Rapid Onset Impulsive Loading: Three Dynamical Case Studies

by

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Abstract

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In this dissertation I present research undertaken on three dynamical systems, which I term ‘case-studies’. The most interesting dynamics in these systems were largely precipitated by abbreviated periods where impulsive loading dominated.

The first case-study features an impulsively-loaded wave energy converter (WEC) for which mass modulation schemes have been proposed which take advantage of the ambient water motion. Experimental results for a pair of passively and impulsively initiated schemes are presented and one of them is shown to be effective in increasing the energy harvesting potential of a WEC; numerical analysis of the model also shows the potential benefits of the mass-modulation scheme and, moreover, validates the benefits of harnessing impulsively applied fluid pressures which are often neglected in the design of a WEC.

The second case-study examines the accidental untying of a shoelace while walking. In this case-study, I discuss the series of events that lead to a shoelace knot becoming untied. Slow-motion video footage and a series of experiments show the failure of the knot happens in a matter of seconds, often without warning, and is catastrophic. Controlled experiments show that increasing inertial effects of the swinging laces leads to increased rate of knot untying, that the directions of the impact and swing influence the rate of failure, and that the knot structure has a profound influence on a knot’s tendency to untie under cyclic impact loading.

The final case-study concerns the ground-up development of prototyping techniques for a soft-robot modeled after the common caterpillar. I sought to suggest an analysis path for rapid prototyping of a SMA based, caterpillar inspired soft-robot to undergo undulatory motion. Analysis of the kinematics and dynamics of the caterpillar are structured through simple models which yield estimates of forces and energetics that would be extremely difficult to determine directly, in addition to suggestions for open-loop motion patterns. Simultaneously, simple experiments and optical tracking of SMA segments were performed to yield properties input directly into a robust numerical solver used to simulate a prototype soft-robot’s undulatory motion and reveal facets leading to the success or failure of its motion.
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Chapter 1

Introduction

Engineering is a field concerned with the application of science and mathematics to the analysis, understanding, and modification of systems. Although the degree to which a given subset of the field tends towards application or pure theory varies, the essential idea of the ‘engineering approximation’ is ever-present, in some capacity. That is, to what extent can a phenomenon be approximated, generalized, or reframed to aid in analysis of the system at hand? As a researcher in the subset of mechanical engineering that can be vaguely termed dynamics, the extent of engineering approximations in dynamical research was always of paramount interest to me. My field is one where one can, depending on one’s prerogatives, be immersed in the tools and language of mathematics in all but name – or be lodged in the mechanic’s workshop, tinkering and developing, ne’er a theoretical model (seriously) considered.

![Diagram of cyclic impulsive (fluid) loading impinging on a wave-energy collector (WEC); flaps open and close twice per cycle, harnessing the ‘added-mass effect’ due to the surrounding fluid – the complex dynamics of such a system are investigated numerically and experimentally in Chapter 2.](image)

**Fig. 1.1** Cyclic impulsive (fluid) loading impinging on a wave-energy collector (WEC); flaps open and close twice per cycle, harnessing the ‘added-mass effect’ due to the surrounding fluid – the complex dynamics of such a system are investigated numerically and experimentally in Chapter 2.
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Even in the case of phenomena which are readily appreciable and utilizable by both sides of this spectrum, there is always the question of to what extent the modeling of the system should be simplified whilst still capturing the spirit of the system. Rapid onset impulsive loading can be so appreciated – a large force, experienced over only a short period of time, acting on an object or system. The proper consideration of impulsive loading, of the extent of engineering approximation necessary, is endemic to the specific systems concerned, and this has always fascinated me. In line with this fascination, in-depth research was undertaken on three dynamical systems, which I term ‘case-studies’, whose most interesting dynamics were largely precipitated by abbreviated periods where impulsive loading dominated, each differently. More specifically, a short interval of time (relative to the overall cycle of respite and commencement in all three) wherein the appearance of a large impulse (relative to the forces otherwise experienced) impinged on each system and motivated each in fascinating ways. In the chapters I shall shortly describe, the impulsive forces are used to convert energy, destroy, and enable locomotion, respectively. Solving for the dynamics of such systems is generally non-trivial and often actively challenging: for the following, in many cases it proved difficult to form the hypothesis – that is, the question – for which I sought an answer. Each case-study is a multi-bodied non-smooth (or discontinuous) dynamical system, often with frictional and unilateral contacts, and as such are examples in the wider study of such systems presented in the literature [1, 8, 29, 30, 58].

