), there is some existential fact true about \( y' \) which is not true of \( y \). Let \( \bar{u}' \) be tuple of elements witnessing these existential formulas for each \( z' \in \bar{z}' \). Then there are \( \bar{z} \) and \( \bar{u} \) such that \( \bar{z}' \bar{u}' \leq_0 \bar{z} \bar{u} \). So \( \bar{z}' \bar{z} \leq_0 \bar{z} \bar{z} \) and each \( z \in \bar{z} \) satisfies an existential formula which no \( y \in \bar{y} \) satisfies. Hence \( \bar{z} \) is disjoint from \( \bar{y} \).

\textit{Proof of Lemma 6.6.12.} Once again we will simplify the notation. For \( z = x \) or \( z = y \), let \( G(z) \) be the game \( H(\bar{b}_1, \bar{e}_1; \ldots; \bar{b}_r, \bar{e}_r; \bar{c}z, \bar{v}u) \).

Let \( T \) be a winning strategy for I in \( G(x) \). We need to find a winning strategy \( S \) for I in \( G(y) \). We will use the fact that

\[ G\bar{b}_1 \cdots \bar{b}_r \bar{c}x \leq_1 G\bar{b}_1 \cdots \bar{b}_r \bar{c}y \]

to convert \( T \) into the desired strategy \( S \).

Let \( H_0 \supset G\bar{b}_1 \cdots \bar{b}_r \bar{c}x \) be the initial play for I according to \( T \). Let \( \bar{u} \) be the tuple of elements such that \( H_0 = G\bar{b}_1 \cdots \bar{b}_r \bar{c}x \bar{u} \).

Let \( I = \{i_1, \ldots, i_n\} \) be a maximal set of indices in \( \bar{u} \) such that there is a tuple \( \bar{v} \) such that

\[ G\bar{b}_1 \cdots \bar{b}_r \bar{c}x \bar{u} \bar{t} \leq_1 G\bar{b}_1 \cdots \bar{b}_r \bar{c}y \bar{v} \]

where \( \bar{u} \bar{t} \) is \( (u_{i_1}, \ldots, u_{i_n}) \). Let \( J \) be the rest of the indices.

The first play according to \( S \) will be \( G_0 = G\bar{b}_1 \cdots \bar{b}_r \bar{c}y \bar{v} \). We have \( H_0^I \leq_1 G_0 \) where \( H_0^I \) denotes \( H_0 \) with the entries at indices in \( J \) removed.

Now suppose that \( \Pi \) responds with \((d_0, \alpha_1)\). If \( \alpha_1 = \sigma_{s+1} \), then I can play \( G_0 \bar{d}_0 \) in response to win (by property (B1) of basic lists). Otherwise, \( \alpha_1 \neq \sigma_{s+1} \).

Using the fact that \( H_0^I \leq_1 G_0 \), choose \( \bar{e}_0 \) such that \( G_0 \bar{d}_0 \leq_0 H_0^I \bar{e}_0 \). By the previous lemma, \( \bar{e}_0 \) is disjoint from \( H_0^I \). Let \( H_1 \) be the response, according to \( T \), to \((\bar{e}_0, \alpha_1)\). Let \( s \) be the permutation which moves those entries of \( H_1 \) with indices in \( J \) to the end (the permutation \( s \) fixes anything not in the domain of \( H_1 \), so that for example if \( H' \supset H_1 \), then applying the permutation \( s \) to \( H' \) does not move the indices \( J \) to the end of \( H' \), but rather to somewhere in the middle). Then \( S \) will tell I to respond to \((\bar{d}_0, \alpha_1)\) with \( H_1^s \), the application of the permutation \( s \) to \( H_1 \).

To any further play \((\bar{e}_1, \alpha_2), (\bar{e}_2, \alpha_3), \ldots\), \( S \) will respond in the same way as \( T \), except that it will again apply the permutation \( s \).

The games \( G(x) \) and \( G(y) \) are the same, except for the initial move; otherwise, the ways in which I can lose are the same. The permutation \( s \) does not affect any of the conditions, since it only permutes indices which do not do any coding (i.e. above everything in the image of \( L_s \), and also above the position of \( x \) and \( y \) in \( G_0 \) and \( H_0 \) respectively). Since \( G_0 \bar{d}_0 \leq_0 H_0^I \bar{e}_0 \leq_0 H_1^s \), and if \( H_i \leq_0 H_{i+1} \) then \( H_i^s \leq_0 H_{i+1}^s \), \((\text{At}^{\dagger})\) still holds. And as none of \( \alpha_1, \ldots, \alpha_r \) are \( \sigma_{s+1} \), \((\text{Ext1}^{\dagger})\) is not affected by changing the change from \( H_0 \) to \( G_0 \), and \((\text{Ext2}^{\dagger})\) is also not affected by the application of the permutation \( s \). Thus I has a winning strategy for \( G(y) \). \( \square \)
6.7 Further Questions

In this section we will list some of the unresolved questions from our investigation. This is a new investigation and so there are many questions to be answered. We have seen that there are many nice properties that degree spectra on a cone must have. Many degree spectra are well-known classes of degrees and satisfy many “fullness” properties. But on the other hand, there are some interesting degree spectra that are not so nicely behaved, like the incomparable degree spectra from Theorem 6.1.6 and the relation on \((\omega, <)\) from 6.5.19. The general question is: to what extent do the degree spectra avoid pathological behaviour? Of course, there are many specific questions about particular types of pathological behaviour and many questions arising directly out of results in this paper. We will take the opportunity to list some of them here.

In Section 6.4 we gave a condition, involving d-free tuples, which was equivalent to being intrinsically of c.e. degree, but the equivalence only held for relations which are relatively intrinsically d.c.e. (see Proposition 6.4.5). A relation which is not intrinsically \(\Delta^0_2\) cannot be intrinsically of c.e. degree. We ask:

**Question 6.7.1.** Which intrinsically \(\Delta^0_2\) but not relatively intrinsically d.c.e. relations are intrinsically of c.e. degree?

Also in Section 6.4, we gave an example of two relations with incomparable degree spectra on a cone, but whose degree spectra are strictly contained within the d.c.e. degrees and strictly contain the c.e. degrees (see Theorem 6.1.6, Proposition 6.4.15, and Proposition 6.4.17). We ask whether it is possible to find other such degree spectra:

**Question 6.7.2.** How many different possible degree spectra on a cone are there strictly containing the c.e. degrees and strictly contained in the d.c.e. degrees? How are they ordered?

Many of the degree spectra on a cone have a “name,” that is, some sort of description of degrees which relativizes. For example, the \(\Delta^0_\alpha\) degrees, the \(\Sigma^0_\alpha\) degrees, \(\alpha\)-c.e. degrees, and \(\alpha\)-CEA degrees. We do not know of any such description of the degree spectra from Proposition 6.4.15 and Proposition 6.4.17. In general, one would hope that any degree spectrum on a cone has a nice description of some form.

**Question 6.7.3.** Is there a good degree-theoretic description of the degree spectra from Examples 6.4.13 and 6.4.14?

If one can give a good degree-theoretic description of these degree spectra, then one would have added a new natural class of degrees.

In Section 6.5, we show that every relation on \((\omega, <)\) which is intrinsically \(\alpha\)-c.e. is intrinsically of c.e. degree (see Propositions 6.5.13 and 6.5.14). It might, however, be possible to have a relation which is intrinsically of \(\alpha\)-c.e. degree but not intrinsically \(\alpha\)-c.e.

**Question 6.7.4.** Is there a computable relation on the standard copy of \((\omega, <)\) which is not intrinsically of c.e. degree, but is intrinsically of \(\alpha\)-c.e. degree for some fixed ordinal \(\alpha\)?
In Theorem 6.5.19, we show that there is a relation \( R \) on \((\omega, <)\) such that either the degree spectrum of \( R \) on a cone is strictly contained between the c.e. degrees and the \( \Delta^0_2 \) degrees, or \( R \) has degree spectrum \( \Delta^0_0 \) but not uniformly. It would be interesting to know if either of these behaviours is possible.

**Question 6.7.5.** Is there a relation on \((\omega, <)\) whose degree spectrum on a cone is strictly contained between the c.e. degrees and the \( \Delta^0_2 \) degrees?

**Question 6.7.6.** Is there a relation with \( \text{dgSp}_{rel} = \Delta^0_2 \) but not uniformly? Is there such a relation on \((\omega, <)\)?

In Section 6.6 we proved a “fullness” result by showing that any degree spectra on a cone which strictly contains the \( \Delta^0_2 \) degrees contains all of the 2-CEA degrees. Fullness results are very interesting because they show that degree spectra must contain all of the degrees of a particular type, and hence can provide a good description of the degree spectrum. Our result for 2-CEA degrees is an answer to a question of Ash and Knight from [AK95] and [AK97]. The general question, when stated in our framework, is as follows:

**Question 6.7.7.** If a degree spectrum on a cone strictly contains the \( \Delta^0_3 \) degrees, must it contain the 3-CEA degrees? What about for general \( \alpha \)?

Recall that a positive answer to this question implies a positive answer to the following questions of Montalbán which first appeared in [Wri13]:

**Question 6.7.8 (Montalbán).**

1. Is it true that for any relation \( R \), for all degrees \( d \) on a cone, \( \text{dgSp}(R)_{\leq d} \) has a maximal element?

2. Does the function which takes a degree \( d \) to the maximal element of \( \text{dgSp}(R)_{\leq d} \) satisfy Martin’s conjecture?

We also add the question:

3. Is this function uniformly degree invariant (in the sense of the footnote at the end of Section 6.2)?

In [Sla05], Slaman defines a \( \Sigma \)-closure operator to be a map \( M : 2^\omega \to 2^{2^\omega} \) such that:

1. for all \( X, X \in M(X) \),
2. for all \( X \) and \( Y, Z \in M(X) \), \( Y \oplus Z \in M(X) \),
3. for all \( X \) and \( Y \in M(X) \), if \( Z \equiv_T Y \) then \( Z \in M(X) \),
4. for all \( X \leq_T Y, M(X) \in M(Y) \).
Note that for any relation $R$ on a structure $A$, the map which takes a set $X$ to $\text{dgSp}(A, R)_{\leq d}$ (where $d$ is the degree of $X$) satisfies (1), (3), and (4) of this definition.

Slaman shows:

**Theorem 6.7.9** (Slaman [Sla05, Theorem 5.2]). Let $M$ be a Borel $\Sigma$-closure operator. If, on a cone, $M(X) \notin \Delta_0^0(X)$, then there is a cone on which $M(X)$ contains all of the sets which are CEA in $X$.

One can see this as an analogue of Harizanov’s Theorem 6.1.2. We ask whether the degree spectrum on a cone is a $\Sigma$-closure operator:

**Question 6.7.10.** On a cone, do the degree spectra form an upper semi-lattice under joins?

In Section 6.2, we defined the alternate degree spectrum

$$\text{dgSp}^*(A, R)_{\leq d} = \{d(R^B) : (B, R^B) \text{ is an isomorphic copy of } (A, R) \text{ with } B \leq_T d\}$$

and used this to define the alternate degree spectrum on a cone, $\text{dgSp}_{rel}^*$. In Section 6.3, we showed that Harizanov’s Theorem 6.1.2 on c.e. degrees (in the form of Corollary 6.2.7) holds for $\text{dgSp}_{rel}^*$. The general question is whether anything we can prove about $\text{dgSp}_{rel}^*$ is also true of $\text{dgSp}_{rel}^*$. More formally:

**Question 6.7.11.** Is it always the case that restricting $\text{dgSp}^*(A, R)_{\leq d}$ to the degrees above $d$ gives $\text{dgSp}(A, R)_{\leq d}$?
Chapter 7

Computable Categoricity

The results presented in this chapter appeared in [CHT]. They are joint work with Barbara Csima and appear here with her permission.

7.1 Introduction

In this paper, we will consider the complexity of computing isomorphisms between computable copies of a structure after relativizing to a cone. By relativizing to a cone, we are able to consider natural structures, that is, those structures which one might expect to encounter in normal mathematical practice. The main result of this paper is a complete classification of the natural degrees of categoricity: the degrees of categoricity of natural computable structures. Unless otherwise stated, all notation and conventions will be as in the book by Ash and Knight [AK00]. We consider countable structures over at most countable languages.

Recall that a computable structure is said to be computably categorical if any two computable copies of the structure are computably isomorphic. As an example, consider the rationals as a linear order; the standard back-and-forth argument shows that the rationals are computably categorical. It is easy to see, however, that not all computable structures are computably categorical. The natural numbers as a linear order is one example.

There has been much work in computable structure theory dedicated to characterizing computable categoricity for various classes of structures (e.g., a linear order is computably categorical if and only if it has at most finitely many successivities [GD80], [Rem81]). For those structures that are not computably categorical, what can we say about the isomorphisms between computable copies, or more generally, about the complexities of the isomorphisms relative to that of the structure?

We can extend the definition of computable categoricity as follows.

Definition 7.1.1. A computable structure $\mathcal{A}$ is $d$-computably categorical if for all computable $\mathcal{B} \cong \mathcal{A}$ there exists a $d$-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.
It is easy to see, for example, that the natural numbers as a linear order, \( \mathbb{N} \), is \( 0' \)-computably categorical. Indeed, it is also easy to construct a computable copy \( \mathcal{A} \) of \( \mathbb{N} \) such that every isomorphism between \( \mathcal{A} \) and \( \mathbb{N} \) computes \( 0' \). Thus \( 0' \) is the least degree \( d \) such that \( \mathbb{N} \) is \( d \)-computably categorical. This motivates the following definitions.

**Definition 7.1.2.** We say a computable structure \( \mathcal{A} \) has *degree of categoricity* \( d \) if

1. \( \mathcal{A} \) is \( d \)-computably categorical.
2. If \( \mathcal{A} \) is \( c \)-computably categorical, then \( c \geq d \).

**Definition 7.1.3.** We say that a Turing degree \( d \) is a *degree of categoricity* if there exists a computable structure \( \mathcal{A} \) with degree of categoricity \( d \).

The notion of a degree of categoricity was first introduced by Fokina, Kalimullin and R. Miller [FKM10]. They showed that if \( d \) is d.c.e. (difference of c.e.) in and above \( 0^{(n)} \), then \( d \) is a degree of categoricity. They also showed that \( 0^{(\omega)} \) is a degree of categoricity. For the degrees c.e. in and above \( 0^{(n)} \), they exhibited rigid structures capturing the degrees of categoricity. In fact, all their examples had the following, stronger property.

**Definition 7.1.4.** A degree of categoricity \( d \) is a *strong* degree of categoricity if there is a structure \( \mathcal{A} \) with computable copies \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) such that \( d \) is the degree of categoricity for \( \mathcal{A} \), and every isomorphism \( f : \mathcal{A}_0 \to \mathcal{A}_1 \) satisfies \( \deg(f) \geq d \).

In [CFS13], Csima, Franklin and Shore showed that for every computable ordinal \( \alpha \), \( 0^{(\alpha)} \) is a strong degree of categoricity. They also showed that if \( \alpha \) is a computable successor ordinal and \( d \) is d.c.e. in and above \( 0^{(\alpha)} \), then \( d \) is a strong degree of categoricity.

In [FKM10] it was shown that all strong degrees of categoricity are hyperarithmetic, and in [CFS13] it was shown that all degrees of categoricity are hyperarithmetic. There are currently no examples of degrees of categoricity that are not strong degrees of categoricity. Indeed, we do not even have an example of a structure that has a degree of categoricity but not strongly.

All known degrees of categoricity satisfy \( 0^{(\alpha)} \leq d \leq 0^{(\alpha+1)} \) for some computable ordinal \( \alpha \). So in particular, all known non-computable degrees of categoricity are hyperimmune. In [AC16], Anderson and Csima showed that no non-computable hyperimmune-free degree is a degree of categoricity. They also showed that there is a \( \Sigma^0_2 \) degree that is not a degree of categoricity, and that if \( G \) is 2-generic (relative to a perfect tree), then \( \deg(G) \) is not a degree of categoricity. The question of whether there exist \( \Delta^0_2 \) degrees that are not degrees of categoricity remains open.

Turning to look at the question of degree of categoricity for a given structure, R. Miller showed that there exists a field that does not have a degree of categoricity [Mil09], and Fokina, Frolov, and Kalimullin [FFK16] showed that there exists a rigid structure with no degree of categoricity.
In this paper, we claim that the only natural degrees of categoricity are the $\Delta^0_\alpha$-complete degrees for some computable ordinal $\alpha$. By a natural degree of categoricity, we mean the degree of categoricity of a natural structure.

What do we mean by natural? By a natural structure, we mean one which might show up in the normal course of mathematics; we will not include a structure which has been constructed, say via diagonalization, to have some computability-theoretic property as a natural structure. So, for example, we will not consider a structure which is computably categorical but not relatively computably categorical to be a natural structure. On the other hand, the infinite-dimensional vector space is a natural structure. Of course, this is not a rigorous definition. Instead, we note that arguments involving natural structures tend to relativize, and so a natural structure will have property $P$ if and only if it has property $P$ on a cone (i.e., there is a Turing degree $d$ such that for all $c \geq d$, $P$ holds relative to $c$). Thus by considering arbitrary structures on a cone, we can prove results about natural structures.

The second author previously considered degree spectra of relations on a cone [HTb]. McCoy [McC02] has also shown that on a cone, every structure has computable dimension 1 or $\omega$. Here, we give an analysis of degrees of categoricity along similar lines.

Our main theorem is:

**Theorem 7.1.5.** Let $\mathcal{A}$ be a countable structure. Then, on a cone: $\mathcal{A}$ has a strong degree of categoricity, and this degree of categoricity is $\Delta^0_\alpha$-complete.

There are three important parts to this theorem: first, that every natural structure has a degree of categoricity; second, that this degree of categoricity is a strong degree of categoricity; and third, that the degree of categoricity is $\Delta^0_\alpha$-complete. The ordinal $\alpha$ is the least ordinal $\alpha$ such that $\mathcal{A}$ is $\Delta^0_\alpha$ categorical on a cone. This is related to the Scott rank of $\mathcal{A}$ under an appropriate definition of Scott rank [Mon15a]: $\alpha$ is the least ordinal $\alpha$ such that $\mathcal{A}$ has a $\Sigma^0_{\alpha+2}$ Scott sentence if $\alpha$. (While $\alpha$ may not be computable, every ordinal is computable on some cone. The reader may be uncomfortable with talking about $\Delta^0_\alpha$-complete degrees on a cone when $\alpha$ is not computable; precisely what we mean will be clarified in Section 7.2.)

The construction of a structure with degree of categoricity some d.c.e. (but not c.e.) degree uses a computable approximation to the d.c.e. degree; this requires the choice of a particular index for the approximation, and hence the argument that the resulting structure has degree of categoricity d.c.e. but not c.e. does not relativize. By our theorem, there is no possible construction which does relativize. Moreover, our theorem says something about what kinds of constructions would be required to solve the open problems about degrees of categoricity, for example whether there is a 3-c.e. but not d.c.e. degree of categoricity, or whether there is a degree of categoricity which is not a strong degree of categoricity—the proof must be by constructing a structure which is not natural, using a construction which does not relativize.

The proof of Theorem 7.1.5 also gives an effectiveness condition which, if it holds of some computable structure, means that the conclusion of the theorem is true of that structure without relativizing to a cone. See, for example, the definition of the degree $e$ in Theorem
7.6.2. If $A$ is a computable structure, $\alpha$ is a computable ordinal and is least such that $A$ is $\Delta^0_\alpha$ categorical, and one can take $e = 0$ (which, in particular, means that it is effectively witnessed that $\alpha$ is the least ordinal such that $A$ is $\Delta^0_\alpha$ categorical), then $A$ has strong degree of categoricity $\Delta^0_\alpha$.

**Corollary 7.1.6.** The degrees of categoricity on a cone are the $\Delta^0_\alpha$-complete degrees for some $\alpha$.

Indeed, each $\Delta^0_\alpha$-complete degree is a degree of categoricity on a cone. To see this, examine the proof of Theorem 3.1 of [CFS13] showing that each $\Delta^0_\alpha$-complete degree is a degree of categoricity, and note that the proof relativizes.

In 2012, Csima, Kach, Kalimullin and Montalbán worked out a proof of Theorem 7.1.5 in the case where $A$ is $\Delta^0_\alpha$ categorical on a cone. That is, they showed that if $A$ is $\Delta^0_\alpha$ categorical on a cone, but not computably categorical on a cone, then $A$ has $\Delta^0_\alpha$-complete strong degree of categoricity on a cone. They also conjectured the general result at that time. The work was not written up. The result was later independently suggested by the second author. The proof of the general result require not only the machinery of $\alpha$-systems but also some new ideas. The proof of the special case is quite similar to a result of Harizanov [Har91, Theorem 2.5], who answered an analogous question for degree spectra of relations; the corresponding general case for degree spectra of relations is still open (though some more general results are proved in [HTb]). On the other hand, our proof of the general result for categoricity uses, in an integral way, certain facts about automorphisms (which were not used in the case of a $\Delta^0_2$ categorical structure), and so our proof does not work for degree spectra. We discuss in Section 7.6 the new difficulties which arise in the general case.

The second result of this paper concerns the difficulty of computing isomorphisms between two given copies $A$ and $B$ of a structure. We show that, on a cone, there is an isomorphism of least degree between $A$ and $B$, and that it is of c.e. degree.

**Theorem 7.1.7.** Let $A$ be a countable structure. Let $\alpha$ be such that $A$ is $\Delta^0_\alpha$ categorical on a cone. Then, on a cone: for every copy $B$ of $A$, there is a degree $d$ that is $\Sigma^0_{\alpha-1}$ in $B$ if $\alpha$ is a successor, or $\Delta^0_\alpha$ in $B$ if $\alpha$ is a limit, such that $d$ computes an isomorphism between $A$ and $B$ and such that all isomorphisms between $A$ and $B$ compute $d$.

The degree $d$ is the least degree of an isomorphism between $A$ and $B$.

We begin in Section 7.2 by giving the technical definitions for what we mean by “on a cone.” In Section 7.3 we prove Theorem 7.1.7. In Section 7.4 we prove a stronger version of Theorem 7.1.5 in the restricted case of structures which are $\Delta^0_2$ categorical on a cone; it will follow that the only possible degrees of categoricity on a cone for such structures are $\Delta^0_1$-complete or $\Delta^0_2$-complete. In order to prove the general case of Theorem 7.1.5, we need to use the method of $\alpha$-systems. These were introduced by Ash, see [AK00]. Montalbán [Mon14] introduced $\eta$-systems, which are similar to Ash’s $\alpha$-systems but give more control. They also deal with limit ordinals in a different way. We need the extra control of Montalbán’s $\eta$-systems, but we need to deal with limit ordinals as in Ash’s $\alpha$-systems. So in Section 7.5
we introduce a modified version of Montalbán’s \( \eta \)-systems. We conclude in Section 7.6 with a complete proof of Theorem 7.1.5.

### 7.2 Relativizing to a Cone

A *cone of Turing degrees* is a set \( C_d = \{ c : c \geq d \} \). Martin [Mar68] showed that under set-theoretic assumptions of determinacy, every set of Turing degrees either contains a cone or is disjoint from a cone. Noting that every countable intersection of cones contains a cone, we see that we can form a \( \{0, 1\} \)-valued measure on sets of degrees by assigning measure one to those sets which contain a cone. In this paper, all of the sets of degrees which we will consider arise from Borel sets, and by Borel determinacy [Mar75], such sets either contain or are disjoint from a cone.

If \( P \) is a statement which relativizes to any degree, we say that \( P \) holds on a cone if there is a degree \( d \) (the base of the cone) such that for all \( c \geq d \), \( P \) holds relative to \( c \). Thus a statement holds on a cone if and only if it holds almost everywhere relative to the Martin measure. In the rest of this section, we will relativize the definitions we are interested in.

**Definition 7.2.1.** The structure \( \mathcal{A} \) is *computably categorical on the cone above \( d \)* if for all \( c \geq d \), whenever \( B \) and \( C \) are \( c \)-computable copies of \( A \), there exists a \( c \)-computable isomorphism between \( B \) and \( C \). More generally, a structure is *\( \Delta^0_\alpha \) categorical on the cone above \( d \)* if for all \( c \geq d \) whenever \( B \) and \( C \) are \( c \)-computable copies of \( A \), there exists a \( \Delta^0_\alpha(c) \)-computable isomorphism between \( B \) and \( C \).

Note that even if \( \alpha \) is not computable, there is a cone on which \( \alpha \) is computable, and for \( c \) on this cone, \( \Delta^0_\alpha(c) \) makes sense. In a similar way, we do not have to assume that the structure \( \mathcal{A} \) is computable. If \( A \) is \( \Delta^0_\alpha \)-categorical on a cone, there is a degree \( d \) which computes \( A \) and \( \alpha \), and \( A \) is \( \Delta^0_\alpha \)-categorical on the cone above \( d \).

Recall that a computable structure \( \mathcal{A} \) is relatively \( \Delta^0_\alpha \) categorical if for all \( B \cong A \), some isomorphism from \( A \) onto \( B \) is \( \Delta^0_\alpha(B) \), and that there exist structures that are \( \Delta^0_\alpha \) categorical but not relatively so [Gon77, GHK'05, DKL'15]. If we were to modify the definition of relatively \( \Delta^0_\alpha \) categorical to be on a cone, it would be equivalent to Definition 7.2.1. That is, there is no difference between relatively \( \Delta^0_\alpha \) categorical on a cone and \( \Delta^0_\alpha \) categorical on a cone.

The notion of relatively \( \Delta^0_\alpha \) categoricity is intimately related to that of a Scott family.

**Notation 7.2.2.** All formulas in this paper will be infinitary formulas, that is, formulas in \( L_{\omega_1\omega} \). See Chapter 6 of [AK00] for background on infinitary formulas and computable infinitary formulas. We will denote by \( \Sigma^\inf_\alpha \) the infinitary \( \Sigma_\alpha \) formulas and by \( \Sigma^c_\alpha \) the computable \( \Sigma_\alpha \) formulas.

**Definition 7.2.3.** A *Scott family* for a structure \( \mathcal{A} \) is a countable family \( \Phi \) of formulas over a finite parameter such that
for each $\bar{a} \in A$, there exists $\varphi \in \Phi$ such that $A \models \varphi(\bar{a})$

- if $\varphi \in \Phi$, $A \models \varphi(\bar{a})$, and $A \models \varphi(\bar{b})$, then there is an automorphism of $A$ taking $\bar{a}$ to $\bar{b}$.

It follows from work of Scott [Sco65] (see [AK00]) that every countable structure has a Scott family consisting of $\Sigma^\inf_\alpha$ formulas for some countable ordinal $\alpha$.

**Theorem 7.2.4** (Ash-Knight-Manasse-Slaman [AKMS89] and Chisholm [Chi90]). A computable structure $A$ is relatively $\Delta^0_\alpha$ categorical if and only if it has a Scott family which is a c.e. set of $\Sigma^\alpha_\alpha$ formulas.

Now we can see the power of working on a cone.

**Remark 7.2.5.** Let $A$ be a countable structure. Then $A$ has a Scott family consisting of $\Sigma^\inf_\alpha$ formulas for some countable ordinal $\alpha$. Let $d$ be such that $A$ and $\alpha$ are $d$-computable, and such that the Scott family for $A$ is c.e. and consists of $\Sigma^\alpha_\alpha$ formulas relative to $d$. Then $A$ is $\Delta^0_\alpha$ categorical on the cone above $d$. That is, every countable structure is $\Delta^0_\alpha$ categorical on a cone for some $\alpha$.

There is also an analogue of Theorem 7.2.4 for (non-relative) $\Delta^0_\alpha$ categoricity. Historically, this came first; the $\alpha = 1$ case is due to Goncharov [Gon75] and the general case is due to Ash [Ash87].

We now recall some definitions from [AK00].

**Definition 7.2.6** (Back-and-forth relations). For a structure $A$ tuples $\bar{a}, \bar{b} \in A$ of the same length

- $\bar{a} \preceq_\alpha \bar{b}$ if and only if for every quantifier-free formula $\varphi(\vec{x})$ with Gödel number less than $\text{length}(\bar{a})$, if $A \models \varphi(\bar{a})$ then $B \models \varphi(\bar{b})$.

- for $\alpha > 0$, $\bar{a} \preceq_\alpha \bar{b}$ if and only if, for each $\bar{d} \in A$ and each $0 \leq \beta < \alpha$, there exists $\bar{c}$ in $A$ such that $\bar{b}, \bar{d} \preceq_\beta \bar{a}, \bar{c}$.

**Definition 7.2.7** (p. 269 Ash-Knight). For tuples $\bar{c}$ and $\bar{a}$ in $A$, we say that $\bar{a}$ is $\alpha$-free over $\bar{c}$ if for any $\bar{a}_1$ and for any $\beta < \alpha$, there exist $\bar{a}'$ and $\bar{a}_1'$ such that $\bar{c}, \bar{a}, \bar{a}_1 \preceq_\beta \bar{c}, \bar{a}', \bar{a}_1'$ and $\bar{c}, \bar{a}, \bar{a}_1' \not\preceq_\alpha \bar{c}, \bar{a}$.

**Definition 7.2.8** (p. 241 Ash-Knight). A structure $A$ is $\alpha$-friendly if for $\beta < \alpha$, the standard back-and-forth relations $\preceq_\beta$ are c.e. uniformly in $\beta$.

There is a version of Theorem 7.2.4 for the non-relative notion of categoricity. It comes in two parts:

**Proposition 7.2.9** (Prop 17.6 from [AK00]). Let $A$ be a computable structure. Suppose $A$ is $\alpha$-friendly, with computable existential diagram. Suppose that there is a tuple $\bar{c}$ in $A$ over which no tuple $\bar{a}$ is $\alpha$-free. Then $A$ has a formally $\Sigma^\alpha_\alpha$ Scott family, with parameters $\bar{c}$.
Theorem 7.2.10 (Theorem 17.7 from [AK00]). Let $A$ be $\alpha$-friendly. Suppose that for each tuple $\bar{c}$ in $A$, we can find a tuple $\bar{a}$ that is $\alpha$-free over $\bar{c}$. Finally, suppose that the relation $\not\preceq_\alpha$ is c.e. Then there is a computable $B \cong A$ with no $\Delta^0_\alpha$ isomorphism from $A$ to $B$.

Corollary 7.2.11. Suppose that $A$ is not $\Delta^0_\alpha$ categorical on any cone. Then for any $\bar{c}$ in $A$, there is some $\bar{a} \in A$ that is $\alpha$-free over $\bar{c}$.

We now give the definitions needed to discuss degrees of categoricity on a cone.

Definition 7.2.12. The structure $A$ has degree of categoricity $d$ relative to $c$ if $d$ can compute an isomorphism between any two $c$-computable copies of $A$, and moreover $d \geq c$ is the least degree with this property. If in addition to this there exist two $c$-computable copies of $A$ such that for every isomorphism $f$ between them, $f \oplus c \geq_T d$, then we say $A$ has strong degree of categoricity $d$ relative to $c$.

Definition 7.2.13. We say that a structure $A$ has a (strong) degree of categoricity on a cone, if there is some $d$ such that for every $c \geq d$, $A$ has a (strong) degree of categoricity relative to $c$.

Definition 7.2.14. We say that a structure $A$ has $\Delta^0_\alpha$-complete (strong) degree of categoricity on a cone, if there is some $d$ such that for every $c \geq d$, $A$ has $\Delta^0_\alpha$-complete (strong) degree of categoricity relative to $c$.

7.3 Isomorphism of C.E. Degree

Theorem 7.1.7 follows from the following more technical statement.

Theorem 7.3.1. Let $A$ be a structure. Suppose that $A$ is $\Delta^0_\alpha$ categorical on a cone. Then there is a degree $c$ such that for every copy $B$ of $A$, there is a degree $d$ that is $\Sigma^0_{\alpha-1}$ in and above $B \oplus c$ if $\alpha$ is a successor ordinal, or $\Delta^0_\alpha$ in and above $B \oplus c$ if $\alpha$ is a limit ordinal, such that

1. $d$ computes some isomorphism between $A$ and $B$ and

2. for every isomorphism $f$ between $A$ and $B$, $f \oplus c \geq_T d$.

Before giving the proof, we consider two motivating examples.

Example 7.3.2. Let $N$ be the standard presentation of $(\omega, \prec)$. If $A$ is any other presentation, let $\text{Succ}(A)$ be the successor relation in $A$. Then the unique isomorphism between $N$ and $A$ has the same Turing degree as $\text{Succ}(A)$. Note that $\text{Succ}(A)$ is $\Pi^0_1$. 


Example 7.3.3. Let \( V \) be an infinite-dimensional \( \mathbb{Q} \)-vector space with a computable basis. If \( W \) is any other presentation of \( V \), let \( \text{Indep}(W) \) be the independence relation in \( W \), as a subset of \( W^{\omega} \). Then any isomorphism between \( V \) and \( W \) computes \( \text{Indep}(W) \), and \( \text{Indep}(W) \) computes a basis for \( W \) and hence an isomorphism between \( V \) and \( W \). Note that \( \text{Indep}(W) \) is \( \Pi^0_1 \).

Theorem 7.1.7 says that this is the general situation for natural structures.

Proof of Theorem 7.3.1. Let \( c \) be a degree such that \( A \) is \( c \)-computable and \( \Delta^0_\alpha \)-categorical on the cone above \( c \). By increasing \( c \) to absorb the effectiveness conditions of Proposition 7.2.9 and Theorem 7.2.10, \( A \) has a c.e. Scott family \( S \) consisting of \( \Sigma^c_\alpha \) formulas relative to \( c \). Increasing \( c \), we may assume that \( S \) consists of formulas of the form \( (\exists x)\varphi \) where \( \varphi \) is \( \Pi^0_\beta \) relative to \( c \) for some \( \beta < \alpha \). Further increasing \( c \), we may assume that \( c \) can decide whether two formulas from \( S \) are satisfied by the same elements. Then we can replace \( S \) by a Scott family in which every tuple from \( A \) satisfies a unique formula from \( S \). Finally, by replacing \( c \) with a higher degree, we may assume that \( c \) can compute, for an element of \( A \), the unique formula of \( S \) which it satisfies, and can decide, for each tuple of the appropriate arity, whether or not it is a witness to the existential quantifier in that formula. This is the degree \( c \) from the statement of the theorem.

Let \( B \) be a copy of \( A \). Consider the set

\[
S(B) = \{ (\bar{b}, \varphi) : B \vDash \varphi(\bar{b}), \varphi \in S \}.
\]

Let \( d \) be the degree of \( S(B) \oplus B \oplus c \). First, note that the set

\[
S(A) = \{ (\bar{a}, \varphi) : A \vDash \varphi(\bar{a}), \varphi \in S \}
\]

is \( c \)-computable. If \( f \) is an isomorphism between \( A \) and \( B \), then \( f \oplus c \) computes \( S(A) \). Then using \( f \) and \( S(A) \), we can compute \( S(B) \). Thus

\[
f \oplus c \geq_T S(B) \oplus B \oplus c \equiv_T d
\]

for every isomorphism \( f \) between \( A \) and \( B \).

On the other hand, \( c \) computes \( S(A) \). Using \( S(B) \) and \( S(A) \) we can compute an isomorphism between \( A \) and \( B \). So there is an isomorphism \( f \) between \( A \) and \( B \) such that

\[
f \oplus c \equiv_T S(B) \oplus B \oplus c \equiv_T d.
\]

We now introduce a related set \( T(B) \). We will show that \( T(B) \oplus B \oplus c \equiv_T d \). If \( \alpha \) is a successor ordinal, then \( T(B) \) will be \( \Pi^0_{\alpha-1} \) in \( B \oplus c \), and if \( \alpha \) is a limit ordinal then \( T(B) \) will be \( \Delta^0_\alpha \) in \( B \oplus c \). Thus \( d \) will be a degree of the appropriate type. We may consider the elements of \( B \) to be ordered, and hence order tuples from \( B \) via the lexicographic order. Let \( T(B) \) be the set of tuples \( (\bar{a}, \bar{b}, \varphi) \) where:

1. \( \varphi(\bar{x}, \bar{y}) \) is a \( c \)-computable \( \Pi^0_\beta \) formula, for some \( \beta < \alpha \),
(2) \((\exists y)\varphi(\bar{x}, y)\) is in \(S\), and

(3) \(B \models \varphi(\bar{a}, \bar{c})\), for some \(\bar{c} \leq \bar{b}\) in the lexicographical ordering of tuples from \(B\).

It is easy to see that if \(\alpha\) is a successor ordinal, then \(T(B)\) is \(\Pi^0_{\alpha+1}\) in \(B \oplus c\), and if \(\alpha\) is a limit ordinal then \(T(B)\) is \(\Delta^0_\alpha\) in \(B \oplus c\). Now we will argue that \(T(B) \oplus B \oplus c \equiv_T S(B) \oplus B \oplus c\).

Suppose we want to check whether \((\bar{a}, \bar{b}, \varphi) \in T(B)\) using \(S(B) \oplus B \oplus c\). Using \(c\), we first compute whether (1) and (2) hold for \(\varphi\). Then using \(S(B) \oplus B \oplus c\) we can compute an isomorphism \(f: B \to A\). Now for each \(\bar{c} \leq \bar{b}\) in \(B\), \(B \models \varphi(\bar{a}, \bar{c})\) if and only if \(A \models \varphi(f(\bar{a}), f(\bar{c}))\). In \(A\), using \(c\) we can decide whether \(A \models \varphi(f(\bar{a}), f(\bar{c}))\).

On the other hand, to see whether \((\bar{a}, (\exists y)\varphi(\bar{x}, y))\) is in \(S(B)\) using \(T(B)\), look for \(\bar{b}\) and \(\psi\) such that \((\bar{a}, \bar{b}, \psi) \in T(B)\). Some such \(\psi\) and witness \(\bar{b}\) must exist, since \(\bar{a}\) satisfies some formula from \(S\). Then \((\bar{a}, (\exists y)\varphi(\bar{x}, y)) \in S(B)\) if and only if \(\varphi = \psi\) (recall that we assumed that each element of \(A\) satisfied a unique formula from the Scott family).

\[\square\]

### 7.4 Not Computably Categorical on any Cone

This section is devoted to the proof of Theorem 7.1.5 for structures which are \(\Delta^0_\alpha\) categorical on a cone. The general case of the theorem will require the \(\eta\)-systems developed in the next section, and will be significantly more complicated, so the proof of this simpler case should be helpful in following the proof in the general case, and in fact, we have a slightly stronger theorem in this case.

**Theorem 7.4.1.** Let \(A\) be a countable structure. If \(A\) is not computably categorical on any cone, then there exists an \(e\) such that for all \(d \geq e\), if \(c\) is c.e. in and above \(d\), then there exists a \(d\)-computable copy \(B\) of \(A\) such that

(1) there is a \(c\)-computable isomorphism between \(A\) and \(B\) and

(2) for every isomorphism \(f\) between \(A\) and \(B\), \(f \oplus d\) computes \(c\).

**Proof.** Suppose \(A\) is not computably categorical on any cone. Before we begin, note that since \(A\) is not computably categorical on any cone, for any tuple \(\bar{c}\) in \(A\), there exist a tuple \(\bar{a}\) in \(A\) that is 1-free over \(\bar{c}\). Let \(e\) be such that:

(1) \(A\) is \(e\)-computable,

(2) \(e\) computes a Scott family for \(A\) where each tuple satisfies a unique formula, and \(e\) can compute which which formula a tuple of \(A\) satisfies,

(3) \(A\) is 1-friendly relative to \(e\), and

(4) given \(\bar{c}\), \(e\) can compute the least tuple \(\bar{a}\) that is 1-free over \(\bar{c}\).
Let \( d \geq e \), and let \( c \) be c.e. in and above \( d \). Let \( C \in c \) be such that we have a \( d \)-computable approximation to \( C \) where at most one number is enumerated at each stage, and there are infinitely many stages when nothing is enumerated.

We will build \( B \) with domain \( \omega \) by a \( d \)-computable construction. We will build a bijection \( f: \omega \rightarrow A \) and \( B \) will be the pullback, along \( f \), of \( A \). At each stage \( s \), we will have a finite approximation \( f_s \) to \( f \), and \( B[s] \) a finite part of the diagram of \( B \) so that \( f_s \) is a partial isomorphism between \( B[s] \) and \( A \). Once we put something into the diagram of \( B \), we will not remove it, and so \( B \) will be \( d \)-computable. While the approximation \( f_s \) will be \( d \)-computable, \( f \) will be \( C \)-computable.

We will have distinguished tuples \( a_0 \in A \) and \( b_0 \in B \), such that for any isomorphism \( g: B \rightarrow A \), we will have \( 0 \notin C \) if and only if \( g(b_0) \) is isomorphic to \( a_0 \) in \( A \). For \( n > 0 \) the strategy for coding whether \( n \notin C \) will be the same, but our \( a_n \) and \( b_n \) will be re-defined each time some \( m < n \) is enumerated into \( C \). When \( n \) is enumerated into \( C \), we will be able to redefine \( f \) on \( b_n \) and on all greater values. At each stage \( s \), we have current approximations \( a_n[s] \) and \( b_n[s] \) to these values. The tuple \( b_n[s] \) will be a series of consecutive elements of \( \omega \); by \( B \upharpoonright b \) we mean the elements of \( B \) up to, and including, those of \( b \), and by \( B \upharpoonright \tilde{b} \) we mean those up to, but not including, \( \tilde{b} \).

At each stage, if \( n \notin C \), for those \( a_n \) and \( b_n \) which are defined at that stage we will have \( f(b_n) \) is 1-free over \( f(B \upharpoonright b_n) \); otherwise, we will have \( f(B \upharpoonright b_n) f(b_n) \neq f(B \upharpoonright \tilde{b_n})a_n \).

**Construction.**

**Stage 0:** Let \( a_0[0] \) be the least tuple of \( A \) that is 1-free, and let \( b_0[0] \) be the first \( |a_0| \)-many elements of \( \omega \). Define \( f_0 \) to be the map \( b_0[0] \mapsto a_0[0] \). Let \( B[0] \) be the pullback, along \( f_0 \), of \( A \), using only the first \( |a_0[0]| \)-many symbols from the language.

**Stage \( s + 1 \):** Suppose \( n \) enters \( C \) at stage \( s + 1 \). Let \( \tilde{b} = B[s] \upharpoonright b_n[s] \). Let \( \tilde{b'} \) be those elements of \( B[s] \) which are not in \( \tilde{b} \) or \( b_n[s] \). Then, since \( a_n[s] \) is 1-free over \( f(\tilde{b}) \), there are \( a, a' \in A \) such that

\[
\left\{ f(\tilde{b}), a_n[s], f(\tilde{b'}) \leq_0 f(\tilde{b}), a, a', \text{ but } f(\tilde{b}), a \neq f(\tilde{b}), a_n[s]. \right. 
\]

Define \( f_{s+1} \) to map \( \tilde{b}, b_n[s], b' \) to \( f(\tilde{b}), a, a' \). For \( m \leq n \), let \( a_m[s + 1] = a_m[s] \) and \( b_m[s + 1] = b_m[s] \). For \( m > n \), \( a_m[s + 1] \) and \( b_m[s + 1] \) are undefined.

If nothing enters \( C \) at stage \( s + 1 \), let \( n \) be least such that \( a_n[s] \) is undefined. For \( m < n \), let \( a_m[s + 1] = a_m[s] \) and \( b_m[s + 1] = b_m[s] \). Let \( a_n[s + 1] \) be the least tuple that is 1-free over \( \text{ran}(f_s) \). Extend \( f_s \) to \( f_{s+1} \) with range \( A \upharpoonright a_n[s + 1] \) by first mapping new elements \( b_n[s + 1] \) of \( \omega \) to \( a_n[s + 1] \), and then mapping more elements to the rest of \( A \upharpoonright a_n[s + 1] \). If \( n \notin C \), we must modify \( f_{s+1} \) as described above in the case \( n \) entered \( C \).

In all cases, let \( B[s + 1] \) be the pullback, along \( f_{s+1} \), of \( A \). We have \( B[s] \subseteq B[s + 1] \).

**End of construction.**

Since \( a_n \) and \( b_n \) are only re-defined when there is an enumeration of some \( m \leq n \) into \( C \), it is easy to see that for each \( n \), \( a_n \) and \( b_n \) eventually reach a limit. Moreover, since the \( a_n \) and \( b_n \) form infinite sequences in \( A \) and \( B \), respectively, and since \( f \) is not re-defined on \( B \upharpoonright \tilde{b_n} \).
unless there is an enumeration of \(m \leq n\) into \(C\), we see that \(f\) is an isomorphism between \(B\) and \(A\). Moreover, \(C\) can compute a stage when \(\bar{a}_n\) and \(\bar{b}_n\) have reached their limit, and hence \(f\) is \(c\)-computable.

Now suppose \(g : B \to A\) is an isomorphism. To compute \(C\) from \(g \oplus d\), proceed as follows. Compute \(g(\bar{b}_0)\). Ask \(d\) whether \((A, g(\bar{b}_0)) \cong (A, \bar{a}_0)\). If yes, then \(0 \notin C\). We also know that \(\bar{b}_1 = \bar{b}_1[0]\) and that \(\bar{a}_1 = \bar{a}_1[0]\). If \((A, g(\bar{b}_0)) \not\cong (A, \bar{a}_0)\), then \(0 \in C\). Compute \(s\) such that \(0 \in C[s]\). Then \(\bar{b}_1 = \bar{b}_1[s]\) and \(\bar{a}_1 = \bar{a}_1[s]\). Continuing in this way, given \(\bar{b}_n\) and \(\bar{a}_n\), we ask \(d\) whether \((A, g(\bar{b}_n)) \cong (A, \bar{a}_n)\), using the answer to decide whether \(n \in C\) and to compute \(\bar{b}_{n+1}\) and \(\bar{a}_{n+1}\).

Using Knight’s theorem on the upwards closure of degree spectra [Kni86], we get a slight strengthening of the above theorem.

**Corollary 7.4.2.** Let \(A\) be a countable structure. If \(A\) is not computably categorical on any cone, then there exists an \(e\) such that for all \(d \geq e\), if \(c\) is c.e. in and above \(d\), then there exists a \(d\)-computable copy \(B\) of \(A\) such that every isomorphism between \(A\) and \(B\) computes \(c\), and such that there exists a \(c\)-computable isomorphism between \(A\) and \(B\).

*Proof.* Take \(e\) as guaranteed by the theorem, with \(e\) computing \(A\), and fix \(d \geq e\), and let \(c\) be c.e. in \(d\). Let \(C\) be as guaranteed by the theorem. Since \(C\) is \(d\)-computable, by the proof of Knight’s upward closure theorem [Kni86] (and noting that a “trivial” structure is computably categorical on a cone), there exists \(B\) such that \(\deg(B) = d\) and such that there exists a \(d\)-computable isomorphism \(h : B \cong C\). Now since \(A\) is \(e\)-computable and \(\deg(B) = d\), any isomorphism \(g : A \cong B\) computes \(d\). Since \(d\) computes \(h\), \(g\) computes the isomorphism \(g \circ h : C \cong A\) and hence it computes \(c\). On the other hand, since \(c\) computes \(d\) and hence \(h\), and since \(c\) computes an isomorphism between \(A\) and \(C\), we have that \(c\) computes an isomorphism between \(A\) and \(B\). □

**Corollary 7.4.3.** On a cone, a structure cannot have degree of categoricity which is \(\Delta^0_2\) but not \(\Delta^0_0\) or \(\Delta^0_2\)-complete. That is, if \(A\) is not computably categorical on any cone, and if \(A\) has a degree of categoricity on a cone, then there is some \(e\) such that for all \(d \geq e\), the degree of categoricity of \(A\) relative to \(d\) is at least \(d'\).

**Corollary 7.4.4.** If \(A\) is \(\Delta^0_2\) categorical on a cone then \(A\) has \(\Delta^0_1\)-complete or \(\Delta^0_2\)-complete degree of categoricity on a cone.

### 7.5 A Version of Ash’s Metatheorem

The goal of the remainder of the paper is to prove Theorem 7.1.5. Our main tool will be a version of Ash’s metatheorem for priority constructions which was first introduced in [Ash86a, Ash86b, Ash90]. Ash and Knight’s book [AK00] is a good reference. Montalbán [Mon14] has recently developed a variant of Ash’s metatheorem using computable approximations. Montalbán’s formulation of the metatheorem also provides more control over the
construction; for the proof of Theorem 7.1.5, we will require this extra control. However, Montalbán's version of the metatheorem, as written, only covers $0^{(\eta)}$-priority constructions for $\eta$ a successor ordinal. In this section, we will introduce the metatheorem and expand it to include the case of limit ordinals.

Fix a computable ordinal $\eta$ for which we will define $\eta$-systems and the metatheorem for constructions guessing at a $\Delta^0_\eta$-complete function. Here our notation differs from Montalbán's but corresponds to Ash's original notation. What we call an $\eta$-system corresponds to what Ash would have called an $\eta$-system, but what Montalbán calls an $\eta$-system we will call an $\eta + 1$-system. This will allow us to consider, for limit ordinals $\eta$, what Montalbán might have called a $\prec \eta$-system.

### 7.5.1 Some $\Delta^0_\xi$-Complete Functions, Their Approximations, and True Stages

Before defining an $\eta$-system and stating the metatheorem, we discuss some $\Delta^0_\xi$-complete functions and their approximations as introduced by Montalbán [Mon14]. We will introduce orderings on $\omega$ to keep track of our beliefs on the correctness of the approximations.

For each computable ordinal $\xi \leq \eta$, Montalbán defines a $\Delta^0_\xi$-complete function $\nabla^\xi \in \omega^\omega$, and for each stage $s \in \omega$ a computable approximation $\nabla^\xi_s$ to $\nabla^\xi$. $\nabla^\xi_s$ is a finite string which guesses at an initial segment of $\nabla^\xi$. The approximations are all uniformly computable in both $s$ and $\xi$. Montalbán shows that the approximation has the following properties (see Lemmas 7.3, 7.4, and 7.5 of [Mon14]):

(N1) For every $\xi$, the sequence of stages $t_0 < t_1 < t_2 < \cdots$ for which $\nabla^\xi_t$ is correct is an infinite sequence with $\nabla^\xi_{t_0} \subseteq \nabla^\xi_{t_1} \subseteq \cdots$ and $\bigcup_{i \in \omega} \nabla^\xi_{t_i} = \nabla^\xi$.

(N2) For each stage $s$, there are only finitely many $\xi$ with $\nabla^\xi_s \neq \{\}$, and these $\xi$s can be computed uniformly in $s$.

(N3) If $\gamma \leq \xi$, $s \leq t$, and $\{\} \neq \nabla^\xi_s \subseteq \nabla^\xi_t$, then $\nabla^\gamma_s \subseteq \nabla^\gamma_t$.

We say that $s$ is a true stage or $\eta$-true stage if $\nabla^\eta_s \subseteq \nabla^\eta$.

Montalbán defines relations $(\leq_\xi)_{\xi < \eta}$ on $\omega$, to be thought of as a relation on stages in an approximation. We will define relations $(\leq_\xi)_{\xi < \eta}$ which are almost, but not exactly, the same as Montalbán's (we leave the definition of these relations, and the proofs of their properties, to Lemma 7.5.3). An instance $s \leq_\xi t$ of the relation should be interpreted as saying that, from the point of view of $t$, $s$ is a $\xi$-true stage. A relation $s \leq_\xi t$ is almost, but not exactly, equivalent to saying that for all $\gamma \leq \xi + 1$, $\nabla^\gamma_s \subseteq \nabla^\gamma_t$. The problem is that we require the property (B4) below.

**Definition 7.5.1.** Let $s \leq t$ if and only if, for all $\xi \leq \eta$, $\nabla^\xi_s \subseteq \nabla^\xi_t$. 
We can interpret $s \leq t$ as saying that $s$ appears to be a true stage (or $\eta$-true stage) from stage $t$. This relation is computable by (N2) above.

We will see that the relations $\leq_{\xi}$ satisfy the following properties:

(B0) $\leq_0$ is the standard ordering on $\omega$.

(B1) The relations $\leq_{\xi}$ are uniformly computable.

(B2) Each $\leq_{\xi}$ is a preorder (i.e., reflexive and transitive).

(B3) The sequence of relations is nested (i.e., if $\gamma \leq \xi$ and $s \leq_{\xi} t$, then $s \leq_{\gamma} t$).

(B4) The sequence of relations is continuous (i.e., if $\lambda$ is a limit ordinal, then $\leq_{\lambda} = \bigcap_{\xi < \lambda} \leq_{\xi}$).

(B5) For every $s < t$ in $\omega$, if $s \leq_{\xi} t$ then $\nabla_{\gamma}^{\xi+1} s \subseteq \nabla_{\gamma}^{\xi+1} t$.

(B6) The sequence $t_0 < t_1 < \ldots$ of true stages satisfies $t_0 \leq_{\eta} t_1 \leq_{\eta} \ldots$ and $\bigcup_{i \in \omega} \nabla_{i \omega}^{\eta} = \nabla^{\eta}$. We call the sequence of true stages the true path.

(B7) For $s \in \omega$, we can compute $H(s) = \max\{\xi < \eta \mid \nabla_{\xi} s \neq \emptyset\}$. $H(s)$ has the property that if $t > s$ and $s \notin t$, then $s \notin H(s) t$. We call $H(s)$ the height of $s$.

(B8) For every $\xi$ with $\xi < \eta$, and $r < s < t$, if $r \leq_{\xi} t$ and $s \leq_{\xi} t$, then $r \leq_{\xi} s$. Moreover, if $\xi$ is a successor ordinal, then it suffices to assume that $s \leq_{\xi-1} t$.

(B9) $s \leq t$ if and only if for all $\xi < \eta$, $s \leq_{\xi} t$.

(B10) If $t$ is a true stage and $s \leq t$, then $s$ is also a true stage.

Properties (B0)-(B5) are as in Montalbán [Mon14]. Our (B6) is a modification of Montalbán’s (B6). (B7), (B9) and (B10) are new properties. (B8) is Montalbán’s (♦) together with his Observation 2.1.

We will define, for convenience, the relations $\leq_{\xi}$ for $\xi < \eta$.

Definition 7.5.2. Let $s \leq_{\xi} t$ if for all $\gamma \leq \xi + 1$, $\nabla_{\gamma} s \subseteq \nabla_{\gamma} t$.

These relations are uniformly computable because by (N2), we only need to check whether $\nabla_{\gamma} s \subseteq \nabla_{\gamma} t$ for finitely many $\gamma$.

Following Montalbán, we will construct the desired relations $(\leq_{\xi})_{\xi < \eta}$.

Proposition 7.5.3. There is a sequence $(\leq_{\xi})_{\xi < \eta}$ satisfying (B0)-(B10).

In order to prove this proposition, we will use a number of lemmas from [Mon14], as well as properties (N1)-(N3).

Lemma 7.5.4 (Lemma 7.3 of [Mon14]). For each $\xi$, there is a subsequence $\{t_i : i \in \omega\}$ such that $\bigcup_{i \in \omega} \nabla_{t_i}^{\xi} = \nabla^{\xi}$.
Lemma 7.5.5 (Lemma 7.6 of [Mon14]). Let \( \lambda \leq \eta \) be a limit ordinal, and \( s < t \in \omega \). Suppose that \( \nabla^\lambda_s = \emptyset \). Then \( \nabla^\lambda_s \subseteq \nabla^\lambda_t \) if and only if \( (\forall \xi < \lambda) \nabla^\xi_s \subseteq \nabla^\xi_t \).

Lemma 7.5.6 (Lemma 7.7 of [Mon14]). \( (\leq_\xi)_{\xi \in \omega} \) is a nested computable sequence of preorderings satisfying:

(\( \star \)) For every \( \xi < \eta \), and every \( r < s < t \), if \( r \leq_{\xi+1} t \) and \( s \leq_\eta t \), then \( r \leq_{\xi+1} s \).

Lemma 7.5.7. We have:

(\( \triangledown \)) For every limit ordinal \( \xi \leq \eta \), and every \( r < s < t \), if \( \nabla^\xi_r \subseteq \nabla^\xi_t \) and \( \nabla^\xi_s \subseteq \nabla^\xi_t \), then \( \nabla^\xi_r \subseteq \nabla^\xi_s \).

Proof. Fix \( \xi \leq \eta \) and \( r < s < t \) such that \( \nabla^\xi_r \subseteq \nabla^\xi_t \) and \( \nabla^\xi_s \subseteq \nabla^\xi_t \). For each \( \gamma < \xi \), \( r \leq_{\gamma+1} t \) and \( s \leq_\gamma t \), so that by (\( \star \)) we have \( r \leq_{\gamma+1} s \). Then, by Lemma 7.5.5, \( \nabla^\xi_r \leq \nabla^\xi_s \).

In verifying that the relations \( (\leq_\xi)_{\xi < \omega} \) have the desired properties, we will also need to use several facts which Montalbán uses without proof (and without explicitly isolating them as, say, a lemma). We will isolate these in the following lemma, and prove them. Unfortunately, the proofs require notation that is introduced in [Mon14] which we have not introduced here (and which would require repeating most of [Mon14] in order to introduce). We suggest that the reader either take these statements for granted, or if the reader is interested in the proofs of these statements, we suggest that they consult [Mon14] for the required background and definitions.

Lemma 7.5.8.

(i) \( \nabla^1_s \) is the string of \( s \)'s.

(ii) Fix \( s < t \) and \( \xi < \eta \). If \( \nabla^\xi_s = \nabla^\xi_t \), then \( \nabla^{\xi+1}_s = \nabla^{\xi+1}_t \).

(iii) Fix \( s < t \), \( r \) and \( \xi < \eta \). If \( \nabla^\xi_s \subseteq \nabla^\xi_t \subseteq \nabla^\xi_r \) and \( \nabla^{\xi+1}_s \subseteq \nabla^{\xi+1}_r \), then \( \nabla^{\xi+1}_s \subseteq \nabla^{\xi+1}_r \).

(iv) Let \( \xi \) be a limit ordinal. There is an increasing sequence \( \gamma_1, \gamma_2, \gamma_3, \ldots \) with limit \( \xi \) such that for all \( s \),

\[
\nabla^\xi_s = (\nabla^{\gamma_1}_s(0), \nabla^{\gamma_2}_s(0), \ldots, \nabla^{\gamma_n}_s(0))
\]

where \( n_s \) is the greatest such that \( \nabla^{\gamma_n}_s(0) \neq \emptyset \).

(v) Given \( s < t \), if \( \nabla^{\gamma_n_s}_s(0) = \nabla^{\gamma_n}_t(0) \), then \( \nabla^\xi_s \subseteq \nabla^\xi_t \).

Proof. All of the notation in this proof is as in [Mon14].

(i). This is just Definition 7.2 of [Mon14].

(ii). Since \( \xi + 1 \) is a successor ordinal, if we unwrap Definitions 6.15 and 7.2 of [Mon14], we find that \( \nabla^{\xi+1}_s = J(\nabla^\xi_s) \) and \( \nabla^{\xi+1}_t = J(\nabla^\xi_t) \).\(^1\) If \( \nabla^\xi_s = \nabla^\xi_t \), then \( \nabla^{\xi+1}_s = \nabla^{\xi+1}_t \).

\(^1\)This requires some effort to check.
(iii) Once again we have that $\nabla_{\xi+1} = J(\nabla_{\xi})$, $\nabla_{t+1} = J(\nabla_{t})$, and $\nabla_{r+1} = J(\nabla_{r})$. Then Lemma 6.4 of [Mon14] gives the desired conclusion.

(iv) Let $\xi = 1 + \eta(n_0, \ldots, n_k)$. Then, if $n_k > 0$,

$$\nabla_{\xi} = J_{\omega_{\eta}, n_k}(\nabla_{\xi}) = J_{\omega_{\eta}, n_k, n_k}(\nabla_{\xi}).$$

Now

$$J_{\omega_{\eta}, n_k}(\sigma) = (J_{\omega_{\eta}, n_k, n_k}(\sigma)(0), J_{\omega_{\eta}, n_k, n_k}(\sigma)(0), \ldots, J_{\omega_{\eta}, n_k, n_k}(\sigma)(0))$$

where $j$ is greatest such that $J_{\omega_{\eta}, n_k}(\sigma) \neq ()$. Recall that

$$J_{\omega_{\eta}, n_k, n_k}(\sigma) = J_{\omega_{\eta}, n_k, n_k}(\sigma)(0) = J_{\omega_{\eta}, n_k, n_k}(\sigma)(0).$$

Thus $\nabla_{\xi}(n)$ is

$$J_{\omega_{\eta}, n_k, n_k}(\nabla_{\xi})(0) = \nabla_{\xi}(0).$$

If $n_k = 0$, then

$$\nabla_{\xi} = J_{\omega_{\eta}, n_k}(\nabla_{\xi}) = J_{\omega_{\eta}, n_k, n_k}(\nabla_{\xi}).$$

In this case, we get that $\nabla_{\xi}(n)$ is

$$J_{\omega_{\eta}, n_k, n_k}(\nabla_{\xi})(0) = \nabla_{\xi}(0).$$

(v) Let $\xi = 1 + \eta(n_0, \ldots, n_k)$. In (iv), we showed that for each $s$,

$$\nabla_{\xi} = (\nabla_{s+1+\eta(n_0, \ldots, n_k, 0)}(0), \ldots, \nabla_{s+1+\eta(n_0, \ldots, n_k, j-1)}(0))$$

where $j$ is the greatest such that $\nabla_{s+1+\eta(n_0, \ldots, n_k, j-1)}(0) \neq ()$. So given $s < t$, we have

$$\nabla_{\xi} = (\nabla_{t+1+\eta(n_0, \ldots, n_k, l-1)}(0)).$$

If $s \leq 1+\eta(n_0, \ldots, n_k, j-1)$, then

$$\nabla_{s+1+\eta(n_0, \ldots, n_k, i)}(0) = \nabla_{t+1+\eta(n_0, \ldots, n_k, i)}(0)$$

for $0 \leq i < j$. So $\nabla_{\xi} \subseteq \nabla_{t+\xi}$. 

\qed
CHAPTER 7. COMPUTABLE CATEGORICTY

Now we will show how to construct the order $(\leq_\xi)_{\xi<\eta}$ and prove Proposition 7.5.3.

**Proof of Proposition 7.5.3.** The proof of this proposition is very similar to the proof of Lemma 7.8 of [Mon14]. The definition of our relations $\leq_\xi$ is the same as Montalbán’s, except for one small change. Let $C$ be the set of tuples $(\lambda, u, v)$ where $\lambda < \eta$ is a limit ordinal, $\nabla_\lambda^u \nleq \nabla_\lambda^v$, $\nabla_\lambda^u \nleq \nabla^+_{\lambda+1}$, and if there is $r$ with $\nabla^+_{\lambda + r} \nleq \nabla^+_{\lambda+1}$ then $\nabla^+_{\lambda+1} \nleq \nabla^+_{\lambda+1}$. (The only change here is that we require that $\lambda < \eta$.) Let $\gamma_{\lambda,v}$ be such that the last entry of $\nabla^+_{\lambda}$ is $\nabla^+_{\lambda,v}(0)$. (Some such $\gamma_{\lambda,v}$ exists by Lemma 7.5.8 (iv).)

For $\xi < \eta$, define

$$s \leq_\xi t \iff s \leq_\xi t \text{ and } \neg \exists (\lambda, u, v) \in C(\gamma_{\lambda,v} < \xi \text{ and } u \leq s < v \leq_{\gamma_{\lambda,v}} t).$$

Except for the difference in the definition of $C$, this is the same as Montalbán’s definition.

We must now verify that $\leq_\xi$ satisfies (B0)-(B10). For many of the properties the verification is very similar to, or exactly the same as, Montalbán’s, but we will reproduce them here for completeness.

(B0) We can see that $\leq_0 \leq_0$ as $\nabla^0_s$ is the sequence of $s$ zeros (Lemma 7.5.8 (i)).

(B1) The relations $\leq_0$ are uniformly computable as the relations $\leq_k$ are, $\gamma_{\lambda,v}$ is computable in $\lambda$ and $v$ by (N2), and the existential quantifier is bounded, as $u,v \leq t$ and by (N2), there are only finitely many $\lambda$’s with $\nabla^+_{\lambda,v} \nleq ()$.

(B2) Fix $\xi$ and $s$. Then note that $s \nleq_\xi s$, and there is no $v$ with $s < v \nleq_{\gamma_{\lambda,v}} s$. Hence $s \leq_\xi s$.

Now for transitivity, suppose that $s \leq_\xi t \leq_\xi r$, but that $s \not\leq_\xi r$. Since $\leq_0$ is transitive, $s \leq_0 r$, and so it must be that there is $(\lambda, u, v) \in C$ such that $\gamma_{\lambda,v} < \xi$ and $u \leq s < v \nleq_{\gamma_{\lambda,v}} r$. If $t < \lambda$, then $u \leq t < v \nleq_{\gamma_{\lambda,v}} r$ and so $(\lambda, u, v)$ witnesses that $t \not\leq_\xi r$, a contradiction. Thus it must be that $v \leq t$. Now $v \leq_{\gamma_{\lambda,v}} r$, so by Lemma 7.5.8 (v), $\nabla^+_{\lambda} \nleq \nabla^+_{\lambda,v}$. By (N3), $v \leq_{\gamma_{\lambda,v}+1} r$. Also, $t \leq_\xi r$. Since $\xi$ is greater than $\gamma_{\lambda,v}$, by (\lambda), $v \leq_{\gamma_{\lambda,v}} t$. Then $u \leq s < v \nleq_{\gamma_{\lambda,v}} t$ and so $(\lambda, u, v)$ witnesses that $s \nleq_\xi t$. This is again a contradiction. So $\leq_\xi$ is transitive.

(B3) Suppose that $\gamma \leq_\xi \xi$ and $s \leq_\xi t$. We claim that $s \leq_\gamma t$. Since $s \leq_\xi t$, $s \leq_\xi t$, and so $s \leq_\gamma t$ as $\leq$ is nested. We must show that there is no $(\lambda, u, v) \in C$ with $\gamma_{\lambda,v} < \gamma$ and $u \leq s \leq s < v \nleq_{\gamma_{\lambda,v}} t$. If there was, then since $\gamma < \xi$, $(\lambda, u, v)$ witnesses that $s \nleq_\xi t$. Since in fact $s \leq_\xi t$, $s \leq_\xi t$.

(B4) Suppose to the contrary that for some limit ordinal $\alpha < \eta$, $s \nleq_\alpha t$, but that for all $\xi < \alpha$, $s \leq_\xi t$. If $s \nleq_\alpha t$ due to the existence of some $(\lambda, u, v) \in C$ with $\gamma_{\lambda,v} < \alpha$ and $u \leq s < v \nleq_{\gamma_{\lambda,v}} t$, then $(\lambda, u, v)$ also witnesses that $s \nleq_{\gamma_{\lambda,v}+1} t$, and $\gamma_{\lambda,v} + 1 < \alpha$, contrary to our initial assumption. So it must be that $s \nleq_\alpha t$ because $s \nleq_\alpha t$. Now, for all $\xi < \alpha$, $s \leq_\xi t$, and so by Lemma 7.5.5 it must be that $\nabla^\alpha_s \nleq \nabla^\alpha_t$, but $\nabla^\alpha_{\beta+1} \nleq \nabla^\alpha_t$. Let $\beta$ be the least such that $\nabla^\alpha_s \nleq \nabla^\alpha_t$ and $\nabla^\alpha_{\beta+1} \nleq \nabla^\alpha_{\beta+1}$. Some such $\beta$ exists because, by Lemma 7.5.8 (ii), if $\nabla^\alpha_s = \nabla^\alpha_t$, then $\nabla^\alpha_{\beta+1} = \nabla^\alpha_{\beta+1}$. Then $(\alpha, s, v) \in C$. And $v \nleq_{\gamma_{\alpha,v}} t$ by Lemma 7.5.5 because $() \nleq \nabla^\alpha_s \nleq \nabla^\alpha_t$. So $s \nleq_{\gamma_{\alpha,v}+1} t$ contradicting our assumptions.

(B5) Fix $s, t \in \omega$ with $s \leq_\xi t$. Then $s \leq_\xi t$, and so $\nabla^\xi_{s+1} \nleq \nabla^\xi_{t+1}$ by definition.

(B6) Let $t_0 < t_1 < \cdots$ be the true stages. Then, for each $\xi$, this is a subsequence of the sequence from (N1), and so $\nabla^\xi_{t_0} \nleq \nabla^\xi_{t_1} \nleq \cdots$ and $\bigcup_{i \in \omega} \nabla^\xi_{t_i} = \nabla^\xi$. Thus $t_0 \leq t_1 \leq t_2 \leq \cdots$. Also, by Lemma 7.5.4, we get that $\nabla^\xi_{t_0} \nleq \nabla^\xi_{t_1} \nleq \cdots$. So $t_0, t_1, \ldots$ is a subsequence of the sequence from (N1) for $\xi = \eta$, and so $\bigcup_{i \in \omega} \nabla^\eta_{t_i} = \nabla^\eta$. 


(B7) Fix $s$. By (N2) there are only finitely many $\xi$ with $\nabla^\xi_s \neq \emptyset$, and we can compute $H(s) = \{ \xi < \eta \mid \nabla^\xi_{s+1} \neq \{ \} \}$. Suppose that $t > s$ and $s \not\leq_t t$. Then, for some $\xi < \eta$, $\nabla_{s+1}^\xi \notin \nabla^\xi_t$. Since we must have $\nabla_{s+1}^\xi \notin \{ \}$, $\xi \leq H(s)$. Thus $s \not\leq_t H(s)$. 

(B8) First, we will prove the successor case. Suppose that $r < s < t$, $r \leq_{t+1} t$, and $s \leq_t t$. Suppose towards a contradiction that $r \not\leq_{t+1} s$. By (\.), we get that $r \not\leq_{t+1} s$. So it must be that there is some $(\lambda, u, v) \in C$ which witnesses that $r \not\leq_{t+1} s$. So $v \leq_{r,\lambda,v} s \leq_t t$, and so since $\xi + 1 > \gamma_{\lambda,v}$, $v \leq_{\gamma_{\lambda,v}} t$. Thus $(\lambda, u, v)$ witnesses that $r \not\leq_{t+1} t$.

Now we will show the limit case. This is the content of Observation 2.1 of [Mon14]. Suppose that $r < s < t$, $r \leq_{t} t$, and $s \leq_{t} t$. If $\xi$ is a successor, then we just use the previous case and the fact that $s \leq_{t-1} t$. For the limit case, for every $\gamma < \xi$, we have $r \leq_{\gamma_{t+1}} s$ and $s \leq_{\gamma_{t+1}} t$ and so by the previous case we have $r \leq_{\gamma+1} s$. But then, by (B4), we get $r \leq_{t} s$.

(B9) If, for all $\xi < \eta$, $s \leq_{t} t$, then for all $\xi < \eta$, $\nabla^s_{\xi+1} \subseteq \nabla^s_{\xi}$, and so $s \leq_{t} t$. On the other hand, suppose that $s \leq_{t} t$. Fix $\xi < \eta$. Then $s \not\leq_{t} t$, so to show that $s \leq_{t} t$, it suffices to show that there is no $(\lambda, u, v) \in C$ with $\gamma_{\lambda,v} < \xi$ and $u \leq s < v \leq_{\gamma_{\lambda,v}}$. Suppose to the contrary that there was such a $(\lambda, u, v)$. Since $v \leq_{\gamma_{\lambda,v}}$, $\nabla^\lambda_v(0) = \nabla^\lambda_{\gamma_{\lambda,v}}(0)$, and since $\nabla^\lambda_{\gamma_{\lambda,v}}(0)$ is the last entry of $\nabla^\lambda_v$, by Lemma 7.5.8 (v) we have $\nabla^\lambda_v \subseteq \nabla^\lambda_{t}$. Since $s \leq t$, $\nabla^\lambda_s \subseteq \nabla^\lambda_{s}$. Since $(\lambda, u, v) \in C$, $\nabla^\lambda_s \subseteq \nabla^\lambda_{u}$. Since $\lambda$ is a limit ordinal, applying Lemma 7.5.5 and using (\&) we get that $\nabla^\lambda_{u} \subseteq \nabla^\lambda_{s}$ and $\nabla^\lambda_u \subseteq \nabla^\lambda_{t}$. So $\nabla^\lambda_u \subseteq \nabla^\lambda_{s} \subseteq \nabla^\lambda_v \subseteq \nabla^\lambda_{t}$. By the minimality of $v$, we get $\nabla^\lambda_{u+1} \subseteq \nabla^\lambda_{s+1}$, and so since $\nabla^\lambda_{u+1} \subseteq \nabla^\lambda_s \subseteq \nabla^\lambda_v \subseteq \nabla^\lambda_{t}$ and $\nabla^\lambda_{s+1} \subseteq \nabla^\lambda_v$, by Lemma 7.5.8 (ii) $\nabla^\lambda_{s+1} \subseteq \nabla^\lambda_{u+1}$. This is a contradiction (as $s \leq t$), and so $s \leq_{t} t$.

(B10) Suppose that $t$ is a true stage, and $s \not\leq_{t} t$. If $\eta$ is a successor ordinal, say $\eta = \xi + 1$, then $\nabla^\eta_s \subseteq \nabla^\eta_{s+1}$. If $\eta$ is a limit ordinal, then by Lemma 7.5.5, $\nabla^\eta_s \subseteq \nabla^\eta_{t}$.

\[\Box\]

### 7.5.2 $\eta$-Systems and the Metatheorem

We are now ready to define an $\eta$-system. The definition is essentially the same as for Montalbán, except that what Montalbán would have called an $\eta$-system, we call an $\eta + 1$-system.

**Definition 7.5.9.** An $\eta$-system is a tuple $(L, P, (\leq^L_\xi)_{\xi<\eta}, E)$ where:

1. $L$ is a c.e. subset of $\omega$ called the set of **states**.
2. $P$ is a c.e. subset of $L^{\omega}$ called the **action tree**.
3. $(\leq^L_\xi)_{\xi<\eta}$ is a nested sequence of c.e. pre-orders on $L$ called the **restraint relations**.
4. $\ell \leq^L \ell'$ is c.e., where we define $\ell \leq^L \ell'$ if and only if $\ell \leq^L_\xi \ell'$ for all $\xi < \eta$.
5. $E \subseteq L \times \omega$ is a c.e. set called the **enumeration function**, and is interpreted as $E(l) = \{ k \in \omega : (l, k) \in E \}$. We require that for $\ell_0, \ell_1 \in L$ with $\ell_0 \leq^L \ell_1$, $E(\ell_0) \subseteq E(\ell_1)$.
\textbf{Definition 7.5.10.} A 0-run for \((L, P, (\leq_L^\xi_{\xi<\eta}, E))\) is a sequence \(\pi = (\ell_0, \ell_1, \ldots)\), which is in \(P\) if it is a finite sequence, or is a path through \(P\) if it is an infinite sequence, such that for all \(s, t < |\pi|\) and \(\xi < \eta\),

\[s \leq_{\xi} t \Rightarrow \ell_s \leq_L^{\xi} \ell_t.\]

If \(\pi\) is a 0-run, let \(E(\pi) = \bigcup_{s<|\pi|} E(\ell_s)\).

Given an infinite 0-run \(\ell_0, \ell_1, \ldots\) of an \(\eta\)-system \((L, P, (\leq_L^\xi_{\xi<\eta}, E))\), let \(t_0 \leq t_1 \leq t_2 \leq \cdots\) be the true stages. Then by the properties of \(E\) above, \(E(\pi) = \bigcup_{i\in\omega} E(\ell_{t_i})\). So \(E(\pi)\) is c.e., but it is determined by the true stages.

Montalbán defines an extendability condition and a weak extendability condition. For our extendability condition, we weaken Montalbán’s extendability condition even further (as well as modifying it slightly to allow limit ordinals). In order to define our extendability condition, we need the following definition.

\textbf{Definition 7.5.11.} To any stage \(s > 0\), we effectively associate a sequence of stages and ordinals as follows.

Choose \(t^* < s\) greatest such that \(t^* \leq s\). Some such \(t^*\) exists as \(0 \leq s\). Now for each \(\xi < \eta_0\), let \(t_{\xi} < s\) be the largest such that \(t_{\xi} \leq_{\xi} s\). Note that \(t^* \leq t_\xi\) for each \(\xi\) as by \((B8)\) \(t^* \leq_{\xi} s\).

There may be infinitely many \(\xi < \eta\), but there are only finitely many possible values of \(t_\xi\) since they are bounded by \(s\). Since the \(\leq_{\xi}\) are nested \((B3)\), if \(\gamma \leq \xi < \eta\), then \(t_{\xi} \leq \gamma\). Now we will effectively define stages \(t^* = s_k < \cdots < s_0 = s - 1\) so that \(\{s_0, \ldots, s_k\} = \{t_\xi : \xi < \eta\}\) as sets. Let \(s_0 = t_0 = s - 1\). Suppose that we have defined \(s_i\). If \(s_i \leq s\), then \(k = i\) and we are done. Otherwise, let \(\xi_i < \eta\) be the greatest such that \(s_i = t_{\xi_i}\). By definition of \(s_i\), it is of the form \(t_{\xi}\) for some \(\xi\). We can find the greatest such by computably searching for \(\xi_i\) such that \(s_i \leq_{\xi_i} s\) but \(s_i \notin_{\xi_i+1} s\); some such \(\xi_i\) exists since the relations are continuous and nested. Let \(s_{i+1} = t_{\xi_i+1}\). Since \(s_i \notin_{\xi_i+1} s\), \(s_{i+1} < s_i\). This completes the definition of \(s_k < \cdots < s_0 = s - 1\) and \(\xi_0 < \cdots < \xi_{k-1} < \eta\).

By \((B8)\), for \(i < k\), since \(s_{i+1} \leq_{\xi_i+1} s\) and \(s_i \leq_{\xi_i} s\), \(s_{i+1} \leq_{\xi_i+1} s_i\).

\textbf{Definition 7.5.12.} We say that an \(\eta\)-system \((L, P, (\leq_L^\xi_{\xi<\eta}, E))\) satisfies the extendability condition if: whenever we have a finite 0-run \(\pi = (\ell_0, \ldots, \ell_{s-1})\) such that for all \(i < k\), \(\ell_{si+1} \leq_L^{\xi_{i+1}} \ell_{si}\), where \(s_k < s_{k-1} < \cdots < s_0 = s - 1\) and \(\xi_0 < \xi_1 < \cdots < \xi_{k-1} < \eta\) are the associated sequences of stages and ordinals to \(s\) as in Definition 7.5.11, then there exists an \(\ell \in L\) such that \(\pi^i \ell \in P\), \(\ell_{s_k} \leq_L \ell\), and for all \(i < k\), \(\ell_{si} \leq_L \ell\).
Now we are ready for the metatheorem.

**Theorem 7.5.13.** For every $\eta$-system $(L, P, (\leq^L_\xi)_{\xi<\eta}, E)$ with the extendability condition, there is a computable infinite 0-run $\pi$. A 0-run can be built uniformly in the $\eta$-system.

*Proof of Theorem 7.5.13.* The proof is essentially the same as the proof of Theorem 3.2 in [Mon14]. By the trivial case of the extendability condition, there is $\ell_0 \in L$ with $(\ell_0) \in P$. Now suppose that we have a 0-run $\pi = (\ell_0, \ldots, \ell_{s-1})$. We want to define $\ell_s \in L$ such that $\pi \ell_s \in P$, and such that for every $\xi < \eta$, if $t \leq \xi s$, then $\ell_t \leq^L_{\xi} \ell_s$.

Let $\{t_\xi | \xi < \eta\}$, $s_k < \ldots s_0 = s - 1$, and $\xi_1 < \ldots < \xi_k$ be as in Definition 7.5.11. If $t \leq \xi s$, then $t \leq t_{\xi}$, and by (B8), $t \leq \xi T_{\xi}$, so since $\pi$ is a 0-run $\ell_t \leq^L_{\xi} \ell_{t_{\xi}}$. So it is sufficient to find $\ell$ with $\pi \ell \in P$ such that, for $\xi < \eta$, $\ell_{t_{\xi}} \leq^L_{\xi} \ell$. That is, we must find an $\ell$ with $\pi \ell \in P$, $\ell_{s_k} \leq^L_{\xi} \ell$ and $\ell_{s_i} \leq^L_{\xi} \ell$ for $0 \leq i < k$.

By (B8), for $i \leq k$, since $s_{i+1} \leq \xi_{i+1} s$ and $s_i \leq \xi_i$, $s$, $s_{i+1} \leq \xi_{i+1} s_i$. Since $p$ is a 0-run, $\ell_{s_{i+1}} \leq^L_{\xi_{i+1}} \ell_{s_i}$. By the extendability condition, there is $\ell \in L$ with $p \ell \in P$, $\ell_{s_k} \leq^L_{\xi} \ell$, and $\ell_{s_i} \leq^L_{\xi} \ell$ for $i < k$. We can find such an $\ell$ effectively, since we have described how to compute the $s_i$ and since the relations $\leq^L_{\xi}$ and $\leq^L$ are computable. \qed

### 7.6 Proof of Theorem 7.1.5

In this section, we will give the proof of Theorem 7.1.5. The proof will use the $\eta$-systems as developed in the previous section, together with a strategy expanding on that in the proof of Theorem 7.4.1. It is not sufficient to simply combine the techniques of Theorem 7.4.1 with the $\alpha$-system construction. Consider a $\Sigma^0_3$ set $C$. The difficulty is that in the approximation of $C$, an element $x$ may enter $C$, exit $C$ and then later exit $C$ again (and may continue to enter and exit $C$ infinitely many times). Each time $x$ enters $C$, we will have to code this in a way that can be distinguished from each other time that $x$ entered $C$. To do this, we will use that fact that given a tuple $\bar{a}$ in a structure of sufficient length, we can pick a tuple $\bar{b}$ which is automorphic to $\bar{a}$ (coding that $x$ is not in $C$), or we can pick a tuple $\bar{b}$ which is not isomorphic to $\bar{a}$ (coding that $x$ is in $C$). In the latter case, we will distinguish between...
how many times $x$ has entered $C$ by choosing $b$ to be in a different automorphism orbit each time. Of course, we must also code whether or not $x+1$ is in $C$. But the actions that we take towards coding $x$ can interfere with those that we take to code $x+1$, and because $x$ can both enter and exit $C$, the interactions between the two become much more complicated than they were in the case of Theorem 7.4.1; in that case, if $x$ entered $C$, we simply started coding $x+1$ in a new place. Now, if $x$ later exits $C$, we must return to where we were coding $x+1$ beforehand, and if $x$ enters $C$ again, then we must code $x+1$ in another new place because we may have interfered with the previous coding locations of $x+1$ (and we must have the coding of $x$ tell us where to look for the coding of $x+1$).

To begin, we prove the following lemma which we will use for coding.

**Lemma 7.6.1.** Let $A$ be a countable structure. Let $x$ be a tuple from $A$. Let $\alpha_1 > \beta_1, \ldots, \alpha_n > \beta_n$ be computable ordinals with $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$. Let $\bar{u}_1, \ldots, \bar{u}_n$ and $\bar{v}_1, \ldots, \bar{v}_n$ be tuples from $A$ such that $|\bar{u}_{i+1}| = |\bar{u}_i| + |\bar{v}_i|$ and such that $\bar{v}_i$ is $\alpha_i$-free over $\bar{u}_i$. Then there is a tuple $\bar{y}$ from $A$ such that, for each $i = 1, \ldots, n$,

1. $\bar{x} \upharpoonright |\bar{u}_i| = \bar{y} \upharpoonright |\bar{u}_i|$, 
2. $\bar{x} \upharpoonright |\bar{u}_i| + |\bar{v}_i| \leq \beta_i \bar{y} \upharpoonright |\bar{u}_i| + |\bar{v}_i|$, 
3. $\bar{y} \upharpoonright |\bar{u}_i| + |\bar{v}_i| \not\bar{x} \upharpoonright |\bar{u}_i| + |\bar{v}_i|$, 

**Proof.** We will inductively define tuples $\bar{x}_0, \ldots, \bar{x}_n$, so that taking $\bar{y} = \bar{x}_n$ satisfies the lemma. Begin with $\bar{x}_0 = \bar{x}$, so $\bar{x}_0$ satisfies (1) and (2).

Given $\bar{x}_m$ satisfying (1) and (2) for all $i$, and (3) for $i = 1, \ldots, m$, define $\bar{x}_{m+1}$ as follows. If $\bar{x}_m$ already satisfies (3) for $i = m+1$, set $\bar{x}_{m+1} = \bar{x}_m$. Otherwise, $\bar{x}_m \upharpoonright |\bar{u}_{m+1}| + |\bar{v}_{m+1}| \not\bar{x}_m$. Since $\bar{v}_{m+1}$ is $\alpha_{m+1}$-free over $\bar{u}_{m+1}$, there is $\bar{x}_{m+1}$ with $\bar{x}_m \leq_{\beta_{m+1}} \bar{x}_{m+1}$, $\bar{x}_m \upharpoonright |\bar{u}_{m+1}| = \bar{x}_{m+1} \upharpoonright |\bar{u}_{m+1}|$, and $\bar{x}_{m+1} \upharpoonright |\bar{u}_{m+1}| + |\bar{v}_{m+1}| \not\bar{u}_{m+1} \bar{v}_{m+1}$. So $\bar{x}_{m+1}$ satisfies (3) for $i = m+1$. Note that since $\bar{x}_m \upharpoonright |\bar{u}_{m+1}| = \bar{x}_{m+1} \upharpoonright |\bar{u}_{m+1}|$, we have $\bar{x}_{m+1} \upharpoonright |\bar{u}_i| + |\bar{v}_i| = \bar{x}_m \upharpoonright |\bar{u}_i| + |\bar{v}_i|$ for $i \leq m$, so that $\bar{x}_{m+1}$ satisfies (1) and satisfies (2) and (3) for $1 \leq i \leq m$. Since $\bar{x}_m \leq_{\beta_{m+1}} \bar{x}_{m+1}$, and for $i \geq m+1$, $\beta_i \leq_{\beta_{m+1}}$, we have $\bar{x} \upharpoonright |\bar{u}_i| + |\bar{v}_i| \leq_{\beta_i} \bar{x}_m \upharpoonright |\bar{u}_i| + |\bar{v}_i| \leq_{\beta_i} \bar{x}_{m+1} \upharpoonright |\bar{u}_i| + |\bar{v}_i|$ for such $i$. So (2) holds for $\bar{x}_{m+1}$. \hfill \Box

**Theorem 7.6.2.** Let $A$ be a countable structure. If $\eta$ is an ordinal and $A$ is not $\Delta^0_\beta$ categorical on any cone for any $\beta < \eta$, then there exists an $e$ such that for all $d \geq e$, there exists a $d$-computable copy $B$ of $A$ such that

1. there is a $\Delta^0_\eta(d)$-computable isomorphism between $A$ and $B$ and

2. for every isomorphism $f$ between $A$ and $B$, $f \oplus d$ computes $\Delta^0_\eta(d)$.

**Proof.** Suppose $A$ is not $\Delta^0_\beta$ categorical on any cone for any $\beta < \eta$. Let $e$ be such that:
(i) $A$ and $\eta$ are $e$-computable, and $e$ computes a Scott family for $A$ in which each tuple satisfies a unique formula and also computes, for tuples in $A$, which formula in the Scott family they satisfy,

(ii) $A$ is $\eta + 1$-friendly relative to $e$,

(iii) given a tuple $\bar{a}$ and $\beta < \eta$, $e$ can decide whether a tuple $\bar{b}$ is $\beta$-free over $\bar{a}$. (Such a tuple is guaranteed to exist by Corollary 7.2.11 since $A$ is not $\Delta^0_\beta$-categorical on any cone.)

Fix $d \geq e$ and $D \in d$. Our argument involves a $D$-computable $\eta$-system. To ease notation, we make no further mention of $D$ (e.g., whenever we write $\nabla^\beta$ we really mean $\nabla^\beta(D)$, we will say computable when we mean $d$-computable, etc.).

We will define our $\eta$-system. Let $B$ be a computable set of constant symbols not occurring in $A$. Let $L$ be the set of sequences

$$\langle p; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \ldots, (\bar{a}_r, \bar{b}_r) \rangle$$

where:

(L1) $p$ is a finite partial bijection $B \to A$,

(L2) $\bar{a}_n, \bar{b}_n \in A$ are tuples with $|\bar{a}_{n+1}| = |\bar{a}_n| + |\bar{b}_n|$,\n
(L3) $|\text{ran}(p)| = |\bar{a}_r| + |\bar{b}_r|$,\n
(L4) $\text{dom}(p)$ and $\text{ran}(p)$ include the first $r$ elements of $B$ and $A$ respectively,\n
(L5) $\bar{b}_n$ is $\alpha$-free over $\bar{a}_n$, where $\alpha = \max_{m \leq n} H(m)$ (see (B7)).

Note that (L1)-(L4) are clearly computable, and that (L5) is computable by (iii).

If $\ell$ has first coordinate $p$, and $\ell'$ has first coordinate $p'$, then for $\xi < \eta$, we set $\ell \leq^\ell \ell'$ if and only if $p \leq^\xi p'$, that is, if and only if $\text{ran}(p) \leq^\xi \text{ran}(p')$ as substructures of $A$ under the usual back-and-forth relations.

Then $(\leq^\ell)_\xi \leq^\eta$ is nested since the usual back-and-forth relations are, and $(\leq^\ell)^\eta$ are computable by (ii).