Fig. 1.2 Untying of a common shoelace knot translated into an experimental setup; initial observational experiments, which led to the formulation of an hypothesis for the dynamic failure of a shoelace knot (impact and inertia induced) and the development of the setup pictured above are described in Chapter 3.

The first case-study I concerned myself with was the theoretical modeling, numerical simulation and experimental validation of an impulsively-loaded wave energy converter (WEC), which is presented in Chapter 2. In series of contemporary works, mass modulation schemes have been proposed for a class of WECs; the goal of the schemes is to improve the energy harvesting capabilities of these devices by taking advantage of the ambient water. However, this improvement comes at the cost of increased system complexity
and possibly detrimental impulsive loadings at the instances where the mass changes (cf. Figure 1.1). In this case-study, experimental results for a pair of passively and impulsively initiated schemes are presented and one of them is shown to be effective in increasing the energy harvesting potential of a WEC. Building and testing prototype WECs are costly and challenging and so, in order to examine as wide a range of parameters and designs as possible, a detailed two degree-of-freedom model is developed for a WEC equipped with a mass-modulation scheme. Numerical analysis of the model also shows the potential benefits of the mass-modulation scheme and, moreover, validates the benefits of harnessing impulsively applied fluid pressures which are often neglected in the design of a WEC.

The second case-study, presented in Chapter 3, similarly interested me as a dynamicist because of a cyclic impulsive loading: specifically, the untying of a common shoelace knot. The accidental untying of a shoelace while walking occurs often and typically with minimal prior warning. In this case-study, I discuss the series of events that lead to a shoelace knot becoming untied. First, and primarily, the repeated impulsive impact of the shoe on the floor during walking serves to loosen the knot (cf. Figure 1.2). This enables tension forces at the base of the free ends, provided by the inertia of the whipping motions of the free ends of the laces, caused by leg swing, to produce slipping of the laces. This leads to eventual

Fig. 1.3 Development of a bio-inspired soft robot to undergo undulatory motion, described in Chapter 4.
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runaway untangling of the knot. As demonstrated using slow-motion video footage and a series of experiments, the failure of the knot happens in a matter of seconds, often without warning, and is catastrophic. The controlled experiments showed that increasing inertial effects of the swinging laces leads to increased rate of knot untying, that the directions of the impact and swing influence the rate of failure, and that the knot structure has a profound influence on a knot’s tendency to untie under cyclic impact loading.

The final case-study, presented in Chapter 4, concerns the ground-up development of prototyping techniques for a soft-robot whose motion is modeled after that of the common caterpillar. Like the other two case-studies, impulsive loading (here experienced during brief periods of robot actuation) is the primary motivator – in this case, of undulatory motion. The creation of a bio-inspired robot is, in general, a non-trivial task, with no set heuristic for the modeling and analysis of the biological inspiration that will absolutely lead to a successful and demonstrative robot. In contrast to their conventional rigid counterparts, soft machines and robots are elastically-deformable bodies capable of extreme changes in shape and functionality. Despite their potentially extraordinary advantages, the elastic deformability of soft bio-inspired robots yields an infinite degree-of-freedom system that is significantly more difficult to model and control than a discrete system. Through analysis of a specific soft-robot, I sought to suggest an analysis path for rapid prototyping of a SMA based, caterpillar inspired soft-robot to undergo undulatory motion (cf. Figure 1.3). My approach was two-pronged: analysis of the kinematics and dynamics of the caterpillar are structured through simple, string and elastica based models which yield estimates of forces and energetics that would be extremely difficult to determine directly, in addition to suggestions for open-loop motion patterns. Simultaneously, simple experiments and optical tracking of SMA segments were performed to yield properties input directly into a robust numerical solver used to simulate a prototype soft-robot’s undulatory motion and reveal facets leading to the success or failure of its undulatory motion.
Chapter 2

Impulsive Mass Modulation Schemes for a Class of Wave Energy Converters: Experiments, Models, and Efficacy

2.1 Introduction

A subset of ocean wave energy converters (WEC) feature heaving buoys, utilizing methods of energy extraction originally proposed in the mid 1970s [22, 23, 50, 63]. These methods form the basis of much current research in the area of ocean wave energy harvesting where part of the power of the incident waves is converted to electrical power. As with many other energy harvesting devices that exploit resonance, these devices have a relatively narrow bandwidth [66, 68]. Consequently, accurately predicting and, if possible, actively modifying the resonant behavior in response to changes in the frequency of the incident waves is critical. Similar to energy harvesting devices in other application areas (see, e.g., the recent review by [11]), any improvement in the response in the neighborhood of a resonance relative to an unmodified harvester will improve the appeal of the device. As a result, central issues in the type of WEC of interest in this paper are modeling the dynamics of these devices (e.g., [24]) and the development of control strategies which either actively tune a resonant frequency of these devices to the frequency of the dominant incident waves (as in [18]) or latches the motion of the WEC to that of the incident waves (as in [3, 26]).

To improve the energy extraction capabilities of a buoy WEC that exploits resonance phenomena, I have taken an approach different than the aforementioned schemes (which either alter a resonant frequency or involve latching control). Instead, I (in line with previous collaborators) have proposed a scheme to vary the effective mass of the device within each wave cycle (see Figure 2.1). This modulation can be achieved by either trapping water and/or varying the hydrodynamic added mass (via an impulsive loading provided by the surrounding fluid) of the WEC in a manner that produces a state-dependent switching of the mass parameter. The goal of this variation is to increase the velocity of the relative
heaving motion of the system thereby improving energy harvesting. Part of the inspiration for the method came from the response amplification observed in the resonant behavior of parametrically excited systems (see, e.g., [60, 62, 75]).

In earlier works, a heaving buoy WEC design was proposed that was distinguished from contemporary WECs (see, e.g., [32, 49, 73, 76, 77]) by the aforementioned mass modulation scheme. This WEC design was analyzed in [55, 56] and an experimental prototype tested in [54]. The analysis of [55, 56] demonstrated that significant improvement in energy harvesting could be achieved using the innovative impulsive mass modulation scheme. Furthermore, in [54], it was found that improvements in the energy harvested could be achieved, but significant momentum losses which were neither anticipated nor modeled in [55, 56] were also observed. These losses were determined to be associated with momentum conservation at several discrete stages of the mass modulation. The observed momentum loss necessitated exploration of alternative mass modulation schemes and the development of a new set of models.

The research contained herein extends earlier work [17] where new mass modulation schemes were presented, and a simple one degree-of-freedom model was used to explore their efficacy. In the present paper, experimental results for the optimal mass modulation scheme, known as Scheme III, proposed in [17] are presented. These experiments verify a potential improvement in energy harvesting. However, the testing facility and prototype WEC allow only limited adjustment of key design and environmental parameters. As these parameters affect the optimality of any one given mass modulation scheme, an enhanced model was sought which could be used to determine the system’s sensitivity to a wider range of parameters. The resulting two-degree-of-freedom model accounts for the momentum loss and incorporates a more realistic representation of hydrodynamic interactions. The main insight this model provides is to show that the impulsive mass modulation Scheme III will work for a wide variety of parameter choices. Particularly important is a parameter related to momentum loss: designing a WEC where the momentum loss associated with the mass modulation scheme is minimized is shown to be key.

An outline of the chapter is as follows. First, a broader discussion of the mass modulation schemes is presented. Next, a brief overview of the experiments on a system similar to the first model described in [55] in Section 2.3.1 and how they motivated the creation of the second model described in [17], in addition to exploration of new mass modulation schemes, is given. One such scheme, which is referred to as Scheme III, was considered to be optimal for the one degree-of-freedom model used in [17]. In Section 2.3.2, results from a series of experiments on a prototype WEC equipped with a power-take-off element (PTO) are presented. Next, in Section 2.4, the two-degree-of-freedom model for the WEC prototype which accommodates hydrodynamic effects that were absent in earlier single degree-of-freedom models is established. To facilitate using this model to explore parameter regimes for a mass-modulated WEC, an energy harvesting metric is defined in Section 2.5. Discussion of the effect on the metric of varying individual parameters is presented in Section 2.6 and conclusions in Section 4.4.

Work in this chapter was accomplished with the advice and assistance of Ömer Savaş, Carolyn Q. Judge and Oliver O’Reilly, and was built off of concepts previously investigated
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Fig. 2.1 (a) Schematic of a heaving buoy type WEC featuring modulation of the mass $M_1$. The motion of $M_1$ relative to $M_2$ is used by the PTO to generate electricity. In the interests of clarity, several details, including the tethering mechanism, are not shown. In (b), a two degree-of-freedom model for this WEC is shown with various (hydrodynamic and mechanical) dampings and stiffnesses. Note that the mass $M_1$ in this model varies with time.

by Bayram Orazov and Xuance Zhou; publications resulting from work presented here include [16, 17].