Let $P$ consist of the sequences $\ell_0, \ldots, \ell_r$ such that

(P1) if

$$\ell_n = \langle p; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \ldots, (\bar{a}_n, \bar{b}_n) \rangle$$

then

$$\ell_{n+1} = \langle p^*; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \ldots, (\bar{a}_n, \bar{b}_n), (\bar{a}_{n+1}, \bar{b}_{n+1}) \rangle$$

with $\text{dom}(p) \subseteq \text{dom}(p^*)$, 
CHAPTER 7. COMPUTABLE CATEGORICITY

Lemma 7.6.3. Then \( p \leq \eta \) if and only if \( i \leq n \),

(P2) for each \( n \), if

\[
\ell_n = \langle p, (a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n) \rangle
\]

then for each \( i \), \( \text{ran}(p \upharpoonright (a_i, b_i)) \cong \beta_i \) if and only if \( i \leq n \),

(P3) if \( m \leq n \), \( \ell_m \) has first coordinate \( p_m \), and \( \ell_n \) has first coordinate \( p_n \), then \( p_m \leq p_n \).

Note that (P1) and (P3) are computable, and that (P2) is computable by (i).

Given

\[
\ell_n = \langle p, (a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n) \rangle,
\]

let \( E(\ell) \) be the partial atomic diagram on \( B \) obtained by the pullback along \( p \) (using only the first \(|p| \) logical symbols).

Note that \( E(\ell) \) is computable, and if \( \ell_0 \leq \ell_1 \) with first coordinates \( p_0 \) and \( p_1 \), respectively, then \( p_0 \leq p_1 \), so that \( E(\ell_0) \leq E(\ell_1) \).

Thus we have an \( \eta \)-system \( (L, P, (\leq^L_\xi), \leq^\eta, E) \).

Lemma 7.6.3. The \( \eta \)-system \( (L, P, (\leq^L_\xi), \leq^\eta, E) \) has the extendability condition.

Proof. Suppose we have a finite \( \delta \)-run \( \pi = \langle \ell_0, \ldots, \ell_{s-1} \rangle \), and let \( s_k = s_{k-1} \ldots s_0 = s - 1 \), and \( \xi_0 < \xi_1 < \ldots < \xi_{k-1} < \eta \) be the associated sequences of stages and ordinals to \( s \), as in Definition 7.5.11. Suppose that for each \( i \), the first coordinate of \( \ell_{s_i} \) is \( q_{s_i} \).

Claim 7.6.4. There exists \( \xi < \xi \) such that \( q_{s_i} \leq \xi \) for \( 0 \leq i \leq k \).

Proof. We construct \( p \) inductively as follows. We let \( q_{s_0} = q_{s_0} \), and for \( 0 \leq i \leq k \), let \( q_{s_{i+1}} \geq q_{s_{i+1}} \) be such that \( q_{s_i} \leq \xi_i \), \( q_{s_{i+1}} \). This is possible since \( q_{s_{i+1}} \leq \xi_{i+1} \) \( q_{s_i} \) and since \( q_{s_i} \geq q_{s_i} \). Let \( p = q_{s_k} \).

Then certainly \( q_{s_i} \leq \xi_i \) \( q_{s_{i+1}} \) and \( q_{s_{i+1}} \leq \xi_{i+1} \), it follows inductively that each \( q_{s_i} \leq \xi_i \).

Since \( q_{s_k} \geq q_{s_i} \), we have \( q_{s_i} \leq \xi_i \) \( q_{s_k} \) as desired.

Let

\[
\ell_{s_0} = \ell_{s-1} = \langle \ell_{s-1}; (a_0, b_0), (a_1, b_1), \ldots, (a_{s-1}, b_{s-1}) \rangle.
\]

Claim 7.6.5. There exists \( \xi < \xi \) such that \( q_{s_i} \leq \xi \) \( q_{s_k} \) for \( 0 \leq i < k \) and such that \( \text{ran}(p \upharpoonright (a_n, b_n)) \neq \beta_n \) for \( s_k < n \leq s_0 = s - 1 \).

Proof. Let \( p \geq q_{s_k} \) be as in the previous claim. We will use Lemma 7.6.1. Let \( \bar{x} = \text{ran}(p) \) and \( n = s_0 - s_k \). For \( i = 1, \ldots, n \), let \( \bar{u}_i = a_{s_k+i} \) and \( \bar{v}_i = b_{s_k+i} \). For \( i = 1, \ldots, n \), let \( \alpha_i = \max_{1 \leq j \leq s_k+i} H(j) \) and let \( \beta_i = \xi_j \) where \( j \) is such that \( s_{j+1} < i \leq s_j \). Note that by (L5), \( \bar{v}_i \) is \( \alpha_i \)-free over \( \bar{u}_i \) and that \( \beta_1 \geq \beta_2 \geq \cdots \). Also, if \( s_{j+1} < i \leq s_j \), then since \( s_{j+1} = t_{\xi_{j+1}}, i \notin \xi_{j+1} \text{ s s} \). So \( \alpha_i \geq H(i) \geq \xi_j + 1 > \xi_j = \beta_i \). Let \( \bar{y} \) be the tuple we get by applying Lemma 7.6.1 and let \( p^* \) map the domain of \( p \) to \( \bar{y} \). Then

\[
p^* \upharpoonright (a_{s_k}, b_{s_k}) = p \upharpoonright (a_{s_k}, b_{s_k}) \geq q_{s_k}
\]

and so \( p^* \geq q_{s_k} \). Also,

\[
q_{s_i} \leq \xi_i, p \upharpoonright (a_{s_i}, b_{s_i}) \leq \xi_i, p^* \upharpoonright (a_{s_i}, b_{s_i})
\]

and so \( q_{s_i} \leq \xi_i^* \). Finally, for \( i = s_k + 1, \ldots, s_0 \), \( p^* \upharpoonright (a_i, b_i) \neq \beta_i \).
We claim that \( \ell \) is an isomorphism \( \text{Lemma 7.6.7.} \). Let \( \bar{a}_s = \text{ran}(p^*) \), and let \( \bar{b}_s \) be \( \alpha \)-free over \( \bar{a}_s \) where \( \alpha = \max_{t \leq s} H(t) \), and such that \( \bar{a}_s \bar{b}_s \) contains the first \( s \)-many elements of \( A \). Let \( \bar{c} \) be a new set of constants in \( B \) and let \( p^{**} = p^* \cup \{ \bar{c} \mapsto \bar{b}_s \} \). Let

\[
\ell_s = (p^{**}; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \ldots, (\bar{a}_{s-1}, \bar{b}_{s-1}), (\bar{a}_s, \bar{b}_s)).
\]

We claim that \( \ell_0, \ldots, \ell_s \) is in \( P \). That (L1), (L2), and (L3) hold is clear. (L4) and (L5) follow from the choice of \( \bar{b}_s \). (P1) is also clear. (P3) follows from the fact that \( p^{**} \supseteq q_{s_k} \) and \( s_k \) was maximal with \( s_k \subseteq s \).

For (P2), if \( i \leq s_k \), then since \( p^{**} \supseteq q_{s_k} \) and (P2) held at stage \( s_k \), \( \text{ran}(p^{**} \upharpoonright_{[\bar{a}_i, \bar{b}_i]}) \cong \bar{a}_i \bar{b}_i \) if and only if \( i \leq s_k \) and since \( s_k \leq s \), \( i \leq s_k \) if and only if \( i \leq s \) by (B8) and (B9). If \( s_k < i < s \), then since \( s_k \) is maximal with \( s_k \leq s \), \( i \leq s \) and by choice of \( p^* \) in the second claim above, \( \text{ran}(p^{**} \upharpoonright_{[\bar{a}_i, \bar{b}_i]}) \cong \bar{a}_i \bar{b}_i \). The case \( i = s \) is clear. Hence \( \pi \upharpoonright \ell_s \in P \).

Since \( p^{**} \supseteq q_{s_k} \), \( s_{k} \leq \xi \), \( p^* \subseteq p^{**} \). This completes the proof of the extendability condition. \( \square \)

By the metatheorem, there is a computable 0-run \( \pi = \ell_0 \ell_1 \ldots \) for \( (L, P, (\leq^L_i)_{i \leq \eta}, E) \). \( E(\pi) \) is the diagram of a structure on \( B \). For each \( j \), let

\[
\ell_j = (p_j; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \ldots, (\bar{a}_j, \bar{b}_j)).
\]

Then, along the true stages, by (P3) the \( p_i \) are nested, and by (L4) they form a bijection \( B \to A \). By definition of \( E \), they are an isomorphism \( B \to A \).

**Lemma 7.6.6.** Let \( f : B \to A \) be an isomorphism. Then \( f \geq_T \Delta^0_\eta \).

**Proof.** Using \( f \) we will compute the true path \( i_1 \leq i_2 \leq \ldots \). Then we can compute \( \nabla^\eta = \bigcup_{i_n} \nabla^\eta_{i_n} \). We claim that \( \ell_j \) is a true stage if and only if

\[
(\ast) \quad \text{ran}(f \upharpoonright_{[\bar{a}_j, \bar{b}_j]}) \cong \bar{a}_j \bar{b}_j.
\]

Note that (\( \ast \)) is computable in \( f \), and so this will complete the proof.

If \( j \) is a true stage, then \( p_j \) extends to an isomorphism \( B \to A \). Since \( f \) is also an isomorphism, there is an automorphism of \( A \) taking \( \text{ran}(f \upharpoonright_{\text{dom}(p_j)}) \), as an ordered tuple, to \( \text{ran}(p_j) \). By (P2), we have \( \text{ran}(p_j \upharpoonright_{[\bar{a}_j, \bar{b}_j]}) \cong \bar{a}_j \bar{b}_j \) and so we have (\( \ast \)).

If \( j \) satisfies (\( \ast \)), then we claim that \( j \) is a true stage. Suppose not, and let \( p = \bigcup_{i_n} p_{i_n} \) be the isomorphism \( B \to A \) along the true path. Let \( i_n \) be such that \( j < i_n \). Then by (B10), \( j \not\equiv i_n \), and so \( \text{ran}(p_{i_n} \upharpoonright_{[\bar{a}_j, \bar{b}_j]}) \not\cong \bar{a}_j \bar{b}_j \). Since \( p_{i_n} \subseteq p \) and \( f \) is also an isomorphism \( B \to A \), we have

\[
\text{ran}(f \upharpoonright_{[\bar{a}_j, \bar{b}_j]}) \cong \text{ran}(p_{i_n} \upharpoonright_{[\bar{a}_j, \bar{b}_j]}) \not\cong \bar{a}_j \bar{b}_j.
\]

This contradicts (\( \ast \)). So \( j \) is a true stage. \( \square \)

**Lemma 7.6.7.** There is an isomorphism \( f : B \to A \) with \( \Delta^0_\eta \geq_T f \).
CHAPTER 7. COMPUTABLE CATEGORICITY

Proof. Using $\Delta^0_\eta$ we can compute the true path $i_1 \leq i_2 \leq \ldots$. Then along this path we compute an isomorphism $f = \bigcup_n p_{i_n}$ from $B \rightarrow A$.

This completes the proof.

As before, we can improve the statement of the theorem slightly as follows using Knight’s theorem on the upwards closure of degree spectra.

Corollary 7.6.8. Let $A$ be a countable structure. If $\eta$ is an ordinal and $A$ is not $\Delta^0_\beta$ categorical on any cone for any $\beta < \eta$, then there exists an $e$ such that for all $d \geq e$, there exists a $d$-computable copy $B$ of $A$ such that $\Delta^0_\eta(d)$ computes an isomorphism between $A$ and $B$, and every such isomorphism computes $\Delta^0_\eta(d)$.

Proof. Take $e$ as guaranteed by the theorem, with $e$ computing $A$ and $\eta$, and fix $d \geq e$. Let $B$ be as guaranteed by Theorem 7.1.5. Since $B$ is $d$-computable, by the proof of Knight’s upward closure theorem [Kni86], there exists $C$ such that $\text{deg}(C) = d$ and such that there exists a $d$-computable isomorphism $h : C \simeq B$. Now since $A$ is $e$-computable and $\text{deg}(C) = d$, any isomorphism $g : A \simeq C$ computes $d$. Since $d$ computes $h$, $g$ computes the isomorphism $g \circ h : B \simeq A$ and hence $\Delta^0_\eta(d)$. Moreover, $d$ computes an isomorphism between $A$ and $B$, and hence between $A$ and $C$.

It is now simple to extract Theorem 7.1.5 from the above result.

Proof of Theorem 7.1.5. Let $A$ be a countable structure. By Remark 7.2.5, there is an ordinal $\alpha$ such that $A$ is $\Delta^0_\alpha$ categorical on a cone. Let $\alpha \geq 1$ be the least such. By Corollary 7.6.8, there is a cone on which $A$ and $\alpha$ are computable such that for every $d$ in the cone, there exists a $d$-computable copy $B$ of $A$ such that every isomorphism between $A$ and $B$ computes $\Delta^0_\alpha(d)$. Thus $A$ has $\Delta^0_\alpha$-complete strong degree of categoricity on this cone.
Part III

Functors and Interpretations
Chapter 8

Computable Functors and Effective Interpretations

The results presented in this chapter appeared in [HTMMM]. They are joint work with Alexander Melnikov, Russell Miller, and Antonio Montalbán and appear here with their permission.

8.1 Introduction

The main purpose of this paper is to establish a connection between two standard methods of computable structure theory for reducing one structure into another one. One of this methods, effectively interpretability (Definition 8.1.2), is purely syntactical and is an effective version of the classical notion of interpretability in model theory. It is equivalent to the well-studied notion of $\Sigma$-reducibility. The other method is purely computational: it involves computing copies of one structure from copies of the other, and computing isomorphisms between copies of the first structure from isomorphisms between copies of the second. For this we use computable functors (Definition 8.1.4), a notion initiated just recently in [MPSS].

In computable structure theory we study complexity issues related to mathematical structures. One of the objectives of the subject is to measure the complexity of structures. There are three commonly used methods to compare the complexity of structures: Muchnik reducibility, Medvedev reducibility, and $\Sigma$-reducibility. The first two are computational, in the sense that they are about copies of a structure computing other copies; while the third one is purely syntactical. They are listed from weakest to strongest and none of the implications reverse (as proved by Stukachev [Stu08] and Kalimullin [Kal09]).

Effective Interpretability

Informally, a structure $\mathcal{A}$ is effectively interpretable in a structure $\mathcal{B}$ if there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$ (as in model theory [Mar02, Definition 1.3.9]), but where the domain
of the interpretation is allowed to be a subset of $B^\omega$ (while in the classical definition it is required to be a subset of $B^n$ for some $n$), and where all sets in the interpretation are required to be “computable within the structure” (while in the classical definition they need to be first-order definable).\footnote{We remark that this definition is slightly different from what the fourth author called effective-interpretability in [Mon13a, Definition 1.7], as we now allow the domain to be a subset of $B^\omega$ rather than $B^n$ for some $n$, and we do not allow parameters in the definitions.} Here, by “computable within the structure” we mean uniformly relatively intrinsically computable (see Definition 8.1.1). Effective interpretability is among the strongest notions of reducibility between structures that are usually considered. It gives a very concrete way of producing the structure $A$ from the structure $B$, and hence implies that essentially any kind of information encoded in $A$ is also encoded in $B$.

Effective interpretability is equivalent to the parameterless version of the notion of $\Sigma$-definability, introduced by Ershov [Ers96] and widely studied in Russia over the last twenty years (for instance [Puz09, Stu07, Stu08, Stu13, MK08, Kal09]). The standard definition of $\Sigma$-definability is quite different in format: it uses the first-order logic over $\text{HF}(B)$, the structure of hereditarily finite sets over $B$, instead of the computably infinitary language over $B^\omega$. For a more detailed discussion of the equivalence between effective interpretability and $\Sigma$-definability see [Mon12a, Section 4].

Before giving the formal definition, we need to review one more concept.

**Definition 8.1.1.** A relation $R$ on $A^\omega$ is said to be **uniformly relatively intrinsically computably enumerable (u.r.i.c.e.)** if there is a c.e. operator $W$ such that for every copy $(B, R^B)$ of $(A, R)$, $R^B = W^{D(B)}$. A relation $R$ on $A^\omega$ is said to be **uniformly relatively intrinsically computable (u.r.i. computable)** if there is a computable operator $\Psi$ such that for every copy $(B, R^B)$ of $(A, R)$, $R^B = \Psi^{D(B)}$.

(Here $D(B)$ refers to the atomic diagram of $B$; it is an infinite binary sequence that encodes the truth of all the atomic facts about $\vec{B}$. See [Mon12a, Section 2], for instance, for a formal definition).

These relations are the analogues of the c.e. and computable subsets of $\omega$ when we look at relations on a structure. They are computability theoretic notions, but they can be characterized in purely syntactical terms: It follows from the results in Ash, Knight, Manasse and Slaman [AKMS89], and Chisholm [Chi90] that a relation $R$ is u.r.i.c.e. if and only if it can be defined by a computably infinitary $\Sigma_1$ formula without parameters; a relation $R$ is u.r.i. computable if both it and its complement can be defined by computably infinitary $\Sigma_1$ formulas without parameters. (We will use $\Sigma^n_i$ to denote the computably infinitary $\Sigma_1$ formulas, and the same for $\Delta^n_i$, $\Pi^n_i$, etc.) (We will use $\Sigma^c_i$ to denote the computably infinitary $\Sigma_1$ formulas, and the same for $\Delta^c_i$, $\Pi^c_i$, etc.) These theorems were originally proved for $R \subseteq A^n$ for some $n$, but they also hold for $R \subseteq A^\omega$ (see [Mon12a, Theorem 3.14]). In this latter case, we say that $R$ is $\Sigma^n_i$-definable if there is a computable list $\varphi_1, \varphi_2, \varphi_3, \ldots$ of $\Sigma^n_i$ formulas defining $R \cap A^1$, $R \cap A^2$, $R \cap A^3$, ... respectively. The use of $A^\omega$ is not just to be able to take subsets of the different $A^n$ at the same time. Traditionally, computability theory is usually developed...
by considering subsets of \( \omega \) and this is workable because every finite object can be coded by a natural number. In the same way, when we are talking about computability over a structure, \( \mathcal{A}^{\omega} \) is the simplest domain where we can develop computability without losing generality. For instance, it is not hard to see that we can easily encode subsets of \((\mathcal{A}^{\omega}) \times \omega\) by subsets of \(\mathcal{A}^{\omega}\) in an effective way\(^2\) so that we can talk about r.i.e. subsets of \((\mathcal{A}^{\omega}) \times \omega\), etc. Thus, we say that a sequence of relations \((R_i : i \in \omega)\) where \(R_i \subseteq \mathcal{A}^{\omega}\) is r.i.e. or \(\Sigma^0_1\)-definable if it is as a subset of \((\mathcal{A}^{\omega}) \times \omega\).

Throughout the rest of the paper, we assume that all our structures have a computable language. Without loss of generality, we may further assume that all languages considered are relational.

**Definition 8.1.2.** We say that a structure \( \mathcal{A} = (A; P_0^A, P_1^A, \ldots) \) (where \( P_i^A \subseteq A^{a(i)} \)) is **effectively interpretable** in \( \mathcal{B} \) if there exist a \( \Delta^0_1 \)-definable (in the language of \( \mathcal{B} \), without parameters) sequence of relations \((\text{Dom}_{\mathcal{B}}^A, \sim, R_0, R_1, \ldots)\) such that

1. \( \text{Dom}_{\mathcal{B}}^A \subseteq \mathcal{B}^{\omega} \)
2. \( \sim \) is an equivalence relation on \( \text{Dom}_{\mathcal{B}}^A \)
3. \( R_i \subseteq (\mathcal{B}^{\omega})^{a(i)} \) is closed under \( \sim \) within \( \text{Dom}_{\mathcal{B}}^A \)

and there exists a function \( f_{\mathcal{A}}^B : \text{Dom}_{\mathcal{B}}^A \rightarrow A \) which induces an isomorphism:

\[
(\text{Dom}_{\mathcal{B}}^A / \sim; R_0 / \sim, R_1 / \sim, \ldots) \cong (A; P_0^A, P_1^A, \ldots),
\]

where \( R_i / \sim \) stands for the \( \sim \)-collapse of \( R_i \).\(^3\)

As important as the notions of reducibility between structures are the notions of equivalence between structures. Despite extensive study of effective interpretability and \( \Sigma \)-definability, over the last couple of decades, the associated notion of bi-interpretability has not been considered until recently [Monb, Definition 5.2]. Let us remark that the notion of \( \Sigma \)-equivalence between structures, which says that two structures are \( \Sigma \)-definable in each other, has been studied ([Stu13]), but the notion of bi-interpretability we are talking about is much stronger. Informally: two structures \( \mathcal{A} \) and \( \mathcal{B} \) are **effectively bi-interpretable** if they are effectively interpretable in each other, and furthermore, the compositions of the interpretations are \( \Delta^0_1 \)-definable in the respective structures. In other words, when two structures

---

\(^2\)For example, \((b_0, \ldots, b_k, m)\) can be coded by the definable class of tuples of the form \((b', b_0, \ldots, b_k, b', \ldots (m \text{ times}) \ldots, b')\) where \( b' \neq b_i \) for all \( i \). Different choices of \( b' \) will code the same tuple, but we can identify all such codes later when we introduce a definable equivalence relation upon the domain.

\(^3\) In previous definitions in the literature, \( \text{Dom}_{\mathcal{A}}^B \) was asked to be \( \Sigma^0_1 \)-definable instead of \( \Delta^0_1 \)-definable (see for instance [Mon13a, Definition 1.7] and [Monb, Definition 5.1]). But in fact one can demonstrate these definitions are equivalent. Indeed, given a \( \Sigma \)-interpretation, with the domain consisting of the tuples \( \bar{x} \) satisfying a countable disjunction of formulas \( \exists \bar{s} \varphi_1(\bar{x}, \bar{s}) \), we find a new domain consisting of the tuples \( (\bar{x}, \bar{s}, i) \) with \((\bar{x}, \bar{s}) \) satisfying \( \varphi_i \), and we have \((\bar{x}, \bar{s}, i)\) equivalent to \((\bar{y}, \bar{t}, j)\) iff \( \bar{x} \) and \( \bar{y} \) are equivalent in the \( \Sigma \)-interpretation.
interpret each other, we have that $A$ can be interpreted as a structure inside $B^{\omega}$, and that $B^{\omega}$ can be interpreted as a structure inside $(A^{\omega})^{\omega}$. Thus, we have an interpretation of $A$ inside $(A^{\omega})^{\omega}$. For bi-interpretability, we require that the isomorphism between $A$ and its interpretation inside $(A^{\omega})^{\omega}$ be $\Delta^0_1$-definable, and the same for the isomorphism between $B$ and its interpretation inside $(B^{\omega})^{\omega}$.

**Definition 8.1.3.** Two structures $A$ and $B$ are **effectively bi-interpretable** if there are effective interpretations of each structure in the other as in Definition 8.1.2 such that the compositions $f_B^A \circ f_A^B : \text{Dom}_B(\text{Dom}_A^B) \rightarrow B$ and $f_A^B \circ f_B^A : \text{Dom}_A(\text{Dom}_B^A) \rightarrow A$ are u.r.i. computable in $B$ and $A$ respectively. (Here $\text{Dom}_B(\text{Dom}_A^B) \subseteq (\text{Dom}_A^B)^{\omega}$, and the function $f_A^B : (\text{Dom}_A^B)^{\omega} \rightarrow A^{\omega}$ is the obvious extension of $f_A^B : \text{Dom}_A^B \rightarrow A$ mapping $\text{Dom}_B(\text{Dom}_A^B)$ to $\text{Dom}_A^B$.)

When two structures are effectively bi-interpretable, they look and feel the same from a computability point of view. In [Monb, Lemma 5.3] the fourth author shows that if $A$ and $B$ are effectively bi-interpretable then: they have the same degree spectrum; they have the same computable dimension; they have the same Scott rank; their index sets are Turing equivalent (assuming the structures are infinite); $A$ is computably categorical if and only if $B$ is; $A$ is rigid if and only if $B$ is; $A$ has the c.e. extendability condition if and only if $B$ does; for every $R \subseteq A^{\omega}$, there is a $Q \subseteq B^{\omega}$ which has the same relational degree spectrum, and vice-versa; and the jumps of $A$ and $B$ are effectively bi-interpretable too.

**Computable Functors**

One of the most common ways of describing the computational complexity of a structure is by its degree spectrum. Associated with the degree spectrum is the notion of Muchnik reducibility: A structure $A$ is **Muchnik reducible** to a structure $B$ if every copy of $B$ computes a copy of $A$, or (equivalently for non-trivial structures) if $\text{dgSp}(A) \subseteq \text{dgSp}(B)$. The uniform version of this reducibility is called Medvedev reducibility: A structure $A$ is **Medvedev reducible** to a structure $B$ if there is a Turing functional $\Phi$ that, given a copy of $B$ as an oracle, outputs a copy $\Phi^B$ of $A$. It is easy to see that if $A$ is effectively interpretable in $B$, we can use the interpretation to build a Turing functional giving a Medvedev reduction from $B$ to $A$.

Stukachev [Stu08] and Kalimullin [Kal09] showed that this implication cannot be reversed.

Here we consider a strengthening of Medvedev reducibility that is equivalent to effective interpretability. In [MPSS], Poonen, Schoutens, Shlapentokh, and the third author introduced **computable functors**, which are Medvedev reductions in which the Turing functional $\Phi$ is also required to preserve isomorphisms, in the following sense. Given an isomorphism between two copies of $B$, we want an effective way to compute an isomorphism between the two copies of $A$ that we get by applying $\Phi$. Following [MPSS], we will define this more precisely using the language of category theory.
Throughout the paper, we write $\text{Iso}(A)$ for the isomorphism class of a countably infinite structure $A$: 
\[ \text{Iso}(A) = \{ \hat{A} : \hat{A} \cong A \& \text{dom}(\hat{A}) = \omega \}. \]
We will regard $\text{Iso}(A)$ as a category, with the copies $\hat{A}$ of the structure $A$ as its objects and the isomorphisms among them as its morphisms.

**Definition 8.1.4** (cf. [MPSS], Definition 3.1). By a functor from $A$ to $B$ we mean a functor from $\text{Iso}(A)$ to $\text{Iso}(B)$, that is, a map $F$ that assigns to each copy $\hat{A}$ in $\text{Iso}(A)$ a structure $F(\hat{A})$ in $\text{Iso}(B)$, and assigns to each morphism $f: \hat{A} \to \hat{A}$ in $\text{Iso}(A)$ a morphism $F(f): F(\hat{A}) \to F(\hat{A})$ in $\text{Iso}(B)$ so that the two properties hold below:

(N1) $F(\text{id}_{\hat{A}}) = \text{id}_{F(\hat{A})}$ for every $\hat{A} \in \text{Iso}(A)$, and

(N2) $F(f \circ g) = F(f) \circ F(g)$ for all morphisms $f, g$ in $\text{Iso}(A)$.

A functor $F: \text{Iso}(A) \to \text{Iso}(B)$ is computable if there exist two computable operators $\Phi$ and $\Phi_*$ such that

(C1) for every $\hat{A} \in \text{Iso}(A)$, $\Phi^{D(\hat{A})}$ is the atomic diagram of $F(\hat{A}) \in \text{Iso}(B)$;

(C2) for every morphism $f: \hat{A} \to \hat{A}$ in $\text{Iso}(A)$, $\Phi_*^{D(\hat{A})} \circ f \circ D(\hat{A}) = F(f)$.

Recall that $D(\hat{A})$ denotes the atomic diagram of $\hat{A}$. We will often identify a computable functor with the pair $(\Phi, \Phi_*)$ of Turing operators witnessing its computability.

Notice that $\Phi$, without $\Phi_*$, gives a Medvedev reduction from $\text{Iso}(A)$ to $\text{Iso}(B)$. From the examples in the literature of Medvedev reducibilities, some turn out to be effective functors, but not all. A principal goal of the work in [MPSS] was to give a computable functor from the category $\mathcal{C}$ of countable graphs (on the domain $\omega$, with isomorphisms as the morphisms) to the category $\mathcal{D}$ of countable fields (defined similarly), with another computable functor as its left-inverse. The authors there accomplished this highly non-trivial task, and the functors in [MPSS] provide the best existing example of our Definition 8.1.10 below.

Our first main result connects computable functors and effective interpretability.

**Theorem 8.1.5.** Let $A$ and $B$ be countable structures. Then $A$ is effectively interpretable in $B$ if and only if there exists a computable functor from $B$ to $A$.

We prove Theorem 8.1.5 in Section 8.2. It is well-known in model theory that an elementary first-order interpretation of one structure in another gives rise to a functor. One can find a treatment of this fact in Hodges’s book [Hod93, pp. 216–218]. The corresponding direction in Theorem 8.1.5—from left to right—is rather straightforward, and the only new thing is to consider the effectiveness of the functor. The interesting direction is to build an interpretation out of a functor.

Our proof of Theorem 8.1.5 not only shows the existence of such an interpretation, but actually it builds a correspondence between functors and interpretations. This last
observation, which we will discuss in Proposition 8.1.7 and Section 8.3, is quite important. For instance, when \( A \) has a computable copy Theorem 8.1.5 is trivial and Proposition 8.1.7 is still meaningful: in this case we always have an effective interpretation of \( A \) into \( B \) which ignores the structure in \( B \), and also a functor from \( B \) to \( A \) that always outputs the same computable copy of \( A \) and the identity isomorphism on it without consulting the oracle.

Let us now explain how is that Proposition 8.1.7 extends Theorem 8.1.5. Suppose we have a computable functor \( F: \text{Iso}(B) \to \text{Iso}(A) \) whose effectiveness is witnessed by \((\Phi, \Phi_\ast)\). The backward direction of Theorem 8.1.5 says that \( A \) must be effectively interpretable in \( B \). Applying the forward direction of Theorem 8.1.5 to this effective interpretation, we get a computable functor based on this interpretation, denoting this new functor by \( \mathcal{I}^F \) (here \( \mathcal{I} \) stands for ‘interpretation’). We will show that these functors are isomorphic even in an effective way. The appropriate notion of equivalence is the following.

**Definition 8.1.6.** A functor \( F: \text{Iso}(B) \to \text{Iso}(A) \) is **effectively naturally isomorphic** (or just **effectively isomorphic**) to a functor \( G: \text{Iso}(B) \to \text{Iso}(A) \) if there is a computable Turing functional \( \Lambda \) such that for every \( \tilde{B} \in \text{Iso}(B) \), \( \Lambda_{\tilde{B}} \) is an isomorphism from \( F(\tilde{B}) \) to \( G(\tilde{B}) \), and the following diagram commutes for every \( \tilde{B}, \tilde{B}' \in \text{Iso}(B) \) and every morphism \( h: \tilde{B} \to \tilde{B}' \):

\[
\begin{array}{ccc}
F(\tilde{B}) & \xrightarrow{\Lambda_{\tilde{B}}} & G(\tilde{B}) \\
F(h) \downarrow & & \downarrow G(h) \\
F(\tilde{B}') & \xrightarrow{\Lambda_{\tilde{B}'}} & G(\tilde{B}')
\end{array}
\]

**Proposition 8.1.7.** Let \( F: \text{Iso}(B) \to \text{Iso}(A) \) be a computable functor. Then \( F \) and \( \mathcal{I}^F \) (defined above) are effectively isomorphic.

We prove Proposition 8.1.7 in Section 8.3.

Suppose that \( F \) and \( G \) are functors, and \( F \circ G \) and \( G \circ F \) are effectively isomorphic to the identity. The witness to \( G \circ F \) being effectively isomorphic to the identity functor is a Turing functional \( \Lambda_A \) which gives, for any \( \tilde{A} \in \text{Iso}(A) \), a map \( \Lambda^\tilde{A}_A: \tilde{A} \to G(F(\tilde{A})) \). Thus, applying the functor \( F \), we get a map \( F(\Lambda^\tilde{A}_A): F(\tilde{A}) \to F(G(F(\tilde{A}))) \). There is also a map \( \Lambda^F_{\tilde{A}}: F(\tilde{A}) \to F(G(F(\tilde{A}))) \) which is obtained from the Turing functional \( \Lambda_B \) which witnesses that \( F \circ G \) is effectively isomorphic to the identity functor. If these two maps \( F(\tilde{A}) \to F(G(F(\tilde{A}))) \) agree for every \( \tilde{A} \in \text{Iso}(A) \), and similarly with the roles of \( A \) and \( B \) switched, then we say that \( F \) and \( G \) are **pseudo-inverses**.

**Definition 8.1.8.** Two structures \( A \) and \( B \) with domain \( \omega \) are **computably bi-transformable** if there exist computable functors \( F: \text{Iso}(A) \to \text{Iso}(B) \) and \( G: \text{Iso}(B) \to \text{Iso}(A) \) which are pseudo-inverses.

**Theorem 8.1.9.** Let \( A \) and \( B \) be countable structures. Then \( A \) and \( B \) are effectively bi-interpretable iff \( A \) and \( B \) are computably bi-transformable.

We prove Theorem 8.1.9 in Section 8.4.
Effective Transformations of Classes

There has been much work in the last few decades analyzing which classes of structures can be reduced to others, and which are universal in the sense that the class of all structures reduces to them. The meaning of “reduces” has varied. The intuition is that one class reduces to another if every structure in the first class can be somehow encoded by a structure in the second class, and usually we want the encoding structure to have similar complexity as the structure being coded. For instance, a class is universal for degree spectra if every degree spectrum realized by some structure is realized by a structure in the class. The most celebrated paper in this direction was written by Hirschfeldt, Khoussainov, Shore and Slinko [HKSS02]. They defined what it means for a class to be complete with respect to degree spectra of nontrivial structures, effective dimensions, expansion by constants, and degree spectra of relations. Then they showed that undirected graphs, partial orderings, lattices, integral domains of arbitrary characteristic (and in particular rings), commutative semigroups, and 2-step nilpotent groups are all complete in these sense. Their definition is rather cumbersome and does not seem to be equivalent to our definitions below, but the definitions appear rather close in spirit.

Our intention is to apply the proofs of Theorems 8.1.5 and 8.1.9 to the situation in which one class $\mathcal{C}$ of countable structures is effectively interpretable in another class $\mathcal{D}$.

In what will follow, a class is a category of countable structures upon the domain $\omega$ and morphisms are permutations of $\omega$ that induce isomorphisms, and we also assume our classes are closed under such isomorphisms. (That is, if $A$ and $B$ are objects in the class, every isomorphism between them is a morphism in the class.) We can extend the definition of a computable functor to arbitrary classes (not necessarily of the form $\text{Iso}(A)$) by simply allowing the oracles of $\Phi$ and $\Phi_*$ to range over the objects and morphisms of an arbitrary class.

**Definition 8.1.10.** Say that a class $\mathcal{C}$ is uniformly transformally reducible to a class $\mathcal{D}$ there exist a subclass $\mathcal{D}'$ of $\mathcal{D}$ and computable functors $F: \mathcal{C} \rightarrow \mathcal{D}'$, $G: \mathcal{D}' \rightarrow \mathcal{C}$ such that $F$ and $G$ are pseudo-inverses.

The syntactical counterpart of the above definition is:

**Definition 8.1.11 ([Momb]).** Say that a class $\mathcal{C}$ is reducible via effective bi-interpretability to a class $\mathcal{D}$ if for every $C \in \mathcal{C}$ there is a $D \in \mathcal{D}$ such that $C$ and $D$ are effectively bi-interpretable and furthermore the formulae defining the interpretations and the isomorphisms do not depend on the concrete choice of $C$ or $D$.

We have:

**Theorem 8.1.12.** A class $\mathcal{C}$ is reducible via effective bi-interpretability to a class $\mathcal{D}$ iff $\mathcal{C}$ is uniformly transformally reducible to a class $\mathcal{D}$.

**Proof.** The proof of Theorem 8.1.9 is uniform in both directions. \qed
The interpretations defined by Hirschfeldt, Khoussainov, Shore and Slinko [HKSS02] yield the following: undirected graphs, partial orderings, and lattices are on top (or universal) for effective bi-interpretability (see [Monb, Section 5.2]). If we add a finite set of constants to the languages of integral domains, commutative semigroups, or 2-step nilpotent groups, they become on top for effective bi-interpretability too. The results in [MPSS] by B. Poonen, H. Schoutens, A. Shlapentokh, and the third author, which inspired the questions explored in this article, show that fields are also universal for effective bi-interpretability.

We propose in future work to explore extensions of our results to cover all interpretations of one structure in another via $\Sigma^c_{n+1}$ formulas. This should be feasible using the notion of the jump of a structure, as defined in [Mon09] by Montalbán and independently in [SS09] by Soskova and Soskov. Indeed, an interpretation of $A$ in $B$ via $\Sigma^c_{n+1}$ formulas is just an interpretation of $A$ in the $n$-th jump $B^{(n)}$ of $B$ via $\Sigma^c_1$ formulas. Thus Theorem 8.1.5 shows the existence of such an interpretation to be equivalent to the existence of a computable functor from $B^{(n)}$ into $A$ (that is, from $\text{Iso}(B^{(n)})$ into $\text{Iso}(A)$). Notice that the category $\text{Iso}(B^{(n)})$ is exactly the same category as $\text{Iso}(B)$; the only difference is that the atomic diagram of an object in the former category provides more information than the atomic diagram of the same object in the latter category, making it easier to give a computable functor from $\text{Iso}(B^{(n)})$ into $\text{Iso}(A)$. Likewise, Theorems 8.1.9 and 8.1.12 should both carry over to interpretations via $\Sigma^c_{n+1}$ formulas in a similar way, and presumably even to interpretations via $\Sigma^c_{\alpha+1}$ formulas as well, for all $\alpha < \omega^CK$.

It would be natural to use these connections to investigate various recent results in which elements of one category of structures are built from elements of another using such an interpretation. This was done in [Mell2] by Melnikov for ordered abelian groups, in [MM] by Marker and Miller to build differentially closed fields from arbitrary graphs, and in [Oca14] by Ocasio to build real closed fields from arbitrary linear orders. In most of these constructions, more attention was paid to building the structure than to computing isomorphisms: in [MM], for example, it was only required that the differential fields resulting from two graphs should be isomorphic if and only if the graphs themselves were isomorphic. (The works [Mell2] and [Mell10] do consider isomorphisms more closely, since they examine effective categoricity as well as Turing degree spectra of structures.) Our results in this article imply that the difficulty of computing an isomorphism between the resulting differential fields, given an isomorphism between the graphs, is an essential piece of the puzzle and should be examined more closely.

### 8.2 Proof of Theorem 8.1.5

We split the proof into two propositions, one proposition for each direction of Theorem 8.1.5. We start by quickly disposing of the easy direction.

**Proposition 8.2.1.** If $A$ is effectively interpretable in $B$, then there exists a computable functor from $\text{Iso}(B)$ to $\text{Iso}(A)$. 
CHAPTER 8. COMPUTABLE FUNCTORS

Proof. Suppose that \( \mathcal{A} \) is interpreted in \( \mathcal{B} \) via \( \text{Dom}_{\mathcal{A}}^B \sim \), and \( (R_i)_{i \in \omega} \) as in Definition 8.1.2. Given \( \overline{B} \in \text{Iso}(\mathcal{B}) \), we first define \( \overline{A} = F(\overline{B}) \) upon the domain \( \omega \) as follows. Notice that since the sequence of relations \( \text{Dom}_{\mathcal{A}}^B \sim \), and \( (R_i)_{i \in \omega} \) is \( \Delta_i \) definable in \( \mathcal{B} \), the respective interpretations in \( \overline{B} \) are uniformly computable from the open diagram \( D(\overline{B}) \) of \( \overline{B} \). Since \( \overline{B} \) has domain \( \omega \), we have that \( \text{Dom}_{\mathcal{A}}^B \subset \omega^{\omega} \) and using a fixed enumeration of \( \omega^{\omega} \) we get a bijection \( \overline{\tau} \):
\[
\overline{\tau} : \omega \rightarrow \text{Dom}_{\mathcal{A}}^B / \sim .
\]
Note that \( \overline{\tau} \) is uniformly computable from \( D(\overline{B}) \). Using \( \overline{\tau} \), we define relations \( P_i \) on \( \omega \) via the pull-back from \( (\text{Dom}_{\mathcal{A}}^B / \sim ; R_0^B, R_1^B, \ldots) \) along \( \overline{\tau} \), and let the resulting structure be \( F(\overline{B}) = \overline{A} \).

Also, given an isomorphism \( f : \overline{B} \rightarrow \overline{B} \), we need to define an isomorphism \( F(f) : F(\overline{B}) \rightarrow F(\overline{B}) \). Using the respective bijections \( \overline{\tau} \) and \( \overline{\tau} \) as above, and extending \( f \) to the domain \( \overline{B}^{\omega} \) in the obvious way, we define
\[
F(f) = \overline{\tau}^{-1} \circ f \circ \overline{\tau} : \overline{A} \rightarrow \overline{A}
\]
It is straightforward to check that the above definition of \( F \) gives a functor from \( \text{Iso}(\mathcal{B}) \) to \( \text{Iso}(\mathcal{A}) \).

We now move on to the more interesting direction.

Proposition 8.2.2. Suppose there exists a computable functor from \( \text{Iso}(\mathcal{B}) \) to \( \text{Iso}(\mathcal{A}) \). Then \( \mathcal{A} \) is effectively interpretable in \( \mathcal{B} \).

Proof. Let \( F = (\Phi, \Phi_*) \) be a computable functor from \( \text{Iso}(\mathcal{B}) \) into \( \text{Iso}(\mathcal{A}) \). We will produce \( \Sigma_i \) formulas for an effective interpretation of \( \mathcal{A} \) in \( \mathcal{B} \). We begin by introducing some notation and conventions. We will then define \( \text{Dom}_{\mathcal{A}}^B \) and \( \sim \) formally and prove several useful lemmas about them. After that we define \( R_i \) and show that our definitions suffice.

Notation and conventions. We identify a function \( f : \omega \rightarrow \omega \) with its graph, using \( \lambda \) to denote the identity function on \( \omega \). If \( \overline{x} = (x_0, \ldots, x_n) \) and \( \sigma \) is a permutation of \( \{0, \ldots, n\} \), then \( (\overline{x})_\sigma \) is the tuple \( (x_{\sigma(0)}, \ldots, x_{\sigma(n)}) \). For \( b \in \mathcal{B} \) we view \( \overline{b} \) as a partial map which takes the tuple \( (0, \ldots, |\overline{b}|-1) \) to \( (b_0, b_1, \ldots, b_{|\overline{b}|-1}) \). Viewing \( \overline{x} \) as a partial map, note that \( (\overline{x})_\sigma = \overline{x} \circ \sigma \).

If \( f \) is a map from \( \omega \) to the domain of \( \mathcal{B} \), then we can “pull back” the structure on \( \mathcal{B} \) along \( f \) to get a structure \( \mathcal{B}_f \) on \( \omega \) such that \( f : \mathcal{B}_f \rightarrow \mathcal{B} \) is an isomorphism. Given a tuple \( \overline{b} \in \mathcal{B} \) and \( f \circ \overline{b} \), we write \( D(\overline{b}) \) to denote the partial atomic diagram of \( (0, 1, \ldots, |\overline{b}|-1) \) in \( \mathcal{B}_f \) that mentions only the first \( |\overline{b}| \)-many relations. This partial atomic diagram will be typically identified with the finite binary string that, under some fixed Gödel numbering of the atomic formulas, encodes \( D(\overline{b}) \). Thus, \( D(\overline{b}) \) is an initial segment of the atomic diagram \( D(\mathcal{B}_f) \) of \( \mathcal{B}_f \). Furthermore, \( D(\mathcal{B}_f) = \bigcup_{n \in \omega} D(f \upharpoonright n) \). Note that \( D(\overline{b}) \) does not really depend on a particular choice of \( f \) as long as \( f \circ \overline{b} \); it only depends on what atomic formulas hold of \( \overline{b} \).

Finally, for finite tuples \( \overline{b} \) and \( \overline{c} \), we write \( \overline{c} \smallsetminus \overline{b} \) for the set of elements that occur in \( \overline{c} \) but not in \( \overline{b} \).
Definitions of $\text{Dom}^B_A$ and $\sim$. Recall that $\mathcal{B}^{<\omega} \times \omega$ can be easily coded by elements of $\mathcal{B}^{<\omega}$. We define the domain $\text{Dom}^B_A$ and the equivalence relation $\sim$ upon that domain as follows:

$\text{Dom}^B_A$: Define $\text{Dom}^B_A$ to be the set of pairs $(\bar{b}, i) \in \mathcal{B}^{<\omega} \times \omega$ such that

$$\Phi^{D(b) \oplus \lambda \downarrow \overline{b} \oplus D(b)}_*(i) \downarrow \downarrow i.$$ 

$\sim$: For $(\bar{b}, i), (\bar{c}, j) \in \text{Dom}^B_A$, let $(\bar{b}, i) \sim (\bar{c}, j)$ if there exists a finite tuple $\bar{d}$ that does not mention elements from $\bar{b}$ and $\bar{c}$, such that if we let $\bar{c}'$ and $\bar{b}'$ list the elements in $\bar{c} \setminus \bar{b}$ and $\bar{b} \setminus \bar{c}$ respectively, and let $\sigma$ be the finite permutation with $(\bar{b}c'd) = (\bar{c}b'd)_\sigma$, then

$$\Phi^{D(bc'd)\oplus \sigma \oplus D(bc'd)}_*(i) \downarrow j \quad \text{and} \quad \Phi^{D(bc'd)\oplus \sigma^{-1} \oplus D(bc'd)}_*(j) \downarrow \downarrow i.$$ 

Intuitively, given $\bar{b} \in f$, we have that $D(\bar{b}) \in D(\mathcal{B}_f)$, and hence $\Phi^{D(\bar{b})}$ is a finite initial segment of $\Phi^{D(\mathcal{B}_f)}$ which is isomorphic to $\mathcal{A}$. The idea is that $(\bar{b}, i)$ will represent the $i$th element in the presentation $\Phi^{D(\mathcal{B}_f)}$ of $\mathcal{A}$. Of course, there are many possible $f: \omega \rightarrow \mathcal{B}$ extending $\bar{b}$, and the element $i$ on the different presentations $\Phi^{D(\mathcal{B}_f)}$ may correspond to different elements of $\mathcal{A}$. As we will see later, the condition we are imposing to have $(\bar{b}, i) \in \text{Dom}^B_A$ will guarantee that this $i$th element always corresponds to the same element in $\mathcal{A}$.

The intuition behind $\sim$ is that the partial diagrams $D(bc'd)$, $D(\bar{c}b'd)$, and the isomorphism between them are enough information for $\Phi_*$ to recognize that the element $i$ of $\Phi^{B_f}$ should be paired with the element $j$ of $\Phi^{B_g}$ for any $f \supset bc'd$ and $g \supset \bar{c}b'd$. We note that $\sigma \subseteq g^{-1} \circ f: \mathcal{B}_f \rightarrow \mathcal{B}_g$.

Properties of $\text{Dom}^B_A$ and $\sim$. Before we proceed, we verify that our definitions of $\text{Dom}^B_A$ and $\sim$ satisfy the nice properties that one would expect from the “right” definitions of $\text{Dom}^B_A$ and $\sim$.

Lemma 8.2.3. The set $\text{Dom}^B_A$ and its complement are both definable in the language of $\mathcal{B}$ by $\Sigma^1_i$-formulas without parameters.

Proof. One can simply observe that $\text{Dom}^B_A$ is $\text{u.r.i.}$ computable in $\mathcal{B}$, and hence $\Delta^1_i$-definable without parameters. However, let us also include a more syntactical proof to give the reader a better idea of what is going on. We can enumerate the diagrams $D(\bar{b})$ for which $\Phi^{D(\bar{b})\oplus \lambda \downarrow \overline{b} \oplus D(\bar{b})}_*(i)$ converges and is equal to $i$, and we can also compute the diagrams for which the computation diverges or does not equal $i$. Each of these finite partial diagrams corresponds to a quantifier-free formula about $\bar{b}$. (Notice that here “divergence” does not mean that the computation runs forever; indeed, $\Phi^{D(\mathcal{B})\oplus \lambda \oplus D(\mathcal{B})}_*$ must be total. Rather, we say that the computation diverges on an input if it demands information about $D(\mathcal{B})$ or about $\lambda$ that the finite oracle does not include, in which case we will recognize that the computation has diverged in this sense. If it fails to diverge in this sense, then it must in fact halt.) Then $\text{Dom}^B_A$ is defined by the computable disjunction of those formulas corresponding to diagrams.
where the computation converges and is equal to \( i \), and its complement is defined by the disjunction of the other formulas (i.e., where the computation diverges or is not equal to \( i \)).

To ensure that the same computable disjunction works for every structure \( \overline{B} \in \text{Iso}({\mathcal{B}}) \), we include in the disjunction every finite string \( \delta \) for which \( \Phi^d_{\delta}(i) \downarrow = i \) (where \( k \) is the length of the tuple about which \( \delta \) could be a fragment of an atomic diagram). After all, the functional \( \Phi_\ast \) has no particular idea which copy of \( {\mathcal{B}} \) it has for its oracle. Likewise, the computable disjunction defining the complement of \( \text{Dom}_A^\overline{B} \) includes every finite \( \delta \) for which \( \Phi^d_{\delta}(i) \) either converges to a value \( \neq i \), or diverges by demanding more information than \( \delta \) or \( \lambda \uparrow k \) contains (as described above). These are both \( \Sigma^c_1 \) disjunctions: there may exist certain \( \delta \) for which \( \Phi^d_{\delta}(i) \) neither converges nor demands too much information, but because \( (\Phi, \Phi_\ast) \) is assumed to be a computable functor, such a \( \delta \) cannot be an initial segment of the atomic diagram of any copy of \( {\mathcal{B}} \).

\[ \square \]

**Lemma 8.2.4.** The binary relation \( \sim \) and its complement are both definable in the language of \( B \) by \( \Sigma^c_1 \)-formulae without parameters.

**Proof.** It is clear that \( \sim \) has a \( \Sigma^c_1 \)-definition (the same argument as in Lemma 8.2.3). We claim that the complement of \( \sim \) also has a \( \Sigma^c_1 \)-definition, but this has a more complicated proof. Aiming for a definition of the complement of \( \sim \) (and slightly abusing notations), we define a new binary relation \( \sim \) as follows. Let \( (\overline{b}, i) + (\overline{c}, j) \) if there exist \( \overline{d} \) as in the definition of \( \sim \) except that

\[ \Phi^d_{\overline{b}}(i) \downarrow \neq j \text{ or } \Phi^d_{\overline{c}}(j) \downarrow \neq i. \]

If we show that \( + \) is equal to the complement of \( \sim \) (as the notation suggests) then we are done, since \( + \) clearly has a \( \Sigma^c_1 \)-definition. Thus, it is sufficient to prove that for \( (\overline{b}, i), (\overline{c}, j) \in \text{Dom}_A^\overline{B} \), we have exactly one of \( (\overline{b}, i) + (\overline{c}, j) \) and \( (\overline{b}, i) \sim (\overline{c}, j) \).

First, we will show that at least one of \( (\overline{b}, i) \sim (\overline{c}, j) \) or \( (\overline{b}, i) + (\overline{c}, j) \) holds. Let \( \overline{b}' \) and \( \overline{c}' \) be tuples consisting of the elements of \( \overline{b} \) but not in \( \overline{c} \), and in \( \overline{c} \) but not in \( \overline{b} \), respectively. Let \( \sigma \) be the map that matches the elements of \( \overline{b}, \overline{c}' \) with their natural copies in \( \overline{c}, \overline{b}' \), so that \( (\overline{b}, \overline{c'}) = (\overline{c}, \overline{b}')_{\sigma} \). Let \( f, g : \omega \to {\mathcal{B}} \) be bijections extending \( \overline{b}, \overline{c}' \) and \( \overline{c}, \overline{b}' \) respectively, and which coincide on all inputs \( i \geq |\overline{b}, \overline{c'}| = |\overline{c}, \overline{b}'| \). Thus, \( h = g^{-1} \circ f \) is a permutation of \( \omega \) extending \( \sigma \) which is constant on all inputs \( i \geq |\sigma| \). Recall that \( {\mathcal{B}}_f \) and \( {\mathcal{B}}_g \) are the structures in \( \text{Iso}({\mathcal{B}}) \) that we get by pulling back \( f \) and \( g \). Observe that \( h \) is an isomorphism from \( {\mathcal{B}}_f \) to \( {\mathcal{B}}_g \). Thus, for the choice of \( \Phi_\ast \), we must have

\[ \Phi^d_{\overline{b}}(i) \downarrow \neq j' \text{ and } \Phi^d_{\overline{c}}(j) \downarrow \neq i' \]

for some \( i', j' \in \omega \). Let us now consider an initial segment of these oracles where these computations still converge. That is, for some \( \overline{d} \) with \( \overline{b} \overline{c}' \overline{d} \preceq f \) and \( \overline{c} \overline{b}' \overline{d} \preceq g \), and for \( \sigma' \supseteq \sigma \) so that \( (\overline{b} \overline{c}' \overline{d}) = (\overline{c} \overline{b}' \overline{d})_{\sigma'} \), we have

\[ \Phi^d_{\overline{b}}(i) \downarrow \neq j' \text{ and } \Phi^d_{\overline{c}}(j) \downarrow \neq i'. \]
If \( i = i' \) and \( j = j' \), we get \((b, i) \sim (c, j)\), and if either \( i \neq i' \) or \( j \neq j' \), we get \((b, i) \not\sim (c, j)\).

Second, we show that \((b, i) \sim (c, j)\) and \((b, i) \not\sim (c, j)\) do not hold at the same time. Suppose the contrary. Let \( \sigma \) and \( \bar{d}_1 \) witness that \((b, i) \sim (c, j)\), and \( \tau \) and \( \bar{d}_2 \) witness that \((b, i) \not\sim (c, j)\). Without loss of generality, we may assume

\[
\Phi_*^{D(b'c'd_1) \circ \sigma \circ D(b'd_1)}(i) = j \quad \text{but} \quad \Phi_*^{D(b'c'd_2) \circ \tau \circ D(b'd_2)}(i) \neq j.
\]

Choose bijective maps from \( \omega \) to \( B \) such that

\[
f_1 \triangleright b'c'd_1, \quad g_1 \triangleright c'b'_d_1, \quad f_2 \triangleright b'c'd_2, \quad g_2 \triangleright c'b'_d_2.
\]

Then we have isomorphisms

\[
\begin{array}{c}
\xymatrix{
\mathcal{B} \ar[r]^{g_1^{-1} \circ f_1} & \mathcal{B} \ar[r]^{g_2^{-1} \circ g_1} & \mathcal{B} \ar[r]^{f_2^{-1} \circ g_2} & \mathcal{B} \ar[r]^{f_1^{-1} \circ f_1} & \mathcal{B} \\
\bar{b}'c'd_1 \ar@{<->}[rr]_{(\sigma)} & & \bar{c}'b'd_1 \ar@{<->}[rr]_{(\tau)} & & \bar{b}'c'd_2
\end{array}
\]

Since \( F \) is a functor, we have

\[
F(f_2^{-1} \circ f_1) = F(f_2^{-1} \circ g_2) \circ F(g_2^{-1} \circ g_1) \circ F(g_1^{-1} \circ f_1).
\]

Note that \( g_2^{-1} \circ g_1 \supset \lambda \mid \bar{c} \) and \( f_1^{-1} \circ f_2 \supset \lambda \mid \bar{b} \). Now, on the one hand, since \((b, i)\) and \((c, j)\) are in \( \text{Dom}_A^\mathcal{B} \), we have:

\[
F(f_1^{-1} \circ f_2)(i) = \Phi_*^{B_{f_1} \circ D(b'c'd_1)}(i) = \Phi_*^{D(b) \circ \lambda \mid \bar{b} \circ D(b)}(i) = i,
\]

\[
F(g_2^{-1} \circ g_1)(j) = \Phi_*^{B_{g_1} \circ (g_2^{-1} \circ g_1)}(j) = \Phi_*^{D(\bar{c}) \circ \lambda \mid \bar{c} \circ D(\bar{c})}(j) = j.
\]

On the other hand, since \( g_1^{-1} \circ f_1 \supset \sigma \) and \( g_2^{-1} \circ f_2 \supset \tau \) we have:

\[
F(g_1^{-1} \circ f_1)(i) = \Phi_*^{B_{f_1} \circ D(b'c'd_1)}(i) = \Phi_*^{D(b'c'd_1) \circ \sigma \circ D(b')}D(b'c'd_1)(i) = j,
\]

\[
F(g_2^{-1} \circ f_2)(i) = \Phi_*^{B_{f_2} \circ (g_2^{-1} \circ f_2)}(i) = \Phi_*^{D(b'c'd_2) \circ \tau \circ D(b')}D(b'c'd_2)(i) \neq i.
\]

Composing the latter three equation lines, we get that \( F(f_1^{-1} \circ f_2)(i) \neq j \), contradicting the first line.

\[
\square
\]

**Lemma 8.2.5.** On its domain, \( \text{Dom}_A^\mathcal{B} \), the relation \( \sim \) is an equivalence relation.
Proof. It is evident that ∼ is symmetric (use σ⁻¹) and reflexive (since (b, i) ∈ Dom₆). We show that ∼ is transitive. Suppose that (a, i), (b, j), and (c, k) are in Dom₆ and are such that (a, i) ∼ (b, j) and (b, j) ∼ (c, k).

Let b', a', d', σ witness (a, i) ∼ (b, j), and let c'', b'', d'', τ witness (b, j) ∼ (c, k) (see the definition of ∼). Let c'' be a string listing c \ a, and a'' be a string listing a \ c. Choose bijections from ω to B as follows:

\[
\begin{align*}
f_1 &\triangleright a'b'd' \\
f_2 &\triangleright b'a'd'
\end{align*}
\]

\[
\begin{align*}
g_1 &\triangleright b''d'' \\
g_2 &\triangleright c''d''
\end{align*}
\]

\[
\begin{align*}
h_1 &\triangleright a''c'' \\
h_2 &\triangleright b''c''
\end{align*}
\]

where h₁ and h₂ agree outside the initial segment of length |a| + |c''| = |c + a''|.

Since F is a functor, we have

\[F(h_2^{-1} \circ h_1) = F(h_2^{-1} \circ g_2) \circ F(g_2^{-1} \circ f_2) \circ F(g_1^{-1} \circ f_1) \circ F(f_1^{-1} \circ h_1).\]

Note also that (b, j) ∈ Dom₆ and g_1⁻¹ \circ f_2 \triangleright λ \upharpoonright |\tilde{b}| imply F(g_1⁻¹ \circ f_2)(j) = j. Similarly, F(f_1⁻¹ \circ h_1)(i) = i and F(h_2⁻¹ \circ g_2)(k) = k. We also have F(f_2⁻¹ \circ f_1)(i) = j and F(g_2⁻¹ \circ g_1)(j) = k by the choice of f_1, f_2, g_1 and g_2. Thus, F(h_2⁻¹ \circ h_1)(i) = k which must be witnessed by \(\Phi^B_{B_1 \circ (h_2^{-1} \circ h_1) \circ B_{h_2}}(i) = k\). A symmetric argument shows \(\Phi^B_{B_2 \circ (h_1^{-1} \circ h_2) \circ B_{h_1}}(k) = i\). Now recall that h₁ and h₂ agree outside the initial segment of length |a| + |c''| = |c| + |a''|. Thus, for some long enough \(\tilde{c}\) and for \(\rho \in h_2^{-1} \circ h_1\), the permutation mapping \(a''c''\tilde{c}\) to \(a''c''\tilde{c}\), we get a witness for (a, i) ∼ (c, k).

The following two lemmas will be useful later. Their proofs are not difficult and can be skipped in a first reading of the paper.

**Lemma 8.2.6.** For (b, i) ∈ Dom₆ there is an initial segment c = B \ n of B and j ∈ ω such that (b, i) ∼ (c, j).

For B \ n we mean the tuple that corresponds to \((0, 1, \ldots, n - 1)\) in this given presentation B.

**Proof.** Let n be sufficiently large that \(b \in B \ n\). Let σ be a permutation of \{0, \ldots, n - 1\} such that σ(0, \ldots, [b-1]) = b. Extend σ to a permutation f of ω by setting f to be the identity on \{n, n+1, \ldots\}. Then let j be such that F(f)(i) = \(\Phi^B_{\circ f \circ B(B)}(i) = j\). Since F is a functor,
\[ i = F(f^{-1}(j)) = \Phi^D(B) \oplus f^{-1} \oplus D(B)(j) \]. Let \( m > n \) be such that these computations use only the first \( m \) relation symbols and elements of \( B \). Let \( \bar{c}' = f([\bar{b}], \ldots, m-1) \) and \( \bar{c} = B \upharpoonright m \). Then \( \bar{b} \bar{c}' \) and \( \bar{c} \) contain the same elements. Let \( \tau = f \upharpoonright m \), so that \( (\bar{b} \bar{c}') = (\bar{c})_\tau \). Then \( (\bar{b}, i) \sim (\bar{c}, j) \) as witnessed by \( \tau \).

**Lemma 8.2.7.** If \( (\bar{b}, i) \) and \( (\bar{c}, j) \) are in \( \text{Dom}^B_A \), and \( \bar{b} \subseteq \bar{c} \), then \( (\bar{b}, i) \sim (\bar{c}, j) \) if and only if \( i = j \).

**Proof.** Since \( (\bar{b}, i) \in \text{Dom}^B_B \), we have
\[
\Phi^D(\bar{b} \upharpoonright \bar{b}) \oplus D(\bar{b})(i) = i.
\]
Let \( \bar{c}' = \bar{c} \setminus \bar{b} \) and let \( \bar{d} \) and \( \sigma \vdash \lambda \upharpoonright \bar{b} \) witness that either \( (\bar{b}, i) \sim (\bar{c}, j) \) or that \( (\bar{b}, i) \not\sim (\bar{c}, j) \) as in the proof of Lemma 8.2.4. Thus
\[
\Phi^D(\bar{b} \bar{c}') \oplus \sigma \oplus D(\bar{c} \bar{d})(i) = j',
\]
for some \( j' \), and \( (\bar{b}, i) \sim (\bar{c}, j) \) hold if and only if \( j' = j \). But the oracle for this computation extends the oracle \( D(\bar{b}) \oplus \lambda \upharpoonright \bar{b} \oplus D(\bar{b}) \). Therefore \( j' = i \). \( \square \)

**Defining the relations.** For each relation symbol \( P_i \) of arity \( p(i) \) in the language of \( A \) (recall that \( p \) is a computable function), we define a relation \( R_i \) on \( \text{Dom}^B_A \) as follows:

\[ R_i: \text{We let } (\bar{b}_1, k_1), \ldots, (\bar{b}_{p(i)}, k_{p(i)}) \text{ be in } R_i \text{ if there is a tuple } \bar{c} \text{ and } j_1, \ldots, j_{p(i)} \in \omega \text{ such that } (\bar{b}_s, k_s) \sim (\bar{c}, j_s) \text{ for each } 1 \leq s \leq p(i), \text{ and the atomic formula } P_i(j_1, \ldots, j_{p(i)}) \text{ is true in } \Phi^D(\bar{c}). \]

We define a relation \( Q_i \) the same way, except that \( Q_i \) requires \( P_i(j_1, \ldots, j_{p(i)}) \) to be false in \( \Phi^D(\bar{c}) \). (We will show \( Q_i \) is the complement of \( R_i \).)

**Lemma 8.2.4** combined with a standard argument (see, e.g., Lemma 8.2.3) imply that both \( R_i \) and \( Q_i \) are definable by a \( \Sigma^\omega_1 \) formula without parameters, and these formulae can be defined uniformly in \( i \). Alternatively, it is not hard to see they are u.r.i.c.e. The following lemma implies that \( (R_i: i \in \omega) \) is \( \Delta^\omega_1 \)-definable without parameters.

Fix \( i \). We suppress \( i \) in \( R_i, Q_i, \) and \( p(i) \).

**Lemma 8.2.8.** \( Q \) is the complement of \( R \) in \( \text{Dom}^B_A \).

**Proof.** First, we need to show that each \( (\bar{b}_1, i_1), \ldots, (\bar{b}_{p}, i_{p}) \) in \( \text{Dom}^B_A \) is either in \( Q \) or in \( R \). By Lemma 8.2.6 and Lemma 8.2.7, for some sufficiently long initial segment \( \bar{c} \) of the presentation \( B \), there are \( j_1, \ldots, j_p \) such that \( (\bar{b}_k, i_k) \sim (\bar{c}, j_k) \) for each \( 1 \leq k \leq p \). Now \( \Phi^D(\bar{b}) \) determines either that \( (j_1, \ldots, j_p) \) is in \( P \), or that it is not in \( P \). By extending \( \bar{c} \) to the use of this computation and using Lemma 8.2.7, we get that \((\bar{b}_1, i_1), \ldots, (\bar{b}_{p}, i_{p}) \) is either in \( Q \) or in \( R \).
We show that \((\bar{b}_1, i_1), \ldots, (\bar{b}_p, i_p)\) cannot both be in \(Q\) and in \(R\). Aiming for a contradiction, suppose that there are \(\bar{c}\) and \(\bar{d}\), and \(j_1, \ldots, j_p\) and \(k_1, \ldots, k_p\), such that \((\bar{b}_m, i_m) \sim (\bar{c}, j_m)\) and \((\bar{b}_m, i_m) \sim (\bar{d}, k_m)\) for \(1 \leq m \leq p\), and the atomic formula \(P(j_1, \ldots, j_p)\) is in \(\Phi_D^{\bar{c}}\), but \(\neg P(k_1, \ldots, k_p)\) is not in \(\Phi_D^{\bar{d}}\). Note that by the transitivity of \(\sim\), for each \(m\) we have \((\bar{c}, j_m) \sim (\bar{d}, k_m)\).

Let \(f \supseteq \bar{c}\) and \(g \supseteq \bar{d}\) be permutations \(\omega \to \mathcal{B}\). Then, since \(\Phi_D^{\bar{c}}\) says that \(P(j_1, \ldots, j_p)\) holds, and since \(D(\bar{c}) \subseteq D(\mathcal{B}_f)\), in \(F(\mathcal{B}_f)\) the tuple \((j_1, \ldots, j_p)\) belongs to \(P_{\mathcal{B}_f}(\bar{c})\). Similarly, since \(\Phi_D^{\bar{d}}\) says that \(\neg P(k_1, \ldots, k_p)\), the tuple \((k_1, \ldots, k_p)\) is not in \(P_{\mathcal{B}_f}(\bar{d})\).

The map \(g^{-1} \circ f : \mathcal{B}_f \to \mathcal{B}_g\) is an isomorphism. With \((\bar{c}, j_m) \sim (\bar{d}, k_m)\) we must have \(F(g^{-1} \circ f)(j_m) = k_m\), since otherwise \((\bar{c}, j_m) \neq (\bar{d}, k_m)\) as in the proof of Lemma 8.2.4. So the isomorphism \(F(g^{-1} \circ f) : F(\mathcal{B}_f) \to F(\mathcal{B}_g)\) maps \((j_1, \ldots, j_p)\) to \((k_1, \ldots, k_p)\), yielding a contradiction.

Therefore, for each relation symbol \(P_i\) in the language of \(\mathcal{A}\), we get a relation \(R_i\) interpreting \(P\) which is uniformly \(\Delta^r_1\). The corollary below follows from the proof of the previous lemma.

**Corollary 8.2.9.** If \((\bar{b}_1, i_1), \ldots, (\bar{b}_p, i_p)\) and \((\bar{c}_1, j_1), \ldots, (\bar{c}_p, j_p)\) are all in \(\text{Dom}_A^B\), and if \((\bar{b}_m, i_m) \sim (\bar{c}_m, j_m)\) for each \(m\), then \((\bar{b}_1, i_1), \ldots, (\bar{b}_p, i_p)\) is in \(R\) if and only if \((\bar{c}_1, j_1), \ldots, (\bar{c}_p, j_p)\) is in \(R\).

**Defining an isomorphism.** We already know, from Lemma 8.2.5, that \(\sim\) is an equivalence relation, and Corollary 8.2.9 says that \(\sim\) agrees with our definition of \(R_i\). Thus, \((\text{Dom}^B_A / \sim; R_0 / \sim, R_1 / \sim, \ldots)\) is a structure that can be viewed as a structure in the language of \(\mathcal{A}\) (interpreting \(P_i\) as \(R_i / \sim\)). To finalize the proof, we need to define an isomorphism between

\[(\text{Dom}^B_A / \sim; R_0 / \sim, R_1 / \sim, \ldots) \quad \text{and} \quad \mathcal{A} = (A; P_0^A, P_1^A, \ldots).\]

Using our fixed presentation \(\mathcal{B}\), we define \(\mathcal{F} : \mathcal{A} \to \text{Dom}^B_A\) as follows: Given \(i \in \omega = A\), let \(\mathcal{F}(i) = (\bar{c}, i)\) where \(\bar{c} = B \upharpoonright n\) for the least \(n \in \omega\) such that \((\bar{c}, i) \in \text{Dom}^B_A\).

**Lemma 8.2.10.** The function \(\mathcal{F} : \mathcal{A} \to \text{Dom}^B_A\) defined above induces an isomorphism of \((\text{Dom}^B_A / \sim; R_0 / \sim, R_1 / \sim, \ldots)\) onto \((A; P_0^A, P_1^A, \ldots)\).

**Proof.** Lemma 8.2.7 shows \(\mathcal{F}\) to be one-to-one. Lemma 8.2.6 shows it to be onto. That it is an isomorphism follows directly from the definitions of \(R_i\). □

This completes the proof of the proposition and thus of Theorem 8.1.5. □

Abusing terminology, we will often refer to maps such as \(\mathcal{F} : \mathcal{A} \to \text{Dom}^B_A\) in Lemma 8.2.10 as *isomorphisms*, although in fact they only induce isomorphisms. Likewise, a relation on \(A \times \text{Dom}^B_A\) may be called an isomorphism from \(\mathcal{A}\) onto \(\text{Dom}^B_A\) if it becomes one after modding out on the right by the equivalence \(\sim\). Finally, a composition of such "isomorphisms" may also be called an isomorphism, as when we have maps between \(\mathcal{A}\) and \(\text{Dom}^B_A\).
8.3 Effective Uniqueness.

This section is devoted to a further analysis of Theorem 8.1.5. We will prove Proposition 8.1.7, which describes more explicitly what we actually get from the proof of Theorem 8.1.5. Recall that Proposition 8.1.7 states that if $F: \text{Iso}(B) \to \text{Iso}(A)$ is a computable functor, then it is effectively isomorphic to $I^F$, where $I^F$ is the functor we get by transforming $F$ into an effective interpretation as in the proof of Proposition 8.2.2 and then transforming it back into a computable functor using Proposition 8.2.1.

**Proof of Proposition 8.1.7.** For a presentation $B$, set $A = F(B)$. We will define

$$\Lambda^B: F(B) \to I^F(B).$$

On the one hand, note that the map $\mathfrak{S}: F(B) \to \text{Dom}_A^B$ from Lemma 8.2.10 can be computed uniformly from a presentation of $B$. To be more explicit, we denote it by $\mathfrak{S}^B$. On the other hand, recall from the proof of Proposition 8.2.1 that we build $I^F(B)$ out of the interpretation of $A$ within $B$ by pulling back through a bijection $\tau: \omega \to \text{Dom}_A^B$. Let us call this bijection $\tau^B$; it gives a well-defined isomorphism from $I^F(B)$ to $\text{Dom}_A^B/\sim$. We define

$$\Lambda^B = (\tau^B)^{-1} \circ \mathfrak{S}^B: F(B) \to I^F(B).$$

We need to show that $\Lambda$ is a natural isomorphism. It is clear that $\Lambda(B)$ is an isomorphism. We must prove that, for all $\bar{B}, \bar{B} \in \text{Iso}(B)$ and all isomorphisms $h: \bar{B} \to \bar{B}$, the following diagram commutes.

\[
\begin{array}{ccc}
F(\bar{B}) & \xrightarrow{\mathfrak{S}^B} & \text{Dom}_A^B \\
| \downarrow F(h) | & & | \downarrow h | \\
F(\bar{B}) & \xrightarrow{\Lambda^B} & I^F(\bar{B}) \\
| \downarrow \mathfrak{S}^B | & & | \downarrow I^F(h) | \\
\text{Dom}_A^B & \xleftarrow{\tau^B} & I^F(\bar{B})
\end{array}
\]

where $h: \text{Dom}_A^\bar{B} \to \text{Dom}_A^\bar{B}$ is the restriction of $h: \bar{B}^\omega \to \bar{B}^\omega$, which is the extension of $h: \bar{B} \to \bar{B}$.

The right-hand square commutes by definition of $I^F(h)$. To show that the left-hand square commutes, take $i \in F(\bar{B})$ and $j = F(h)(i) \in F(\bar{B})$. Let $(\bar{a}, i) = \mathfrak{S}^\bar{B}(i) \in \text{Dom}_A^\bar{B}$ and $(\bar{b}, j) = \mathfrak{S}^\bar{B}(j) \in \text{Dom}_A^\bar{B}$. We need to show that $h(\bar{a}, i) \sim^B (\bar{b}, j)$. Observe that $h(\bar{a}, i) = (h(\bar{a}), i)$.

With $\Phi_*^{D(\bar{B}) \oplus h \oplus D(\bar{B})}(i) = j$, we can make $\Phi_*^{D(\bar{B}) \oplus h \oplus D(\bar{B})}(i) = j$ by extending $\bar{a}$ and $\bar{b}$. Since $D_B(\bar{a}) = D_B(h(\bar{a}))$, we get that that $(h(\bar{a}), i) \sim (\bar{b}, j)$ in $\bar{B}$ as needed. \qed
8.4 Proof of Theorem 8.1.9

Before proving Theorem 8.1.9, we will prove an alternate characterization of bi-interpretations which is independent of the choice of $f_B^A$ and $f_A^B$. Throughout this section we will use the following convention. Given a map $h$ with domain $A$, $h$ induces a map on tuples, and hence a map on $\text{Dom}_A^B$. We will denote this induced map by $\bar{h}$, and the map induced on $(\text{Dom}_A^B)$ by $\tilde{h}$. For example, if $h: \text{Dom}_A^B \rightarrow A$ is a map, then $\bar{h}$ is a map $\text{Dom}_A^B \rightarrow \text{Dom}_B^A$.

**Proposition 8.4.1.** Let $A$ and $B$ be computable structures. Suppose that $A$ is effectively interpretable in $B$ and $B$ is effectively interpretable in $A$, and let $F$ and $G$ be the functors obtained from these interpretations. Then the following are equivalent.

(1) $A$ and $B$ are effectively bi-interpretable using the interpretations above.

(2) There are u.r.i. computable isomorphisms $g: \text{Dom}_A^B \rightarrow A$ and $h: \text{Dom}_B^A \rightarrow B$, along with isomorphisms $\alpha: \text{Dom}_A^B \rightarrow A$ and $\beta: \text{Dom}_B^A \rightarrow B$, such that $\alpha \circ h \circ \bar{h}^{-1} = g$ and $\beta \circ g \circ \tilde{h}^{-1} = h$.

(3) There are u.r.i. computable isomorphisms $g: \text{Dom}_A^B \rightarrow A$ and $h: \text{Dom}_B^A \rightarrow B$ such that, for all isomorphisms $\alpha: \text{Dom}_B^A \rightarrow A$ and $\beta: \text{Dom}_A^B \rightarrow B$, we have $\alpha \circ \bar{h} \circ h^{-1} = g$ and $\beta \circ \tilde{h} \circ \bar{h}^{-1} = h$.

In (2) and (3), one may always take $g$ and $h$ to be the u.r.i. computable maps from the bi-interpretation in (1).

**Proof.** (1)$\Rightarrow$(2). Suppose that $A$ and $B$ are effectively bi-interpretable; then the compositions

$$f_B^A \circ \tilde{f}_A^B: \text{Dom}_B^A \rightarrow B \quad\text{and}\quad f_A^B \circ \tilde{f}_B^A: \text{Dom}_A^B \rightarrow A$$

are u.r.i. computable. Take $g = f_B^A \circ \tilde{f}_A^B$ and $h = f_A^B \circ \tilde{f}_B^A$. Let $\alpha = f_B^A$ and $\beta = f_A^B$. Then

$$\alpha \circ \bar{h} \circ h^{-1} = f_B^A \circ \tilde{f}_B^A \circ \tilde{f}_A^B = f_B^A \circ \tilde{f}_B^A = g$$

and similarly $\beta \circ \tilde{h} \circ \bar{h}^{-1} = h$.

(2)$\Rightarrow$(3). Let $g: \text{Dom}_A^B \rightarrow A$ and $h: \text{Dom}_B^A \rightarrow B$ be as in (2), with isomorphisms $\alpha: \text{Dom}_B^A \rightarrow A$ and $\beta: \text{Dom}_A^B \rightarrow B$ such that $\alpha \circ h \circ \bar{h}^{-1} = g$ and $\beta \circ g \circ \tilde{h}^{-1} = h$. Let $\alpha': \text{Dom}_A^B \rightarrow A$ and $\beta': \text{Dom}_B^A \rightarrow B$ be arbitrary isomorphisms. Let $\delta: \text{Dom}_A^B \rightarrow \text{Dom}_B^A$ be such that $\alpha' \circ \delta = \alpha$. Then

$$g = \alpha \circ \bar{h} \circ (\bar{h})^{-1} = \alpha' \circ \delta \circ \bar{h} \circ (\bar{h})^{-1} \circ (\alpha')^{-1}.$$

We claim that $\delta \circ \bar{h} \circ (\bar{h})^{-1} = \bar{h}$, and hence that $g = \alpha' \circ \bar{h} \circ (\bar{h})^{-1}$. Using $h$, we can extend $\delta$ to an automorphism $\gamma = h \circ \delta \circ h^{-1}$ of $B$, and we show below that $\tilde{\gamma} = \delta$. Now, since $h$ is u.r.i.
computable, $\gamma(\Gamma_h) = \Gamma_h$ where $\Gamma_h$ is the graph of $h$. But this means that $\gamma \circ h \circ (\tilde{\gamma})^{-1} = h$. Taking tildes of both sides then shows that $\delta \circ \tilde{h} \circ (\tilde{\delta})^{-1} = \tilde{h}$ as required.

To see that $\tilde{\gamma} = \delta$, notice that

$$\text{id} = \gamma^{-1} \circ h \circ \delta^{-1} = h \circ (\tilde{\gamma})^{-1} \circ \delta \circ h^{-1},$$

so $\tilde{\gamma} = \delta$. Let

$$\tilde{\gamma} = (\alpha)^{-1} \circ g \circ \tilde{\alpha} : \text{Dom}_B^{\text{Dom}_B^A} \to \text{Dom}_A^B.$$

Now $\tilde{g}$ must be u.r.i. computable in $B$, since $g$ is (in $A$) and since the structure of $\text{Dom}_B^A$ is $\Sigma_1^1$-defined in $B$. This yields

$$\tilde{\gamma} = \tilde{g} \circ \tilde{\gamma} \circ (\tilde{g})^{-1} = \tilde{g} \circ \tilde{\delta} \circ (\tilde{g})^{-1} = \delta,$$

since $\delta$ induces (via $\alpha$) an automorphism of $A$, which fixes the graph $\Gamma_g$.

A similar argument shows that $h = \beta' \circ \tilde{g} \circ (\tilde{\beta'})^{-1}$.

$(3) \Rightarrow (1)$. Let $g : \text{Dom}_A^{(\text{Dom}_B^A)} \to A$ and $h : \text{Dom}_B^{(\text{Dom}_B^A)} \to B$ be as in $(3)$. Fix an isomorphism $f^A_B : \text{Dom}_B^A \to B$. Let $f^A_B : \text{Dom}_B^A \to A$ be $g \circ (f^A_B)^{-1}$, so that $f^B_A \circ f^A_B = g$. Then

$$h = f^A_B \circ \tilde{g} \circ (\tilde{f}^A_B)^{-1} = f^A_B \circ \tilde{f}^B_A \circ \tilde{f}^A_B \circ (\tilde{f}^A_B)^{-1} = f^A_B \circ \tilde{f}^B_A.$$

Thus $f^B_A \circ \tilde{f}^A_B$ and $f^A_B \circ \tilde{f}^B_A$ are u.r.i. computable.

Recall theorem 8.1.9 that says that $A$ and $B$ are effectively bi-interpretable iff $A$ and $B$ are computably bi-transformable.

**Proof of Theorem 8.1.9.** Suppose $A$ and $B$ are effectively bi-interpretable. From the interpretation of $B$ in $A$, we get a computable functor $F = (\Phi, \Psi_*)$ from $\text{Iso}(A)$ to $\text{Iso}(B)$ which arises by exactly the process described in the proof of Proposition 8.2.1. Recall again from the proof of Proposition 8.2.1 that for each $\tilde{A} \in \text{Iso}(A)$ we build $F(\tilde{A})$ out of the interpretation of $B$ within $A$ by pulling back through a bijection $\tilde{\tau} : \omega \to \text{Dom}_B^A$. Then $\tilde{\tau}$ is an isomorphism $F(\tilde{A}) \to \text{Dom}_B^A / \sim$ and we remarked that it was given by a computable functional in $\tilde{A}$. So there is a computable functional $\Omega$ with $\Omega : \text{Dom}_B^A \to F(\tilde{A})$ (note that $\Omega$ gives the inverse of $\tilde{\tau}$). Similarly, there is a computable functor $G = (\Psi, \Psi_*)$ from $\text{Iso}(B)$ to $\text{Iso}(A)$ and a computable functional $\Gamma$ with $\Gamma : \text{Dom}_B^A \to G(\tilde{B})$. We will show that $F$ and $G$ are pseudo-inverses. We begin by showing that $G \circ F : \text{Iso}(A) \to \text{Iso}(A)$ is effectively isomorphic to the identity functor.

The u.r.i. computable map $f^B_A \circ \tilde{f}^A_B : \text{Dom}_A^{\text{Dom}_B^A} \to A$ gives rise to a computable functional $\Theta$ which gives isomorphisms $\Theta(\tilde{A}) : \text{Dom}_A^{\text{Dom}_B^A} \to \text{Dom}_A^{\text{Dom}_B^A}$. 

**End of Proof.**
Given \( \tilde{A} \in \text{Iso}(A) \), define \( \Lambda^{\tilde{A}} \) as follows. We have the following maps:

\[
\begin{array}{c}
\tilde{A} & \xleftarrow{\Theta^{\tilde{A}}} & \text{Dom}_{B}^{\tilde{A}} & \xleftarrow{\Omega^{\tilde{A}}} & \text{Dom}_{A}^{\tilde{A}}
\end{array}
\]

\[
\begin{array}{c}
F(\tilde{A}) & \xleftarrow{\Omega^{\tilde{A}}} & \text{Dom}_{A}^{F(\tilde{A})} & \xleftarrow{\tilde{\Omega}^{\tilde{A}}} & \text{Dom}_{B}^{F(\tilde{A})}
\end{array}
\]

\[
\begin{array}{c}
G(F(\tilde{A})) & \xleftarrow{\tilde{\Theta}^{\tilde{A}}} & \text{Dom}_{A}^{\tilde{A}} & \xleftarrow{\tilde{\Omega}^{\tilde{A}}} & \text{Dom}_{B}^{\tilde{A}}
\end{array}
\]

where \( \tilde{\Omega}^{\tilde{A}} \) is the extension of \( \Omega^{\tilde{A}} \) to tuples. Let \( \Lambda^{\tilde{A}} \) be the composition

\[
\Lambda^{\tilde{A}} = \Gamma^{F(\tilde{A})} \circ \tilde{\Omega}^{\tilde{A}} \circ \Theta^{\tilde{A}}.
\]

We will show that \( \Lambda \) is the Turing functional which witnesses that \( G \circ F \) is effectively isomorphic to the identity functor. We must show that the diagram from Definition 8.1.6 commutes.

Now given \( j : \tilde{A} \to \tilde{A} \), we have maps as shown in the following diagram (which has not yet been seen to commute):

\[
\begin{array}{c}
\text{Dom}_{A}^{\tilde{A}} & \xleftarrow{\Theta^{\tilde{A}}} & \tilde{A} & \xrightarrow{j} & \tilde{A} & \xrightarrow{\Theta^{\tilde{A}}} & \text{Dom}_{B}^{\tilde{A}}
\end{array}
\]

\[
\begin{array}{c}
\text{Dom}_{B}^{\tilde{A}} & \xleftarrow{\tilde{\Omega}^{\tilde{A}}} & \text{Dom}_{A}^{\tilde{A}} & \xleftarrow{\tilde{\Omega}^{\tilde{A}}} & \text{Dom}_{B}^{\tilde{A}} & \xleftarrow{\tilde{\Omega}^{\tilde{A}}} & \text{Dom}_{A}^{\tilde{A}}
\end{array}
\]

\[
\begin{array}{c}
\text{Dom}_{A}^{F(\tilde{A})} & \xleftarrow{F_{i}} & \tilde{A} & \xrightarrow{F} & \tilde{A} & \xrightarrow{F} & \text{Dom}_{A}^{F(\tilde{A})}
\end{array}
\]

\[
\begin{array}{c}
\text{Dom}_{B}^{F(\tilde{A})} & \xleftarrow{\tilde{G}_{i}} & \tilde{A} & \xrightarrow{\tilde{F}} & \tilde{A} & \xrightarrow{\tilde{F}} & \text{Dom}_{B}^{F(\tilde{A})}
\end{array}
\]

\[
\begin{array}{c}
G(F(\tilde{A})) & \xrightarrow{\tilde{G}_{i}} & \text{Dom}_{A}^{\tilde{A}} & \xleftarrow{\tilde{\Omega}^{\tilde{A}}} & \text{Dom}_{B}^{\tilde{A}}
\end{array}
\]

\[
\begin{array}{c}
G(F(\tilde{A})) & \xleftarrow{\tilde{G}_{i}} & \text{Dom}_{A}^{\tilde{A}} & \xrightarrow{\tilde{\Omega}^{\tilde{A}}} & \text{Dom}_{B}^{\tilde{A}}
\end{array}
\]

By definition (see Proposition 8.2.1) we have that

\[
G(F(j)) = \Gamma^{F(\tilde{A})} \circ \tilde{F}(j) \circ (\Gamma^{F(\tilde{A})})^{-1}
\]

and

\[
F(j) = \Omega^{\tilde{A}} \circ \tilde{j} \circ (\Omega^{\tilde{A}})^{-1}.
\]

Hence

\[
G(F(j)) \circ \Gamma^{F(\tilde{A})} \circ \tilde{\Omega}^{\tilde{A}} = \Gamma^{F(\tilde{A})} \circ \tilde{\Omega}^{\tilde{A}} \circ \tilde{j}.
\]
Also, since \( \Theta \) is u.r.i. computable on \( A \), for any isomorphism \( j : \tilde{A} \rightarrow \tilde{\tilde{A}} \), we have that \( \tilde{j} \circ \Theta \tilde{A} = \Theta \tilde{A} \circ j \). Hence

\[
G(F(j)) \circ \Gamma^F(\tilde{A}) \circ \tilde{\Omega} \tilde{A} \circ \Theta \tilde{A} = \Gamma^F(\tilde{A}) \circ \tilde{\Omega} \tilde{A} \circ \Theta \tilde{A} \circ j.
\]

Using the definition of \( \Lambda \tilde{A} \), we have

\[
G(F(j)) \circ \Lambda \tilde{A} = \Lambda \tilde{A} \circ j.
\]

Thus \( G \circ F \) is effectively isomorphic to the identity functor via \( \Lambda \).

By a similar argument, \( F \circ G \) is effectively isomorphic to the identity functor. Denote the \( \Lambda \) obtained for \( G \circ F \) as \( \Lambda_A \), and that for \( F \circ G \) as \( \Lambda_B \). Let \( \Upsilon \) be the Turing functional which arises from the u.r.i. computable isomorphism \( f_B^A \circ f_A^B \), so \( \Upsilon_B : B \rightarrow \text{Dom}_B \text{Dom}_A^\text{Dom}_A^\text{Dom}_A^B \). Then

\[
\Lambda_B^\text{Dom}_B^\text{Dom}_A^\text{Dom}_A^B = \Omega^G(\Upsilon_B) \circ \tilde{\Upsilon}_B \circ \tilde{\Omega}_B \circ (\Omega_B)^{-1}.
\]

Then

\[
F(\Lambda_A^\text{Dom}_A^\text{Dom}_A^\text{Dom}_A^\text{Dom}_A^B) = \Omega^{G(F(\tilde{A}))} \circ \tilde{\Omega}^F(\tilde{A}) \circ \tilde{\Omega} \tilde{A} \circ (\Omega \tilde{A})^{-1}.
\]

Now by Proposition 8.4.1 with \( h^{-1} = \Upsilon^F(\tilde{A}) \), \( g^{-1} = \Theta \tilde{A} \), and \( \beta = \Omega \tilde{A} \), we have:

\[
\Upsilon^F(\tilde{A}) = \tilde{\Omega} \tilde{A} \circ \Theta \tilde{A} \circ (\Omega \tilde{A})^{-1}
\]

and so

\[
F(\Lambda_A^\text{Dom}_A^\text{Dom}_A^\text{Dom}_A^\text{Dom}_A^B) = \Lambda_B^\text{Dom}_B^\text{Dom}_A^\text{Dom}_A^\text{Dom}_A^B.
\]

A similar argument shows that

\[
G(\Lambda_B^\text{Dom}_B^\text{Dom}_A^\text{Dom}_A^\text{Dom}_A^B) = \Lambda_A^\text{Dom}_A^\text{Dom}_A^\text{Dom}_A^\text{Dom}_A^B.
\]

Now suppose that we have computable functors \( F \) and \( G \) which give a computable bi-transformation between \( A \) and \( B \). Let \( \Lambda \tilde{A} : \tilde{A} \rightarrow G(F(\tilde{A})) \) witness that \( G \circ F \) is effectively isomorphic to the identity. From \( F \) and \( G \) we get interpretations of \( A \) in \( B \) and of \( B \) in \( A \), and Turing functionals \( \Omega \) and \( \Gamma \) as before. For any \( \tilde{A} \in \text{Iso}(A) \), we get an isomorphism

\[
\Theta \tilde{A} = (\tilde{\Omega} \tilde{A})^{-1} \circ (\Gamma^F(\tilde{A}))^{-1} \circ \Lambda \tilde{A} : \tilde{A} \rightarrow \text{Dom}_A \text{Dom}_A^\text{Dom}_A^\text{Dom}_A^B.
\]

We can view \( \Theta \tilde{A} \) as a subset of \( \tilde{A} \times \text{Dom}_A \text{Dom}_A^\text{Dom}_A^\text{Dom}_A^B \).