2.2 Mass-Modulation Schemes

Heaving buoy ocean wave energy converters (WECs) harvest energy by exploiting the oscillation of the buoy in response to incident waves. In order to optimize its energy harvesting capabilities, the WEC is designed so that one of its resonant frequencies is close to the dominant frequency of the incident waves. Designing and operating a buoy WEC is challenging. Among the many challenges, WECs operate in a harsh environment where maintenance can be difficult and the necessary design and wave tank testing of scaled prototypes is expensive and time consuming. Clearly, any scheme that can effectively improve the energy harvesting of a WEC is worth examining.

One scheme that can improve the energy harvesting capabilities of a WEC is a mass modulation scheme. In such a scheme, the mass of an element of a WEC is varied in time. This variation can take several forms. One option is to actively change the geometry of one of the floats of the WEC and thereby change its hydrodynamic added mass. Another option is to periodically enclose and later release a volume of water in one of the floats,
leveraging the fluid’s impulsive load on a section/sections of the WEC. Such modulation takes advantage of the fluid environment and incident waves and can be viewed as a form of parametric excitation. The latter is a well-known phenomenon that has been used to amplify the response of oscillating systems [62], in such varied settings as MEMS oscillators and oscillating water columns [53, 60].

The first such mass modulation scheme was examined in [55] and was further investigated and improved in [17] and [56]. Referring to Figure 2.1, in the present context of WECs, the mass of the inner float of a buoy WEC is modulated in time. If the inner float is completely submerged as shown in the figure, then the impulsive mass modulation can be entirely attributed to the change in the hydrodynamic added mass of the inner float as the flaps are opened and closed.

There are no theoretical restrictions on the frequency and duration of mass modulation that may be specified. Indeed, there exist infinitely many mass modulation schemes that can be concocted. Generally, parametric excitation is most successful when it occurs at around twice the fundamental, or natural, frequency of an oscillator. The naval architecture of a heaving buoy WEC is such that its fundamental frequency is in the neighborhood of the commonly encountered incident wave frequency. Thus, guided by the literature [53, 60, 62] on parametric excitation, the initial work on these schemes in [55, 56] focused on modulation at approximately twice the frequency of wave forcing.

![Fig. 2.2](image-url) Three examples of mass modulation schemes denoted respectively as (a), Scheme I, (b) Scheme II, and (c) Scheme III. Scheme I is featured in the experimental work in [54], Scheme II is the case considered in [55, 56] and [61], and Scheme III is an optimal energy harvesting scheme considered here and in [17].

While it is possible to conceive of a control system which would use the fluid surrounding the WEC to modulate the mass in any conceivable manner, the focus here has been on the development and implementation of mass modulation schemes that can be realized passively. In early designs, it was found that passive mass modulation could be implemented by alternatively allowing water flow, through some section of the WEC, and blocking water flow through that same section. When the flow is blocked, an impulsive pressure associated with the impeded flow and the inertial force of moving the newly trapped volume of water are created, relative to the case where the flow was unimpeded. For a given velocity, these effects are assumed to be proportional only to the acceleration of the mass, and as such
they may be grouped with a mass term. This results in an increase in effective mass - an impulsive mass modulation.

Three such schemes are shown in Figure 2.2. The schemes feature modulation by a fraction \( \mu \) of the mass \( M_{1}^{\text{Off}} \) of an inner float \( M_1 \) when the mass is not being modulated and the variables \( x_1 \) and \( \dot{x}_1 \) denote the respective (heaving or vertical) displacement and velocity of \( M_1 \). Thus the mass of \( M_1 \) changes from

\[
M_1^{\text{On}} = M_1^{\text{Off}} + \mu M_1^{\text{Off}}
\]

(2.1)

to \( M_1^{\text{Off}} \). It is critical to note here that the mass modulation \( \mu M_1^{\text{Off}} \) is dependent on the state of the system and so the models for the WEC will feature state switching and impulsive loading when the mass of \( M_1 \) changes from \( M_1^{\text{Off}} \) changes to \( M_1^{\text{On}} \) and vice versa.

For example, Figure 2.3 illustrates two possible implementations of the mass modulation scheme shown in Figure 2.2(c). The masses in this figure are connected by a power-take-off system which is not shown and both heave in response to the incident waves. The mass \( M_1 \) is equipped with a mechanism that enables it change its effective mass by an amount \( \mu M_1^{\text{Off}} \). The realization shown in Figure 2.3(i) only involves changes to the hydrodynamic added mass while the design shown in Figure 2.3(ii) involves changes both to the hydrodynamic added mass of \( M_1 \) and an additional change in \( M_1 \) due to the enclosed mass of water:

(i) For the realization shown in Figure 2.3(i), \( \mu M_1^{\text{Off}} \) is equal to the difference in the hydrodynamic added mass of \( M_1 \) when the flaps are in the open state compared to the closed state.

(ii) For the realization shown in Figure 2.3(ii), \( \mu M_1^{\text{Off}} \) is equal to the change in hydrodynamic added mass of \( M_1 \) plus the effect of the enclosed mass.

It is again reiterated that the modulation scheme is precipitated by an impulsive loading on the mass \( M_1 \) which alters the dynamics of the WEC – although it enables the switching described above, depending on a variety of factors it may also have some detrimental effects on overall harvesting capabilities. Experimental results for Schemes I & III will be discussed in Section 2.3. I also emphasize that the realization emphasized in this paper is the one shown in Figure 2.3(i).

It is not immediately apparent that a mass modulation scheme should be effective in improving the energy harvesting capabilities of a WEC. Such schemes are difficult to faithfully model, are hindered in effectiveness by the increased damping and impulsive loading introduced by the mass modulation mechanism, and are difficult to implement in a prototype. The current chapter will show how Scheme III above overcomes many of the technical challenges of the earlier Schemes I and II and successfully yields improvements to energy harvesting. This improvement is illuminated with the help of a model that is far more realistic than the ones used in earlier works [17, 54, 55, 56] and by experimental testing of a prototype.
Fig. 2.3 Illustration of the mass modulation scheme III. In the top part of the figure, the idealized modulated mass is a fraction $\mu$ of the baseline mass $M_1^{\text{off}}$ and is a function of position $x_1$ and velocity $\dot{x}_1$ of the inner float. In the bottom portion of the figure, one sees two means of achieving the mass modulation $\mu M_1^{\text{off}}$: in i. utilizing only hydrodynamic added mass and in ii. utilizing both hydrodynamic added mass and entrapped water mass. Elements of note are an outer float of mass $M_2$, a (primarily) submerged inner float (featuring a hollow open ended cylinder) of mass $M_1$, (a) mass modulation control flaps, (b) impeded water flow, (c) unimpeded water flow, (d) water surface, and (e) trapped water mass.
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2.3 Experimental Results

![UC Berkeley Tow Tank facility in Richmond, CA. Schematic (top) and image of the facility with the wavemaker mechanism seen in blue in the foreground (bottom).](image)

While it is straightforward to propose a mass modulation scheme, significant challenges arise when attempting to implement a scheme in a prototype. The first scheme for which a prototype was designed and tested was Scheme I and the results of this testing are partially recorded in Orazov’s dissertation [54]. Here, these results are more fully analyzed in Section 2.3.1 because they serve to illuminate features of the mass modulation that weren’t considered in earlier models. These results also bear contrast to those for Scheme III which are reported in Section 2.3.2.

2.3.1 Experimental Results: Scheme I

Initial experimental work involved the construction of a small scale prototype WEC to verify the previously proposed [55, 56] mass modulation Scheme II at the UC Berkeley Tow Tank facility in Richmond, CA (Figure 2.4). This facility employed a hydraulically actuated and electronically controlled vertical-flap wave maker to independently set incident wave frequency and amplitude.

As can be seen in Figure 2.5, a scale prototype of the WEC was constructed of three parts: an outer float and guides, an inner float which was concentric to the outer float, and the water entrapment mechanism, which is rigidly attached to the inner float. Two deviations were made from the earliest model in [55, 56] and this experiment: no power-take-off
Fig. 2.5  Images of the original scale prototype WEC. (a) Assembly: outer float, inner float and the water entrapment system; (b) outer float with the inner float visible (marked by yellow tape); (c) 3-axis accelerometers rigidly mounted to the outer (top of the T-beam) and inner (center of the upper cap) floats. The signal cables are connected to a land-based Arduino microcontroller.

system, or power-take-off modeling damper, was implemented, and a spring loaded latching mechanism that was to initiate the twice-per-period mass modulation was simplified to a once-per-period mass modulation due to difficulties in realizing the switching in the experimental setup. In terms of Figure 2.2, Scheme I was tested instead of Scheme II.