First, let \( j : A \rightarrow \tilde{A} \) be any isomorphism. We show that the graph of \( \Theta \tilde{A} \) is the image, under \( j \), of the graph of \( \Theta \tilde{A} \), i.e. that \( \Theta \tilde{A} \circ j = \tilde{j} \circ \Theta \tilde{A} \). This is very similar to the argument above. By the properties of \( \Lambda \) we have

\[
\Lambda \tilde{A} \circ j = G(F(j)) \circ \Lambda \tilde{A} = \Gamma^F(\tilde{A}) \circ \tilde{\Omega} \tilde{A} \circ \tilde{j} \circ (\tilde{\Omega} \tilde{A})^{-1} \circ (\Gamma^F(\tilde{A}))^{-1} \circ \Lambda \tilde{A}.
\]

Then

\[
(\tilde{\Omega} \tilde{A})^{-1} \circ (\Gamma^F(\tilde{A}))^{-1} \circ \Lambda \tilde{A} \circ j = \tilde{j} \circ (\tilde{\Omega} \tilde{A})^{-1} \circ (\Gamma^F(\tilde{A}))^{-1} \circ \Lambda \tilde{A}
\]
which gives $\Theta \tilde{A} \circ j = \tilde{j} \circ \Theta A$.

This argument shows first that $\Theta A$ is fixed under automorphisms $j : A \to A$, hence $\mathcal{L}_{\omega_1 \omega}$-definable. The same argument also shows (with $j : A \to \tilde{A}$ any isomorphism) that the same formula also defines $\Theta \tilde{A}$. But $\Theta$ is a Turing functional, so membership in $\Theta \tilde{A}$ is always computable below $\tilde{A}$, and so $\Theta A$ is u.r.i. computable.
Chapter 9

Borel Functors and Infinitary Interpretations

The results presented in this chapter appeared in [HTMMb]. They are joint work with Russell Miller and Antonio Montalbán and appear here with their permission.

9.1 Introduction

Constructions that build new structures out of old ones are common throughout mathematics. For instance, given an integral domain $\mathcal{B}$, we might consider its fraction field or its polynomial ring. In model theory, a common way of performing such constructions is using interpretations, where one structure is defined using tuples from the other, and the operations and relations of the new structure are defined using the operations and relations of the old one. For instance, the fraction field of an integral domain $\mathcal{B}$ can be defined as a set of pairs of elements in $\mathcal{B}$ quotiented out by some definable equivalence relation, with the operations on the pairs defined using the operations in $\mathcal{B}$. Interpretations are useful because they preserve some model theoretic properties of the structures or of their theories. For instance, if a structure $\mathcal{A}$ can be interpreted within a structure $\mathcal{B}$, then there is a homomorphism from the automorphism group of $\mathcal{B}$ to the automorphism group of $\mathcal{A}$ ([Hod93, Theorem 5.3.5]). Furthermore, if we assume these structures are countable, we get a function that maps copies of $\mathcal{B}$ with domain $\omega$ to copies of $\mathcal{A}$ with domain $\omega$, and one that maps isomorphisms between copies of $\mathcal{B}$ to isomorphisms between the respective copies of $\mathcal{A}$, preserving compositions. We naturally view such a pair of functions as a functor from $\mathcal{B}$ to $\mathcal{A}$ (Definition 9.1.7). Functors induced by interpretations are always Borel. In turn, a Borel functor (even a Baire-measurable one) from $\mathcal{B}$ to $\mathcal{A}$ induces a continuous homomorphism from the automorphism group of $\mathcal{B}$ to that of $\mathcal{A}$. However, there are many such functors that do not come from elementary first-order interpretations.\(^1\) In the case of the polynomial

\(^1\)For example, there is the functor which maps every copy of the trivial structure $\mathcal{B}$, with a countable domain and no relations, to the single structure $\mathcal{A} = (\omega, 0, 1, +, \cdot)$, and maps all isomorphisms between copies...
ring, we can easily build a functor that maps copies of a ring $B$ to copies of its polynomial ring $B[X]$ in such a way that isomorphisms between copies of $B$ translate to isomorphisms between the respective copies of $B[X]$. However, $B[X]$ cannot be interpreted in $B$, as we need tuples of arbitrary large size from $B$ to code the polynomials in $B[X]$. In this paper we consider a more general notion of interpretation that we call *infinitary interpretation*, where the sets used in the interpretation need only be $\mathcal{L}_{\omega_1\omega}$-definable and where, instead of using tuples of a fixed size for the interpretation, we allow tuples of different sizes (see Definition 9.1.1 below). We only consider countable structures, and so, whenever we refer to a structure, we assume it is countable and with domain $\omega$. These new interpretations still generate Borel functors from the interpreting structure to the interpreted structure exactly as above, and also continuous homomorphisms between their automorphism groups. Our main theorem is the reversal: Each Borel (even Baire measurable) functor from the copies of $B$ to the copies of $A$ is naturally isomorphic to one induced by an infinitary interpretation (Theorem 9.1.9), and each continuous homomorphism $\text{Aut}(B) \to \text{Aut}(A)$ is induced by an infinitary interpretation (Theorem 9.1.3). Furthermore, the quantifier complexity of the interpretation is the same as the Borel complexity of the functor. In a sense, this shows that infinitary interpretations are the most general kind of interpretations, at least if we restrict ourselves to countable structures. One can view our result as saying that if one has a way of building $A$ from $B$, then a copy of $A$ must already exist inside of $B$.

Continuing this line of investigation, we obtain results towards the following question: What can we tell about a structure by looking at its automorphism group? The first question along these lines that we consider is whether there is a syntactical condition on structures that is equivalent to them having the same automorphism group. The answer is *infinitary bi-interpretability*: Two structures $A$ and $B$ are infinitarily bi-interpretable if each can be infinitarily interpreted in the other and the isomorphism taking $A$ to the copy of $A$ inside the copy of $B$ inside $A$, and the similar isomorphism with $A$ and $B$ reversed, are infinitarily definable in the respective structures (Definition 9.1.5). This is equivalent to the existence of a continuous isomorphism between the automorphism groups of the structures (Theorem 9.1.6). (For the particular case of $\aleph_0$-categorical structures, this was already known from a paper of Ahlbrandt and Ziegler [AZ86].) To show this, we prove that infinitary bi-interpretations correspond naturally (and bijectively) to Borel adjoint equivalences (Definition 9.1.10) of the categories of copies of the structures (Theorems 9.1.11 and 9.1.12). The second question is whether, and how, the existence of a set of indiscernibles within a structure is reflected in the automorphism group of the structure. We will show that a structure has an infinitarily definable set of indiscernible equivalence classes on its tuples if and only if there is a continuous homomorphism from its automorphism group onto $S_\infty$ (Theorem 9.1.4).

This work grew out of a previous paper [HTMMM] by the three authors and Alexander Melnikov, which gives a one-to-one correspondence between effective interpretations and effective functors. (This paper appears as the previous chapter in this thesis.) Effective inter-
pretation is the right notion of interpretability needed for computability theory, and is exactly like infinitary interpretation as we define it in Definition 9.1.1, but using only computable infinitary $\Sigma_1$-formulas. This particular definition was introduced in [Mon13a, Monb], but it is equivalent to the notion of $\Sigma$-definability without parameters, widely studied in Russia. On the other hand, the precise definition of computable functor was introduced in [MPSS], where it was shown to show all structures can be effectively coded by fields. Both effective bi-interpretations and computable functors were introduced to formalize a longstanding idea from [HKSS02] that certain classes of structures are universal for computability-theoretic properties. Some time later, we realized that with some more work, and via the use of forcing, we could extend our results from [HTMMM] through the Borel hierarchy. We then noticed we could apply our results to homomorphisms between automorphism groups and infinitary indiscernibles.

## 9.1.1 Infinitary Interpretations

Let us now formally define the notion of infinitary interpretation. Throughout this article, all signatures are relational and computable: there is a computable function giving the arity of each of the countably many predicates $P_0, P_1, \ldots$. (It seems fairly clear that one could extend our arguments to noncomputable countable signatures by relativizing everything to the Turing degrees of the signatures.)

**Definition 9.1.1.** A structure $A = (A; P^A_0, P^A_1, \ldots)$ (where $P^A_i \subseteq A^{\alpha(i)}$) is **infinitarily interpretable in $B$** if there are relations $\text{Dom}^B_A; \sim, R_0, R_1, \ldots$, each $L_{\omega_1, \omega}$-definable without parameters in the language of $B$, such that

1. $\text{Dom}^B_A \subseteq B^{<\omega}$,
2. $\sim$ is an equivalence relation on $\text{Dom}^B_A$,
3. $R_i \subseteq (\text{Dom}^B_A)^{\alpha(i)}$ is closed under $\sim$,

and there exists a function $f^B_A: \text{Dom}^B_A \to A$ which induces an isomorphism:

$$f^B_A((\text{Dom}^B_A/\sim; R_0/\sim, R_1/\sim, \ldots) \cong (A; P^A_0, P^A_1, \ldots),$$

where $R_i/\sim$ stands for the $\sim$-collapse of $R_i$.

In the definition above, when we refer to an $L_{\omega_1, \omega}$-definable subset $S \subseteq B^{<\omega}$ we mean a countable sequence $\{S_1, S_2, \ldots\}$ of $L_{\omega_1, \omega}$-definable subsets $S_i \subseteq B^i$. We refer the reader to [AK00, Chapters 6 and 7] for background on the infinitary language and its effective version. Our notation $\Sigma^c_\alpha$ refers to computable infinitary $\Sigma_\alpha$ formulas (necessarily with $\alpha < \omega_1^{CK}$), and likewise for $\Delta^c_\alpha$. We also sometimes use $\Sigma^i_\alpha$, simply to emphasize that infinitary (not necessarily computable) formulas are included.

We only deal with countable structures in this paper, and for a relation on a countable structure, being $L_{\omega_1, \omega}$ definable is equivalent to being invariant under automorphisms
(\cite{Kue68, Mak69}). One might then say that this is not really a syntactical definition. However, we will also be interested in the complexity of the interpretations defined in terms of the syntactic complexity of the formulas. We say that an interpretation is $\Delta^a \in$, or $\Delta^a c$, if all the relations $\text{Dom}^A_\sim, R_0, R_1, \ldots$ are. (In the lightface case, when we refer to a $\Delta^c \in$-definable subset $S \subseteq B^\omega$ we mean a computable sequence (of indices) $\{s_1, s_2, \ldots\}$ of $\Delta^c \in$-definable subsets $S_i \subseteq B^i$. Similarly, when we refer to a $\Delta^c \in$-definable sequence $\{S_k : k \in \omega\}$ of subsets of $B^\omega$, we mean that the sequence of indices for the $S_k$’s is computable.)

Notice that given a presentation of a structure $A$ with domain $\omega$, we get a presentation of $\text{Aut}(A)$ as a subgroup of $S_\infty$. The automorphism group of a different presentation would be a different subgroup of $S_\infty$, although these two subgroups would be conjugated by the isomorphism between the presentations. Given fixed copies of $A$ and $B$ with domain $\omega$, an infinitary interpretation induces a map between their automorphism groups in an obvious way.

**Definition 9.1.2.** To each interpretation $I$ of $A$ in $B$ as in Definition 9.1.1, we associate a homomorphism $G_I : \text{Aut}(B) \rightarrow \text{Aut}(A)$ as follows:

$$G_I(f) = f_A^B \circ \tilde{f} \circ (f_A^B)^{-1}.$$  

Here $\tilde{f}$ permutes $\text{Dom}^B_A$ as defined by the given $f$, and hence preserves $\sim$.

Throughout this paper, we use $\tilde{f}$ (where $f$ is a map with domain $B$) to denote the induced map on tuples from $\text{Dom}^B_A$.

It is not hard to see that $G_I$ is a continuous homomorphism. (Let us remark that every Baire-measurable homomorphism between Polish groups is continuous (see \cite[Theorem 2.3.3]{Gao09}) and all automorphism groups are Polish (see \cite[Exercise 2.4.7]{Gao09}).) One of the main results of this paper is that all continuous homomorphisms between automorphism groups are induced by infinitary interpretations.

**Theorem 9.1.3.** Let $A$ and $B$ be countable structures. Every continuous homomorphism from $\text{Aut}(B)$ into $\text{Aut}(A)$ is of the form $G_I$ for some infinitary interpretation $I$ of $A$ in $B$.

Note that there do exist structures whose automorphism groups are isomorphic as groups, but not as topological groups \cite{EH90}. So we cannot drop the hypothesis of continuity. On the other hand, there are models of ZF + DC such that every homomorphism between polish groups is continuous \cite{Sol70, She84} and so we might expect such examples to be the exception.

As a corollary of this theorem we give a characterization, in terms of the automorphism group of a structure $A$, for $A$ to have an absolutely indiscernible set of $\mathcal{L}_{\omega_1\omega}$-imaginary elements (i.e., an absolutely indiscernible set of equivalence classes under some $\mathcal{L}_{\omega_1\omega}$-definable equivalence relation).

**Theorem 9.1.4.** Let $A$ be a countable structure. The following are equivalent:

(1) There is a continuous homomorphism from $\text{Aut}(A)$ onto $S_\infty$. 


(2) There is an \(n\), an \(L_{\omega_1\omega}\)-definable \(D \subset A^n\), and an \(L_{\omega_1\omega}\)-definable equivalence relation \(E \subset D^2\) with infinitely many equivalence classes and such that the \(E\)-equivalence classes are absolutely indiscernible, in the sense that every permutation of the \(E\)-equivalence classes extends to an automorphism of \(A\).

The theorem above shows the connections behind the new proof by Baldwin, Friedman, Koerwein, and Laskowski [BFKL16] and the original proof of a result of Hjorth [Hjo07] that states that if there is a counterexample to Vaught’s conjecture, there is one with no copies of size \(\aleph_2\). For this, Hjorth’s proof started by considering a structure whose automorphism group divides \(S_\infty\) (i.e., there is an onto continuous homomorphism from a closed subgroup of the automorphism group onto \(S_\infty\)) and then used descriptive set theoretic tools. This proof is hard to visualize for those outside of descriptive set theory, and so Baldwin, Friedman, Koerwein, and Laskowski found another proof starting from a structure that has a set of absolute indiscernibles. It is suggested in [BFKL16] that the use of absolute indiscernibles is in a sense the model theoretic version of the use of the divisibility of \(S_\infty\) by the automorphism group. The theorem above makes this sense precise.

In general it is necessary that we look at equivalence classes to find the indiscernibles, as it was shown in [HTIK] that every structure is bi-interpretable with one that has no triple of indiscernibles.

We also show (Theorem 9.4.1) that a structure has absolute order indiscernibles if and only if there is a continuous homomorphism from \(\text{Aut}(A)\) onto \(\text{Aut}(\mathbb{Q})\).

We will also consider bi-interpretations. Two structures are bi-interpretable if they are each interpretable in the other, and the compositions are definable:

**Definition 9.1.5.** Two structures \(A\) and \(B\) are *infinitarily bi-interpretable* if there are interpretations of each structure in the other as in Definition 9.1.1 such that the compositions

\[
f^A_B \circ \tilde{f}^B_A : \text{Dom}^{(\text{Dom}^B_A)}_B \to B \quad \text{and} \quad f^B_A \circ \tilde{f}^A_B : \text{Dom}^{(\text{Dom}^B_A)}_A \to A
\]

are \(L_{\omega_1\omega}\)-definable in \(B\) and \(A\) respectively. (Here, we have \(\text{Dom}^{(\text{Dom}^B_A)}_B \subseteq (\text{Dom}^B_A)^\omega\), and the map \(\tilde{f}^B_A : (\text{Dom}^B_A)^\omega \to A^\omega\) is the obvious extension of \(f^B_A : \text{Dom}^B_A \to A\) mapping \(\text{Dom}^{(\text{Dom}^B_A)}_B\) to \(\text{Dom}^A_B\).

Two structures which are bi-interpretable behave in the same way. In particular, we get a continuous isomorphism of the automorphism groups of the two structures. For this, the fact that the two \(L_{\omega_1\omega}\)-definable isomorphisms are of the form \(f^B_A \circ \tilde{f}^B_A\) and \(f^B_A \circ \tilde{f}^A_B\) for some \(f^B_A\) and \(f^A_B\) is vital.

**Theorem 9.1.6.** Two countable structures \(A\) and \(B\) are infinitarily bi-interpretable if and only if their automorphism groups are Baire-measurably isomorphic. Furthermore, every continuous isomorphism from \(\text{Aut}(B)\) onto \(\text{Aut}(A)\) is of the form \(G_I\) for some infinitary bi-interpretation \(I\) of \(A\) in \(B\).
CHAPTER 9. BOREL FUNCTORS

9.1.2 Functors

Throughout the paper, we write Iso\((\mathcal{A})\) for the isomorphism class of a countably infinite structure \(\mathcal{A}\):

\[
\text{Iso}(\mathcal{A}) = \{\mathcal{A} : \mathcal{A} \cong \mathcal{A} \& \text{dom}(\mathcal{A}) = \omega\}.
\]

We will regard Iso\((\mathcal{A})\) as a category, with the copies of the structure as its objects and the isomorphisms among them as its morphisms.

**Definition 9.1.7.** By a functor from \(\mathcal{A}\) to \(\mathcal{B}\) we mean a functor from Iso\((\mathcal{A})\) to Iso\((\mathcal{B})\), that is, a map \(F\) that assigns to each copy \(\mathcal{A}\) in Iso\((\mathcal{A})\) a structure \(F(\mathcal{A})\) in Iso\((\mathcal{B})\), and assigns to each morphism \(f: \mathcal{A} \to \mathcal{A}\) in Iso\((\mathcal{A})\) a morphism \(F(f): F(\mathcal{A}) \to F(\mathcal{A})\) in Iso\((\mathcal{B})\) so that the two properties below hold:

1. \((\mathrm{N}1)\) \(F(\text{id}_\mathcal{A}) = \text{id}_{F(\mathcal{A})}\) for every \(\mathcal{A} \in \text{Iso}(\mathcal{A})\), and
2. \((\mathrm{N}2)\) \(F(f \circ g) = F(f) \circ F(g)\) for all morphisms \(f, g\) in Iso\((\mathcal{A})\).

\(F\) is \(\Delta^0_\alpha\) (or \(\Delta^0_\alpha\)) if it is given by a pair of \(\Delta^0_\alpha\) (resp. \(\Delta^0_\alpha\)) operators \(2^\omega \to 2^\omega\). It is Borel if it is given by Borel operators, and Baire-measurable if it is given by Baire-measurable operators.

Every interpretation \(I\) of a structure \(\mathcal{A}\) in a structure \(\mathcal{B}\) induces an functor, \(F_I\), from \(\mathcal{B}\) to \(\mathcal{A}\). There is only one small technicality in the definition of \(F_I\), which has to do with making the domain of \(F_I(\mathcal{B})\) equal to \(\omega\). Using the interpretation we can associate, to each copy \(\mathcal{B}\) of \(\mathcal{B}\), a copy of \(\mathcal{A}\) whose domain consists of the \(\sim\)-equivalence classes of \(\text{Dom}^\mathcal{B}_\mathcal{A} \subseteq \omega^{<\omega}\); Using a bijection \(\tau^\mathcal{B}\) between \(\omega\) and \(\text{Dom}^\mathcal{B}_\mathcal{A}/\sim\) (defined in some canonical way using an effective bijection between \(\omega\) and \(\omega^{<\omega}\), so that we can compute \(\tau^\mathcal{B}\) from \(\text{Dom}^\mathcal{B}_\mathcal{A}\)), we then define \(F_I(\mathcal{B})\) to be the pull-back of this structure through \(\tau^\mathcal{B}\). If \(h\) is an isomorphism \(\mathcal{B} \to \mathcal{B}\), we define \(F_I(h): F_I(\mathcal{B}) \to F_I(\mathcal{B})\) by \(F_I(h) = (\tau^\mathcal{B})^{-1} \circ h \circ \tau^\mathcal{B}\).

Our main theorem states that every functor from \(\mathcal{B}\) to \(\mathcal{A}\) is of the form \(F_I\) up to natural isomorphism.

**Definition 9.1.8.** A functor \(F:\text{Iso}(\mathcal{B}) \to \text{Iso}(\mathcal{A})\) is naturally isomorphic (or just isomorphic) to a functor \(G:\text{Iso}(\mathcal{B}) \to \text{Iso}(\mathcal{A})\) if for every \(\mathcal{B} \in \text{Iso}(\mathcal{B})\), there is an isomorphism \(\eta_\mathcal{B}: F(\mathcal{B}) \to G(\mathcal{B})\), such that the following diagram commutes for every \(\mathcal{B}, \mathcal{B} \in \text{Iso}(\mathcal{B})\) and every morphism \(h: \mathcal{B} \to \mathcal{B}\):

\[
\begin{array}{ccc}
F(\mathcal{B}) & \xrightarrow{\eta_\mathcal{B}} & G(\mathcal{B}) \\
F(h) \downarrow & & \downarrow G(h) \\
F(\mathcal{B}) & \xrightarrow{\eta_\mathcal{B}} & G(\mathcal{B})
\end{array}
\]

An isomorphism is Borel (or \(\Delta^0_\alpha\), or \(\Delta^0_\alpha\)) if \(\eta\) is given by a Borel (resp. \(\Delta^0_\alpha\) or \(\Delta^0_\alpha\)) operator.

The following is the key result of the paper and Section 9.3 is dedicated to proving it.
Theorem 9.1.9. Let $B$ and $A$ be countable structures, possibly in different countable languages. For each Baire-measurable functor $F: \text{Iso}(B) \to \text{Iso}(A)$ there is an infinitary interpretation $I$ of $A$ within $B$, such that $F$ is naturally isomorphic to the functor $F_I$ associated to $I$. Furthermore, if $F$ is $\Delta^0_\alpha$ in the lightface Borel hierarchy, then the interpretation can be taken to be $\Delta^c_\alpha$ and the isomorphism between $F$ and $F_I$ can be taken to be $\Delta^0_\alpha$.

We also get a similar way of moving between bi-interpretations and a category-theoretic equivalent. For bi-interpretations, we must consider adjoint equivalences of categories.

Definition 9.1.10. An adjoint equivalence of categories consists of two functors, from one category to the other and back, such that their compositions are both naturally isomorphic to the identity functors, and furthermore, these two natural isomorphisms are mapped to each other via these two functors. More formally, functors $F: \text{Iso}(B) \to \text{Iso}(A)$ and $G: \text{Iso}(A) \to \text{Iso}(B)$, together with families of isomorphisms $\epsilon_{\vec{A}}: \vec{A} \to F(G(\vec{A}))$ and $\eta_{\vec{B}}: \vec{B} \to G(F(\vec{B}))$ for $\vec{A} \in \text{Iso}(A)$ and $\vec{B} \in \text{Iso}(B)$, form an adjoint equivalence of categories if

$$F(\eta_{\vec{B}}) = \epsilon_{F(\vec{B})} \text{ and } G(\epsilon_{\vec{A}}) = \eta_{G(\vec{A})}.$$  

An adjoint equivalence of categories is Borel if $F$, $G$, $\eta$, and $\epsilon$ are Borel operators.

For bi-interpretations, both directions—producing an equivalence of categories from a bi-interpretation, and vice versa—are non-trivial.

Theorem 9.1.11. Let $B$ and $A$ be countable structures. For every infinitary bi-interpretation $(I, J)$ of $A$ and $B$, $F_I$ and $F_J$ form a Borel adjoint equivalence of categories of $\text{Iso}(B)$ and $\text{Iso}(A)$. Furthermore, complexities are maintained.

Theorem 9.1.12. Let $B$ and $A$ be countable structures. For every Borel adjoint equivalence of categories $(F, G)$ between $\text{Iso}(B)$ and $\text{Iso}(A)$ there is an infinitary bi-interpretation $(I, J)$ between $A$ and $B$, such that $F$ and $G$ are naturally isomorphic to the functors $F_I$ and $F_J$ associated to $I$ and $J$ respectively. Furthermore, complexities are maintained.

9.2 Homomorphisms of Automorphism Groups

Our main result, Theorem 9.1.9, shows the connection between functors and interpretations. In this section, we discuss the connection between homomorphisms of automorphism groups and functors, which we will then be able to connect to interpretations once we prove Theorem 9.1.9.

Theorem 9.2.1. For every continuous homomorphism $H: \text{Aut}(B) \to \text{Aut}(A)$, there is a Borel functor $G: \text{Iso}(B) \to \text{Iso}(A)$ with $G(B) = A$ and whose restriction to $\text{Aut}(B)$ is $H$. 
Proof. Let \( \Gamma \) be a map that assigns, to each copy \( \widehat{B} \) of \( B \), an isomorphism \( \Gamma^B : \widehat{B} \to B \) with \( \Gamma^B = \text{id}_B \). Let us first show how will use \( \Gamma \), and then show how we can choose it to be Borel.

Using \( \Gamma \) and \( H \) we define \( G \) as follows. First, the action of the functor on the copies of \( B \) is trivial: For every copy \( \widehat{B} \) of \( B \), we let \( G(\widehat{B}) = A \). The action of \( G \) on the isomorphisms is a bit more interesting: If \( f : \widehat{B} \to \widehat{B} \) is an isomorphism, then \( \Gamma^B \circ f \circ \Gamma^B^{-1} \) is an automorphism of \( B \), and we can define \( G(f) = H(\Gamma^B \circ f \circ \Gamma^B^{-1}) \in \text{Aut}(A) \). It is not hard to check that \( G \) is a functor. If \( f \in \text{Aut}(B) \), then since \( \Gamma^B = \text{id}_B \), \( G(f) = H(f) \). Moreover, the continuity of \( H \) and the fact that \( \Gamma \) is Borel ensure that \( G \) is Borel.

Let us now build \( \Gamma \) in a Borel way. Let \( \alpha \) be the Scott rank of \( \widehat{B} \) in the sense of \[\text{Mon15b}\]. So, by \[\text{Mon15b}, \text{Theorem 1.1}\], \( B \) is uniformly \( \Delta^0 \) relatively categorical on a cone, say the cone above \( X \). Let \( \Gamma \) be the operator witnessing this uniformity. Note that we can choose \( \Gamma^B = \text{id}_B \). \( \square \)

Corollary 9.2.2. Every Baire-measurable functor \( F : \text{Iso}(B) \to \text{Iso}(A) \) is naturally isomorphic to a Borel one.

Proof. Fix the presentations of \( A \) and \( B \), with \( A = F(B) \). When restricted to the automorphisms of \( B \), \( F \) is a Baire-measurable homomorphism from \( \text{Aut}(B) \) to \( \text{Aut}(A) \). As we mentioned earlier, such a homomorphism must be continuous (see \[\text{Gao09, Theorem 2.3.3}\]). We can then apply the previous theorem to get a Borel functor \( G : \text{Iso}(B) \to \text{Iso}(A) \) which coincides with \( F \) on \( \text{Aut}(B) \). Write \( H = F \upharpoonright \text{Aut}(B) \) as above. Then \( F \) and \( G \) are isomorphic: Let \( \Gamma \) be a map that assigns, to each copy \( \widehat{B} \) of \( B \), an isomorphism \( \Gamma^B : \widehat{B} \to B \), as in the previous theorem. Given a copy \( \widehat{B} \) of \( B \), let \( \eta_B = F(\Gamma^B) : F(\widehat{B}) \to G(\widehat{B}) \) (recall that \( G(\widehat{B}) = A = F(B) \)). We claim that \( \eta \) is a natural isomorphism between \( F \) and \( G \). Given an isomorphism \( f : \widehat{B} \to \widehat{B} \), and using the fact that \( G \upharpoonright \text{Aut}(B) = F \upharpoonright \text{Aut}(B) = H \), we have

\[
G(f) \circ \eta_B = H(\Gamma^B \circ f \circ \Gamma^B^{-1}) \circ F(\Gamma^B) \\
= F(\Gamma^B \circ f \circ \Gamma^B^{-1}) \circ F(\Gamma^B) \\
= F(\Gamma^B) \circ F(f) \\
= \eta_B \circ F(f). \quad \quad \quad \quad \square
\]

Theorem 9.2.1 together with our main Theorem 9.1.9 provides a proof of Theorem 9.1.3, that each homomorphism between automorphism groups is induced by an infinitary interpretation. We can then use this to define a measure of complexity for homomorphisms between automorphism groups.

Definition 9.2.3. Given a continuous homomorphism \( H : \text{Aut}(B) \to \text{Aut}(A) \), we define the rank of \( H \) to be the least \( \alpha \) such that there is a \( \Delta^0 \) functor from \( B \) to \( A \) coinciding with \( H \) on \( \text{Aut}(B) \), or equivalently, a \( \Delta^1 \) interpretation \( I \) of \( A \) within \( B \) with \( H = G_I \) as in Definition 9.1.2. From the proof of Theorem 9.2.1 we get that the rank of \( H \) is at most the Scott rank of \( B \).
Note that the rank of a homomorphism depends on the underlying structures $A$ and $B$, and not just on their automorphism groups. We will not develop this notion of rank any further in this paper, but it seems so natural that we think it deserves further study.

We now turn to the connection between isomorphisms of automorphism groups and adjoint equivalences of categories.

**Theorem 9.2.4.** Let $F: \text{Iso}(B) \to \text{Iso}(A)$, $G: \text{Iso}(A) \to \text{Iso}(B)$, $\eta$, and $\epsilon$ form a Borel adjoint equivalence of categories between $\text{Iso}(A)$ and $\text{Iso}(B)$ with $F(B) = A$. Then $F$, restricted to $\text{Aut}(B)$, gives an isomorphism between $\text{Aut}(B)$ and $\text{Aut}(A)$.

**Proof.** Let $H_1: \text{Aut}(B) \to \text{Aut}(A)$ be defined by $H_1(h) = F(h)$, and let $H_2: \text{Aut}(A) \to \text{Aut}(B)$ be defined by $H_2(g) = \eta_B^{-1} \circ G(g) \circ \eta_B$. Then

$$H_1 \circ H_2(h) = F(\eta_B^{-1}) \circ F(G(h)) \circ \eta_B = \epsilon_A^{-1} \circ F(G(h)) \circ \epsilon_A = h$$

and

$$H_2 \circ H_1(g) = \eta_B^{-1} \circ G(F(g)) \circ \eta_B = g. \qed$$

**Theorem 9.2.5.** For every continuous isomorphism $H: \text{Aut}(B) \to \text{Aut}(A)$, there is a Borel adjoint equivalence of categories $F: \text{Iso}(B) \to \text{Iso}(A)$ with $F(B) = A$ and whose restriction to $\text{Aut}(B)$ is $H$.

Note that the inverse of $H$ is also continuous [Gao09, Exercise 2.3.5].

**Proof.** Define $F$ as before: Let $\Gamma$ be a map that assigns, to each copy $\bar{B}$ of $B$, an isomorphism $\Gamma^\bar{B}: \bar{B} \to B$ with $\Gamma^B = \text{id}_B$, and (overloading notation a bit) assigns to each copy $\bar{A}$ of $A$, an isomorphism $\Gamma^{\bar{A}}: \bar{A} \to A$ with $\Gamma^A = \text{id}_A$. For every copy $\bar{B}$ of $B$, we let $F(\bar{B}) = A$, and if $h: \bar{B} \to \bar{B}$ is an isomorphism, then $F(h) = H(\Gamma^\bar{B} \circ h \circ \Gamma^B^{-1})$. Define $G$ in a similar way: For every copy $\bar{A}$ of $B$, we let $G(\bar{A}) = B$, and if $h: \bar{A} \to \bar{A}$ is an isomorphism, then $G(h) = H^{-1}(\Gamma^{\bar{A}} \circ h \circ \Gamma^{\bar{A}}^{-1})$.

First, we want to show that $F$ and $G$ are inverse equivalences, via the natural isomorphisms $\eta$ and $\epsilon$ defined by

$$\eta_B = \Gamma^\bar{B}: \bar{B} \to B = G(F(\bar{B}))$$

and

$$\epsilon_A = \Gamma^{\bar{A}}: \bar{A} \to A = F(G(\bar{A})).$$

We have, by definition, $G(F(\bar{B})) = B$ and $G(F(\bar{A})) = A$. Now let $h: \bar{B} \to \bar{B}$ be an isomorphism. Then

$$G \circ F(h) = G(H(\Gamma^\bar{B} \circ h \circ \Gamma^B^{-1}))$$

$$= H^{-1}(\Gamma^A \circ H(\Gamma^\bar{B} \circ h \circ \Gamma^B^{-1}) \circ \Gamma^{A^{-1}})$$

$$= \Gamma^B \circ h \circ \Gamma^B^{-1}.$$

(Above, recall that $\Gamma^A = \text{id}_A$.) So

$$G(F(h)) \circ \eta_B = \eta_B \circ h.$$
Similarly, for an isomorphism $h: \tilde{A} \rightarrow \tilde{A}$

$$F(G(h)) \circ \epsilon_{\tilde{A}} = \epsilon_{\tilde{A}} \circ h.$$ 

Thus $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity.

Note that

$$F(\eta_{\tilde{B}}) = H(\Gamma^B \circ \Gamma^{\tilde{B}} \circ \Gamma^{\tilde{B}}^{-1}) = H(id_B) = \Gamma_A = \eta_{F(\tilde{B})}.$$ 

Similarly, $G(\epsilon_{\tilde{A}}) = \epsilon_{F(\tilde{A})}$. Thus $F$, $G$, $\eta$, and $\epsilon$ form an adjoint equivalence of categories.

### 9.3 The Construction

In this section, we prove Theorem 9.1.9. Let $A$ and $B$ be countable structures, and let $F: Iso(B) \rightarrow Iso(A)$ a Baire-measurable functor. By Corollary 9.2.2, up to natural isomorphism we may assume that $F$ is Borel.

The proof will involve a forcing: we will build multiple mutually generic structures and consider how the functor acts on the maps between these structures. The definability, in $B$, of our forcing notion will give the formulas of our interpretation.

#### 9.3.1 The Forcing Notion

Let $B^*$ be the set of finite one-to-one tuples from $B$. Since the domain of $B$ is $\omega$, this is the same as finite tuples from $\omega$. We view $B^*$ as a forcing notion, extension of tuples being extension of conditions. Thus, generics for these forcing notions are one-to-one functions $\omega \rightarrow B$ respectively. A small amount of genericity guarantees these functions are onto and hence bijections.

Often in computable structure theory, forcing is used to build a single generic copy $B_g$ of $B$. Given a generic function $g: \omega \rightarrow B$, $B_g$ is the pullback of $B$ along $g$. Here, we will want to build several generic copies and thus we will work with product forcing. Thus, given $\ell \in \omega$, we will define the product forcing $(B^*)^\ell$. We write $p$ for a forcing condition in $(B^*)^\ell$; $p$ is of the form $(\tilde{b}_1, \ldots, \tilde{b}_n)$.

We will want the forcing relation to be definable in $B$. Often in computability theory, this is accomplished by taking as the forcing language $\mathcal{L}_{\omega_1\omega}$ formulas about $B$. Here, we will want to force statements of the form $F(B_{g_1}, g_2^{-1} \circ g_1, B_{g_2})(i) = j$. Thus we will be required to force statements of the form $g_2^{-1} \circ g_1(i) = j$. This leads us to the definition of our forcing language.

**Definition 9.3.1** (Forcing language). The finitary formulas in the forcing language for $(B^*)^\ell$ are built up as follows:

- $\hat{g}_i^{-1} \circ \hat{g}_j(m) = n$ and $\hat{g}_i^{-1} \circ \hat{g}_j(m) \neq n$ where $m, n \in \omega$,
- $R^{B_{g_1}}(a_1, \ldots, a_n)$ and $\neg R^{B_{g_1}}(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n \in \omega$ and $R$ is a relation symbol in the language for $B$, 


• finite conjunctions and finite disjunctions,
• \( \hat{g}_i(m) = n \) and \( \hat{g}_i(m) \neq n \) where \( m, n \in \omega \).

The forcing language \( \mathcal{L} \) is built up from the finitary formulas by taking countable conjunctions and disjunctions. A formula is \( X \)-computable if the conjunctions and disjunctions are over \( X \)-c.e. sets of indices. By neg(\( \varphi \)), we mean the formal negation within the forcing language (flipping conjunctions and disjunctions, and negating the basic formulas).

We will also consider the restricted language \( \mathcal{L}' \subset \mathcal{L} \) where we do not allow terms of the form \( \hat{g}_i(m) = n \) or \( \hat{g}_i(m) \neq n \).

We use \( \hat{g}_i \) as a formal symbol; the idea is that we will substitute a generic \( g_i \) for \( \hat{g}_i \). We will only get the definability of forcing within \( \mathcal{B} \) for the restricted language \( \mathcal{L}' \); the whether or not the other sentences are forced depends on the presentation of \( \mathcal{B} \).

We want to be able to express certain statements about \( F \) by formulas in our forcing language. If we consider \( F \) as a Borel functional, \( F(\mathcal{B}_g) \) reads from its oracle statements about relations holding or not holding in \( \mathcal{B}_g \) — these are all in the forcing language — and then computes its values using infinitary conjunctions and disjunctions. Thus, for \( P \) a relation in the language of \( \mathcal{A} \), we can express

\[
F(\mathcal{B}_g) \equiv P(j_1, \ldots, j_{p(i)j})
\]

using infinitary formulas in the forcing language. Similarly, we can express

\[
F(\mathcal{B}_{g_1}, g_2^{-1} \circ g_1, \mathcal{B}_{g_2})(i) = j.
\]

If \( F \) is \( \Delta_0^\alpha \), then we can express these as \( \Delta_\alpha^\xi \) formulas (i.e., as \( \Sigma_\alpha^\xi \) formulas and also as \( \Pi_\alpha^\xi \) formulas). Similarly, if \( F \) is \( \Delta_0^\alpha \), then we can express these as \( \Delta_\alpha^{\infty} \) formulas. Using conjunctions and disjunctions of such statements, we can also express more complicated statements such as

\[
F(\mathcal{B}_{g_z}, g_1^{-1} \circ g_2, \mathcal{B}_{g_z}) = F(\mathcal{B}_{g_1}, g_2^{-1} \circ g_1, \mathcal{B}_{g_2})^{-1}
\]

and

\[
F(\mathcal{B}_{g_2}, g_3^{-1} \circ g_2, \mathcal{B}_{g_3}) \circ F(\mathcal{B}_{g_1}, g_2^{-1} \circ g_1, \mathcal{B}_{g_2}) = F(\mathcal{B}_{g_1}, g_3^{-1} \circ g_1, \mathcal{B}_{g_3})
\]

in the forcing language. These formulas are all in the restricted language \( \mathcal{L}' \). In the language \( \mathcal{L} \), we can express

\[
\hat{F}(\mathcal{B}, g_1^{-1}, \mathcal{B}_{g_1})(i) = j.
\]

If \( F \) is \( \Delta_0^\alpha \), then this is a \( \hat{\mathcal{B}} \)-computable \( \Delta_\alpha^\xi \) formula.

**Definition 9.3.2 (Definition of Forcing).** Let \( p = (\bar{b}_1, \ldots, \bar{b}_\ell) \) be a forcing condition for \( (\mathcal{B}^*)' \). We define \( p \Vdash_{(\mathcal{B}^*)'} \varphi \) for \( \varphi \) a sentence of the forcing language. We begin with the finitary formulas.

• if \( \varphi \equiv \hat{g}_i^{-1} \circ \hat{g}_j(m) = n \), then \( p \Vdash_{(\mathcal{B}^*)'} \varphi \) if and only if \( \bar{b}_i(n) \) and \( \bar{b}_j(m) \) are defined and equal.
CHAPTER 9. BOREL FUNCTORS

- if \( \varphi \equiv \hat{g}_i^{-1} \circ \hat{g}_j(m) \neq n \), then \( p \models (B^*)^\varphi \) if and only if either:
  - \( \bar{b}_i(n) \) and \( \bar{b}_j(m) \) are defined and distinct, or
  - there is \( m' \neq m \) such that \( \bar{b}_i(n) = \bar{b}_j(m') \), or
  - there is \( n' \neq n \) such that \( \bar{b}_i(n') = \bar{b}_j(m) \).

- if \( \varphi \equiv R^{B_i}(a_1, \ldots, a_n) \), then \( p \models (B^*)^\varphi \) if and only if \( \bar{b}_i(a_1), \ldots, \bar{b}_i(a_n) \) are all defined and \( B \models R(\bar{b}_i(a_1), \ldots, \bar{b}_i(a_n)) \).

- if \( \varphi \equiv \neg R^{B_i}(a_1, \ldots, a_n) \), then \( p \models (B^*)^\varphi \) if and only if \( \bar{b}_i(a_1), \ldots, \bar{b}_i(a_n) \) are all defined and \( B \models \neg R(\bar{b}_i(a_1), \ldots, \bar{b}_i(a_n)) \).

- if \( \varphi \equiv \hat{g}_i(m) = n \), then \( p \models (B^*)^\varphi \) if and only if \( \bar{b}_i(m) = n \).

- if \( \varphi \equiv \hat{g}_i(m) \neq n \), then \( p \models (B^*)^\varphi \) if and only if either \( \bar{b}_i(m) \neq n \), or for some \( m' \neq m \), \( \bar{b}_i(m') = n \).

- if \( \varphi \equiv \psi_1 \lor \cdots \lor \psi_n \), then \( p \models (B^*)^\varphi \) if and only if \( p \models \psi_i \) for some \( i \).

- if \( \varphi \equiv \psi_1 \land \cdots \land \psi_n \), then \( p \models (B^*)^\varphi \) if and only if \( p \models \psi_i \) for each \( i \).

Now for infinitary formulas:

- if \( \varphi \equiv \bigvee_n \psi_n \), then \( p \models (B^*)^\varphi \bigvee_n \psi_n \) if and only if there is \( n \) such that \( p \models (B^*)^\varphi \psi_n \).

- if \( \varphi \equiv \bigwedge_n \psi_n \), then \( p \models (B^*)^\varphi \bigwedge_n \psi_n \) if for all \( n \) and \( q \supseteq p \), there is \( r \supseteq q \) such that \( r \models (B^*)^\varphi \psi_n \).

Given an injection \( g: \omega \to \omega \), we can define a structure \( B_g \) using the pullback of \( B \) along \( g \). That is, \( R^{B_g}(a_1, \ldots, a_n) \) if and only if \( R^B(g(a_1), \ldots, g(a_n)) \). If \( g \) is a bijection, then \( B_g \) is isomorphic to \( B \) via \( g: B_g \to B \).

Given \( \varphi \) a sentence in the forcing language for \( (B^*)^\ell \), and \( g_1, \ldots, g_\ell \) functions \( \omega \to B \), we say that \( \varphi[g_1, \ldots, g_\ell] \) holds if \( \varphi \) becomes true under the natural interpretation, substituting \( g_i \) for \( \hat{g}_i \).

**Lemma 9.3.3.** If \( p \models (B^*)^\varphi \), and \( q \supseteq p \), then \( q \models (B^*)^\varphi \).

**Proof.** The proof is by induction on the complexity of \( \varphi \). The lemma is clear for the finitary formulas. If \( \varphi \equiv \bigvee_n \psi_n \) and \( p \models (B^*)^\varphi \), then there is \( n \) such that \( p \models (B^*)^\varphi \psi_n \). By the induction hypothesis, \( q \models (B^*)^\varphi \psi_n \). If \( \varphi \equiv \bigwedge_n \psi_n \), then for all \( n \) and \( r \supseteq q, r \supseteq p \), and so there is \( r' \supseteq r \) such that \( r' \models (B^*)^\varphi \psi_n \). Thus \( q \models (B^*)^\varphi \psi_n \). 

**Lemma 9.3.4.** For every \( p \) and \( \varphi \), there is \( q \supseteq p \) such that \( q \) decides \( \varphi \).
CHAPTER 9. BOREL FUNCTORS

Proof. The proof is by induction. It is easy to see that the lemma holds when \( \varphi \) is a finitary formula. If \( \varphi \equiv \forall_n \psi_n \), then if there are \( n \) and \( q \geq p \) such that \( q \models (B^*)^{\ell} \psi_n \), then we are done. Otherwise, for all \( n \) and \( q \geq p \), \( q \models (B^*)^{\ell} \psi_n \). By the induction hypothesis, there is \( r \geq q \) such that \( r \) decides \( \psi_n \); by the previous lemma, \( r \models (B^*)^{\ell} \neg \psi_n \). Thus \( p \models (B^*)^{\ell} \neg \psi_n \). The same argument works if \( \varphi \equiv \bigwedge_n \neg \psi_n \).

Lemma 9.3.5. It is not the case that \( p \models (B^*)^{\ell} \varphi \) and \( p \models (B^*)^{\ell} \neg \varphi \).

Proof. The lemma is easy to check for finitary formulas. If \( \varphi \equiv \forall_n \psi_n \) or \( \varphi \equiv \bigwedge_n \psi_n \), and \( p \models (B^*)^{\ell} \varphi \) and \( p \models (B^*)^{\ell} \neg \varphi \), then there is \( n \) such that \( p \models (B^*)^{\ell} \psi_n \). Also, there is \( q \geq p \) such that \( q \models (B^*)^{\ell} \neg \psi_n \). This contradicts the induction hypothesis.

Definition 9.3.6. Let \( X \subseteq \omega \). By an \( X \)-generic for \((B^*)^{\ell}\) we mean a tuple \( g = (g_1, \ldots, g_\ell) \) of mutually \((X \oplus B)\)-hyperarithmetically generic functions \( \omega \rightarrow B \).

It is clear that for any particular \( X \)-computable sentence of the forcing language, the forcing relation is \((X \oplus B)\)-hyperarithmetic. Thus, by Lemma 9.3.4, an \( X \)-generic \( g \) for \((B^*)^{\ell}\) has the property that it forces every \( X \)-computable sentence or its negation. We also get that \((g_1, g_2)\) is \((B^*)^{\ell_1 + \ell_2}\)-generic if and only if \( g_1 \) is \((B^*)^{\ell_1}\)-generic and \( g_2 \) is \((X \oplus g_1)\)-generic for \((B^*)^{\ell_2}\).

Lemma 9.3.7 (Restriction). If \( \varphi \) is a computable sentence of the forcing language which does not involve \( g_i \), then \((b_1, \ldots, b_\ell) \models (B^*)^{\ell} \varphi \) if and only if \((b_1, \ldots, b_{i-1}, b_i+1, \ldots, b_\ell) \models (B^*)^{\ell-1} \varphi \).

Proof. This is a simple induction argument.

Lemma 9.3.8 (Forcing Lemma). Let \( \varphi \) be an \( X \)-computable sentence of the forcing language for \((B^*)^{\ell}\).

1. For \( X \)-generic \( g \), \( \varphi[g] \) holds if and only if for some \( p \triangleleft g \), \( p \models (B^*)^{\ell} \varphi \).

2. If \( \varphi \) starts with a \( \bigwedge \), then \( p \models (B^*)^{\ell} \varphi \) if and only if for every \( X \)-generic \( g \triangleright p \), \( \varphi[g] \) holds.

Proof. For (1), first suppose that for some \( p = (b_1, \ldots, b_\ell) \subseteq g = (g_1, \ldots, g_\ell) \), \( p \models \varphi \). We argue by induction. For the finitary formulas, everything is simple:

- if \( \varphi \equiv g_i^{-1} \circ g_j(m) = n \), then \( b_i(n) = b_j(m) \) and so \( g_i^{-1} \circ g_j(m) = n \).
- if \( \varphi \equiv g_i^{-1} \circ g_j(m) \neq n \), then either:
  - \( b_i(n) \neq b_j(m) \) and so \( g_i^{-1} \circ g_j(m) \neq n \),
  - there is \( m' \neq m \) such that \( b_i(n) = b_j(m') \), and so since \( g_j \) is injective, \( g_i^{-1} \circ g_j(m) \neq n \),
  - or
  - there is \( n' \neq n \) such that \( b_i(n') = b_j(m) \), and so since \( g_i \) is injective, \( g_i^{-1} \circ g_j(m) \neq n \).
• if $\varphi \equiv R^{B_{\ell}}(a_1, \ldots, a_n)$, then $B \models R(b_1(a_1), \ldots, b_1(a_n))$ and so $B_{g_1} \models R(a_1, \ldots, a_n)$.
• if $\varphi \equiv \neg R^{B_{\ell}}(a_1, \ldots, a_n)$, then $B \models \neg R(b_1(a_1), \ldots, b_1(a_n))$ and so $B_{g_1} \models \neg R(a_1, \ldots, a_n)$.
• if $\varphi \equiv \dot{g}_1(m) = n$, then $\dot{b}_1(m) = n$ and so $g_1(m) = n$.
• if $\varphi \equiv \dot{g}_1(m) \neq n$, then either $\dot{b}_1(m) \neq n$, or for some $m = m'$, $\dot{b}_1(m') = n$; thus $g_1(m) \neq n$.
• if $\varphi \equiv \psi_1 \lor \cdots \lor \psi_n$, then $p \models (B^*)^\ell \psi_i$ for some $i$ and so $\psi_i[g]$ holds for some $i$.
• if $\varphi \equiv \psi_1 \land \cdots \land \psi_n$, then $p \models (B^*)^\ell \psi_i$ for all $i$ and hence for all $i$, $\psi_i[g]$ holds.

Now for infinitary formulas:
• if $\varphi \equiv \forall \gamma \psi_n$, then $p \models (B^*)^\ell \psi_i$ for some $i$ and so $\psi_i[g]$ holds for some $i$.
• if $\varphi \equiv \exists \gamma \psi_n$, then for all $n$ and $q \geq p$, there exists $r \geq q$ such that $r \models (B^*)^\ell \psi_n$. Fix $n$.

Since $g$ is generic, there is $q \in g$ such that $q$ decides $\psi_n$. We may assume, by Lemma 9.3.3, that $q \geq p$. So there is $r \geq q$ such that $r \models (B^*)^\ell \psi_n$. By Lemma 9.3.5, $q \models (B^*)^\ell \psi_n$.

By the induction hypothesis, $\psi_n[g]$ holds. Since this was true for all $n$, $\varphi[g]$ holds.

Now suppose that $\varphi[g]$ holds. There is $p \in g$ such that $p$ decides $\varphi$. If $p \models (B^*)^\ell \neg \varphi$, then $\neg \varphi[g]$ holds. This is a contradiction. Hence $p \models (B^*)^\ell \varphi$.

For (2), suppose that $p \models (B^*)^\ell \varphi$. Let $g \supseteq p$ be $X$-generic. By (1), $\varphi[g]$ holds.

For the other direction, suppose that for all $X$-generic $g \supseteq p$, $\varphi[g]$ holds. Then for all $q \supseteq p$, $q \models (B^*)^\ell \neg \varphi$; if we did have $q \models (B^*)^\ell \neg \varphi$, then for some $X$-generic $g \supseteq q$, $\neg \varphi[g]$ would hold, a contradiction. Now if $\varphi$ begins with $\exists$, say $\varphi \equiv \exists \gamma \psi_n$, then $\neg \varphi \equiv \forall \gamma \neg \psi_n$. So for all $n$ and $q \supseteq p$, $q \models (B^*)^\ell \neg \psi_n$. Now by Lemma 9.3.4 there is $r \supseteq q$ such that $r$ decides $\psi_n$; we cannot have $r \models (B^*)^\ell \neg \psi_n$ (since $r \supseteq p$) and so $r \models (B^*)^\ell \psi_n$.

Thus $p \models (B^*)^\ell \varphi$.

Lemma 9.3.9 (Definability of Forcing). For $\alpha \geq 1$, given a $\Sigma_\alpha$ formula $\varphi$ in the restricted language $L^\ell$, the set $\{ p \in (B^*)^\ell : p \models \varphi \}$ is $\Sigma_\alpha$-definable in $B$, and if $\varphi$ is $\Pi_\alpha$, $\{ p \in (B^*)^\ell : p \models \varphi \}$ is $\Pi_\alpha$-definable. This also relativizes.

Proof. We argue by induction. For finitary formulas $\varphi$, it is easy to see from Definition 9.3.2 that the set $\{ p \in (B^*)^\ell : p \models \varphi \}$ is definable in $B$ by a finitary formula. The key is to note that the tuples $\dot{b}$ and $\dot{c}$ such that $\dot{b}(n) = \dot{c}(m)$, or the tuples $\dot{b}$ such that $B \models R(b(a_1), \ldots, b(a_n))$, are definable in $B$ by atomic formulas.

Now we consider infinitary formulas. If $\varphi \equiv \forall \gamma \psi_n$, then $p \models (B^*)^\ell \forall \gamma \psi_n$ if and only if for some $n$, $p \models (B^*)^\ell \psi_n$. Since, for each $n$, $p \models (B^*)^\ell \psi_n$ is $\Pi^\ell\beta$-definable in $B$ for some $\beta < \alpha$, this is $\Sigma^\ell\alpha$-definable in $B$.

If $\varphi \equiv \exists \gamma \psi_n$, then note that by Lemmas 9.3.3, 9.3.4, and 9.3.5, $p \models \exists \gamma \psi_n$ if and only if for all $q \supseteq p$, $q \models (B^*)^\ell \neg \psi_n$; $q \models (B^*)^\ell \neg \psi_n$ is $\Sigma^\ell\beta$-definable in $B$ for some $\beta < \alpha$, and so $p \models (B^*)^\ell \exists \gamma \psi_n$ is $\Pi^\ell\alpha$-definable in $B$. \qed
9.3.2 The Definition of the Interpretation

Recall that $F: \text{Iso}(B) \to \text{Iso}(A)$ is a Borel functor. As everything will relativize, we will assume from now on that it is a lightface Borel operator.

**Definition 9.3.10.** We define the domain of interpretation, $\mathcal{D}om_B^A$, as a subset of $B^* \times \omega$ as follows: For $(b, i) \in B^* \times \omega$, let

$$\text{dom}_B \equiv (b, \bar{b}) \leadsto (b, \bar{b}) \vdash (B_{g_1}, \hat{g}_2^{-1} \circ \hat{g}_1, B_{g_2})(i) = \tilde{b}.$$

Recall that subsets of $B^{\omega \times \omega}$ can be effectively coded by subsets of $B^{\omega \times \omega}$. Next, we define a relation $\sim$ on $\mathcal{D}om_B^A$ which we will later prove is an equivalence relations. For $(b, i), (\bar{b}, j) \in \mathcal{D}om_B^A$, let

$$(b, i) \sim (\bar{b}, j) \equiv (b, \bar{b}) \vdash (B_{g_1}, \hat{g}_2^{-1} \circ \hat{g}_1, B_{g_2})(i) = \tilde{b}.$$

Last, we need to interpret the relation symbols. For each relation symbol $P_i$ of arity $p(i)$ in the language of $A$, we define a relation $R_i$ on $\mathcal{D}om_B^A$ as follows: For $(b, 1), \ldots, (b_p(i), k_p(i)) \in \mathcal{D}om_B^A$, let

$$(b_1, k_1), \ldots, (b_p(i), k_p(i)) \in R_i \iff (\exists \bar{c} \in B^*) (\exists j_1, \ldots, j_p(i) \in \omega) (\bigwedge_{s=1}^{p(i)} (b_s, k_s) \sim (\bar{c}, j_s)) \& (\bar{c} \vdash (B_{g_1}, \hat{g}_2^{-1} \circ \hat{g}_1, B_{g_2})(i) = \tilde{b}).$$

By Lemma 9.3.9, these are all defined by formulas of $L_{\omega_1 \omega}$ since they can be expressed in $L'$.

9.3.3 Verifications

The first thing to observe before starting the verifications is that since $F$ is a functor that works for all copies of $B$, all its properties are forced by the empty conditions. For instance,

$$(\emptyset, \emptyset, \emptyset) \vdash (B_{g_2}, \hat{g}_3^{-1} \circ \hat{g}_2, B_{g_3}) \circ (B_{g_1}, \hat{g}_2^{-1} \circ \hat{g}_1, B_{g_2}) = (B_{g_2}, \hat{g}_3^{-1} \circ \hat{g}_1, B_{g_3}).$$

**Lemma 9.3.11.** $\sim$ is an equivalence relation on $\mathcal{D}om_B^A$.

**Proof.** Reflexivity follows from the definition of $\mathcal{D}om_B^A$. Symmetry holds because

$$(\emptyset, \emptyset) \vdash (B_{g_2}, \hat{g}_3^{-1} \circ \hat{g}_2, B_{g_3}) = (B_{g_2}, \hat{g}_3^{-1} \circ \hat{g}_1, B_{g_3})^{-1}.$$ 

Transitivity follows from the fact that

$$(\emptyset, \emptyset, \emptyset) \vdash (B_{g_2}, \hat{g}_3^{-1} \circ \hat{g}_2, B_{g_3}) \circ (B_{g_1}, \hat{g}_2^{-1} \circ \hat{g}_1, B_{g_2}) = (B_{g_2}, \hat{g}_3^{-1} \circ \hat{g}_1, B_{g_3}).$$

The next objective of this subsection is to define a map $\mathcal{F}: A \to \mathcal{D}om_B^A$ which gives an isomorphism between $A$ and its interpretation within $B$. Remember we are fixing a copy of $B$, and that $A = F(B)$.
CHAPTER 9. BOREL FUNCTORS

Let $g : \omega \to B$ be generic; before defining $\mathfrak{g}$, we define a map $\mathfrak{g}_g : F(B) \to \text{Dom}_A^B$ also intended to be an isomorphism (Lemma 9.3.15). Given $i$, we let $\mathfrak{g}_g(i)$ be the least tuple of the form $(\bar{c}, i)$ for $\bar{c} \subset g$, and with $(\bar{c}, i) \in \text{Dom}_A^B$ (we prove such a tuple exists in Lemma 9.3.12). We will need to show that all of this works (Lemmas 9.3.13 and 9.3.14). Then, to define $\mathfrak{g}$, we simply compose $\mathfrak{g}_g : F(B) \to \text{Dom}_A^B$ with $F(B, g^{-1}, B_g) : A \to F(B)$). We will also need to show that this definition is independent of the choice of $g$ (Lemma 9.3.16).

The first lemma shows that $\mathfrak{g}_g(i)$ is defined for every $i$.

**Lemma 9.3.12.** For every generic $g : \omega \to B$ and every $i \in \omega$ there exists $n \in \omega$ such that $(g \upharpoonright n, i) \in \text{Dom}_A^B$.

**Proof.** Let $g_1$ be generic with respect to $g = g_1$ so that $(g_1, g_2)$ is generic for $(B^*)^2$. Let $j = F(B_{g_1}, g_1^{-1} \circ g_1, B_{g_2})$.

For some $\bar{b} \subset g_1$ and some $\bar{c} \subset g_2$, $(\bar{b}, \bar{c}) \equiv (B^*)^2 F(B_{g_1}, g_1^{-1} \circ g_1, B_{g_2})$.

Notice that we also have $(\bar{b}, \bar{c}) \equiv (B^*)^2 F(B_{g_2}, g_2^{-1} \circ g_2, B_{g_2})$.

It then follows that

$${(\bar{b}, \bar{c}, \bar{b}) \equiv (B^*)^3 F(B_{g_1}, g_1^{-1} \circ g_1, B_{g_2})(i) = j \& F(B_{g_2}, g_2^{-1} \circ g_2, B_{g_2})(j) = i},$$

and hence

$${(\bar{b}, \bar{c}, \bar{b}) \equiv (B^*)^3 F(B_{g_1}, g_1^{-1} \circ g_1, B_{g_2})(i) = i}.$$

Since $g_2$ does not appear in the formula above, by Lemma 9.3.7 we get

$${(\bar{b}, \bar{b}) \equiv (B^*)^2 F(B_{g_1}, g_1^{-1} \circ g_1, B_{g_2})(i) = i},$$

and hence that $(\bar{b}, i) \in \text{Dom}_A^B$. 

The second lemma shows that $\mathfrak{g}_g$ is onto the set of $\sim$-equivalence classes.

**Lemma 9.3.13.** For every generic $g : \omega \to B$ and every $(\bar{c}, j) \in \text{Dom}_A^B$ there exists $n \in \omega$ and $i \in \omega$ such that $(g \upharpoonright n, i) \sim (\bar{c}, j)$.

**Proof.** The proof is similar to that of the lemma above. Let $g_2 \supseteq \bar{c}$ be generic with respect to $g = g_1$, and let $j = F(B_{g_1}, g_1^{-1} \circ g_1, B_{g_2})$.

There are $\bar{b} \subset g_1$ and $\bar{c} \subset g_2$ such that $(\bar{b}, \bar{c}) \equiv (B^*)^3 F(B_{g_1}, g_1^{-1} \circ g_1, B_{g_2})$.

Since $g_2$ does not appear in the formula above, $(\bar{b}, \bar{c}) \equiv (B^*)^3 F(B_{g_1}, g_1^{-1} \circ g_1, B_{g_2})$.

The third lemma shows that $\mathfrak{g}_g$ is one-to-one on $\sim$-equivalence classes.
Lemma 9.3.14. For \((\bar{c}, i), (\bar{d}, j) \in \text{Dom}_{\mathcal{A}}^B\) with \(\bar{c} \subseteq \bar{d}\) we have that \((\bar{c}, i) \sim (\bar{d}, j)\) if and only if \(i = j\).

Proof. By definition, \((\bar{c}, i) \sim (\bar{d}, j)\) if and only if \((\bar{c}, \bar{d}) \equiv_{(B^*)^2} F(B_{\hat{g}_1}, \hat{g}_2^{-1} \circ \hat{g}_1, B_{\hat{g}_2})(i) = j\). But since \((\bar{c}, i) \in \text{Dom}_{\mathcal{A}}^B\), we know \((\bar{c}, \bar{c}) \equiv_{(B^*)^2} F(B_{\hat{g}_1}, \hat{g}_2^{-1} \circ \hat{g}_1, B_{\hat{g}_2})(i) = i\). With \((\bar{c}, \bar{d})\) extending \((\bar{c}, i)\), we get that \((\bar{c}, i) \sim (\bar{d}, j)\) if and only if \(i = j\). \(\square\)

So we have that \(\mathfrak{F}_g\) is a bijection from \(\omega\) onto \(\text{Dom}_{\mathcal{A}}^B/\sim\). We now show that it is an isomorphism from \(F(B_g)\) to \((\text{Dom}_{\mathcal{A}}^B/\sim; R_0/\sim, R_1/\sim, \ldots)\).

Lemma 9.3.15. For every relation symbol \(P_i\), and \((j_1, \ldots, j_{p(i)}) \in \omega^{p(i)}\),

\[
F(B_g) = P_i(j_1, \ldots, j_{p(i)}) \iff (\mathfrak{F}_g(j_1), \ldots, \mathfrak{F}_g(j_{p(i)})) \in R_i.
\]

Proof. First suppose that \(F(B_g) = P_i(j_1, \ldots, j_{p(i)})\). Then there is \(\bar{c} \subseteq g\) such that \(\bar{c} \equiv_{BO} F(B_g) = P_i(j_1, \ldots, j_{p(i)})\); by Lemma 9.3.12 we may also assume that \((\bar{c}, j_s) \in \text{Dom}_{\mathcal{A}}^B\) for each \(s\). Then by Lemma 9.3.14, \(\mathfrak{F}_g(j_s) \sim (\bar{c}, j_s)\). Hence, by definition of \(R_i\), \((\mathfrak{F}_g(j_1), \ldots, \mathfrak{F}_g(j_{p(i)})) \in R_i\).

On the other hand, suppose that \((\mathfrak{F}_g(j_1), \ldots, \mathfrak{F}_g(j_{p(i)})) \in R_i\). Then there are \(\bar{c} \subseteq B^*\) and \(k_1, \ldots, k_{p(i)}\) such that for each \(s\), \(\mathfrak{F}_g(j_s) \sim (\bar{c}, k_s)\) and \(\bar{c} \equiv_{BO} F(B_{\hat{g}_1}, \hat{g}_2^{-1} \circ \hat{g}_1, B_{\hat{g}_2})(k_s) = j_s\). Then \((\bar{c}, \bar{d}) \equiv_{(B^*)^2} F(B_{\hat{g}_1}, \hat{g}_2^{-1} \circ \hat{g}_1, B_{\hat{g}_2})(\bar{d}) = j\). So \(\bar{d} \subseteq \bar{g}\) such that for each \(s\), \((\bar{c}, \bar{d}) \equiv_{(B^*)^2} F(B_{\hat{g}_1}, \hat{g}_2^{-1} \circ \hat{g}_1, B_{\hat{g}_2})(\bar{d}) = j\). Hence, by definition of \(R_i\), \((\mathfrak{F}_g(j_1), \ldots, \mathfrak{F}_g(j_{p(i)})) \in R_i\). But \(\bar{d} \equiv \bar{g}\) is generic, so \(F(B_g) = P_i(j_1, \ldots, j_{p(i)})\). \(\square\)

Last, we need to show that \(\mathfrak{F}\), defined as \(\mathfrak{F}_g \circ F(B, g^{-1}, B_g)\), is independent of the choice of the generic \(g\).

Lemma 9.3.16. For \(i \in \omega\) and \((\bar{c}, j) \in \text{Dom}_{\mathcal{A}}^B\),

\[
\mathfrak{F}(i) \sim (\bar{c}, j) \iff \bar{c} \equiv_{BO} F(B, \hat{g}_1^{-1}, B_{\hat{g}_1})(i) = j.
\]

Proof. Let \(g_2 \supseteq \bar{c}\) be generic relative to \(g = g_1\). Let \(k = F(B, \hat{g}_1^{-1}, B_{\hat{g}_1})(i)\) and \((\bar{d}, k) = \mathfrak{F}(i) = \mathfrak{F}_{g_1}(k)\). For some \(\bar{d}' \supseteq \bar{d}, \bar{d}' \equiv_{BO} F(B, \hat{g}_1^{-1}, B_{\hat{g}_1})(i) = k\).

Suppose that \(\mathfrak{F}(i) \sim (\bar{c}, j)\). Then \((\bar{d}, k) \sim (\bar{c}, j)\) and so \((\bar{d}, \bar{c}) \equiv_{(B^*)^2} F(B_{\hat{g}_1}, \hat{g}_2^{-1} \circ \hat{g}_1, B_{\hat{g}_2})(k) = j)\). We see that \((\bar{d}', \bar{c}) \equiv_{(B^*)^2} F(B, \hat{g}_1^{-1}, B_{\hat{g}_1})(i) = j\), and since \(\hat{g}_1\) does not appear in this formula, \(\bar{c} \equiv_{BO} F(B, \hat{g}_1^{-1}, B_{\hat{g}_1})(i) = j\) as desired.

Now suppose that \(\bar{c} \equiv_{BO} F(B, \hat{g}_1^{-1}, B_{\hat{g}_1})(i) = j\). Then since \(\bar{d}' \equiv_{BO} F(B, \hat{g}_1^{-1}, B_{\hat{g}_1})(i) = k, (\bar{d}', \bar{c}) \equiv_{(B^*)^2} F(B_{\hat{g}_1}, \hat{g}_2^{-1} \circ \hat{g}_1, B_{\hat{g}_2})(k) = j)\). Thus \((\bar{c}, j) \sim (\bar{d}', k)\) and \((\bar{d}', k) \sim (\bar{d}, k)\) by Lemma 9.3.14. \(\square\)

In fact, given any \(\bar{B} \supseteq \bar{B}\), we can define \(\mathfrak{F}_{\bar{B}}: F(\bar{B}) \to \text{Dom}_{\mathcal{A}}^B\) by

\[
\mathfrak{F}_{\bar{B}}(i) \sim (\bar{c}, j) \iff \bar{c} \equiv_{BO} F(\bar{B}, \hat{g}_1^{-1}, \bar{B}_{\hat{g}_1})(i) = j.
\]

This gives an isomorphism from \(F(\bar{B})\) to \((\text{Dom}_{\mathcal{A}}^B/\sim; R_0/\sim, R_1/\sim, \ldots)\). If \(F\) is \(\Delta^0\), then \(\mathfrak{F}_{\bar{B}}\) is \(\Delta^0(\bar{B})\) uniformly in \(\bar{B}\).
**Lemma 9.3.17.** There is a natural isomorphism between $F$ and $F_T$.

**Proof.** Recall that given $\tilde{B} \cong B$, we build $F_T(\tilde{B})$ out of the interpretation of $A$ within $\tilde{B}$ by pulling back through a bijection $\tau: \omega \to \text{Dom}_{\tilde{A}}^B/\sim$. Let us call this bijection $\tau^\tilde{B}$; it gives a well-defined isomorphism from $F_T(\tilde{B})$ to $\text{Dom}_{\tilde{A}}^B/\sim$. We define

$$\eta_{\tilde{B}} = (\tau^\tilde{B})^{-1} \circ \tilde{F} : F(\tilde{B}) \to F_T(\tilde{B}).$$

We need to show that $\eta$ is a natural isomorphism. It is clear that $\eta_{\tilde{B}}$ is an isomorphism. We must prove that, for all $\tilde{B}, \tilde{B} \in \text{Iso}(B)$ and all isomorphisms $h: \tilde{B} \to \tilde{B}$, the following diagram commutes:

Here, $h: \text{Dom}_{\tilde{A}}^B \to \text{Dom}_{\tilde{A}}^B$ is the restriction of $h: \tilde{B}^\omega \to \tilde{B}^\omega$, which is the extension of $h: \tilde{B} \to \tilde{B}$.

The right-hand square commutes by definition of $F_T(h)$. To show that the left-hand square commutes, take $i \in F(\tilde{B})$ and $j = F(h)(i) \in F(\tilde{B})$. Let $(\check{c}, i') = \tilde{F}(i) \in \text{Dom}_{\tilde{A}}^B$. We must show that $\tilde{F}(j) = h(\check{c}, i') = (h(\check{c}), i')$. Since $(\check{c}, i') = \tilde{F}(i)$,

$$\tilde{c} \models_B^* F(\tilde{B}, g^{-1}, \tilde{B}_g)(i) = i'.$$

We claim that

$$h(\check{c}) \models_B^* F(\tilde{B}, g^{-1}, \tilde{B}_g)(j) = i'$$

from which it follows that $\tilde{F}(j) = (h(\check{c}), i')$. Let $g \supset h(\check{c})$ be $\tilde{B} \oplus \tilde{B} \oplus h$-generic. Then $h^{-1} \circ g \supset \check{c}$ is also $\tilde{B} \oplus \tilde{B} \oplus h$-generic. So

$$F(\tilde{B}, g^{-1} \circ h, \tilde{B}_{h^{-1}g})(i) = i'.$$

Then

$$F(\tilde{B}, g^{-1} \circ h, \tilde{B}_{h^{-1}g}) \circ F(\tilde{B}, h^{-1}, \tilde{B}) \circ F(\tilde{B}, h, \tilde{B})(i) = i'.$$

Simplifying this, and using the fact that $F(\tilde{B}, h, \tilde{B})(i) = j$, we get

$$F(\tilde{B}, g^{-1}, \tilde{B}_g)(j) = i'.$$

Since $g$ was chosen arbitrarily,

$$h(\check{c}) \models_B^* F(\tilde{B}, g^{-1}, \tilde{B}_g)(F(h)(i)) = i'.$$
Remark 9.3.18. In the proof of the previous lemma, we saw that the following diagram commutes:

\[
\begin{array}{ccc}
F(\mathcal{B}) & \xrightarrow{\delta^B} & \text{Dom}^B_A \\
F(h) \downarrow & & \downarrow \tilde{h} \\
F(\tilde{\mathcal{B}}) & \xrightarrow{\delta^{\tilde{B}}} & \text{Dom}^{\tilde{B}}_A
\end{array}
\]

We will use this fact later.

The last thing we need to verify is the complexity claim.

**Proposition 9.3.19.** For any $\Delta^0_\alpha$ functor $F: \text{Iso}(\mathcal{B}) \to \text{Iso}(\mathcal{A})$ there is a $\Delta^\mathcal{A}_\alpha$ interpretation, $\mathcal{I}$, of $\mathcal{A}$ within $\mathcal{B}$, such that $F$ is naturally isomorphic to the functor $F_\mathcal{I}$ associated to $\mathcal{I}$. Furthermore, the isomorphism between $F$ and $F_\mathcal{I}$ can be taken to be $\Delta^0_\alpha$.

**Proof.** That $\mathcal{I}$ is a $\Delta^\mathcal{A}_\alpha$ interpretation follows immediately from Lemma 9.3.9, the definition of the interpretation, and our remark that if $F$ is a $\Delta^0_\alpha$ functor, then the formulas involved in the definition of the interpretation are all $\Delta^\mathcal{A}_\alpha$. That the isomorphism between $F$ and $F_\mathcal{I}$ is $\Delta^0_\alpha$ follows from the fact that determining whether \[\bar{c} \equiv_{\mathcal{B}} F(\mathcal{B}, \bar{g}^{-1}, \mathcal{B}_\bar{g})(i) = j\] is $\Delta^0_\alpha(\mathcal{B})$ uniformly in $\mathcal{B}$.

**9.3.4 Bi-Interpretations**

**Proof of Theorem 9.1.12.** Let $F: \text{Iso}(\mathcal{B}) \to \text{Iso}(\mathcal{A})$ and $G: \text{Iso}(\mathcal{A}) \to \text{Iso}(\mathcal{B})$ be a Borel adjoint equivalence of categories, as in the statement of the theorem, with $\eta: \text{id}_{\text{Iso}(\mathcal{B})} \to GF$ and $\epsilon: \text{id}_{\text{Iso}(\mathcal{A})} \to FG$. Assume that $\mathcal{A} = F(\mathcal{B})$.

Let $\mathcal{I}$ and $\mathcal{J}$ be the interpretations using the method described earlier. Recall that just before Lemma 9.3.17 we defined an operator $\mathfrak{F}$ which, for each $\mathcal{B}$, gives an isomorphism $\mathfrak{F}^B: F(\mathcal{B}) \to \text{Dom}^B_A$. We get such an operator for each of $F$ and $G$, denoting them by $\mathfrak{F}^B: F(\mathcal{B}) \to \text{Dom}^B_A$ and $\mathfrak{G}^A: G(\mathcal{A}) \to \text{Dom}^A_B$.

Consider the isomorphism

\[\mathfrak{F}^B \circ \mathfrak{G}^F(\mathcal{B}) \circ \eta^B: \mathcal{B} \to \text{Dom}^B_A.\]

Let $h: \mathcal{B} \to \tilde{\mathcal{B}}$ be an isomorphism. Then we get maps

\[
\begin{array}{cccc}
\mathcal{B} & \xrightarrow{\eta^B G(F(\mathcal{B}))} & \text{Dom}^F_B & \xrightarrow{\delta^B} \text{Dom}^{\tilde{B}}_A \\
\downarrow h & & \downarrow G(F(h)) & \downarrow \tilde{h} \\
\tilde{\mathcal{B}} & \xrightarrow{\eta^{\tilde{B}} G(F(\tilde{\mathcal{B}}))} & \text{Dom}^F_{\tilde{B}} & \xrightarrow{\delta^{\tilde{B}}} \text{Dom}^{\text{Dom}^B_A}
\end{array}
\]
The first square commutes because \( \eta \) is a natural isomorphism \( \text{id}_{\ Iso(B)} \to GF \), and the remaining two squares commute by Remark 9.3.18.

First, take \( \mathcal{B} = \mathcal{B} = \mathcal{B} \) and \( h \) an automorphism of \( \mathcal{B} \). We see from the fact that the diagram above commutes that \( \tilde{\mathcal{S}}^B \circ \mathcal{G}(F(B)) \circ \eta_B \) is invariant under automorphisms of \( \mathcal{B} \), and so it is \( L_{\omega_1 \omega} \)-definable.

Now we claim that if \( F \) is \( \Delta^0_\alpha \) (or \( \Delta^0_\alpha \)), then \( \tilde{\mathcal{S}}^B \circ \mathcal{G}(F(B)) \circ \eta_B \) is relatively intrinsically \( \Delta^0_\alpha \) (resp. \( \Delta^0_\alpha \)) and hence definable by a \( \Delta^0_\alpha \) (resp. \( \Delta^0_\alpha \)) formula. Consider the commutative diagram above, with \( \mathcal{B} = \mathcal{B} \) and \( \mathcal{B} \) some other copy of \( \mathcal{B} \), with an isomorphism \( h: \mathcal{B} \to \mathcal{B} \). We see that

\[
\tilde{\mathcal{S}}^B \circ \mathcal{G}(F(B)) \circ \eta_B : \mathcal{B} \to \text{Dom}_{\mathcal{B}} \circ \mathcal{G}(A)
\]

is defined within \( \mathcal{B} \) by the same formula which defines \( \tilde{\mathcal{S}}^B \circ \mathcal{G}(F(B)) \circ \eta_B \) in \( \mathcal{B} \). Moreover, since \( F, G \), and \( \eta \) are \( \Delta^0_\alpha \) (resp. \( \Delta^0_\alpha \)) operators, \( \tilde{\mathcal{S}}^B \circ \mathcal{G}(F(B)) \circ \eta_B \) is \( \Delta^0_\alpha \) (resp. \( \Delta^0_\alpha \)) in \( \mathcal{B} \). Thus \( \tilde{\mathcal{S}}^B \circ \mathcal{G}(F(B)) \circ \eta_B \) is relatively intrinsically \( \Delta^0_\alpha \) (resp. \( \Delta^0_\alpha \)). A similar argument works for \( \mathcal{G}(A) \circ \mathcal{G}(A) \circ \epsilon_A \).

Define \( g^B_A = \tilde{\mathcal{S}}^B : \mathcal{A} \to \text{Dom}_{\mathcal{B}} \) and \( g^A_B = \mathcal{G}(F(B)) \circ \eta_B : \mathcal{B} \to \text{Dom}_{\mathcal{B}} \). We know that these are isomorphisms. The maps \( g^B_A \) and \( g^A_B \) go in the opposite direction as the maps from Definition 9.1.5. Letting \( f^B_A \) and \( f^A_B \) be their inverses, we get the maps required for a bi-interpretation. We just have to show that the compositions of these maps are the \( L_{\omega_1 \omega} \)-definable isomorphisms from the previous paragraph.

We have

\[
g^B_A \circ g^A_B = \tilde{\mathcal{S}}^B \circ \mathcal{G}(F(B)) \circ \eta_B : \mathcal{B} \to \text{Dom}_{\mathcal{B}} \circ \mathcal{G}(A).
\]

Also, by Remark 9.3.18 (With \( h = \eta_B \)), and the fact that \( F(\eta_B) = \epsilon_{F(B)} \),

\[
\tilde{\eta}_B \circ \tilde{\mathcal{S}}^B = \mathcal{G}(F(B)) \circ F(\eta_B) = \mathcal{G}(F(B)) \circ \epsilon_{F(B)}.
\]

Then, using the fact that \( \mathcal{A} = F(\mathcal{B}) \),

\[
\tilde{\mathcal{S}}^A \circ \mathcal{G}(A) \circ \epsilon_A = \tilde{\mathcal{S}}^A \circ \tilde{\eta}_B \circ \tilde{\mathcal{S}}^B = g^A_B \circ g^B_A.
\]

**Proof of Theorem 9.1.11.** Let \( \mathcal{I} \) and \( \mathcal{J} \) be as in the statement of the theorem: \( \mathcal{I} \) is an interpretation of \( \mathcal{A} \) inside of \( \mathcal{B} \) and \( \mathcal{J} \) is an interpretation of \( \mathcal{B} \) inside of \( \mathcal{B} \). From these bi-interpretations we get functors \( F = F_B: \text{Iso}(\mathcal{B}) \to \text{Iso}(\mathcal{A}) \) and \( G = F_J: \text{Iso}(\mathcal{A}) \to \text{Iso}(\mathcal{B}) \). These functors were defined so that, for \( \mathcal{B} \in \text{Iso}(\mathcal{B}) \) and \( \mathcal{A} \in \text{Iso}(\mathcal{A}) \), there are isomorphisms \( \tau^\mathcal{B}: F(\mathcal{B}) \to \text{Dom}_{\mathcal{B}} \) and \( \rho^\mathcal{A}: G(\mathcal{A}) \to \text{Dom}_{\mathcal{A}} \). Moreover, given an isomorphism \( h: \mathcal{B} \to \mathcal{B} \), \( F(h) = \tau^\mathcal{B} \circ h \circ \tau^\mathcal{B} \) and given \( h: \mathcal{A} \to \mathcal{A} \), \( G(h) = \rho^\mathcal{A} \circ h \circ \rho^\mathcal{A} \).

Recall that \( f^B_A \circ f^A_B \) is \( L_{\omega_1 \omega} \)-definable as a subset of \( \mathcal{B}^{\omega_1} \). In any copy \( \mathcal{B} \) of \( \mathcal{B} \), let \( \varphi^\mathcal{B}: \text{Dom}_{\mathcal{B}} \to \mathcal{B} \) be defined by this formula. Similarly, let \( \psi^\mathcal{A}: \text{Dom}_{\mathcal{A}} \to \mathcal{A} \) be defined by the same formula which defines \( f^B_A \circ f^A_B \) in \( \mathcal{A} \).
Essentially what we want to do is to identify $\mathcal{D}om_{\mathcal{B}}^{\mathcal{A}}$ with $G(F(\mathcal{B}))$ and $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$ with $F(G(\mathcal{A}))$ and use $\varphi$ and $\psi$ as our natural isomorphisms. We make these identifications using $\tau$ and $\rho$.

Define
\[ \eta_{\mathcal{B}} = \varphi_{\mathcal{B}} \circ \tau_{\mathcal{B}} \circ \rho^{\mathcal{B}} : G(F(\mathcal{B})) \to \mathcal{B} \]
and
\[ \epsilon_{\mathcal{A}} = \psi_{\mathcal{A}} \circ \rho^{\mathcal{A}} \circ \tau^{G(\mathcal{A})} : F(G(\mathcal{A})) \to \mathcal{A}. \]

We begin by showing that $\eta$ is a natural isomorphism between $GF$ and $id_{\mathcal{B}}$. (Note that it is more convenient here to have $\eta$ and $\epsilon$ mapping in the opposite direction as in Definition 9.1.10.) Let $h: \mathcal{B} \to \mathcal{B}$ be an isomorphism. Then since $\varphi$ is $\mathcal{L}_{\omega_1\omega}$-definable,
\[
\begin{align*}
h \circ \eta_{\mathcal{B}} &= h \circ \varphi_{\mathcal{B}} \circ \tau_{\mathcal{B}} \circ \rho^{\mathcal{B}} \\
&= \varphi_{\mathcal{B}} \circ \tau_{\mathcal{B}} \circ \rho^{\mathcal{B}}.
\end{align*}
\]
By definition of $F$ and $G$,
\[
\varphi_{\mathcal{B}} \circ \tau_{\mathcal{B}} \circ \rho^{\mathcal{B}} = \varphi_{\mathcal{B}} \circ \tau_{\mathcal{B}} \circ F(h) \circ \rho^{\mathcal{B}}
\]
\[
= \varphi_{\mathcal{B}} \circ \tau_{\mathcal{B}} \circ \rho^{\mathcal{B}} \circ G(F(h))
\]
\[
= \eta_{\mathcal{B}} \circ G(F(h)).
\]
Similarly, $\epsilon$ is a natural isomorphism between $FG$ and $id_{\mathcal{A}}$. Thus we have shown that $F$, $G$, $\eta$, and $\epsilon$ give an equivalence of categories.

Now we must show that this is equivalence is an adjoint equivalence by showing that given $\mathcal{B}$ and $\mathcal{A}$, $F(\eta_{\mathcal{B}}) = \epsilon_{F(\mathcal{B})}$ and $G(\epsilon_{\mathcal{A}}) = \eta_{G(\mathcal{A})}$. To begin, we prove two claims which give identities of compositions of isomorphisms.

**Claim 9.3.20.** $\varphi_{\mathcal{B}} \circ \tau_{\mathcal{B}} = \tau_{\mathcal{B}} \circ \psi^{F(\mathcal{B})}$

**Proof.** Let $h: \mathcal{A} \to F(\mathcal{B})$ be an isomorphism. Then since $h^{-1} \circ \psi^{F(\mathcal{B})} \circ \tilde{h} = \psi^{\mathcal{A}}$, we just need to show that
\[
\tilde{f}_{B}^{A} \circ \tilde{f}_{B}^{A} \circ \tilde{f}_{B}^{A} \circ \tilde{h} = \tau_{\mathcal{B}} \circ h \circ \tilde{f}_{B}^{A} \circ \tilde{f}_{B}^{A}.
\]

Consider the isomorphisms $f_{A}^{B}: \mathcal{D}om_{\mathcal{B}}^{\mathcal{A}} \to \mathcal{A}$ and $\tau_{\mathcal{B}} \circ h: \mathcal{A} \to \mathcal{D}om_{\mathcal{B}}^{\mathcal{A}}$. Let $\alpha = f_{A}^{B} \circ \tau_{\mathcal{B}} \circ h$. Then $\alpha$ is an automorphism of $\mathcal{A}$. Since $f_{A}^{B} \circ \tilde{f}_{B}^{A}$ is definable, we have
\[
\tilde{f}_{B}^{A} \circ \tilde{f}_{B}^{A} \circ \tilde{f}_{B}^{A} \circ \tilde{h} = f_{A}^{-1} \circ f_{B} \circ \tilde{f}_{B}^{A} \circ \tilde{h}
\]
\[
\tilde{f}_{B}^{A} \circ \tilde{f}_{B}^{A} \circ \tilde{f}_{B}^{A} \circ \tilde{h} = \tau_{\mathcal{B}} \circ h \circ \tilde{f}_{B}^{A} \circ \tilde{f}_{B}^{A}
\]
as desired. \qed
The next claim replaces $B$ in the claim above by an arbitrary copy $\tilde{B}$ of $B$.

**Claim 9.3.21.** $\tilde{\varphi}^\tilde{B} \circ \tilde{\tau}^\tilde{B} = \tau^B \circ \psi^F(\tilde{B})$.

**Proof.** Let $h: \tilde{B} \rightarrow B$ be an isomorphism. By definition of $F$, we have

$$\tilde{h} \circ \tau^\tilde{B} = \tau^B \circ F(h).$$

Since $\varphi$ and $\psi$ are given by $L_{\omega_1 \omega}$ definitions,

$$h \circ \varphi^B = \varphi^\tilde{B} \circ \tilde{h} \text{ and } F(h) \circ \psi^F(\tilde{B}) = \psi^F(B) \circ \tilde{F}(h).$$

From the previous claim, we get

$$\tilde{\varphi}^\tilde{B} \circ \tilde{\tau}^\tilde{B} = \tau^B \circ \psi^F(B).$$

Composing both sides with $\tilde{F}(h)$ on the right, we get

$$\tilde{\varphi}^\tilde{B} \circ \tilde{\tau}^\tilde{B} \circ \tilde{F}(h) = \tau^B \circ \psi^F(\tilde{B}) \circ \tilde{F}(h)$$

$$\varphi^B \circ \tilde{h} \circ \tilde{\tau}^\tilde{B} = \tau^B \circ F(h) \circ \psi^F(\tilde{B})$$

$$\tilde{h} \circ \varphi^\tilde{B} \circ \tilde{\tau}^\tilde{B} = \tilde{h} \circ \tau^B \circ \psi^F(B)$$

Applying $\tilde{h}^{-1}$ to both sides, we complete the claim. \hfill $\square$

To see that $F(\eta_B) = \epsilon_{F(\tilde{B})}$, note that

$$F(\eta_B) = \tau^B^{-1} \circ \tilde{\varphi}^\tilde{B} \circ \tilde{\tau}^\tilde{B} \circ \tilde{\rho}^F(\tilde{B}) \circ \tau^G(F(\tilde{B})); F(G(F(\tilde{B}))) \rightarrow F(\tilde{B})$$

and

$$\epsilon_{F(\tilde{B})} = \psi^F(\tilde{B}) \circ \tilde{\rho}^F(\tilde{B}) \circ \tau^G(F(\tilde{B})); F(G(F(\tilde{B}))) \rightarrow F(\tilde{B}).$$

Then it follows from the previous claim that these are equal. Similarly, $G(\epsilon_{\tilde{A}}) = \eta_{G(\tilde{A})}$. \hfill $\square$

### 9.4 Indiscernibles

In this section we prove Theorem 9.1.4, which says that, for a structure $A$, there is a continuous homomorphism from $\text{Aut}(A)$ onto $S_\infty$ if and only if $A$ has an infinite definable set of absolutely indiscernible definable equivalence classes.

**Proof of Theorem 9.1.4.** The direction $(2) \Rightarrow (1)$ is easy to see. For the other direction, suppose that there is a continuous homomorphism $H$ from $\text{Aut}(A)$ onto $S_\infty$. Let $B$ be the trivial structure with a countable domain and no relations; then $\text{Aut}(B) = S_\infty$. By Theorem 9.1.3, there is an interpretation $I$ of $B$ in $A$ such that $H = G_I \upharpoonright \text{Aut}(A)$. 
Let \( D = \operatorname{Dom}_B \subseteq A^\omega \), and let \( E \) be the relation \( \sim \). Let \( h \) be a permutation of the \( E \)-equivalence classes. Then \( h \) induces an automorphism \( f_B^A \circ h \circ f_B^{A^{-1}} \) of \( B \). Then, since \( H \) is onto, there is an automorphism \( g \) of \( A \) with \( H(g) = f_B^A \circ h \circ f_B^{A^{-1}} \). But then \( G_T(g) = H(g) = f_B^A \circ h \circ f_B^{A^{-1}} \), and so, by definition of \( G_T \), \( g \) extends \( h \).

Above, we chose \( D \subseteq A^\omega \); we need to choose \( D \subseteq A^n \) for some \( n \). It suffices to show that for some \( n \), \( D' = D \cap A^n \) and \( E' = E \cap (D' \times D') \) have infinitely many equivalence classes. Let \( n \) be such that \( D \cap A^n \) is non-empty: say it contains some element \( \bar{a} \). Let \( x \in B \) be \( f_B^A(\bar{a}) \). Let \( y_1, y_2, \ldots \) be infinitely many elements of \( B \) distinct from \( x \), and let \( h_1, h_2, \ldots \) be automorphisms of \( B \) such that \( h_i(x) = y_i \). Then since \( H \) is onto, there are automorphisms \( g_i \) of \( A \) with \( H(g_i) = h_i \). Then \( h_i = G_T(g_i) = f_B^A \circ g_i \circ f_B^{A^{-1}} \). Since \( h_i(x) = y_i \) and \( f_B^A(\bar{a}) = x \), \( f_B^A \circ g_i(\bar{a}) = y_i \). Thus \( g_i(\bar{a}) \) must be in a different \( E \)-equivalence class from \( \bar{a} \), and also from \( g_j(\bar{a}) \) for \( i \neq j \); but since \( \bar{a} \in A^n \), \( g_i(\bar{a}) \in A^n \). Thus there are infinitely many \( E \)-equivalence classes in \( D \cap A^n \).

A similar argument proves the following theorem.

**Theorem 9.4.1.** Let \( A \) be a countable structure. The following are equivalent:

1. There is a continuous homomorphism from \( \operatorname{Aut}(A) \) onto \( \operatorname{Aut}(\mathbb{Q},<) \).
2. There is an \( n \), an \( \mathcal{L}_{\omega_1 \omega} \)-definable \( D \subseteq A^n \), an \( \mathcal{L}_{\omega_1 \omega} \)-definable equivalence relation \( E \subseteq D^2 \) with infinitely many equivalence classes, and an \( \mathcal{L}_{\omega_1 \omega} \)-definable order, such that the \( E \)-equivalence classes are order indiscernible, in the sense that each order-preserving permutation of the \( E \)-equivalence classes extends to an automorphism of \( A \).

By considering isomorphisms, we also get:

**Theorem 9.4.2.** Let \( A \) be a countable structure. The following are equivalent:

1. There is a continuous isomorphism between \( \operatorname{Aut}(A) \) and \( S_\infty \).
2. There is an \( n \), an \( \mathcal{L}_{\omega_1 \omega} \)-definable \( D \subseteq A^n \), and an \( \mathcal{L}_{\omega_1 \omega} \)-definable equivalence relation \( E \subseteq D^2 \) with infinitely many equivalence classes and such that the \( E \)-equivalence classes are absolutely indiscernible, and every other element is definable from this set. In other words, if we add relations naming each of these equivalence classes, then every element of the structure is \( \mathcal{L}_{\omega_1 \omega} \)-definable.

**Theorem 9.4.3.** Let \( A \) be a countable structure. The following are equivalent:

1. There is a continuous isomorphism between \( \operatorname{Aut}(A) \) and \( \operatorname{Aut}(\mathbb{Q},<) \).
2. There is an \( n \), an \( \mathcal{L}_{\omega_1 \omega} \)-definable \( D \subseteq A^n \), an \( \mathcal{L}_{\omega_1 \omega} \)-definable equivalence relation \( E \subseteq D^2 \) with infinitely many equivalence classes, and an \( \mathcal{L}_{\omega_1 \omega} \)-definable order, such that the \( E \)-equivalence classes are order indiscernible, and every other element is definable from this set.
Part IV

Computable Algebra
Chapter 10

Extensions of Embeddings of Fields

The results presented in this chapter appeared in [HTMMa]. They are joint work with Alexander Melnikov and Russell Miller and appear here with their permission.

10.1 Introduction

This article is a contribution to effective field theory, where the main objects of study are computable fields. Recall that an algebraic structure is computable if the elements of its domain are associated with natural numbers in such a way that the operations become computable functions upon this domain [Mal61, Rab60]. There are a number of classical results which say that maps between fields can be extended to maps between their algebraic closures. We consider when this can be done effectively. That is, if all of the fields involved are computable, and we are given a computable map, must there exist a computable extension to the algebraic closures? We obtain both necessary and sufficient conditions on a computable field $F$ which ensure that these classical theorems hold effectively for the field $F$. We also apply our results to computable fields with a distinguished (computable) automorphism; such fields are known as difference fields. We investigate the problem of effectively embedding difference fields into computable difference-closed fields (these are existentially closed difference fields, to be discussed). As we will see, the most naive analogy of the well-known results of Rabin [Rab60] and Harrington [Har74] fails for computable difference fields, in all characteristics. Nonetheless, we will find a broad class of fields (including abelian extensions of a prime field) for which a stronger version of the analogous result holds.

10.1.1 Embeddings into Algebraically Closed Fields

In the pioneering paper [Rab60], Rabin proved that every computable field $F$ can be embedded into a computable presentation $\mathcal{E}$ of its algebraic closure by a computable map $\iota: F \to \mathcal{E}$. Provided that $\mathcal{E}$ is algebraic over the image $\iota(F)$, we call such an embedding $\iota$ a Rabin embedding of $F$ into $\mathcal{E}$, writing $\overline{F}$ for $\mathcal{E}$ since $\mathcal{E}$ may thus be regarded as an algebraic closure
of $\mathcal{F}$. In what follows it will be important that, in general, the image of $\mathcal{F}$ under the Rabin embedding $\iota$ does not have to be a computable subset of $\overline{\mathcal{F}}$. Rabin [Rab60] showed that the problem of deciding the $\iota$-image of $\mathcal{F}$ in $\overline{\mathcal{F}}$ is fully captured by the notion of the splitting set. Recall that the splitting set $S_\mathcal{F}$ of $\mathcal{F}$ is the set of all polynomials $p \in \mathcal{F}[X]$ which are reducible over $\mathcal{F}$. If the splitting set of $\mathcal{F}$ is computable, then we say that $\mathcal{F}$ has a splitting algorithm. Rabin [Rab60] showed that for each computable field $\mathcal{F}$, and for each Rabin embedding $\iota$ of $\mathcal{F}$, the image $\iota(\mathcal{F})$ of $\mathcal{F}$ in $\overline{\mathcal{F}}$ is Turing equivalent to the splitting set of $\mathcal{F}$, which may be undecidable [Rab60]. We note that splitting algorithms had been studied long before Rabin. For instance, in 1882, Kronecker [Kro82] analyzed splitting algorithms for finitely generated extensions of $\mathbb{Q}$.

10.1.2 The First Main Result

It is well known that every isomorphic embedding $\alpha$ of a field $\mathcal{F}$ into an algebraically closed $\mathcal{K}$ extends to an embedding $\beta$ of the algebraic closure of $\mathcal{F}$ into $\mathcal{K}$. Since we are interested in effective embeddings, we ask whether $\beta$ can always be chosen to be effective. In our notation, with a fixed Rabin embedding $\iota$ and an arbitrary computable $\alpha$, we ask for a computable $\beta$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\overline{\mathcal{F}} & \overset{\beta}{\longrightarrow} & \mathcal{K} \\
\downarrow{\iota} & & \downarrow{\alpha} \\
\mathcal{F} & & \\
\end{array}
$$

i.e., $\alpha = \beta \circ \iota$. If a computable solution to the diagram above exists for every choice of $\alpha$ and of the computable algebraically closed field $\mathcal{K}$, then we say that $(\mathcal{F}, \iota)$ has the computable extendability of embeddings property. Notice, however, that if some Rabin embedding $\iota$ of a particular $\mathcal{F}$ has the computable extendability of embeddings property, then so does every other Rabin embedding $j$ of $\mathcal{F}$ (into any computable presentation of $\overline{\mathcal{F}}$): just apply the computable extendability of embeddings property for $\iota$, with $j$ as the $\alpha$, to get an embedding $\beta_j$ which extends $j \circ \iota^{-1}$ (and must be an isomorphism). Then, given any other $\alpha$, the computable extendability of embeddings property for $\iota$ yields a $\beta$ such that $\beta \circ \beta_j^{-1}$ satisfies the computable extendability of embeddings property for $j$ and this $\alpha$. Therefore, we usually simply say that $\mathcal{F}$ itself has the computable extendability of embeddings property.

The first problem that we address in the paper is:

*Find a necessary and sufficient condition for a computable $\mathcal{F}$ to have the computable extendability of embeddings property.*

Before we give a necessary and sufficient condition, we discuss a subtlety that would not occur in the classical case. The desired extension $\beta$ clearly depends on the choice of the Rabin embedding $\iota$. Classically, the dependence on $\iota$ is often suppressed, since we can identify $\mathcal{F}$ with its $\iota$-image. However, as noted above, such an identification is generally
impossible effectively: the membership problem for \( \iota(F) \) may be undecidable. To emphasize the dependence on the embedding \( \iota : F \to \bar{F} \), we say that \( \beta \) \( \iota \)-extends \( \alpha \) if it is a solution to the diagram above. Later in the paper we will allow \( \iota \) to vary, but for now we fix a concrete choice of a Rabin embedding \( \iota \).

We may further restrict ourselves and ask for a uniform procedure (i.e., a Turing functional) that takes the open diagram of an algebraically closed field \( K \) and an embedding \( \alpha : F \to K \) and outputs an embedding of \( \bar{F} \) into \( K \) \( \iota \)-extending \( \alpha \). For uniform extendability we do not require \( K \) or \( \alpha \) to be computable, but we still fix \( \iota \). The reader may find it somewhat unexpected that this uniform version is equivalent to the computable extendability of embeddings property:

**Theorem 10.1.1.** Let \( F \) be a computable field together with a computable embedding \( \iota : F \to \bar{F} \) of \( F \) into its algebraic closure. Then the following are equivalent:

1. \( F \) has a splitting algorithm,
2. \( F \) has the computable extendability of embeddings property,
3. There exists a Turing functional which, given as its oracle the open diagram of an algebraically closed field \( K \) and an embedding \( \alpha : F \to K \), computes an embedding of \( \bar{F} \) into \( K \) \( \iota \)-extending \( \alpha \).

The property captured by Theorem 10.1.1 above is also equivalent to an a priori weaker uniform extendability condition, namely the existence of a uniform procedure that takes indices of computable \( K \) and \( \alpha : F \to K \) and outputs an index of a computable \( \beta : \bar{F} \to K \) extending \( \alpha \). Indeed, this weaker uniform property follows from the uniform extendability condition in Theorem 10.1.1 and implies the computable extendability property.

In the language of reverse mathematics, Theorem 10.1.1 would say that in the \( \omega \)-model \( REC \) consisting of the computable sets, a field has a unique algebraic closure if and only if that field has a splitting algorithm. Thus, while \( RCA_0 \) proves that every field with a splitting algorithm has a unique algebraic closure, it is consistent that every other field has more than one algebraic closure. We note that it was already known from work in reverse mathematics (and is easy to see) that in the situation described above there is always a low \( \iota \)-extension of \( \alpha \), and in characteristic zero if \( F \) has a splitting algorithm then there is a computable extension of \( \alpha \) (see [DHS13, Theorem 9] and [FSS83, Theorem 3.3]). In our result we do not restrict ourselves to fields of characteristic 0; the issue that we face in the case of a positive characteristic will be circumvented using purely inseparable extensions (to be defined). We remark that the essential part of our proof of Theorem 10.1.1 is based on a certain preservation strategy combined with a variation of the Henkin construction; such a combination has not yet been seen in effective algebra.
10.1.3 The Second Main Result

Another classical result says that every automorphism of a field \( F \) extends to an automorphism of its algebraic closure. In our notation, the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\imath} & \overline{F} \\
\alpha \downarrow & & \downarrow \beta \\
F & \xrightarrow{\imath} & \overline{F}
\end{array}
\]

always has a solution \( \beta \) such that the diagram commutes, i.e., \( \imath \circ \alpha = \beta \circ \imath \). Once again this is dependent on the embedding \( \imath : F \to \overline{F} \), and slightly abusing our terminology we say that \( \beta \ \imath \text{-extends} \ \alpha \). We ask when \( \beta \) can be computed effectively. In the setting of automorphisms, it is natural to look at normal algebraic extensions of the prime field (as we will see in Proposition 10.4.2). In this case, we can apply Theorem 10.1.1 to fully characterize existence of such \( \imath \)-extensions in terms of a splitting algorithm; the exact statement will be given in Corollary 10.4.1. Although the reader may find Corollary 10.4.1 interesting on its own right, the discussed above dependence on \( \imath \) makes it somewhat unsatisfying. Also, as we will discuss in the next subsection, we would like to apply our results to difference fields, and there this dependence on \( \imath \) is an obstacle. Therefore, in contrast to the situation of the computable extendability of embeddings property above, we would like to allow the embedding \( \imath \) to vary.

**Definition 10.1.2.** We say that a computable field \( F \) has the **computable extendability of automorphisms property** if for every computable automorphism \( \alpha : F \to F \) there is a Rabin embedding \( \imath : F \to \overline{F} \) and a computable automorphism \( \beta : \overline{F} \to \overline{F} \) which \( \imath \)-extends \( \alpha \).

The second problem we address in the paper is:

*Find a necessary and sufficient condition for a computable \( F \) to have the computable extendability of automorphisms property.*

As we mentioned above, the computable extendability of automorphisms property is the property which is of interest in constructing embeddings of difference fields into difference closed fields (as we will see in Theorem 10.1.5). It is not hard to see that if a normal extension \( F \) of the prime field has a splitting algorithm, then \( F \) has the computable extendability of automorphisms property. Is having a splitting algorithm implied by computable extendability of automorphisms property? Although we don’t know if this is true in general (and we conjecture that perhaps not), we give a condition on the Galois group of \( F \) over the prime field—the **non-covering property**—under which the computable extendability of automorphisms property is equivalent to having a splitting algorithm.

**Definition 10.1.3.** We say that a group \( G \) has the **non-covering property** if for all finite index normal subgroups \( M \lhd N \) of \( G \) and \( g \in G \), there is \( h \in gN \) such that for all \( x \in G \), \( x^{-1}hx \notin gM \).
In Lemma 10.4.5 we will give an equivalent condition in the language of field extensions, using Galois correspondence.

Before we state our second main result, we note that groups with the non-covering property include abelian and simple groups, and the class of profinite groups with the non-covering property is closed under direct products.

**Theorem 10.1.4.** Let $\mathcal{F}$ be a computable normal extension of $\mathbb{F}_p$, for some prime $p$, such that $\text{Gal}(\mathcal{F}/\mathbb{F}_p)$ has the non-covering property. The following are equivalent:

1. $\mathcal{F}$ has a splitting algorithm,
2. $\mathcal{F}$ has the computable extendability of automorphisms property,
3. $\mathcal{F}$ has the uniform extendability of automorphisms property.

In characteristic $p > 0$, all Galois groups are abelian, and so every Galois group has the non-covering property. Thus, in characteristic $p > 0$ the computable extendability of automorphisms property is equivalent to having a splitting algorithm.

### 10.1.4 Applications to Difference Closed Fields

Rabin [Rab60] showed that every computable field can be computably embedded into its computable algebraic closure, and Harrington [Har74] later showed that every computable differential field can be computably embedded into a differential closure. We consider the possibility of such a result for fields with a distinguished automorphism; such structures are called *difference fields* [CH99]. An existential closure of such a structure analogous to an algebraically closed field exists and is called a *difference closed field*. (We note that there is no such a thing as the difference closure since there might be no “smallest” difference closed field containing a given difference field. The formal definitions will follow later.) In what follows next, we refer to this hypothetical analogous result as the Rabin-Harrington theorem.

We note that a difference field $(\mathcal{F}, \sigma)$ may distinguish a rather boring automorphism $\sigma$, e.g., the identity, for which the Rabin-Harrington theorem clearly holds. On the other hand, we will see that there exist computable difference fields that do not embed into any computable difference closed field. Thus, the same field may have two different automorphisms, one witnessing the Rabin-Harrington theorem, and the other witnessing its failure, and finding a satisfactory characterization in this setting seems rather hopeless (yet the reader may try to find one). On the other hand, we are mostly interested in the properties of the underlying field which make the Rabin-Harrington theorem hold, and we are not that much concerned with the properties of some “pathological” automorphism that may witness the failure of the Rabin-Harrington theorem. Thus, we arrive at the third main question addressed in the paper:

*For which $\mathcal{F}$ does $(\mathcal{F}, \sigma)$ satisfy the Rabin-Harrington theorem for all $\sigma$?*
Here of course $\mathcal{F}$ is a computable field and $\sigma$ ranges over all computable automorphisms of $\mathcal{F}$. We show in Theorem 10.5.1 that the Rabin-Harrington Theorem holds for difference fields with underlying field $\mathcal{F}$ if and only if $\mathcal{F}$ has the computable extension of automorphisms property. Using our results on extending automorphisms, namely the second main result of the paper (Theorem 10.1.4), we can find a large class of difference fields which satisfy the Rabin-Harrington theorem for any interpretation of the distinguished automorphism:

**Theorem 10.1.5.** Let $\mathcal{F}$ be a computable normal extension of $\mathbb{F}_p$, for some prime $p$, such that $\text{Gal}(\mathcal{F}/\mathbb{F}_p)$ has the non-covering property. Then the following are equivalent:

1. $\mathcal{F}$ has a splitting algorithm,
2. for any computable $\sigma$, $(\mathcal{F}, \sigma)$ can be computably embedded into a computable difference closed field.

Even without the non-covering property, (1) implies (2).

In particular, this theorem gives a complete answer to the third main question of the paper in the case of a normal extension of $\mathbb{F}_p$ for any $p > 0$. On the other hand, Theorem 10.1.5 will be used to produce various examples of computable difference fields that cannot be embedded into computable difference closed fields. We conclude that the most naive attempt to generalize the results of Rabin and Harrington fails. On the other hand, if we allow the automorphism to vary, we get a complete characterization for a large class of fields.

### 10.1.5 The Non-Covering Property

Since our main results refer to the non-covering property of Galois groups, we would like to know more about the class of groups having this property. In Subsection 10.4.4 we study the class of groups that have the non-covering property, with an emphasis on profinite groups. It is not hard to see that abelian groups and simple groups have the non-covering property (see Lemma 10.4.5). However, it takes a lot more effort to prove:

**Theorem 10.1.6.** Let $\{G_i : i \in I\}$ be a collection of profinite groups, each of which has the non-covering property. Then $\prod_{i \in I} G_i$ has the non-covering property.

The proof of this theorem might be of some independent interest to the reader. It filters through Goursat’s lemma [Gou89] (to be stated in the proof of Theorem 10.1.6). We note that our proof uses profiniteness to reduce the case of arbitrarily many direct factors to just two factors, and the proof of the case of just two factors (Lemma 10.4.9) does not use profiniteness. We leave open whether one can use profiniteness to simplify our proof of Lemma 10.4.9. We also note that some groups do not have the non-covering property (to be discussed).
10.1.6 The Structure of the Paper

We will begin in §10.2 by giving some background on computable fields and difference fields. In §10.3 we will consider embeddings into algebraically closed fields and the computable extendability of embeddings property, and prove the first main result, Theorem 10.1.1. In §10.4 we will consider automorphisms and the computable extendability of automorphisms property. We begin in §10.4.1 by considering a strengthening of the computable extendability of automorphisms property. In §10.4.2, we prove the second main result, Theorem 10.1.4. In §10.4.3, we study the class of groups with the non-covering property, and in §10.4.4 we give some applications of Theorem 10.1.4. In §10.5 we consider applications to difference fields and the Rabin-Harrington theorem.

10.2 Preliminaries

10.2.1 Separable and Purely Inseparable Extensions

If \( F \) is a field, a polynomial \( f \in F[X] \) is called \textit{separable} if it has no repeated roots. A element \( a \in E \) of an algebraic field extension \( E/F \) is called \textit{separable over} \( F \) if its minimal polynomial over \( F \) is a separable polynomial. An algebraic field extension \( E/F \) is called \textit{separable} if every element of \( E \) is separable over \( F \). Recall that if \( F \) is finite or characteristic zero, then it is \textit{perfect}, i.e., every algebraic extension is a separable extension.

An algebraic field extension \( E/F \) is called \textit{purely inseparable} if \( E \setminus F \) contains no separable elements. Equivalently, \( E \) is a field of characteristic \( p > 0 \) and every element of \( E \) is the unique root of a polynomial \( X^p - a = 0 \) with \( a \in F \). Given an algebraic field extension \( E/F \), the set

\[ F^s = \{ a \in E : a \text{ is separable over } F \} \]

is the maximal separable extension of \( F \) inside of \( E \) and is called the \textit{separable closure} of \( F \) in \( E \). The field extension \( E/F^s \) is purely inseparable. In the special case where \( E = F \) is the algebraic closure of \( F \), \( F^s \) is called the \textit{separable closure} of \( F \) and is the maximal separable extension of \( F \).

An algebraic field extension \( E/F \) is \textit{normal} if every irreducible polynomial in \( F[X] \) that has a root in \( E \) factors completely in \( E[X] \). A normal separable extension \( E/F \) is called a Galois extension and has associated to it the Galois group \( \text{Gal}(E/F) \) of automorphisms of \( E \) fixing \( F \). Recall that the Galois group obeys the fundamental theorem of Galois theory: the normal subgroups \( H \leq \text{Gal}(E/F) \) correspond to the intermediate normal field extensions.

10.2.2 Computable Fields

Recall that the \textit{splitting set} \( S_F \) of \( F \) is the set of all polynomials \( p \in F[X] \) which are reducible over \( F \). The splitting set of a field is not necessarily computable (see [Mil08, Lemma 7]), but it is always c.e. If the splitting set of \( F \) is computable, then we say that \( F \) has a \textit{splitting}
algorithm. Finite fields and algebraically closed fields trivially have splitting algorithms. Kronecker [Kro82] showed that \( \mathbb{Q} \) has a splitting algorithm, and also that many other field extensions also have a splitting algorithm:

**Theorem 10.2.1** (Kronecker [Kro82]; see also [vdW70]). The field \( \mathbb{Q} \) has a splitting algorithm. If a computable field \( \mathcal{F} \) has a splitting algorithm, and \( a \) is transcendental over \( \mathcal{F} \), then \( \mathcal{F}(a) \) has a splitting algorithm. If \( a \) is separable and algebraic over \( \mathcal{F} \), then \( \mathcal{F}(a) \) has a splitting algorithm. Moreover, the splitting algorithm for \( \mathcal{F}(a) \) is uniform in the minimal polynomial for \( a \) over \( \mathcal{F} \).

Given a field \( \mathcal{F} \) and an element \( a \) which is either transcendental over \( \mathcal{F} \), or separable and algebraic over \( \mathcal{F} \), we know that \( \mathcal{F}(a) \) has a splitting algorithm. However, the algorithm depends on whether \( a \) is transcendental or algebraic. To find a splitting algorithm uniformly, we must know which is the case.

Rabin [Rab60] showed that every computable field \( \mathcal{F} \) has a computable algebraic closure \( \overline{\mathcal{F}} \), and moreover there is a computable embedding \( \iota: \mathcal{F} \to \overline{\mathcal{F}} \). We call such an embedding a Rabin embedding. Moreover, he characterized the image of \( \mathcal{F} \) under this embedding:

**Theorem 10.2.2** (Rabin [Rab60]). Let \( \mathcal{F} \) be a computable field. Then there is a computable algebraically closed field \( \overline{\mathcal{F}} \) and a computable field embedding \( \iota: \mathcal{F} \to \overline{\mathcal{F}} \) such that \( \overline{\mathcal{F}} \) is algebraic over \( \iota(\mathcal{F}) \). Moreover, for any such \( \overline{\mathcal{F}} \) and \( \iota \), the image \( \iota(\mathcal{F}) \) of \( \mathcal{F} \) in \( \overline{\mathcal{F}} \) is Turing equivalent to the splitting set of \( \mathcal{F} \).

A computable field \( \mathcal{F} \) has a dependence algorithm if given \( a \) and \( b_1, \ldots, b_n \), we can compute whether \( a \) is algebraically independent over \( b_1, \ldots, b_n \). A field has a dependence algorithm if and only if it has a computable transcendence base (see, for example, [HTMM15, Proposition 2.2]). In particular, fields of finite transcendence degree have a dependence algorithm.

**Convention.** By an extension \( \mathcal{E}/\mathcal{F} \) of computable fields, we mean that there is a computable embedding of \( \mathcal{F} \) into \( \mathcal{E} \).

### 10.2.3 Difference Fields

Difference fields were first studied by Ritt in the 1930s. A good reference on the classical algebraic theory of difference fields is the book by Cohn [Coh65]. A difference field is a field \( \mathcal{F} \) together with an embedding \( \sigma: \mathcal{F} \to \mathcal{F} \). If \( \sigma \) is onto, \( (\mathcal{F}, \sigma) \) is called inverse. As every difference field has a unique inverse closure up to isomorphism, we lose nothing by assuming that all of our difference fields are inverse.

A difference field \( (\mathcal{F}, \sigma) \) is called a difference closed field if it is existentially closed in the language of difference fields. Difference closed fields arose in the model theoretic study of difference fields (see [Mac97] and [CH99]). \( \mathcal{F} \) is difference closed if and only if:

(i) \( \sigma \) is an automorphism of \( \mathcal{F} \);

(ii) \( \mathcal{F} \) is algebraically closed;
(iii) For every variety $U$, every affine variety $V \subseteq U \times \sigma(U)$ which projects generically onto $U$ and $\sigma(U)$, and every algebraic set $W \not\subseteq V$, there is an $F$-rational point $a \in U(F)$ such that $(a, \sigma(a)) \in V \setminus W$.

The condition (iii) may be viewed as saying that certain systems of equations and inequations have solutions in $F$. Conditions (i), (ii), and (iii) axiomatize the theory $ACFA$ of difference closed fields. $ACFA$ is decidable, and moreover the theories $ACFA_p$ of difference closed fields of characteristic $p$ are also decidable for any $p$, including $p = 0$ [(1.4) of CH99]. $ACFA$ is the model companion of the theory of difference fields [(1.4) of CH99] and hence every formula is equivalent, modulo $ACFA$, to an existential formula [(1.6) of CH99]. Thus, we have:

**Fact 10.2.3.** Every computable difference closed field has a computable (full) elementary diagram.

We call a structure with a computable elementary diagram *decidable*; thus every difference closed field is decidable.

### 10.3 Extending Embeddings into the Algebraic Closure

We begin by showing that if $\mathcal{F}$ is any computable field with a splitting algorithm, $\nu: \mathcal{F} \to \overline{\mathcal{F}}$ is a Rabin embedding, and $\alpha: \mathcal{F} \to \mathcal{K}$ is a computable embedding of $\mathcal{F}$ into an algebraically closed field $\mathcal{K}$, then there is a computable embedding of $\overline{\mathcal{F}}$ into $\mathcal{K}$ extending $\alpha$. In particular, the new results here are in the case of characteristic $p > 0$. The new issue we have to deal with in characteristic $p > 0$ is that Theorem 10.2.1 fails for non-separable extensions. We begin by finding the separable closure of a field $\mathcal{F}$ within its algebraic closure $\overline{\mathcal{F}}$.

**Lemma 10.3.1.** Let $\mathcal{F}$ be a computable field. Then the separable closure of $\mathcal{F}$ is c.e. If $\mathcal{F}$ has a splitting algorithm, then the separable closure $\mathcal{F}^s$ of $\mathcal{F}$ in $\overline{\mathcal{F}}$ is computable (so that $\mathcal{F}^s$ has a splitting algorithm).

**Proof.** Embed $\mathcal{F}$ in its algebraic closure $\overline{\mathcal{F}}$. An element $a \in \overline{\mathcal{F}}$ is separable if and only if there is a polynomial $p(X) \in \mathcal{F}[X]$ of degree $m$ with $p(a) = 0$ and with $m$ distinct roots in $\mathcal{F}$. Thus the separable closure of $\mathcal{F}$ is c.e. If $\mathcal{F}$ has a splitting algorithm, then given $a \in \overline{\mathcal{F}}$ we can find the minimal polynomial $p$ of $a$ over $\mathcal{F}$. Then $a$ is separable over $\mathcal{F}$ if and only if $p$ has no repeated roots, which happens if and only if $p'(a) \neq 0$. (Here, $p'(X)$ is the derivative of $p(X)$ with respect to $X$, treating the coefficients as constants.) So the separable closure of $\mathcal{F}$ is computable.

We are now ready to extend an embedding from a field with a splitting algorithm. The main idea is to break the embedding into two steps; first to extend an embedding $\alpha: \mathcal{F} \to \mathcal{K}$ to an embedding $\beta: \mathcal{F}^s \to \mathcal{K}$ of the separable closure of $\mathcal{F}$ into $\mathcal{K}$, and second to note that $\beta$ extends to a unique embedding of $\overline{\mathcal{F}}$ into $\mathcal{K}$ and that this extension is computable from $\beta$.
Theorem 10.3.2. Let \( \mathcal{F} \) be a computable field and \( \nu : \mathcal{F} \to \overline{\mathcal{F}} \) a Rabin embedding of \( \mathcal{F} \) into its algebraic closure. Suppose that \( \mathcal{F} \) has a splitting algorithm. Then there is a Turing functional \( \Phi \) such that whenever \( \alpha : \mathcal{F} \to \mathcal{K} \) is an embedding of \( \mathcal{F} \) into an algebraically closed field \( \mathcal{K} \), \( \Phi^{\alpha \circ \mathcal{K}} : \overline{\mathcal{F}} \to \mathcal{K} \) is an embedding of \( \overline{\mathcal{F}} \) into \( \mathcal{K} \) \( \nu \)-extending \( \alpha \).

Proof. Since \( \mathcal{F} \) has a splitting algorithm, the image \( \iota(\mathcal{F}) \) of \( \mathcal{F} \) in \( \overline{\mathcal{F}} \) is computable. We may identify \( \mathcal{F} \) with its image. By Lemma 10.3.1 the separable closure \( \mathcal{F}^s \) of \( \mathcal{F} \) is computable as a subset of \( \overline{\mathcal{F}} \) and has a splitting algorithm.

Let \( \mathcal{K} \) be an algebraically closed field and \( \alpha : \mathcal{F} \to \mathcal{K} \) a field embedding. We will begin by describing a procedure to extend \( \alpha \) to an embedding \( \beta : \mathcal{F}^s \to \mathcal{K} \). Let \( \{a_1, a_2, \ldots\} \) be an enumeration of the elements \( \mathcal{F}^s \). Start with \( \beta \) defined only on \( \mathcal{F} \) and \( \nu \)-extending \( \alpha \). Using the splitting algorithm for \( \mathcal{F} \), find the minimal polynomial \( P_1 \in \mathcal{F}[X] \) of \( a_1 \) over \( \mathcal{F} \). Find a solution \( b_1 \in \mathcal{K} \) to \( \alpha(P_1) \). Then define \( \beta \) on \( \mathcal{F}(a_1) \) by mapping \( a_1 \) to \( b_1 \). Since \( a_1 \) is algebraic and separable over \( \mathcal{F} \) (and we know its minimal polynomial), we have a splitting algorithm for \( \mathcal{F}(a_1) \). The separable closure of \( \mathcal{F}(a_1) \) is \( \mathcal{F}^s \). Now find the minimal polynomial \( P_2 \in \mathcal{F}[X] \) of \( a_2 \) over \( \mathcal{F}(a_1) \), and a solution \( b_2 \) to \( \alpha(P_2) \). Define \( \beta \) on \( \mathcal{F}(a_1, a_2) \) by mapping \( a_2 \) to \( b_2 \). Note that \( a_2 \) is separable over \( \mathcal{F}(a_1) \) since

\[
\mathcal{F} \subseteq \mathcal{F}(a_1) \subseteq \mathcal{F}(a_1, a_2) \subseteq \mathcal{F}^s
\]

and \( \mathcal{F}^s \) is a separable algebraic extension of \( \mathcal{F} \). Since \( a_2 \) is algebraic and separable over \( \mathcal{F}(a_1) \), we have a splitting algorithm for \( \mathcal{F}(a_1, a_2) \). Its separable closure is still \( \mathcal{F}^s \). Continuing in this way, we define an embedding \( \beta : \mathcal{F}^s \to \mathcal{K} \) which \( \nu \)-extends \( \alpha : \mathcal{F} \to \mathcal{K} \).

In characteristic zero, we are done since \( \mathcal{F}^s = \overline{\mathcal{F}} \). In characteristic \( p > 0 \), we can extend \( \beta \) to an embedding \( \overline{\mathcal{F}} \to \mathcal{K} \) in the following manner. Given \( b \in \overline{\mathcal{F}} \), find the minimal polynomial \( P \in \mathcal{F}^s[X] \) of \( b \) over \( \mathcal{F}^s \) (recalling that \( \mathcal{F}^s \) has a splitting algorithm). Then \( P(X) \) is of the form \( X^{p^r} - r = 0 \) with \( r \in \mathcal{F} \). Note that \( b \) is the unique solution of \( p(X) = 0 \), and we can find the unique solution \( c \) to \( \beta(p)(X) = 0 \). Map \( b \) to \( c \). This is the unique embedding of \( \overline{\mathcal{F}} \) into \( \mathcal{K} \) extending \( \beta \).

The construction was uniform in \( \alpha \) and \( \mathcal{K} \), and so we get the desired Turing functional \( \Phi \).

We are now ready to prove Theorem 10.1.1, which says that a field \( \mathcal{F} \) has a splitting algorithm if and only if it has the computable (or uniform) extendability of embeddings property.

Proof of Theorem 10.1.1. The implication \( (1) \Rightarrow (2) \) is Theorem 10.3.2. The implication \( (2) \Rightarrow (3) \) is immediate. It remains to show the implication \( (3) \Rightarrow (1) \).

Fix \( \iota : \mathcal{F} \to \overline{\mathcal{F}}, \) a computable embedding of \( \mathcal{F} \) into a computable presentation \( \overline{\mathcal{F}} \) of its algebraic closure. Suppose that every computable embedding of \( \mathcal{F} \) into a computable algebraically closed field \( \mathcal{K} \) \( \nu \)-extends to a computable embedding of \( \overline{\mathcal{F}} \) into \( \mathcal{K} \).

We will attempt to construct a computable field \( \mathcal{K} \) and a computable embedding \( \alpha : \mathcal{F} \to \mathcal{K} \) while attempting to diagonalize against all potential computable extensions \( \varphi_c : \overline{\mathcal{F}} \to \mathcal{K} \) (by
having $\alpha(a) \neq \varphi_e(i(a))$ for some $a \in \mathcal{F}$). We know that the construction must fail, and from this we will conclude that $\mathcal{F}$ has a splitting algorithm.

We construct $\mathcal{K}$ by an effective Henkin-style construction. The Henkin construction will be similar to one that can be used to prove Rabin’s theorem that every field embeds into a computable presentation of its algebraic closure. See, for example, [FSS83, Theorem 2.5] where this construction is carried out in reverse mathematics. (Rabin’s original proof constructed the algebraic closure using a quotient of a polynomial ring with infinitely many variables.) Let $\mathcal{L}_F$ be the language of fields with constant symbols for the elements of $\mathcal{F}$, and let $T$ be the consistent theory of algebraically closed fields together with the atomic diagram of $\mathcal{F}$. By quantifier elimination for the theory of algebraically closed fields, $T$ is a complete theory and hence is decidable. We want to construct a decidable prime model of the theory $T$, which gives an algebraic closure $\mathcal{K}$ of $\mathcal{F}$ together with an embedding of $\mathcal{F}$ into $\mathcal{K}$. The embedding $\alpha : \mathcal{F} \to \mathcal{K}$ will be built as part of the Henkin construction. Constructing a prime model requires a slight modification of the Henkin construction, which is possible in this case—we must also omit the type of an element that is transcendental over $\mathcal{F}$ (see [Mil83] for the general theorem on effectively omitting types).

Let $C = \{c_0, c_1, \ldots\}$ be the new constant symbols for the Henkin construction. The domain of $\mathcal{K}$ will be the equivalence classes of some computable equivalence relation on $C$. Let $\varphi_e : \mathcal{F} \to C$ be a list of partial computable functions which we interpret as the possible computable embeddings $\mathcal{F} \to \mathcal{K}$. Let $\{a_0, a_1, a_2, \ldots\}$ be a computable enumeration of $\mathcal{F}$. We use $\bar{a}_s$ to denote the constant symbol associated with $a_s \in \mathcal{F}$.

**Construction.** At each stage $s$, we define formulas $\delta_0, \ldots, \delta_s$ in the language $\mathcal{L}_{\mathcal{F} \cup C}$ which form the partial diagram of $\mathcal{K}$ at stage $s$. The theory $\Delta = \{\delta_0, \delta_1, \ldots\}$ will be a complete theory extending $T$ which is the complete diagram of the model $\mathcal{K}$ (with the domain of $\mathcal{K}$ being the equivalence classes in $C$ by the equivalence relation $c \sim d \Leftrightarrow \Delta \vdash c = d$). At stage $s$, let $\psi_s = \delta_0 \land \cdots \land \delta_{s-1}$. We can arrange the construction so that the only constant symbols from $\mathcal{F}$ that appear in $\delta_s$ are $\bar{a}_0, \ldots, \bar{a}_s$.

At stage 0, let $\delta_0$ be $c_0 = \bar{c}_0$. At stage $s = 4t + 1$, we try to diagonalize against a $\varphi_e$ for $e \leq t$. Search for an $e \leq t$ and an $i < s + 5$ such that $\varphi_{e,t}(t(a_i)) = c_j$ and (where $\bar{c} = (c_0, c_1, \ldots)$ is the sequence of constants from $C$ that appear in $\psi_s$): $T \not\vdash \forall \bar{x}(\psi_s[\bar{x}/\bar{c}] \Rightarrow \bar{a}_i = x_j)$.

By $\psi_s[\bar{x}/\bar{c}]$, we mean that the variables $\bar{x} = (x_0, x_1, \ldots)$ have been substituted for the constants $\bar{c} = (c_0, c_1, \ldots)$. This is a bounded search since $T$ is decidable and we only have to search through finitely many $\bar{a}_i$. If such an $e$ exists, choose the least $e$ such that we have not yet diagonalized against $\varphi_e$. Then set $\delta_s$ to be the formula $\bar{a}_i \neq c_j$ for that $e$. If no such $e$ exists, set $\delta_s$ to be the formula $c_0 = c_0$.

At stages $s = 4t + 2$, $s = 4t + 3$, and $s = 4t + 4$, we act as in the standard method of constructing a computable prime model (e.g., Theorems 5.1 and 7.1 of Harizanov’s survey [Har98]), as follows:
At stage \( s = 4t + 2 \), we add a Henkin witness for \( \delta_i \). If \( \delta_i \) is of the form \((\exists x) \varphi(x)\), then let \( c_i \) be a constant which does not appear in \( \psi_s \) and let \( \delta_s \) be \( \varphi(c_i) \). Otherwise, set \( \delta_s \) to be the formula \( c_0 = c_0 \).

At stage \( s = 4t+3 \), we satisfy the completeness requirement for the sentence \( \chi_t \) from some fixed listing \((\chi_t)_{t\in\omega}\) of the sentences in the language \( L_{\mathcal{F},C} \). Let \( \bar{c} \) be the constants from \( C \) which appear in \( \psi_s \) and \( \chi_t \). Check whether

\[
T \models \forall \bar{x}(\psi_s [\bar{x}/\bar{c}] \Rightarrow \chi_t)[\bar{x}/\bar{c}].
\]

If this is the case, let \( \delta_s \) be \( \chi_t \). Otherwise, let \( \delta_s \) be \( \neg \chi_t \).

At stage \( s = 4t+4 \), we omit the type of an element transcendental over \( \mathcal{F} \). We will have \( c_t \) satisfy some polynomial over \( \mathcal{F} \). Let \( \bar{c} \) be the constants from \( C \) which appear in \( \psi_s \), except for \( c_t \). Search for a polynomial \( p(x) \in \mathcal{F}[[X]] \) such that

\[
T \not\models \forall x \forall \bar{z}(\psi_s [x\bar{z}/c_t\bar{c}] \Rightarrow p(x) \neq 0).
\]

Set \( \delta_s \) to be the formula \( p(c_t) = 0 \). Some such polynomial \( p \) must exist as the type of a transcendental over \( \mathcal{F} \) is a non-principal type.

**Verification.** By the standard Henkin construction arguments, we get a decidable prime model \( \mathcal{K} \) whose domain consists of equivalence classes from \( C \). We get a computable embedding \( \alpha \) of \( \mathcal{F} \) into \( \mathcal{K} \) by mapping \( a \in F \) to the element of \( \mathcal{K} \) labeled by the symbol \( a \). Then \( \alpha \) extends to an embedding \( \beta \) of \( \overline{\mathcal{F}} \) into \( \mathcal{K} \), which we may represent as a computable map \( \varphi_e : \overline{\mathcal{F}} \to C \) (by, say, choosing \( \varphi_e(a) \) to be the least element of \( C \) in the equivalence class of \( \beta(a) \), which we can do computably since the equivalence classes are computable). There is a stage \( s_0 \) after which we never diagonalize against an \( e' < e \). We never diagonalize against \( e \).

**Claim.** Let \( b \in \overline{\mathcal{F}} \) and \( t \) be a stage such that \( \varphi_{e,t}(b) \downarrow = c_j \) for some \( j \in \omega \). Let \( s = 4t+1 \). Then \( b \in \iota(\mathcal{F}) \) if and only if there is some \( i \) such that

\[
T \models \forall \bar{x}(\psi_s [\bar{x}/\bar{c}] \Rightarrow a_i = x_j). \tag{\ast}
\]

**Proof.** Given \( \ast \), in \( \mathcal{K} \) the constant symbol \( a_i \) is interpreted as the equivalence class of \( c_j \). Thus \( \alpha \) maps \( a_i \) to the equivalence class of \( c_j \). Since \( \beta \) extends \( \alpha \) and is one-to-one, \( \iota(a_i) = b \).

On the other hand, suppose that \( b \in \iota(\mathcal{F}) \), say \( b = a_i \), and suppose to the contrary that \( \ast \) does not hold. We have two cases. First, if \( i < s + 5 \), then we set \( \delta_s \) to be the formula \( a_i \neq c_j \). Then \( \alpha(a_i) \neq c_j = \varphi_{e,t}(a_i) \), which is a contradiction. Second, if \( i \geq s + 5 \), then let \( s' > s \) be the first stage of the form \( s' = 4t' + 1 \) with \( i < s' + 5 \). We have \( i > s' \) (as if \( i \leq s' \) we could have chosen \( s' - 4 \)). Since the only constant symbols from \( \mathcal{F} \) that appear in \( \psi_{s'} \) are \( a_0, \ldots, a_{s'} \), and \( i > s', a_i \) does not appear in \( \psi_{s'} \). Then we have

\[
T \not\models \forall \bar{x}(\psi_{s'} [\bar{x}/\bar{c}] \Rightarrow a_i = x_j).
\]

We set \( \delta_{s'} \) to be the formula \( a_i \neq c_j \) which again yields a contradiction. Hence \( \ast \) holds. \( \square \)
The claim gives us a decision procedure for $\iota(F) \subseteq \overline{F}$. At any stage $s$, there are only finitely many constants $c \in C$ mentioned in $\psi$, and hence only finitely many $a_i$ such that we might possibly have $(\ast)$. So given $b \in \overline{F}$, compute $s = 4t + 1 \geq s_0$ and $j$ such that $\varphi_{e_i}(b) \models c_j$, and then check $(\ast)$ for the finitely many possible $a_i$ to decide whether $b \in \iota(F)$.

It was important in Theorem 10.1.1 that we allow the field $K$ to vary. This is because if $F$ is a field of infinite transcendence degree, there may be computable algebraically closed fields of infinite transcendence degree into which $F$ does not effectively embed. For example, if $F$ does not have a dependence algorithm but $K$ does, then there is no computable embedding of $F$ into $K$. If we restrict to the case where $F$ is an algebraic field, then $F$ has a computable embedding into every computable algebraically closed field $K$. In this particular case we get the following corollary, which we use in §10.4, where the field $K$ is fixed:

**Corollary 10.3.3.** Let $F$ be a computable algebraic field and $\iota: F \to \overline{F}$ a computable embedding of $F$ into a computable presentation of its algebraic closure. Let $K$ be a computable algebraically closed field. Then the following are equivalent:

1. $F$ has a splitting algorithm,
2. There is a Turing functional $\Phi$ which takes an embedding $\alpha: F \to K$ to an embedding $\Phi^\alpha$ of $\overline{F}$ into $K$ extending $\alpha$,
3. Every computable embedding of $F$ into $K$ $\iota$-extends to a computable embedding of $\overline{F}$ into $K$.

**Proof.** By Theorem 10.1.1, it suffices to show that (3) in the statement implies that $F$ has the computable extendability property with respect to $\iota: F \to \overline{F}$. Let $\alpha: F \to L$ be a computable embedding of $F$ into a computable algebraically closed field $L$. We can enumerate, in $L$, the algebraic closure of the prime field and this contains the image $\alpha(F)$ of $F$. So we may assume that $L$ is the algebraic closure of its prime field.

We can compute an embedding $\psi: L \to K$ and let $\alpha^*: F \to K$ be $\psi \circ \alpha$. By (3), there is an embedding $\beta^*: \overline{F} \to K$ which $\iota$-extends $\alpha$.

Since $F$ and $\overline{L}$ are both algebraic closures of the prime field, the image of $\beta^*$ is the same as the image of $\psi$. So there is an embedding $\beta: \overline{F} \to L$ such that $\psi \circ \beta = \beta^*$. Then $\beta^*$ $\iota$-extends $\alpha$. 

\[ \begin{array}{c}
F \\
\mapright{\iota} \\
\overline{F} \\
\mapright{\beta} \\
\beta^* \\
\mapright{\alpha^*} \\
K \\
\mapright{\psi} \\
\mapright{\beta} \\
\overline{L} \\
\mapright{\alpha} \\
F \\
\end{array} \]
10.4 Extending Automorphisms of Normal Extensions of the Prime Field

10.4.1 Strong Extendability of Automorphisms Property

In the setting of automorphisms, it is natural to look at normal algebraic extensions of the prime field (see Proposition 10.4.2). When \( F \) is such an extension, we get the following corollary of Theorem 10.1.1, with two strengthenings of the computable extendability of automorphisms property. (We denote the prime field by \( \mathbb{F}_p \) even in the case of characteristic \( p = 0 \).)

**Corollary 10.4.1.** Let \( F \) be a computable normal algebraic extension of the prime field and \( \iota: F \to \overline{F} \) an embedding of \( F \) into a computable presentation of its algebraic closure. The following are equivalent:

(1) \( F \) has a splitting algorithm.

(2) For every computable automorphism \( \alpha: F \to F \) of \( F \), there is a computable automorphism \( \beta: \overline{F} \to \overline{F} \) which \( \iota \)-extends \( \alpha \).

(3) There is a uniform procedure which, given any computable automorphism \( \alpha: F \to F \) of \( F \), outputs a computable automorphism \( \beta: \overline{F} \to \overline{F} \) which \( \iota \)-extends \( \alpha \).

**Proof of Corollary 10.4.1.** Suppose that \( F \) is a computable normal algebraic field, and \( \iota: F \to \overline{F} \) is a Rabin embedding. If \( F \) has a splitting algorithm, then by Corollary 10.3.3, any automorphism \( \alpha \) of \( F \) extends to an automorphism of \( \overline{F} \) (taking \( K = \overline{F} \) in the statement of the corollary). Indeed, \( \iota \circ \alpha \) is a computable embedding of \( F \) into \( \overline{F} \) and hence there is an automorphism \( \beta \) of \( \overline{F} \) which \( \iota \)-extends \( \iota \circ \alpha \); that is, \( \beta \circ \iota = \iota \circ \alpha \). So \( \beta \) \( \iota \)-extends \( \alpha \).

On the other hand, suppose that every automorphism of \( F \) extends to an automorphism of \( \overline{F} \). We will check (3) of Corollary 10.3.3 with \( K = \overline{F} \). Let \( \alpha: F \to \overline{F} \) be an embedding. Since \( F \) is normal, \( \alpha(F) = \iota(F) \). Then \( \iota^{-1} \circ \alpha: F \to F \) is an automorphism of \( F \), and hence extends to an automorphism \( \beta \) of \( \overline{F} \). We have the following diagram:

\[
\begin{array}{c}
\mathcal{F} \xrightarrow{\beta} \overline{\mathcal{F}} \\
\downarrow \iota \quad \downarrow \iota \\
\iota(\mathcal{F}) \xrightarrow{\alpha \circ \iota^{-1}} \iota(\mathcal{F}) \\
\downarrow \iota \quad \downarrow \iota \\
\mathcal{F} \xrightarrow{\iota^{-1} \circ \alpha} \mathcal{F}
\end{array}
\]

Note that \( \beta: \overline{F} \to \overline{F} \) \( \iota \)-extends the embedding \( \alpha \) of \( F \) into \( \overline{F} \). \( \square \)
Note that we had to use Corollary 10.3.3 rather than Theorem 10.1.1, because we needed to fix $K = \overline{F}$ instead of letting $K$ be arbitrary.

In Corollary 10.4.1 we asked for $F$ to be a normal extension of $\mathbb{F}_p$. This is required in order to prove the theorem—we will construct an algebraic field which demonstrates that we need the hypothesis of normality in the preceding results. A rigid field automatically satisfies (2) of Corollary 10.4.1.

**Proposition 10.4.2.** There is a rigid computable algebraic field $F$ of characteristic zero with no splitting algorithm.

**Proof.** Let $p_0, p_1, \ldots$ list the primes greater than two. Let $F = \mathbb{Q}(a_n : n \in \varnothing')$ where $a_n$ is the unique real $p_n$-th root of 2, and $\varnothing'$ is the Turing jump of the empty set. Since $F \subseteq \mathbb{R}$, for each $n \in \varnothing'$, $a_n$ is the only $p_n$-th root of 2 in $F$. So every automorphism of $F$ fixes the $a_n$, and hence fixes $F$. Hence $F$ is rigid.

We can use an enumeration of $\varnothing'$ to give a computable presentation of $F$: $F$ can be embedded as a c.e. subfield of $\overline{\mathbb{Q}}$ and from this we get a computable presentation of $F$.

We need to argue that for $n \notin \varnothing'$, $a_n \notin F$. We claim that if $n \notin I$, $a_n \notin \mathbb{Q}(a_i : i \in I)$. Suppose not; then we can find a finite set $I$ and $n \notin I$ such that $a_n \in \mathbb{Q}(a_i : i \in I)$ and for each $j \in I$, $a_j \notin \mathbb{Q}(a_i : i \in I \setminus \{j\})$. Then $\mathbb{Q}(a_i : i \in I)$ is a finite extension of $\mathbb{Q}$ of degree $d = \prod_{i \in I} p_i$. Since $p_n$ does not divide $d$, $\mathbb{Q}(a_n)$ is not a subfield of $\mathbb{Q}(a_i : i \in I)$. This contradicts the assumption that $a_n \in \mathbb{Q}(a_i : i \in I)$. Thus for $n \notin \varnothing'$, $a_n \notin F$.

No computable presentation of $F$ can have a splitting algorithm, as a splitting algorithm would allow us to compute $\varnothing'$.

\[\square\]

### 10.4.2 Computable Extendability of Automorphisms Property

In Corollary 10.4.1, we fixed an embedding $\iota : F \to \overline{F}$ and considered only $\iota$-extensions. Now we allow $\iota$ to vary. Note that while every computable presentation of $\overline{F}$ is isomorphic, there may be different computable embeddings $\iota : F \to \overline{F}$ which are not equivalent up to a computable automorphism of $\overline{F}$. By Corollary 10.3.3, if $F$ is an algebraic field with no splitting algorithm, there are embeddings $\iota$ and $j$ of $F$ into $\overline{F}$ such that there is no computable automorphism $\sigma$ of $\overline{F}$ with $\sigma \circ \iota = j$.

In the introduction, we said that $F$ had the **computable extendability of automorphisms property** if each computable automorphism of $F$ had such an extension to $\overline{F}$. Recall that our interest in the computable extendability of automorphisms property comes from its role in an analogue of Rabin’s theorem in the context of difference closed fields; see Theorem 10.5.1 which we will prove in the following section.

We can already produce examples of fields without the computable extendability of automorphisms property. We use the fact that every noncomputable c.e. set is the union of two disjoint, computably inseparable c.e. subsets. This is a theorem of Yates, who saw that it followed from a construction of Friedberg; the theorem was subsequently published by Cleave [Cle70] in 1970.
Proposition 10.4.3. For each noncomputable c.e. set \( C \), the field \( \mathcal{F} = \mathbb{Q}(\sqrt{p_n} : n \in C) \) (with \( p_n \) the \( n \)th prime) does not have the computable extendability of automorphisms property.

Proof. Let \( A \) and \( B \) be disjoint computably inseparable c.e. sets with \( A \cup B = C \). Recalling the classic result (originally due to Besicovitch [Bes40]) that if \( r \) and \( q_1, \ldots, q_\ell \) are distinct primes, then \( \sqrt{r} \notin \mathbb{Q}(\sqrt{q_1}, \ldots, \sqrt{q_\ell}) \), we define an automorphism \( \alpha \) of \( \mathcal{F} \) by letting \( \alpha \) fix the two square roots of \( p_n \) if \( n \in A \), but interchange them if \( n \in B \). This \( \alpha \) is computable, but if \( \iota \) is any Rabin embedding of \( \mathcal{F} \) into some presentation \( \overline{\mathcal{F}} \) of its algebraic closure and \( \beta \) is an automorphism of \( \overline{\mathcal{F}} \) \( \iota \)-extending \( \alpha \), then \( \{ n \in \omega : \beta(\sqrt{p_n}) = \sqrt{p_n} \} \) is a \( \beta \)-computable separation of \( A \) from \( B \).

However, we would like a more complete description of which fields have, and which do not have, the computable extendability of automorphisms property. We do not obtain a complete description, but we give a characterization in terms of a splitting algorithm for many fields. The idea will be to isolate certain normal extensions \( \mathcal{K}/\mathbb{F}_p \) whose subfield structure behaves sufficiently like the field \( \mathcal{F} \) in Proposition 10.4.3 above, allowing us to make a particular diagonalization argument. In diagonalizing against the computable extendability of automorphisms property, we do not have access to a Rabin embedding \( \iota \) (and it does not seem possible to diagonalize against all possible computable Rabin embeddings). So rather than defining \( \alpha \) to diagonalize against \( \beta \) using the image under a fixed \( \iota \), we must define \( \alpha \) to diagonalize against all possible images under all possible \( \iota \). In Proposition 10.4.3, we do this using the fact that if \( \alpha \) fixes the square roots of \( p_n \), then so does \( \beta \) for any \( \iota \)-extension of \( \alpha \) under any Rabin embedding \( \iota \), and similarly if \( \alpha \) interchanges the roots of \( p_n \). In general, we want to have some finite subfield \( \mathcal{E} \) of \( \mathcal{F} \), and to define \( \alpha \) on \( \mathcal{E} \) so that there is no embedding \( \iota \) under which \( \beta \) \( \iota \)-extends \( \alpha \). We may have already defined \( \alpha \) on some subfield of \( \mathcal{E} \), so we do not have a completely free choice of \( \alpha \). There are some fields where this argument will always work successfully: those with the non-covering property from Definition 10.1.3. In many other fields, we can find some appropriate subfield which satisfies the required condition, allowing the argument to go through—see Examples 10.4.15 and 10.4.17.

Using Galois theory, there is also a field-theoretic characterization of the field extensions whose Galois group has the non-covering property, and it is this characterization that we will use in the proof of Theorem 10.4.6 (though, in applying the theorem, it will usually be easier to use the group-theoretic characterization). In what follows, it will be helpful to use the language of difference fields to talk about field automorphisms.

Remark 10.4.4. Let \( \mathcal{F}/\mathcal{E} \) be a field extension, \( \alpha \) an automorphism of \( \mathcal{E} \), and \( \beta \) an automorphism of \( \mathcal{F} \). Let \( \iota : \mathcal{E} \to \mathcal{F} \) be a field embedding of \( \mathcal{E} \) into \( \mathcal{F} \). Then \( \beta \) \( \iota \)-extends \( \alpha \) if and only if \( \iota \) is an embedding of \( (\mathcal{E},\alpha) \) into \( (\mathcal{F},\beta) \) as difference fields.

Proof. Both are equivalent to having \( \beta \circ \iota = \iota \circ \alpha \). \( \square \)

Lemma 10.4.5. Let \( \mathcal{E}/\mathcal{F} \) be a separable normal extension. The following are equivalent:

1. \( \text{Gal}(\mathcal{E}/\mathcal{F}) \) has the non-covering property.
(2) For all finite normal subextensions $K_1/F$ and $K_2/F$ with $K_2 \not\subset K_1$, and every pair of automorphisms $\sigma$ of $K_1$ and $\tau$ of $K_2$ fixing $F$, there is an automorphism $\alpha$ of $E$ extending $\sigma$ and incompatible with $\tau$ (i.e., $(K_2, \tau)$ does not embed into $(E, \alpha)$ as a difference field).

The second point is related to the monadic and incompatible extensions of difference fields studied by Cohn [Coh52], Babbitt [Bab62], and Evanovich [Eva73].

**Proof.** We begin by showing (1)$\Rightarrow$(2). Let $K_1$ and $K$ be as in (2). Let $\sigma$ and $\tau$ be automorphisms of $K_1$ and $K_2$ respectively fixing $F$. Let $G = \text{Gal}(E/F)$. Let $M$ be the normal subgroup of automorphisms fixing $K_2$, and $N$ the normal subgroup of automorphisms fixing $K_1$. Since $K_1$ and $K_2$ are finite extensions, $M$ and $N$ are of finite index. We also have $N \not\subset M$. Let $g_1 \in G$ be an automorphism of $E$ extending $\sigma$, and $g_2$ an automorphism of $E$ extending $\tau$.

We will argue that there is an $h \in g_1 N$ such that for all $z \in G$, $z^{-1}hz \not\in g_2 M$. Such an $h$ is an automorphism of $E$ extending $\sigma$, and $g_2 M$ is the set of automorphisms of $E$ extending $\tau$. Since, for all $z \in G$, $x^{-1}g_1 hx \not\in g_2 M$, $(E, \alpha)$ is not isomorphic as a difference field to $(E, \beta)$ for any extension $\beta$ of $\tau$.

First, we argue that it suffices to assume $M \not\subset N$. Suppose that there is $h' \in g_1 NM$ such that for all $z \in G$, $z^{-1}hz \not\in g_2 M$. Then write $h' = g_1 nm$. Suppose for some $z \in G$ that $z^{-1}g_1 nz \in g_2 M$, say $z^{-1}g_1 nz = g_2m'$ with $m' \in M$. Let $m'' \in M$ be such that $z^{-1}mz = m''$. Then

$$z^{-1}g_1 m z = g_2 m' m''^{-1} \in g_2 M.$$ 

This contradicts the choice of $h' = g_1 nm$. So for all $z \in G$, $z^{-1}g_1 nz \not\in g_2 M$. Then $h = g_1 n \in g_1 N$ is the automorphism of $E$ that we desire. So we may replace $N$ by $NM$.

Now we have two cases. First, suppose that there is no $z \in G$ such that $z^{-1}g_1 z \in g_2 M$. Then $h = g_1$ is as desired.

Second, suppose that for some $c \in M$ and $z \not\in G$, $z^{-1}g_1 z = g_2 c$. Then $g_1 = zg_2 cz^{-1} = zg_2 z^{-1} c'$ for some other $c' \in M$ since $M$ is a normal subgroup. So $zg_2 z^{-1} = g_1 m$, where $m = (c')^{-1}$. Using the fact that $G$ has the non-covering property, choose $h \in N$ such that for all $x \in G$, $x^{-1}g_1 hx \not\in g_1 M$. We claim that for all $x \in G$, $x^{-1}g_1 hx \not\in g_2 M$. Suppose to the contrary that there is $x \in G$ such that $x^{-1}g_1 hx \in g_2 M$. Since $x^{-1}g_1 hx \in g_2 M$ and $M$ is a normal subgroup, $g_1 h \in xg_2 x^{-1} M$. We have

$$xg_2 x^{-1} = (xz^{-1}) zg_2 z^{-1} (xz^{-1})^{-1} = (xz^{-1})g_1 m (xz^{-1})^{-1}.$$ 

Let $y = (xz^{-1})$. Since $m \in M$ is a normal subgroup, $yg_1 my^{-1} = yg_1 y^{-1} m'$ for some other $m' \in M$. Thus $g_1 h \in yg_1 y^{-1} M$ and so $y^{-1}g_1 hy \in g_1 M$. This contradicts the choice of $h$. So for all $x \in G$, $x^{-1}g_1 hx \not\in g_2 M$.

The direction (2)$\Rightarrow$(1) proceeds simply by the Galois correspondence. Fix finite index normal subgroups $M \not\subset N$ of $G = \text{Gal}(E/F)$ and $g \in G$. Let $K_1$ and $K_2$ be the fields fixed by $N$ and $M$ respectively; we have $K_1 \not\subset K_2$. The $\sigma$ be the restriction of $g$ to $K_1$ and $\tau$ its restriction to $K_2$. There is an automorphism $\alpha$ of $E$ extending $\sigma$ and not compatible with
We will restrict our attention to field extensions whose Galois group has the non-covering property, but we will allow the base field to be an extension of $\mathbb{F}_p$ with a splitting algorithm. The main theorem of this section is as follows. (We state it in a slightly more general form than it appears in the introduction.)

**Theorem 10.4.6.** Let $\mathcal{E}$ be a computable normal extension of $\mathbb{F}_p$ and let $\mathcal{F} \subseteq \mathcal{E}$ be a subfield of $\mathcal{E}$ with a splitting algorithm which is also a normal extension of $\mathbb{F}_p$. Suppose that $\text{Gal}(\mathcal{E}/\mathcal{F})$ has the non-covering property. The following are equivalent:

1. $\mathcal{E}$ has a splitting algorithm,
2. $\mathcal{E}$ has the computable extendability of automorphisms property,
3. $\mathcal{E}$ has the uniform extendability of automorphisms property.

Many applications of this theorem will have $\mathcal{F} = \mathbb{F}_p$, but the freedom to choose $\mathcal{F}$ will allow us to apply the theorem in situations where $\text{Gal}(\mathcal{E}/\mathbb{F}_p)$ does not have the non-covering property. Producing an example where the theorem cannot be applied seems to be a non-trivial task, and we do not know of any such examples. See §10.4.4 for some applications of the theorem.

**Proof of Theorem 10.4.6.** We already know that the implications $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$ are true even given a fixed embedding of $\mathcal{E}$ into $\overline{\mathcal{E}}$. (3) clearly implies (2). We must show $(2) \Rightarrow (1)$.

Suppose that every computable automorphism of $\mathcal{E}$ extends to a computable automorphism of $\overline{\mathcal{E}}$ (via some embedding of $\mathcal{E}$ into $\overline{\mathcal{E}}$). We will attempt to construct a computable automorphism $\alpha \in \text{Gal}(\mathcal{E}/\mathcal{F})$ while diagonalizing against possible computable automorphisms $\varphi_e : \overline{\mathcal{E}} \to \overline{\mathcal{E}}$ by making sure that the difference field $(\mathcal{E}, \alpha)$ does not embed into the difference field $(\overline{\mathcal{E}}, \varphi_e)$. It suffices to ensure that some difference subfield of $(\mathcal{E}, \alpha)$ does not embed into $(\overline{\mathcal{E}}, \varphi_e)$. We know that the construction must fail, and from this we will conclude that $\mathcal{E}$ has a splitting algorithm.

Note that the field $\mathcal{F}$ has a splitting algorithm and is perfect (since it is an algebraic extension of a perfect field), so any finite algebraic extension of $\mathcal{F}$ has a splitting algorithm which we can determine effectively from a generating set for the extension.

We will require a special enumeration $\{a_1, a_2, \ldots\}$ of $\mathcal{E}$ with the following properties:

1. for each $n$, $\mathcal{F}(a_1, \ldots, a_n)$ is a normal extension of $\mathbb{F}_p$, and
2. for each $n$, there are no normal extensions of $\mathbb{F}_p$ which are strictly contained between $\mathcal{F}(a_1, \ldots, a_n)$ and $\mathcal{F}(a_1, \ldots, a_n, a_{n+1})$. 
We can find such an enumeration using the primitive element theorem and Galois theory, as follows. Suppose that we have already defined \(a_1, \ldots, a_n\). Given a new element \(x\) of \(\mathcal{E}\), first check whether \(x \in \mathcal{F}(a_1, \ldots, a_n)\) using the splitting algorithm for this field. If \(x\) is in \(\mathcal{F}(a_1, \ldots, a_n)\), we can safely set \(a_{n+1} = x\). Otherwise, compute the conjugates \(x = x_1, \ldots, x_\ell\) of \(x\) over \(\mathbb{F}_p\). Search for a single element \(y\) such that

\[
\mathcal{F}(y) \subseteq \mathcal{F}(a_1, \ldots, a_n, x_1, \ldots, x_\ell).
\]

Such an element exists by the primitive element theorem as \(\mathcal{F}(a_1, \ldots, a_n, x_1, \ldots, x_\ell)\) is a finite separable extension of \(\mathcal{F}\). Now we can compute the Galois group \(\text{Gal}(\mathcal{F}(y)/\mathcal{F})\) as each automorphism of \(\mathcal{F}(y)\) is determined by where it maps \(y\). We can compute the normal subgroups and hence the normal extensions of \(\mathcal{F}\) contained between \(\mathcal{F}(a_1, \ldots, a_n)\) and \(\mathcal{F}(a_1, \ldots, a_n, a_{n+1})\). Let

\[
\mathcal{F}(a_1, \ldots, a_n) \subseteq \mathcal{K}_1 \subseteq \cdots \subseteq \mathcal{K}_m = \mathcal{F}(a_1, \ldots, a_n, a_{n+1})
\]

be a maximal chain of normal extensions of \(\mathbb{F}_p\). We can compute for each \(\mathcal{K}_i\) a primitive generator over \(\mathcal{F}\) and add these to the enumeration in order (with \(y\) chosen as the primitive generator of \(\mathcal{K}_m = \mathcal{F}(a_1, \ldots, a_n, a_{n+1})\)).

Construction. At each stage \(s\), we will have defined an embedding \(\alpha_s : \mathcal{F}(a_1, \ldots, a_s) \to \mathcal{E}\) fixing \(\mathcal{F}\) such that \(\alpha_0 \subseteq \alpha_1 \subseteq \cdots \subseteq \alpha_s\). Begin with \(\alpha_0 : \mathbb{F}_p \to \mathcal{F}\).

At stage \(s + 1\), we are given \(\alpha_s\). Use the splitting algorithm for \(\mathcal{F}(a_1, \ldots, a_s)\) to check whether \(a_{s+1} \in \mathcal{F}(a_1, \ldots, a_s)\). If it is, set \(\alpha_{s+1} = \alpha_s\). Otherwise, check whether there is \(e \leq s\) against which we have not yet diagonalized such that

1. \(a_i \in \mathcal{F}(a_1, \ldots, a_s, a_{s+1}) \setminus \mathcal{F}(a_1, \ldots, a_s)\) and
2. for each \(x \in \mathcal{E}\) which satisfies the same minimal polynomial over \(\mathcal{F}\) as \(a_i\), \(\varphi_{e, s}(x) \downarrow c\) for some \(c \in \mathcal{E}\).

This is a computable search. We have splitting algorithms for \(\mathcal{F}(a_1, \ldots, a_s, a_{s+1})\) and also for \(\mathcal{F}(a_1, \ldots, a_s)\), so we can check (1) for a given \(a_i\). Also, \(\varphi_{e, s}(x)\) converges only for \(x\) among the first \(s\)-many elements of \(\mathcal{E}\), and we can use our splitting algorithms to compute the finite set of \(a_i\) satisfying (1) and also satisfying the same minimal polynomial as some such \(x\).

If there is such an \(e\), choose the least one. Let \(x_1, \ldots, x_n\) be the conjugates of \(a_i\) over \(\mathcal{F}\). By property (ii) of the enumeration,

\[
\mathcal{F}(a_1, \ldots, a_s, a_{s+1}) = \mathcal{F}(a_1, \ldots, a_s, x_1, \ldots, x_n).
\]

Now we can extend \(\varphi_{e, s}\) in a unique way to a computable automorphism of \(\mathcal{F}(x_1, \ldots, x_n)\). If this automorphism is not the identity on \(\mathcal{F}\), then since \(\mathcal{F}\) is normal, \(\varphi_{e, s}\) will be incompatible with \(\alpha\) no matter how we define \(\alpha\). Suppose that \(\varphi_{e, s}\) is the identity on \(\mathcal{F}\). Since \(\text{Gal}(\mathcal{E}/\mathcal{F})\) has the non-covering property, by Lemma 10.4.5 we can extend \(\alpha_s\) to an automorphism of \(\alpha_{s+1}\) of \(\mathcal{F}(a_1, \ldots, a_s, a_{s+1})\) which is incompatible with the automorphism \(\varphi_{e, s}\).
on $\mathcal{F}(x_1, \ldots, x_n)$, in the sense that $(\mathcal{F}(x_1, \ldots, x_n), \varphi_e)$ does not embed as a difference field into $(\mathcal{F}(a_1, \ldots, a_{s}, a_{s+1}), \alpha_{s+1})$. We can do all of this computably by looking at the actions of the automorphisms on the generators of the fields.

**Verification.** We get an automorphism $\alpha = \bigcup_s \alpha_s$ of $\mathcal{E}$ which fixes $\mathcal{F}$. Now we know that for some $e$, $\varphi_e$ is an automorphism of $\overline{\mathcal{E}}$ such that $(\mathcal{E}, \alpha)$ embeds into $(\overline{\mathcal{E}}, \varphi_e)$ as a difference field. We claim that $\mathcal{E}$ has a splitting algorithm. The proof will be to show that we can compute the image of $\mathcal{E}$ in $\overline{\mathcal{E}}$ (since $\mathcal{E}$ is a normal extension of $\mathbb{F}_p$, this image is unique; we may fix some embedding $\iota: \mathcal{E} \to \overline{\mathcal{E}}$ and show that the image of $\mathcal{E}$ under $\iota$ is computable in $\overline{\mathcal{E}}$).

Let $s$ be a stage after which we never diagonalize against an $e' \leq e$. Fix $x \in \overline{\mathcal{E}}$, and let $x = x_1, x_2, \ldots, x_n$ be the conjugates of $x$ over $\mathcal{F}$. Let $t \geq s$ be a stage by which $\varphi_e(x_i)$ has converged for each $i$. Since $\mathcal{F}(a_1, \ldots, a_t)$ has a splitting algorithm, we can compute its image $\iota(\mathcal{F}(a_1, \ldots, a_t))$ in $\overline{\mathcal{E}}$.

**Claim.** $x \in \iota(\mathcal{E})$ if and only if $x \in \iota(\mathcal{F}(a_1, \ldots, a_t))$.

**Proof.** If $x \in \iota(\mathcal{F}(a_1, \ldots, a_t))$ then $x \in \iota(\mathcal{E})$. On the other hand, suppose that $x \in \iota(\mathcal{E})$, say $x = \iota(a_i)$, and suppose to the contrary that $a_i \notin \mathcal{F}(a_1, \ldots, a_t)$. Now, for some $t' > t$, we have

$$a_i \in \mathcal{F}(a_1, \ldots, a_{t'}+1) \setminus \mathcal{F}(a_1, \ldots, a_{t'}).$$

Then at stage $t' + 1$, we define $\alpha_{t'+1} \subset \alpha$ such that $(\mathcal{F}(x_1, \ldots, x_n), \varphi_e)$ does not embed into $(\mathcal{F}(a_1, \ldots, a_{t'+1}), \alpha_{t'+1})$ as a difference field. Since $\mathcal{F}(x_1, \ldots, x_n)$ and $\mathcal{F}(a_1, \ldots, a_{t'+1})$ are both normal extensions of $\mathbb{F}_p$ (with the former contained in the latter), $(\mathcal{E}, \alpha)$ cannot embed into $(\overline{\mathcal{E}}, \varphi_e)$. \qed

From the claim we get a decision procedure for $\iota(\mathcal{E})$. Given $x \in \overline{\mathcal{E}}$, compute a stage $t \geq s$ at which $\varphi_e$ converges when applied to all of the conjugates of $x$ over $\mathbb{F}_p$. Using the splitting algorithm for $\mathcal{F}(a_1, \ldots, a_t)$, we check whether $x \in \iota(\mathcal{F}(a_1, \ldots, a_t))$ and hence whether $x \in \iota(\mathcal{E})$. \qed

**10.4.3 The Non-Covering Property**

To apply Theorem 10.4.6, we need a field extension whose Galois group has the non-covering property. We now give some examples of groups with the non-covering property before giving an example of an application of Theorem 10.4.6.

**Lemma 10.4.7.** The following groups have the non-covering property:

1. abelian groups,
2. simple groups,
3. the quaternion group.
Proof. (1) Let $G$ be an abelian group. Let $M \varsubsetneq N$ be normal subgroups of finite index, and fix $g \in G$. Let $h$ be an element of $g(N \setminus M)$. Then for all $x \in G$, $x^{-1}hx = h \notin gM$.

(2) Let $G$ be a simple group. Let $M \varsubsetneq N$ be normal subgroups of finite index, and $x \in G$. Let $h$ be an element of $g(N \setminus M)$. Then for all $x \in G$, $x^{-1}hx = h \notin gM$.

(3) Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group. The normal subgroups are $\{1\}$, $\{1, -1\}$, $\{1, -1, i, -i\}$, $\{1, -1, j, -j\}$, $\{1, -1, k, -k\}$, and $G$. The conjugacy classes are $\{1\}$, $\{-1\}$, $\{i, -i\}$, $\{j, -j\}$, and $\{k, -k\}$. It is easy to see that every coset is a disjoint union of conjugacy classes. Thus, given normal subgroups $M \varsubsetneq N$ and $g \in G$, there is a conjugacy class in $gN$ which is not in $gM$; let $h$ be in this conjugacy class.

The example from Proposition 10.4.3 has an abelian Galois group $\prod_{n \in \omega} C_2$, and hence Proposition 10.4.3 follows immediately from Theorem 10.4.6. Also, in characteristic $p > 0$ we have:

**Theorem 10.4.8.** Let $E$ be a computable normal extension of $\mathbb{F}_p$ in characteristic $p > 0$. The following are equivalent:

1. $E$ has a splitting algorithm,
2. $E$ has the computable extendability of automorphisms property,
3. $E$ has the uniform extendability of automorphisms property.

**Proof.** The Galois group of every normal extension $K/F_p$ in characteristic $p > 0$ is abelian and hence has the non-covering property. Theorem 10.4.6 finishes the proof.

We can also take arbitrary products of Galois groups with the non-covering property and produce another group with the non-covering property. We must assume that the groups are profinite, but as every Galois group is profinite, this is not a restriction. See [FJ08] for an introduction to profinite groups.

**Theorem 10.1.6.** Let $\{G_i : i \in I\}$ be a collection of profinite groups, each of which has the non-covering property. Then $\prod_{i \in I} G_i$ has the non-covering property.

**Proof.** We reduce to the case of a product of two groups. If $M \varsubsetneq N$ are normal subgroups of $\prod_{i \in I} G_i$ of finite index, then $M$ contains a finite intersection of the groups

$$\hat{G}_i = \{(x_j)_{j \in I} : x_i = e\}.$$ 

The intersection of all of the $\hat{G}_i$ is the trivial group, so $\cap \hat{G}_i \subseteq M$. Moreover, it is easy to check that these groups are open in the profinite topology of the profinite group $\prod_{i \in I} G_i$ (which is just the product topology) and hence they are closed as well. As the profinite topology is compact, $M$ contains $G_{i_1} \cap \cdots \cap G_{i_n}$ for some $i_1, \ldots, i_n$. 


Let $M', N' \subseteq G_{i_1} \times \cdots \times G_{i_2}$ be the projection of $M$ and $N$ to these indices; $M'$ and $N'$ are normal subgroups. Then

$$\left( \prod_{i \in I} G_i \right)/M \cong \left( G_{i_1} \times \cdots \times G_{i_n} \right)/M'.$$

We will prove in the following lemma that $G_{i_1} \times \cdots \times G_{i_n}$ has the non-covering property, and we can use this to check (for $M$ and $N$) that $\prod_{i \in I} G_i$ has the non-covering property.

**Lemma 10.4.9.** Let $G$ and $H$ be groups which both have the non-covering property. Then $G \times H$ has the non-covering property.

**Proof.** Let $M \varsubsetneq N$ be normal subgroups of $G \times H$. Let $\pi_1$ and $\pi_2$ be the projections onto $G$ and onto $H$ respectively.

**Case 1.** We have $\pi_1(M) \varsubsetneq \pi_1(N)$.

Let $a = (a_1, a_2) \in G \times H$ and $b = (b_1, b_2) \in G \times H$ be arbitrary. Choose $g = a_1 g' \in a_1 \pi_1(N)$ such that for all $x \in G$, $x^{-1} g x \notin b_1 \pi_1(M)$. Let $h' \in H$ be such that $(g', h') \notin N$, and let $h = a_2 h'$. Then $f = (g, h) \in aN$ is such that for all $z = (x, y) \in G \times H$, $z^{-1} f z \notin bM$.

**Case 2.** We have $\pi_2(M) \varsubsetneq \pi_2(N)$.

Similar to Case 1.

**Case 3.** $\pi_1(M) = \pi_1(N)$ and $\pi_2(M) = \pi_2(N)$.

Define $M_1 \subseteq G$ and $M_2 \subseteq H$ by

$$M_1 = \{ x \in G : (x, e) \in M \} \quad \text{and} \quad M_2 = \{ y \in H : (e, y) \in M \}.$$ 

Then $M_1 \times M_2 \subseteq M$. Define $N_1$ and $N_2$ similarly. We have $M_1 \subseteq N_1$ and $M_2 \subseteq N_2$.

**Claim 10.4.10.** $M_1$ and $N_1$ are normal subgroups of $G$ and $M_2$ and $N_2$ are normal subgroups of $H$.

**Proof.** We show that $M_1$ is a normal subgroup of $G$. Let $m \in M_1$ and $x \in G$. Let $x' = (x, e)$ and $m' = (m, e)$. Then, since $M$ is a normal subgroup of $G$, $(x^{-1} mx, e) = x'^{-1} m' x' \in M$. Hence $x^{-1} mx \in M_1$. \hfill \Box

**Claim 10.4.11.** $M_1 \varsubsetneq N_1$ and $M_2 \varsubsetneq N_2$.

*Proof.** We use Goursat’s lemma:

**Lemma ([Gou89]).** Let $G_1$ and $G_2$ be groups. Let $H$ be a subgroup of $G_1 \times G_2$ such that the projections $\pi_1 : H \to G_1$ and $\pi_2 : H \to G_2$ are surjective. Let $N_1$ and $N_2$ be the kernels of $\pi_2$ and $\pi_1$ respectively; $N_1$ can be identified as a normal subgroup of $G_1$, and $N_2$ as a normal subgroup of $G_2$. Then the image of $H$ in $G_1/N_1 \times G_2/N_2$ is isomorphic to the graph of an isomorphism between $G_1/N_1$ and $G_2/N_2$. 


By Goursat’s lemma, the image of $M$ in $\pi_1(M)/M_1 \times \pi_2(M)/M_2$ is the graph of an isomorphism $\pi_1(M)/M_1 \cong \pi_2(M)/M_2$. The same is true with $M$ replaced by $N$. Since $\pi_1(M) = \pi_1(N)$, $\pi_2(M) = \pi_2(N)$, and $M \not\subseteq N$, we must have $M_1 \not\subseteq N_1$ and $M_2 \not\subseteq N_2$. 

Claim 10.4.12. $[G, \pi_1(M)] \subseteq M_1$ and $[H, \pi_2(M)] \subseteq M_2$. Thus $[G \times H, \pi_1(M) \times \pi_2(M)] \subseteq M_1 \times M_2$.

**Proof.** Let $g \in G$ and $m \in \pi_1(M)$. Let $g' = (g, e)$ and $m' = (m, e)$. Since $M$ is a normal subgroup of $G \times H$, $[g', m'] = ([g, m], e) \in M$. Thus $[g, m] \in M_1$. 

Fix $g \in G \times H$ for which we will show that there is $h \in gN$ such that for all $x \in G \times H$, $x^{-1}hx \notin gM$. This will finish the proof of the proposition. Since $N_2 \not\subseteq M_2$, we can choose $b \in \pi_2(g)M_2$ such that for all $y \in H$, $y^{-1}by \notin \pi_2(g)M_2$. Choose $a = \pi_1(g)$. Then $(a, b) \in gN$. Suppose that $(x, y) \in G \times H$ is such that $(x^{-1}ax, y^{-1}by) \in gM$. Let $m \in M$ be such that $x^{-1}ax = \pi_1(gm)$.

Claim 10.4.13. $\pi_1(m) \in M_1$.

**Proof.** Suppose to the contrary that $\pi_1(m) \notin M_1$. Let $m_1 = \pi_1(m)$ and $g_1 = a = \pi_1(g)$. We have $x^{-1}g_1x = g_1m_1$. Let $K$ be the subgroup of $G$ generated by $M_1$ and $m_1$. Since $M_1$ is a normal subgroup of $G$, each element of $K$ can be written in the form $km_1^\ell$ for some $k \in M_1$ and $\ell \in \mathbb{N}$. $K$ is a normal subgroup of $G$ since $[G, m_1] \subseteq M_1$. If $m_1 \notin M_1$, then $M_1$ is a proper subgroup of $K$. So there is $h \in K$ such that for all $z \in G$, $z^{-1}g_1hz \notin g_1M_1$. Let $r$ be such that $m_1^r = e$ and let $h = km_1^\ell$ with $k \in M_1$ and $\ell < r$. Then since $[x, m_1] \in M_1$,

$$x^{-(r-\ell)}g_1hx^{r-\ell} \in x^{-(r-\ell)}g_1x^{r-\ell}m_1\ell M_1 = g_1m_1^rM_1 = g_1M_1.$$

This is a contradiction which proves the claim. 

Since $\pi_1(m) \in M_1$, we have $(e, \pi_2(m)) = m - (\pi_1(m), e) \in M$, and so $\pi_2(m) \in M_2$. But $y^{-1}by = \pi_2(gm) \notin \pi_2(g)M_2$, a contradiction. This completes the proof of the lemma.

### 10.4.4 Examples

We can apply Theorem 10.1.6 to construct groups having the non-covering property from the groups in Lemma 10.4.7. In all cases, we know that if the field $\mathcal{E}$ has a splitting algorithm, then it has the computable extendability of automorphisms property.

We begin by noting that there exist groups without the non-covering property:

**Proposition 10.4.14.** The following groups do not have the non-covering property: $S_3$, $D_8$, and $A_4$. 
Proof. For $S_3$, let $M = \{e\}$ and $N$ the normal subgroup of rotations. Let $g$ be a reflection. Then $gN$ is the set of all reflections, and all reflections are conjugate.

Write $D(8) = \{e, a, a^2, a^3, x, ax, a^2x, a^3x\}$. Let $M = \{e\}$, $N = \{e, a^2\}$, and $g = a$. Then $aM = \{a\}$ and $aN = \{a, a^3\}$. We have $x^{-1}ax = a^3$.

For $A_4$, let $M = \{e\}$ and $N$ the normal subgroup of $A_4$ isomorphic to $C_2 \times C_2$. Let $g$ be the permutation $(1, 2, 3)$. Then $gN$ consists of $(1, 2, 3)$, $(1, 4, 2)$, $(2, 4, 3)$, and $(1, 3, 4)$ all of which are conjugate.

Even if $\text{Gal}(\mathcal{E}/\mathbb{F}_p)$ does not have the non-covering property, we can still sometimes apply Theorem 10.4.6 either by finding the right field $\mathcal{F}$ as in the statement of the theorem, or using Lemma 10.4.16 below with a subfield $\mathcal{F}$ and applying Theorem 10.4.6 to the field extension $\mathcal{F}/\mathbb{F}_p$. The following two examples illustrate these methods. We begin with a field extension $\mathcal{E}/\mathbb{Q}$ whose Galois group does not have the non-covering property, but we can use the freedom in choosing the field $\mathcal{F}$ in the statement of Theorem 10.4.6 to apply the theorem.

Example 10.4.15. Let $\mathcal{E} = \mathbb{Q}(\omega, \sqrt{p_n} : n \in \mathbb{N})$ where $\omega$ is a primitive cube root of unity. Note that $\text{Gal}(\mathcal{E}/\mathbb{Q})$ does not have a forking lattice of subgroups for the same reason as $S_3$, because its Galois group is

$$\text{Gal}(\mathcal{E}/\mathbb{Q}) = \prod_{i \in \omega} C_3 \rtimes C_2$$

with $C_2$ acting on $C_3$ by inverting elements. Here, we need to know that the intersection of the fields $\mathbb{Q}(\omega, \sqrt{p_n} : n \in U)$ and $\mathbb{Q}(\omega, \sqrt{p_n} : n \in V)$ for $U$ and $V$ disjoint is the field $\mathbb{Q}(\omega)$. See [Mor53].

Let $\mathcal{F} = \mathbb{Q}(\omega)$. Then $\mathcal{F}$ has a splitting algorithm. $\text{Gal}(\mathcal{E}/\mathcal{F}) = \prod_{i \in \omega} C_3$ which is abelian. Since $\mathcal{E}$ does not have a splitting algorithm, by Theorem 10.4.6 it does not have the computable extension of automorphisms property.

The following lemma will allow us to consider a subextension of $\mathcal{E}$; this will be useful when the Galois group of the extension does not have the non-covering property, but it has a quotient which does.

Lemma 10.4.16. Let $\mathcal{E} \supseteq \mathcal{F} \supseteq \mathbb{F}_p$ be computable algebraic extensions such that $\mathcal{E}$ is a normal extension of $\mathbb{F}_p$. Suppose that given $x \in \mathcal{E}$, we can compute the minimal polynomial of $x$ over $\mathcal{F}$. Then if $\mathcal{E}$ has the computable extendability of automorphisms property, $\mathcal{F}$ does as well.

Proof. This follows from the fact that we can computably extend an automorphism of $\mathcal{F}$ to an automorphism of $\mathcal{E}$ in the style of Theorem 10.3.2 and uses the fact that $\mathcal{F}$ is a perfect field.

We now have an example where we apply this lemma together with Theorem 10.4.6.

Example 10.4.17. This example is quite complicated. The idea is to produce a field extension whose Galois group is \( \prod_{n \in \omega} S_3 \), but which does not have a splitting algorithm.

Let $q_0, q_1, \ldots$ be a list of infinitely many distinct primes in the arithmetic progression $4n + 27$, and let $a_n$ be such that $4a_n + 27 = q_n$. Let $\mathcal{E}$ be the splitting field, over $\mathbb{Q}$, of
The polynomials \( \{x^3 + a_n x + n : n \in \varnothing\} \). Let \( \omega_n \) be a primitive element for the splitting field of \( x^3 + a_n x + a_n \), so that \( \mathcal{E} = \mathbb{Q}(\omega_n : n \in \varnothing') \). Each of these polynomials has discriminant \( D_n = -4a_n^3 - 27a_n^2 = -a_n^2 q_n < 0 \), and hence \( \mathbb{Q}(\omega_n) \) has Galois group \( S_3 \). We claim that the Galois group of \( \mathcal{E} \) is \( \Pi_{n \in \varnothing'} S_3 \). It suffices to show that given \( m \) and \( n_1, \ldots, n_l \) all distinct that \( \mathbb{Q}(\omega_m) \) and \( \mathbb{Q}(\omega_{n_1}, \ldots, \omega_{n_l}) \) are disjoint. Suppose not; then there is a non-trivial subfield \( \mathcal{K} \) of \( \mathbb{Q}(\omega_m) \) which is contained in \( \mathbb{Q}(\omega_{n_1}, \ldots, \omega_{n_l}) \). We may assume that \( \mathcal{K} = \mathbb{Q}(\sqrt{D_m}) = \mathbb{Q}(\sqrt{-q_m}) \). Then \( \sqrt{D_m} \in \mathbb{Q}(\sqrt{D_{n_1}}, \ldots, \sqrt{D_{n_l}}) \), a contradiction since \( q_m, q_{n_1}, \ldots, q_{n_l} \) are distinct primes. \( \mathcal{E} \) does not have a splitting algorithm, but \( \Pi_{n \in \varnothing'} S_3 \) does not have a forking lattice of subgroups.