Figure 2.6 shows two phase portraits, one simulation generated using a one degree-of-freedom model in (a) and one physically realized (from this experiment) in (b). In both one may observe a downward jump in velocity when the upper flaps close and mass is effectively added, and it is notable that good agreement is seen between the modified theory and the experiment generally. I invite the reader to contrast this jump in velocity with the corresponding portrait shown in Figure 2.7 where no flaps are installed and the mass is unmodulated: no velocity jump is observed when the flaps are absent. This is a consequence of the conservation of momentum, which at the time of the experiments reported in [54] was not accounted for in the one degree-of-freedom model. The jump’s appearance in Figure 2.6 provoked a reassessment of the original theory and the creation of an updated single mass model discussed in [17, Eqns. (1)–(4)]. This updated model was used here to generate Figure 2.6(a), which is in good agreement with the measurements shown in Figure 2.6(b).
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Fig. 2.6  (a) Dimensional output of simulation of one degree-of-freedom model in [17, Eqns. (1)–(4)] with switching boundary momentum loss; specific parameter values utilized are: $f = 0.25$, $\omega_f = 0.68$, $\omega_n = 6$, $\mu = 0.45$, $\delta = 0.07$, $\varepsilon$ (momentum loss coefficient) = 0.8, $\ell = 0.1$. (b) Experimental data from [54] of the inner float position $x_i$ versus the inner float velocity $v_i$ for a wave excitation frequency 0.68 Hz, with flaps installed (i.e., with added mass effects). The dashed vertical lines mark $\pm 5$cm, corresponding to the 10 cm peak-to-peak input wave amplitude.

Agreement between experimental results and theory, as modified by momentum considerations in [17], was encouraging, but several questions provoked by the experiment needed to be addressed. Power-proportional damping was absent in the experimental setup. Lack of this damping has the potential to alter the dynamics from those specified by the model. Moreover, updated boundary conditions implemented in light of momentum considerations mentioned above were incorporated into [17]. These conditions corresponded to Scheme III, and this scheme warranted its own investigation.

2.3.2 Experimental Results: Scheme III

Improved experimental realization of the mass-modulated WEC required the addition of a power-proportional damper. To model the damping provided by a PTO, a stepper motor, rotated by the relative motion between the inner and outer floats via a rack-and-pinion mechanism, was installed on the prototype. A discussion of this type of power generator and its applicability to this model follows in Section 2.3.2. Additionally, testing of Scheme III necessitated a new mechanism to modulate the mass. A simple, passive mechanism was designed which is briefly presented in Section 2.3.2.
**Fig. 2.7** Experimental data of inner float position $x_i$ vs. inner float velocity $v_i$ for a wave excitation frequency of 0.68 Hz without flaps installed (i.e., without mass modulation). The dashed vertical lines mark ± 5cm, corresponding to the 10 cm peak-to-peak wave amplitude.

The Power-Take-Off Element

A simple PTO is constructed by placing a resistive load $R_L$ between two leads of a properly configured common stepper motor. The rack-and-pinion mechanism translates the relative heaving motion into a rotary motion at the shaft of the stepper motor, thus generating power. Determination of the damping applied between the inner and outer floats at a given relative velocity is dependent on the effective moment arm of the pinion gear ($r_P$) and the torque $\tau_M$ generated by the motor at a given rotational speed ($\tau_M = \tau_M(\theta; R_L)$). As the torque $\tau_M = \tau_M(\theta; R_L)$ is generally not an analytic function, it was experimentally determined for a variety of resistive loads $R_L$ and rotational speeds $\theta$. Two motors - a Slo-Syn M062-FC-404B and a Slo-Syn M092-FD-416E - were characterized by measuring the power applied to a third motor to rotate them at a known speed with a known resistive load. A sample of the characterization data may be seen in Figure 2.8.

It is instructive to recall here from the literature on single degree-of-freedom mass-spring-damper oscillators of the form $m \ddot{u} + (b_e + c) \dot{u} + ku = A \cos(\omega t)$ that are equipped with a PTO modeled as $b_e \dot{u}$ that optimal harvesting is achieved when $\omega^2 = \frac{k}{m}$ and $b_e = c$ (see, e.g., [63, 66]). While the corresponding optimal conditions are not known for the models employed in this paper, these results for the simpler model serve as useful benchmarks. Referring to Figure 2.8, it is readily apparent that the damping provided by the stepper motor PTO is quite different than the linear proportional damping that is typically considered in models for PTOs [63, 66]. In the latter works, the power $P$ extracted by the PTO is assumed proportional to the relative velocity $v_{rel}$ between the heaving bodies:

$$P = F_{PTO} \cdot v_{rel} \approx c_{PTO} v_{rel}^2$$

(2.2)

where $c_{PTO}$ is a positive constant. Similarly in models here, the damping force is assumed to be positively proportional to the relative velocity between the floats, through the pinion
CHAPTER 2. IMPULSIVE MASS MODULATION SCHEMES FOR A CLASS OF WAVE ENERGY CONVERTERS: EXPERIMENTS, MODELS, AND EFFICACY

\[ \theta \text{(radians/second)} \]
\[ \tau \text{ } M \text{ (N} \cdot \text{m)} \]
\[ \begin{array}{ccccccc}
0 & 10 & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90 \\
0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 \\
\end{array} \]

\[ R_L = 1 \Omega \]
\[ R_L = 2 \Omega \]
\[ R_L = 4.6 \Omega \]
\[ R_L = 9.9 \Omega \]
\[ R_L = 14.4 \Omega \]

Fig. 2.8 Angular speed-torque \((\dot{\theta}, \tau_M(\dot{\theta}; R_L))\) curves for the Slo-Syn M062-FC-404B stepper motor used as a PTO in the experiment, for a variety of different load resistances \(R_L\). Restricting attention to the area of low rotational speeds \((\approx< 10 \text{ radians/second})\), one observes that the torque \(\tau_M\) decreases, and the magnitude of the torque/rotational speed proportionality constant increases, with increasing \(R_L\).

As such, the optimal damping for single degree-of-freedom oscillators mentioned above could not be exactly recreated here. Regardless, this new PTO was used, as it would affect the mass modulated and unmodulated versions of the prototype identically and would be far more representative of a real system than that considered in the previous experimental work. Additionally, one may note a few methods of enhancing this PTO.
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Fig. 2.9 Images of the current scale prototype WEC with updates. (a) The implemented PTO; (b) Accelerometers mounted to the outer and inner floats, as before, with addition of PTO and necessary counterweights; (c) Assembly, showing addition of PTO and updated flap mechanism.

of the torque/rotational speed proportionality increases, with increasing $R_L$. For a direct rack-and-pinion setup - one absent adjustable gearing - $r_p$ may easily be selected for these sort of rotational speeds. This suggests that, if desired, there is the potential that the resistive load $R_L$ could be adjusted via a control scheme, on some larger timescale, to achieve a desired effective damping. If a variable gearing and clutch setup were incorporated to interface the relative motion of the masses to the PTO, $r_p$ could be a function of various parameters and even more general control schemes could be constructed to tune the optimal damping in real time.

Additional study of the new type of damping, and determination of its optimal value, may reveal a great deal about the desired PTO method for a mass modulated WEC, but such an investigation is beyond the scope of the work presented in this chapter. In the end, only the smaller M062-FC-404B was used, as it was experimentally determined that the larger M092-FD-416E provided excessive damping for the prototype, effectively attenuating almost all relative motion.
Fig. 2.10 Flap mechanisms used in experiments to approximate (a) Scheme I and (b) Scheme III. In the former, a spring force resists the fluid pressure until mass modulation is desired. In the latter, contact force between the rubber at the edges of the flaps and the cylinder wall resists the fluid pressure until mass modulation is desired.

Scheme III Mass-Modulation Mechanism

Updated optimal mass modulation boundary conditions in [17] necessitated a new mechanism and a passive, flap-based mechanism was desired for ease of construction. In such a case Scheme III requires closing of the flaps, resulting in the onset of the modulated-mass regime, at the points of zero velocity of the inner float, and the opening of the flaps, resulting in onset of the unaltered mass regime, at the points of maximum absolute velocity. These latter points coincide in time with those where the maximum pressure is exerted by the impulsive load of the surrounding fluid on the flaps if they are in the closed position. It was hypothesized that one could set the force resisting this pressure by the flaps - for the maximum inner float velocities encountered in the previous experiments - to be just defeated at these points of maximum velocity, closely approximating the desired mass modulation.