Now let \( \mathcal{F} = \mathbb{Q}(\sqrt{-q_n} : n \in \varnothing') \). \( \mathcal{F} \) does not have a splitting algorithm. By Theorem 10.4.6, \( \mathcal{F} \) does not have the computable extension of isomorphisms property, and hence by Lemma 10.4.16, \( \mathcal{E} \) does not have the computable extension of automorphisms property.

We do not know of any examples in which one cannot use either a direct application of Theorem 10.4.6 or one of the methods in these two examples.

### 10.5 Applications to Difference Closed Fields

We will conclude this paper by applying our results to difference closed fields. The main idea will be to note that \((\mathcal{F}, \sigma)\) embeds into a computable difference closed field if and only if there is an embedding \( \iota \) of \( \mathcal{F} \) into \( \overline{\mathcal{F}} \) and an automorphism \( \tau \) of \( \overline{\mathcal{F}} \) such that \( \tau \iota \)-extends \( \sigma \). In the one direction, this will follow from an effective Henkin construction, while on the other hand it will follow from the fact that the algebraic closure of the prime field can be enumerated in any difference closed field.

**Theorem 10.5.1.** Let \( \mathcal{F} \) be a computable extension of \( \mathbb{F}_p \), and \( \sigma \) a computable automorphism of \( \mathcal{F} \). Then the following are equivalent:

1. \((\mathcal{F}, \sigma)\) embeds computably into a computable difference closed field.
2. There is a computable embedding \( \iota : \mathcal{F} \to \overline{\mathcal{F}} \) of \( \mathcal{F} \) into a computable presentation of its algebraic closure and a computable automorphism \( \tau \) of \( \overline{\mathcal{F}} \) which \( \iota \)-extends \( \sigma \).

**Proof.** We begin by proving \((1) \Rightarrow (2)\). Suppose that there is a computable difference closed field \((\mathcal{K}, \rho)\) into which \((\mathcal{F}, \sigma)\) embeds. We can enumerate in \( \mathcal{K} \) the algebraic closure \( \overline{\mathcal{F}} \) of \( \mathcal{F} \) (which is also the algebraic closure of the prime field) and the restriction \( \tau \) of \( \rho \) to \( \overline{\mathcal{F}} \) (recall that every computable presentation of the algebraic closure of \( \mathcal{F} \) is computable isomorphic to every other computable presentation). Then, since \((\mathcal{F}, \sigma)\) embeds into \((\mathcal{K}, \rho)\) and is algebraic over \( \mathbb{F}_p \), its image is in \((\overline{\mathcal{F}}, \tau)\). Then \( \tau \) is an extension of \( \sigma \) to \( \overline{\mathcal{F}} \) via this embedding.

We now prove \((1) \Rightarrow (2)\). The completions of \( ACFA \) are given by the possible actions of the automorphism \( \sigma \) on the algebraic closure of the prime field \( \overline{\mathbb{F}}_p \) (see [(1.4) of CH99]). Let \( \iota \) be a computable embedding of \( \mathcal{F} \) into \( \overline{\mathcal{F}} \) and \( \tau \) an \( \iota \)-extension of \( \sigma \) to \( \overline{\mathcal{F}} \). Let \( \mathcal{L}_{\overline{\mathcal{F}}} \) be the language of difference fields together with names for the constants of \( \overline{\mathcal{F}} \). Let \( \mathcal{T} \) be the consistent theory axiomatized by \( ACFA \) together with the existential diagram of
Then $T$ contains a completion of $ACFA$, and since every formula is equivalent to an existential formula modulo $ACFA$, $T$ is complete. Moreover, $T$ is recursively axiomatizable and hence computable. So $T$ has a decidable model $(K, \rho)$. Using the embedding $\nu : \mathcal{F} \to \overline{\mathcal{F}}$, we get an embedding of the difference field $(\mathcal{F}, \sigma)$ into $(K, \rho)$.

We can use this, together with the examples from the previous section, to see that Rabin’s Theorem on the existence of computable algebraic closures (and its analogue in differentially closed fields due to Harrington [Har74]) does not hold in the context of difference closed fields:

**Corollary 10.5.2.** There exist computable difference fields which cannot be effectively embedded into any computable difference closed field. Moreover, there is a counterexample in every characteristic.

**Proof.** In characteristic zero, apply the previous corollary to the field from Proposition 10.4.3, and in characteristic $p > 0$, by Corollary 10.4.8, we can use any normal extension of $\mathbb{F}_p$ with no splitting algorithm.

**Corollary 10.5.3.** The analogue of Rabin’s Theorem holds for difference fields with underlying field $\mathcal{F}$ if and only if $\mathcal{F}$ has the computable extension of automorphisms property.

A set is **low** if its Turing jump is as low as possible, i.e., Turing equivalent to $\emptyset'$. We note that every computable difference field does embed into a **low** difference closed field:

**Fact 10.5.4** (essentially Friedman, Simpson, Smith [FSS83]). Every computable difference field embeds (by a map of low degree) into a low difference closed field.

**Proof.** Let $(\mathcal{F}, \sigma)$ be a computable difference field. Let $\nu : \mathcal{F} \to \overline{\mathcal{F}}$ be a computable embedding of $\mathcal{F}$ into its algebraic closure. Then there is a low automorphism $\tau$ of $\overline{\mathcal{F}}$ extending $\sigma$ (see [FSS83]). The theory $ACFA$ together with the action of $\tau$ on $\overline{\mathcal{F}}$ is a complete low theory, and an effective Henkin construction produces a low model as in Theorem 10.5.1.

In Theorem 10.4.6, we showed that for a field whose Galois group has the non-covering property, having a splitting algorithm is equivalent to the computable extendability of automorphisms property. We do not know in general whether these are equivalent. We leave open:

**Question 10.5.5.** For a normal extension $\mathcal{F}$ of $\mathbb{Q}$, is the computable extendability of automorphisms property equivalent to having a splitting algorithm?
Chapter 11

Notions of Independence

The results presented in this chapter appeared in [HTMM15]. They are joint work with Alexander Melnikov and Antonio Montalbán and appear here with their permission.

11.1 Introduction

The main objects of this paper are computable algebraic structures. A countably infinite algebraic structure $A$ is computable (Mal'cev [Mal61] and Rabin [Rab60]) if it admits a labeling of its domain by natural numbers so that the operations on $A$ become Turing computable upon the respective labels. Such a numbering is called a computable presentation, a computable copy, or a constructivization of $A$. Without loss of generality, we restrict ourselves to countable structures with domain $\omega$ (the natural numbers). Examples of computably presented structures include recursively presented groups with decidable word problem (Higman [Hig61]) and explicitly presented fields (van der Waerden [vdW30] and Fröhlich and Shepherdson [FS56]).

11.1.1 Independence With Applications

Mal'cev and his mathematical descendants were perhaps the first to realize the fundamental role of various notions of independence in effective algebra, especially in the study of the number of computable presentations of structures. In his pioneering paper [Mal62], Mal'cev made an important observation:

The additive group $\mathbb{V}^\infty \cong \bigoplus_{i \in \mathbb{N}} \mathbb{Q}$ has two computable presentations that are not computably isomorphic.

This effect had never been seen before, since algorithms had mostly been applied to finitely generated structures whose presentations are effectively unique. Mal'cev noted that $\mathbb{V}^\infty$ clearly has a “good” computable presentation $\mathcal{G}$ that is built upon a computable basis, and he constructed a “bad” computable presentation $\mathcal{B}$ of $\mathbb{V}^\infty$ that has no computable basis.
Clearly, $G$ is not computably isomorphic to $B$ (written $G \not\cong_{\text{comp}} B$). Essentially the same argument applies to the algebraically closed field $\mathbb{U}$ of infinite transcendence degree $[\text{MN79}]$. Similarly, manipulations with bases were used in the study of the number of computable copies in the contexts of torsion-free abelian groups $[\text{Dob83, Nur74, Gon82}]$ and ordered abelian groups $[\text{GLS03}]$ of infinite rank (though for ordered abelian groups, the existence of a “bad” copy is a new result appearing in this paper). The latter two examples are nontrivial, since the existence of a “good” copy is not evident. Nonetheless, in all these examples the “good” copy $G$ and the “bad” copy $B$ are isomorphic relative to the halting problem, or $\Delta^0_2$-isomorphic. Goncharov $[\text{Gon82}]$ showed that $G \not\cong_{\text{comp}} B$ and $G \cong_{\Delta^0_2} B$ imply there exist infinitely many computable presentations of the structure up to computable isomorphism. Thus, in each case discussed above we get infinitely many effectively different presentations.

Notions of independence play a central role in the study of the combinatorial properties of effectively presented vector spaces and for other structures with an appropriate notion of independence. Such studies were quite popular in the 70’s and 80’s; the standard reference is the fundamental paper of Metakides and Nerode $[\text{MN77}]$, see also $[\text{Dek69, Dek71a, Dek71b, Sho78, Dow84}]$ and, for applications to reverse mathematics, $[\text{Sim99}]$. Many results on subspaces of effectively presented vector spaces go thorough in the abstract setting of computable pregeometries (to be defined) – see the survey $[\text{DR98}]$ of Downey and Remmel. A number of results true of vector spaces go through for an arbitrary pregeometry if we have access to a “good” presentation $G$ with a computable basis. See, e.g., a recent paper of Conidis and Shore $[\text{CS14}]$ for a non-elementary illustration of this phenomenon.

11.1.2 The Mal’cev Property

We would like to unify the known results and extend them to other classes of structures. The definition below is central to the paper. Throughout the paper, $\mathcal{K}$ stands for a class of computable structures that admits a notion of independence (we will shortly clarify the notions of “independence” and “basis”).

**Definition 11.1.1.** A class $\mathcal{K}$ has the Mal’cev property if each member $\mathcal{M}$ of $\mathcal{K}$ of infinite dimension has a computable presentation $G$ with a computable basis and a computable presentation $B$ with no computable basis such that $B \cong_{\Delta^0_2} G$.

Of course, we can also talk about a single structure having the Mal’cev property, but it will be more natural to consider only classes since all of our applications will be to an entire class of structures.

In this paper we address:

**Question:** Which common algebraic classes have the Mal’cev property?

Note that, if $\mathcal{K}$ has the Mal’cev property, then every $\mathcal{M} \in \mathcal{K}$ of infinite dimension has infinitely many computable copies up to computable isomorphism $[\text{Gon82}]$. The standard abstraction for independence is the notion of a pregeometry that we briefly discuss in the next subsection.
11.1.3 Effective Pregeometries

A pregeometry on $M$ is a function $\text{cl} : \mathcal{P}(M) \rightarrow \mathcal{P}(M)$, where $\text{cl}(A)$ should be thought as the “span” of $A$ [Whi35]. A pregeometry must satisfy several natural properties (e.g., $\text{cl}(\text{cl}(A)) = \text{cl}(A)$) that will be stated in the preliminaries (see §11.2.1). Until then, the reader may safely rely on her/his intuition. For instance, we can define the notions of dimension, rank and basis in terms of $\text{cl}$. Computable pregeometries have been intensively studied, see survey [DR98]. We, however, will be working in the weaker context of pregeometries that can merely be effectively enumerated. This phenomenon is typically captured in terms of relatively intrinsically computably enumerable (r.i.c.e.) families of relations (to be defined—see Definition 11.2.1). It is crucial for us that being r.i.c.e. is equivalent to being definable in $\mathcal{L}_{\omega_1^\omega}$ logic in the language of the structure by an infinitary computable $\Sigma_1$-formula [AKMS89, Chi90]. Thus, we will have a syntactical definition of $\text{cl}$, and in fact this definition will induce a pregeometry on every member of a class $K$ of computable algebras. For convenience, we say that $\text{cl}$ is a r.i.c.e. pregeometry on $K$.

11.1.4 The Meta-Theorem

Let $M$ be a computable structure, $(M, \text{cl})$ a r.i.c.e. pregeometry. The independence diagram $I_M(\bar{c})$ of $\bar{c}$ in $M$ is the collection of all existential formulas true of tuples independent over $\bar{c}$. We also say that independent tuples are locally indistinguishable if for every $\exists$-formula true of one independent tuple we can find independent witnesses for this formula within the $\text{cl}$-span of any independent tuple (we will give the definition later—see Definition 11.2.4—but intuitively, it means that we would not be able to “code” a computably enumerable (c.e.) set using existential formulas into the $\text{cl}$-span of independent tuples, since the $\text{cl}$-span of any two independent tuples is in some sense the same). We will prove that the condition below is sufficient for producing a “good” computable presentation of $M$ with a computable basis:

**Condition G:** Independent tuples are locally indistinguishable in $M$ and for each $M$-tuple $\bar{c}$, $I_M(\bar{c})$ is computably enumerable uniformly in $\bar{c}$.

For convenience, we will be using one more term which should be thought of as saying that independent types are non-principal. We say that dependent elements are dense in $M$ if, whenever $M \models \exists \bar{y} \psi(\bar{c}, \bar{y}, a)$ for a quantifier-free formula $\psi$, non-empty tuple $\bar{c}$, and $a \in M$, there is a $b \in \text{cl}(\bar{c})$ such that $M \models \exists \bar{y} \psi(\bar{c}, \bar{y}, b)$. We may also assume that $\bar{c}$ contains at least $m$ independent elements, for some fixed $m$. This corresponds to localizing the pregeometry at a finite set. The next property is sufficient for having a “bad” computable presentation of $M$ with no computable basis:

**Condition B:** Dependent elements are dense in $M$.

Furthermore, our methods allow us to keep the isomorphisms $\Delta^0_2$. We summarize the above results in a theorem:
Theorem 11.1.2. Let $\mathcal{K}$ be a class of computable structures that admits a r.i.c.e. pregeometry cl. If each $\mathcal{M}$ in $\mathcal{K}$ of infinite dimension satisfies Conditions $G$ and $B$, then $\mathcal{K}$ has the Mal'cev property.

The (meta-)theorem above is the central technical tool of the paper that allows us to separate our proofs into an algebraic part and a part consisting of the effective combinatorics of r.i.c.e. pregeometries. We prove the metatheorem in §11.3.

11.1.5 Applications

That vector spaces and algebraically closed fields have the Mal'cev property is obvious. We discuss several non-trivial applications below. As we will note in the conclusion, there are indeed many more potential applications of our metatheorem. To keep the paper short, we give only five applications that (we think) are the most important ones.

Differentially Closed Fields

A differential field [Rit50] is a field $K$ together with a derivation operator $\delta : K \to K$. A differentially closed field is an existentially closed differential field. Robinson and Blum [Rob59, Blu68] came up with an elegant first-order axiomatization for differentially closed fields of characteristic zero ($DCF_0$). There has been a lot of work on model theory of differentially closed fields as described in [MMP06]. It is known that $DCF_0$ is complete, decidable, and has quantifier elimination.

In contrast to algebraically closed fields, we don’t know any “natural” example of a non-trivial differentially closed field. However Harrington [Har74] showed that every computable differential field can be computably embedded into a computable presentation of its differential closure. Thus, if we start with some differential field that we fully understand, we at least can effectively construct its differential closure. Differentially closed fields have some further nice computability-theoretic properties including the low property (see, e.g., [MM]).

Differential fields admit a natural notion of independence called $\delta$-independence (to be defined). The first new application of Theorem 11.1.2, which we prove in §11.4.1, is:

Theorem 11.1.3. The class of computable differentially closed fields of characteristic zero has the Mal'cev property with respect to $\delta$-independence.

Our result, in a way, improves the result of Harrington since every computable differential closed field has an even “nicer” presentation in which $\delta$-dependence is decidable (however, we may lose the computable embedding of the original differential field). Furthermore, if the $\delta$-rank of $\mathcal{M}$ is infinite, then we can produce infinitely many effectively non-isomorphic computable presentations of $\mathcal{M}$. 
CHAPTER 11. NOTIONS OF INDEPENDENCE

Difference Closed Fields

A difference field [Coh65] is a field together with a distinguished automorphism $\sigma$. A difference closed field is an existentially closed difference field. The theory $ACFA$ of difference closed fields is first-order axiomatizable, and the theories $ACFA$, $ACFA_0$, and $ACFA_p$ are decidable, see [CH99, (1.4) of]. For a detailed exposition of the model theory of difference fields, see [CH99]. The authors are unaware of any "naturally defined" algebraic example of a difference closed field. Nonetheless, it is not hard to effectively construct an example of a difference closure of a given algebraically closed field with an automorphism. It can be done using, say, an effective variation of the Henkin construction [AK00] or Ershov’s Kernel Theorem [EG00]. Difference fields admit a natural notion of transformal independence (to be defined). In §11.4.2 we prove:

**Theorem 11.1.4.** The class of computable difference closed fields has the Mal’cev property with respect to transformal independence.

We emphasize that we also get a new corollary on the number of computable copies of difference closed fields of infinite transformal rank.

Real Closed Fields

We assume that the reader is familiar with the definition of an ordered field. Real closed fields are existentially closed ordered fields. Tarski [Tar48] showed that the theory $RCF$ is complete, decidable, and has quantifier elimination. Model-theoretic features of real closed fields admit a generalization called o-minimality, see e.g. [vdD98]. We note that o-minimality has recently been applied to solve an open problem in pure number theory [PW06].

Computability-theoretic properties of ordered and real closed fields have been investigated in [EG00, KL13, Lev16]. In §11.4.3 we prove:

**Theorem 11.1.5.** The class of computable real closed fields has the Mal’cev property with respect to the standard field-theoretic (or, equivalently, model-theoretic) algebraic independence.

Torsion-Free Abelian Groups

The results of Nurtazin [Nur74], Dobrica [Dob83], and Goncharov [Gon82] mentioned above can be summarized in one theorem:

**Theorem 11.1.6.** The class of computable torsion-free abelian groups has the Mal’cev property with respect to linear independence over $\mathbb{Z}$.

In contrast to the number of computable copies and existence of a “good” copy, existence of a “bad” copy is a very recent fact that can be found in [Mel14]. We give a proof, using our metatheorem, in §11.4.4.
CHAPTER 11. NOTIONS OF INDEPENDENCE

Archimedean Ordered Abelian Groups

Using an involved combinatorial argument, Goncharov, Lempp, and Solomon [GLS03] showed that every computable Archimedean ordered abelian group has a computable copy with a computable base and that in the case of infinite rank it has infinitely many effectively distinct computable presentations. Using the model-theoretic properties of $({\mathbb{R}},+,{\leq})$, we extend their results in §11.4.5:

**Theorem 11.1.7.** The class of computable Archimedean ordered abelian groups has the Mal'cev property with respect to linear independence over $\mathbb{Z}$.

We note that the existence of a computable presentation with no computable basis is a new result.

11.2 Preliminaries

11.2.1 Pregeometries

A dependence relation often induces a **pregeometry**. Let $X$ be a set and $\text{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ a function on $\mathcal{P}(X)$. We say that $\text{cl}$ is a **pregeometry** if:

1. $A \subseteq \text{cl}(A)$ and $\text{cl}(\text{cl}(A)) = \text{cl}(A)$,

2. $A \subseteq B \Rightarrow \text{cl}(A) \subseteq \text{cl}(B)$,

3. (finite character) $\text{cl}(A)$ is the union of the sets $\text{cl}(B)$ where $B$ ranges over finite subsets of $A$, and

4. (exchange principle) if $a \in \text{cl}(A \cup \{b\})$ and $a \notin \text{cl}(A)$, then $b \in \text{cl}(A \cup \{a\})$.

An operation which satisfies the first two properties is called a **closure operator**. Let $(X, \text{cl})$ be a pregeometry, and $A \subseteq X$. We say that:

i. $A$ is **closed** if $A = \text{cl}(A)$;

ii. $A$ **spans** a set $B \subseteq A$ if $B \subseteq \text{cl}(A)$;

iii. $A \subseteq X$ is **independent** if for all $a \in A$, $a \notin \text{cl}(A\setminus\{a\})$, and $A$ is **dependent** otherwise;

iv. $B$ is a **basis** for $Y \subseteq X$ if $B$ spans $Y$ and is independent.

One can show that $B$ is a basis for $Y$ if, and only if, $B$ is a maximal independent set contained in $Y$. A standard argument shows that every set has a basis, and that every basis for $Y$ has the same cardinality. This cardinality is the **dimension** of $Y$, written $\text{dim}(Y)$. We can also generalize iii and define the associated notion of independence over a subset $C$ of $X$. We cite [Pil96] for more information about pregeometries.
Because a pregeometry has finite character, all of the information of the pregeometry is captured in the relations (for each \( n \)) \( x \in \text{cl}(y_1, \ldots, y_n) \) on finite tuples. We will say that \( \text{cl} \) is \emph{c.e. (computable)} if each of the relations \( x \in \text{cl}(\{y_1, \ldots, y_n\}) \) are computably enumerable (computable, respectively) uniformly in \( n \).

**Definition 11.2.1.** A pregeometry \( \text{cl} \) on a structure \( \mathcal{M} \) is relatively intrinsically computably enumerable (r.i.c.e) if the relations \( x \in \text{cl}(\{y_1, \ldots, y_n\}) \) are c.e. uniformly in \( n \) within, and relative to, any presentation of \( \mathcal{M} \).

All standard examples of pregeometries (vector spaces, fields, etc.) are r.i.c.e. pregeometries. As we have already mentioned in the introduction, it follows from \cite{AKMS89} and \cite{Chi90} that there is a tuple \( \bar{d} \) in \( \mathcal{M} \) such that these relations are uniformly defined by computable infinitary \( \Sigma^1_n \) formulas with parameters \( \bar{d} \) (the uniformity comes from a small modification to the same proof – see, for example, \cite{Mon12a}). See \cite{AK00} for a background on computable infinitary logic \( \mathcal{L}^{c}_{\omega_1\omega} \). The easy proposition below may help the reader to develop some intuition.

**Proposition 11.2.2.** Let \((X, \text{cl})\) be a pregeometry, with \( X \subseteq \omega \) a computable set and \( \text{cl} \) c.e. Then \((X, \text{cl})\) has a computable basis if and only if \( \text{cl} \) is computable.

**Proof.** A standard argument shows that a computable pregeometry has a computable basis. Now suppose \( B \) is a computable basis for \((X, \text{cl})\). To decide whether \( x_1, \ldots, x_n \) are dependent or not, find a finite set \( A \subseteq B \) such that \( \{x_1, \ldots, x_n\} \subseteq \text{cl}(A) \). Then \( x_1, \ldots, x_n \) are independent if, and only if, there exist \( a_1, \ldots, a_n \in A \) such that

\[
a_1, \ldots, a_n \in \text{cl}(\{x_1, \ldots, x_n\} \cup (A \setminus \{a_1, \ldots, a_n\})).
\]

The latter is a c.e. property. Since for \( \{x_1, \ldots, x_n\} \) being dependent is c.e. as \( \text{cl} \) is c.e., we conclude that determining whether or not a tuple is dependent is computable. From this, we can compute the closure relation. \( \square \)

### 11.2.2 The Notion of \( k \)-Dependence

We will be using a computable way of approximating a r.i.c.e. pregeometry. We fix a r.i.c.e. pregeometry \((\mathcal{M}, \text{cl})\) upon a computable infinite structure \( \mathcal{M} \). Since the relations \( x \in \text{cl}(\{y_1, \ldots, y_n\}) \) are r.i.c.e., we have

\[
x \in \text{cl}(\{y_1, \ldots, y_n\}) \iff \mathcal{M} = \bigvee_{k \in S_n} (\exists z) \phi_k(\bar{d}, x, y_1, \ldots, y_n, z)
\]

where \( S_n \) are c.e. sets of indices for open formulas with parameters \( \bar{d} \), given uniformly (using a standard forcing argument one can show that all the formulas \( \phi_k \) indeed share the same tuple of parameters \( \bar{d} \)). To simplify our notation, we suppress \( \bar{d} \) in \( \phi_k \). We approximate the relations \( x \in \text{cl}(\{y_1, \ldots, y_n\}) \) as follows.
Definition 11.2.3. We say that \( x \) is \( k \)-dependent on \( \{y_1, \ldots, y_n\} \), which we denote by \( x \in \text{cl}_k(\{y_1, \ldots, y_n\}) \), if \( x \) comes from among the first \( k \)-many elements of \( M \) and
\[
M \models \bigvee_{i \leq k, i \in S_n} (\exists z \in M \upharpoonright k) \phi_i(d, x, y_1, \ldots, y_n, z)
\]
(i.e., the witnesses \( z \) come from among the first \( k \) elements of \( M \)).

Similarly, we say that a set \( X \) is \( k \)-dependent if for some \( x \in X \), \( x \in \text{cl}_k(X \setminus \{x\}) \), and otherwise we say that \( X \) is \( k \)-independent. It is clear that \( \text{cl}_k \) is a computable operator which, however, typically is not a pregeometry. Moreover, \( \text{cl}_k(X) \) is finite for every set \( X \).

11.2.3 Locally Indistinguishable Tuples

The definition below clarifies the informal discussion from the introduction. We say that a tuple \( \bar{x} \) is independent if the set of its components is independent. Independence over a tuple is defined similarly.

Definition 11.2.4. We say that independent tuples in \( M \) are locally indistinguishable if for every tuple \( \bar{c} \) in \( M \) and \( \bar{u}, \bar{v} \) independent tuples over \( \bar{c} \), for each existential formula \( \phi \) such that \( M \models \phi(\bar{c}, \bar{u}) \), there exists a tuple \( \bar{w} \) that is independent over \( \bar{c} \), has \( M \models \phi(\bar{c}, \bar{w}) \), and (with \( \bar{w} = (w_1, \ldots, w_n) \) and \( \bar{v} = (v_1, \ldots, v_n) \)) we have \( w_i \in \text{cl}(\bar{c}, v_1, \ldots, v_i) \) for \( i = 1, \ldots, n \).

Thus, even if \( \bar{u} \) and \( \bar{v} \) can be separated by an \( \exists \)-formula, we can always find independent witnesses for any \( \exists \)-formula true of \( \bar{u} \) within the \( \text{cl} \)-span of \( \bar{v} \). The conditions \( \bar{w} \) is independent over \( \bar{c} \) and \( w_i \in \text{cl}(\bar{c}, v_1, \ldots, v_i) \) are equivalent to \( w_i \) and \( v_i \) being interdependent over \( \bar{c}, v_1, \ldots, v_{i-1} \) (and hence also over \( \bar{c}, w_1, \ldots, w_{i-1} \)).

11.3 Proof of the Metatheorem

We fix a computable \( M \) and a r.i.e. pregeometry \( \text{cl} \) on \( M \). This section is devoted to a proof of Theorem 11.1.2. We first prove that Condition G implies that \( M \) has a computable copy with a computable basis, and then we show that Condition B guarantees the existence of a copy with no computable basis.

11.3.1 A Computable Copy with a Computable Basis

Recall that the independence diagram \( I_M(\bar{c}) \) of \( \bar{c} \) in \( M \) is the collection of all existential first-order formulas \( \exists \bar{y} \psi(\bar{c}, \bar{y}, \bar{x}) \) such that \( M \models \exists \bar{y} \psi(\bar{c}, \bar{y}, \bar{a}) \) for some tuple \( \bar{a} \in M \) independent over \( \bar{c} \). By our assumption, \((M, \text{cl})\) satisfies:

**Condition G:** Independent tuples are locally indistinguishable in \( M \) and for each \( M \)-tuple \( \bar{c} \), \( I_M(\bar{c}) \) is computably enumerable uniformly in \( \bar{c} \).

\(^1\)In general, we will use \( w_i \) to denote the entries of a tuple \( \bar{w} \) in this way.
Proposition 11.3.1. Suppose that $\mathcal{M}$ is a computable structure, and let $\text{cl}$ be a r.i.c.e. pre-geometry on $\mathcal{M}$. If $(\mathcal{M},\text{cl})$ satisfies Condition G, then there exists a computable copy $\mathcal{G}$ of $\mathcal{M}$ having a computable basis. Furthermore, $\mathcal{G} \cong \Delta^0_2 \mathcal{M}$.

Proof. It is sufficient to prove the proposition in the case where the dimension of $\mathcal{M}$ is infinite. We may also assume that the language is relational as any $\Sigma_1$ formula involving function symbols can be replaced by a $\Sigma_1$ formula which involves only the relation symbols for the graphs of those functions.

At stage $s$ of the construction we will define three things: a finite one-to-one map $\tau_s$ from an initial segment of $\omega$ to $\mathcal{M}$, a finite tuple $a_0, \ldots, a_s$ inside the domain of $\tau_s$ and a number $t_s > s$. We will also define a finite structure $\mathcal{G}_s$ by pulling back the structure $\mathcal{M}$ through $\tau_s$ (and, if the language is infinite, at stage $s$ we consider only the first $s$ many relations in the language). This structure will never change, and thus we will have $\mathcal{G}_s \subseteq \mathcal{G}_{s+1}$ and at the end we will get that $\mathcal{G} = \bigcup_s \mathcal{G}_s$ is a computable structure. The elements $a_i$ will never change either, and we will make sure they end up forming a computable basis for $\mathcal{G}$. The partial isomorphisms $\tau_s$ will change from stage to stage, but they will stabilize pointwise and hence they will have a $\Delta^0_2$ limit, $\tau$, which will be an isomorphism $\tau: \mathcal{G} \to \mathcal{M}$. The number $t_s$ represents how much we have looked into $\mathcal{M}$ to guess which elements are independent and which are not. Of course we would like to be able to pick $\tau_s(a_0), \ldots, \tau_s(a_s)$ so that they are independent in $\mathcal{M}$ from the beginning, but since this is a $\Pi^0_1$ property, we will not be always correct. For starters, we choose $\tau_s(a_0), \ldots, \tau_s(a_s)$ to be $t_s$-independent (see Definition 11.2.3). But to make sure we can recover from our mistakes we will need to ask for some extra assurances. At each stage $s$, $\tau_s$, $a_0, \ldots, a_s$, and $t_s$ will satisfy properties (P1)-(P6) which we describe below. We start with the most obvious ones.

Basic Properties

Let us start with independence:

1. $\tau_s(a_0), \ldots, \tau_s(a_s)$ are $t_s$-independent.

Once we show that the values of $\tau_s(a_i)$ eventually stabilize, property (P1) guarantees that their limits $\tau(a_0), \tau(a_1), \ldots$ are independent in $\mathcal{M}$. Therefore, $a_0, a_1, \ldots$ will end up being independent in $\mathcal{G}$.

The second property is about the range of $\tau_s$.

2. $\{0, \ldots, s-1\} \subseteq \text{range}(\tau_s) \subseteq \text{cl}_{t_s}(\tau_s(a_0), \ldots, \tau_s(a_s))$.

As one might expect, property (P2) will be useful to show that $\tau$ is onto $\mathcal{M}$. Also, the right inclusion will help us to keep things tight when we want to prove that the $\tau_s$’s stabilize pointwise.
**Compatibility**

Usually, in this kind of construction, one requires that the atomic facts true in $G_{s-1}$ remain unchanged in $G_s$ and unchanged for the rest of the construction. In our case, we also want that whenever some tuple is $t_{s-1}$-dependent, it will stay dependent forever. Since $t_{s-1}$-dependence is witnessed by some $\Sigma_1$ formula being true in $M_{t_{s-1}}$, we will ask for all such formulas to be preserved.

(3) For every tuple $\bar{b}$ in the domain of $\tau_{s-1}$ and every $\Sigma_1$ formula $\varphi(\bar{x})$ (using only the first $t_{s-1}$ relation symbols) we have that

$$M_{t_{s-1}} \models \varphi(\tau_{s-1}(\bar{b})) \Rightarrow M_{t_s} \models \varphi(\tau_s(\bar{b})).$$

In particular, we get that if

$$\tau_{s-1}(b) \in \text{cl}_{t_{s-1}}(\tau_{s-1}(a_0), \ldots, \tau_{s-1}(a_i)),$$

then $\tau_s(b) \in \text{cl}_{t_s}(\tau_s(a_0), \ldots, \tau_s(a_i))$.

**Stabilization**

To make sure that the sequence of $\tau_s$ converges pointwise, we will require that once the elements $\tau_s(a_0), \ldots, \tau_s(a_i)$ are really independent, they never change again. Also, nobody that appears to be in their closure will change either. More formally: for $s > 0$,

(4) Suppose that $b \in \text{dom}(\tau_{s-1})$ and that $\tau_{s-1}(b) \in \text{cl}_{t_{s-1}}(\tau_{s-1}(a_0), \ldots, \tau_{s-1}(a_i))$. Then, we can only have $\tau_s(b) \neq \tau_{s-1}(b)$, if $\tau_{s-1}(a_0), \ldots, \tau_{s-1}(a_i)$ are $t_s$-dependent.

In particular, if $\tau_{s-1}(a_0), \ldots, \tau_{s-1}(a_i)$ stay $t_s$-independent at stage $s$, then their values won’t change at $s$. And if $\tau_{s-1}(a_0), \ldots, \tau_{s-1}(a_i)$ are actually independent, their values will never change. Therefore, once we show that $\tau_s(a_0), \ldots, \tau_s(a_i)$ are eventually going to be independent, we get that their values eventually stabilize.

Furthermore, by condition (P2), for every $b$ in the domain of $\tau_s$, we will have $\tau_s(b) \in \text{cl}_{t_s}(\tau_s(a_0), \ldots, \tau_s(a_s))$. By condition (P3) this fact will never change at future stages. So, once $\tau_r(a_0), \ldots, \tau_r(a_s)$ become independent at some future stage $r > s$, condition (P4) implies that $\tau_r(\bar{b})$ will never change again.

**Safeness**

The idea of condition (P5) below is that before choosing values for $\tau_s(a_0), \ldots, \tau_s(a_s)$ we will not only ask for them to be $t_s$-independent, but also that whatever we have committed about them is consistent with the corresponding independence diagrams, as if they were actually independent. This way, if we later realize they were not independent, we can find some other (potentially) independent tuple which is compatible with our construction thus far.
We start by defining the formulas that describe the commitments we have made so far. Suppose we have \((\tau_s; a_0, \ldots, a_s)\) as above. Fix \(i < s\). For each \(j\), let \(u_j = \tau_s(a_j)\). Let \(\bar{c}\) be the tuple of elements in the range of \(\tau_s\) which belong to \(\text{cl}_{t_s}(u_0, \ldots, u_i)\). (These are the elements that, by condition (P4), we do not want to move if \(u_0, \ldots, u_i\) were to be independent.) Let \(\bar{v}\) be the elements of \(M_{t_s}\) which are neither in \(\bar{c}\) nor in \(\bar{u}\), that is, those elements which are not in \(\bar{u}\), and which are not in the \(\text{cl}_{t_s}\)-closure of \(u_0, \ldots, u_i\), but (by P2) are in the closure of \(\bar{u} = (u_0, \ldots, u_s)\). So \(M_{t_s} = \{\bar{c}\} \cup \{\bar{v}\} \cup \{u_{i+1}, \ldots, u_s\}\). Let \(\theta(\bar{c}, \bar{v}, u_{i+1}, \ldots, u_s)\) be the conjunction of the atomic diagram of \(M_{t_s}\) (using only the first \(t_s\) relation symbols). Define

\[
\psi_i(\bar{c}, x_{i+1}, \ldots, x_s) \equiv \exists \bar{y}\theta(\bar{c}, \bar{y}, x_{i+1}, \ldots, x_s).
\]

Note that \(\psi_i\) is an \(\exists\)-formula with parameters \(\bar{c}\), and with \(s - i\) indeterminates \(x_{i+1}, \ldots, x_s\).

(5) For each \(i < s\), \(\psi_i(\bar{c}, \bar{x})\) belongs to the independence diagram of \(\bar{c}\).

Recall that Condition G tells us that these independence diagrams are uniformly c.e., and hence we can always wait to see if a formula shows up in one. Thus condition (P5) is \(\Sigma^0_1\). All the previous conditions were computable.

Also, let us remark that if \(u_0, \ldots, u_s\) are independent in \(M\), then these formulas do belong to the corresponding independence diagrams, and hence (P5) holds.

**Least Span**

We need this last property to guarantee that, as we are moving the values of \(\tau_s(a_0), \ldots, \tau_s(a_i)\) around, we will eventually fall on one that is actually independent. Also, it will help us ensure that they form a basis at the end.

If we had a 0′ oracle, we could build what we call the \(\omega\)-least basis for \(M\) as follows: once we have defined \(m_0, m_1, \ldots, m_{s-1}\), we define the next element, \(m_s\), be the \(\omega\)-least element of \(M\) which is not in the closure of \(m_0, m_1, \ldots, m_{s-1}\). (Recall that the domain of \(M\) is \(\omega\).) We will not be able to get \(\tau(a_0), \tau(a_1), \ldots\) to be this particular basis, but we can get close. We say that a tuple \(u_0, \ldots, u_s\) in \(M\) has \(\omega\)-least span if, for every \(i \leq s\), \(\text{cl}(u_0, \ldots, u_i) = \text{cl}(m_0, \ldots, m_i)\), or equivalently, if \(\text{cl}(u_0, \ldots, u_i)\) contains the \(\omega\)-least element of \(M\) which is not in \(\text{cl}(u_0, \ldots, u_{i-1})\). By the exchange principle and the fact that \(m_0, m_1, \ldots\) are independent, \(m_i \in \text{cl}(u_0, \ldots, u_i)\) and \(u_i \in \text{cl}(u_0, \ldots, u_{i-1}, m_i)\). Note that if an infinite subset of \(M\) has \(\omega\)-least span then it is a basis.

At each stage \(t\) we can only approximate this property. At stage \(t\), let \(n_0[t]\) be the least element not in \(\text{cl}_t(\emptyset)\), \(n_1[t]\) the least element not in \(\text{cl}_t(n_0[t])\), \(n_2[t]\) the least element not in \(\text{cl}_t(n_0[t], n_1[t])\), and so on. We say that a tuple \(u_0, \ldots, u_s\) in \(M\) has \(\omega\)-least span at \(t\) if for every \(i \leq s\), \(\text{cl}_t(u_0, \ldots, u_i)\) contains \(n_i[t]\) and \(\text{cl}_t(u_0, \ldots, u_{i-1}, n_i[t])\) contains \(u_i\).

(6) \(\tau_s(a_0), \ldots, \tau_s(a_s)\) has \(\omega\)-least span at \(t_s\)
Note that the $n_i[t]$ are computable and are increasing (in the lexicographic order) as $t$ increases, with limit $m_i$ (the elements of the $\omega$-least basis). Suppose that at stage $t$ we have that $n_0[t], \ldots, n_i[t]$ are part of the $\omega$-least basis, i.e., they are equal to $m_0, \ldots, m_i$. If $u_0, \ldots, u_i$ has $\omega$-least span at stage $t$, then the exchange principle implies that $u_0, \ldots, u_i$ are independent.

On the other hand, if $u_0, \ldots, u_i$ are independent and have $\omega$-least span at stage $t$, then since $u_0, \ldots, u_i \in \text{cl}(n_0[t], \ldots, n_i[t])$, $n_0[t], \ldots, n_i[t]$ are independent and hence equal to $m_0, \ldots, m_i$. Then $u_0, \ldots, u_i$ have $\omega$-least span at stage $t$.

Therefore, we get that, for every $i \in \omega$, there is $t$ large enough such that, once $t_0 > t$, Condition (P6) implies that $\tau_s(a_0), \ldots, \tau_s(a_i)$ are independent in $M$. As we have argued above, this then implies that the sequence $\tau_s$ converges pointwise to a bijection $\tau : \omega \to M$ getting that $G \equiv M$. Condition (P6) also implies that $\tau(a_0), \tau(a_1), \ldots$ is a basis of $M$ and hence that $a_0, a_1, \ldots$ is a basis of $G$.

Construction

At stage 0, define $G_0 = \{a_0\}$ and let $\tau_0(a_0)$ be the $\omega$-least element of $M$ that is independent over $\emptyset$. Without loss of generality, we assume $\tau_0(a_0) = 0$. After stage $s$ is finished, we will have defined $\tau_s, a_0, \ldots, a_s \in G_{t_s}$ and $t_s$ satisfying (P1)-(P6). At stage $s+1$ define $\tau_{s+1}$, $a_0, \ldots, a_{s+1} \in G_{t_{s+1}}$ and $t_{s+1}$ to be the first ones we find satisfying (P1)-(P6).

The last step in the proof is to show that Condition G guarantees such objects exist.

Claim 11.3.2. Given $\tau_s, a_0, \ldots, a_s$ and $t_s$ satisfying (P1)-(P6), there exist $\tau_{s+1}, a_0, \ldots, a_{s+1}$ and $t_{s+1}$ also satisfying (P1)-(P6).

Proof. To simplify the notation let $u_i = \tau_s(a_i)$ for each $i$.

Suppose now that $u_0, \ldots, u_s$ are not independent. Let $i$ be the largest such that the elements $\tau_s(a_0), \ldots, \tau_s(a_i)$ are independent. We noted above that this implies that they also have $\omega$-least span. Let $\bar{c}$ be the tuple of elements in the range of $\tau_s$ which belong to $\text{cl}(u_0, \ldots, u_i)$. We will keep $\tau_s$ fixed on (the pre-image of) $\bar{c}$ so that condition (P4) is satisfied, and change the rest. Let $\psi_i(\bar{c}, x_{i+1}, \ldots, x_s)$ be as in the subsection on Safeness. Since $\psi_i$ is in the independence diagram of $\bar{c}$ we know it is true of some independent tuple. We are now ready to apply Condition G. Let $v_{i+1}, \ldots, v_s$ be independent over $\bar{c}$ such that $u_0, \ldots, u_i, v_{i+1}, \ldots, v_s$ has $\omega$-least span. By Condition G there exist $w_{i+1}, \ldots, w_s$ independent over $\bar{c}$ such that $M \models \psi_i(\bar{c}, w_{i+1}, \ldots, w_s)$ and such that for every $j \leq s$, $w_j \in \text{cl}(\bar{c}, v_{i+1}, \ldots, v_j)$ (and, by the exchange principle, $\text{cl}(\bar{c}, v_{i+1}, \ldots, v_j) = \text{cl}(\bar{c}, w_{i+1}, \ldots, w_j)$ for each $i < j \leq s$).

Thus $u_0, \ldots, u_i, w_{i+1}, \ldots, w_s$ has $\omega$-least span as well.

Re-define $\tau_{s+1}(a_j) = w_j$ for $j > i$. For all the other elements in the domain of $\tau_s$ use the fact that $M \models \psi_i(\bar{c}, w_{i+1}, \ldots, w_s)$ to define $\tau_{s+1}$ in a way their existential diagrams within $M_{t_s}$ remain unchanged, so that (P3) holds. Let $t_{s+1}$ be larger than all of the elements in the image of $\tau_{s+1}$.

Since $u_0, \ldots, u_i, w_{i+1}, \ldots, w_s$ is independent we get conditions (P1) and (P5). By increasing $t_{s+1}$, we can have $n_j[t_{s+1}] = m_j$ for $0 \leq j \leq s$. For every $i < j \leq s$, $m_j \in \text{cl}(\bar{c}, w_{i+1}, \ldots, w_j)$
Recall that dependent elements are dense in $\mathcal{M}$ if, whenever $\psi$ is quantifier-free and $\mathcal{M} \models \exists \bar{y}\psi(\bar{c}, \bar{y}, a)$ for a non-empty tuple $\bar{c}$ and $a \in \mathcal{M}$, there is $b \in \text{cl}(\bar{c})$ and $\mathcal{M} \models \exists \bar{y}\psi(\bar{c}, \bar{y}, b)$. We now prove:

**Proposition 11.3.4.** Let $\mathcal{M}$ be a computable structure, and let cl be a r.i.c.e. pregeometry upon $\mathcal{M}$. If the cl-dimension of $\mathcal{M}$ is infinite and dependent elements are dense in $\mathcal{M}$ (this is Condition B), then $\mathcal{M}$ has a computable presentation $\mathcal{B} \equiv_{\Delta_2} \mathcal{M}$ that has no computable basis.

**Proof.** Because the proof is a standard finite injury construction we will only give a sketch. Suppose $\mathcal{M}$ has a computable basis $a_0, a_1, \ldots$ (otherwise there is nothing to prove). We construct $\mathcal{B}$ by stages. We meet the requirements:

$$R_e : \varphi_e \text{ is not a dependence algorithm for } \mathcal{B},$$

where $\varphi_0, \varphi_1, \ldots$ is the standard effective listing of all partial recursive functions. Initially we will attempt to copy $\mathcal{M}$ so that the images of $a_i$ are special elements $b_i$ of $\mathcal{B}$.

The strategy for $R_e$ will have a witness $b_e$; it waits until $\varphi_e$ declares $b_e$ independent of $b_0, \ldots, b_{e-1}$. If it ever does, we make $b_e$ dependent on $b_0, \ldots, b_{e-1}$ using that dependent elements are dense in $\mathcal{M}$. We also rearrange the map from $\mathcal{M}$ to $\mathcal{B}$; to do that we introduce a new image for $a_e$ in $\mathcal{B}$.

Now this does not immediately prevent $\mathcal{B}$ from having a computable basis, but it does ensure that the closure operation on $\mathcal{B}$ is not computable. By Proposition 11.2.2, this is actually equivalent. 

Theorem 11.1.2 follows from Propositions 11.3.1 and 11.3.4.
11.4 Applications

It is not difficult to see that computable vector spaces and algebraically closed fields both have the Mal'cev property and indeed satisfy Conditions B and G (essentially Mal'cev [Mal62], Metakide-Nerode [MN79], and Goncharov [Gon82]). Since we do not really need the metatheorem for these trivial cases, we skip these examples and leave the (elementary) verification to the reader. We concentrate on non-elementary applications.

11.4.1 Differentially Closed Fields

Recall that a differential field is a field with a differential operator $\delta$. In this section we look at existentially closed differential fields of characteristic zero. The first-order theory $DCF_0$ of differentially closed fields is complete, axiomatizable, decidable, and has quantifier elimination. It is the model companion of differential fields. A more detailed exposition can be found [MMP06].

In differential fields, there are analogs of polynomial rings and algebraic independence. The ring $K\{X_1, \ldots, X_n\}$ of $\delta$-polynomials over $K$ is the polynomial ring

$$K[X_1, \ldots, X_n, \delta(X_1), \ldots, \delta^2(X_1), \ldots]$$

where each of these are indeterminates. We extend the derivation to this ring in the natural way, by setting $\delta(\delta^n(X_i)) = \delta^{n+1}(X_i)$.

**Definition 11.4.1.** We say that a set $A$ is $\delta$-dependent over $B$ if there are $a_1, \ldots, a_n$ in $A$ and a nonzero $\delta$-polynomial $f \in \mathbb{Q}(B)\{X_1, \ldots, X_n\}$ which they satisfy. Otherwise, we say that $A$ is $\delta$-independent over $B$. A $\delta$-transcendence base is a maximal $\delta$-independent subset of $K$.

We get a finitary closure operator by defining $\text{cl}(B)$ to be the set of elements which are $\delta$-dependent\(^\text{2}\) over $B$. It is not hard to verify that differential closure induces a r.i.c.e. pregeometry. We prove:

**Theorem 11.1.3.** The class $DCF_0$ of computable differential closed fields of characteristic zero has the Mal'cev property with respect to $\delta$-independence.

**Proof.** By Theorem 11.1.2, it suffices to check that any $\mathcal{M} \models DCF_0$ of infinite dimension satisfies Conditions G and B.

Differentially closed fields have unique independent types. That is, for any fixed tuple $\vec{c}$, and $\vec{a}$, $\vec{b}$ both independent over $\vec{c}$, $\text{tp}(\vec{a}/\vec{c}) = \text{tp}(\vec{b}/\vec{c})$. Let $p(\vec{x})$ be this type with parameters $\vec{c}$. This type $p(\vec{x})$ is generated over the theory $DCF_0$ by all of the quantifier-free formulas true about $\vec{c}$ together with the formulas which say that $\vec{x}$ satisfies no non-trivial differential

\(^2\)Warning: In a differentially closed field, $\delta$-dependence is always different from model-theoretic algebraic dependence as the equation $\delta(x) = 0$ defines an infinite set.
CHAPTER 11. NOTIONS OF INDEPENDENCE

polynomial with coefficients from $\mathbb{Q}\{\bar{c}\}$ [Mar02, see Section 3 of]. Since $M$ is computable, we can list all valid quantifier-free formulas true of $\bar{c}$, and thus $p(\bar{x})$ is computable uniformly in $\bar{c}$.

Now we will check Condition G. Given $\bar{c}$, we can compute the type $p(\bar{x})$ of an independent tuple of some arity over $\bar{c}$. Then we can decide whether or not any existential formula is in this type. Similarly, independent tuples are locally indistinguishable, because if $\bar{c}$ is a tuple, and $\bar{u}$ and $\bar{v}$ are both independent over $\bar{c}$, then $tp(\bar{u}/\bar{c}) = tp(\bar{v}/\bar{c})$.

We check Condition B. We first observe that independent types are non-principal in $DCF_0$. Indeed, it is well-known that if $K$ is a differential field and $L$ a differentially closed field containing $K$, then the differential closure of $K$ in $L$ is a differentially closed field which omits the type of a $\delta$-transcendental element over $K$. Now suppose $M = \exists\bar{y}\psi(\bar{c}, \bar{y}, a)$ for a non-empty tuple $\bar{c}$ and $a \in M$. If $a \in cl(\bar{c})$ then we are done. Otherwise, consider an embedding of the prime model $K$ over $\bar{c}$ in $M$ (since $DCF_0$ is $\omega$-stable, it has prime models over any set). Since $K \preceq M$, $K = \exists b \exists\bar{y}\psi(\bar{c}, \bar{y}, b)$. Any such $b$ will be differentially algebraic over $\bar{c}$. 

11.4.2 Difference Closed Fields

A difference field is a field together with a distinguished automorphism $\sigma$. Difference fields have a model companion $ACFA$. The theory $ACFA$ of difference closed fields is first-order axiomatizable. The theories $ACFA$, $ACFA_0$, and $ACFA_p$ (the subscript denoting the characteristic) are decidable, see [CH99, (1.4) of]. Note that the field-theoretic and model-theoretic algebraic closures coincide, see [CH99, (1.7) of]. For a more detailed exposition of the model theory of difference fields, see [CH99].

The difference polynomial ring $K(X_1, \ldots, X_n)$ is the polynomial ring

$$K[X_1, \ldots, X_n, \sigma(X_1), \ldots, \sigma(X_n), \sigma^2(X_1), \ldots]$$

with the natural extension of $\sigma$.

**Definition 11.4.2.** Let $A$ be a subset of $K$, and let $E$ be the difference field generated by $A$. We say that $a_1, \ldots, a_n$ are transformally dependent over $A$ if there is a nontrivial difference polynomial $f \in E(X)$ which they satisfy. Otherwise, we say that $a_1, \ldots, a_n$ are transformally independent over $A$.

We get a finitary closure operator by defining $cl(B)$ to be the set of elements which are transformally dependent over $B$. This notion of dependence induces a r.i.e.c. pregeometry. We prove:

**Theorem 11.1.4.** The class $ACFA$ of computable difference closed fields has the Mal’cev property with respect to transformal independence.

**Proof.** By Theorem 11.1.2, it suffices to check that any $M \models ACFA$ of infinite dimension satisfies Conditions G and B.
Let $\mathcal{M}$ be a computable model of $ACFA$. The complete theory of $\mathcal{M}$ is given by the axioms of $ACFA$, the characteristic, and the action of the automorphism on the algebraic closure of the prime field. Every formula is equivalent, modulo $ACFA$, to an existential formula [CH99, see (1.6) of]. So the elementary diagram of $\mathcal{M}$ is decidable, since for any formula $\varphi$, we will eventually find that either $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \neg \varphi$.

We check Condition G. Difference closed fields have unique independent types over algebraically closed subfields (in fact, as we will see, over any parameters, but the types over arbitrary parameters may be more complicated). Thus, for any algebraically closed $E \subseteq \mathcal{M}$ and $\overline{a}, \overline{b} \in \mathcal{M}$ both independent over $E$,

$$tp(\overline{a}/E) = tp(\overline{b}/E).$$

Moreover, this type $p(\overline{x})$ is generated by all of the quantifier-free formulas true about $E$ together with the formulas which say that $\overline{x}$ satisfies no non-trivial difference polynomial with coefficients from $\mathbb{Q}\{E\}$ [CH99, see Proposition 2.10 of]. Note that if $\overline{c}$ is any tuple, then there is a unique type of an independent element over $acl(\overline{c})$, and hence over $\overline{c}$. Thus, independent tuples are locally indistinguishable. Given $\overline{c}$, we can enumerate $acl(\overline{c})$ and compute the type $p(\overline{x})$ of an independent tuple over $acl(\overline{c})$. We can restrict ourselves to formulas about $\overline{c}$ and compute the type of an independent tuple over $\overline{c}$. Then we can decide whether or not any existential formula is in this type.

We check Condition B. As in the case of $DCF_0$, independent types are non-principal in $ACFA$. This can be derived from the fact that if $E$ is a difference field contained in a difference closed field $K$, then the set $K_0$ of elements transformally algebraic over $E$ is a difference closed field with $K_0 \preceq K$ [CH99, see, e.g., the remark after Theorem 1.1 of]. If $\overline{e}$ is a tuple in some difference closed field $K$, and $E$ is the difference field generated by $\overline{e}$, then the corresponding $K_0 \preceq K$ omits the type of an independent tuple over $\overline{e}$. The rest can be done just as in the case of $DCF_0$. 

### 11.4.3 Real Closed Fields

We assume that the reader is familiar with the definition of an ordered field. Real closed fields are existentially closed ordered fields. Equivalently, a field $F$ is real closed if every positive element has a square root in $F$ and every polynomial of odd degree with coefficients in $F$ has a root in $F$. These give axioms for the theory $RCF$ of real closed fields. Tarski [Tar48] showed that the theory $RCF$ is complete, decidable, and has quantifier elimination. Using quantifier elimination, it is easy to see that the definable sets in a single variable consist of finitely many points (the solutions to certain polynomial equations) plus finitely many (possibly unbounded) intervals. The standard generalization of this phenomenon is $o$-minimality, see e.g. [vdD98].

Let $F$ be a real closed field. The model-theoretic algebraic closure agrees with the algebraic closure as a pure field, as any finite set of points which is definable can be defined without the ordering. In general, Pillay and Steinhorn [PS86] remarked that for $o$-minimal
structures, the model-theoretic algebraic closure agrees with the model-theoretic definable closure, and that this is always a pregeometry.

Unlike algebraically closed fields, transcendental types are not unique. For example, in \( \mathbb{R} \), \( \pi \) and \( e \) have different types over \( \mathbb{Q} \) and yet are both transcendental. Our proof that real closed fields have the Mal’cev property will use cell decomposition.

**Definition 11.4.3.** The collections of *cells* is defined recursively by:

1. If \( X \) is a single point in \( F^n \), then \( X \) is a 0-dimensional cell.
2. Every open interval \( (a,b) \) in \( F \) is a 1-dimensional cell (with \( a \in F \cup \{-\infty\} \) and \( b \in F \cup \{\infty\} \)).
3. If \( X \subseteq F^n \) is an \( m \)-dimensional cell, and \( f : X \to F \) is a continuous definable function, then
   \[
   Y = \{(\bar{x}, f(\bar{x})) : \bar{x} \in X\}
   \]
   is an \( m \)-dimensional cell.
4. If \( X \subseteq F^n \) is an \( m \)-dimensional cell, and \( f \) and \( g \) are either both continuous functions \( X \to F \), or \( f \) is possibly the constant function \(-\infty\), or \( g \) is possibly the constant function \( \infty \), and \( f(\bar{x}) < g(\bar{x}) \) for all \( \bar{x} \in X \), then
   \[
   Y = \{(\bar{x}, y) : \bar{x} \in X \text{ and } f(\bar{x}) < y < g(\bar{x})\}
   \]
   is an \( m + 1 \)-dimensional cell.

Every definable set can be built up from cells using the following theorem:

**Theorem 11.4.4 (Cell Decomposition, [KPS86]).** Given a set \( X \) definable over \( \bar{a} \), we can write \( X \) as a finite union of disjoint cells. The functions and endpoints of the intervals are all definable over \( \bar{a} \).

We note that a cell decomposition is uniquely described by its rank, end points, and the definable functions used in its definition.

**Theorem 11.1.5.** The class of computable real closed fields has the Mal’cev property with respect to algebraic independence.

**Proof.** We first observe that cell decompositions can be computed.

**Claim 11.4.5.** Let \( \mathcal{M} \) be a computable real closed field. There exists a uniform procedure that, given a first-order \( \phi \) defining \( X_\phi \subseteq \mathcal{M}^n \) with parameters \( \bar{a} \in \mathcal{M} \), outputs the cell decomposition of \( X_\phi \).
CHAPTER 11. NOTIONS OF INDEPENDENCE

Proof. Since \( Th(\mathcal{M}) \) admits elimination of quantifiers, the type of \( \bar{a} \) is (uniformly) computable. There exists a first-order formula stating that these points, intervals and continuous functions correspond to a cell decomposition of a \( X_\phi \). We can list all formulas of this form. Thus, we will eventually find the right formula. We will then extract the definitions of the functions and end-points from the formula.

The next claim provides us with a condition for a definable set to contain an independent tuple.

**Claim 11.4.6.** Let \( \mathcal{M} \) be a real closed field of infinite transcendence degree. Let \( X \subseteq \mathcal{M}^n \) be a definable set with parameters \( \bar{a} \). Then \( X \) contains a tuple algebraically independent over \( \bar{a} \) if and only if the cell decomposition of \( X \) contains a cell of dimension \( n \).

**Proof.** It is easy to see that any tuple in a cell of dimension strictly less than \( n \) is algebraically dependent over \( \bar{a} \). This is because such a cell must be built using, at some point, (1) or (3) from the definition of cell decomposition. If it uses (1), then this point is definable. If it uses (3), then any tuple \( (x_1, \ldots, x_n) \in X \) has \( x_i = f(x_1, \ldots, x_{i-1}) \) for some \( i \) and some function \( f \) definable over \( \bar{a} \). Then \( x_i \) is definable over \( x_1, \ldots, x_{i-1}, \bar{a} \) and hence algebraically dependent over them.

Now suppose that the cell decomposition of \( X \) contains a cell \( D \) of dimension \( n \). Suppose that the cell \( D \) is built up from an interval \((c, d)\) using functions

\[
(f_2, g_2), \ldots, (f_n, g_n).
\]

We will assume that the functions are bounded but the case when some of them are \( \pm \infty \) requires just a simple modification. Let \( b_1, \ldots, b_n \) be algebraically independent over \( \bar{a} \). We may assume that each \( b_i \) satisfies \( 0 < b_i < 1 \) (by replacing \( b_i \) by \(-b_i, b_i^{-1}, \) or \(-b_i^{-1} \) if necessary). Let \( b'_1 = c + (d-c)b_1 \); so \( b'_1 \in (c, d) \). Then, for \( i = 1, \ldots, n - 1, \) let

\[
b'_{i+1} = f_{i+1}(b'_1, \ldots, b'_i)(b_{i+1}) + g_{i+1}(b'_1, \ldots, b'_i)(1 - b_{i+1}).
\]

Note that \((b'_1, \ldots, b'_n)\) is in the open cell \( D \). Also, \( b'_i \) is interdefinable with \( b_i \) over \( c \) and \( d \), and since \( c \) and \( d \) are \( \bar{a} \)-definable, \( b_1 \) and \( b'_1 \) are interalgebraic over \( \bar{a} \). Similarly, since each \( f_i \) and \( g_i \) is \( \bar{a} \)-definable, \( b_2 \) and \( b'_2 \) are interalgebraic over \( \bar{a}, b'_1 \). In general, \( b_{i+1} \) and \( b'_{i+1} \) are interalgebraic over \( \bar{a}, b'_1, \ldots, b'_i \). Since \( b_1, \ldots, b_n \) are independent over \( \bar{a}, b'_1, \ldots, b'_n \) are independent over \( \bar{a} \).

By Theorem 11.1.2, it suffices to check that any \( \mathcal{M} \models RCF \) of infinite dimension satisfies Conditions G and B.

We now argue that every computable real closed \( \mathcal{M} \) of infinite transcendence degree satisfies Condition G. It follows from Claims 11.4.5 and 11.4.6 above that we can effectively and uniformly list the independence diagram of each \( \bar{a} \in \mathcal{M} \). It follows at once from the proof of Claim 11.4.6, where we had no restrictions on the choice of \( b_1, \ldots, b_n \) (and hence can take them to be \( u_1, \ldots, u_n \)), that independent tuples are locally indistinguishable in \( \mathcal{M} \).
We show that $\mathcal{M}$ satisfies Condition B. Let $\bar{c}$ be a tuple from $\mathcal{M}$ and let $\bar{a}$ be independent over $\bar{c}$. Then any formula $\varphi(\bar{c}, \bar{x})$ with parameters $\bar{c}$ true of $\bar{a}$ defines, by Claim 11.4.6, a $\bar{c}$-definable set of dimension $n$. Such a set contains a definable open cell whose endpoints and functions are $\bar{c}$-definable, and hence it contains some point $\bar{b} \in \mathbb{Q}(\bar{c})$. For example, if the open cell is built up from an interval $(p, q)$ using bounded functions $(f_2, g_2), \ldots, (f_n, g_n)$ then let $b_1$ be the midpoint of the interval $(p, q)$, $b_2$ the midpoint of the interval $(f_2(b_1), g_2(b_1))$, and so on. So $\bar{b} \in \text{cl}(\bar{c})$ satisfies $\varphi(\bar{c}, \bar{x})$.

11.4.4 Torsion-Free Abelian Groups

Recall that an abelian group is torsion-free if it has no non-zero elements of finite order. We cite Fuchs [Fuc70, Fuc73] for background on infinite abelian groups. The notion of independence is the usual linear independence, but the coefficients are taken from $\mathbb{Z}$. We will refer to parts of the proof below when we consider ordered abelian groups in the following section.

Theorem 11.1.6. The class of computable torsion-free abelian groups has the Mal’cev property with respect to $\mathbb{Z}$-independence.

Proof. Recall that a subgroup $H \leq A$ of an abelian group $A$ is pure if for any integer $m$ and each $h \in H$,

$$(\exists g \in G) \ mg = h \Rightarrow (\exists w \in H) \ mw = h.$$ 

It is well-known (see [Kap69]) that a finitely generated pure subgroup of an abelian group $A$ detaches as a direct summand of $A$. The $\mathbb{Z}$-dimension of an abelian group $A$ is often called the rank of $A$, but to be consistent with our notation for pregeometries we will call it the dimension of $A$, $\dim(A)$.

Let $\mathcal{M}$ a computable torsion-free abelian group of infinite dimension.

Claim 11.4.7. $\mathcal{M}$ satisfies Condition B.

Proof of Claim. Suppose $\mathcal{M} \models \exists \bar{y} \psi(\bar{c}, \bar{a}, \bar{y})$, where $\psi$ is a conjunction of linear equations and negations of linear equations. By a linear equation we mean a linear equation over $\mathbb{Z}$. We must find an element $b$ that is $\mathbb{Z}$-dependent over $\bar{c}$ and satisfies $\exists \bar{y} \psi(\bar{c}, \bar{b}, \bar{y})$.

Fix any tuple $\bar{w}$ witnessing the existential quantifier $\exists \bar{y}$. Consider the subgroup $\mathcal{X}$ of $\mathcal{M}$ generated (as a subgroup, rather than as a pure subgroup) by $\bar{c}, \bar{a}, \bar{w}$, and let $C$ be the least pure subgroup of $\mathcal{X}$ that contains $\bar{c}$. Note that $C = \text{cl}(\bar{c}) \cap \mathcal{X}$. Since $\mathcal{X}$ is finitely generated and torsion-free, it is free abelian and thus is isomorphic to a direct sum of finitely many copies of $\mathbb{Z}$ ([Fuc70]). Furthermore, since $C$ is pure in $\mathcal{X}$ and is finitely generated (since it is contained in the finitely generated group $\mathcal{X}$), it detaches as a direct summand:

$$\mathcal{X} = C \oplus W$$

for some $W$. We can choose generators $\bar{g} \bar{h}$ of $\mathcal{X}$ so that $C = \langle \bar{g} \rangle$ and $W = \langle \bar{h} \rangle$. Moreover, we may choose these generators to be linearly independent (since $C$ and $W$ decompose as
CHAPTER 11. NOTIONS OF INDEPENDENCE

327
direct sums of copies of \( \mathbb{Z} \)). From now on, we will assume that every generating set that we consider is such a linearly independent generating set.

Replace \( \bar{c}, a, \bar{y} \) in \( \psi \) by the respective linear combinations of \( \bar{g}h \) and denote the resulting formula by \( \phi(\bar{g}, \bar{h}) \). It is equal to \( \phi_0(\bar{g}, \bar{h}) \land \phi_1(\bar{g}, \bar{h}) \), where \( \phi_0(\bar{g}, \bar{h}) \) is a conjunction of \( \mathbb{Z} \)-linear equations and \( \phi_1(\bar{g}, \bar{h}) \) is a conjunction of \( \mathbb{Z} \)-linear inequations. Write \( \bar{h} = (h_1, \ldots, h_k) \) and choose any non-zero \( u \in C \). We claim that there is a natural number \( m \) such that replacing \( h_k \) by \( h'_k = mu \) preserves the validity of \( \phi \) in \( \mathcal{X} \) (and thus in \( \mathcal{M} \) since \( \phi \) is quantifier-free).

To see why this is the case, note that \( \bar{g}, \bar{h} \) is an independent set, and therefore \( \phi_0 \) must necessarily be a conjunction of trivial equations. If we replace \( h_k \) by any other value, \( \phi_0 \) will still trivially be true. Now we turn to \( \phi_1 \). Replace \( h_k \) in \( \phi_1 \) by a new indeterminate \( z \). There are only finitely many linear equations that could potentially witness the failure of the formula \( \phi_1 \) (viewed as a formula of \( z \)). Since the group is torsion-free, each of these equations has at most one solution (in \( z \)), since if a solution exists then it can be uniquely expressed as a linear combination of \( \bar{g} \) and the rest of the \( h_i \) (possibly with rational coefficients). Since \( \mathcal{M} \) and hence \( \mathcal{X} \) is torsion-free, \( (u) \cong \mathbb{Z} \) is infinite, and so there is some \( m \) such that \( mu \) is not a solution to any of these equations. We conclude that replacing \( h_k \) by \( h'_k = mu \) preserves the validity of the formula \( \phi \). We can repeat the process described above for \( h_{k-1}, h_{k-2}, \ldots \), each time replacing them by \( h'_k \in C \). So we get a tuple \( \bar{h}' \in C \) such that \( \phi(\bar{g}, \bar{h}') \) holds in \( \mathcal{X} \).

We also note that the replacement of the parameters \( \bar{c}, a, \bar{w} \) in \( \psi \) by \( \bar{g}, \bar{h} \) described above uniquely induces a replacement of the original parameters \( \bar{c}, a, \bar{w} \) in \( \psi \) by new parameters \( \bar{c}, \bar{b}, \bar{w}' \) by writing \( b \) and \( \bar{w}' \) as a linear combination of \( \bar{g}, \bar{h}' \) with the same coefficients as when we wrote \( a \) and \( \bar{w} \) as a linear combination of \( \bar{g}, \bar{h} \). Note that \( \bar{c} \) stays untouched since \( \bar{c} \in \langle \bar{g} \rangle \). This replacement preserves the validity of \( \psi \), since each elementary equation or inequation still holds after the replacement. Since \( \bar{h}' \in C \), \( b \) is dependent on \( \bar{c} \).

\[ \Box \]

Claim 11.4.8. \( \mathcal{M} \) satisfies Condition G.

Proof of Claim. We will begin by proving that independent tuples are locally indistinguishable. Suppose \( \bar{u} = (u_0, \ldots, u_n) \) and \( \bar{v} = (v_0, \ldots, v_n) \) are independent over \( \bar{c} \). We may assume that \( \bar{c} \) contains at least one non-zero element.

Suppose that \( \mathcal{M} \models \exists \bar{y}\psi(\bar{c}, \bar{u}, \bar{y}) \) where \( \psi \) is quantifier-free. Let \( \bar{w} \) be a tuple in \( \mathcal{M} \) witnessing the existential quantifier.

Let \( \mathcal{X} \) be the least subgroup of \( \mathcal{M} \) that contains \( \bar{c}, \bar{u} \) and \( \bar{w} \). Let \( C \) be the smallest pure subgroup of \( \mathcal{X} \) containing \( \bar{c} \) and \( U \) the smallest pure subgroup of \( \mathcal{X} \) containing \( \bar{u} \). The only elements of \( U \cap C \) are those which are linearly dependent over both \( \bar{c} \) and \( \bar{u} \), and since \( \bar{u} \) is independent over \( \bar{c} \) and \( \mathcal{X} \) is torsion-free, \( U \cap C = 0 \). Hence we can decompose \( \mathcal{X} \) as

\[ \mathcal{X} = C \oplus U \oplus W. \]

By a similar argument to the previous claim, using the fact that \( \bar{c} \) contains some non-zero element we can replace \( \bar{w} \) with some \( \bar{w}' \in C \oplus U \). So we may assume that

\[ \mathcal{X} = C \oplus U. \]
Let $\bar{g}$ and $\bar{h}$ be independent tuples which generate $C$ and $U$ respectively. Moreover, we may assume that $\bar{h} = (h_1, \ldots, h_n)$ and that there are integers $k_i$ such that $u_i = k_i h_i$. To see this, let $U_i$ be the pure closure of $u_i$ in $U$. Since the $u_i$ are independent, for each $i$ we have

$$U_i \cap \bigoplus_{j \neq i} U_j = 0.$$ 

Thus $U = U_1 \oplus \cdots \oplus U_n$. Each $U_i$ is isomorphic to $Z$, so we can take $h_i$ to be a generator for $U_i$.

Define $\tau : \mathcal{X} \to C \oplus \langle v_0, \ldots, v_n \rangle$ to be the unique homomorphism such that $\tau(g_i) = g_i$ and $\tau(h_j) = v_j$ for all $i$ and $j$. Obviously $\tau$ is a homomorphism, but note that it is in fact an isomorphism: it is clearly onto, and it is also one-to-one since it maps a linearly independent generating set of $\mathcal{X}$ to a linearly independent set in the image. Thus $\tau(\bar{u})$ is independent over $\mathcal{X}$ and satisfies $\exists \bar{y} \psi(\bar{c}, \tau(\bar{u}), \bar{y})$ with $\tau(\bar{w})$ being the witness for the $\exists$-quantifier. Finally, since for each $i$ we know that $u_i$ is a multiple of $h_i$, $\tau(u_i)$ is a multiple of $v_i$. Thus $\tau(u_i)$ and $v_i$ are interdependent.

We will now explain how we list the independence diagram of a tuple $\bar{c}$. We begin with some preliminary remarks. Given a formula $\phi(\bar{c}, \bar{x}) = \exists \bar{y} \psi(\bar{c}, \bar{x}, \bar{y})$ which holds for some $\bar{a}$ independent over $\bar{c}$, then we can find a finitely generated subgroup $G$ of $M$ containing $\bar{a}$ and $\bar{c}$ so that $G \models \exists \bar{y} \psi(\bar{c}, \bar{a}, \bar{y})$.

Using the same argument as above, we can write

$$G = C \oplus W,$$

where $C$ is the pure subgroup of $G$ generated by $\bar{c}$, $W$ is the pure subgroup of $G$ generated by $\bar{a}$, and the tuple $\bar{w}$ witnessing the existential quantifier belongs to $G$. Choose independent generators $\bar{g}$ and $\bar{h}$ of $C$ and $W$ respectively. Then we can write $\bar{c}$ and $\bar{a}$ as $Z$-linear combinations of $\bar{g}$ and $\bar{h}$. Let $p = |\bar{g}|$ and $q = |\bar{h}|$. Since $\bar{g}$ and $\bar{h}$ are independent, the elements of $G$ are in one-to-one correspondence with the $Z$-module $\mathbb{Z}^p \oplus \mathbb{Z}^q$ with $\bar{g}$, $\bar{h}$ mapping to the standard basis elements. We call the image of some $g \in G$ in $\mathbb{Z}^p \oplus \mathbb{Z}^q$ the formal representation of $g$. Then the formal representations of $\bar{a}$ are $Z$-linearly independent over the formal representations of $\bar{c}$, and hence over $\mathbb{Z}^p$.

Observe that there is a finite partial subgroup $H \subseteq G$ containing $\bar{c}$ and $\bar{a}$ such that $H \models \exists \bar{y} \psi(\bar{c}, \bar{a}, \bar{y})$ and such that $H$ contains only elements of the form

$$\sum_i m_i g_i + \sum_j n_j h_j,$$

with $|m_i|, |n_j| \leq k$ for some $k$. By a partial group, we mean that $H$ is not necessarily closed under the group operations, but that the operations on $H$ agree with those on $G$ where possible. By $H \models \exists \bar{y} \psi(\bar{c}, \bar{a}, \bar{y})$, we mean that there is a tuple $\bar{w}$ in $H$ such that $\psi(\bar{c}, \bar{a}, \bar{w})$ holds with all of the intermediate terms occurring in $H$ (e.g., if $\psi$ is $c + a - 2w = 0$, then all of the terms $c + a$, $2w$, and so on appear in $H$). This partial group is isomorphic to a direct sum of partial additive groups $\mathbb{Z}/k$ upon $\{-k, \ldots, -1, 0, 1, \ldots, k\}$, since the $g_i$ and $h_j$
are independent (we believe that the notion of a direct sum is self-explanatory when applied to partial structures).

We claim that the formula \( \exists \bar{y} \psi(\bar{c}, \bar{x}, \bar{y}) \) is in the independence diagram of \( \bar{c} \) if and only if there exists a finite partial subgroup \( H \) containing \( \bar{c} \) and a tuple \( \bar{a} \) such that \( H \models \exists \bar{y} \psi(\bar{c}, \bar{a}, \bar{y}) \) and \( H \) has the form as described above, i.e.:

1. \( H \) is direct sum of partial subgroups \( C \) and \( W \) generated by \( \bar{g} \) and \( \bar{h} \) as above,
2. \( \bar{c} \in C, \bar{a}, \bar{w} \in H \),
3. both \( C \) and \( W \) are direct sums of partial groups of the form \( \mathbb{Z}/k \) for some \( k \), and
4. the formal representations of \( \bar{a} \) are linearly independent over the formal representations of \( \bar{c} \).

We have already checked that such a partial subgroup \( H \) exists for any \( \exists \bar{y} \psi(\bar{c}, \bar{x}, \bar{y}) \) in the independence diagram of \( \bar{c} \). Conversely, suppose such a finite partial \( H \) exists. Since the dimension of \( M \) is infinite, we can find a tuple \( \bar{u} \) in \( M \) independent over \( \bar{g} \) (and thus, over \( \bar{c} \)). The map \( h_j \to u_j \) can be uniquely extended, via linear combinations, to an isomorphic embedding \( \tau \) of \( H \) into \( M \) that fixes \( \bar{g} \) componentwise. Since \( \psi \) is quantifier-free, we have \( M \models \psi(\bar{c}, \tau(\bar{a}), \tau(\bar{w})) \) is preserved under this embedding. Furthermore, since the formal representation of \( \bar{a} \) with respect to \( \bar{g}, \bar{h} \) is the same as that of \( \tau(\bar{a}) \) with respect to \( \bar{g}, \bar{u} \), since \( \bar{u} \) is independent over \( \bar{g} \), we conclude that the image of \( \bar{a} \) is independent over \( \bar{c} \) as desired.

It remains to apply Theorem 11.1.2.

11.4.5 Archimedean Ordered Abelian Groups

Recall that an ordered abelian group \( A \) is Archimedean if for every non-zero \( a, b \in A \) there exists an \( m \in \mathbb{Z} \) such that \( m|a| > |b| \) and \( m|b| > |a| \), where \( |x| = x \) if \( x > 0 \) and \( |x| = -x \) otherwise. See Kokorin and Kopytov [KK74] and Fuchs [Fuc63] for more algebraic background on ordered groups.

**Theorem 11.1.7.** The class of computable Archimedean ordered abelian groups has the Mal'cev property with respect to \( \mathbb{Z} \)-independence.

**Proof.** Suppose \( M \) is a computable Archimedean ordered abelian group of infinite dimension. We will use properties of the ordered field \( \mathbb{R} \) throughout by embedding \( M \) into \( \mathbb{R} \).

**Claim 11.4.9.** \( M \) satisfies Condition B.

**Proof.** Suppose \( M \models \exists \bar{y} \psi(\bar{c}, a, \bar{y}) \), where \( a \) is \( \mathbb{Z} \)-independent over \( \bar{c} \). We also assume \( \bar{c} \) contains at least two linearly independent elements. Fix any tuple \( \bar{w} \) witnessing \( \exists \bar{y} \). As in the proof for torsion-free abelian groups, consider the free abelian group \( \mathcal{X} \) spanned by \( \bar{c}, a, \bar{w} \) and let \( C \) be the least pure subgroup of \( \mathcal{X} \) that contains \( \bar{c} \). We have

\[ \mathcal{X} = C \oplus W \]
(group-theoretically) and thus we can choose generators \( \bar{g}h \) of \( \mathcal{X} \) so that \( C = \langle \bar{g} \rangle \) and \( W = \langle \bar{h} \rangle \). Choose a formula \( \phi(\bar{g}, \bar{h}) \) by replacing \( \bar{c}, a, \bar{w} \) with \( \bar{g}, \bar{h} \) as we did for torsion-free abelian groups.

We use the well-known fact that every Archimedean ordered abelian group can be isomorphically embedded into the ordered group of reals \( (\mathbb{R}, +, \leq) \) [Höl96]. We identify \( \mathcal{X} \) with its image under this embedding. Let \( Y \) be the subset of \( \mathbb{R}^n \) isolated by \( \phi(\bar{g}, \bar{x}) \). Since \( \bar{g}, \bar{h} \) are independent, the formula \( \phi \) can contain only trivial linear equations in \( \bar{g}, \bar{h} \). Thus, each component of \( \bar{h} \) is contained in \( Y \) together with some interval (similar to Claim 11.4.6). It is also well-known [AB98, Exercise 21 of Section 4 of] that if \( H \subseteq (\mathbb{R}, +) \) and \( \dim(H) \geq 2 \), then \( H \) is dense in \( (\mathbb{R}, <) \). By our assumption, \( \dim(C) \geq 2 \), so we can choose \( \bar{h}' \in C \) which are contained in these intervals and hence satisfy \( \phi(\bar{g}, \bar{h}') \). Then, as with unordered groups, we can find \( b \) and \( \bar{w}' \) which are linear combinations of \( \bar{g} \) and \( \bar{h} \) which satisfy \( \psi(\bar{g}, b, \bar{w}') \). \( \square \)

Claim 11.4.10. \( \mathcal{M} \) satisfies Condition G.

Proof. We first show that independent tuples in \( \mathcal{M} \) are locally indistinguishable. Suppose \( \bar{u} = (u_0, \ldots, u_n) \) and \( \bar{v} = (v_0, \ldots, v_n) \) are independent over \( \bar{c} \). We assume that the dimension of \( \bar{c} \) is at least 1. Let \( \theta \) be an existential formula such that \( \mathcal{M} \models \theta(\bar{c}, \bar{u}) \). We show that there exists a tuple \( \bar{z} = (z_0, \ldots, z_n) \in \operatorname{cl}(\bar{c}, \bar{v}) \) independent over \( \bar{c} \) such that \( \mathcal{M} \models \theta(\bar{c}, \bar{z}) \).

As we have already noted above, \( \mathcal{M} \) can be identified with a subgroup of \( \mathbb{R} \). Since \( \bar{u} \) is independent over \( \bar{c} \), there exists an open neighborhood \( U \) of \( \bar{u} \) in the respective power of \( \mathbb{R} \) such that every tuple from \( U \) satisfies \( \theta \). The argument is essentially the same as before. We can as before pass to the smallest pure subgroup of \( \mathcal{M} \) containing \( \bar{c}, \bar{u} \) and the witnesses for the existential quantifier. Then we re-write the formula in terms of the generators of this finitely generated free abelian group. We then conclude that all the equations in the formula must become trivial after the re-writing, and thus each generator is contained in a definable open interval such that any choices of elements from these intervals satisfy the formula. Since \( \bar{u} \) is a linear combination of the generators, it is also contained in a definable open set \( U \) around with the property that every tuple from \( U \) satisfies \( \theta(\bar{c}, x) \).

Since \( \bar{v} \) is independent over \( \bar{c} \), for every \( i \) and any non-zero \( c \in \operatorname{cl}(\bar{c}) \) the set \( \{ sc + tv_i : s, t \in \mathbb{Z} \} \) is dense in \( (\mathbb{R}, <) \) since it has dimension at least 2. Thus, for an arbitrary non-zero \( c \in \operatorname{cl}(\bar{c}) \) we can find \( s_i, t_i \) such that the tuple \( \bar{z} = (z_0, \ldots, z_n) \) with

\[
z_i = s_i c + t_i v_i, \quad i = 0, \ldots, n,
\]

belongs to \( U \). Then \( \bar{z} \) satisfies the desired properties.

Now we will describe a method of enumerating the independence diagram of \( \bar{c} \in \mathcal{M} \). Every existential formula \( \theta(\bar{c}, \bar{x}) \) in the independence diagram of \( \bar{c} \) is witnessed by a quantifier-free formula \( \psi(\bar{c}, \bar{a}, \bar{w}) \) in the open diagram of \( \mathcal{M} \), where \( \bar{a} \) is independent over \( \bar{c} \) and where \( \bar{w} \) are the witnesses to the existential quantifier from \( \theta \).

We follow the second half of the proof of Claim 11.4.8 closely. We claim that \( \theta(\bar{c}, \bar{a}) \) is in the independence diagram of \( \bar{c} \) if and only if \( \psi(\bar{c}, \bar{a}, \bar{w}) \) is satisfied in a finite partial ordered subgroup \( H \) of \( \mathcal{M} \) that:
11.5 Conclusion

We suspect that our metatheorem holds for many other algebraic classes and classes of theories. For instance, the theory $T_{\text{exp}}$ of $\mathbb{R}$ as a field with the exponential function would be an appropriate candidate. Wilkie and Macintyre [MW96] showed that, assuming that Schanuel’s conjecture\(^3\) is true, $T_{\text{exp}}$ is a decidable theory. This is still unresolved. Recently, Jones and Servi [JS11] and Miller [Mil] gave examples of decidable theories expanding the theory of real closed fields. We suspect that the corresponding algebraic classes might have the Mal’cev property. We also conjecture that some other field-like structures perhaps including algebraically closed valued fields and the like [HHM08] have the Mal’cev property.

Although we did not include the case of an ordered abelian group with finitely many Archimedean classes, we conjecture that our methods can be applied to simplify the proof in [GLS03]. Our ideas may be useful in covering the case of infinitely many Archimedean classes (this is an open problem [GLS03]), but perhaps some adjustments and new ideas will be necessary.

We also conjecture that our metatheorem can be applied to arbitrary computable abelian groups with respect to $\mathbb{Z}$-independence (note that torsion elements are “dependent on themselves”), see [Gon80a, Khi98].

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\(^3\)Schanuel’s conjecture is as follows: suppose that $n \geq 1$ and $c_1, \ldots, c_n \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$; then $e^{c_1}, \ldots, e^{c_n}$ have transcendence degree at least $n$ over $\mathbb{Q}$. 

We suspect that our metatheorem has relativized and generalized versions that could be applied to, say, completely decomposable groups [DM14, Khi02] and other structures where the notions of independence are not r.i.c.e. but are relatively intrinsically $\Sigma^0_n$ for some $n > 1$.

We also note that the spiritually related $p$-basic tree problem [AK00, Mel14] seems to require new ideas since the corresponding notion of independence is not a pregeometry. Finally, we would like to find some non-trivial applications to non-commutative structures.
Chapter 12

Computable Valued Fields

The results presented in this chapter appeared in [HTa].

12.1 Introduction

Recently there has been interest in studying, from the perspective of computability theory, various types of fields which arise in model theory. Marker and Miller [MM] studied the degree spectra of differentially closed fields, while Miller, Ovchinnikov, and Trushin [MOT14] have looked at generalizations of splitting algorithms for differential fields. Real closed fields have been studied by Calvert [Cal04], Ocasio [Oca14], Knight and Lange [KL13], and Igusa, Knight, and Schweber [IKS]. Generalizations to difference fields of Rabin’s theorem on embeddings into algebraic closures have been studied by Melnikov, Miller, and the author [HTMMa]. This article is a study of valued fields from the perspective of computable algebra. Variations of Rabin’s theorem for valued fields were previously studied by Smith [Smi81]; some of our results extend those of that paper.

Definition 12.1.1. A valued field is a field $K$ together with a valuation $v$ on $K$, that is, a map $K \to \Gamma \cup \{\infty\}$ from $K$ to an ordered abelian group $\Gamma$, such that

1. $v(x) = \infty$ if and only if $x = 0$,
2. $v(xy) = v(x) + v(y)$, and
3. $v(x + y) \geq \min(v(x), v(y))$ (with equality if $v(x) \neq v(y)$).

$\Gamma$ is called the value group. We will always assume that the valuation is surjective.

Standard examples of valued fields are the $p$-adic valuations on $\mathbb{Q}$ and their completions, the $p$-adic fields $\mathbb{Q}_p$.

In computable algebra, we consider computable presentations of algebraic structures. A computable valued field is a field whose underlying domain is a computable set $K \subseteq$
\(\omega\), equipped with computable functions \(+_K\) and \(\times_K\) giving the addition and multiplication operations, and with a computable valuation, i.e. a computable function \(v : K \to \Gamma\) where \(\Gamma\) is a computable group (a computable subset of \(\omega\) with a computable group operation). There are a number of equivalent ways of presenting a valued field (see Section 12.2.3), but this method is most faithful to the classical definition of a valued field. Two computable valued fields may be classically isomorphic but not computably isomorphic.

One objective of computable algebra is to see which classical theorems hold in the effective setting, considering only computable objects. For example, it is a classical result that every valued field embeds into an algebraically closed valued field. The same is true in the effective setting: every computable valued field effectively embeds into a computable algebraically closed valued field. Similarly, every valued field has a Henselization, and every computable valued field effectively embeds into a computable presentation of its Henselization.

On the other hand, a slight variation of this does not hold. If we fix an embedding of a valued field \((\mathbb{K}, v)\) into its algebraic closure \((\overline{\mathbb{K}}, w)\) with an extension of the valuation, the Henselization of \(\mathbb{K}\) in \(\overline{\mathbb{K}}\) is unique. In the effective setting, we assume that these fields \((\mathbb{K}, v)\) and \((\overline{\mathbb{K}}, w)\) are computable and that the embedding is effective. In this case, we cannot compute the Henselization of \(\mathbb{K}\) inside of \(\overline{\mathbb{K}}\), even if we assume that \(\mathbb{K}\) has a splitting algorithm (an algorithm for finding the minimal polynomial over \(\mathbb{K}\) of an element of \(\overline{\mathbb{K}}\), or equivalently, for deciding which elements of \(\overline{\mathbb{K}}\) are actually in \(\mathbb{K}\)). Thus there is no effective criteria to decide, for a given \(a \in \overline{\mathbb{K}}\), and using only the minimal polynomial of \(a\) over \(\mathbb{K}\) and the valuations of various elements, whether or not \(a\) is in the Henselization of \(\mathbb{K}\).

### 12.1.1 Extending Valuations

In [HTMMa] the author, together with Melnikov and Miller, considered the problem of extending an automorphism of a field \(F\) to an automorphism of an algebraic extension \(K\) of \(F\) (with a fixed computable embedding of \(F\) in \(K\)). In this article, we consider the related problem of extending a valuation of \(F\) to a valuation of \(K\). Smith [SmI81] proved several results along these lines, most importantly that every valued field embeds into an algebraically closed field with an extension of the valuation, but that one cannot do this with a fixed embedding into a fixed algebraically closed field. Our main result is as follows:

**Theorem 12.1.2.** Let \((\mathbb{K}, v)\) be a computable algebraic valued field. Then the following are equivalent:

1. for every computable embedding \(\iota : \mathbb{K} \to \mathbb{L}\) of \(\mathbb{K}\) into a field \(\mathbb{L}\) algebraic over \(\mathbb{K}\), there is a computable extension of \(v\) to a computable valuation \(w\) on \(\mathbb{L}\),
2. the Hensel irreducibility set

\[
H_K := \{ f = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 \in O_K[x] : f \text{ is irreducible over } \mathbb{K}, \, v(a_{n-1}) = 0, \text{ and } v(a_{n-2}), \ldots, v(a_0) > 0 \}
\]

of \((\mathbb{K}, v)\) is computable.
12.1.2 $p$-adically Closed Fields

Among the most important examples of valued fields are the $p$-adics $\mathbb{Q}_p$. The theory of $p$-adically closed fields is the theory of $\mathbb{Q}_p$. Just as the theory of real closed fields is the model companion of the formally real fields, the theory of $p$-adically closed fields is the model companion of a class of fields called the formally $p$-adic fields. Classically, every formally $p$-adic embeds into a $p$-adic closure. The effective analogue is false:

**Theorem 12.1.3.** There is a computable formally $p$-adic field which does not embed into a computable $p$-adic closure.

The issue is that we can construct a formally $p$-adic field in which the divisibility relation on the value group is not computable. If we have an algorithm to compute the divisibility relation on the value group of a formally $p$-adic field, then we can effectively embed that field into a computable $p$-adic closure.

**Theorem 12.1.4.** Let $(K, v)$ be a computable formally $p$-adic valued field with value group $\Gamma$. Suppose that we can compute, for each $\gamma \in \Gamma$ and $k \in \mathbb{N}$, whether $\gamma$ is divisible by $k$. Then there is a computable embedding of $K$ into a computable $p$-adic closure $(L, w)$.

12.1.3 Copies with Computable and Non-Computable Transcendence Bases

Many algebraic structures admit a notion of independence, such as algebraic independence in field, linear independence in vectors spaces, $\mathbb{Z}$-linear independence in abelian groups, and differential independence in differential fields. In the 1960’s, Mal’cev noticed that there are two non-computably-isomorphic computable presentations of the infinite-dimensional $\mathbb{Q}$-vector space, one with a computable basis, and the other with no computable basis, and that the two were $\Delta^0_2$-isomorphic. Many other structures have been found to have the same property, such as algebraically closed fields, torsion-free abelian groups [Nur74, Dob83, Gon82], Archimedean ordered abelian groups [GLS03], differentially closed fields, real closed fields, and difference closed fields [HTMM15]. In [HTMM13], the author together with Melnikov and Montalbán formally characterized this phenomenon (which they named the Mal’cev property) using the notion of a r.i.c.e. pregeometry, and presented a metatheorem unifying all of these examples. Here we will apply the metatheorem to algebraically closed valued fields and $p$-adically closed valued fields.

**Theorem 12.1.5.** Every computable algebraically closed valued field or $p$-adically closed valued field $K$ of infinite transcendence degree has a computable copy $G \cong_{\Delta^0_2} K$ with a computable transcendence base and a computable copy $B \cong_{\Delta^0_2} K$ with no computable transcendence base.

Note that by a theorem of Goncharov [Gon82], every such structure has computable dimension $\omega$. 

12.2 Preliminaries

12.2.1 Splitting Algorithms

Recall that the splitting set $S_F$ of $F$ is the set of all polynomials $p \in F[X]$ which are reducible over $F$. The splitting set of a field is not necessarily computable (see [Mil08, Lemma 7]), but it is always c.e. If the splitting set of $F$ is computable, then we say that $F$ has a splitting algorithm. Finite fields and algebraically closed fields trivially have splitting algorithms. Kronecker [Kro82] showed that $\mathbb{Q}$ has a splitting algorithm, and also that many other field extensions also have splitting algorithms:

**Theorem 12.2.1** (Kronecker [Kro82]; see also [vdW70]). The field $\mathbb{Q}$ has a splitting algorithm. If a computable field $F$ has a splitting algorithm, and $a$ is transcendental over $F$ (or separable and algebraic over $F$), then $F(a)$ has a splitting algorithm. Moreover, in the case that $a$ is algebraic over $F$, the splitting algorithm for $F(a)$ can be found uniformly in the splitting algorithm for $F$ and the minimal polynomial of $a$ over $F$. If $a$ is transcendental over $F$, then the splitting algorithm can be found uniformly in the splitting algorithm for $F$.

Given a field $F$ with a splitting algorithm and an element $a$ which is either transcendental over $F$, or separable and algebraic over $F$, we know that $F(a)$ has a splitting algorithm. However, the algorithm depends on whether $a$ is transcendental or algebraic. To find a splitting algorithm uniformly, we must know which is the case.

Rabin [Rab60] showed that every computable field $F$ has a computable algebraic closure $\overline{F}$, and moreover there is a computable embedding $\iota: F \to \overline{F}$. We call such an embedding a Rabin embedding. Moreover, he characterized the image of $F$ under this embedding:

**Theorem 12.2.2** (Rabin [Rab60]). Let $F$ be a computable field. Then there is a computable algebraically closed field $\overline{F}$ and a computable field embedding $\iota: F \to \overline{F}$ such that $\overline{F}$ is algebraic over $\iota(F)$. Moreover, for any such $\overline{F}$ and $\iota$, the image $\iota(F)$ of $F$ in $\overline{F}$ is Turing equivalent to the splitting set of $F$.

12.2.2 Valued Fields

The valuation ring $\mathcal{O}_{K,v}$ of $K$ is the subring consisting of all elements $a$ with $v(a) \geq 0$. $\mathcal{O}_{K,v}$ is a local ring with maximal ideal $m_{K,v} = \{x : v(x) > 0\}$. The residue field $k_{K,v}$ is the quotient $\mathcal{O}_{K,v}/m_{K,v}$. When the valuation $v$ is clear from the context, we write $\mathcal{O}_K$, $m_K$, and $k_K$. Given $a \in \mathcal{O}_K$, we denote by $\bar{a}$ its image in the residue field. For a comprehensive reference on valued fields, see [EP05].

**Definition 12.2.3.** A valued field $(K,v)$ is Henselian if it satisfies one of the following equivalent properties (see [EP05, Theorem 4.1.3]):

1. $v$ has a unique extension to every algebraic extension $L$ of $K$, 

...
(2) given \( f \in \mathcal{O}_K[x] \) and \( a \in \mathcal{O}_K \) such that \( v(f(a)) > 2v(f'(a)) \), there is a unique \( b \in \mathcal{O} \) such that \( f(b) = 0 \) and \( v(a - b) > v(f'(a)) \),

(3) given \( f \in \mathcal{O}_K[x] \) and \( a \in \mathcal{O}_K \) such that \( f(\bar{a}) = 0 \) and \( f'(\bar{a}) \neq 0 \), there is a \( b \in \mathcal{O}_K \) with \( f(b) = 0 \) and \( \bar{a} = b \),

(4) every polynomial \( x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 \in \mathcal{O}_K[x] \) with \( v(a_{n-1}) = 0 \) and \( v(a_{n-2}), \ldots, v(a_0) > 0 \) has a solution in \( K \).

Every valued field has a Henselization, that is, a minimal Henselian field into which it embeds. The Henselization of a field is algebraic over that field, and every Henselization of a given field is isomorphic. Moreover, after fixing an embedding of the field into its algebraic closure, the Henselization is unique. We denote by \( K^h \) the Henselization of a field \( K \).

If \((L, w)\) is a valued field extension of \((K, v)\), then we may view the value group \( \Gamma_K \) as a subgroup of \( \Gamma_L \) and the residue field \( k_K \) as a subfield of \( k_L \). We call \( e(w/v) = [\Gamma_L : \Gamma_K] \) the ramification index of the extension and \( f(w/v) = [k_L : k_K] \) the residue degree of the extension. An extension is called immediate if the ramification index and the residue degree are both 1. If we consider a field \( L \) which is an extension (as a field) of the valued field \((K, v)\), we can ask about extensions of \( v \) to \( L \). There may in general be many possible extensions, but the number is limited by the degree \([L : K]\) of the extension according to the following theorem.

**Theorem 12.2.4** (Theorems 3.3.4 and 3.3.5 of [EP05]). Let \( L/K \) be a finite extension of fields and \( v \) a valuation on \( K \). Let \( w_1, \ldots, w_n \) be the distinct extensions of \( v \) to \( L \). Then

\[
\sum_{i=1}^{n} e(w_i/v) f(w_i/v) \leq [L : K].
\]

If the extension \( L/K \) is separable and the value group of \( K \) is \( \mathbb{Z} \), then we have equality.

This inequality is known as the fundamental inequality. In the case that we have equality, i.e., when the extension is separable and the value group is \( \mathbb{Z} \), we call this the fundamental equality. All of the extensions \( w_1, \ldots, w_n \) in the theorem are conjugate by an automorphism of \( L \) over \( K \).

The following theorem will allow us to represent extensions of a valuation across a finite extension \( L/K \) of fields by elements of \( L \). It is a restatement of Theorem 3.2.7 (3) of [EP05] for finite extensions of fields, using Lemma 3.2.8 to see that the hypotheses of Theorem 3.2.7 can be simplified in this case.

**Theorem 12.2.5.** Let \( L/K \) be a finite extension of fields and \( v \) a valuation on \( K \). Let \( w_1, \ldots, w_n \) be distinct valuations on \( L \) extending \( v \). Then given \( a_1, \ldots, a_n \in L \) such that \( w_i(a_i) \geq 0 \) for all \( i \), there is \( a \in L \) such that \( w_i(a) \geq 0 \) for all \( i \) and \( w_i(a - a_i) > 0 \) for all \( i \).

Let \((K, v)\) be a valued field. If \((K, v)\) has no proper separable immediate extensions, then \( K \) is Henselian. We call such a \( K \) algebraically maximal. The converse is only true if
K is finitely ramified: if the residue field has characteristic zero, or if it has characteristic p and there are only finitely many elements of the value group between 0 and 1 = v(p).

**Theorem 12.2.6** (Theorem 4.1.10 of [EP05]). Suppose that (K, v) is finitely ramified. Then (K, v) is Henselian if and only if it is algebraically maximal.

### 12.2.3 Computable Valued Fields

There are many natural languages in which to talk about valued fields [Cha11]. Three of them are:

1. Macintyre’s language \( \mathcal{L}_{\text{div}} \) which adds a binary relation \( a \mid b \) to the ring language, with \( a \mid b \) interpreted as \( v(a) \leq v(b) \).

2. Robinson’s two-sorted language \( \mathcal{L}_{\text{Rob}} \) which has a sort for the value group (as an ordered group) and contains the valuation function \( v : K \to \Gamma \cup \infty \).

3. The three-sorted language \( \mathcal{L}_{\Gamma,k} \) which extends \( \mathcal{L}_{\text{Rob}} \) by adding the residue field and residue map.

A computable valued field is a computable field (i.e., the domain is a computable set, and the operations of addition and multiplication are computable) together with a computable valuation. By this we mean, in \( \mathcal{L}_{\text{div}} \), that the relation \( a \mid b \) is computable; in \( \mathcal{L}_{\text{Rob}} \), that there is a computable group \( \Gamma \) and that the valuation map \( v \) is computable; and in \( \mathcal{L}_{\Gamma,k} \), that in addition the residue field \( k \) and the residue map are computable. It follows from the proof of the following proposition that all three ways of presenting a valued field are effectively bi-interpretable (see [HTMMM]), and hence it does not matter which we choose.

**Proposition 12.2.7.** Let \( (K, v) \) be a computable valued field in the language \( \mathcal{L}_{\text{div}} \). There is a computable presentation \( \Gamma \) of the value group of \( K \) and a computable presentation \( k \) of the residue field of \( K \) so that the valuation map \( v : K \to \Gamma \) and the reduction map \( \mathcal{O}_K \to k \) are computable.

**Proof.** The value group \( \Gamma \) is the quotient of \( K^\times \) by the computable equivalence relation

\[
a \sim b \iff (a \mid b) \land (b \mid a).
\]

The group operation is given by \([a] + [b] = [ab]\). The ordering on the value group is that induced by \( a \mid b \). The valuation map \( v : K \to \Gamma \) is just the quotient map.

We can compute, inside \( K \), the valuation ring \( \mathcal{O}_K \). The residue field is the quotient of \( \mathcal{O}_K \) by its maximal ideal \( m = \{ a \in \mathcal{O}_K : v(a) > 0 \} \). So we can present the residue field as a quotient of the valuation ring by the computable equivalence relation

\[
a \sim b \iff v(a - b) > 0.
\]

\( \square \)
12.2.4 Algebraically Closed Valued Fields

The theory ACVF of algebraically closed valued fields is axiomatized by saying that \((K, v)\) is a valued field which is algebraically closed as a field (and recalling that we assumed that the valuation map is surjective). For a reference on algebraically closed valued fields, see [Cha11]. The theory is complete (after naming the characteristic and the characteristic of the residue field), decidable, and admits quantifier elimination. ACVF is the model completion of the theory of valued fields.

12.2.5 \(p\)-adically Closed Valued Fields

A valued field \((K, v)\) extending \(\mathbb{Q}\) is formally \(p\)-adic if:

1. \(v\) extends the \(p\)-adic valuation on \(\mathbb{Q}\),
2. the residue field is \(\mathbb{F}_p\), and
3. \(v(p)\) is the least positive element of the value group.

\(K\) is \(p\)-adically closed if in addition:

4. \(K\) is Henselian and
5. the value group is elementarily equivalent to \(\mathbb{Z}\), i.e., a model of Presburger arithmetic.\(^1\)

This axiomatizes the complete theory \(p\text{CF}\) of \(p\)-adically closed fields, which is the theory of the \(p\)-adics \(\mathbb{Q}_p\). See [PR84] for a reference on formally \(p\)-adic fields.

In a formally \(p\)-adic field, we can identify \(\mathbb{Z}\) with the convex subgroup of the value group \(\Gamma\) generated by \(v(p)\). The coarse valuation \(\bar{v}\) is the composition of \(v\) with the quotient map \(\Gamma \to \Gamma/\mathbb{Z}\). Then \(\Gamma\) is elementarily equivalent to \(\mathbb{Z}\) if and only if \(\Gamma/\mathbb{Z}\) is divisible. We call \(\Gamma/\mathbb{Z}\) the coarse value group.

Every formally \(p\)-adic field embeds into a \(p\)-adic closure, that is, an algebraic extension which is \(p\)-adically closed. The \(p\)-adic closure is not necessarily unique. The theory \(p\text{CF}\) is the model companion of the theory of formally \(p\)-adic fields, and hence every formula is equivalent, modulo \(p\text{CF}\), to an existential formula. In fact, \(p\text{CF}\) eliminates quantifiers after adding the predicate \(P_n\) which picks out the \(n\)th powers [Mac76]. Thus the elementary diagram of any computable model of \(p\text{CF}\) is decidable. We denote by \(P_n^*\) the non-zero \(n\)th powers. The theory \(p\text{CF}\) also admits definable Skolem functions [vdD84]. Finally, there is a cell decomposition theorem for definable sets in a \(p\)-adically closed field (see [Den86, SvdD88, Mou09]).

**Definition 12.2.8.** The collections of cells in \(K\) is defined recursively by:

\[1\] The models of Presburger arithmetic are the discrete ordered abelian semigroups with a zero and a least element 1, such that for all \(x\) and \(n\) there is \(y\) such that \(x = ny + r\) for some \(r = 0, \ldots, n - 1\).
(1) If $X$ is a single point in $K^n$, then $X$ is a $(0)$-cell.

(2) If $\square_1$ and $\square_2$ are either $<, \leq$, or no condition, $\gamma_1, \gamma_2 \in \nu(K) \cup \{-\infty, \infty\}$, $c \in K$, $k \in \omega$, and $\lambda \in K^*$, then
\[
\{x \in K : \gamma_1 \square_1 \nu(x - c) \square_2 \gamma_2 \text{ and } P_k^*(\lambda(x - c))\}
\]
is a $(1)$-cell.

(3) If $f$ is a definable continuous function from a $(i_1, \ldots, i_n)$-cell $C$ to $K$, then the graph of $f$ is a $(i_1, \ldots, i_n, 0)$-cell.

(4) If $B$ is a $(i_1, \ldots, i_n)$-cell, $f$, $g$, and $h$ are definable continuous functions from $B$ to $K$, $\lambda \in K^*$, and $\square_1$ and $\square_2$ are either $<, \leq$, or no condition, then
\[
C = \{(\bar{x}, y) \in B \times K : \nu(f(\bar{x})) \square_1 \nu(y - g(\bar{x})) \square_2 \nu(h(\bar{x})) \text{ and } P_k^*(\lambda(y - g(\bar{x})))\}
\]
is a $(i_1, \ldots, i_n, 1)$-cell.

**Theorem 12.2.9** (Cell decomposition for $pCF$). Let $(K, v)$ be a $p$-adically closed valued field. Let $S \subseteq K^n$ be a definable set. Then $S$ can be partitioned into finitely many cells. Moreover, the parameters over which the cells are defined are all definable over the parameters of $S$.

### 12.3 Extending Valuations

We begin this section by showing that we can effectively embed valued fields into their Henselizations and into algebraically closed valued fields. This result appeared in [Smi81] and we repeat the proof here as we will later build on these ideas.

**Proposition 12.3.1** (Theorem 3 of [Smi81]). Let $(K, v)$ be a computable valued field. There is a computable embedding of $K$ into a computable presentation $\overline{K}$ of its algebraic closure and a computable extension of $v$ to $\overline{K}$.

**Proof.** If $v$ is the trivial valuation, then extend it to the trivial valuation on $\overline{K}$ under any computable embedding of $K$ into its algebraic closure. Otherwise, the theory $ACVF \cup \text{Diag}_{\text{sat}}(K)$ is complete, hence decidable. So it has a computable model $(L, w)$ by an effective Henkin construction (see, for example, [Har98]), and we get a computable embedding of $K$ into $L$ by mapping $x \in K$ to the interpretation of the constant representing $x$ in $L$. In $L$, we can enumerate the algebraic closure $\overline{K}$ of $K$ and hence construct a computable presentation. \(\square\)

A consequence of this is that every computable non-trivially-valued field $K$ embeds into a model of $ACVF$ whose underlying field is algebraic over $K$. 

Lemma 12.3.2. Let \((K,v)\) be a computable finite extension of valued fields of \(\mathbb{Q}\) with the \(p\)-adic valuation. Given \(K(a)\) a finite field extension of \(K\), we can compute a list of all of the extensions of \(v\) to \(K(a)\), with no duplication, as well as the ramification indices and residue degrees of these extensions. We can also compute the residue fields and the value groups of these extensions as subsets of \(\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}\) and \(\mathbb{Q} = v(\mathbb{Q})\) respectively. This computation is uniform in the generators for \(K\) over \(\mathbb{Q}\).

**Proof.** We argue by induction on the number of generators of \(K\). Since we know the generators for \(K\), \(K\) has a splitting algorithm. Embedding \((K,v)\) into \((\overline{K},w) = (\overline{\mathbb{Q}},w)\) via the previous lemma, we can compute the image of \(K\) in \(\overline{K}\). We can compute the minimal polynomial of \(a\) over \(K\), and use it to find the embeddings of \(K(a)\) into \(\overline{K}\) over \(K\). By restricting \(w\) to \(K(a)\) under each of these embeddings, we get a list of the possible extensions of \(v\) to \(K(a)\), possibly containing duplicates.

Given \(u_1, \ldots, u_n\) valuations on \(K(a)\) extending \(v\), we claim that we can tell in a c.e. way that they are a complete list, without duplicates, of the extensions of \(v\) to \(K(a)\). To see that there are no duplicates in the list, we just have to find elements of \(K(a)\) on which they differ. Since \(K\) is a finite extension of \(\mathbb{Q}\), \(v(K) \cong \mathbb{Z}\), and so by Theorem 12.2.4, if \(u_1, \ldots, u_n\) is a complete list of the extensions of \(v\) to \(K(a)\), then

\[
\sum_{i=1}^{n} e(u_i/v) f(u_i/v) = [K(a):K].
\]

Note that we can compute \([K(a):K]\) using the splitting algorithm for \(K\). Inductively, we can compute the value group and residue field of \(K\) as subsets of the value group \(\mathbb{Q}\) and the residue field \(\mathbb{F}_p\) of \(\mathbb{Q}\) respectively. Since they are finitely generated substructures and we know the residue degree and ramification index of \((K,v)\) over \(\mathbb{Q}\), we can compute finite sets of generators for the value group and residue field of \((K,v)\). So for each valuation \(u\) from among \(u_1, \ldots, u_n\), we can compute the value group and residue field of \(u\) as c.e. subsets of \(\mathbb{Q}\) and \(\mathbb{F}_p\). So we can compute increasing sequences with limits \(e(u/v)\) and \(f(u/v)\).

We always have, for any such list with no duplication,

\[
\sum_{i=1}^{n} e(u_i/v) f(u_i/v) \leq [K(a):K].
\]

So \(u_1, \ldots, u_n\) is a complete list if and only if the increasing approximations to \(e(u_i/v)\) and \(f(u_i/v)\) we computed above eventually give equality.

When we compute, in this way, a complete list of the extensions of \(v\) to \(K(a)\), we also get their ramification indices and residue degrees. Using these values, we can compute the value groups and residue fields of these extensions as subsets of \(\mathbb{Q}\) and \(\mathbb{F}_p\). \(\square\)

Let \((K,v)\) be a computable valued field with a splitting algorithm. Given an element \(a\) algebraic over \(K\), one can use Newton polygons to decide what possible valuations \(a\) can take under an extension of \(v\) to \(K'(a)\). Even if \(a\) always has a unique valuation, \(K(a)\) may admit multiple distinct extensions of \(v\). The following lemma shows that in the general case
(i.e., when $K$ is not finitely generated) there is no way to decide in a computable way, from the minimal polynomial of $a$ over $K$, how many extension of $v$ there are.

**Proposition 12.3.3.** There is a computable algebraic valued field $(K, v)$ with a splitting algorithm such that there is no way to (uniformly in $a$) compute the number of extensions of $v$ to an algebraic extension $K(a)$.

*Proof.* Assume that $0_0 = \emptyset$, and that at each subsequent stage, exactly one element enters $0'$. Fix a presentation $\overline{K}$ of the algebraic closure of $K$ and a computable Rabin embedding of $K$ into $\overline{K}$.

Fix an odd prime $r$. Let $p_1, p_2, \ldots$ be a list of the infinitely many primes $p \neq r$. Begin at stage $0$ with $K_0 = \mathbb{Q}$ with $v_0$ the $r$-adic valuation.

Suppose that at stage $s$, $0'_s = \{a_1, \ldots, a_s\}$. We will have already defined

$$K_s = \mathbb{Q}((r q_i)^{\frac{1}{p_i}} : i = 1, \ldots, s)$$

with the unique extension $v_s$ of the $r$-adic valuation to $K_s$ (the fact that this extension of the valuation is unique follows from the fundamental inequality). Here, $q_1, \ldots, q_s$ are distinct primes $q \equiv 1 \mod r$. Let $a_{s+1} = b$ be the element which enters $0'$ at stage $s + 1$. Search for a prime $q_{s+1} \equiv 1 \mod r$ which is not $r$ such that, as subsets of $\overline{K}$ with domain $\omega$,

$$K_s((r q_{s+1})^{\frac{1}{p_b}}) \cap \{0, \ldots, s\} = K_s \cap \{0, \ldots, s\}.$$ 

Let $K_{s+1} = K_s((r q_{s+1})^{\frac{1}{p_b}})$. As $K_s$ is an extension of $\mathbb{Q}$ of degree $p_a \cdots p_{a_s}$ and $p_b$ is coprime to this, for any two distinct primes $q$ and $q'$,

$$K_s((r q)^{\frac{1}{p_b}}) \cap K_s((r q')^{\frac{1}{p_b}}) = K_s \text{ or } K_s((r q)^{\frac{1}{p_b}}) = K_s((r q')^{\frac{1}{p_b}}).$$

Thus we can find a $q_{s+1}$ as desired. Extend $v_s$ to the unique valuation $v_{s+1}$ on $K_{s+1}$. Let $(K, v) = \bigcup_s (K_s, v_s)$. Note that $K$ has a splitting algorithm: to decide whether a given $s \in K_s$ is in $K$, one can simply check whether $s \in K_s$. Also, $v$ is the unique extension of the $r$-adic valuation from $\mathbb{Q}$ to $K$.

We claim that if $a \in 0'$, then the valuation $v$ on $K$ has more than one extension to $K((r q_a)^{\frac{1}{p_a}})$, and if $a \notin 0'$, then $v$ has a unique extension to $K((r q_a)^{\frac{1}{p_a}})$.

First suppose that $a$ enters $0'$ at stage $s$. Then we have a tower of extensions

$$\mathbb{Q} \subset \mathbb{Q}(1 + q_s^{\frac{1}{p_a}}) \subset K((r q_a)^{\frac{1}{p_a}}).$$

Note that $1 + q_s^{\frac{1}{p_a}}$ has minimal polynomial

$$(x - 1)^{p_a} - q_s = \left(\frac{p_a}{0}\right)x^{p_a} - \left(\frac{p_a}{1}\right)x^{p_a-1} + \left(\frac{p_a}{2}\right)x^{p_a-2} + \cdots \pm \left(\frac{p_a}{p_a - 1}\right)x - (q_s - 1).$$
Since \( q_s \equiv 1 \mod r \), \( r \mid q_s - 1 \). Also, since \( p_a \neq r \), \( r + \left( \frac{p_a}{p_{a-1}} \right) = p_a \), \( r + \left( \frac{p_0}{0} \right) = 1 \), and \( r + \left( \frac{p_1}{1} \right) = p \). Thus, by looking at the Newton polygon of this minimal polynomial, we see that there are multiple distinct extensions of the \( r \)-adic valuation on \( \mathbb{Q} \) to \( \mathbb{Q}(1 + q_s^{\frac{1}{r}}) \). So there are multiple distinct extensions of the \( r \)-adic valuation on \( \mathbb{Q} \) to \( K(r^{\frac{1}{pa}}) \). Since \( v \) was the unique extension of the \( r \)-adic valuation to \( K \), there are multiple extensions of \( v \) to \( K(r^{\frac{1}{pa}}) \).

Now suppose that \( a \notin 0' \). Then consider the tower of extensions

\[
\mathbb{Q} \subset \mathbb{Q}(r^{\frac{1}{pa}}) \subset K_1(r^{\frac{1}{pa}}) \subset K_2(r^{\frac{1}{pa}}) \subset \ldots.
\]

Since \( a \notin 0' \), each of these extensions has ramification index equal to its degree as a field extension. By the fundamental inequality, there is a unique extension of the valuation for each field extension. \( \square \)

We can also embed every valued field into a computable presentation of its Henselization.

**Proposition 12.3.4** (Proposition 6 of [Smi81]). Let \((K,v)\) be a computable valued field. There is a computable embedding of \( K \) into a computable valued field \((L,w)\) such that \((L,w)\) is the Henselization of \( K \).

**Proof.** It is enough to show that if \((K,v) \to (\overline{K},v)\) is an embedding of \( K \) into a computable presentation of its algebraic closure, then we can enumerate \( K^h \) in \( \overline{K} \). We can close under applications of Hensel’s lemma, say in the version (2) of Definition 12.2.3 above, to enumerate the Henselization of \( K \). Note that the solutions in (2) are unique. \( \square \)

Smith also showed that Henselizations are recursively unique [Smi81].

Given an embedding of a valued field into its algebraic closure, we might want to decide which elements of the algebraic closure are in the Henselization, rather than just enumerating the elements of the Henselization. We show that this can be done for the Henselization of \( \mathbb{Q} \) inside any fixed presentation of \( \overline{\mathbb{Q}} \).

**Proposition 12.3.5.** Let \((\mathbb{Q},v)\) be a computable valued field with the \( p \)-adic valuation. Fix a Rabin embedding of \( \mathbb{Q} \) into \( \overline{\mathbb{Q}} \). Then the Henselization \( \mathbb{Q}^h \subset \overline{\mathbb{Q}} \) of \( \mathbb{Q} \) is computable inside \( \overline{\mathbb{Q}} \).

**Proof.** Since \( \mathbb{Q} \) is finitely ramified, it has value group \( \mathbb{Z} \). By Theorem 12.2.6, the Henselization of \( \mathbb{Q} \) is the smallest algebraically maximal valued field containing \( \mathbb{Q} \); that is, the minimal extension of \( \mathbb{Q} \) with no immediate extensions. The Henselization of \( \mathbb{Q} \) is unique inside the fixed presentation of \( \overline{\mathbb{Q}} \).

Given \( a, a \in \mathbb{Q}^h \) if and only if \( \mathbb{Q}(a) \), together with the induced valuation \( v \) coming from the valuation on \( \overline{\mathbb{Q}} \), is an immediate extension of \( \mathbb{Q} \). If \( a \in \mathbb{Q}^h \), then \( \mathbb{Q}(a) \) is an immediate extension of \( \mathbb{Q} \). On the other hand, if \( \mathbb{Q}(a) \) is an immediate extension of \( \mathbb{Q} \), since \( \mathbb{Q}(a)^h \) is an immediate extension of \( \mathbb{Q}(a) \) we know that \( \mathbb{Q}(a)^h \) is an immediate extension of \( \mathbb{Q}^h \). Since \( \mathbb{Q}^h \) has no proper immediate extensions, \( \mathbb{Q}(a)^h = \mathbb{Q}^h \). Thus \( a \in \mathbb{Q}^h \).
To check whether $\mathbb{Q}(a)$ is an immediate extension of $\mathbb{Q}$, we need to compute the ramification index and residue degree of the extension of $v$ to $\mathbb{Q}(a)$. We can do this uniformly in $a$ by Lemma 12.3.2.

This lemma is not true for an arbitrary algebraic valued field. The following proposition shows that there is a computable algebraic valued field $(K,v)$, with a splitting algorithm, so that we cannot decide whether or not an element $a$ is in the Henselization of $K$. As a consequence, there is no computable way to decide, from a minimal polynomial of $a$ over $K$, whether or not $a$ is in the Henselization of $K$.

**Proposition 12.3.6.** There is a computable algebraic valued field $(K,v)$ with a splitting algorithm whose Henselization is not computable as a subset of $\overline{\mathbb{Q}}$.

**Proof.** Fix a prime $r$ and a computable list $p_1, p_2, \ldots$ of the primes not equal to $r$. In a similar way to Proposition 12.3.3, construct a computable valued field

$$(K, v) = \mathbb{Q}((r q_i)^{\frac{1}{p_i}} : i \in 0')$$

with a splitting algorithm. As before, for each $i$, $q_i \equiv 1 \mod r$. The primes $q_i$ do not necessarily form a computable sequence in $i$. The valuation $v$ is the unique extension of the $r$-adic valuation to $K$.

Then for each $i \in 0'$, $q_i^{\frac{1}{p_i}}$ is in the Henselization of $\mathbb{Q}$, and hence in the Henselization of $(K,v)$. This is because $1^{p_i} \equiv q \mod r$ but $p_i 1^{p_i-1} \equiv p_i \not\equiv 0 \mod r$. So $r^{\frac{1}{p_i}}$ is in the Henselization of $(K,v)$.

On the other hand, suppose that $i \not\in 0'$. We will show that $r^{\frac{1}{p_i}}$ is not in the Henselization of $(K,v)$. Note that the value group of $K$ is $\mathbb{Z}(\frac{1}{p_j} : j \in 0')$. Then this is also the value group of the Henselization of $K$, and so $r^{\frac{1}{p_i}}$ is not in the Henselization. 

We now come to the main result of this section. We showed above that we can embed a valued field $(K,v)$ into an algebraically closed valued field, constructing the algebraic closure $\overline{K}$ at the same as we construct the extension of the valuation. But what if we have a fixed embedding of $K$ into a presentation of its algebraic closure, and we want to extend the valuation $v$ to $\overline{K}$ via that particular embedding? Theorem 4 of [Smi81] shows that one cannot always do this.

If $\iota$ is an embedding of $K$ into $\overline{K}$, by an $(\iota)$-extension of the valuation $v$ to the field $\overline{K}$ we mean a valuation $w$ on $\overline{K}$ with $w \circ \iota = v$. The following theorem gives a necessary and sufficient condition for a valuation $v$ on an algebraic field $K$ to extend to every algebraic extension.

**Theorem 12.3.7.** Let $(K,v)$ be a computable algebraic valued field. Then the following are equivalent:

1. for every computable embedding $\iota: K \to L$ of $K$ into a field $L$ algebraic over $K$, there is a computable extension of $v$ to a computable valuation $w$ on $L$,
(2) the Hensel irreducibility set

\[ H_K := \{ f = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 \in \mathcal{O}_K[x] : f \text{ is irreducible over } K, v(a_{n-1}) = 0, \text{ and } v(a_{n-2}), \ldots, v(a_0) > 0 \} \]

of \((K, v)\) is computable.

Note the relation between the set \(H_K\) and (4) of Definition 12.2.3. Indeed, Smith showed that given a Henselian computable field, and a fixed embedding in an algebraic closure, one can extend the valuation (see Proposition 5 of [Smi81]); our result can be seen as a significant generalization of this, as a Henselian field trivially has computable Hensel irreducibility set.

Proof. (2) \(\Rightarrow\) (1). Fix \(\overline{\mathbb{Q}}\) a computable presentation of the algebraic closure of \(\mathbb{Q}\), and \(\iota\) an embedding of \(K\) into \(\overline{\mathbb{Q}}\). Using this embedding, we can view \(K\) as a c.e. subset of \(\overline{\mathbb{Q}}\). Begin by defining \(w_0\) to be the \(p\)-adic valuation on \(F_0 = \mathbb{Q}\).

We begin by showing that we can find a sequence

\[ F_0 = \mathbb{Q} \subseteq F_1 = F_0(a_0) \subseteq F_2 = F_1(a_1) \subseteq \ldots \]

of fields, such that each \(F_s\) is a normal extension of \(F_0\), and so that \(\overline{\mathbb{Q}}\) is the union of these fields. Given \(F_s\) a finite normal extension of \(F_0\), and a splitting algorithm for \(F_s\), \(F_s\) is a computable subset of \(\overline{\mathbb{Q}}\). Let \(a\) be the first element of \(\overline{\mathbb{Q}}\) which is not in \(F_s\). Search for an element \(a_s\) such that \(a \in F_i(a_s)\), and all of the conjugates of \(a_s\) over \(\mathbb{Q}\) are in \(F_s(a_s)\). By Theorem 12.2.1, \(F_s(a_s)\) has a splitting algorithm, so we can check this computably. Some such \(a_s\) exists by the primitive element theorem. Then let \(F_{s+1} = F_s(a_s)\). We have, uniformly in \(s\), a splitting algorithm for \(F_s\).

Suppose that we have defined \(w_s\) on \(F_s\), with the property that there is a common extension of \(v\) and \(w_s\) to \(\overline{\mathbb{Q}}\). We will show how to extend \(w_s\) to a valuation \(w_{s+1}\) on \(F_{s+1}\) such that \(w_s\) and \(v\) have a common extension to \(\overline{\mathbb{Q}}\).

By Lemma 12.3.2 we can find all of the extensions of \(w_s\) to \(F_{s+1}\). If there is only one extension, let \(w_{s+1}\) be this extension. Otherwise, let \(u_1, \ldots, u_m\) be the distinct valuations on \(F_{s+1}\) extending the \(p\)-adic valuation on \(\mathbb{Q}\).

For each \(i\), we will search for evidence that \(u_i\) is not compatible with \(v\). If \(u_i\) is not compatible with \(v\), then (by König’s Lemma, since there are only finitely many valuations on a finitely generated algebraic extension of \(\mathbb{Q}\)) there is some finitely generated subfield \(K'\) of \(K\) such that \(v | K'\) and \(u_i\) are not compatible on \(K'F_{s+1}\). For each \(K'\), \(K'F_{s+1}\) is a finite degree extension of \(\mathbb{Q}\), and so by Lemma 12.3.2 we can find all of the valuations on \(K'F_{s+1}\). If \(u_i\) and \(v | K'\) do not have a common extension to \(K'F_{s+1}\), then every valuation on \(K'F_{s+1}\) will differ from either \(u_i\) or \(v | K'\) when applied to some element. So if \(u_i\) and \(v\) are not compatible, we will discover this in a c.e. way.

On the other hand, using Theorem 12.2.5 with \(a_i = 1\) and \(a_j = 0\) for \(i \neq j\), there is \(\beta_i \in F_{s+1}\) such that \(u_i(\beta_i - 1) > 0\) and \(u_j(\beta_i) > 0\) for \(j \neq i\). Note that \(u_i(\beta_i) = 0\). We can choose such a
\(\beta_i\) for each \(i\). We claim that if \(u_i\) and \(v\) have a common extension, say \(w\), to \(KF_{s+1}\), then we can eventually find the minimal polynomial of \(\beta_i\) over \(K\). Let

\[ f_i = x^n + n_{n-1}x^{n-1} + n_{n-2}x^{n-2} + \cdots + a_0 \]

be the minimal polynomial of \(\beta_i\) over \(K\). Let \(\beta_i = \beta_i^1, \ldots, \beta_i^n\) be the conjugates of \(\beta_i\) over \(K\). Since \(F_{s+1}\) is a normal extension of \(Q\), and \(\beta_i \in F_{s+1}\), each of these conjugates is in \(F_{s+1}\). Each of \(\beta_i^2, \ldots, \beta_i^n\) is a conjugate of \(\beta_i\) over \(Q\). Among \((u_j)_{i \leq j}\) are the conjugates of the valuation \(u_i\) over \(Q\). Since \(u_j(\beta_i) > 0\) for each \(i \neq j\), \(u_i(\beta_i^2), \ldots, u_i(\beta_i^n) > 0\). Then

\[ f_i = (x - \beta_i^1)(x - \beta_i^2)\cdots(x - \beta_i^n) \]

and so

\[ v(a_{n-1}) = w(a_{n-1}) = w(-\beta_i^1 - \cdots - \beta_i^n) = w(-\beta_i) = u_i(\beta_i) = 0. \]

For \(k = 0, \ldots, n-2\), we can write \(a_k\) as a sum of products of \(\beta_i^1, \ldots, \beta_i^n\), where each term of the sum has at least two factors, and each of \(\beta_i^1, \ldots, \beta_i^n\) shows up at most once in each product. Thus the \(w\)-value of each term is strictly positive, and so \(v(a_k) = w(a_k) > 0\). To find the minimal polynomial of \(\beta_i\) over \(K\), we search for an irreducible polynomial

\[ f = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 \]

with \(f(\beta_i) = 0\), \(v(a_{n-1}) = 0\), and \(v(a_{n-2}), \ldots, v(a_0) > 0\). Note that we can check whether such a polynomial is irreducible. We can perform this search whether or not \(u_i\) and \(v\) have a common extension to \(KF_{s+1}\). If \(u_i\) and \(v\) do have a common extension to \(KF_{s+1}\), then we will eventually find the minimal polynomial of \(\beta_i\). We can also find all of the conjugates \(\beta_i = \beta_i^1, \ldots, \beta_i^n\) of \(\beta_i\) over \(K\).

Suppose that \(u_i\) and \(v\) have a common extension to \(KF_1(a)\), and \(u_j\) and \(v\) have a common extension to \(KF(a)\). Since any two extensions of \(v\) to \(KF_{s+1}\) are conjugate over \(K\), \(u_i\) and \(u_j\) are conjugate over \(K\). Thus \(u_j(\beta_i^k) = 0\) for some \(k\).

On the other hand, suppose that \(u_j(\beta_i^k) = 0\) for some \(k\). Note that \(u_j\) is conjugate over \(K\) to a valuation \(u_j'(\beta_i) = 0\). By choice of \(\beta_i\), \(u_j' = u_i\). Thus \(u_i\) and \(u_j\) are conjugate over \(K\). Then \(u_i\) and \(v\) have a common extension to \(KF_{s+1}\) if and only if \(u_j\) and \(v\) do.

Eventually, we will find, for some \(i\), the minimal polynomial

\[ f = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 \]

of \(\beta_i\) over \(K\), and conjugates \(\beta_i = \beta_i^1, \ldots, \beta_i^n\) of \(\beta_i\) over \(K\). Some of the \(u_j\), for \(j \neq i\), will be found to be incompatible with \(v\). The rest of the \(u_j\) will have \(u_j(\beta_i^k) = 0\) for some \(k\). Since at least one of the \(u_j\) has a common extension with \(v\) to \(KF_{s+1}\), it must be that \(u_i\) and all of the \(u_j\) with \(u_j(\beta_i^k) = 0\) have such an extension. In particular, \(u_i\) and \(v\) have a common extension to \(KF_{s+1}\). Take \(w_{s+1} = u_i\).

For \((1) \Rightarrow (2)\), let \(\bar{Q}\) be a computable presentation of the algebraic closure of \(Q\). Let \(a_0, a_1, a_2, \ldots\) enumerate the elements of \(K\). We will define, at stage \(s + 1\), an embedding
\[ 
Q \ni \sum_{k=0}^{2^m-1} f_k \cdot b_k = 0.
\]

Now \( f \) splits over \( \overline{Q} \), say \( f = g_1 \cdots g_r \) with \( g_1, \ldots, g_r \) irreducible over \( \overline{Q}(a_0, \ldots, a_s) \). Given the valuation \( v \) on \( \overline{Q}(a_0, \ldots, a_s) \subseteq K \), there is exactly one \( j \) for which \( g_j \) can have a solution with valuation \( 0 \) with respect to a valuation extending \( v \); we can find such a \( j \) computably by looking at the values of the coefficients of the \( g_j \). Without loss of generality, let \( j = 1 \). Note also that \( g_1, \ldots, g_r \) are conjugate over \( \overline{Q}(a_0, \ldots, a_s) \). Thus, we can extend \( \iota_s \) to \( \iota_{s+1}: \overline{Q}(a_0, \ldots, a_s) \rightarrow \overline{R} \) such that \( c_1 \) is not a solution of \( \iota(g_1) \). Then if \( w = \varphi_i \) is a valuation on \( \overline{Q} \) extending \( v \), \( w(c_1) = 0 \) and so \( c_1 \) must be a root of \( \iota(g_1) \); but this is not the case, and so \( w = \varphi_i \) is not a valuation on \( \overline{Q} \) extending \( v \). Thus \( R_i \) is satisfied.
End construction.

We built an embedding $\iota = \bigcup_s \iota_s$ of $K$ into $\overline{\mathbb{Q}}$. By assumption, there is a computable valuation $w$ on $\overline{\mathbb{Q}}$ extending the valuation $v$ on $K$. Let $w$ be given by $\phi_i$. Given a polynomial $f \in K[x]$, with $f = x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_0$ where $v(b_{n-1}) = 0$ and $v(b_{n-2}), \ldots, v(b_0) > 0$, let $c_1, \ldots, c_m$ be the solutions of $\iota(f)$ in $\overline{\mathbb{Q}}$. Let $t$ be a stage such that:

1. no $R_j$, for $j < i$, acts after stage $t$,
2. $\phi_{i,t}(c_j)$ is defined for each $j$,
3. $f \in \mathbb{Q}(a_0, \ldots, a_{t-1})[x]$.

Note that (1) is independent of $f$, and depends only on the stage $i$. The following claim will finish the proof.

Claim 12.3.8. $f$ is irreducible over $K$ if and only if $f$ is irreducible over $\mathbb{Q}(a_0, \ldots, a_t)$.

Proof. The left to right direction is obvious. Suppose that $f$ is irreducible over $\mathbb{Q}(a_0, \ldots, a_t)$. Then suppose that $f$ is not irreducible over $K$. Then $f$ splits over $\mathbb{Q}(a_0, \ldots, a_s)$ for some least $s > t$. Then, by choice of $t$, in the construction we satisfy the requirement $R_t$ at stage $s + 1$. But then $w$ does not extend $v$, a contradiction. So $f$ is irreducible over $K$.

Given $f$, we can compute $t$ as required, and then to check whether $f$ is irreducible over $K$ it suffices to check whether it is irreducible over $\mathbb{Q}(s_0, \ldots, s_t)$. Since this is a finite algebraic extension of $\mathbb{Q}$, we have a splitting algorithm for this field.

12.4 $p$-adic Closures

It was easy to see by an effective Henkin construction in Proposition 12.3.1 that every valued field embeds effectively into a computable algebraically closed valued field. The same argument does not work to show that every computable formally $p$-adic field embeds effectively into a computable $p$-adic closure, because the theory $pCF$ is not the model completion of formally $p$-adic fields: if $K$ is a $p$-adic field, the elementary diagram of $K$ together with the theory $pCF$ is not complete. Indeed, there is a computable formally $p$-adic field which does not computably embed into a $p$-adic closure. This uses ideas from the proof that a formally $p$-adic field whose value group is not a $\mathbb{Z}$-group embeds into two non-isomorphic $p$-adic closures (Theorem 3.2 of [PR84]).

Theorem 12.4.1. There is a computable formally $p$-adic field which does not computably embed into a $p$-adic closure.

Proof. We will construct a formally $p$-adic field $E$ by diagonalizing against computable embeddings $f_i$ into $p$-adic closures $(K_i, v_i)$. Let $q_i$ be the $i$th prime. Begin at stage 0 with $E_0 = \mathbb{Q}(t)$ a transcendental extension of $\mathbb{Q}$, together with the valuation $v$ with
$v(t) > \mathbb{Z} = v(\mathbb{Q})$. At stage \( s + 1 \), we will have built \( E_0 \subseteq E_1 \subseteq \cdots \subseteq E_s \) a chain of embeddings of computable valued fields, with each extension algebraic. Let \( i < s \) be the least \( i \) against which we have not yet diagonalized such that at stage \( s \) there is an element \( a \) among the first \( s \) elements of \( K_i \) with \( q_i \cdot v(a) = v(p^m f_i(t)) \) for some \( 0 \leq m < q_i \) (i.e., \( f_{i,s}(t) \) converges, and enough of the diagram of \( K_i \) converges to decide that \( q_i \cdot v(a) = v(p^m f_i(t)) \)). We will diagonalize against this \( i \). Let \( E_{s+1} = E_s(b) \), where \( b \) is such that \( b^\omega = p^{m+1} f_i(t) \). Extend the valuation to \( E_{s+1} \) (again, by abuse of notation, calling it \( v \)).

Now \( E_s \) will be an extension of degree \( q_1 \cdots q_i \) of \( E_0 \), where \( i_1, \ldots, i_n \) are the requirements which we have already diagonalized against. \( E_0(b) \) is an extension of \( E_0 \) of degree \( q_i \), and so since \( q_i \) is coprime to \( q_1, q_2, \ldots, q_n \), \( E_{s+1} \) is an extension of \( E_s \) of degree \( q_i \).

The value group of \( E_0 \) is \( \mathbb{Z}(r) \), where \( r = v(t) > \mathbb{Z} \). Then the value group of \( E_s \) will be

\[
v(E_s) = \mathbb{Z}(r, \frac{r + m_1 + 1}{q_i}, \ldots, \frac{r + m_n + 1}{q_n}).
\]

The value group of \( E_{s+1} \) will contain

\[
G = \mathbb{Z}(r, \frac{r + m_1 + 1}{q_i}, \ldots, \frac{r + m_n + 1}{q_n}, \frac{r + m + 1}{q_i}).
\]

Since \( q_i \) is coprime to \( q_1, \ldots, q_n \), \( v(E_s) \) is a subgroup of \( G \) index \( q_i \). By the fundamental inequality, \( G \) is the value group of \( E_{s+1} \), and the residue degree is \( 1 \). Note also that since \( q_1, \ldots, q_n, q_i \) are coprime, \( 1 = v(p) \) is still the minimal element of the value group. So \( E_s(b) \) is formally \( p \)-adic. The extension of \( v \) from \( E_i \) to \( E_{s+1} \) is unique.

Let \( E = \cup E_i \), with valuation \( v \). Then \( E \) is a formally \( p \)-adic field. Suppose towards a contradiction that \( E \) computably embeds into a computable \( p \)-adic closure; let \( i \) be an index such that the embedding is \( f_i \) into the \( p \)-adic closure \( (K_i, v_i) \). Since the value group of \( K_i \) is a \( \mathbb{Z} \)-group, there is \( r \in \Gamma_{K_i} \) such that \( q_i \cdot v(r) = v(f_i(t)) + m \) for some \( 0 \leq m < q_i \). Then, at some stage \( s \) we have diagonalized against every \( j < i \) which we will ever diagonalize against, there is \( a \) among the first \( s \) elements of \( K_i \) with \( q_i \cdot v(a) = v(p^m f_i(t)) \) for some \( 0 \leq m < q_i \), \( f_{i,s}(t) \) has converged, and enough of the diagram of \( K_i \) has converged to decide that \( q_i \cdot v(a) = v(p^m f_i(t)) \). Then at stage \( s + 1 \), we will diagonalize against \( K_i \) by putting into \( E_{s+1} \) an element \( b \) with \( v(b^\omega) = v(t) + m + 1 \). Then, in the value group of \( K_i \), we have

\[
q_i \cdot v\left( \frac{f_i(b)}{a} \right) = q_i \cdot v(f_i(b)) - q_i \cdot v(a) = v(f_i(c)) - q_i \cdot v(a) = v(f_i(t)) + m + 1 - v(f_i(t)) - m = 1.
\]

But then \( q_i \) divides \( 1 \) in the value group, and hence \( K_i \) is not formally \( p \)-adic. So \( K_i \) cannot be the \( p \)-adic closure of \( E \).

The problem with the field from the previous theorem which prevents us from embedding it into a \( p \)-adic closure is that we cannot decide, for a given element of the value group and \( n \in \omega \), whether or not it is divisible by \( n \). Theorem 12.4.4 below will show that this is the only obstacle.
**Definition 12.4.2.** Let \( G \) be a torsion-free abelian group. The *dividing set* of \( G \) is

\[
\text{div}(G) = \{(x,n) \in G \times \omega : n \text{ divides } x\}.
\]

The set \( \text{div}(G) \) should be viewed as analogous to the splitting set for a field. The following lemma is the analogue of Theorem 12.2.1 of Kronecker’s.

**Lemma 12.4.3.** Suppose that \( G \) is a computable torsion-free abelian group with \( \text{div}(G) \) computable. Let \( H = G(a) \) be a computable group where \( a \) is a new element with \( na = b \in G \). Then \( \text{div}(H) \) is computable uniformly in \( \text{div}(G) \), \( n \), and \( b \).

Note that since \( G \) is torsion-free, \( a \) is uniquely determined by \( b \) and \( n \).

**Proof.** We begin by finding \( a \in H \) with \( na = b \). Using \( \text{div}(G) \), we may suppose that \( b \) is not divisible in \( G \) by any prime factor of \( n \); if it is, find such a divisor, and replace \( b \) by that divisor. Thus \( ma \in G \) for any \( 0 \leq m < n \). Given \( x \in H \), write \( x = ma + g \) with \( 0 \leq m < n \) and \( g \in G \). We want to decide whether \( x \) is divisible by some number \( r \). It suffices to decide whether \( x \) is divisible by a prime \( q \); if it is, then we can find such a divisor and repeat the process, noting that since \( G \) is torsion-free, divisors are unique.

If \( q \) and \( n \) are coprime, then we claim that \( q \) divides \( x \) if and only if \( q \) divides \( mb + ng \). If \( q \) divides \( x \), then \( q \) divides \( mb + ng = nx \). For the other direction, suppose \( q \) divides \( mb + ng \), say \( qh = mb + ng \). Since \( q \) and \( n \) are coprime, let \( r \) and \( s \) be such that \( qr = 1 + ns \). Then

\[
q(rx - sh) = qr x - qsh = (1 + ns)x - nsx = x.
\]

So \( q \) divides \( x \). Since \( mb + ng \) is in \( G \), we can decide whether \( q \) divides \( mb + ng \), and hence whether \( q \) divides \( x \).

On the other hand, suppose that \( q \) and \( n \) are not coprime, so that \( q \mid n \). If \( q \) divides \( g \) and \( q \mid m \), then \( q \) divides \( x = ma + g \). For the other direction, suppose that \( q \) divides \( x = ma + g \). Let \( y = m'a + g' \), with \( g' \in G \), be such that \( qy = x \). Then

\[
(qm' - m)a = g - qq'
\]

Thus \( n \mid qm' - m \), and so \( q \mid m \). Since \( q \) divides \( ma \) and \( q \) divides \( ma + g \), \( q \) divides \( g \). So \( q \) divides \( x = ma + g \) in \( H \) if and only if \( q \mid m \) and \( q \) divides \( g \) in \( G \). Thus \( \text{div}(H) \) is computable.

We are now ready to show that when we can compute the dividing set of the value group of a formally \( p \)-adic valued field, we can effectively embed the field into a \( p \)-adic closure.

**Theorem 12.4.4.** Let \((K,v)\) be a computable formally \( p \)-adic valued field with value group \( \Gamma \). Suppose that \( \text{div}(\Gamma) \) is computable. There is a computable embedding of \( K \) into a computable \( p \)-adic closure \((L,w)\).
Proof. We will construct a sequence \((K_0, v_0) = (K, v)^h \subseteq (K_1, v_1) \subseteq (K_2, v_2) \subseteq \cdots\) of computable Henselian valued fields such that \((L, w) = \bigcup_i (K_i, v_i)\) is a computable \(p\)-adic closure of \((K, v)\). If \((a_i, q_i)\) is an enumeration of all of the pairs of elements \(a\) from \(L\) and primes \(q\) with \(a_i \in K_i\), we will ensure at stage \(s+1\) that \(q_s\) divides one of \(w(a_s), w(a_s) + 1, \ldots, w(a_s) + q_s - 1\). Note that we must construct the sequence \((a_i, q_i)\) concurrently with the \(K_i\). For each \(i\), \(\text{div}(v_i(K_i))\) will be computable.

At stage \(s+1\), ask \(\text{div}(v_s(K_s))\) whether \(q_s\) divides one of \(v_s(a_s), v_s(a_s) + 1, \ldots, v_s(a_s) + q_s - 1\). If it does, then just set \(K_{s+1} = K_s\). Otherwise, let \(b\) be an \(q_s\)th root of \(a_s\) and let \(E = K_s(b)\); since \(K_s\) was Henselian, there is only extension \(v'\) of \(v_s\) to \(E\). Note that \(q_s\) divides \(a_s\) in \(v'(E)\). By Lemma 12.4.3, \(\text{div}(v(E))\) is computable uniformly. Let \(K_{s+1}\) be a Henselization of \(E\). Then the value group of \(K_{s+1}\) is the same as that of \(E\).

To see that \(E\) (and hence \(K_{s+1}\)) is formally \(p\)-adic, we must show that the residue field is still \(\mathbb{F}_p\) and that \(1 = v(p)\) is still the least positive element of the value group. First, since \(q_s\) is prime and \(v(a_s)\) is not divisible by \(q_s\), \([v'(E) : v_s(K_s)] = q_s = [E : K_s]\). By the fundamental inequality, the residue degree of \(v'\) over \(v_s\) is one. Thus the residue field of \(E\) is again \(\mathbb{F}_p\).

Each element of \(E\) can be written in the form
\[d = c_{q_1} b_1^{q_1} + c_{q_2} b_2^{q_2} + \cdots + c_1 b + c_0\]
with the \(c_i \in K_s\). We want to show that \(v'(d)\) is not strictly in between 0 and 1 = \(v(p)\). Suppose to the contrary that \(d\) has valuation strictly between 0 and 1. Note that as \(v'(b), \ldots, v'(b^{q_1})\) are all distinct and not in \(\Gamma_{K_s}\), that \(v'(d) = \min_{0 \leq i < q_1} v'(c_i b^i)\). Since \(c_0 \in K_s\) does not have valuation strictly between 0 and 1, \(v'(d) = v'(c_i b^i)\) for some \(i \geq 1\). Then \(0 < q_s v'(c_i b^i) < q_s\). Note that \(q_s v'(c_i b^i) = q_s v(c_i) + iv(a_s)\) is in \(\Gamma_{K_s}\). Let \(\gamma = v(c_i) \in \Gamma_{K_s}\). Thus, for some \(j, 1 \leq j < q_s\), \(q_s \gamma + iv(a_s) = j\). Since \(1 \leq i < q_s, \gcd(q_s, i) = 1\). Let \(m, n\) be such that \(mi + nq_s = 1\). Then
\[mq_s \gamma + v(a_s) = mq_s \gamma + miv(a_s) + nq_s v(a_s) = mj + nq_s v(a_s)\]
Since \(q_s \nmid m, j\), we can write \(mj = q_s d - r\), where \(1 \leq r < q_s\). Then
\[v(a_s) + r = q_s (m \gamma + d + nv(a_s))\]
This is a contradiction, as \(q_s\) does not divide \(v(a_s) + r\) in \(\Gamma_{K_s}\). So no element of \(E\) has valuation strictly between 0 and 1. Thus \(E\) is formally \(p\)-adic.

Now \((L, w) = \bigcup_i (K_i, v_i)\) is a computable valued field into which \((K, v)\) embeds computably, and \((L, w)\) is algebraic over \((K, v)\). Moreover, \((L, w)\) is a model of \(p\text{CF}\): it is formally \(p\)-adic as the union of formally \(p\)-adic valued fields, it is Henselian as the union of Henselian fields, and we ensured that the value group was a model of Presburger arithmetic.

\[\square\]

### 12.5 The Mal’cev Property

We begin by recalling the metatheorem from [HTM13]. The metatheorem is stated using the general notion of a pregeometry, but for the purposes of this paper, the pregeometry
will always be algebraic independence in fields, and the reader need not know the general definition of a pregeometry.

**Definition 12.5.1.** A class $\mathcal{K}$ has the Mal'cev property if each member $\mathcal{M}$ of $\mathcal{K}$ of infinite dimension has a computable presentation $\mathcal{G}$ with a computable basis and a computable presentation $\mathcal{B}$ with no computable basis such that $\mathcal{B} \equiv_{\Delta^0_2} \mathcal{M} \equiv_{\Delta^0_2} \mathcal{G}$.

In [HTMM15], two conditions were isolated which imply the Mal'cev property. We require some definitions before we state these conditions and the metatheorem.

**Definition 12.5.2.** The independence diagram $\mathcal{I}_\mathcal{M}(\bar{c})$ of $\bar{c}$ in $\mathcal{M}$ is the collection of all existential formulas true of tuples independent over $\bar{c}$.

**Definition 12.5.3.** We say that dependent elements are dense in $\mathcal{M}$ if, whenever $\mathcal{M} \models \exists \bar{y} \psi(\bar{c}, \bar{y}, a)$ for a quantifier-free formula $\psi$, non-empty tuple $\bar{c}$, and $a \in \mathcal{M}$, there is a $b \in \text{cl}(\bar{c})$ such that $\mathcal{M} \models \exists \bar{y} \psi(\bar{c}, \bar{y}, b)$. We may also assume that $\bar{c}$ contains at least $m$ independent elements, for some fixed $m$.

**Definition 12.5.4.** We say that independent tuples in $\mathcal{M}$ are locally indistinguishable if for every tuple $\bar{c}$ in $\mathcal{M}$ and $\bar{u}$, $\bar{v}$ independent tuples over $\bar{c}$, for each existential formula $\phi$ such that $\mathcal{M} \models \phi(\bar{c}, \bar{u})$, there exists a tuple $\bar{w}$ that is independent over $\bar{c}$, has $\mathcal{M} \models \phi(\bar{c}, \bar{w})$, and (with $\bar{w} = (w_1, \ldots, w_n)$ and $\bar{v} = (v_1, \ldots, v_n)$) we have $w_i \in \text{cl}(\bar{c}, v_1, \ldots, v_i)$ for $i = 1, \ldots, n$.

The two conditions are as follows:

**Condition G:** Independent tuples are locally indistinguishable in $\mathcal{M}$ and for each $\mathcal{M}$-tuple $\bar{c}$, $\mathcal{I}_\mathcal{M}(\bar{c})$ is computably enumerable uniformly in $\bar{c}$.

**Condition B:** Dependent elements are dense in $\mathcal{M}$.

**Theorem 12.5.5** (Theorem 1.2 of [HTMM15]). Let $\mathcal{K}$ be a class of computable structures that admits a r.i.c.e. pregeometry $\text{cl}$.

2 If each $\mathcal{M}$ in $\mathcal{K}$ of infinite dimension satisfies Conditions $G$ and $B$, then $\mathcal{K}$ has the Mal'cev property.

### 12.5.1 The Mal'cev Property for ACVF

We will now use the metatheorem to show that algebraically closed valued fields have the Mal'cev property. Note that in ACVF, algebraic dependence is the same as model-theoretic acl.

**Theorem 12.5.6.** Algebraically closed valued fields have the Mal'cev property.

\[2\text{Recall that here this will just be algebraic independence.}\]
Proof. Let \((K, v)\) be an algebraically closed valued field of infinite transcendence degree. We begin by checking that independent types are locally indistinguishable. Let \(S \subseteq K^n\) be a definable set with parameters \(\bar{c}\) which contains a tuple \(\bar{a} = (a_1, \ldots, a_n) \in K^n\) independent over \(\bar{c}\). We may assume that some element of the tuple \(\bar{c}\) is non-trivially valued. Using quantifier elimination in \(ACVF\) and writing \(S\) in disjunctive normal form, we may, without loss of generality, take \(S\) to be the disjunct which contains \(\bar{a}\). Since \(S\) contains \(\bar{a}\) which is independent over \(\bar{c}\), \(S\) is defined by a conjunction of formulas of the form \(v(f(\bar{x}, \bar{c})) \leq v(g(\bar{x}, \bar{c}))\) (or such a formula with \(\leq\) replaced by \(<\), or \(\neq\)). The subfield \(\mathbb{Q}(\bar{c})_{\text{alg}}\) is a model of \(ACVF\), and by model completeness, an elementary submodel of \(K\). Hence it contains an element \(\bar{u} = (u_1, \ldots, u_n)\) which is in \(S\). Note that \(S\) is open in the valuation topology, and so it contains an open ball

\[ B(\bar{u}, \epsilon) = \{ \bar{x} : v(u_i - x_i) \geq \epsilon \} \]

around \(\bar{u}\), with \(\epsilon \in \Gamma(\mathbb{Q}(\bar{c})_{\text{alg}})\). There is also some \(\bar{v} \neq \bar{u}\) with \(\bar{v} \in B(\bar{u}, \epsilon) \cap \mathbb{Q}(\bar{c})_{\text{alg}}\). Write \(\bar{v} = (v_1, \ldots, v_n)\). Let \(\bar{b} = (b_1, \ldots, b_n)\) be an arbitrary tuple from \(K\) independent over \(\bar{c}\). Possibly replacing each \(b_i\) with \(b_i^{-1}\), we may assume that \(v(b_i) \geq 0\). Let \(\bar{b}' = (b_i v_i - (b_i - 1) u_i)_{i=1}^n\).

Note that \(b_i\) and \(b'_i\) are interalgebraic over \(\bar{c}\). Then

\[ v(u_i - b_i v_i + (b_i - 1) u_i) = v(b_i u_i - b_i v_i) = v(b_i) + v(u_i - v_i) \geq \epsilon. \]

So \(\bar{b}' \in B(\bar{u}, \epsilon) \subseteq S\). We have shown that independent types are locally indistinguishable.

A similar argument works to show that independent types are non-principal. Let \(S \subseteq K^n\) be a definable set with parameters \(\bar{c}\), again assuming that some element of the tuple \(\bar{c}\) is non-trivially valued. Then \(\mathbb{Q}(\bar{c})_{\text{alg}}\) is a model of \(ACVF\) and by model completeness there is a tuple \(\bar{a} \in \mathbb{Q}(\bar{c})_{\text{alg}}\) which is contained in \(S\). The tuple \(\bar{a}\) is algebraic over \(\bar{c}\).

We showed above that a definable set \(S\) with parameters \(\bar{c}\) contains a tuple independent over \(\bar{c}\) if and only if it contains, as a disjunct, a non-empty definable set defined by a conjunction of formulas of the form \(v(f(\bar{x}, \bar{c})) \leq v(g(\bar{x}, \bar{c}))\) (or with \(\leq\) replaced by \(<\) or \(\neq\)). Together with the decidability of the theory \(ACVF\), this fact allows us to enumerate the independence diagram of \(K\).

By Theorem 12.5.5, \(ACVF\) has the Mal’cev property. \(\square\)

### 12.5.2 The Mal’cev Property for \(p\)CF

Now we will apply the metatheorem to \(p\)-adically closed fields. Once again, the pregeometry will be algebraic independence which is the same as model-theoretic \(\text{acl}\). Our proof will use the cell decomposition for \(p\)-adically closed fields. We begin with a lemma which we will use to check that independent tuples are locally indistinguishable.

**Lemma 12.5.7.** Given a cell

\[ C = \{ (\bar{x}, y) \in B \times K : v(f(x)) \sqcap v(y - g(\bar{x})) \sqcap v(h(x)) \text{ and } P^*_k(\lambda(x - g(y)) \} \]
and $a \in B$, $b$ algebraically independent from $a$, $\lambda$, and the coefficients of $f$ and $g$, with 
$(a, b) \in C$, and $c$ is algebraically independent from $a$, there is $c'$ interalgebraic with $c$ over $a$
with $(a, c') \in C$.

Proof. Since $b$ is algebraically independent from $a$, we know that $k \neq 0$. Assume that $\Box_1$ and
$\Box_2$ are $\leq$, so that

$$C = \{(\bar{x}, y) \in B \times K : v(f(x)) \leq v(x - g(y)) \leq v(h(x)) \mbox{ and } P_k^*(\lambda(x - g(y)))\}.$$ 

The other cases are similar. It suffices to find $c''$ interalgebraic with $c$ over $a$ such that 
v$(\lambda f(\bar{a})) \leqkv(c'') \leq kv(\lambda h(\bar{a}))$, as then $c' = (c'')^k/\lambda + g(\bar{a})$ has $(\bar{a}, c') \in C$. We may replace $\lambda f$
by $\hat{f}$ and similarly with $h$ and $\hat{h}$ to get $v(\hat{f}(\bar{a})) \leq kv(c'') \leq v(\hat{h}(\bar{a}))$. Now $K = (\exists y)v(\hat{f}(\bar{a})) \leq
kv(y) \leq v(\hat{h}(\bar{a}))$, and so since we have definable Skolem functions, there is $a'$ algebraic over $\bar{a}$
satisfying this. Moreover, we can choose $a' \neq 0$.

If $v(c) = 0$, then we have $v(ca') = v(a')$ and so we can take $c'' = ca'$. Otherwise, by
replacing $c$ by $c^{-1}$ if necessary, we may assume that $v(c) > 0$. Then $v(1 + c) = 0$, and so
$v(a' + ca') = v(a')$. Then we can take $c'' = a' + ca'$. $lacksquare$

Theorem 12.5.8. $p$-adically closed fields have the Mal'cev property.

Proof. Let $(K, v)$ be a model of $pCF$ of infinite transcendence degree. We begin by checking
that independent tuples are locally indistinguishable. Let $S$ be a set definable over parameters $\bar{c}$, containing a tuple $\bar{a} = (a_1, \ldots, a_n)$ independent over $\bar{c}$. Let $\bar{b} = (b_1, \ldots, b_n)$ be another
tuple independent over $\bar{c}$. The set $S$ has a cell decomposition with parameters definable over $\bar{c}$.
Some cell must contain $\bar{a}$, and this cell must be of type $(1, \ldots, 1)$ since $\bar{a}$ is independent
over $\bar{c}$. By repeated applications of Lemma 12.5.7, we get $\bar{b}'$ in $S$ as required.

Suppose that $S \subseteq K^n$ is a definable set over parameters $\bar{c} \in K^m$. Models of $pCF$ have
definable Skolem functions, so there is a definable function $f: K^m \to K^n$ (without parameters)
with $f(\bar{c}) \in S$. Then $f(\bar{c})$ is definable over $\bar{c}$, and hence algebraic over $\bar{c}$. So independent
types are non-principal.

Finally, we have to enumerate the independence diagram of $K$. We showed above that
there is an independent tuple in a cell if and only if it is of type $(1, \ldots, 1)$. Using the
decidability of the elementary diagram of $K$, we can enumerate the definable sets which contain such a cell.

By Theorem 12.5.5, $pCF$ has the Mal'cev property. $lacksquare$
Chapter 13

Left-Orderable Computable Groups

The results presented in this chapter appeared in [HTe].

13.1 Introduction

A left-ordered group is a group $G$ together with a linear order $\leq$ such that if $a \leq b$, then $ca \leq cb$. $G$ is right-ordered if instead whenever $a \leq b$, $ac \leq bc$, and bi-ordered if $\leq$ is both a left-order and a right-order. A group which admits a left-ordering is called left-orderable, and similarly for right- and bi-orderings. A group is left-orderable if and only if it is right-orderable. Some examples of bi-orderable groups include torsion-free abelian groups and free groups [Shi47, Vin49, Ber90]. The group $(x, y : x^{-1}yx = y^{-1})$ is left-orderable but not bi-orderable. For a reference on orderable groups, see [KM96].

In this paper, we will consider left-orderable computable groups. A computable group is a group with domain $\omega$ whose group operation is given by a computable function $\omega \times \omega \to \omega$. Downey and Kurtz [DK86] showed that a computable group, even a computable abelian group, which is orderable need not have a computable order. If a computable group does admit a computable order, we say that it is computably orderable. Of course, by the low basis theorem, every orderable computable group has a low ordering.

For an abelian group, any left-ordering (or right-ordering) is a bi-ordering. An abelian group is orderable if and only if it is torsion-free. Given a computable torsion-free abelian group $G$, Dobritsa [Dob83] showed that there is another computable group $H$, which is classically isomorphic to $G$, which has a computable $\mathbb{Z}$-basis. Note that $H$ need not be computably isomorphic to $G$. Solomon [Sol02] noted that a $\mathbb{Z}$-basis for a torsion-free abelian group computes an ordering of that group. Hence every orderable computable abelian group is classically isomorphic to a computably orderable group.

Downey and Kurtz asked whether this is the case even for non-abelian groups:

**Question 13.1.1** (Downey and Kurtz [DR00]). Is every orderable computable group classically isomorphic to a computably orderable group?
If one takes “orderable” to mean “left-orderable” then we give a negative answer to this question. (We leave open the question for bi-orderable groups.)

**Theorem 13.1.2.** There is a computable left-orderable group which has no presentation with a computable left-ordering.

Our strategy is to build a group

\[ G = \mathbb{N} \rtimes H/R \]

and code information into the finite orbits of certain elements of \( N \) under inner automorphisms given by conjugating by elements of \( H/R \). This strategy cannot work to build a bi-orderable group, as in a bi-orderable group there is no generalized torsion—i.e., no product of conjugates of a single element can be equal to the identity—and hence no inner automorphism has a non-trivial finite orbit. We leave open the case of bi-orderable groups.

### 13.2 Notation

We will use calligraphic letter such as \( G \), \( N \), and \( H \) to denote groups. For free groups, we will use upper case latin letters such as \( A, B, C, U, V \), and \( W \) to denote words, while using lower case letters such as \( a, b, c \) to denote letter variables. We use \( \epsilon \) for the empty word, \( 0 \) for the identity element of abelian groups, and \( 1 \) for the identity element of non-abelian groups (except for free groups, where we use \( \epsilon \)).

### 13.3 The Construction

Fix \( \psi \) a partial computable function which we will specify later (see Definition 13.3.6). Let \( p_i, q_i, \) and \( r_i \) be a partition of the odd primes into three lists.\(^1\) Let \( H \) be the free abelian group on \( \alpha_i, \beta_i, \) and \( \gamma_i \) for \( i \in \omega \). We write \( H \) additively. Let \( R \) be the set of relations

\[ R = \{ R_{i,t}: \psi_{at}(i) \downarrow \} \]

where

\[ R_{i,t} = \begin{cases} p_t^i \alpha_i = q_t^i \beta_i & \text{if } \psi_{at}(i) = 0 \\ p_t^i \alpha_i = -q_t^i \beta_i & \text{if } \psi_{at}(i) = 1 \end{cases} \]

By \( \psi_{at}(i) = 0 \), we mean that the computation \( \psi(i) \) has converged exactly at stage \( t \) (but not before) and equals zero.

The idea is that these relations force, for any ordering \( \leq \) on \( H/R \), that if \( \psi(i) = 0 \) then \( \alpha_i > 0 \iff \beta_i > 0 \) (and if \( \psi(i) = 1 \) then \( \alpha_i > 0 \iff \beta_i < 0 \)). The strategy is, in a very general sense, to use \( \psi \) to diagonalize against computable orderings of \( H/R \). The semidirect product

\(^1\)We use the fact that 2 does not appear in these lists in Lemma 13.5.3.
will add enough structure to allow us to find $\alpha_i$ and $\beta_i$ within a computable copy of $G$. (One cannot find $\alpha_i$ and $\beta_i$ within a copy of $H/R$, since $H/R$ is a torsion-free abelian group.) Note that
\[
H/R = \left( \bigoplus_i (\alpha_i, \beta_i) / R_i \right) \oplus (\bigoplus_i (\gamma_i))
\]
where $R_i = R_{i,t}$ if $\psi_{at}(i) \downarrow$ for some $t$, or no relation otherwise. Define
\[
V_i = R \cup \{ p_i \alpha_i = 0 \} \\
W_i = R \cup \{ q_i \beta_i = 0 \} \\
X_i = R \cup \{ r_i \gamma_i = 0 \}
\]
Let $N$ be the free (non-abelian) group on the letters
\[
\{ u_i : i \in \omega \} \cup \{ v_{i,g} : g \in H/V_i, i \in \omega \} \cup \{ w_{i,g} : g \in H/W_i, i \in \omega \} \\
\cup \{ x_{i,g} : g \in H/X_i, i \in \omega \} \cup \{ y_{i,g} : g \in H/Y_i, i \in \omega \} \cup \{ z_{i,g} : g \in H/Z_i, i \in \omega \}.
\]
Let $G = N \times (H/R)$, with $g \in H/R$ acting on $N$ via the automorphism $\varphi_g$ as follows:
\[
\varphi_g(u_i) = u_i \\
\varphi_g(v_{i,h}) = v_{i,g+h} \\
\varphi_g(w_{i,h}) = w_{i,g+h} \\
\varphi_g(x_{i,h}) = x_{i,g+h} \\
\varphi_g(y_{i,h}) = y_{i,g+h} \\
\varphi_g(z_{i,h}) = z_{i,g+h}.
\]
Here, $\bar{g}$ is the image of $g$ under the quotient map $H/R \to H/V_i$ (or $H/W_i$, $H/X_i$, etc.). Recall that the semidirect product $G = N \times (H/R)$ is the group with underlying set $N \times (H/R)$ with group operation
\[
(n, g)(m, h) = (n \varphi_g(m), g + h).
\]
Note that $\varphi_g$ permutes the letters of $N$, and so given a word $A \in N$, $\varphi_g(A)$ is a word of the same length as $A$. We write $G$ multiplicatively.

**Lemma 13.3.1.** $H/R$ has a computable presentation.

**Proof.** It suffices to show that we can decide whether or not a relation of the form
\[
\sum_{i=1}^{k} \ell_i \alpha_i + \sum_{i=1}^{k} m_i \beta_i + \sum_{i=1}^{k} n_i \gamma_i = 0
\]
holds. This sum is equal to zero if and only if each $n_i = 0$ and for each $i$ we have $\ell_i \alpha_i + m_i \beta_i = 0$. So it suffices to decide, for a given $\ell$ and $m$ in $\mathbb{Z}$, whether $\ell \alpha_i = m \beta_i$.

Looking at $R$, $\ell \alpha_i = m \beta_i$ if and only if either
\[
(1) \text{ for some } t, \psi_{at}(i) = 0 \text{ and there is } s \in \mathbb{Z} \text{ such that } \ell = sp_i^t \text{ and } m = sq_i^t \text{ or }
\]
\[
(2) \text{ for some } t, \psi_{at}(i) = 1 \text{ and there is } s \in \mathbb{Z} \text{ such that } \ell = sp_i^t \text{ and } m = -sq_i^t.
\]
If $t > |\ell|$ or $t > |m|$ then neither of these can hold. So we just need to check, for each $t \leq |\ell|, |m|$, whether $\psi_{at}(i)$ converges. \qed
Lemma 13.3.2. $G$ has a computable presentation.

Proof. We just need to check that $\mathcal{H}/\mathcal{V}_i$, $\mathcal{H}/\mathcal{W}_i$, and so on have computable presentations. We will see that the embeddings of the computable presentation (from the previous lemma) of $\mathcal{H}/\mathcal{R}$ into these presentations are computable. Then the action $\varphi$ of $\mathcal{H}/\mathcal{R}$ on $\mathcal{N}$ is computable. We can construct a computable presentation of $G$ as the semidirect product $\mathcal{N} \rtimes (\mathcal{H}/\mathcal{R})$ under this computable action.

We need to decide whether in $\mathcal{H}/\mathcal{V}_i$ we have a relation

$$\sum_{j=1}^{k} \ell_j \alpha_j + \sum_{j=1}^{k} m_j \beta_j + \sum_{j=1}^{k} n_j \gamma_j = 0.$$ 

It suffices to decide, for a given $j$, whether

$$\ell \alpha_j + m \beta_j + n \gamma_j = 0.$$ 

If $j \neq i$, this is just as in the previous lemma. Otherwise, this holds if and only if $p_i$ divides $\ell$, $q_i$ divides $m$ for some $t$ with $\psi_{at}(i)$ 1, and $n = 0$. As before, we can check this computably.

The other cases—for $\mathcal{H}/\mathcal{W}_i$, $\mathcal{H}/\mathcal{X}_i$, and so on—are similar.

Lemma 13.3.3. $\mathcal{H}/\mathcal{R}$ is a torsion-free abelian group.

Proof. $\mathcal{H}/\mathcal{R}$ is abelian as $\mathcal{H}$ was abelian. Recall that

$$\mathcal{H}/\mathcal{R} = \left( \bigoplus_i (\alpha_i, \beta_i)/\mathcal{R}_i \right) \oplus \left( \bigoplus_i (\gamma_i) \right)$$

where $\mathcal{R}_i = \mathcal{R}_{i,t}$ if $\psi_{at}(i) \downarrow$ for some $t$, or no relation otherwise. So it suffices to show that $(\alpha_i, \beta_i)/\mathcal{R}_i$ is torsion-free. If $\mathcal{R}_i$ is no relation, then this is obvious. So now suppose that $\psi_{at}(i) = 0$ and that

$$k(m \alpha_i + n \beta_i) = \ell(p_i \alpha_i - q_i \beta_i)$$

in $(\alpha_i, \beta_i)$. Since $\mathcal{H}$ is torsion-free, we may assume that $\gcd(k, \ell) = 1$. Then $km = \ell p_i$ and $kn = -\ell q_i$. So we must have $k = \pm 1$, in which case $m \alpha_i + n \beta_i$ is already zero in $(\alpha_i, \beta_i)/\mathcal{R}_i$. Thus $(\alpha_i, \beta_i)/\mathcal{R}_i$ is torsion-free. The case where $\psi_{at}(i) = 1$ is similar.

Lemma 13.3.4. $G$ is left-orderable.

Proof. Since $\mathcal{H}/\mathcal{R}$ is a torsion-free abelian group, it is bi-orderable. $\mathcal{N}$ is bi-orderable as it is a free group. Then by the following claim, $G$ is left-orderable (see Theorem 1.6.2 of [KM96]).

Claim 13.3.5. Let $A \rtimes B$ be a semi-direct product of left-orderable groups. Then $A \rtimes B$ is left-orderable.
Proof. Let $\varphi$ be the action of $B$ on $A$. Let $\leq_A$ and $\leq_B$ be left-orderings on $A$ and $B$ respectively. Define $\leq$ on $A \times B$ as follows: $(a, b) \leq (a', b')$ if $b <_B b'$ or $b = b'$ and $\varphi_{b^{-1}}(a) \leq_A \varphi_{b'^{-1}}(a')$. This is clearly reflexive and symmetric. We must show that it is transitive and a left-ordering.

Suppose that $(a, b) \leq (a', b') \leq (a'', b'')$. Then $b \leq_B b'$, $b' \leq_B b''$. If $b <_B b''$, then $(a, b) \leq (a'', b'')$, so suppose that $b = b' = b''$. Then

$$\varphi_{b^{-1}}(a) \leq_A \varphi_{b^{-1}}(a') = \varphi_{b'^{-1}}(a') \leq_A \varphi_{b'^{-1}}(a'') = \varphi_{b^{-1}}(a'').$$

So $\varphi_{b^{-1}}(a) \leq_A \varphi_{b^{-1}}(a'')$ and so $(a, b) \leq (a'', b'')$. Thus $\leq$ is transitive.

Given $(a, b) \leq (a', b')$ we must show that $(a'', b'')(a, b) \leq (a'', b'')(a', b')$. We have that

$$(a'', b'')(a, b) = (a'', \varphi(b)(a), b'b)$$

and

$$(a'', b'')(a', b') = (a'', \varphi(b)(a'), b'b').$$

If $b <_B b'$, then $b'' b <_B b'' b'$, and so $(a'', b'')(a, b) \leq (a'', b'')(a', b')$. Otherwise, if $b = b'$ and $\varphi_{b^{-1}}(a) \leq_A \varphi_{b'^{-1}}(a')$, then $b'' b = b'' b'$ and

$$\varphi_{(\psi_{b''})^{-1}}(a'' \varphi(b)(a)) = \varphi_{(\psi_{b''})^{-1}}(a'' \varphi_{b^{-1}}(a))$$

$$\leq_A \varphi_{(\psi_{b''})^{-1}}(a'' \varphi_{b'^{-1}}(a'))$$

and

$$= \varphi_{(\psi_{b''})^{-1}}(a'' \varphi_{b'^{-1}}(a')).$$

So $(a'', b'')(a, b) \leq (a'', b'')(a', b')$. \qed

Note that if $\leq$ is any left-ordering on $G$, if $\psi_{\text{ht}}(i) = 0$ then $(\varepsilon, \alpha_i) > 1$ if and only if $(\varepsilon, \beta_i) > 1$. On the other hand, if $\psi_{\text{ht}}(i) = 1$ then $(\varepsilon, \alpha_i) > 1$ if and only if $(\varepsilon, \beta_i) < 1$. Later, in Definition 13.4.8, we will define existential formulas Same$(i)$ and Different$(i)$ (with no parameters) in the language of ordered groups. We would like to have that for any left-ordering $\leq$ on $G$, $(G, \leq) \models$ Same$(i)$ if and only if $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) < 1$, and $(G, \leq) \models$ Different$(i)$ if and only if $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) < 1$. We will not quite get this for every ordering $\leq$, but this will be true for those against which we want to diagonalize (see Lemma 13.3.7).

Definition 13.3.6. Fix a list $G_i \leq \omega$ of the (partial) computable structures in the language of ordered groups. Let $\psi$ be a partial computable function with $\psi(i) = 0$ if $(G_i, \leq_i) \models$ Different$(i)$ and $\psi(i) = 1$ if $(G_i, \leq_i) \models$ Same$(i)$. It is possible, a priori, that we have both $(G_i, \leq_i) \models$ Same$(i)$ and $(G_i, \leq_i) \models$ Different$(i)$; in this case, let $\psi(i)$ be defined according to whichever existential formula we find to be true first.

In fact, we will discover from the following lemma that we cannot have both $(G_i, \leq_i) \models$ Same$(i)$ and $(G_i, \leq_i) \models$ Different$(i)$.

Lemma 13.3.7. Fix $i$. Suppose that $G_i$ is isomorphic to $G$ and $\leq_i$ is a computable left-ordering of $F_i$. Let $\leq$ be an ordering on $G$ such that $(G, \leq) \cong (F_i, \leq_i)$. Then:

1. $(G, \leq) \models$ Same$(i)$ if and only if $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1$.
Lemma 13.4.1. \( (G, \leq) \equiv \text{Different}(i) \iff (\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) < 1. \)

This lemma will be proved later. We will now show how to use Lemma 13.3.7 to complete proof.

Lemma 13.3.8. \( \mathcal{G} \) has no computable presentation with a computable ordering.

Proof. Let \( i \) be an index for \( (\mathcal{F}_i, \leq_i) \) a computable presentation of \( \mathcal{G} \) with a computable left-ordering. Let \( \leq \) be an ordering on \( \mathcal{G} \) such that \( (\mathcal{G}, \leq) \cong (\mathcal{F}_i, \leq_i) \). Now by Lemma 13.3.7 either \( (\mathcal{G}, \leq) \equiv \text{Same}(i) \) or \( (\mathcal{G}, \leq) \equiv \text{Different}(i) \) (but not both). Suppose first that \( (\mathcal{G}, \leq) \equiv \text{Same}(i) \). So \( (\mathcal{F}_i, \leq_i) \equiv \text{Same}(i) \). By definition, \( \psi(i) = 1 \), say \( \psi_{\text{at } i}(i) = 1 \). Then, in \( \mathcal{H}/\mathcal{R} \), \( p_i^t \alpha_i = -q_i^t \beta_i \). So \( (\varepsilon, \alpha_i) > 1 \) if and only if \( (\varepsilon, \beta_i) < 1 \), contradicting Lemma 13.3.7 and the assumption that \( (\mathcal{G}, \leq) \equiv \text{Same}(i) \). The case of \( (\mathcal{G}, \leq) \equiv \text{Different}(i) \) is similar. Thus \( \mathcal{G} \) has no computable copy with a computable left-ordering.

All that remains to prove Theorem 13.1.2 is to define \( \text{Same}(i) \) and \( \text{Different}(i) \) and to prove Lemma 13.3.7.

13.4 \text{Same}(i), Different(i), and the Proof of Lemma 13.3.7

To define \( \text{Same}(i) \), we would like to come up with an existential formula which says that \( (\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1 \). A first attempt might be to try to find an existential formula defining \( (\varepsilon, \alpha_i) \) and an existential formula defining \( (\varepsilon, \beta_i) \). This cannot be done, but it will be helpful to think about how we might try to do this.

We will consider the problem of recognizing \( \alpha_i \) and \( \beta_i \) inside of \( \mathcal{H}/\mathcal{R} \) by their actions on \( \mathcal{N} \). Note that \( \alpha_i \) has the property that \( \varphi_{\alpha_i}(v_{i,0}) = v_{i,0} \), but \( \varphi_{p_i \alpha_i}(v_{i,0}) = v_{i,0 - i} \). So \( \alpha_i \) acts with order \( p_i \) on some element of \( \mathcal{N} \). In fact, it is not hard to see that the only elements which act with order \( p_i \) on an element of \( \mathcal{N} \) are the multiples \( n\alpha_i \) of \( \alpha_i \) where \( p_i + n \). (Note that if \( \alpha_i \) acts with order \( p_i \) on a word in \( \mathcal{N} \), then it either fixes or acts with order \( p_i \) on each letter in that word, and it acts with order \( p_i \) on at least one letter.)

One difficulty we have is that \( \mathcal{H}/\mathcal{R} \) and \( \mathcal{N} \) are not existentially definable inside of \( \mathcal{G} \). The problem is that if some element of \( \mathcal{G} \) satisfies a certain existential formula, then every conjugate of \( \mathcal{G} \) does as well. So it is only possible to define subsets of \( \mathcal{G} \) which are closed under conjugation. Given \( S \subseteq \mathcal{G} \), let \( S^\mathcal{G} \) be the set of all conjugates of \( S \) by elements of \( \mathcal{G} \).

In this section, we will take for granted the following lemma about existential definability in \( \mathcal{G} \). It will be proved in the following section. The lemma says that we can find \( \mathcal{H}/\mathcal{R} \) inside of \( \mathcal{G} \), up to conjugation, by an existential formula.

Lemma 13.4.1. \( (\mathcal{H}/\mathcal{R})^\mathcal{G} \) is \( \exists \)-definable within \( \mathcal{G} \) without parameters.

The different conjugates of \( \mathcal{H}/\mathcal{R} \) cannot be distinguished from each other. Instead, we will try to always work inside a single conjugate of \( \mathcal{H}/\mathcal{R} \). The following lemma tells us when we can do this.
Lemma 13.4.2. Suppose that \( r, s \in (\mathcal{H}/\mathcal{R})^g \) and \( rs \in (\mathcal{H}/\mathcal{R})^g \). Then there is \( A \in \mathcal{N} \) and \( g, h \in \mathcal{H}/\mathcal{R} \) such that

\[
\begin{align*}
  r &= (A, 0)(\varepsilon, g)(A^{-1}, 0) \\
  s &= (A, 0)(\varepsilon, h)(A^{-1}, 0).
\end{align*}
\]

Thus \( r \) and \( s \) commute.

The following remarks will be helpful not only here, but throughout the rest of the paper. They can all be checked by an easy computation.

Remark 13.4.3. If \( r \in (\mathcal{H}/\mathcal{R})^g \), then for some \( A \in \mathcal{N} \) and \( f \in \mathcal{H}/\mathcal{R} \) we can write \( r \) in the form

\[
  r = (A, 0)(\varepsilon, f)(A^{-1}, 0).
\]

Remark 13.4.4. Let \( r = (A, f) \) be an element of \((\mathcal{H}/\mathcal{R})^g\). If \( K \subseteq \mathcal{H}/\mathcal{R} \), then \( r \in K^g \) if and only if \( f \in K \).

Remark 13.4.5. If \( \varphi_g(B) = B \), then

\[
(AB, 0)(\varepsilon, g)(AB, 0)^{-1} = (A, 0)(\varepsilon, g)(A, 0)^{-1}.
\]

Proof of Lemma 13.4.2. Using Remark 13.4.3, let

\[
\begin{align*}
  r &= (A, 0)(\varepsilon, g)(A^{-1}, 0) \\
  s &= (B, 0)(\varepsilon, h)(B^{-1}, 0) \\
  rs &= (C, 0)(\varepsilon, g + h)(C^{-1}, 0).
\end{align*}
\]

By conjugating \( r \) and \( s \) by some further element of \( \mathcal{G} \) (and noting that the conclusion of the lemma is invariant under conjugation), we may assume that \( A^{-1}B \) is a reduced word, that is, that \( A \) and \( B \) have no common non-trivial initial segment. Using Remark 13.4.5, we may assume that \( A\varphi_g(A^{-1}) \), \( B\varphi_h(B^{-1}) \), and \( C\varphi_{g+h}(C^{-1}) \) are reduced words. Indeed, if, for example, \( A\varphi_g(A^{-1}) \) was not a reduced word, then we could write \( A = A'B \) where \( B \) is a word which is fixed by \( \varphi_g \), and such that \( A'\varphi_g(A'^{-1}) \) is a reduced word. Then, by Remark 13.4.5,

\[
(A, 0)(\varepsilon, g)(A, 0)^{-1} = (A'B, 0)(\varepsilon, g)(A'B, 0)^{-1} = (A', 0)(\varepsilon, g)(A', 0)^{-1}.
\]

So we may replace \( A \) by \( A' \).

We have

\[
(A, 0)(\varepsilon, g)(A^{-1}, 0)(B, 0)(\varepsilon, h)(B^{-1}, 0) = (C, 0)(\varepsilon, g + h)(C^{-1}, 0).
\]

Multiplying out the first coordinates, we get

\[
A\varphi_g(A^{-1})\varphi_g(B)\varphi_{g+h}(B^{-1}) = C\varphi_{g+h}(C^{-1}).
\]
CHAPTER 13. LEFT-ORDERABLE COMPUTABLE GROUPS

By the assumptions we made above, both sides are reduced words. A is an initial segment of the left hand side, so it must be an initial segment of the right hand side, and hence an initial segment of C. On the other hand, taking inverses of both sides, we get

$$\varphi_{g+h}(B)\varphi_g(B^{-1})\varphi_g(A)A^{-1} = \varphi_{g+h}(C)C^{-1}.$$ 

Once again both sides are reduced words, and $\varphi_{g+h}(B)$ is an initial segment of the left hand side, and hence of $\varphi_{g+h}(C)$. But then B is an initial segment of C. So it must be that A is an initial segment of B or vice versa. This contradicts one of our initial assumptions unless A or B (or both) is the trivial word. Suppose it was A (the case of B is similar). Then

$$\varphi_g(B)\varphi_{g+h}(B^{-1}) = C\varphi_{g+h}(C^{-1})$$

and both sides are reduced words. Then we get that $C = B$ and $C = \varphi_g(B)$. So

$$r = (\varepsilon, g) = (B, 0)(\varepsilon, g)(B, 0)^{-1}$$

by Remark 13.4.5. □

Above, we noted that the set $\{n\alpha_i : p_i \perp n\}$ is the set of elements of $H/R$ which act with order $p_i$ on an element of $N$. Our next goal is to show that if we close under conjugation, then this set (and a few other similar sets) are definable. The key is the following remark which follows easily from Lemma 13.4.2.

Remark 13.4.6. Fix $r, s_1, s_2 \in (H/R)^G$. Suppose that $rs_1 \in (H/R)^G$ and $rs_2 \in (H/R)^G$ but $s_1$ and $s_2$ do not commute. By Lemma 13.4.2 we can write

$$r = (A, 0)(\varepsilon, f)(A^{-1}, 0) = (B, 0)(\varepsilon, f)(B^{-1}, 0)$$

$$s_1 = (A, 0)(\varepsilon, g)(A^{-1}, 0)$$

$$s_2 = (B, 0)(\varepsilon, h)(B^{-1}, 0).$$

Then there is some element of $N$ which is fixed by $\varphi_f$ but which is not fixed by $\varphi_g$.

Indeed, since $(A, 0)(\varepsilon, f)(A^{-1}, 0) = (B, 0)(\varepsilon, f)(B^{-1}, 0)$, we see that

$$B^{-1}A = \varphi_f(B^{-1}A).$$

Suppose for the sake of contradiction that $\varphi_g$ also fixes $B^{-1}A$. Then

$$s_1 = (A, 0)(A^{-1}B, 0)(\varepsilon, g)(B^{-1}A, 0)(A^{-1}, 0) = (B, 0)(\varepsilon, g)(B^{-1}, 0).$$

So $s_1$ and $s_2$ would commute. This is a contradiction. So there is some element of $N$ which is fixed by $\varphi_f$ but which is not fixed by $\varphi_g$.

Lemma 13.4.7. There are $3$-formulas which express each of the following statements about an element $a$ in $G$:
(1) \( a \in \{n\alpha_i : p_i + n\}^\mathcal{G} \).

(2) \( a \in \{n\beta_i : q_i + n\}^\mathcal{G} \).

(3) \( a \in \{n\gamma_i : r_i + n\}^\mathcal{G} \).

(4) \( a \in \{n(\alpha_i - \gamma_i) : p_i, r_i + n\}^\mathcal{G} \).

(5) \( a \in \{n(\beta_i - \gamma_i) : q_i, r_i + n\}^\mathcal{G} \).

Proof. For (1), we claim that \( a \in \{n\alpha_i : p_i + n\}^\mathcal{G} \) if and only if \( a \in (\mathcal{H}/\mathcal{R})^\mathcal{G} \) and there is \( b \in (\mathcal{H}/\mathcal{R})^\mathcal{G} \) such that \( a^n b \in (\mathcal{H}/\mathcal{R})^\mathcal{G} \) but \( a \) and \( b \) do not commute. This is expressed by an \( \exists \)-formula by Lemma 13.4.1.

Suppose that \( a \) satisfies this \( \exists \)-formula, as witnessed by \( b \). Let \( a = (A, f) \) and \( b = (B, g) \). Then by Remark 13.4.6 (taking \( r = a^n \), \( s_1 = a \), and \( s_2 = b \)), there is an element of \( \mathcal{N} \) which is fixed by \( \varphi_{pi,f} \) but not by \( \varphi_f \). Thus we see that \( p_i f = 0 \) but \( f \neq 0 \) in \( \mathcal{H}/\mathcal{V}_i \), and \( f = n\alpha_i \) for some \( n \) with \( p_i \neq n \). (It must be in \( \mathcal{H}/\mathcal{V}_i \), because this cannot happen in any of \( \mathcal{H}/\mathcal{V}_j \) for \( j \neq i \), or \( \mathcal{H}/\mathcal{W}_j \), \( \mathcal{H}/\mathcal{X}_j \), \( \mathcal{H}/\mathcal{Y}_j \), or \( \mathcal{H}/\mathcal{Z}_j \).) Thus by Remark 13.4.4, \( a \in \{n\alpha_i : p_i + n\}^\mathcal{G} \).

On the other hand, suppose that \( a \in \{n\alpha_i : p_i + n\}^\mathcal{G} \). Write

\[
a = (A, 0)(\varepsilon, n\alpha_i)(A^{-1}, 0).
\]

with \( p_i \) not dividing \( n \). Then let \( b = (Av_i, 0)(\varepsilon, n\alpha_i)((Av_i)^{-1}, 0) \). By Remark 13.4.5, since \( \varphi_{np_i,\alpha_i}(v_i, 0) = v_i,0 \), we have

\[
a^n = (A, 0)(\varepsilon, np_i\alpha_i)(A^{-1}, 0) = (Av_i, 0)(\varepsilon, np_i\alpha_i)((Av_i)^{-1}, 0).
\]

So \( a^n b \in (\mathcal{H}/\mathcal{R})^\mathcal{G} \). On the other hand,

\[
ab = (Av_i, 0)\varphi_{n\alpha_i}(v_i, 0)^{-1}\varphi_{2n\alpha_i}(A^{-1}, 2n\alpha_i)
\]

and

\[
ba = (Av_i, 0)\varphi_{n\alpha_i}(v_i, 0)^{-1}\varphi_{2n\alpha_i}(A^{-1}, 2n\alpha_i).
\]

So \( a \) does not commute with \( b \) since \( \varphi_{n\alpha_i}(v_i, 0) = v_i, n\alpha_i \neq v_i, 0 \). The proofs of (2) and (3) are similar.

For (4), we claim that \( a \in \{n(\alpha_i - \gamma_i) : p_i, r_i + n\}^\mathcal{G} \) if and only if there are \( b_1 \in \{n\alpha_i : p_i + n\}^\mathcal{G} \), \( b_2 \in \{n\gamma_i : r_i + n\}^\mathcal{G} \), and \( c \in (\mathcal{H}/\mathcal{R})^\mathcal{G} \) such that \( a = b_1 b_2^{-1} \), \( ac, ab \in (\mathcal{H}/\mathcal{R})^\mathcal{G} \), and \( c \) does not commute with \( b_1 \).

Suppose that there are such \( b_1 \), \( b_2 \), and \( c \). We can write \( b_1 = (B_1, m\alpha_i) \) with \( p_i + m \) and \( b_2 = (B_2, n\gamma_i) \) with \( r_i + \gamma_i \). Thus we can write \( a = b_1 b_2^{-1} = (A, m\alpha_i - n\gamma_i) \). By Remark 13.4.6 (with \( r = a \), \( s_1 = b_1 \), and \( s_2 = c \)), \( \varphi_{m\alpha_i-n\gamma_i} \) fixes some element of \( \mathcal{N} \) which is not fixed by \( \varphi_{m\alpha_i} \). Thus, in one of \( \mathcal{H}/\mathcal{V}_j \), \( \mathcal{H}/\mathcal{W}_j \), \( \mathcal{H}/\mathcal{X}_j \), \( \mathcal{H}/\mathcal{Y}_j \), or \( \mathcal{H}/\mathcal{Z}_j \) for some \( j \) we have \( m\alpha_i - n\gamma_i = 0 \) but \( m\alpha_i \neq 0 \). Since \( p_i + m \), it must be in \( \mathcal{H}/\mathcal{Y}_i \). So \( n = m \). Note that \( p_i \) and \( r_i \) do not divide \( n \).
On the other hand, suppose that \( a \in \{ n(\alpha_i - \gamma_i) : p_i, r_i + n \}^G \). Then write
\[
a = (A, 0)(\varepsilon, n\alpha_i - n\gamma_i)(A^{-1}, 0).
\]
with \( p_i \) and \( r_i \) not dividing \( n \). Let
\[
b_1 = (A, 0)(\varepsilon, n\alpha_i)(A^{-1}, 0) \quad \text{and} \quad b_2 = (A, 0)(\varepsilon, n\gamma_i)(A^{-1}, 0)
\]
and let
\[
c = (Ay_i, 0, 0)(\varepsilon, n\alpha_i)((Ay_i, 0)^{-1}, 0).
\]
Then \( a = b_1b_2^{-1} \). Clearly \( ab_1 \in (\mathcal{H}/\mathcal{R})^G \). Also, since \( \varphi_{n\alpha_i - n\gamma_i}(y_i, 0) = y_i, 0 \),
\[
ac = ca = (Ay_i, 0, 0)(\varepsilon, 2n\alpha_i - n\gamma_i)((Ay_i, 0)^{-1}, 0).
\]
So \( ac \in (\mathcal{H}/\mathcal{R})^G \) and \( a \) and \( c \) commute. On the other hand, \( b_1 \) does not commute with \( c \) since \( \varphi_{r_i}(y_i, 0) = y_i, t_{r_i} \neq y_i, 0 \) as \( p_i \) does not divide \( \ell \).

We will now define Same(i) and Different(i).

**Definition 13.4.8.** Same(i) says that there are \( a, b, \) and \( c \) such that:

1. \( a, b, c, \) and \( ab \) are in \( (\mathcal{H}/\mathcal{R})^G \),
2. \( a > 1 \iff b > 1 \),
3. \( a \in \{ n\alpha_i : p_i + n \}^G \),
4. \( b \in \{ n\beta_i : q_i + n \}^G \),
5. \( c \in \{ n\gamma_i : r_i + n \}^G \),
6. \( ac^{-1} \in \{ n(\alpha_i - \gamma_i) : p_i, r_i + n \}^G \),
7. \( bc^{-1} \in \{ n(\beta_i - \gamma_i) : q_i, r_i + n \}^G \).

Different(i) is defined in the same way as Same(i), except that in (2) we ask that \( a > 1 \) if and only if \( b < 1 \).

Suppose, for simplicity, that \( a, b, \) and \( c \) are all in \( \mathcal{H}/\mathcal{R} \). Then we would have that \( a = (\varepsilon, \ell\alpha_i), b = (\varepsilon, m\beta_i), \) and \( c = (\varepsilon, n\gamma_i) \). Now \( ac^{-1} = (\varepsilon, \ell\alpha_i - n\gamma_i) \) is a power of \( (\varepsilon, \alpha_i - \gamma_i) \), and so \( \ell = n \). Similarly, \( bc^{-1} = (\varepsilon, m\beta_i - n\gamma_i) \) is a power of \( (\varepsilon, \beta_i - \gamma_i) \), and so \( m = n \). Thus \( \ell = m \). Since \( (\varepsilon, \ell\alpha_i) > 1 \iff (\varepsilon, \ell\beta_i) > 1 \), \( (\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1 \). Checking that this works even if \( a, b, \) and \( c \) are conjugates of \( \mathcal{H}/\mathcal{R} \) is the heart of Lemma 13.4.9.

**Lemma 13.4.9.** Let \( \leq \) be a left-ordering on \( G \). Then:

1. If \( (\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1 \), then \( (G, \leq) \models \text{Same}(i) \).
(2) If \((\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) < 1\), then \((G, \leq) \models \text{Different}(i)\).

(3) If \(\psi(i) \downarrow\), then \((\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1\) if and only if \((G, \leq) \models \text{Same}(i)\).

(4) If \(\psi(i) \downarrow\), then \((\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) < 1\) if and only if \((G, \leq) \models \text{Different}(i)\).

Proof. First, for (1), suppose that \((\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1\). Then \((G, \leq) \models \text{Same}(i)\) as witnessed by \(c = (\varepsilon, \alpha_i), c = (\varepsilon, \beta_i), \) and \(c = (\varepsilon, \gamma_i)\). (2) is similar.

Now for (3), suppose that \((G, \leq) \models \text{Same}(i)\) as witnessed by \(a, b, \) and \(c\), and that \(\psi(i) \downarrow\). Let \(f, g, \) and \(h\) be the second coordinates of \(a, b, \) and \(c\) respectively. Write \(f = \ell\alpha_i\) with \(p_i + \ell, g = m\beta_i\) with \(q_i + m, \) and \(h = n\gamma_i\) with \(r_i + h\). Then since \(f - h\) is a multiple of \(\alpha_i - \gamma_i, \ell = n\). Similarly, \(m = n, \) and so \(\ell = m\).

Since \(ab \in (H/R)^G\) and \(a\) and \(b\) commute, by Lemma 13.4.2 we can write

\[
a = (B, 0)(\varepsilon, \ell\alpha_i)(B, 0)^{-1}
\]

and

\[
b = (B, 0)(\varepsilon, \ell\beta_i)(B, 0)^{-1}.
\]

Now since \(\psi(i) \downarrow,\) in \(H/R\) either \(p_i^t\alpha_i = q_i^t\beta_i\) or \(p_i^t\alpha_i = -q_i^t\beta_i\) for some \(t\). In the second case, \(a^{\ell^t} = b^{\ell^t}\) which contradicts the fact that \(a > 1 \iff b > 1\). Thus \(p_i^t\alpha_i = q_i^t\beta_i,\) and so \((\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1\).

(4) is proved similarly.

Proof of Lemma 13.3.7. We will prove (1): \((G, \leq) \models \text{Same}(i)\) if and only if \((\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1\). The proof of (2) is similar. The right to left direction follows immediately from (1) of Lemma 13.4.9. For the left to right direction, suppose that \((F_i, \leq) \models \text{Same}(i)\). Then \(\psi(i) \downarrow\). Then the lemma follows from (3) of Lemma 13.4.9.

13.5 An Existential Definition of \((H/R)^G\)

The goal of this section is to prove Lemma 13.4.1, which says that \((H/R)^G\) is definable within \(G\) by an existential formula. To prove this lemma, we will first have to give a detailed analysis of which elements of \(G\) commute with each other.

The first lemma is the analogue of the following well-known fact about free groups: two elements \(a\) and \(b\) in a free group commute if and only if there is \(c\) such that \(a = c^m\) and \(b = c^n\) (see [LS01, Proposition 2.17]).

Lemma 13.5.1. Let \(r, s \in G\) commute. Then there are \(W, V \in N, x, y, z \in H/R,\) and \(k, \ell \in \mathbb{Z}\) such that

\[
r = (W, 0)(V, x)^k(\varepsilon, y)(W, 0)^{-1}
\]

and

\[
s = (W, 0)(V, x)^\ell(\varepsilon, z)(W, 0)^{-1}.
\]

If \(k \neq 0\) then \(\varphi_r(V) = V,\) and if \(\ell \neq 0\) then \(\varphi_s(V) = V.\)
It is easy to check that two such elements commute.

**Proof.** Suppose that \( rs = sr \). Let \( r = (A, g) \) and \( s = (B, h) \). Then we find that

\[
rs = (A, g)(B, h) = (A\varphi_g(B), g + h)
\]

\[
sr = (B, h)(A, g) = (B\varphi_h(A), g + h).
\]

So \( A\varphi_g(B) = B\varphi_h(A) \) in \( N \). Write

\[
A = a_0\cdots a_{m-1} \quad \text{and} \quad B = b_0\cdots b_{n-1}
\]

as reduced words. So

\[
a_0\cdots a_{m-1}\varphi_g(b_0)\cdots\varphi_g(b_{n-1}) = b_0\cdots b_{n-1}\varphi_h(a_0)\cdots\varphi_h(a_{m-1}).
\]

We divide into several cases.

**Case 1.** \( A \) is the trivial word.

We must have \( B = \varphi_g(B) \). Then \( r = (\varepsilon, g) \) and \( s = (B, h) \). Take \( W = \varepsilon, V = B, x = h, y = g, z = 0, k = 0, \) and \( \ell = 1 \).

**Case 2.** \( B \) is the trivial word.

We must have \( A = \varphi_h(A) \). Then \( r = (A, g) \) and \( s = (\varepsilon, h) \). Take \( W = \varepsilon, V = A, x = g, y = 0, z = h, k = 1, \) and \( \ell = 0 \).

**Case 3.** Neither \( A \) nor \( B \) is the trivial word, and both \( A\varphi_g(B) \) and \( B\varphi_h(A) \) are reduced words.

We have \( A\varphi_g(B) = B\varphi_h(A) \) as reduced words. Assume without loss of generality that \( |A| = m \geq n = |B| \). Then \( n, m > 0 \) and

\[
a_0\cdots a_{m-1}\varphi_g(b_0)\cdots\varphi_g(b_{n-1}) = b_0\cdots b_{n-1}\varphi_h(a_0)\cdots\varphi_h(a_{m-1})
\]

as reduced words. So

\[
a_i = b_i \quad \text{for} \quad 0 \leq i < n
\]

\[
a_i = \varphi_h(a_{i-n}) \quad \text{for} \quad n \leq i < m
\]

\[
\varphi_g(b_i) = \varphi_h(a_{m-n+i}) \quad \text{for} \quad 0 \leq i < n.
\]

Let \( d = \gcd(m, n) \). (This is where we use the fact that \( m, n > 0 \).) Let \( n' = n/d \) and \( m' = m/d \).

Given \( p, q \geq 0 \), write \( i = qn - pm + r \) with \( 0 \leq r < d \) and assume that \( 0 \leq i < m \). Note that every \( i, 0 \leq i < m, \) can be written in such a way. We claim that

\[
a_i = \varphi_{qh - pg}(a_r).
\]
We argue by induction, ordering pairs \((q,p)\) lexicographically. For the base case \(p = q = 0\) we note that \(a_r = \varphi_0(a_r)\). Otherwise, if \(n \leq i < m\), then we must have \(q > 0\). By the induction hypothesis, \(a_{i-n} = \varphi_{(q-1)h-pg}(a_r)\). So

\[
\begin{align*}
    a_i &= \varphi_h(a_{i-n}) = \varphi_{qh-pg}(a_r).
\end{align*}
\]

If \(0 \leq i < n\), and \((q,p) \neq (0,0)\), then \(q > 0\) and \(p > 0\). Note that \(a_{m-n+i} = \varphi_{(q-1)h-(p-1)g}(a_r)\) by the induction hypothesis and so

\[
\begin{align*}
    a_i &= b_i = \varphi_{h-g}(a_{m-n+i}) = \varphi_{qh-pg}(a_r).
\end{align*}
\]

This completes the induction.

Write \(d = qn - pm\) with \(p,q \geq 0\). Let \(f =qh - pg\). Then each \(i, 0 \leq i < m\), can be written as \(i = kd + r\) with \(0 \leq r < d\), and so \(a_i = \varphi_kf(a_r)\).

Let \(C = a_0a_d\). Then

\[
A = C\varphi_f(C)\cdots\varphi_{(m-1)f}(C)
\]

and so

\[
r = (A,g) = (C,f)^{m'}(\varepsilon,g-m'f).
\]

Since for \(0 \leq i < n\), \(a_i = b_i\), we have

\[
s = (B,h) = (C,f)^{m'}(\varepsilon,h-n'f).
\]

This is in the desired form: take \(W = \varepsilon\), \(V = C\), \(x = f\), \(y = g-m'f\), \(z = h-n'f\), \(k = m'\), and \(\ell = n'\).

We still have to show that \(\varphi_y(V) = \varphi_z(V) = V\). Noting that

\[
(n'q-1)n - (n'p)m = n'(qn-pm) - n = n'd - n = 0
\]

we have, for all \(0 \leq r < d\),

\[
a_r = \varphi_{(n'q-1)h-n'pg}(a_r) = \varphi_{n'f-h}(a_r).
\]

Similarly,

\[
a_r = \varphi_{m'f-g}(a_r).
\]

Hence \(\varphi_{g-m'f}(C) = \varphi_{h-n'f}(C) = C\).

**Case 4.** Neither \(A\) nor \(B\) is the trivial word, and both \(B^{-1}A\) and \(\varphi_h(A)\varphi_g(B)^{-1}\) are reduced words.

Note that \(B^{-1}A = \varphi_h(A)\varphi_g(B)^{-1}\). We can make a transformation to reduce this to the previous case. Let

\[
A' = B^{-1} \quad B' = \varphi_h(A) \quad g' = -h \quad h' = g.
\]
Then \( A'\varphi_g(B') = B'\varphi_h(A') \) and these are reduced words. Hence by the previous case there are \( C \in \mathcal{N}, f \in \mathcal{H}/\mathcal{R} \), and \( m, n \in \mathbb{Z} \) such that

\[
(A', g') = (C, f)^m(\varepsilon, g' - mf)
\]

and

\[
(B', h') = (C, f)^n(\varepsilon, h' - nf)
\]

and such that \( \varphi_{g'-mf}(C) = C \) and \( \varphi_{h'-nf}(C) = C \). Now

\[
(A, g) = (\varepsilon, h)(\varphi_h(A), g)(\varepsilon, h)
= (\varepsilon, h)(B', h')(\varepsilon, h)
= (\varepsilon, h)(C, f)^n(\varepsilon, h' - nf)(\varepsilon, h)
= (\varphi_{-h}(C), f)^n(\varepsilon, g - nf).
\]

Note that \( \varphi_{g-nf}(C) = \varphi_{h'-nf}(C) = C \), and so \( \varphi_{g-nf}(\varphi_{-h}(C)) = \varphi_{-h}(C) \). Similarly,

\[
(B, h) = (\varepsilon, h)(B^{-1}, -h)^{-1}(\varepsilon, h)
= (\varepsilon, h)(A', g')^{-1}(\varepsilon, h)
= (\varepsilon, h)(\varepsilon, g' - mf)^{-1}(C, f)^{-m}(\varepsilon, h)
= (\varepsilon, mf)(C, f)^{-m}(\varepsilon, h)
= (\varphi_{mf}(C), f)^{-m}(\varepsilon, h + mf).
\]

Since \( \varphi_{h+mf}(C) = \varphi_{g'-mf}(C) = C \), \( \varphi_{mf}(C) = \varphi_{-h}(C) \). So

\[
(B, h) = (\varphi_{-h}(C), f)^{-m}(\varepsilon, h + mf).
\]

This completes this case, taking \( W = \varepsilon, V = \varphi_{-h}(C), x = f, y = g - nf, z = h + mf, k = n \), and \( \ell = -m \).

**Case 5.** \(|A| = 1, B \) is not the trivial word, and neither \( A\varphi_g(B) = B\varphi_h(A) \) nor \( B^{-1}A = \varphi_h(A)\varphi_g(B^{-1}) \) are reduced words.

Let \( A = a \). Then \( a^{-1} = \varphi_g(b_0) \) and \( b_{n-1} = \varphi_h(a^{-1}) \). Recall that \( B = b_0 \cdots b_{n-1} \). From the non-reduced words \( A\varphi_g(B) = B\varphi_h(A) \), we get, as reduced words,

\[
\varphi_g(b_1)\varphi_g(b_2)\cdots\varphi_g(b_{n-1}) = b_0b_1\cdots b_{n-2}.
\]

Then, for \( 0 \leq i < n - 1 \) we get \( \varphi_g(b_{i+1}) = b_i \). Thus \( a = \varphi_{ng+h}(a) \). Also, letting \( C = b_0, r = (\varphi_g(C)^{-1}, g) = (C, g)^{-1} \).

and

\[
s = (C, g)^n(\varepsilon, h + ng)
\]

Note that \( \varphi_{h+ng}(C) = \varphi_{h+ng}(b_0) = b_0 \) since \( a = \varphi_{ng+h}(a) \) and \( b_0 = \varphi_g(a^{-1}) \).

So in this case we take \( W = \varepsilon, V = C, x = g, y = 0, z = h + ng, k = -1 \), and \( \ell = n \).
**Case 6.** \(|B| = 1\), \(A\) is not the trivial word, and neither \(A\varphi_g(B) = B\varphi_h(A)\) nor \(B^{-1}A = \varphi_h(A)\varphi_g(B^{-1})\) are reduced words.

This case is similar to the previous case.

**Case 7.** \(|A|, |B| \geq 2\) and neither \(A\varphi_g(B) = B\varphi_h(A)\) nor \(B^{-1}A = \varphi_h(A)\varphi_g(B^{-1})\) are reduced words.

We have \(b_{n-1} = \varphi_h(a_0)^{-1}\) and \(\varphi_h(a_{m-1}) = \varphi_g(b_{n-1})\) and so

\[
\varphi_g(a_0) = \varphi_g(a_0)^{-1} = \varphi_{g^{-1}}(b_{n-1})^{-1} = a_{m-1}^{-1}.
\]

Letting

\[
A' = a_1 \cdots a_{m-2} = a_0^{-1}A\varphi_g(a_0)
\]

and

\[
B' = a_0^{-1}b_0b_1 \cdots b_{n-2} = a_0^{-1}B\varphi_h(a_0)
\]

we have

\[
B'\varphi_h(A')\varphi_g(B')^{-1} = B'b_{n-1}\varphi_h(a_0)\varphi_h(A')\varphi_h(a_{m-1})\varphi_g(b_{n-1})^{-1}\varphi_g(B')^{-1}
\]

\[
= a_0^{-1}B\varphi_h(A)\varphi_g(B)^{-1}a_{m-1}
\]

\[
= a_0^{-1}Aa_{m-1}^{-1}
\]

\[
= A'.
\]

So \((A', g)\) and \((B', h)\) still commute.

Note that \(|A'| < |A|\) and \(|B'| \leq |B|\). So we only have to repeat this finitely many times until we are in one of the other cases. Thus, for some word \(D\) we get reduced words

\[
A' = DA\varphi_g(D^{-1})
\]

and

\[
B' = DB\varphi_h(D^{-1})
\]

which fall into one of the other cases. So

\[
(A', g) = (C, f)^m(\varepsilon, g - mf)
\]

and

\[
(B', h) = (C, f)^n(\varepsilon, h - nf).
\]

Thus

\[
r = (DA'\varphi_g(D^{-1}), g) = (D, 0)(A', g)(D^{-1}, 0)
\]

and

\[
s = (DB'\varphi_h(D^{-1}), h) = (D, 0)(B', h)(D^{-1}, 0)
\]

are in the desired form.
The next lemma gives a criterion for knowing that an element \( r \) is in \((\mathcal{H}/\mathcal{R})^G\), but it requires knowing that two particular elements \( s_1 \) and \( s_2 \) are not in \((\mathcal{H}/\mathcal{R})^G\). This does not seem useful yet, but in Lemma 13.5.4 we will show that any three elements \( s_1, s_2, \) and \( s_3 \), such that \( r \) commutes with each of them but \( s_1, s_2, \) and \( s_3 \) pairwise do not commute, give rise to two such elements which are not in \((\mathcal{H}/\mathcal{R})^G\).

**Lemma 13.5.2.** Let \( r, s_1, s_2 \in G \). Suppose that \( r \) commutes with \( s_1 \) and \( s_2 \), but \( s_1 \) and \( s_2 \) do not commute. If \( s_1, s_2 \notin (\mathcal{H}/\mathcal{R})^G \), then \( r \in (\mathcal{H}/\mathcal{R})^G \).

**Proof.** Suppose to the contrary that \( r \notin (\mathcal{H}/\mathcal{R})^G \). Since \( r \) and \( s_1 \) commute, and \( r \) and \( s_2 \) commute, by Lemma 13.5.1 we can write

\[
\begin{align*}
    r &= (A, 0)(C, f_1)^{m_1}(\varepsilon, g_1)(A^{-1}, 0) = (B, 0)(D, f_2)^{m_2}(\varepsilon, g_2)(B^{-1}, 0) \\
    s_1 &= (A, 0)(C, f_1)^{m_1}(\varepsilon, h_1)(A^{-1}, 0) \\
    s_2 &= (B, 0)(D, f_2)^{m_2}(\varepsilon, h_2)(B^{-1}, 0)
\end{align*}
\]

Since \( r, s_1, \) and \( s_2 \) are not in \((\mathcal{H}/\mathcal{R})^G \), \( C \) and \( D \) are non-trivial and \( m_1, m_2, n_1, n_2 \neq 0 \). So \( \varphi_{g_1}(C) = \varphi_{h_1}(C) = C \) and \( \varphi_{g_2}(D) = \varphi_{h_2}(D) = D \). Moreover, we will argue that we may assume that

\[ C\varphi_{f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C) \text{ and } D\varphi_{f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D) \]

are reduced words. If the former is not a reduced word, then it must have length at least 2, and we can write \( C = aC'\varphi_{f_1}(a^{-1}) \). Then

\[
C\varphi_{f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C) = aC'\varphi_{f_1}(C')\cdots\varphi_{(m_1-1)f_1}(C')\varphi_{m_1f_1}(a^{-1})
\]

and so, since \( \varphi_{g_1} \) fixes \( C \) and hence \( a \),

\[
r = (Aa, 0)(C', f_1)^{m_1}(\varepsilon, g_1)(a^{-1}A^{-1}, 0).
\]

Similarly,

\[
s_1 = (Aa, 0)(C', f_1)^{m_1}(\varepsilon, h_1)(a^{-1}A^{-1}, 0).
\]

So we may replace \( A \) by \( Aa \) and \( C \) by \( C' \). We can continue to do this until

\[ C\varphi_{f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C) \]

is a reduced word. The same argument works for

\[ D\varphi_{f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D). \]

Rearranging the two expressions for \( r \), we get

\[
(B^{-1}A, 0)(C, f_1)^{m_1}(\varphi_{g_1}(A^{-1}B), g_1) = (D, f_2)^{m_2}(\varepsilon, g_2).
\]
Looking at the first coordinate,
\[ B^{-1}AC\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C)\varphi_{m_1f_1+g_1}(A^{-1}B) = D\varphi_{f_2}(D)\varphi_{2f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D). \]

We claim that we can write \( B^{-1}A = E_2^{-1}E_1 \) where \( \varphi_{g_1}(E_1) = \varphi_{h_1}(E_1) = E_1 \) and \( \varphi_{g_2}(E_2) = \varphi_{h_2}(E_2) = E_2 \). Recall that
\[ C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C) \]
is a non-trivial reduced word. Taking a high enough power \( \ell \), the length of
\[ (C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C))^{\ell} \]
as a reduced word is more than twice the length of \( B^{-1}A \). Then
\[ B^{-1}A(C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C))^{\ell}\varphi_{m_1f_1+g_1}(A^{-1}B) = (D\varphi_{f_2}(D)\varphi_{2f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D))^{\ell}. \]

We can write \( B^{-1}A = E_2^{-1}E_1 \) as a reduced word where \( E_2^{-1} \) appears at the start of the right hand side when it is written as a reduced word, and \( E_1 \) cancels with the beginning of \( (C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C))^{\ell} \). Thus \( E_1 \) is fixed by \( \varphi_{g_1} \) and \( \varphi_{h_1} \) since they fix each letter appearing in the word \( (C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C))^{\ell} \), and \( E_2 \) is fixed by \( \varphi_{g_2} \) and \( \varphi_{h_2} \) since they fix each letter appearing in the right hand side.

Since \( E_2B^{-1} = E_1A^{-1} \),
\[ E_2B^{-1}rBE_2^{-1} = (E_1,0)(C,f_1)^{m_1}(\varepsilon,g_1)(E_1^{-1},0) \]
\[ = (E_2,0)(D,f_2)^{m_2}(\varepsilon,g_2)(E_2^{-1},0) \]
\[ E_2B^{-1}s_1BE_2^{-1} = (E_1,0)(C,f_1)^{m_1}(\varepsilon,h_1)(E_1^{-1},0) \]
\[ = (E_2,0)(D,f_2)^{m_2}(\varepsilon,h_2)(E_2^{-1},0) \]

So, applying the automorphism of \( \mathcal{G} \) given by conjugating by \( E_2B^{-1} \) (and noting that this automorphism fixes \( (\mathcal{H}/\mathcal{R})_G \)) we may assume from the beginning that \( \varphi_{g_1}(A) = \varphi_{h_1}(A) = A \) and \( \varphi_{g_2}(B) = \varphi_{h_2}(B) = B \). Thus
\[ r = (A,0)(C,f_1)^{m_1}(A^{-1},0)(\varepsilon,g_1) = (B,0)(D,f_2)^{m_2}(B^{-1},0)(\varepsilon,g_2) \]
\[ s_1 = (A,0)(C,f_1)^{m_1}(A^{-1},0)(\varepsilon,h_1) \]
\[ s_2 = (B,0)(D,f_2)^{m_2}(B^{-1},0)(\varepsilon,h_2). \]

Now looking at the first coordinate, we have
\[ AC\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C)\varphi_{m_1f_1+g_1}(A^{-1}) = BD\varphi_{f_2}(D)\varphi_{2f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D)\varphi_{m_2f_2}(B)^{-1}. \]
Our next step is to argue that we may assume that these are reduced words. Suppose that there was some cancellation, say $A = A' a$ and $C = a^{-1} C'$. Let $C^* = C' \varphi_{f_1}(a^{-1})$. Then

\[
AC \varphi_{f_1}(C) \varphi_{f_2}(C) \cdots \varphi_{f_{m-1}}(C) \varphi_{f_1}(A)^{-1} = A' C^* \varphi_{f_1}(C^*) \varphi_{f_2}(C^*) \cdots \varphi_{f_{m-1}}(C^*) \varphi_{f_1}(A')^{-1}.
\]

Thus

\[
r = (A', 0)(C^*, f_1)^{m_1}(\varepsilon, g_1)(A', 0)^{-1},
\]

\[
s_1 = (A', 0)(C^*, f_1)^{n_1}(\varepsilon, h_1)(A', 0)^{-1}.
\]

Note that

\[(C^*, f_1)^{m_1} = C^* \varphi_{f_1}(C^*) \varphi_{f_2}(C^*) \cdots \varphi_{f_{m-1}}(C^*)\]

is still a reduced word. If it was not a reduced word, then we would have $m_1 > 0$, $|C^*| > 1$, and $\varphi_{f_1}(a^{-1}) = \varphi_{f_1}(a')^{-1}$, where $a'$ is the first letter of $C^*$. Thus $a' = a$ is the second letter of $C$, which together with the fact that the first letter of $C$ is $a^{-1}$ contradicts our assumption that $C$ is a reduced word. We have reduced the size of $A$, so after finitely many reductions of this form, we get

\[
AC \varphi_{f_1}(C) \varphi_{f_2}(C) \cdots \varphi_{f_{m-1}}(C) \varphi_{f_1}(A)^{-1} = BD \varphi_{f_2}(D) \varphi_{f_2}(D) \cdots \varphi_{f_{m-1}}(D) \varphi_{f_2}(B)^{-1}
\]

and that both sides are reduced words.

Now either $|A| \leq |B|$ or $|B| \leq |A|$. Without loss of generality, assume that we are in the first case. Then $A$ is an initial segment of $B$ (i.e., $B = AB'$ as a reduced word). Then by replacing $r$, $s_1$, and $s_2$ with $A^{-1} r A$, $A^{-1} s_1 A$, and $A^{-1} s_2 A$, we may assume that $A$ is trivial. To summarize the reductions we have made so far, we have

\[
\begin{align*}
r &= (C, f_1)^{m_1}(\varepsilon, g_1) = (B, 0)(D, f_2)^{m_2}(\varepsilon, g_2)(B^{-1}, 0), \\
s_1 &= (C, f_1)^{n_1}(\varepsilon, h_1), \\
s_2 &= (B, 0)(D, f_2)^{n_2}(\varepsilon, h_2)(B^{-1}, 0).
\end{align*}
\]

The automorphisms $\varphi_{g_1}$ and $\varphi_{h_1}$ fix $C$, and the automorphisms $\varphi_{g_2}$ and $\varphi_{h_2}$ fix $D$ and $B$. Both sides of

\[
C \varphi_{f_1}(C) \varphi_{f_2}(C) \cdots \varphi_{f_{m-1}}(C) = BD \varphi_{f_2}(D) \varphi_{f_2}(D) \cdots \varphi_{f_{m-1}}(D) \varphi_{f_2}(B)^{-1}
\]

are reduced words.

Now we will show that either $m_1 = 1$ or $B$ is trivial. Suppose that $B$ was non-trivial, say $B = bB'$. First note that the length of $C$ is greater than one, as otherwise $C = b$ and $\varphi_{f_1}(C) = \varphi_{f_2}(b^{-1})$; but there is no $e \in H/R$ such that $\varphi_e(b) = b^{-1}$. Then we must have
\[ C = bC'\varphi_{m_2 f_2 - (m_1 - 1)f_1}(b^{-1}) \] for some \( C' \). We have \( m_1 f_1 + g_1 = m_2 f_2 + g_2 \). Since \( b \) appears both in \( C \) and in \( B \), it is fixed by both \( \varphi_{g_1} \) and \( \varphi_{g_2} \). Thus \( C = bC'\varphi_{f_1}(b^{-1}) \). But then if \( m_1 > 1 \),
\[ C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1 - 1)f_1}(C) \]
is not a reduced word. So we conclude that either \( m_1 = 1 \) or \( B \) is trivial.

**Case 1.** Suppose that \( m_1 = 1 \).

We have
\[ r = (C, f_1)(\varepsilon, g_1) = (B, 0)(D, f_2)^{m_2}(\varepsilon, g_2)(B^{-1}, 0). \]
Also, as reduced words,
\[ C = BD\varphi_{f_2}(D)\varphi_{2f_2}(D)\cdots\varphi_{(m_2 - 1)f_2}(D)\varphi_{m_2 f_2}(B)^{-1}. \]
Since the right hand side is a reduced word, \( \varphi_{g_1} \) and \( \varphi_{h_1} \) fix \( B \) and \( D \) since each letter in \( B \) and \( D \) appears in \( C \). Thus
\[ s_1 = (C, f_1)^{n_1}(\varepsilon, h_1) = [(B, 0)(D, f_2)^{m_2}(B^{-1}, 0)(\varepsilon, f_1 - m_2 f_2)]^{n_1}(\varepsilon, h_1). \]
Now \( f_1 + g_1 = m_2 f_2 + g_2 \). Since \( \varphi_{g_1} \) and \( \varphi_{g_2} \) fix \( B \) and \( D \), \( \varphi_{f_1 - m_2 f_2} \) also fixes \( B \) and \( D \). Thus
\[ s_1 = (B, 0)(D, f_2)^{m_2 n_1}(\varepsilon, h_1 + n_1(f_1 - m_2 f_2))(B^{-1}, 0) \]
and \( h_1 + n_1(f_1 - m_2 f_2) \) fixes \( D \). Thus \( s_1 \) and \( s_2 \) commute. This is a contradiction.

**Case 2.** \( B \) is trivial.

Let \(|C| = k\) and \(|D| = \ell\). Suppose without loss of generality that \( k \geq \ell \). Let \( d_0, d_1, d_2, \ldots \) be the reduced word
\[ C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1 - 1)f_1}(C) = D\varphi_{f_2}(D)\varphi_{2f_2}(D)\cdots\varphi_{(m_2 - 1)f_2}(D). \]
Then we have
\[ d_i = \varphi_{f_2}(d_{i-\ell}) \quad \text{for} \quad i \geq \ell \]
\[ \varphi_{((m_1 - 1)f_1)}(d_{k-\ell+i}) = \varphi_{((m_2 - 1)f_2)}(d_i) \quad \text{for} \quad 0 \leq i < \ell \]
Let \( e = \gcd(k, \ell) \).

Given \( p, q \geq 0 \), write \( i = q\ell - pk + r \) with \( 0 \leq r < e \) and assume that \( 0 \leq i < m_1 k = m_2 \ell \). Note that every \( i \), \( 0 \leq i < m_1 k = m_2 \ell \), can be written in such a way. We claim that
\[ d_i = \varphi_{qf_2 + p((m_1 - 1)f_1 - m_1 f_2)}(d_r). \]
We argue by induction, ordering pairs \((q, p)\) lexicographically. For the base case \( p = q = 0 \) we note that \( d_r = \varphi_0(d_r) \). If \( \ell \leq i \), then we must have \( q > 0 \). By the induction hypothesis, \( d_{i-\ell} = \varphi_{(q-1)f_2 + p((m_1 - 1)f_1 - m_2 f_2)}(d_r) \). So
\[ d_i = \varphi_{f_2}(d_{i-\ell}) = \varphi_{qf_2 + p((m_1 - 1)f_1 - m_2 f_2)}(d_r). \]
CHAPTER 13. LEFT-ORDERABLE COMPUTABLE GROUPS

If \( 0 \leq i < \ell \), and \((q,p) \neq (\varepsilon,0)\), then \(q > 0\) and \(p > 0\). Note that
\[
d_{k-\ell+i} = \varphi(q-1)f_{2}+(p-1)\lfloor (m_{1}-1)f_{1}-m_{2}f_{2}\rfloor(d_{r}) = \varphi(q_{2}+p\lfloor (m_{1}-1)f_{1}-m_{2}f_{2}\rfloor\rfloor (d_{r})
\]
by the induction hypothesis and so
\[
d_{i} = \varphi(m_{1}-1)f_{1}-(m_{2}-1)f_{2}(d_{i+k-\ell}) = \varphi(q_{2}+p\lfloor (m_{1}-1)f_{1}-(m_{2}-1)f_{2}\rfloor\rfloor (c_{r})
\]
This completes the induction.

Write \( e = q\ell - pk \) with \( p,q \geq 0 \). Let \( f = qf_{2}+p\lfloor (m_{1}-1)f_{1}-m_{2}f_{2}\rfloor \). Then each \( i, 0 \leq i < km_{1}, \)
can be written as \( i = se + r \) with \( 0 \leq r < d \), and so
\[
d_{i} = \varphi_{sf}(d_{r})
\]
Let \( E = d_{1}\cdots d_{e} \). Then
\[
C = E\varphi_{f}(E)\cdots\varphi_{(\ell-1)f}(E).
\]
Similarly,
\[
D = E\varphi_{f}(E)\cdots\varphi_{(\ell-1)f}(E).
\]
Also,
\[
\varphi_{f_{1}}(E) = d_{k}\cdots d_{k-1} = \varphi_{\frac{1}{e}f}(d_{0},\ldots,d_{e-1}) = \varphi_{\frac{1}{e}f}(E)
\]
and
\[
\varphi_{f_{2}}(E) = d_{\ell}\cdots d_{\ell+e-1} = \varphi_{\frac{1}{e}f}(d_{0},\ldots,d_{e-1}) = \varphi_{\frac{1}{e}f}(E).
\]
So \( \varphi_{f_{1}}(C) = \varphi_{\frac{1}{e}f}(C) \) and \( \varphi_{f_{2}}(D) = \varphi_{\frac{1}{e}f}(D) \). Hence
\[
s_{1} = (C,f_{1})^{m_{1}}(\varepsilon,h_{1}) = (E,f)^{-m_{1}k}(e,h_{1} + m_{1}f_{1} - \frac{m_{1}k}{e} f)
\]
and
\[
s_{2} = (D,f_{2})^{m_{1}}(\varepsilon,h_{2}) = (E,f)^{-m_{2}k}(e,h_{2} + m_{2}f_{2} - \frac{m_{2}k}{e} f)
\]
Note that \( \varphi_{h_{1}} \) and \( \varphi_{h_{2}} \) both fix \( E \), since they fix \( C \) and \( D \) respectively. Also, since \( \varphi_{f_{1}}(E) = \varphi_{\frac{1}{e}f}(E) \), \( \varphi_{m_{1}f_{1}-(\frac{m_{1}k}{e})} \) fixes \( E \). Similarly, \( \varphi_{m_{2}f_{2}-(\frac{m_{2}k}{e})} \) fixes \( E \). So \( s_{1} \) and \( s_{2} \) commute. This is a contradiction. \( \square \)

Lemma 13.5.3. Fix \( r \in \mathcal{G} \). If \( r^{2} \in \mathcal{H}/\mathcal{R} \), then \( r \in \mathcal{H}/\mathcal{R} \).

Proof. Write \( r = (A,f) \). We will show that if \( r \notin \mathcal{H}/\mathcal{R} \), i.e. if \( A = \varepsilon \), then \( r^{2} \notin \mathcal{H}/\mathcal{R} \). Since
\[
r^{2} = (A\varphi_{f}(A),2f)
\]
we must show that \( A\varphi_{f}(A) \) is non-trivial. Suppose that it was trivial; then the length of \( A \)
as a reduced word must be even. (If the length of \( A \) was odd, say \( A = A_{1}aA_{2} \) with \( A_{1} \) and \( A_{2} \) of equal lengths, then
\[
A\varphi_{f}(A) = A_{1}aA_{2}\varphi_{f}(A_{1})\varphi_{f}(a)\varphi_{f}(A_{2}) = \varepsilon.
\]
So it must be that $\varphi_f(a) = a^{-1}$, which cannot happen for any letter $a$.) Write $A = BC$, where $B$ and $C$ are each half the length of $A$. Then since $A\varphi_f(A)$ is the trivial word, $C\varphi_f(B)$ is the trivial word; thus $C = \varphi_f(B^{-1})$. So $A = B\varphi_f(B^{-1})$, and

$$A\varphi_f(A) = B\varphi_f(B^{-1})\varphi_f(B)\varphi_2f(B^{-1}) = B\varphi_2f(B^{-1}).$$

Since $A\varphi_f(A)$ is the trivial word, $\varphi_2f(B) = B$. Since $A$ is not the trivial word, $B \neq \varphi_f(B)$. But this is impossible, as $p_i$, $q_i$, and $r_i$ were all chosen to be odd primes. \hfill \square

The next lemma is the heart of the existential definition of $(\mathcal{H}/\mathcal{R})^g$. The proof is to show that under the hypotheses of the lemma, elements not in $(\mathcal{H}/\mathcal{R})^g$ such as in Lemma 13.5.2 must exist.

**Lemma 13.5.4.** Let $r, s_1, s_2, s_3 \in G$. Suppose that $r$ commutes with $s_1$, $s_2$, and $s_3$, but that no two of $s_1$, $s_2$, and $s_3$ commute. Then $r \notin (\mathcal{H}/\mathcal{R})^g$.

**Proof.** If at least two of $s_1$, $s_2$, and $s_3$ are not in $(\mathcal{H}/\mathcal{R})^g$, then this follows immediately by Lemma 13.5.2. Otherwise, without loss of generality suppose that $s_1$ and $s_2$ are in $(\mathcal{H}/\mathcal{R})^g$. By Lemma 13.4.2, $s_1s_2 \notin (\mathcal{H}/\mathcal{R})^g$.

Note that $r$ commutes with $s_1s_2$ and with $s_1(s_2)^2$. Also, $s_1s_2$ does not commute with $s_1(s_2)^2$, since if it did, then

$$s_1s_2s_1s_2 = s_1s_2s_1s_2 \Rightarrow s_1s_2 = s_2s_1.$$  

We claim that $s_1(s_2)^2 \notin (\mathcal{H}/\mathcal{R})^g$. If $s_1(s_2)^2$ was in $(\mathcal{H}/\mathcal{R})^g$, then by Lemma 13.4.2, we could write

$$s_1 = (A, 0)(\varepsilon, g)(A^{-1}, 0) \text{ and } (s_2)^2 = (A, 0)(\varepsilon, h)(A^{-1}, 0).$$

Then let $s_2' = (A^{-1}, 0)s_2(A, 0) = (C, f)$. Then $(s_2')^2 = (\varepsilon, h)$, and so by Lemma 13.5.3, $s_2' = (\varepsilon, f)$. Thus $s_2 = (A, 0)(\varepsilon, f)(A^{-1}, 0)$. So $s_1$ and $s_2$ would commute; since we know that $s_1$ and $s_2$ do not commute, $s_1(s_2)^2 \notin (\mathcal{H}/\mathcal{R})^g$.

By Lemma 13.5.2, with $r$, $s_1s_2$, and $s_1s_2^2$, we see that $r$ is in $(\mathcal{H}/\mathcal{R})^g$. \hfill \square

The existential definition of $(\mathcal{H}/\mathcal{R})^g$ comes from the previous lemma. It remains only to show that if $r \in (\mathcal{H}/\mathcal{R})^g$, then the hypothesis of the previous lemma is satisfied.

**Proof of Lemma 13.4.1.** By the previous lemma, it suffices to show that if $r \in (\mathcal{H}/\mathcal{R})^g$, then there are $s_1$, $s_2$, and $s_3$ such that $r$ commutes with $s_1$, $s_2$, and $s_3$, but no two of these commute with each other. If $r = (A, 0)(\varepsilon, g)(A^{-1}, 0)$, let $s_1 = (A, 0)(u_0, 0)(A^{-1}, 0)$, $s_2 = (A, 0)(u_1, 0)(A^{-1}, 0)$, and $s_3 = (A, 0)(u_2, 0)(A^{-1}, 0)$. Then $r$ commutes with $s_1$, $s_2$, and $s_3$ since $g$ fixes $u_0$, $u_1$, and $u_2$, but no two of $s_1$, $s_2$, and $s_3$ commute with each other as $u_0$, $u_1$, and $u_2$ do not commute with each other. \hfill \square
Part V

Miscellaneous
Chapter 14

The Complexity of Decidable Presentability

The results presented in this chapter appeared in [HTg].

14.1 Introduction

In effective mathematics, we are concerned with computable structures. A mathematical structure—a set together with operations and relations on that set—is computable if the set and the operations and relations on it are all computable. For example, a computable field is one where the domain is a computable set and the operations of addition and multiplication are computable. In a computable structure, we can effectively answer quantifier-free questions, such as, for elements $a$, $b$, and $c$ of a field, whether $a + b = b \cdot c$.

There are many other questions about a structure that we might want to answer in a computable way. For example, in a field, we might want to be able to decide whether a given polynomial has a root. In general, this is undecidable, but sometimes, such as for algebraically closed fields, this can be done. In fact, given a computable algebraically closed field, as a result of quantifier elimination we can decide the answer to any question that can be formulated in elementary first-order logic, i.e., as a logical formula using $\lor$, $\land$, $\neg$, $\rightarrow$, $\forall$, and $\exists$. In general, we say that a computable structure is decidable if there is a method to compute, given elements $a_1, \ldots, a_n$ and a formula $\varphi$ of elementary first-order logic, whether $\varphi$ holds of $a_1, \ldots, a_n$. Every computable algebraically closed field is decidable.

An important phenomenon in computability theory is that there can be computable structures which are isomorphic, but not computably isomorphic, so that we cannot transfer computational properties from one to the other. For example, the standard presentation of the linear order $(\mathbb{N}, <)$ is decidable. However, there is also a computable copy of the same structure in which the successor relation is not computable, and hence this copy is not decidable. (Here, $a$ is a successor of $b$ if and only if $(\forall c)[c < b \lor c > a].$) Though these two computable structures are isomorphic, they are not computably isomorphic.
This paper is about the problem of characterizing those computable structures which have a decidable presentation. This problem was probably first stated by Goncharov, and has more recently been posed for example by Bazhenov at the 2015 Mal’cev Meeting and Fokina at the 2016 ASL meeting in Storrs, CT. We will show that there is no such characterization.

More formally, our main theorem is as follows. Fix an effective list of the diagrams of the (partial) computable structures.

**Theorem 14.1.1.** The index set

\[ I_{\text{d-pres}} = \{ i \mid \text{the } i\text{th computable structure is decidably presentable} \} \]

is \( \Sigma^1_1 \)-complete.

This theorem is proved in Section 14.4.

As a result, there is no possible reasonable characterization of the computable structures with decidable presentations. What we mean is that there is no simpler way to check whether a computable structure \( A \) has a decidable presentation than to ask: *Does there exist a decidable structure \( B \) and a classical isomorphism between \( A \) and \( B \)?* This requires searching through all possible isomorphisms, of which there may be continuum-many, between \( A \) and \( B \). (Contrast this with a very naive, and incorrect, candidate for a characterization: A computable structure \( A \) has a decidable copy if and only if there is a computable listing of the types it realizes. In this case, we must look through the countably many possible computable listings of types, and check whether they list the types in \( A \). This requires only quantifiers over natural numbers, and objects which can be coded by natural numbers.\(^1\)) If there were a simpler characterization of the computable structures with decidable presentations, then one would expect that characterization to yield a simpler way of checking whether a computable structure has a decidable presentation.

A similar approach was taken in \([D KL +15]\), where it was shown that there is no reasonable characterization of computable categoricity, and in \([DM08]\), where it was shown that there is no reasonable classification of abelian groups. This approach originated with \([GK02]\). See also \([LS07, Fok07, CFG+07, FGK+15, GBM15a, GBM15b]\).

### 14.1.1 Decidable Presentability in Familiar Classes

What if we are interested in a specific class of structures, such as fields, graphs, or groups? In the case of some particular classes of structures—which are *universal* in a sense soon to be described—it follows immediately from Theorem 14.1.1 that there is no classification of the decidable presentable structures in that class.

Hirschfeldt, Khoussainov, Shore, and Slinko \([HKSS02]\) showed that many classes of structures—such as graphs and groups—are *universal*. What we mean when we say that

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\(^1\)For some restricted classes of structures, such a characterization might be possible. For example, Andrews \([And14]\) showed that if \( M \) is a model of a decidable \( \omega \)-stable theory with countably many countable models, then \( M \) has a decidable copy if and only if all of the types realized in \( M \) are recursive.
a class $C$ of structures is universal is that for every structure $A$, there is a structure $B \in C$ such that $A$ and $B$ are effectively bi-interpretable. That is, each is interpretable in the other using computable infinitary $\Sigma_1$ formulas, in a compatible way, and so for most computability-theoretic purposes, the two structures are the same. Equivalently—see [HTMMM]—there is a computable bi-transformation between copies of $A$ and copies of $B$, i.e., there is a computable way to turn presentations of $A$ into presentations of $B$, and vice versa, in a functorial way. We note that this exact definition of a universal class did not appear in [HKSS02], but comes from later work in [MPSS, Section 3] and [Monb, Definition 5.4]. (For groups, we need to add finitely many constants to the language.)

When we examine the proofs from [HKSS02], we see that the effective bi-interpretations between a structure $A$ and the corresponding graph (or group) $G_A$, use only elementary first-order formulas. (An effective bi-interpretation is, in general, allowed to use infinitary formulas.) That means that a structure $A$ is decidable if and only if the corresponding graph (or group) $G_A$ is decidable.

Miller, Poonen, Schoutens, and Shlapentokh [MPSS] recently showed that the class of fields is also universal. Again, the bi-interpretations between a structure and the corresponding field use only elementary first-order formulas.

It follows that one cannot characterize which graphs, groups, and fields are decidable presentable.

Theorem 14.1.2. The index sets of the decidably presentable graphs, groups, and fields are $\Sigma^0_1$-complete.

Other familiar classes of structures, such as linear orders and boolean algebras, are not universal, and so a similar argument does not work. It is possible that such classes admit a characterization of the decidably presentable structures in that class. For linear orders in particular, this question has already been raised:

Question 14.1.3 (Moses, see [CLLS00]). Can one characterize the linear orderings which have a decidable copy?

We believe that the Friedman-Stanley [FS89] transformation $T$ of structures into linear orders preserves decidability, in the sense that $A$ is decidable if and only if $T(A)$ is decidable. It would follow that the answer to this question is “no”.

Abelian groups are another class of structures which is not universal. However, it is still an open question whether or not abelian groups are Borel complete.

Question 14.1.4. Can one characterize the torsion-free abelian groups which have a decidable copy?

14.1.2 $n$-Decidable Structures

One can also ask whether a computable structure has an $n$-decidable copy. An $n$-decidable structure is a structure in which we can decide whether a formula $\varphi$, with $n$ alternations of
quantifiers, holds of a tuple $a_1, \ldots, a_\ell$. For each $n$, there are $n$-decidable structures which are have no $n + 1$-decidable copies, and there is a structure which has $n$-decidable copies for all $n$, but no decidable copy [CM98]. We say that a structure is $n$-presentable is it has an $n$-decidable copy. There is no simpler characterization of the $n$-presentable structures.

**Theorem 14.1.5.** For each $n \in \omega$, the index set

$$I_{n\text{-pres}} = \{ i \mid \text{the } i\text{th computable structure is } n\text{-presentable} \}$$

is $\Sigma^1_1$-complete.

The proof of Theorem 14.1.5 will be simpler than the proof of Theorem 14.1.1, and so we will begin by proving Theorem 14.1.5 in Section 14.3. To prove Theorem 14.1.1, we must also guess at $\Sigma^0_2$ facts.

### 14.1.3 Further Questions

In addition to Questions 14.1.3 and 14.1.4 above, there are many more questions to resolve. In [DKL+15], it was shown that the index set of the computably categorical structures is $\Pi^1_1$ complete. In [GBM15b], this was extended to show that the index set of the computable structures with computable dimension $n$ is $\Pi^1_1$-complete, for finite $n$. The case of $n = \omega$ is still open.

**Question 14.1.6.** What is the complexity of the index set of the computable structures with computable dimension $\omega$?

In this paper, we considered structures which have one computable copy which is decidable. One could also consider structures all of whose computable copies are decidable. We call such a structure *intrinsically decidable*. One can, as usual, also define a notion of relative intrinsic decidability: A structure is *relatively intrinsically decidable* if, for every isomorphic copy $A$ of that structure, the elementary diagram of $A$ is computable in $\deg(A)$. By the uniform version of a theorem of Ash, Knight, Manasse, and Slaman [AKMS89], and independently Chisholm [Chi90], a computable structure $A$ is relatively intrinsically decidable if and only if it has a sort of quantifier elimination: Every elementary first-order definable subset of $A$ is (uniformly) definable by a computable infinitary $\Sigma_1$ formula, and also by a computable infinitary $\Pi_1$ formula. One expects there to be structures which are intrinsically decidable but not relatively intrinsically decidable, as there are, for example, structures which are computably categorical but not relatively computably categorical [Gon77]. Note that deciding whether a structure is relatively intrinsically decidable is arithmetic; however, one might guess that intrinsic decidability is actually $\Sigma^1_1$ complete.

**Question 14.1.7.** What is the complexity of the index set of the computable structures all of whose computable copies are decidable?

See also Question 14.4.18 which we state later after providing sufficient context.
14.2 Some Useful Lemmas

14.2.1 A Sequence of Structures

It is well-known that there are computable structures $C_\omega$ such that the index set of the computable structures which are isomorphic to $C_\omega$ is $\Sigma^1_1$-complete. A small modification of the same argument, which we will repeat below in brief, shows that the same is true of decidable structures: There is a decidable structure $C_\omega$ such that the index set of the decidable structures which are isomorphic to $C_\omega$ is $\Sigma^1_1$-complete. We will use these structures in the constructions for Theorems 14.1.1 and 14.1.5.

To build the structure $C_\omega$ we will use the following lemma, which is probably folklore; similar results appear in, for example, [Ash91].

Lemma 14.2.1. Given a computable linear order $L$, we can, uniformly in $L$, build a decidable copy of $\omega^\omega \cdot (1 + L)$.

Proof. It is well-known that there is a decidable copy, which we will call $W$, of $\omega^\omega$; we may also choose $W$ so that $W + W$ is decidable. Define $A = W \cdot (1 + L)$. We represent elements of $A$ as pairs $(l, w)$ with $l \in 1 + L$ and $w \in W$, ordered lexicographically starting with $l$. We claim that $A$ is decidable.

Indeed, given a tuple $\bar{a}$, break up $\bar{a}$ into tuples $a_1, \ldots, a_n$ where each element of $a_i$ is of the form $(l_i, w)$ for some $w \in W$, and $l_1 < \cdots < l_n$. Let $\bar{a}_i$ consist of the elements $a_1^i < \cdots < a_m^i$, and let $w^i_j$ be such that $a_j^i = (l_i, w^i_j)$. Then (see Corollary 13.39 of [Ros82]) the complete type of $\bar{a}$ is determined effectively by the elementary first-order theories of the intervals

$$(-\infty, a_1^1], [a_1^1, a_1^2], \ldots, [a_1^m_1, a_2^1], [a_2^1, a_2^2], \ldots, [a_2^m_2, a_3^1], \ldots, [a_n^m_n, \infty).$$

Each interval $[a_j^i, a_j^{i+1}]$ has the same order type as $[w_j^i, w_j^{i+1}]$ which is decidable, as it is a definable subset of $W$. The order type of $[a_m^i, a_{i+1}^i]$ is $\omega^\omega \cdot [l_i, l_{i+1}] + w_{i+1}^1$, which has the same theory as $\omega^\omega + w_{i+1}^1$ (see Theorem 6.21 of [Ros82]); this theory is decidable. The interval $(-\infty, a_1^1]$ has the same theory as either $w_1^1$ (if $l_1$ is smaller than $L$) or $\omega^\omega + w_1$ (if $l_1 \in L$). Finally, the interval $[a_n^m_n, \infty)$ has the same theory as $\omega^\omega$. Thus the type of $\bar{a}$ is computable in $A$, and so $A$ is decidable.

Lemma 14.2.2. Let $S$ be a $\Sigma^1_1$ set. There is a decidable structure $C_\omega$ and a uniformly decidable sequence of structures $(C_n)_{n \in \omega}$ such that $C_n \cong C_\omega$ if and only if $n \in S$. All of these structures are in the same language.

Proof. Harrison [Har68] constructed a computable linear order $H$ of order type $\omega^\omega \cdot (1 + \mathbb{Q})$. From [CDH08, Lemma 5.2] or [GK02, Theorem 4.4(d)], we get a computable sequence of computable linear orders $(L_n)_{n \in \omega}$ such that $L_n$ is isomorphic to $H$ if and only if $n \in S$. Then letting $C_n$ be a decidable copy of $\omega^\omega \cdot (1 + L_n)$, we get a uniformly decidable sequence of structures $(C_n)_{n \in \omega}$. (We take $C_\omega$ to be a decidable copy of $H$, which is isomorphic to $\omega^\omega \cdot (1 + H)$.) If $L_n$ was well-founded, so is $C_n$, and if $L_n$ was isomorphic to $H$, then so is $C_n$. 

14.2.2 Building Decidable Structures from Disjoint Unions

In this section, we will prove three lemmas about constructing a decidable structure by taking disjoint unions of other decidable structures. We will use these lemmas during the construction.

Lemma 14.2.3. Let $A_1, \ldots, A_k$ be decidable structures. Then the disjoint union $B$ of $A_1, \ldots, A_k$, with relations $R_1, \ldots, R_k$ picking out the domains of $A_1, \ldots, A_k$ respectively, is also decidable. This is uniform.

Proof. It suffices to show that $B$ is decidable with respect to the many-sorted logic with sorts defined by $R_1, \ldots, R_k$. The many-sorted logic has quantifiers which range only over a single sort $R_i$, and the relations of a structure $A_i$ are restricted to the sort $R_i$. Indeed, it is easy to translate any formula in the single-sorted language of $B$ to an equivalent formula in the many-sorted language. In what follows, by an $A_i$-formula we mean a formula involving only the sort $A_i$.

We can easily argue by induction on formulas that each formula $\varphi$ in the many-sorted language of $B$ is equivalent to a boolean combination of $A_i$-formulas. For example, if $\varphi \equiv (\exists x \in R_p) \psi$, and (placing the boolean combination equivalent to $\psi$ in disjunctive normal form)

$$\psi \equiv \bigvee_{i=1}^{r} \bigwedge_{j=1}^{k} \theta_{i,j}$$

where $\theta_{i,j}$ is a $A_j$-formula, we get that

$$\varphi \equiv \bigvee_{i=1}^{r} \bigwedge_{j=1}^{k} \theta'_{i,j}$$

where $\theta'_{i,p} = (\exists x \in R_p) \theta_{i,p}$ and $\theta'_{i,j} = \theta_{i,j}$ if $j \neq p$.

Then given a formula $\varphi$ in the many-sorted language of $B$, write $\varphi$ as a boolean combination of $A_i$-formulas:

$$\varphi \equiv \bigvee_{i=1}^{r} \bigwedge_{j=1}^{k} \theta_{i,j}$$

where $\theta_{i,j}$ is a $A_j$-formula. We can decide the truth of each $\theta_{i,j}$ as each $A_j$ is decidable, and hence we can decide the truth of $\varphi$.\hfill \square

A slightly more complicated argument proves the following similar lemma.

Lemma 14.2.4. Let $A_1, \ldots, A_k$ be decidable structures. Then the disjoint union $B$ of $A_1, \ldots, A_k$, with an equivalence relation $E$ whose equivalence classes pick out the structures $A_i$, is also decidable. This is uniform.

Proof sketch. The structure $B$ is effectively bi-interpretable, using first-order formulas, with the structure from the previous lemma after naming one element from each of the $k$ equivalence classes.\hfill \square
CHAPTER 14. THE COMPLEXITY OF DECIDABLE PRESENTABILITY

The third, and final, lemma allows us to take the disjoint union of infinitely many structures, as long as they are all elementarily equivalent.

Lemma 14.2.5. Let \((A_i)_{i \in \omega}\) be a sequence of uniformly decidable structures. Suppose that for each \(i\) and \(j\), \(A_i \equiv A_j\). Let \(B\) be the disjoint union of the \(A_i\), with an equivalence relation \(E\) whose equivalence classes pick out the structures \(A_i\). Then \(B\) is decidable. This is uniform.

Proof. View the structures as relational structures. Given a formula \(\varphi(x_1, \ldots, x_\ell)\) and \(a_1, \ldots, a_\ell\), we need to decide whether \(B \models \varphi(a_1, \ldots, a_\ell)\). Let \(n\) be the quantifier depth of \(\varphi\). Let \(B^*\) be substructure of \(B\) which consists of those structures \(A_i\) containing \(a_1, \ldots, a_\ell\) and \(n\) other structures \(A_i\). We claim that \(B \models \varphi(a_1, \ldots, a_\ell)\) if and only if \(B^* \models \varphi(a_1, \ldots, a_\ell)\). Since \(B^*\) is decidable, uniformly in \(n\), by the previous lemma, we can decide whether \(B \models \varphi(a_1, \ldots, a_\ell)\). Thus \(B\) is decidable, and this is uniform.

To see that \(B \models \varphi(a_1, \ldots, a_\ell)\) if and only if \(B^* \models \varphi(a_1, \ldots, a_\ell)\), we can play the Ehrenfeucht-Fraissé game with \(\varphi\). Denote by \(M \lesssim N\) that Duplicator has a winning strategy for the Ehrenfeucht-Fraissé game with \(r\) moves, i.e., that \(M\) and \(N\) satisfy the same formulas with quantifier depth \(r\). We want to show that \((B^*; a_1, \ldots, a_\ell) \lesssim (B; a_1, \ldots, a_\ell)\). To prove this, it is more convenient to prove a stronger claim.

Claim 14.2.6. Given \(r\) and \(m\) with \(r + m \leq n + \ell\), tuples \(\bar{x}_1 \in A_{j_1}, \ldots, \bar{x}_m \in A_{j_m}\) all in \(B^*\), and \(\bar{y}_1 \in A_{k_1}, \ldots, \bar{y}_m \in A_{k_m}\) (with no repetition among the lists of the structures), \((B^*; \bar{x}_1, \ldots, \bar{x}_m) \lesssim (B; \bar{y}_1, \ldots, \bar{y}_m)\) if and only if for each \(i\), \((A_{j_i}; \bar{x}_i) \lesssim (A_{k_i}; \bar{y}_i)\).

From this, if we take \(r = n\) and (rearranging \(a_1, \ldots, a_\ell\)) take
\[
(a_1, \ldots, a_\ell) = (\bar{x}_1, \ldots, \bar{x}_m) = (\bar{y}_1, \ldots, \bar{y}_m)
\]
with \(j_i = k_i\) for all \(i\), then we immediately get that \((B^*; a_1, \ldots, a_\ell) \lesssim (B; a_1, \ldots, a_\ell)\) as desired. So the proof of the claim will finish the proof of the lemma.

Proof of claim. The proof of this claim is by induction on \(r\). For \(r = 0\), \(\bar{x}_1, \ldots, \bar{x}_m\) satisfy the same atomic formulas in \(B^*\) as \(\bar{y}_1, \ldots, \bar{y}_m\) do in \(B\) if and only if for each \(i\), \(\bar{x}_i \in A_{j_i}\) satisfies the same atomic formulas in \(A_{j_i}\) as \(\bar{y}_i\) does in \(A_{k_i}\). Given \(r > 0\), it is clear that if \((B^*; \bar{x}_1, \ldots, \bar{x}_m) \lesssim (B; \bar{y}_1, \ldots, \bar{y}_m)\) then for each \(i\), \((A_{j_i}; \bar{x}_i) \lesssim (A_{k_i}; \bar{y}_i)\). For the other direction, suppose that for each \(i\), \((A_{j_i}; \bar{x}_i) \lesssim (A_{k_i}; \bar{y}_i)\). Given \(y' \in B\), we must find \(x' \in B^*\) such that \((B^*; \bar{x}_1, \ldots, \bar{x}_m, x') \lesssim (B; \bar{y}_1, \ldots, \bar{y}_m, y')\). (The other case—finding \(y' \in B\) given \(x' \in B^*\)—is similar and actually easier.)

Case 1. If \(y' \in A_{k_i}\) for some \(i = 1, \ldots, m\), then since \((A_{j_i}; \bar{x}_i) \lesssim (A_{k_i}; \bar{y}_i)\), there is \(x' \in A_{j_i}\) such that \((A_{j_i}; \bar{x}_i; x') \lesssim (A_{k_i}; \bar{y}_i; y')\). Thus, by the induction hypothesis, \((B^*; \bar{x}_1, \ldots, \bar{x}_m, x') \lesssim (B; \bar{y}_1, \ldots, \bar{y}_m, y')\).

Case 2. Otherwise, let \(k_{m+1}\) be such that \(y' \in A_{k_{m+1}}\). Since \(r + m \leq n + \ell\), we can choose \(j_{m+1}\) different from \(j_1, \ldots, j_m\) such that \(A_{j_{m+1}}\) is included in \(B^*\). Since \(A_{km+1} \equiv A_{j_{m+1}}\), we can find \(x' \in A_{j_{m+1}}\) such that \((A_{km+1}; y') \lesssim (A_{j_{m+1}}; x')\). We then have, with \(\bar{x}_{m+1} = x'\) and \(\bar{y}_{m+1} = y'\), that \((A_{j_i}; \bar{x}_i; x') \lesssim (A_{k_i}; \bar{y}_i)\) for \(i = 1, \ldots, m + 1\) and that \((r - 1) + (m + 1) \leq n + \ell\). So \((B^*; \bar{x}_1, \ldots, \bar{x}_m, x') \lesssim (B; \bar{y}_1, \ldots, \bar{y}_m, y')\) by the induction hypothesis. \(\square\)
14.3 1-Presentable Structures

In this section, we will prove the case \( n = 1 \) of Theorem 14.1.5: The index set of 1-presentable structures is \( \Sigma^1_1 \)-complete. The general case is essentially the same, but restricting to the case \( n = 1 \) will make the proof more readable, and, in fact, the case \( n \geq 2 \) will follow from the proof of Theorem 14.1.1. (See Section 14.4.7.)

Fix a \( \Sigma^1_1 \) set \( S \). We must build a uniformly computable sequence of computable structures \( (M_n)_{n \in \omega} \) such that \( M_n \) is 1-presentable if and only if \( n \in S \). Fix, as in Lemma 14.2.2, decidable structures \( C_n \) and \( C_\infty \) such that \( C_n \cong C_\infty \) if and only if \( n \in S \). We will use \( C_n \) and \( C_\infty \) in the construction of \( M_n \). Also fix a computable listing \( (D_i)_{i \in \omega} \) of the (possibly partial) 1-diagrams of the 1-decidable structures.

The structures \( M_n \) will be the disjoint union of infinitely many structures \( (A_i)_{i \in \omega} \), each distinguished in \( M_n \) by some unary relation \( P_i \). (We may assume that each of the structures \( D_i \) is a partial structure of this form.) There are two properties that we want from the construction of \( A_i \):

1. If \( n \in S \), then \( A_i \) will have a 1-decidable presentation uniformly in \( i \).
2. If \( n \notin S \) and \( D_i \) is a 1-decidable structure, then \( A_i \) will not be isomorphic to the structure with domain \( P_i \) in the 1-decidable structure \( D_i \).

Thus, if \( n \in S \), then we can build a 1-decidable presentation of \( M_n \) by building 1-decidable copies of each \( A_i \). On the other hand, if \( n \notin S \), then \( M_n \) is not 1-presentable as it cannot be isomorphic to any 1-decidable structure \( D_i \).

For the remainder of the construction, we can fix \( i \). For simplicity, denote \( A_i \) by \( A \) and let \( D \) be the structure with domain \( P_i \) in \( D_i \). So we want to build \( A \) so that if \( n \in S \), then \( A \) will have a 1-decidable presentation (which we can construct uniformly), and if \( n \notin S \), then \( A \) is not isomorphic to \( D \).

14.3.1 \( \Sigma^0_1 \) Labeling of 1-Decidable Structures

Given a 1-decidable structure \( A \), we will describe how to add labels to \( A \) which are \( \Sigma^0_1 \) over the 1-diagram of \( A \) using a construction which is essentially a Marker extension [Mar89]. Intuitively, what we want to do is as follows. We want to be able to attach labels to elements of \( A \) in a c.e. way—that is, so that at any stage, we can add a label to a node—so that the resulting structure, with the labels attached, is also 1-decidable, and so that in the 1-diagram of an isomorphic copy of \( A \), we can enumerate the labels.

More formally, fix an infinite computable set \( \mathcal{L} \) of labels. Given a sequence of subsets \( X = (X_\ell)_{\ell \in \mathcal{L}} \) of \( A \), we want to define a three-sorted structure \( A^X \), whose first sort is just the structure \( A \), as follows. We will refer to the sorts as \( A, S_1, \) and \( S_2 \). The language of \( A^X \) will be the language of \( A \) augmented with functions \( f: S_1 \rightarrow A \) and \( g: S_2 \rightarrow S_1 \), a unary relation \( U_\ell \subseteq S_1 \) for each \( \ell \in \omega \), and a unary relation \( R \subseteq S_2 \).
For each element $x$ of $A$, there will be infinitely many elements $y$ of the second sort $S_1$ with $f(y) = x$. These will be partitioned into infinitely many disjoint sets $U^\ell$ for $\ell \in \omega$. Each element of $S_1$ will be the pre-image, under $f$, of some $x \in A$.

For each element $y$ of $S_1$, there will be infinitely many elements $z \in S_2$ with $g(z) = y$, and each element of $S_2$ will be the pre-image, under $g$, of some $y \in S_1$.

For every $x \in A$, there will be infinitely many $y \in f^{-1}(x) \cap U^\ell$ such that there are infinitely many $z \in g^{-1}(y)$ with $R(z)$, and infinitely many $z \in g^{-1}(y)$ with $\neg R(z)$. If $x \notin X_\ell$, this will be the case for all $y \in f^{-1}(x) \cap U^\ell$, but if $x \in X_\ell$, then there will also be infinitely many $y \in f^{-1}(x) \cap U^\ell$ such that for all $z \in g^{-1}(y)$, $R(z)$.

The next three lemmas show that this construction does what we want it to do.

**Lemma 14.3.1.** Let $A$ be a structure and let $X = (X_\ell)_{\ell \in \mathbb{L}}$ be subsets of $A$. The sets $X_\ell$ are definable in $A^X$ by $\exists \forall$ formulas, and these formulas are uniform in $\ell$ and independent of $A$ or $X$.

*Proof.* The set $X_\ell$ is definable as the subset of the first sort of $A^X$ defined by $(\exists y \in S_1) [f(y) = x \land U^\ell(y) \land (\forall z \in S_2)(g(z) = y \rightarrow R(z))].$ \hfill \Box

**Lemma 14.3.2.** Let $A$ be a computable structure and let $X = (X_\ell)_{\ell \in \mathbb{L}}$ be a computable sequence of codes for c.e. subsets of $A$. Then, uniformly in $X$ and in the atomic diagram of $A$, we can build a computable copy of $A^X$.

*Proof.* The copy of $A^X$ we build will have the computable copy of $A$ in the first sort, the second sort will contain elements $(x, \ell, s, t)$, and the third sort will contain elements $(x, \ell, s, t, u)$. We define

$$U^\ell = \{(x, \ell, s, t) \in S_1\}$$

$$f:S_2 \rightarrow S_1 \text{ defined by } (x, \ell, s, t, u) \mapsto (x, \ell, s, t)$$

$$g:S_1 \rightarrow A \text{ defined by } (x, \ell, s, t) \mapsto x.$$ 

It only remains to define the relation $R$. Given $s$, $t$, and $u$, we will have $R(x, \ell, s, t, u)$ if and only if $u$ is even or if $u$ is odd and $x$ enters $X_\ell$ exactly at stage $s$. \hfill \Box

**Lemma 14.3.3.** Let $A$ be a 1-decidable structure and let $X = (X_\ell)_{\ell \in \mathbb{L}}$ be a computable sequence of codes for c.e. subsets of $A$. Then, uniformly in $X$ and in the 1-diagram of $A$, we can build a 1-decidable copy of $A^X$.

*Proof.* We can build a 1-decidable copy of $A^X$ by putting the 1-decidable copy of $A$ in the first sort, and defining the second and third sorts as in the previous lemma. Given a tuple $\bar{a} \in A^X$ and an existential formula $(\exists \bar{y}) \varphi(\bar{x}, \bar{y})$, we want to decide whether $A^X \models (\exists \bar{y}) \varphi(\bar{a}, \bar{y})$. First, we may rewrite $\varphi$ in the language where we replace the language of $A$ with the predicates

$$P^{\theta(x_1, \ldots, x_n)} = \{(a_1, \ldots, a_n) \in A^n : A \models \theta(a_1, \ldots, a_n)\}$$
where $\theta$ is an existential formula in the language of $A$. Next, we may assume that $\varphi$ is a conjunction of atomic formulas.

We will show that $(\exists y)\varphi(x, y)$ is equivalent, in $A^X$, to a quantifier-free formula $\psi(x)$ in an expanded language with the predicate

$$Q = \{(x, \ell, s, t) \in S_1 : x \notin X_{\ell, s, t}\}$$

which is only allowed to appear positively. Note that the predicates $Q$ and $P^\theta$ are computable in $A^X$, and so we can decide whether $A^X = \psi(\bar{a})$, and hence whether $A^X = (\exists y)\varphi(\bar{a}, y)$.

Arguing by induction, it suffices to show that if $\bar{a}$ is a tuple from $A^X$, and $\varphi(x_1, \ldots, x_n)$ is a quantifier-free formula in which $Q$ appears only positively, then $(\exists x_n)\varphi(x_1, \ldots, x_n)$ is equivalent in $A^X$ to a formula $\psi(x_1, \ldots, x_{n-1})$ in which $Q$ appears only positively.

Since every element of $A$ is the image of an element of $S_1$ under $g$, and every element of $S_1$ is the image of an element of $S_2$ under $f$, we may assume that $x_1, \ldots, x_n$ are from the sort $S_2$. We may write $\varphi(x_1, \ldots, x_n)$ in the following form:

$$P^\theta(y_1, \ldots, y_n)(f(g(x_1)), \ldots, f(g(x_n))) \land \left[ \bigwedge_{i \in I(U)} U^\ell(g(x_i)) \land \bigwedge_{i \in I(U^c)} \neg U^\ell(g(x_i)) \land \bigwedge_{i \in I(R)} R(x_i) \land \bigwedge_{i \in I(R^c)} \neg R(x_i) \land \bigwedge_{(i, j) \in J^*_1} x_i = x_j \land \bigwedge_{(i, j) \in J^*_1} x_i \neq x_j \land \bigwedge_{(i, j) \in J^*_2} g(x_i) = g(x_j) \land \bigwedge_{(i, j) \in J^*_2} g(x_i) \neq g(x_j) \land \bigwedge_{(i, j) \in J^*_3} f(g(x_i)) = f(g(x_j)) \land \bigwedge_{(i, j) \in J^*_3} f(g(x_i)) \neq f(g(x_j)) \right].$$

So that we can refer to it later, let $\chi(x_1, \ldots, x_n)$ be the part of this formula after

$$P^\theta(f(g(x_1)), \ldots, f(g(x_n))).$$

We may assume that $\varphi$ looks consistent in the sense that $I(U^\ell)$ and $I(\neg U^\ell)$ are disjoint, $I(R)$ and $I(\neg R)$ are disjoint, and so on.

**Case 1.** If $(n, i) \in J^*_1$ for some $i$, then $(\exists x_n)\varphi(x_1, \ldots, x_n)$ is clearly equivalent to the formula $\varphi(x_1, \ldots, x_{n-1}, x_i)$.

**Case 2.** Otherwise, if $(n, i) \in J^*_2$ for some $i$, then $(\exists x_n)\varphi(x_1, \ldots, x_n)$ is equivalent to

$$P^\theta(f(g(x_1)), \ldots, f(g(x_{n-1})), f(g(x_i))) \land Q(g(x_i)) \land \chi'(x_1, \ldots, x_{n-1})$$
if $n \in I(\neg R)$, and

$$P^{y_1,\ldots,y_n}(f(g(x_1)),\ldots,f(g(x_{n-1})),f(g(x_n))) \land \chi'(x_1,\ldots,x_{n-1})$$

otherwise, where $\chi'(x_1,\ldots,x_{n-1})$ is $\chi(x_1,\ldots,x_n)$ with $g(x_n)$ replaced by $g(x_i)$ everywhere, and any term involving only $x_n$ (but not $g(x_n)$, or $f(g(x_n))$) deleted.

**Case 3.** Otherwise, if $\{n,i\} \in J_3$ for some $i$, then $(\exists x_n)\varphi(x_1,\ldots,x_n)$ is equivalent to

$$P^{y_1,\ldots,y_n}(f(g(x_1)),\ldots,f(g(x_{n-1})),f(g(x_i))) \land \chi'(x_1,\ldots,x_{n-1})$$

where $\chi'(x_1,\ldots,x_{n-1})$ is $\chi(x_1,\ldots,x_n)$ with $f(g(x_n))$ replaced by $f(g(x_i))$ everywhere, and any term involving only $x_n$ or $g(x_n)$ (but not $f(g(x_n))$) deleted.

**Case 4.** Otherwise, $(\exists x_n)\varphi(x_1,\ldots,x_n)$ is equivalent to

$$P^{\exists y_n\theta(y_1,\ldots,y_n)}(f(g(x_1)),\ldots,f(g(x_{n-1}))) \land \chi'(x_1,\ldots,x_{n-1})$$

where $\chi'(x_1,\ldots,x_{n-1})$ is $\chi(x_1,\ldots,x_n)$ with any term involving $x_n$, $g(x_n)$, or $f(g(x_n))$ deleted.

\[\square\]

### 14.3.2 Overview of the Construction

Recall that given a structure $D$, we want to build $A$ so that if $n \in S$, then $A$ will have a 1-decidable presentation (which we can construct uniformly), and if $n \notin S$, then $A$ is not isomorphic to $D$.

The structure $A$ will actually be of the form $B^X$ for some sequence of subsets $X = (X_\ell)_{\ell \in \mathbb{L}}$ of $B$. We will build the diagram of $B$ in a computable way while also enumerating the sets $X_\ell$. (Though rather than saying that we put an element $x$ into $X_\ell$, we will say that we put the label $\ell$ on $x$.) By Lemma 14.3.2, $A = B^X$ will be a computable structure. At the end, to see that if $n \in S$ then $A$ has a 1-decidable presentation, we will use Lemma 14.3.3. If $D$ is going to be isomorphic to $A = B^X$, then it will have to be of the form $E^Y$ for a sequence of subsets $Y = (Y_\ell)_{\ell \in \mathbb{L}}$ of $E$; by Lemma 14.3.1, using the 1-diagram of $D$, we can compute $E$ and enumerate the sets $Y_\ell$. So we will diagonalize against the 1-diagram of $E$ together with an enumeration of the sequence $Y$. To keep the construction as intuitive as possible, we will not mention $B$ and $E$. Instead, we will think of $A$ and $D$ as computable structures with c.e. labels.

We will now describe the language and general form of $A$. There will be a set $N$ of nodes. To each node $\nu$, we attach two other structures: a structure in the language of Lemma 14.2.2 with domain $T_\nu$ and a linear order with domain $W_\nu$. $T_\nu$ will be isomorphic to either $C_n$ or $C_\infty$, and $W_\nu$ will be isomorphic to one of $\omega$, $\omega^*$, or $\omega^* + \omega$. We call $T_\nu$ the tag of $\nu$, and we say that the elements of $T_\nu$ are the $T$-elements of $\nu$. To each node $\nu$, we associate the structure consisting of $T_\nu$ and $W_\nu$. We call this structure the $\nu$-component of $A$.

Note that if $n \notin S$, and one node $\nu$ is tagged with $C_n$, and a second node $\nu'$ is tagged with $C_\infty$, then there is no automorphism of $A$ taking $\nu$ to $\nu'$, as $C_n$ and $C_\infty$ are not isomorphic. On
the other hand, if $n \in S$, then the $\nu$-component and the $\nu'$-component might be isomorphic. The linear orders $W_i$ will be used to diagonalize against 1-presentations; in a 1-presentation, a maximal (or minimal) element of a linear order will be distinguished by a universal formula, while in a computable presentation we can always change our mind between building a copy of $\omega$ or $\omega^*$. 

To the nodes $\nu$, and to the $T$-elements, we attach labels which are $\Sigma^0_1$ over the 1-diagram in the sense described in Section 14.3.1. We have infinitely many labels $\ell_k$ and a distinguished label $L$. These labels will be used in the same way that labels are used to build computably categorical structures ([DKL15]) or structures of finite computable dimension ([Gon80b]), and we suggest that it might help the reader who is not familiar with this technique read one of these papers before proceeding. At each stage $s$, each node $\nu$ which is of the form $\rho$ or $\sigma^{\infty}_i$ (these will be defined later), and each of their $T$-elements, will have two labels $\ell_k$ which are unique to them; one label will be the primary label and the other the secondary label. There will be other labels in the bag which hold of every element. The bag will begin empty. The nodes $\tau^{\infty}_{i,s}$, and their $T$-elements, will all be labeled in the same way as $\rho$ was at stage $s$ (except that they may also be labeled with $L$). The nodes $\rho$ and $\sigma^{\infty}_i$, and their $T$-elements, will never be labeled $L$.

While it looks like $D$ is copying $A$, we will periodically add the primary labels of each element to the bag, labeling every element with them, and then give each element a new unique label. What were the secondary labels will become the new primary labels, and the new labels will be the new secondary labels. If infinitely often we add the primary labels to the bag then at the end of the construction every element will be labeled with the same labels—those in the bag. But at every finite stage of the construction, every element will be distinguished.

### 14.3.3 The Construction of $A$

We begin at stage $s = 0$. To start, put into $A$ the distinguished node $\rho$, and the other nodes $(\sigma^{\infty}_i)$, $(\sigma^*_i)$, and $(\tau^*_i)$ and $(\tau^*_0)$. At later stages of the construction, we will add new nodes $(\tau^*_i)$ and $(\tau^*_0)$ for other values of $s$.

For the node $\rho$: Let $T_\rho$ contain a copy of $C_n$, and let $W_\rho$ begin with a single element. For each node $\sigma^i$: Let $T^{\infty}_{\sigma^i}$ contain a copy of $C_\infty$, and let $W^{\infty}_{\sigma^i}$ contain a linear order which depends on $\square$: for $\square = \leftrightarrow$, set $W^{\infty}_{\sigma^i} = \omega^*$; for $\square = \rightarrow$, set $W^{\infty}_{\sigma^i} = \omega$; and for $\square = \rightarrow\leftarrow$, set $W^{\infty}_{\sigma^i} = \omega^* + \omega$. The nodes $\tau^0_{i,0}$ will be the same as the nodes $\sigma^i$, except that $T^0_{\tau^0_{i,0}}$ will contain a copy of $C_n$ instead of $C_\infty$. For every node $\nu$ other than $\rho$, the $\nu$-component of $A$ will be a 1-decidable structure. (Note also that there are no relations that hold between different components.) Indeed, as soon as we add a node $\nu$ (other than $\rho$) to the domain, we will immediately completely decide $T_\nu$ and $W_\nu$. Later, we may add labels to the elements, but since the labels are $\Sigma^0_1$ over the 1-diagram, this is 1-decidable.
Assign, to each of the nodes $\rho$ and $\sigma_i^\square$, and to each of their $T$-elements, two unique labels $\ell_k$. Label the $\tau_{i,s}^\square$, in the same way as $\rho$. It will always be true at each stage $s$ that every node $\rho$ and $\sigma_i^\square$ and each of their $T$-elements will have two unique labels that distinguish them from every other such element. No nodes will be labeled by $L$ at this point. The bag begins empty.

Certain stages will be *expansionary stages*. The expansionary stages are those where we get more evidence that $A$ is isomorphic to $D$. The stage 0 is an expansionary stage by definition. At each expansionary stage $s$, we will have a number $\text{scope}(s)$ which measures how much of the structures $A$ and $D$ we are looking at. Begin with $\text{scope}(0) = 0$.

At each stage $s$, we will have a *target*, target$(s)$, for $\rho$. The target is a node of $D$ which we think is the image, under isomorphism, of $\rho$. We will try to make $W_{\rho}$ different from the target. We do this by choosing a *direction*, direction$(s)$, for $\rho$ at stage $s$, which is either left or right. If the direction is left, then we are trying to build $W_{\rho}$ to be a copy of $\omega^*$; if it is right, then we are trying to build a copy of $\omega$. We will update the target and direction only at expansionary stages. At every stage, expansionary or not, we will add a single element to $W_{\rho}$ depending on the direction at that stage. Thus $W_{\rho}$ will end up being isomorphic to $\omega$, $\omega^*$, or $\omega^* + \omega$.

The general idea of the construction is as follows, when $D$ is a total 1-decidable structure, in each of the two cases $n \in S$ and $n \notin S$. If $n \notin S$, then $C_n$ and $C_\infty$ are not isomorphic. So the node $\rho$ is fixed by every automorphism of $A$. If we can identify the image of $\rho$ in $D$, and have it be our target for all sufficiently large stages, then we will diagonalize against $D$ by making $W_{\rho}$ different from the target in $D$. Of course, the only thing distinguishing $\rho$ from the $\sigma_i^\square$ is that it is tagged with $C_n$ instead of $C_\infty$, and these two structures may look very similar. This is where we use the labels: In $A$, we give $\rho$ a label that distinguishes it from all of the other nodes, and so $D$ must produce a node which looks similar; we use this node as the target. Then, if $D$ copies the labels we put on $A$, we can force it to also tag the target node with $C_n$, making our diagonalization successful. Of course, in the limit, everything ends up with the same labels; and the $r_{i,s}^\square$ are labeled $L$, so that they can be distinguished from $\rho$.

If $n \in S$, then $C_n$ and $C_\infty$ are isomorphic. First, if there are infinitely many expansionary stages, then all of the nodes and $T$-elements end up tagged the same. If $W_{\rho}$ is isomorphic to $\omega$, then the $\rho$-component is isomorphic to each $\sigma_i^\square$-component; so we could have built a copy of $A$ without ever having built the $\rho$-component! The $\sigma_i^\square$-components are actually
1-decidable, since we decide everything about them (except the labels, which are $\Sigma_0^1$ over the 1-diagram) as soon as we add them to the structure. Thus we can build a 1-decidable copy of $A$. The same argument works if $W_\rho$ is isomorphic to $\omega^*$ or to $\omega^* + \omega$. Unfortunately, if there are only finitely many expansionary stages, then the nodes and $T$-elements may end up having different labels. But in this case, after the last expansionary stage $s$, we never add any more labels, and so the $\rho$-component will be isomorphic to each $\tau_{i,s}^{\omega^*}$ or $\tau_{i,s}^{\omega^*+\omega}$-component, and again we could have built a 1-decidable copy of $A$ by not building the $\rho$-component.

Construction at stage $s$. At stage $s$, so far we have built $A[s-1]$. The first thing we do at stage $s$ is to decide whether the stage $s$ is expansionary. Let $s^*$ be the last expansionary stage. Stage $s$ is expansionary if there are:

1. nodes $\nu_0, \ldots, \nu_r$ of $A[s-1]$, containing among them the first scope($s^*$) nodes of $A[s-1]$;
2. $T$-elements $\bar{a}_0 \in T_{\nu_0}, \ldots, \bar{a}_r \in T_{\nu_r}$, containing among them the first scope($s^*$) elements of each of these components;
3. nodes $\mu_0, \ldots, \mu_r$ of $B[s]$, containing among them the first scope($s^*$) nodes of $D[s]$; and
4. $T$-elements $\bar{d}_0 \in T_{\mu_0}, \ldots, \bar{d}_r \in T_{\mu_r}$, containing among them the first scope($s^*$) elements of each of these components

such that

- the atomic types of $\nu_0, \ldots, \nu_r; \bar{a}_0, \ldots, \bar{a}_r$ in $A[s-1]$ and $\mu_0, \ldots, \mu_r; \bar{d}_0, \ldots, \bar{d}_r$ in $D[s]$ are the same, and
- each of the elements from $\nu_0, \ldots, \nu_r; \bar{a}_0, \ldots, \bar{a}_r$ has the same labels in $A[s-1]$ as the corresponding elements from $\mu_0, \ldots, \mu_r; \bar{d}_0, \ldots, \bar{d}_r$ have in $D[s]$.

Otherwise, stage $s$ is not expansionary. If stage $s$ is expansionary, let scope$(s) \geq$ scope$(s^*) + 1$ be large enough that $\nu_0, \ldots, \nu_r$ are among the first scope$(s)$ nodes of $A$, $\bar{a}_0, \ldots, \bar{a}_r$ are among the first scope$(s)$ elements of their components, $\mu_0, \ldots, \mu_r$ are among the first scope$(s)$ nodes of $D$, and $\bar{d}_0, \ldots, \bar{d}_r$ are among the first scope$(s)$ elements of their components.

If stage $s$ is expansionary, then continue by updating the target followed by renewing labels as described below. If the stage $s$ is not expansionary, the target and direction are the same as they were at the last expansionary stage. At all stages, expansionary or not, we finish by adding a new element to $W_\rho$. If direction$(s) = \text{right}$, add the new element to the right of all existing elements. Otherwise, if direction$(s) = \text{left}$, add the new element to the left of the existing ones. In this way we obtain the structure $A[s]$.

Updating the target. In $D[s]$, find the least node, if one exists, which is labeled exactly by the labels of $\rho$ (and so not by $L$). Set $\text{target}(s)$ to be this node. (If no such element exists, $\text{target}(s)$ is undefined and direction$(s) = \text{right}$.)
Now, look at the linear order $W_{\text{target}(s)}$. If it has a greatest element (i.e., an element which the 1-diagram of $D[s]$ says is the greatest element), set $\text{direction}(s) = \text{right}$. Otherwise, set $\text{direction}(s) = \text{left}$. 

**Renewing labels.** Recall that $s^*$ was the previous expansionary stage. First, apply the label $L$ to each node $\tau_{i,s^*}$. Second, each of the nodes $\rho$ and $\sigma^\circ_i$ and their $T$-elements have two labels which only of themselves and which are not in the bag. Add each of the primary labels to the bag. The secondary labels becomes the primary labels. Then, label each of these elements with each label from the bag along with a new unique secondary label.

Build new nodes $\tau_{i,s}^+$ and $\tau_{i,s}^-$ tagged with copies of $C^n$. Attach a copy of $\omega^*$ or $\omega$ to each of these nodes respectively. Label these nodes and their $T$-elements in the same way that $\rho$ and its $T$-elements are currently labeled.

### 14.3.4 The Verification

**Lemma 14.3.4.** $W_{\rho}$ is isomorphic to either $\omega$, $\omega^*$, or $\omega^* + \omega$. These three cases correspond, respectively, to having $\text{direction}(s) = \text{right}$ for all but finitely many $s$, $\text{direction}(s) = \text{left}$ for all but finitely many $s$, and $\text{direction}(s) = \text{right}$ and $\text{direction}(s) = \text{left}$ for infinitely many $s$ each.

**Proof.** At each stage $s$ we add a single element to $W_{\rho}$ on either the left or right hand side, depending on the direction. 

Note that the direction can only change at an expansionary stage, so that if there are only finitely many expansionary stages, $W_{\rho}$ is isomorphic to either $\omega$ or $\omega^*$. This is why we only add nodes $\tau_{i,s}^+$ and $\tau_{i,s}^-$, but not $\tau_{i,s}^\circ$.

**Lemma 14.3.5.** If $\mathcal{A}$ is isomorphic to $\mathcal{D}$, then there are infinitely many expansionary stages.

**Proof.** Suppose to the contrary that there is a last expansionary stage $s^*$, and that $\mathcal{A}$ is isomorphic to $\mathcal{D}$ at the end of the construction, say by an isomorphism $f$. Then after stage $s^*$, we never add any more nodes into $\mathcal{A}$, and we never add any new labels to any elements. Let $\mu_0,\ldots,\mu_r$ be the first scope($s^*$) nodes of $\mathcal{A}$ together with the inverse images, under $f$, of the first scope($s^*$) nodes of $\mathcal{D}$. Let $\bar{a}_0 \in T_{\mu_0},\ldots,\bar{a}_r \in T_{\mu_r}$ be the first scope($s^*$) elements of these components, together with the inverse images, under $f$, of the first scope($s^*$) elements of $T_{f(\mu_0)}\ldots,T_{f(\mu_r)}$. Then, for sufficiently large $s$, $\mu_0,\ldots,\mu_r;\bar{a}_0,\ldots,\bar{a}_r$ and $f(\mu_0),\ldots,f(\mu_r); f(\bar{a}_0),\ldots,f(\bar{a}_r)$ have the same labels in $\mathcal{A}[s-1]$ and $\mathcal{D}[s]$ respectively. Such a stage $s$ is expansionary.

**Lemma 14.3.6.** If there are infinitely many expansionary stages, then every node $\rho$ or $\sigma^\circ_i$ and their $T$-elements have exactly the same labels. Each $\tau_{i,s}^\circ$ is labeled by $L$.

**Proof.** This lemma is easily seen from the way the labels are renewed in the construction.
Lemma 14.3.7. Let \( s \) be an expansionary stage and suppose that \( a \in A \) and \( d \in D \) are nodes which are among the first scope(s) nodes of \( A \) and \( D \) respectively (or \( T \) elements which are among the first scope(s) elements of their components, and are associated to nodes which are among the first scope(s) nodes), and so that \( a \) has the same labels in \( A[s-1] \) as \( d \) does in \( D[s] \). Then for any expansionary stage \( s^* \geq s \), either \( a \) and \( d \) have the same labels in \( A[s^*-1] \) and \( D[s^*] \) respectively, or one of them is labeled \( L \).

Proof. It suffices to show that if \( s^* \geq s \) is an expansionary stage at which \( a \) and \( d \) have the same labels in \( A[s^*-1] \) and \( D[s] \) respectively, and \( s^{**} > s^* \) is the next expansionary stage, then either \( a \) and \( d \) have the same labels in \( A[s^{**}-1] \) and \( D[s^{**}] \) or one of them is labeled \( L \).

Let \( \ell_k \) be the primary label of \( a \) in \( A[s^*-1] \), and let \( \ell_k \) be its secondary label. Then by assumption, \( d \) is also labeled by \( \ell_k \) and \( \ell_k \) in \( D[s^*] \). During stage \( s^* \), \( k \) becomes the primary label of \( a \), and \( d \) gets a new secondary label \( \ell_k \). Now at all stages \( t \), \( s^* < t < s^{**} \), we do not add any labels to elements of \( A \). In \( A[s^{**}-1] \), the only elements labeled \( \ell_k \) are either labeled the same way as \( a \), or labeled \( L \). Since \( s^{**} \) is an expansionary stage, and \( d \) is among the first scope(s) nodes of \( D \) if it is a node (or the first scope(s) elements of its component, which is among the first scope(s) components of \( D \), if \( d \) is a \( T \)-element), there is an element \( a' \in A[s^{**}-1] \) which is labeled in the same way as \( d \). As \( d \) is labeled \( k \), \( a' \) is labeled \( \ell_k \), and so they must both be labeled in the same way as \( a \), or be labeled \( L \).

\[ \square \]

Lemma 14.3.8. Suppose that \( A \) and \( D \) are in fact isomorphic. Let \( s \) be an expansionary stage, and let \( \nu \) and \( \mu \) be nodes of \( A \) and \( D \) respectively, which are among the first scope(s) nodes of those structures, and assume that neither are ever labeled \( L \). If, at stage \( s \), \( \nu \) and \( \mu \) are labeled in the same way in \( A[s-1] \) and \( D[s] \) respectively, then \( \nu \in A \) and \( \mu \in D \) are isomorphic.

Proof. Let \( s_0 = s, s_1, s_2, \ldots \) list the expansionary stages after \( s \). By the previous lemma, at each expansionary stage \( s_i \), \( \nu \) and \( \mu \) are labeled in the same way in \( A[s_i-1] \) and \( D[s_i] \) respectively.

Since \( A \) and \( D \) are isomorphic, by Lemma 14.3.5 there are infinitely many expansionary stages. Given \( i \), define a partial isomorphism \( f_i: T_\nu \to T_\mu \), as follows. Put a \( T \)-element \( a \), which is among the first scope(s) elements of \( T_\nu \), into the domain of \( f_i \) if there is \( d \) a \( T \)-element of \( \mu \), which is among the first scope(s) elements of \( T_\mu \), such that \( a \) and \( d \) have the same labels in \( A[s_i-1] \) and \( D[s] \) respectively. In this case, set \( f_i(a) = d \). (Note that there can be at most one such \( d \) for a given \( a \), as no two elements of the same component of \( A[s_i-1] \) are labeled in the same way.)

Claim 14.3.9. If \( i < i' \), then \( f_i \subseteq f_{i'} \).

Suppose that \( f_i(a) = d \). Then \( a \) and \( d \) are labeled in the same way in \( A[s_i-1] \) and \( D[s_i] \) respectively, and are among the first scope(s) elements of \( T_\nu \) and \( T_\mu \) respectively. Since \( \nu \) and \( \mu \) are never labeled \( L \), neither are \( a \) and \( d \) at the expansionary stage \( s_{i'} \); we will not label \( a \) by \( L \), and if \( d \) was labeled \( L \), then \( s_{i'} \) could not be an expansionary stage. So by the
previous lemma, at the stage $s_i$, $a$ and $d$ are labeled in the same way. Thus we will define $f_i(a) = d$.

Let $f = \bigcup_{i<\omega} f_i$.

**Claim 14.3.10.** $f$ is one-to-one.

If $f$ was not one-to-one, then for some $i$, we would have $f_i(a_1) = f_i(a_2) = d$. So then, in $A[s_i-1]$, $a_1$ and $a_2$ are labeled in the same way; but they are both in the same component, and so this cannot happen.

**Claim 14.3.11.** $f$ is total and onto.

To see that $f$ is total, fix $a \in T_\nu$. For some sufficiently large $i$, $a$ will be among the first scope($s_i$) elements of $T_\nu$. Then, at the next expansionary stage $s_{i+1}$, there will have to be some $\mu';d'$ corresponding (in the sense that they witness that $s_{i+1}$ is a true stage) to $\nu;a$ and $\nu'$ corresponding to $\mu$. Now since $\nu$ and $\mu$ are labeled in the same way, and $\mu$ and $\nu'$ are labeled in the same way, $\nu'$ and $\nu'$ are labeled in the same way in $A[s_{i+1}-1]$. From the construction, we see that $T_\nu$ and $T_\nu'$ are identically either copies of $C_n$ or $C_\infty$. (The nodes $\nu$ and $\nu'$ might be, for example, $\rho$ and $\tau_{0,s_i}$.) Thus there is $a' \in T_\nu'$ which corresponds to $a \in T_\nu$, and since $a$ is among the first scope($s_i$) of $T_\nu$, $a'$ is among the first scope($s_i$) elements of $T_\nu'$. Also, $\nu'$ is among the first scope($s_i$) nodes of $A$. Thus there is $d \in T_\mu$ which is labeled in the same way as $a'$, which is labeled in the same way as $a$; hence we would set $f_{i+1}(a) = d$.

To see that $f$ is onto, a similar but not identical argument works. Fix $d \in T_\mu$. For some sufficiently large $i$, $a$ will be among the first scope($s_i$) elements of $T_\nu$. Then, at the next expansionary stage $s_{i+1}$, there will have to be some $\nu';a'$ corresponding to $\mu;d$ and $\mu'$ corresponding to $\nu$. Now since $\nu$ and $\mu$ are labeled in the same way, and $\mu$ and $\nu'$ are labeled in the same way, $\nu'$ and $\nu'$ are labeled in the same way in $A[s_{i+1}-1]$. From the construction, we see that $T_\nu$ and $T_\nu'$ are identically either copies of $C_n$ or $C_\infty$. Thus there is $a \in T_\nu$ which corresponds to $a' \in T_\nu'$. Then $d$ is labeled the same way as $a'$, which is labeled in the same way as $a$; hence we would set $f_{i+1}(a) = d$.

**Claim 14.3.12.** $f$ is an isomorphism.

It suffices to show that each $f_i$ is a partial isomorphism. At stage $s_i$, let $a_0,\ldots,a_r$ be the elements in the domain of $f_i$, and let $d_0 = f_i(a_0),\ldots,d_r = f_i(a_r)$. Since $s_i$ is an expansionary stage, there must be elements $a'_0,\ldots,a'_r$ of $A[s_i-1]$ which are labeled in the same way, and have the same atomic type as $a_0,\ldots,a_r$ in $D[s_i]$. But then $a'_0,\ldots,a'_r$ are labeled in the same way, in $A[s_i-1]$, as $a_0,\ldots,a_r$. We can see from the construction that $a_0,\ldots,a_r$ and $a'_0,\ldots,a'_r$ must then have the same atomic type in $A[s_i-1]$. (It is possible that $a_0,\ldots,a_r$ are not equal to $a'_0,\ldots,a'_r$, for example if the former are in $T_\rho$ and the latter are in $T_{\rho,0,s_i-1}$.) Hence $f_i$ is a partial isomorphism.

This finished the proof of the lemma.

**Lemma 14.3.13.** If $n \notin S$, then $A$ is not isomorphic to $D$. 

CHAPTER 14. THE COMPLEXITY OF DECIDABLE PRESENTABILITY

394

Proof. Suppose to the contrary that $A$ was isomorphic to $D$ via an isomorphism $f$. Then by Lemma 14.3.5 there are infinitely many expansionary stages.

Note that $\rho$ is the only node of $A$ which is both not labeled $L$ and which is tagged $C_n$. Since $C_n$ and $C_\infty$ are not isomorphic, no node $\sigma^D_i$ is tagged $C_n$, and since there are infinitely many expansionary stages, each $T^{\Box}_{i,s}$ is labeled $L$.

Let $d_0, d_1, d_2, \ldots$ list the elements of $D$, and let $d_i = f(\rho)$. Let $t$ be a stage after which each of $d_0, \ldots, d_{i-1}$, if it is the image, under $f$, of a node $T^{\Box}_{i,s}$ or one of its $T$-elements, is labeled $L$; thus, if one of these elements ever becomes labeled $L_i$ it does so by stage $t$. Suppose that $t$ is also large enough that $\rho$ and $d_i$ are among the first scope($t$) nodes of $A$ and $D$ respectively. We claim that for all expansionary stages $s > t$, target($s$) = $d_i$.

Suppose to the contrary that there is an expansionary stage $s$ at which target($s$) $\neq d_i$. Since $\rho$ is among the first scope($t$) nodes of $A$, there is at least one $d_j \in D_s$ among the first scope($s$) nodes of $D$ which has the same labels as $\rho$ at stage $s$; since target($s$) $\neq d_i$, there is one such $d_j \neq d_i$.

Then either $d_i$ and $\rho$ are labeled differently at stage $s$, or there is a node $d_j$, $j < i$, among the first scope($s$) nodes of of $D$, which is labeled in the same way as $d_i$ at stage $s$ (and hence both are labeled in the same way as $\rho$).

In the first case—if $d_i$ and $\rho$ are labeled differently at stage $s$—then there is another node $\nu \neq \rho$ of $A_s$, which is among the first scope($s$) nodes of $A$, which is labeled in the same way as $d_i$ is in $D_s$. Note that $d_i$ is not labeled $L$, as $f(\rho) = d_i$. So by Lemma 14.3.8, $T_{d_i}$ is isomorphic to $T_\nu$; and, since $\nu \neq \rho$, and $\nu$ is not labeled $L$, $\nu$ is of the form $\sigma^D_i$ and so $T_\nu$ is isomorphic to $C_\infty$. This is a contradiction, as $d_i = f(\rho)$ and $T_\rho$ is isomorphic to $C_n$.

In the second case—if there is a node $d_j$, $j < i$, among the first scope($s$) nodes of $D$, which is labeled in the same way as $d_i$ at stage $s$—by Lemma 14.3.8, $T_{d_j}$ and $T_{d_i}$ are both isomorphic to $T_\nu = C_n$ and not labeled $L$. But then $D$ cannot be isomorphic to $A$, as $\rho$ is the only node $\nu$ of $A$ not labeled $L$ and with $T_\nu$ isomorphic to $C_n$.

So for all expansionary stages $s > t$, target($s$) = $d_i$. If $W_{f(\rho)} = \omega^+$, then at some point the greatest element of $W_{f(\rho)}$ is enumerated into $D$, and the 1-diagram says that this is the greatest element. Then, from some sufficiently large expansionary stage on, the direction is always right. Thus $W_\rho = \omega$. On the other hand, if $W_{f(\rho)} = \omega$ or $\omega^\omega$, then there is never a greatest element of $W_{f(\rho)}$, and so the direction is always left. Then $W_\rho = \omega^\omega$. In all cases, $W_\rho$ is not isomorphic to $W_{f(\rho)}$, a contradiction. \qed

Lemma 14.3.14. If $n \in S$, then $A$ has a 1-decidable presentation which we can construct uniformly.

Proof. Since $n \in S$, $C_n \cong C_\infty$. We claim that if we run the construction without building the node $\rho$ and its component, we get a structure $A^-$ which is 1-decidable and isomorphic to $A$. To see that $A^-$ is isomorphic to $A$, there are two cases. First, if there are infinitely many expansionary stages then, by Lemma 14.3.5, $\rho$ and its $T$-elements, and each node $\sigma^D_i$ and their $T$-elements, all have the same labels. So $\rho$ and its component is actually isomorphic to each of the $\sigma^D_i$ and their components for the appropriate choice of $\Box$. Since there are infinitely many such nodes, removing $\rho$ does not change the isomorphism type.
On the other hand, if there are only finitely many expansionary stages, then let \( s^* \) be the last expansionary stage. After that stage, we never add any more labels. Then \( \rho \) and its component is isomorphic to each of the \( \tau^i_{s^*} \) and their components for some \( \square \in \{\leftrightarrow, \rightarrow, \leftarrow\} \).

Now we will argue that \( A^- \) is 1-decidable. By Lemma 14.3.3, it suffices to show that the reduct of \( A^- \) to the language without the labels is 1-decidable (in fact this reduct is decidable), from which it will follow that \( A^- \) itself, with the labels, is 1-decidable. The rest of the proof of this lemma will be in this smaller language without the labels.

Whenever we add a new node \( \nu \) to \( A^- \), we immediately decide whether \( T_\nu = C_n \) or \( T_\nu = C_\infty \), and whether \( W_\nu \) is isomorphic to \( \omega, \omega^* \), or \( \omega^* + \omega \). These structures—\( C_n, C_\infty, \omega, \omega^* \), and \( \omega^* + \omega \)—all have decidable presentations. So the structure which is the disjoint union of \( T_\nu \) and \( W_\nu \) is decidable, uniformly in \( \nu \), by Lemma 14.2.3. Since this disjoint union is essentially (i.e., up to effective bi-interpretability using finitary \( \Delta_0 \) formulas) the \( \nu \)-component, the \( \nu \)-component is decidable.

By Lemma 14.2.5, the following five structures are decidable:

1. The disjoint union of the \( \sigma^i_{\square} \)-components, for a fixed \( \square \in \{\leftrightarrow, \rightarrow, \leftarrow\} \).

2. The disjoint union of the \( \tau^i_{s^*} \)-components, for a fixed \( \square \in \{\leftrightarrow, \rightarrow, \leftarrow\} \).

Then by Lemma 14.2.4, the disjoint union of these five structures is also decidable. This is effectively bi-interpretable, using finitary \( \Delta_0 \) formulas, to \( A^- \), which is thus decidable.

Lemmas 14.3.13 and 14.3.14 are exactly what we wanted from the construction, and complete the proof of Theorem 14.1.5.

### 14.4 Decidably Presentable Structures

In this section, we will add a guessing argument to the construction from the previous section to show that the index set of decidable presentable structures is \( \Sigma^1_1 \)-complete (Theorem 14.1.1). The new issue that we have to deal with is that the system of labeling which we used previously no longer works with decidable structures, as we cannot make labels which are \( \Sigma_1^0 \) over the elementary diagram. Instead of labeling elements with existential facts, we will label them by the existence of a non-principal type, which is a \( \Sigma_2^0 \) fact over the elementary diagram. Then, when examining the decidable structure \( D \) against which we are diagonalizing, we must guess at the labels.

The argument will also complete the proof of Theorem 14.1.5. See Section 14.4.7.

#### 14.4.1 \( \Sigma^0_2 \) Labeling of Decidable Structures

This subsection will be analogous to Section 14.3.1. Once again, fix an infinite computable set \( L \) of labels. Given a decidable structure \( A \) and a sequence \( X = (X_\ell)_{\ell \in \mathbb{L}} \) of subsets of \( A \), we want to build a two-sorted structure \( A^X \), whose first sort is just the structure \( A \), which codes \( X \) in a \( \Sigma^0_2 \) way over the elementary diagram of \( A \).
We can build $A^X$ as follows. $A^X$ will again be two-sorted, with the first sort consisting of $A$. We will call the second sort $S$. The language of $A^X$ will be the language of $A$ augmented with a function $f : S \rightarrow A$, a unary predicate $U^\ell \subseteq S$ for each label $\ell$, and infinitely many unary relations $R_i \subseteq S$, $i \in \omega$.

The second sort $S$ will be partitioned into the pre-images $f^{-1}(x)$ of the elements $x \in A$, and each fibre $f^{-1}(x)$ will be partitioned into infinitely many disjoint sets $U^\ell$. If $i < i'$, and $R_{i'}$ holds of an element, then $R_i$ will hold of that element, and for each $x$, $\ell$, $i$ there will be infinitely many elements of $f^{-1}(x) \cap U^\ell$ satisfying $R_j$ for $j < i$ but not $R_i$. There is a unique non-principal type $p_i$ in $f^{-1}(x) \cap U^\ell$ of an element satisfying $R_i$ for all $i$.

We will define the relations $R_i$ such that, given $x \in A$ and $\ell$, if $x \in X_\ell$ then there is a single realization of the non-principal type $p_i$ in $f^{-1}(x) \cap U^\ell$, and otherwise there will be no realizations of $p_i$ in $f^{-1}(x) \cap U^\ell$.

**Lemma 14.4.1.** Let $A$ be a structure and let $X = (X_\ell)_{\ell \in \mathbb{L}}$ be a sequence of $\Sigma^0_2$ subsets of $A$. The sets $X_\ell$ are definable in $A^X$ by computable formulas of the form $\exists x \bigwedge_{i \in \mathbb{L}} \psi_i(x, \cdot)$, with the $\psi_i$ quantifier-free. These formulas are computable uniformly in $\ell$, and are independent of $A$ or $X$.

**Proof.** The set $X_\ell$ is definable as the subset of the first sort of $A^X$ defined by

$$\exists y \left[ f(y) = x \land U^\ell(y) \land \bigwedge_i R_i(y) \right]. \quad \square$$

As a result, if $A$ is computable, then the sets $X_\ell$ are uniformly $\Sigma^0_2$.

**Lemma 14.4.2.** Let $A$ be a computable structure and let $X = (X_\ell)_{\ell \in \mathbb{L}}$ be a uniform sequence of indices for $\Sigma^0_2$ subsets of $A$. Then, uniformly in $X$ and in the atomic diagram of $A$, we can build a computable copy of $A^X$.

**Proof.** Let $X_\ell$ be defined by

$$x \in X_\ell \iff (\exists y) \left[ (x, y) \in X^\Pi_1 \right]$$

where $X^\Pi_1$ is $\Pi^0_1$ and, if $x \in X_\ell$, then there is a unique $y$ witnessing this. We can find such a set $X^\Pi_1$ uniformly in a $\Sigma^0_2$ index for $X_\ell$.

The copy of $A^X$ we build will have the decidable copy of $A$ in the first sort, and the second sort will contain elements $(x, \ell, s, t)$ and $(x, \ell, \infty, t)$ with $x$ from the first sort and $\ell$, $s$, and $t$ in $\omega$. We will have $f(x, \ell, s, t) = f(x, \ell, \infty, t) = x$ and $U^\ell(x, m, s, t)$ if and only if $m = \ell$. Given $s$, $t$, and $i$, we will have $R_i(x, \ell, s, t)$ if and only if $s < i$. We will have $R_i(x, \ell, \infty, t)$ if and only if $(x, t) \in X^\Pi_1$ at stage $i$. This defines a computable copy of $A^X$. \square

**Lemma 14.4.3.** Let $A$ be a decidable structure and let $X = (X_\ell)_{\ell \in \omega}$ be a uniform sequence of indices for $\Sigma^0_2$ subsets of $A$. Then, uniformly in $X$ and in the atomic diagram of $A$, we can build a the elementary diagram of a decidable copy of $A^X$. 

CHAPTER 14. THE COMPLEXITY OF DECIDABLE PRESENTABILITY

Proof. We can build a decidable copy of $A^X$ by putting the decidable copy of $A$ in the first sort, and defining the second sort as in the previous lemma. This copy of $A^X$ is decidable.

For each $\ell$, let $A^X[\ell]$ be the reduct of $A^X$ which discards all of the predicates $R_i$ except for $R_0, \ldots, R_\ell$. We claim that $A^X[\ell]$ is decidable uniformly in $\ell$. From this it will follow that $A^X$ is decidable.

These reducts are quite simple structures: Given $x \in A$, there are infinitely many elements $y$ of $f^{-1}(x)$, each of which each have, for each $0 \leq i \leq \ell + 1$, infinitely many elements in $g^{-1}(y)$ with $R_j$ for $j < i$ but not $R_i$. Thus any two such elements $y$ are isomorphic. A simple argument, in the style of Lemma 14.3.3 (or Lemma 14.4.10 to follow) but without having to introduce the predicate $Q$, shows that every formula is equivalent in $A^X[\ell]$ to a quantifier-free formula in the language with the additional predicate

$$P^{\theta(y_1, \ldots, y_n)}(x_1, \ldots, x_n) = \{ (a_1, \ldots, a_n) \in A^n : A \models \theta(a_1, \ldots, a_n) \}$$

where $\theta$ is any formula in the language of $A$. \hfill \qed

14.4.2 The Guesses

In this section, fix a (possibly partial) decidable structure $D$, and a computable sequence $X = (X_\ell)_{\ell \in \mathbb{L}}$ of indices of $\Sigma_2^0$ subsets of $D$, just as one might obtain from a decidable copy of $D^X$ as in Lemma 14.4.1. (Even if $D$ is a partial structure, we can still obtain a sequence of $\Sigma_2^0$ sets in this way.) We will describe a way of guessing at membership in the sets $X_\ell$. Write

$$x \in X_\ell \iff (\exists n)(\forall m)[(x, n, m) \in X_\ell^c]$$

for some uniformly computable predicates $X_\ell^c$. Fix an enumeration of the tuples $(x, \ell, n)$, where $x \in D$, $\ell$ is a label, and $n \in \omega$. Assume that in this enumeration, if $(x, \ell, n)$ comes before $(x, \ell, n')$, then $n < n'$.

At each stage $s$, we will have a guess $G_s$ at which elements look like they are in $X_\ell$, and at what the witnesses are. $G_s$ will be a finite set of tuples $(x, \ell, n)$. For each $(x, \ell, n) \in G_s$, we will have that for all $m < s$, $(x, n, m) \in X_\ell^c$; the converse will not necessarily be true. If, for all $m < s$, $(x, n, m) \in X_\ell^c$, and $n$ is the least such witness, then we say that $x$ appears to be labeled $\ell$ at stage $s$ with witness $n$. Note that if, at some stage, $x$ appears to be labeled $\ell$ with witness $n$, and then at some later stage, $x$ does not appear to be labeled $\ell$ with witness $n$, then $x$ can never again appear to be labeled $\ell$ with witness $n$. It is, however, possible for $x$ to not appear to be labeled $\ell$ with witness $n$, then later to appear to be labeled $\ell$ with witness $n$, and then later to again not appear to be labeled $\ell$ with witness $n$.

Begin with $G_0 = \emptyset$. At stage $s$, we will have defined $G_s^*$ for $s^* < s$. We must now define $G_s$. If there is some $(x, \ell, n) \in G_{s-1}$ so that $x$ does not appear to be labeled $\ell$ at stage $s$ with witness $n$, then we have made a mistake. In this case, let $t < s$ be greatest such that for each $(x, \ell, n) \in G_t$, $x$ appears to be labeled $\ell$ at stage $s$ with witness $n$, and let $G_s = G_t$. Otherwise, if there are no mistakes to correct, let $(x, \ell, n)$ be least (in our fixed enumeration) such that $(x, \ell, n) \notin G_{s-1}$ but $x$ appears to be labeled $\ell$ at stage $s$ with witness $n$. Let
$G_s = G_{s-1} \cup \{(x, \ell, n)\}$. (If no such tuple exists, let $G_s = G_{s-1}$.) Note that there is no other $m \neq n$ with $(x, \ell, m) \in G_{s-1}$.

We will borrow some notation from Ash's $\alpha$-systems [Ash86a, Ash86b] to talk about the true path. Write $s \preceq t$ if and only if $s \leq t$, and $s \preceq t$ if $s \leq t$ and $G_s \subseteq G_t$.

**Lemma 14.4.4.** If $s < t < u$, and $s \preceq u$, then $s \preceq t$.

*Proof.* Suppose to the contrary that $s \not\preceq t$, so that $G_s \not\subseteq G_t$. We may assume that $t$ is the least such. So $G_s \subseteq G_{t-1}$. Since $G_s \not\subseteq G_t$, we can see from the definition of $G_t$ that there is $(x, \ell, n) \in G_s$ so that $x$ does not appear to be labeled $\ell$ at stage $t$ with witness $n$. By choice of $t$, at stage $t-1$, $x$ appeared to be labeled $\ell$ with witness $n$. So, at stage $u$, $x$ cannot appear to be labeled $\ell$ with witness $n$, and so $(x, \ell, n)$ cannot be in $G_u$. So $s \not\preceq u$. \hfill $\Box$

We say that a stage $s$ is a true stage if, for all $t > s$, $s \preceq t$.

**Lemma 14.4.5.** There are infinitely many true stages.

*Proof.* Assume that there is a greatest true stage $s$. There is some least $t$ such that $s+1 \not\preceq t$. Since $s$ is a true stage, $G_s \subseteq G_{s+1}, G_t$. By choice of $t$, $G_{s+1} \not\subseteq G_t$; by the minimality of $t$, $G_{s+1},\ldots, G_{t-1} \not\subseteq G_t$ as well. Then we see from the construction that $G_t = G_s$. Thus $t \preceq u$ for all $u > t$, contradicting the choice of $s$. \hfill $\Box$

We call the sequence $s_0 < s_1 < s_2 < \ldots$ of true stages the true path of the construction.

**Lemma 14.4.6.** If $s$ is a true stage, and $t \preceq s$, then $t$ is also a true stage.

*Proof.* Suppose that $t \preceq s$. Then, by Lemma 14.4.4, $t \preceq s^*$ for all $s^*$ with $t \preceq s^* \preceq s$; and since $s \preceq s^*$ for all $s^* \geq s$, $t \preceq s^*$ for all $s^* \geq t$. \hfill $\Box$

Define $X^s_\ell = \{ x \mid (\exists n) (x, \ell, n) \in G_s \}$. Note that if $s \preceq t$, then $X^s_\ell \subseteq X^t_\ell$. The next lemma will show that the set $X_\ell$ is the union, along the true stages, of the sets $X^s_\ell$.

**Lemma 14.4.7.** $X_\ell = \bigcup_{t \preceq s} X^s_\ell$.

*Proof.* Note that if $x \notin X_\ell$, then for all $n$, there is $m$ such that $(x, n, m) \notin X^s_\ell$. Fix $n$, and let $m$ be such that $(x, n, m) \notin X^s_\ell$. Thus, for all stages $s > m$, $(x, \ell, n) \notin G_s$; so, for any true stage $t$, $(x, \ell, n) \notin G_t$. Since this is true for all $n$, $x \notin X^s_\ell$ for any true stage $s$.

On the other hand, suppose that $x \in X_\ell$, but for all true stages $s$, $x \notin X^s_\ell$. Since $x \in X_\ell$, for some $n$, for all $m$ we have $(x, n, m) \in X^s_\ell$. And since $x \notin X^s_\ell$ for all true stages $s$, $(x, \ell, n) \notin G_s$ for all true stages $s$. We may assume that $(x, \ell, n)$ is the least such tuple. For some true stage $s$, for all $(x', \ell', n')$ less than $(x, \ell, n)$ in our chosen enumeration, we will either have that $x'$ does not appear to be labeled $\ell'$ as witnessed by $n'$ at all true stages after $s$ (and so $(x', \ell', n')$ can never be in $G_t$ for any $t \geq s$) or that $x' \in X^s_\ell$ (with least witness $n'$) and $(x', \ell', n') \in G_s$ (so that $(x', \ell', n') \in G_t$ for all $t > s$). So $x$ appears to be labeled $\ell$ as witnessed by $n$ at all stages after $s$. Then at stage $s+1$, we have $G_{s+1} = G_s \cup \{(x, \ell, n)\}$ and $s+1$ is a true stage. So $x \in X^{s+1}_\ell$, a contradiction. \hfill $\Box$

We will say that a node or $T$-element $x$ from $D[s]$ is labeled $\ell$ (at stage $s$) if $x \in X^s_\ell$. 

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**Chapter 14. The Complexity of Decidable Presentability**

398
14.4.3 \( \exists \forall \) Marker Extensions

Given a structure \( \mathcal{A} \) together with a relation \( X \) on \( \mathcal{A} \), we will describe how to make a certain kind of Marker extension of \( (\mathcal{A}, X) \). We will define a three-sorted structure \( M(\mathcal{A}, X) \) whose first sort is a copy of the structure \( \mathcal{A} \). Let \( n \) be the arity of \( X \). We will refer to the sorts as \( \mathcal{A} \), \( S_1 \), and \( S_2 \). The language of \( M(\mathcal{A}, X) \) will be the language of \( \mathcal{A} \) augmented with functions \( f:S_1 \to \mathcal{A}^n \) and \( g:S_2 \to S_1 \) and a unary relation \( R \in S_2 \).

For each element \( \bar{x} \in \mathcal{A}^n \), there will be infinitely many elements \( y \) of the second sort \( S_1 \) with \( f(y) = \bar{x} \). Each element of \( S_1 \) will be the pre-image, under \( f \), of some \( \bar{x} \in \mathcal{A}^n \). For each element \( y \) of \( S_1 \), there will be infinitely many elements \( z \in S_2 \) with \( g(z) = y \), and each element of \( S_2 \) will be the pre-image, under \( g \), of some \( y \in S_1 \).

For every \( \bar{x} \in \mathcal{A}^n \), there will be infinitely many \( y \in f^{-1}(\bar{x}) \) such that there are infinitely many \( z \in g^{-1}(y) \) with \( R(z) \), and infinitely many \( z \in g^{-1}(y) \) with \( \neg R(z) \). If \( \bar{x} \not\in X \), this will be the case for all \( y \in f^{-1}(\bar{x}) \), but if \( \bar{x} \in X \), then there will also be infinitely many \( y \in f^{-1}(\bar{x}) \) such that for all \( z \in g^{-1}(y) \), \( R(z) \).

**Lemma 14.4.8.** \( X \) is definable in \( M(\mathcal{A}, X) \) by an \( \exists \forall \) formula.

**Proof.** \( X \) is defined by the formula

\[
\bar{x} \in X \iff (\exists y)[f(y) = \bar{x} \land (\forall z)[f(z) = y \to R(z)]] .
\]

**Lemma 14.4.9.** If \( \mathcal{A} \) is computable and \( X \) is \( \Sigma^0_2 \), then we can build a computable copy of \( M(\mathcal{A}, X) \) uniformly in \( \mathcal{A} \) and \( X \).

**Proof.** Let \( X \) be defined by

\[
\bar{x} \in X \iff (\exists y)(\forall z)[(\bar{x}, y, z) \in X^c]
\]

where \( X^c \) is computable and, if \( \bar{x} \in X \), then there are infinitely many \( y \) witnessing this (and, for all \( y \), if there is \( z \) with \((\bar{x}, y, z) \in X^c \), then there are infinitely many such \( z \)). We can find such a set \( X^c \) uniformly in a \( \Sigma^0_2 \) index for \( X \).

The copy of \( M(\mathcal{A}, X) \) we build will have the decidable copy of \( \mathcal{A} \) in the first sort, the second sort will contain elements \( (\bar{x}, s) \), and the third sort will contain the elements \( (\bar{x}, s, t) \). We will have \( f(\bar{x}, s) = \bar{x} \) and \( g(\bar{x}, s, t) = (\bar{x}, s) \). It only remains to define the relation \( R \). Given \( s \) and \( t \), we will have \( R(\bar{x}, s, t) \) if and only if \((\bar{x}, s, t) \in X^c \). This defines a computable copy of \( M(\mathcal{A}, X) \). \( \square \)

**Lemma 14.4.10.** If \((\mathcal{A}, X)\) is decidable, then we can build a decidable copy of \( M(\mathcal{A}, X) \) uniformly in the elementary diagram of \((\mathcal{A}, X)\).

**Proof.** The copy of \( M(\mathcal{A}, X) \) we build will have the decidable copy of \( \mathcal{A} \) in the first sort, the second sort will contain elements \( (\bar{x}, s) \), and the third sort will contain elements \( (\bar{x}, s, t) \). We will have \( f(\bar{x}, s) = \bar{x} \), and \( f(\bar{x}, s, t) = (\bar{x}, s) \). Define \( R(\bar{x}, s, t) \) if \( t \) is odd, or if \( s \) and \( t \) are even and \( \bar{x} \in X \). \( \square \)
We claim that this is decidable. Given a tuple $\bar{a} \in M(A, X)$ and a formula $\varphi(\bar{x})$, we want to decide whether $M(A, X) \models \varphi(\bar{a})$. First, we may rewrite $\varphi$ in the language where we replace the language of $A$ with the predicates

$$P^{\theta(y_1, \ldots, y_n)} = \{(a_1, \ldots, a_n) \in A^n : A \models \varphi(a_1, \ldots, a_n)\}$$

where $\theta$ is a formula, possibly involving quantifiers, in the language of $A$.

We will show that $\varphi(\bar{x})$ is equivalent, in $M(A, X)$, to a quantifier-free formula $\psi(\bar{x})$ in an expanded language with the predicate

$$Q = \{((\bar{x}, s) \in S_1 : s \text{ is odd}, t \text{ is odd}, or \bar{x} \notin X\}$$

and the predicates $P^{\theta(y_1, \ldots, y_n)}$, where $\theta$ is now allowed to contain the predicate $R$. Note that the predicates $Q$ and $P^{\theta}$ are computable in $M(A, X)$, and so we can decide whether $M(A, X) \models \psi(\bar{a})$, and hence whether $M(A, X) \models \varphi(\bar{a})$.

Arguing by induction, it suffices to show that if $\varphi(x_1, \ldots, x_n)$ is a quantifier-free formula possibly involving $Q$ and $P^{\theta}$ (where $\theta$ may involve $R$), $(\exists x_n)\varphi(x_1, \ldots, x_n)$ is equivalent in $M(A, X)$ to a quantifier-free formula $\psi(x_1, \ldots, x_{n-1})$. The argument is essentially the same as Lemma 14.3.3, though $f(g(x_n))$ is now a tuple rather than a single element. 

\[\square\]

14.4.4 Overview of the Construction

As before, fix a $\Sigma^1_1$ set $S$. Given $D$ a 2-decidable structure, we want to build a structure $A$ so that, if $n \in S$, we can uniformly build a decidable copy of $A$, and if $n \notin S$, then $A$ is not isomorphic to $D$. (We could have taken $D$ to be decidable, but by taking it to be 2-decidable we will simultaneously prove the $n \geq 2$ case of Theorem 14.1.1. See Section 14.4.7.)

The structure $A$ we construct will be of the form $[M(B, \preceq)]^X$, where $\preceq$ is a binary relation and $X = (X_\ell)_{\ell \in \mathcal{L}}$ is a sequence of subsets of $B$ (and hence of $M(B, \preceq)$). We will build $B$ in a computable way, with $\preceq$ and the sets $X_\ell$ defined via $\Sigma^0_2$ approximations. By Lemmas 14.4.2 and 14.4.9, $A = [M(B, \preceq)]^X$ will be computable. To see that if $n \in S$ then $A$ has a decidable presentation, we will use Lemmas 14.4.3 and 14.4.10. If $D$ is a total 2-decidable structure which is isomorphic to $A$, then $D = [M(E, \preceq)]^Y$ where $\preceq$ is a binary relation on $E$ and $Y = (Y_\ell)_{\ell \in \mathcal{L}}$ is a sequence of subsets of $E$. Since $D$ is 2-decidable, by Lemma 14.4.8, $\preceq$ is computable. However, the sets $Y_\ell$ may not be computable; so we will have to use the approximations from the Section 14.4.2. Recall from that section that we can find a sequence of computable sets $Y^\ell_s$ such that, if $s_0 < s_1 < s_2 < \cdots$ are the true stages, $Y_\ell = \bigcup_{i \in \omega} Y^\ell_{s_i}$. Recall also that we say that a node or $T$-element $x$ from $D[s]$ is labeled $\ell$ (at stage $s$) if $x \in Y^\ell_s$. Thus, the labels which hold at any true stage are actual labels of elements of $D$.

We will describe how to build the structure $B$, together with $\Sigma^0_2$ approximations of $\preceq$ and the sets $X = (X_\ell)_{\ell \in \mathcal{L}}$: for this latter sequence, we will simply talk about labeling elements of $B$ by a label $\ell$, by which we mean that we put that element into the set $X_\ell$. Then we will set $A = [M(B, \preceq)]^X$. 

\[\square\]
CHAPTER 14. THE COMPLEXITY OF DECIDABLE PRESENTABILITY

The structure \( \mathcal{B} \) will have nodes \( \rho, \sigma_i^\Box \), and \( \tau_i^\Box, \) each of which has attached to it a copy \( T_\nu \) of \( \mathcal{C}_n \) or \( \mathcal{C}_\infty \), and a linear order \( W_\nu \) which is isomorphic to \( \omega, \omega^* \), or \( \omega^*+\omega \). The linear orders \( W_\nu \) will given by the binary relation \( \preceq \) with respect to which we take the Marker extension; thus the linear orders are not themselves in the language of \( \mathcal{B} \), but rather are definable by an \( \exists \forall \) formula in \( M(\mathcal{B}, \preceq) \).

We again have infinitely many labels \( \ell_k \) and a distinguished label \( L \), but now these labels will be \( \Sigma_2^0 \) over the elementary diagram of \( \mathcal{A} \).

14.4.5 Acting for a Guess

At each stage \( s \), our construction will build a partial structure \( \mathcal{B}^s \), together with a binary relation \( \preceq_s \) and labels \( \ell_k^s \) and \( L^s \).

If \( s < t \), then \( \mathcal{B}^t \) will extend \( \mathcal{B}^s \). It will not necessarily be true that if \( x \) is labeled \( \ell \) at stage \( s \), then it will be labeled \( \ell \) at stage \( t \), or that if \( x \preceq_s y \), then \( x \preceq_t y \). If, in fact, \( s \leq_1 t \), then \( \preceq_s \) will extend \( \preceq_t \), and anything labeled \( \ell \) at stage \( s \) will still be labeled \( \ell \) at stage \( t \).

Note that by Lemma 14.4.4, if \( s < t < u \), then \( s \leq_1 t \) and \( s \leq_1 u \). Thus, this last requirement need only be checked at stage \( u \) for the greatest \( t < u \) with \( t \leq_1 u \).

Stage 0. Begin at stage 0 with \( \mathcal{B}[0] \) as follows. In \( \mathcal{B}[0] \), there will be nodes \( \rho \) and \( \sigma_i^\Box \) for \( \Box \in \{ \leftrightarrow, \rightarrow, \leftrightarrow \} \). We have \( T_\rho = \mathcal{C}_n \) and \( T_{\sigma_i^\Box} = \mathcal{C}_\infty \). We put a single element in \( W_\rho \), and in \( W_{\sigma_i^\Box} \) we put a linear order \( \preceq \) isomorphic to either \( \omega \) or \( \omega^* \), depending on whether \( \Box \) is \( \leftrightarrow \) or \( \rightarrow \).

Unlike before, we will not immediately add infinitely many nodes \( \tau_i^\Box_{s,0} \), but rather will “schedule” two such nodes (one for each of \( \Box = \rightarrow \) and \( \Box = \leftrightarrow \)) to be added at each stage. We do, at stage 0, create an infinite reserve of nodes which will, at some later stage, become one of the \( \tau_i^\Box \). To each of these nodes \( \nu \) in the reserve, we have \( T_\nu \) be a copy of \( \mathcal{C}_n \), and \( W_\nu \) a linear order isomorphic to \( \omega \) for half of the nodes, and \( \omega^* \) for the other half.

To each node or \( T \)-element \( x \) associated to a node \( \rho \) or \( \sigma_i^\Box \), we choose two unique labels \( \ell_1 \) and \( \ell_2 \), as primary and secondary labels, and label \( x \) with them.

Set scope(0) = 0.

Action at stage \( s \). Let \( s_1, \ldots, s_n < s \) be the previous stages with \( s_i \leq_1 s \). We say that these stages are the \( s \)-true stages, and if they were expansionary stages, then we say that they are \( s \)-true expansionary stages. Let \( s^* \) be the last \( s \)-true expansionary stage.

At stage \( s \), so far we have built \( \mathcal{B}[s-1], \preceq_{s-1} \) and certain labels \( \ell^{s-1} \) on \( \mathcal{B}[s-1] \). The first thing we need to do is to fix any errors that we may have made since the stage \( s_n \). So we begin stage \( s \) with the order \( \preceq_{s_n} \) and only the labels which held at stage \( s_n \); any changes to \( \preceq \) or the labels after stage \( s_n \) and up to, and including, stage \( s - 1 \) are discarded. Also, return all of the nodes \( \tau_i^\Box_{s,\rho} \), for \( s_n < s' < s \), to the reserve.

Now we need to decide whether the stage \( s \) is expansionary. Stage \( s \) is expansionary if there are:

1. nodes \( \nu_0, \ldots, \nu_r \) of \( \mathcal{A}[s_n-1] \), containing among them the first scope(\( s^* \)) nodes of \( \mathcal{A} \);
(2) \( T \)-elements \( \bar{a}_0 \in T_{v_0}, \ldots, \bar{a}_r \in T_{v_r} \), containing among them the first \( \text{scope}(s^*) \) elements of each of these components;

(3) nodes \( \mu_0, \ldots, \mu_r \) of \( B[s] \), containing among them the first \( \text{scope}(s^*) \) nodes of \( D \); and

(4) \( T \)-elements \( \bar{d}_0 \in T_{\mu_0}, \ldots, \bar{d}_r \in T_{\mu_r} \), containing among them the first \( \text{scope}(s^*) \) elements of each of these components

such that:

- the atomic types of \( v_0, \ldots, v_r; \bar{a}_0, \ldots, \bar{a}_r \) in \( A[s_n - 1] \) and \( \mu_0, \ldots, \mu_r; \bar{d}_0, \ldots, \bar{d}_r \) in \( D[s_n] \) are the same, and

- each of the elements from \( v_0, \ldots, v_r; \bar{a}_0, \ldots, \bar{a}_r \) has the same labels in \( A[s_n - 1] \) as the corresponding elements from \( \mu_0, \ldots, \mu_r; \bar{d}_0, \ldots, \bar{d}_r \) have in \( D[s_n] \).

Otherwise, stage \( s \) is not expansionary. If stage \( s \) is expansionary, let \( \text{scope}(s) \geq \text{scope}(s^*) + 1 \) be large enough that \( v_0, \ldots, v_r \) are among the first \( \text{scope}(s) \) nodes of \( A \), \( a_0, \ldots, a_r \) are among the first \( \text{scope}(s) \) elements of their components, \( \mu_0, \ldots, \mu_r \) are among the first \( \text{scope}(s) \) nodes of \( D \), and \( d_0, \ldots, d_r \) are among the first \( \text{scope}(s) \) elements of their components.

If stage \( s \) is expansionary, then continue by updating the target followed by renewing labels as described below. If the stage \( s \) is not expansionary, the target and direction are the same as they were at the last \( s \)-true expansionary stage.

At all stages, expansionary or not, we finish by adding a new element to the linear order \( \leq \) in \( W_\rho \). In \( B[s_n] \), finitely many of the elements of \( W_\rho \) are bear some relation \( \leq \), and these are linearly ordered. If direction \( (s) = \) right, pick the least element \( x \) of \( W_\rho \) which does not bear any such relation, and put this new element to the right of the linear order we have built so far. Otherwise, if direction \( (s) = \) left, do the same but add the new element to the left. This defines \( \leq_s \).

Let \( s^* \) be the last \( s \)-true expansionary stage. Take two nodes, which we call \( \tau_{s,s^*}^\rightarrow \) and \( \tau_{s,s^*}^\leftarrow \), from the reserve (with \( W_\rho \) isomorphic to \( \omega^* \) and \( \omega \) respectively). Label these with the same labels as \( \rho \).

**Updating the target.** In \( D[s] \), find the least node, if one exists, which is labeled exactly by the labels of \( \rho \) (and not by \( L \)). Set target \((s)\) to be this node. (If no such element exists, target \((s)\) is undefined and direction \((s)\) = right.)

Now, look at the linear order \( W_{\text{target}(s)} \). If it has a greatest element, set direction \((s)\) = right. Otherwise, set direction \((s)\) = left. We can recognize whether an element \( x \) of \( W_{\text{target}(s)} \) is the greatest element by asking whether for all \( y \), \( x \not\leq y \) (where \( x \not\leq y \) is definable by a \( \forall \exists \) formula by Lemma 14.4.8). This is a \( \forall \exists \) fact, and so we can ask the 2-diagram of \( D \).

**Renewing labels.** Recall that \( s^* \) was the previous expansionary stage. First, apply the label \( L \) to each node \( \tau_{l,s^*}^\rightarrow \). Second, each of the nodes \( \rho \) and \( \sigma_l^\rightarrow \) and their \( T \)-elements have two labels which only of themselves and which are not in the bag. Add each of the primary labels to the bag. The secondary labels become the primary labels. Then, label each of these elements with each label from the bag along with a new unique secondary label.
14.4.6 Verification

Let \( s_0 < s_1 < s_2 < \cdots \) be the true path of the approximation of the labels of \( \mathcal{D} \), i.e., of \( Y = (Y_t)_{t \in \mathcal{L}} \). During the construction, we defined a computable structure \( B = \bigcup_s B[s] \). We also defined a \( \Sigma_2^0 \) relation \( \leq = \bigcup_{i \in \omega} \leq_s \) along the true stages, and a sequence of \( \Sigma_2^0 \) subsets \( X = (X_t)_{t \in \mathcal{L}} \) of \( B \), where \( X_t = \bigcup_{i \in \omega} X^i_t \) and \( X^i_t \) consists of the elements of \( B \) which were labeled \( \ell \) at stage \( s \). To see that \( \leq \) and the \( X_t \) are in fact \( \Sigma_2^0 \) sets, note that the set of true stages is a \( \Pi_1^0 \) subset of \( \omega \): \( s \) is a true stage if and only if, for all \( t > s \), \( s \leq_1 t \). Then, for example, \( x \leq y \) if and only if there is a true stage \( s \) such that \( x \leq_s y \). Thus we can define \( A = [M(B, \leq)]^X \).

The construction ends up being essentially the same as the 1-decidable case along the true stages. Note that \( \leq \) and \( X \) were defined so that:

1. An element \( x \in B \) labeled \( \ell \) in \( A \) if and only if it was labeled \( \ell \) at some true stage \( s \).

2. A pair of elements \( x, y \in B \) have \( x \leq y \) if and only if \( x \leq_s y \) at some true stage \( s \).

The proofs of the following lemmas end up being almost exactly the same as proofs of the corresponding lemmas in the 1-decidable case, except that we talk only about true stages. We will repeat the statements of the lemmas, with the modifications to refer only to true stages.

**Lemma 14.4.11.** \((W, \leq)\) is isomorphic to either \( \omega \), \( \omega^* \), or \( \omega^* + \omega \). These three cases correspond, respectively, to having \( \text{direction}(s) = \text{right} \) for all but finitely many true stages \( s \), \( \text{direction}(s) = \text{left} \) for all but finitely many true stages \( s \), and \( \text{direction}(s) = \text{right} \) and \( \text{direction}(s) = \text{left} \) for infinitely many true stages \( s \) each.

We say that a true stage which is also an expansionary stage is a **true expansionary stage**.

**Lemma 14.4.12.** If \( A \) is isomorphic to \( \mathcal{D} \), then there are infinitely many true expansionary stages.

**Lemma 14.4.13.** If there are infinitely many true expansionary stages, then every node \( \rho \) or \( \sigma^0 \) and their \( T \)-elements have exactly the same labels. Each \( \tau^0 \) is labeled by \( L \).

**Lemma 14.4.14.** Let \( s \) be a true expansionary stage and suppose that \( a \in A \) and \( d \in \mathcal{D} \) are nodes which are among the first scope(s) nodes of \( A \) and \( \mathcal{D} \) respectively (or \( T \)-elements which are among the first scope(s) elements of their components, and associated to nodes which are among the first scope(s) nodes), and so that \( a \) has the same labels in \( A[s-1] \) as \( d \) does in \( \mathcal{D}[s] \). Then for any expansionary stage \( s^* \geq s \), either \( a \) and \( d \) have the same labels in \( A[s^*-1] \) and \( \mathcal{D}[s] \) respectively, or one of them is labeled \( L \).

**Lemma 14.4.15.** Suppose that \( A \) and \( \mathcal{D} \) are in fact isomorphic. Let \( s \) be a true expansionary stage, and let \( \nu \) and \( \mu \) be nodes of \( A \) and \( \mathcal{D} \) respectively, which are among the first scope(s) nodes of those structures, and assume that neither are ever labeled \( L \) at a true stage. If, at
stage $s$, $\nu$ and $\mu$ are labeled in the same way in $A[s - 1]$ and $D[s]$ respectively, then $T_\nu \subseteq A$ and $T_\mu \subseteq D$ are isomorphic.

**Lemma 14.4.16.** If $n \notin S$, then $A$ is not isomorphic to $D$.

**Lemma 14.4.17.** If $n \in S$, then $A$ has a decidable presentation which we can construct uniformly.

These last two lemmas complete the proof.

### 14.4.7 The $n \geq 2$ Case of Theorem 14.1.5

Note that we built $A$ while diagonalizing against a 2-decidable structure $D$. So in fact we have shown that

$$ (\Sigma^1_1, \Pi^1_1) \leq_1 (Id-pr, \overline{I_2-pres}). $$

That is, for any $\Sigma^1_1$ set $S$, there is a computable function $f$ such that

$$ n \in S \implies the \ f(n)\text{-th computable structure has a decidable presentation} $$

and

$$ n \notin S \implies the \ f(n)\text{-th computable structure has no 2-decidable presentation}. $$

This proves the $n \geq 2$ case of Theorem 14.1.5.

**Question 14.4.18.** Is it true that $(\Sigma^1_1, \Pi^1_1) \leq_1 (Id-pr, \overline{I_1-pres})$?
Chapter 15

The Gamma Problem for Many-One Degrees

The results presented in this chapter appeared in [HTd].

15.1 Introduction

We give a solution to the Gamma question for many-one degrees by showing that for each \( r \in [0, 1/2] \), there is a many-one degree \( a \) such that \( \Gamma_m(a) = r \).

A set \( A \subseteq \omega \) is \emph{coarsely computable} if, roughly speaking, we have an algorithm for deciding membership in \( A \) which always gives an answer, and the answer is correct except on a set of density zero. By density, we mean asymptotic lower density.

\textbf{Definition 15.1.1.} The \emph{lower density} of a set \( Z \subseteq \omega \) is

\[
\rho(Z) := \liminf_{n \to \infty} \frac{|Z \cap [0, n]|}{n}.
\]

More generally, we can talk about algorithms which are correct half the time, or a third of the time, or almost never. To a set \( A \subseteq \omega \), we can assign a real number which measures the highest density to which it can be approximated by a computable set.

\textbf{Definition 15.1.2} ([HJMS16]). A set \( A \subseteq \omega \) is \emph{coarsely computable at density} \( r \in [0, 1] \) if there is a computable set \( R \) such that \( \rho(A \leftrightarrow R) = r \). Here, \( A \leftrightarrow R \) is the set on which \( A \) and \( R \) agree:

\[
A \leftrightarrow R := \{ x \mid x \in A \iff x \in R \}.
\]

\textbf{Definition 15.1.3} ([HJMS16]). The \emph{coarse computability bound} of a set \( A \subseteq \omega \) is

\[
\gamma(A) := \sup \{ r \mid A \text{ is coarsely computable at density } r \}.
\]

That is, \( \gamma(A) \) is the supremum, over all computable sets \( R \), of \( \rho(A \leftrightarrow R) \).
It is known that for each \( r \in (0, 1] \), there are sets with coarse computability bound \( r \) such that the supremum is obtained, and sets where the supremum is not obtained [HJMS16].

Jockusch and Schupp [JS12] have shown that every non-zero Turing degree contains a set which is not coarsely computable. (This follows from the proof of Proposition 15.1.6 below.) Thus, if \( \Gamma_T(a) = 1 \), then \( a = 0 \). Andrews, Cai, Diamondstone, Jockusch, and Lempp suggested assigning to each Turing degree a real number which measures the extent to which all sets computable in that degree can be coarsely computed.

**Definition 15.1.4 ([ACD+16]).** The coarse computability bound of a Turing degree \( a \) is

\[
\Gamma_T(a) := \inf \{ \gamma(A) \mid A \text{ is } a\text{-computable} \}.
\]

It suffices to take the infimum only over sets in \( a \).

Andrews, Cai, Diamondstone, Jockusch, and Lempp showed that \( \Gamma_T(a) \) can take on the values 0, 1/2, and 1.\(^1\)

**Theorem 15.1.5 ([ACD+16]).** For a Turing degree \( a \):

1. If \( a \) is computable, \( \Gamma_T(a) = 1 \).
2. If \( a \) is computably traceable and non-computable, \( \Gamma_T(a) = 1/2 \).
3. If \( a \) is 1-random and hyperimmune-free, \( \Gamma_T(a) = 1/2 \).
4. If \( a \) is hyperimmune, \( \Gamma_T(a) = 0 \).
5. If \( a \) is PA, \( \Gamma_T(a) = 0 \).

Hirschfeldt, Jockusch, McNicholl, and Schupp showed that \( \Gamma_T(a) \) cannot take on any values in the open interval \((1/2, 1)\). We will repeat the proof here because we will reference it later.

**Proposition 15.1.6 ([HJMS16]).** Let \( a \) be a nonzero Turing degree. Then \( \Gamma_T(a) \leq 1/2 \).

**Proof.** Fix \( A \in a \). We will show that there is \( B \leq_m A \) such that \( \gamma(B) \leq 1/2 \). The idea is that each bit of \( A \) will be copied many times by \( B \), so that if we have a computable approximation to \( B \) which is correct more than half the time, we can correctly guess at the bits of \( A \) with only finitely many errors.

For each \( n \in \omega \), define \( I_n = [n!, (n + 1)!] \). Let

\[
B = \bigcup_{n \in A} I_n.
\]

---

\(^1\)See also [MN15] for a unifying approach to some of these examples.
It is easy to see that $B \leq_m A$. Suppose towards a contradiction that $\gamma(B) > \frac{1}{2}$. Let $R$ be a computable approximation to $B$, with $\rho(B \leftrightarrow R) > \frac{1}{2}$. Fix $N$ and $p$ such that for all $n \geq N$,

$$\frac{|(B \leftrightarrow R) \cap [0,n]|}{n} > p > \frac{1}{2}.$$ 

Increasing $N$, we may assume that $\frac{1}{2} + \frac{1}{N} < p$.

Given $n \geq N$, we will show how to decide computably whether $n \in A$. We claim that $n \in A$ if and only if more than half of the elements of $I_n$ are in $R$. Indeed, suppose that $n \in A$, but at most half of the elements of $I_n$ are in $R$. Then

$$\frac{|(B \leftrightarrow R) \cap [0,(n+1)!]|}{(n+1)!} \leq \frac{n! + \frac{(n+1)!-n!}{2}}{(n+1)!} = \frac{1}{2} + \frac{1}{2(n+1)} < p.$$ 

This is a contradiction. So if $n \in A$, more than half of the elements of $I_n$ are in $R$. A similar argument works when $n \notin A$. 

The Gamma question, from [ACD+16], asks whether the value of $\Gamma_T$ can be strictly between 0 and 1/2. Monin [Mona] has recently given a solution to the Gamma question: The only possible values of $\Gamma_T$ are 0, 1/2, and 1.

Our work grew out of an independent attempt to answer the Gamma question. If we replace Turing reducibility by many-one reducibility, we get a Gamma function on many-one degrees:

**Definition 15.1.7.** The coarse computability bound of an $m$-degree $a$ is

$$\Gamma_m(a) := \inf \{ \gamma(A) \mid A \leq_m a \}.$$ 

It suffices to take the infimum only over sets in $a$.

The proof of Proposition 15.1.6 used a many-one reduction, so it still holds for $m$-degrees. Moreover, the examples in Theorem 15.1.5 yield examples of $m$-degrees with $\Gamma_m$ being 0, 1/2, and 1. Thus, we can ask the Gamma question for $m$-degrees: Can the value of $\Gamma_m$ be strictly between 0 and 1/2? Interestingly, we get the opposite answer from Monin’s: Every $p \in [0,1/2]$ is a possible value of $\Gamma_m$.

**Theorem 15.1.8.** Fix $0 \leq p \leq \frac{1}{2}$. There is an $m$-degree $a$ with $\Gamma_m(a) = p$.

Versions of the Gamma question for weaker reducibilities have already been asked in the literature: In [Hir17], Hirschfeldt asked the Gamma question for truth table degrees. (Monin’s answer to the Gamma question for Turing degrees also yields the same answer for truth table degrees: The value of $\Gamma_{tt}$ cannot be strictly between 0 and 1/2.) An interesting question is what happens for intermediate reductions, such as bounded truth table reductions. Do such reductions have enough computational power to apply the theorems from coding theory used by Monin, or are they sufficiently simple to allow a construction such as the one we use for many-one degrees?
15.2 Background on the Hypergeometric Distribution

The proof of Theorem 15.1.8 will make use of a probabilistic argument about random variables following a hypergeometric distribution. We will quickly review this distribution here. (See [HPS71, p. 52].)

The hypergeometric distribution is the discrete probability distribution of the number of successes in $K$ draws, without replacement, from a population of size $N$ which contains $n$ successes. For example, one might think of red and blue marbles in a box; if there are $N$ marbles, $n$ of which are red, and we randomly select $K$ marbles, the number of red marbles we pick will follow a hypergeometric distribution. We denote the hypergeometric distribution by $H(K, N, n)$ and, if $X \sim H(K, N, n)$, we have

$$
Pr(X = x) = \frac{\binom{n}{x} \binom{N-n}{K-x}}{\binom{N}{K}}.
$$

Our particular application will be to have a set $U$ of size $N$, with a subset $V$ of size $n$. If $p > q$ are real numbers in $[0, 1]$, we will randomly pick from $U$ a set $S$ consisting of about $pN$ elements. We want to choose, in $S$, at least $qn$ elements which are also in $V$. Since $p > q$, it is reasonable to think that we should often get enough elements of $V$. Intuitively, the larger $N$ and $n$ are, the more likely we are to get want we want. Precise bounds are given by the following theorem:

**Theorem 15.2.1** ([Hoe63], see also [Chv79]). Let $X \sim H(K, N, n)$ where $p = K/N > q$. Let $t = p - q$. Then

$$
Pr(X \leq qn) \leq \exp(-2t^2n).
$$

It will be important that this bound does not depend on $N$ (though of course, as $n$ becomes bigger, $N$ will as well).

15.3 Proof of the Main Theorem

We will now prove Theorem 15.1.8.

*Proof of Theorem 15.1.8.* Fix $0 < p < \frac{1}{2}$. We will find a set $A$ whose $m$-degree $a$ has $\Gamma_m(a) = p$. Fix $(C_\ell)_{\ell \in \omega}$ a non-effective list of the computable sets, in which each set is repeated infinitely many times.

We will ensure that $\gamma(A) \leq p$ having, for each computable set $C_\ell$,

$$
\liminf_{n \to 0} \frac{|(A \leftrightarrow C_\ell) \cap [0, n]|}{n} \leq p.
$$

We will accomplish this by making sure that for each $\ell$, there are infinitely many values of $n$ for which we force $A$ to differ from $C_\ell$ on a large portion of $[0, n)$. This will force $\Gamma_m(a) \leq p$. 

In fact, since each computable set appears infinitely many times in the list \((C_\ell)_{\ell \in \omega}\), it suffices to find, for each \(\ell\), a single \(n \geq \ell\) with

\[
\left| (A \leftrightarrow C_\ell) \cap [0, n]\right| \leq p + \epsilon_\ell
\]

where \(\epsilon_\ell \to 0\).

To have \(\Gamma_m(a) \geq p\), we must make sure that for each set \(B\) which is \(m\)-reducible to \(A\) via \(f\), \(\gamma(B) \geq p\). (One such \(B\) will be \(A\) itself via the identity reduction.) We will think of \(A\) as being approximated by \(A^* = \emptyset\). (So we want the bits of \(A\) to be 0’s with density at least \(p\).) Thus we might initially try to approximate \(B\) by \(B^* = \emptyset\), which is what we get by applying the reduction \(f\) to \(A^*\). This will not work, for the reason that \(f\) could be highly non-injective. For example, if \(f\) maps every element to the same element \(y\), then we could have \(A = \{y\}\), which is very well approximated by \(A^*\), but applying the reduction \(f\) we get \(B = \omega\) which is very badly approximated by \(B^*\). This is where we will exploit the fact that \(p \leq 1/2\). Say \(z = f(x) = f(y)\). Then if we put \(x \in B^*\) but \(y \notin B^*\), we are guaranteed to be right about at least one of the two; for if \(z \in A\), then \(x, y \in B\) in which case we were right about \(x\), and if \(z \notin A\), then \(x, y \notin B\), in which case we were right about \(y\). If we manage this in the right way, we will be correct with density \(1/2\). (The proof of Proposition 15.1.6 shows that we must do something like this.)

More formally, let \((f_e : \omega \to \omega)_{e \in \omega}\) be a (non-effective) list of the total many-one reductions. For each \(e\), let

\[
B_e = f_e^{-1}(A) := \{x \mid f_e(x) \in A\}.
\]

We define a computable approximation \(B_e^+\) to \(B_e\) as follows. First, let \(I_1, I_2, I_3, \ldots\) be the consecutive intervals of length one, two, three, and so on. (So \(I_1 = \{0\}, I_2 = \{1, 2\}, I_3 = \{3, 4, 5\}, \text{ etc.}\) For each interval \(I_n\), let \(J_{n,e} = f_e(I_n)\) be the multiset image\(^2\) of \(I_n\) under \(f_e\), and write \(J_{n,e} = J_{n,e}^* \cup 2J_{n,e}^{**}\) where each element has multiplicity one in \(J_{n,e}^*\). So, for example, if \(f_e(3) = 0\), \(f_e(4) = 0\), and \(f_e(5) = 1\), then \(f_e(I_3) = \{0, 0, 1\} = \{0\} \cup 2\{0\}\); if \(J_{8,e} = f_e(I_8) = \{0, 0, 0, 0, 0, 1, 1, 2\}\), then \(J_{8,e} = \{0, 2\} \cup 2\{0, 0, 1\}\). We can write \(I_n\) as a disjoint union \(I_{n,e}^* \cup I_{n,e}^{**} \cup I_{n,e}^{***}\), where \(f_e(I_{n,e}^{**}) = f_e(I_{n,e}^{***}) = J_{n,e}^*\) and \(f_e(I_{n,e}^{*}) = J_{n,e}\). Then let \(B_e^+ = \bigcup_n I_{n,e}^{**}\). The simplicity with which we can describe \(B_e^+\) is where we take advantage of the fact that we are considering many-one reductions rather than Turing reductions. We will have that, for some decreasing positive sequence \(\gamma_n \to 0\), and for all \(n\),

\[
\left| (B_e \leftrightarrow B_e^+) \cap I_n \right| \geq p - \gamma_n.
\]

Since the length of the intervals \(I_n\) are increasing slowly, this will suffice to get \(\gamma(B_e) \geq p\).

**Claim 15.3.1.** Assuming (†), \(\gamma(B_e) \geq p\).

\(^2\)Recall that multisets are a generalization of sets to allow multiple instances of the same element. By the multiset image, we mean that we want to count the number of pre-images of an element in the range.
CHAPTER 15. THE GAMMA PROBLEM FOR MANY-ONE DEGREES

Proof. Since

\[
\frac{|(B_e \leftrightarrow B_{e}^*) \cap I_{m}|}{x + 1} \geq p - \gamma_{m},
\]

for each \(m\), there is \(K_m\) such that for all \(K \geq K_m\),

\[
\frac{|(B_e \leftrightarrow B_{e}^*) \cap (\bigcup_{n \leq K} I_n)|}{\sum_{n \leq K} n} \geq p - \gamma_{m}.
\]

(Assume that the sequence \(K_m\) is strictly increasing in \(m\).) Given \(m\), let \(x \in I_{N+1}\) for some \(N \geq K_m\), with \(N\) sufficiently large that \(\frac{N-1}{N+1}(p - \gamma_{m}) \geq p - 2\gamma_{m}\). Then

\[
|(B_e \leftrightarrow B_{e}^*) \cap (\bigcup_{n \leq N} I_n)| \leq |(B_e \leftrightarrow B_{e}^*) \cap [0, x]|.
\]

So

\[
\frac{|(B_e \leftrightarrow B_{e}^*) \cap [0, x]|}{x + 1} \geq \frac{(\sum_{n \leq N} n)}{x + 1} \frac{|(B_e \leftrightarrow B_{e}^*) \cap (\bigcup_{n \leq N} I_n)|}{(\sum_{n \leq N} n)} \geq \frac{(\sum_{n \leq N} n)}{x + 1} (p - \gamma_{m}).
\]

Now

\[
\frac{(\sum_{n \leq N} n)}{x + 1} \geq \frac{(\sum_{n \leq N} n)}{(\sum_{n \leq N} n)} = \frac{N - 1}{N + 1}.
\]

By choice of \(N\), \(\frac{N-1}{N+1}(p - \gamma_{m}) \geq p - 2\gamma_{m}\), and so

\[
\frac{|(B_e \leftrightarrow B_{e}^*) \cap [0, x]|}{x + 1} \geq p - 2\gamma_{m}.
\]

Thus

\[
\gamma(B_e) = \liminf_{x \to \infty} \frac{|(B_e \leftrightarrow B_{e}^*) \cap [0, x]|}{x} \geq p.
\]

Note that one of the reductions \(f_e\) is the identity reduction, so for that \(e\), \(B_e = A\). Thus \(\gamma(A) = p\). Hence \(\Gamma_m(A) = p\).

We now construct \(A\) as the concatenation of infinitely many finite binary strings: \(A = \alpha_1 \beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3 \ldots\). Each \(\alpha_i\) will consist entirely of 0’s.\(^3\) Note that we have already defined each of the approximations \(B_{e}^*\) computably; our construction of \(A\) can be non-effective. Fix \((\epsilon_{\ell})_{\ell \omega}\) a decreasing sequence of positive reals converging to zero (and assume that the \(\epsilon_{\ell}\) are small relative to \(p\), so that for example \(p - \epsilon_{\ell} > 0\) and \(p + \epsilon_{\ell} < 1/2\)). Given \(\alpha_1 \beta_1 \alpha_2 \beta_2 \ldots \beta_{\ell}\), we must define \(\alpha_{\ell+1}\) and \(\beta_{\ell+1}\). Let \(L\) be the length of \(\alpha_1 \beta_1 \alpha_2 \beta_2 \ldots \beta_{\ell}\).

At stage \(\ell\), we consider only the reductions \(f_e\) for \(e \leq \ell\). We have two competing desires. First, we want to define \(\beta_{\ell+1}\) so that \(A\) has very little agreement with \(C_{\ell}\) on this part of their domain. Second, we want \(\beta_{\ell+1}\) to have many 0’s, particularly on elements in the ranges of the

\(^3\)We should expect to have long sequences of zeros. Since \(\gamma(A) = p \leq 1/2\), \(A \leftrightarrow \omega\) should have density at most \(p\). But that means that the upper destiny of \(A \leftrightarrow \emptyset\) should be at least \(1 - p \geq 1/2\), so there should be initial segments of \(A\) with many 0’s.
for the hypergeometric distribution. More formally, if we about \( \frac{J}{\text{divides.alt0}} \) then we are guaranteed to get agreement of close to 1 at the elements in \( W \). We will have \( \frac{\text{divides.alt0}}{\text{divides.alt0}} \) \( \alpha \). Let \( L \) depend on \( K \) and \( p \). Most \( \alpha \) worry about what we have already chosen for the infinite series \( \ell \). Since \( \text{intervals}. (1) says that \( M \) is big enough to apply the tail bounds for the hypergeometric distribution, and (2) says that \( M \) is large enough compared to \( L \) that we do not have to worry about what we have already chosen for \( \alpha_1 \cdots \alpha_2 \cdots \alpha_\ell+1 \). Our choice of \( \beta_{\ell+1} \) will not affect the reduction on these intervals.

Then choose \( N \) sufficiently large that so that \( \frac{L+K+pN+\epsilon_\ell N}{L+K+2N} \leq p + 2\epsilon_\ell \) and \( \frac{\epsilon_\ell N}{N} - p \geq \epsilon_\ell/2 \). We will have \( |\beta_{\ell+1}| = N \). \( N \) is large enough that, by making \( \beta_{\ell+1} \) agree with \( C_{\ell} \) at density at most \( p \), we can make \( A \) agree with \( C_{\ell} \) at density at most \( p + \epsilon_\ell \). This \( N \) will be the \( N \) in our applications of the tail bounds for the hypergeometric distribution. Because \( K \) was chosen dependent on \( M \), and \( N \) was chosen dependent on \( K \), it is important here that these tail bounds depend only on \( n \) (which will be bounded below by a fixed multiple of \( M \)).

As before, write \( J_{n,e} = f_e(I_n) \) as \( J_{n,e}^* \) where each element has multiplicity one in \( J_{n,e}^* \). Recall that, by choice of \( B_e \), we are guaranteed to have \( B_e \) agree with \( B_e^* \) on half of the elements in \( I_{n,e} \) which map to \( J_{n,e}^* \). If the size of \( J_{n,e}^* \) is very close to the size of \( J_{n,e} \), then we are guaranteed to get agreement of close to \( 1/2 \) on \( I_{n,e} \), and we do not have to worry about \( I_{n,e} \). The other option is that \( J_{n,e}^* \) is very large, in which case we get small tail bounds for the hypergeometric distribution. More formally, if \( n \geq M \), then either \( \frac{|J_{n,e}^*|}{|J_{n,e}|} \geq \frac{1}{2} - \epsilon_\ell \) or \( |J_{n,e}^*| > 2\epsilon_\ell |J_{n,e}| \). Indeed, if we are not in the former case, then

\[
2|J_{n,e}^*| < |J_{n,e}| - 2\epsilon_\ell |J_{n,e}|.
\]

Rearranging, and using the fact that \( |J_{n,e}| = |J_{n,e}| + 2|J_{n,e}^*| \), we get

\[
|J_{n,e}^*| > 2\epsilon_\ell |J_{n,e}|.
\]
Let $\Omega$ index the pairs $n \geq M$ and $e \leq \ell$ for which $|J_{n,e}^*| > 2\epsilon \ell |J_{n,e}|$.

**Claim 15.3.2.** There is a set $S \subseteq [L + K, L + K + N)$ with $|S| < N \leq p + \epsilon \ell$ such that for each $(n, e) \in \Omega$, 

$$\left| \frac{S \cap J_{n,e}^* + |J_{n,e}^* \setminus [L + K, L + K + N]|}{|J_{n,e}^*|} \right| \geq p.$$ 

The set $S$ is a set on which $A$ will be forced to have 0’s. On the other elements of $[L + K, L + K + N)$, $A$ will have the freedom to be different from $C_\ell$. This claim says that we can choose $S$ to simultaneously have $S$ small enough that $A$ can be sufficiently different from $C_\ell$ and large enough that the reductions $f_e$ find sufficiently many 0’s in their ranges.

**Proof.** First, note that if we modify $J_{n,e}^*$ by removing an element which is outside of the interval $[L + K, L + K + N)$ and adding a new element which is inside of this interval, for any fixed set $S \subseteq [L + K, L + K + N)$ the quantity 

$$\left| \frac{S \cap J_{n,e}^* + |J_{n,e}^* \setminus [L + K, L + K + N]|}{|J_{n,e}^*|} \right|$$

can only decrease. Also, if $J_{n,e}^* \supseteq [L + K, L + K + N)$, then for any choice of $S$ with $|S| \geq p$ we will have 

$$\left| \frac{S \cap J_{n,e}^* + |J_{n,e}^* \setminus [L + K, L + K + N]|}{|J_{n,e}^*|} \right| \geq p$$

as desired. So we may assume that, for each $(n, e)$, $J_{n,e}^* \subseteq [L + K, L + K + N)$.

We give a probabilistic argument that the desired set $S$ exists. Imagine that we randomly pick a set $S$ of size $r = \lceil (p + \epsilon \ell) N \rceil$. For each $(n, e) \in \Omega$, let $X_{n,e}$ be the random variable 

$$|S \cap J_{n,e}^*|; \text{ we have } X_{n,e} \sim H(r, N, |J_{n,e}^*|) \text{.}$$

Let $t = \frac{r}{N} - p$. By choice of $N$, we have $t \geq \epsilon \ell / 2$. So by the tail bounds for the hypergeometric distribution, the probability that for some fixed $(e, n) \in \Omega$, $|S \cap J_{n,e}^*| \leq p|J_{n,e}^*|$ is bounded above by 

$$Pr(X_{e,n} \leq p|J_{n,e}^*|) \leq \exp(-2\ell^2|J_{n,e}^*|) \leq \exp(-\epsilon \ell |J_{n,e}^*|) = \exp(-\epsilon^3 \ell n).$$

(Note that this holds even if $p|J_{n,e}^*|$ is not an integer.) So the probability that for all $(e, n) \in \Omega$, $|S \cap J_{n,e}^*| \leq p|J_{n,e}^*|$, is bounded above by 

$$\sum_{(e,n)\in\Omega} \exp(-\epsilon^3 \ell n) \leq \ell \sum_{n \geq M} \exp(-\epsilon^3 \ell n) < 1.$$ 

So there is a non-zero probability that we pick a set $S$ as desired; some such set must exist. 

For $i < N$, when $L + K + i \in S$, set $\beta_{\ell+1}(i) = 0$, and otherwise set $\beta_{\ell+1}(i) \neq C_\ell(L + K + i)$. So for $x \in [L + K, L + K + N)$, if $x \in S$ then $A(x) = 0$ and if $x \notin S$ then $A(x) \neq C_\ell(x)$.

First, we will show that we made $A$ sufficiently different from $C_\ell$. 


Claim 15.3.3. $\gamma(A) \leq p$.

Proof. We have, for each $\ell$, that

\[
\frac{|(A \leftrightarrow C_\ell) \cap [0, L + K + N]|}{L + K + N} \leq \frac{L + K + |\{ x \in S \mid A(i) = C_\ell(i) \}|}{L + K + N} \\
\leq \frac{L + K + |S|}{L + K + N} \\
\leq \frac{L + K + pN + \epsilon_\ell N}{L + K + N}.
\]

Here, $L$, $K$, and $N$ are the values of those variables at stage $\ell$ of the construction. By choice of $N$,

\[
\frac{L + K + pN + \epsilon_\ell N}{L + K + N} \leq p + 2\epsilon_\ell.
\]

So

\[
\frac{|(A \leftrightarrow C_\ell) \cap [0, L + K + N]|}{L + K + N} \leq p + 2\epsilon_\ell.
\]

Then, noting that for each $\ell$ there are infinitely many $\ell'$ with $C_\ell = C_{\ell'}$,

\[
\gamma(A) = \liminf_{n \to \infty} \frac{|(A \leftrightarrow C_\ell) \cap [0, n]|}{n} \leq p.
\]

Second, we will show that $B_\epsilon$ is sufficiently well approximated by $B_\epsilon^*$. The following claim verifies the hypotheses of Claim 15.3.1.

Claim 15.3.4. Fix $e$. Given $n$, let $\ell \geq e$ be such that $M_{\ell+1} > n \geq M_\ell$, where $M_\ell$ is the value of $M$ at stage $\ell$. Then

\[
\frac{|(B_\epsilon \leftrightarrow B_\epsilon^*) \cap I_n|}{n} \geq p - \epsilon_\ell.
\]

The $n$ satisfying $M_{\ell+1} > n \geq M_\ell$ are the sizes of intervals of medium size for $\ell$, that is, those where $\beta_\ell$ is exactly the right length to determine the amount of agreement between $B_\epsilon$ and $B_\epsilon^*$ on the interval $I_n$.

Proof. Write $I_\epsilon = I_{n,e}^{1} \cup I_{n,e}^{2} \cup I_{n,e}^{3} \cup I_{n,e}^{*,1} \cup I_{n,e}^{*,2}$ where $I_{n,e}^{*,1}$ and $I_{n,e}^{*,2}$ are as before, and $f_\epsilon(I_{n,e}^{*,1}) \subseteq [0, L)$, $f_\epsilon(I_{n,e}^{*,2}) \subseteq [L + K, L + K + N)$, and $f_\epsilon(I_{n,e}^{*,3}) \subseteq [L, L + K) \cup [L + K + N, \infty)$. (Again, $L$, $K$, and $N$ are the values at stage $\ell$.)

For $x \in I_{n,e}^{*,3}$, $f_\epsilon(x) \in [L, L + K) \cup (L + K + N, \infty]$, and so, since $n < M_{\ell+1}$, $B_\epsilon^*(x) = 0 = A(f_\epsilon(x)) = B_\epsilon(x)$. Thus

\[
|B_\epsilon \leftrightarrow B_\epsilon^*| \cap I_{n,e}^{*,3} = |I_{n,e}^{*,3}|.
\]

For $I_{n,e}^{*,2}$, if $f(x) \in S$, then $B_\epsilon(x) = B_\epsilon^*(x)$ since $x \notin B_\epsilon^*$ and $f_\epsilon(x) \notin A$, so that $x \notin B_\epsilon$. Thus (recalling that $J_{n,e}^{*,2} = f_\epsilon(I_{n,e}^{*,2})$, and that $f_\epsilon$ is injective on this set):

\[
|B_\epsilon \leftrightarrow B_\epsilon^*| \cap I_{n,e}^{*,2} \geq |S \cap J_{n,e}^{*,2}|.
\]
By choice of $S$, we have
\[
\frac{|S \cap J^*_{n,e}| + |J^*_{n,e} \setminus [L + K, L + K + N]|}{|J^*_{n,e}|} \geq p.
\]
Thus, noting that $|J^*_{n,e} \setminus [L + K, L + K + N]| = |I^*_{n,e}| + |I^*_{n,e,1}|$,
\[
|(B_e \leftrightarrow B_e^*) \cap I^*_{n,e}| + L \geq p|I^*_{n,e}|.
\]
By definition of $B_e^*$,
\[
|(B_e \leftrightarrow B_e^*) \cap I^*_{n,e}| = |I^*_{n,e,1}| = |I^*_{n,e,2}|.
\]
This is because each $x \in I^*_{n,e,1}$ can be paired with a $y \in I^*_{n,e,2}$ with $f_e(x) = f_e(y)$; we have $x \in B_e^*$ and $y \notin B_e^*$. Either $f_e(x) = f_e(y) \in A$, in which case $x, y \in B_e$, or $f_e(x) = f_e(y) \notin A$, in which case $x, y \notin B_e$.

So combining equations ($*$) and ($**$), we get
\[
|(B_e \leftrightarrow B_e^*) \cap I^*_{n,e}| \geq |I^*_{n,e,1}| + p|I^*_{n,e}| - L.
\]
By choice of $M_\ell$,
\[
\frac{|I^*_{n,e,1}| + p|I^*_{n,e}| - L}{|I^*_{n,e}|} \geq p - \epsilon_\ell.
\]
This completes the proof of the theorem, as Claim 15.3.1 now gives that $\gamma(B_e) \geq p$ for each $e$. 

Part VI

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Bibliography


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<tr>
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