One set of flaps was able to provide the mass modulation in both directions, as opposed to the two sets required previously for Scheme II which had caused significant problems. The remaining pair of flaps were made of newly fabricated acrylic semicircles, identical in thickness to those considered previously for Scheme I, cut about 0.75 cm shorter in radius than before. Large rubber semicircles were affixed to these flaps and, by observationally verified trial and error, were cut down to the size corresponding to flap opening due to fluid pressure at the moment of max velocity (approximately 191mm long 85mm wide). Similar to [54], this method of resisting the impulsively applied fluid pressure, and thus affecting added mass, was not remotely adjustable and was still dependent on the characteristics of the incident wave, but it was much closer to the optimal mass modulation Scheme III discussed in [17]. The same mechanism could be made, in a more advanced prototype, to be remotely adjustable through a variety of methods, but such was not necessary for the scope of these experiments.
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Table 2.1 Comparison of the experimental velocities for Scheme III at the six testing frequencies. \( \Delta v = v_i - v_o \), where \( v_i \) is the inner mass velocity and \( v_o \) is the outer mass velocity; additionally, the subscript “no flaps” indicates testing without modulation, and the subscript “flaps” indicates testing with modulation. The superscript “RMS” indicates and average by root-mean-square, or \( f(t)_{\text{RMS}} = \sqrt{\frac{1}{T_f - T_0} \int_{T_0}^{T_f} f(\tau)^2 \, d\tau} \)

<table>
<thead>
<tr>
<th>( f ) (Hz)</th>
<th>( \Delta v_{\text{RMS}} ) no flaps (m/s)</th>
<th>( \Delta v_{\text{RMS}} ) flaps (m/s)</th>
<th>( v_{\text{RMS}} ) no flaps (m/s)</th>
<th>( v_{\text{RMS}} ) flaps (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.650</td>
<td>0.050</td>
<td>0.122</td>
<td>0.177</td>
<td>0.203</td>
</tr>
<tr>
<td>0.675</td>
<td>0.057</td>
<td>0.122</td>
<td>0.193</td>
<td>0.189</td>
</tr>
<tr>
<td>0.725</td>
<td>0.082</td>
<td>0.098</td>
<td>0.229</td>
<td>0.225</td>
</tr>
<tr>
<td>0.750</td>
<td>0.073</td>
<td>0.141</td>
<td>0.207</td>
<td>0.257</td>
</tr>
<tr>
<td>0.800</td>
<td>0.099</td>
<td>0.099</td>
<td>0.254</td>
<td>0.202</td>
</tr>
<tr>
<td>0.850</td>
<td>0.112</td>
<td>0.125</td>
<td>0.251</td>
<td>0.212</td>
</tr>
</tbody>
</table>

Experimental Testing and Results

Apart from the aforementioned changes to the prototype, a few changes to the experimental procedure used in [54] and discussed previously in Section 2.3.1 were also made. First, power collected by the PTO was monitored in real time by an oscilloscope placed in parallel to one of the resistors between the leads of the PTO. Secondly, the prototype was moored by three lines to the sides of the tow tank, rather than the test platform (or carriage), as previously. Thirdly, the range of excitation frequencies was changed from \( 0.60 \text{ Hz} \leq f \leq 0.76 \text{ Hz} \) to \( 0.65 \text{ Hz} \leq f \leq 0.85 \text{ Hz} \): the addition of the PTO was determined to have made the response of the prototype more regular to a larger range of frequencies, and reduced the relative motion at frequencies below 0.65 Hz. Additionally, the excitation wave amplitude was increased from \( 10 \pm 1 \text{ cm} \) to approximately \( 15 \pm 1.5 \text{ cm} \) for all excitation frequencies for the same reason. Beyond these changes, the experimental setup, experimental procedure, and data collection methods are as they were in [54], and I refer readers to this dissertation for more detailed information.

A comparison of important system responses, for the modified prototype both with and without mass modulation, for six selected incident wave forcing frequencies is presented in Table 2.1. For ease of comparison, a table of system responses from the initial testing in Table 2.2 is also attached. Additionally, note that mean values are computed in the root-mean-square (RMS) sense, i.e.

\[
f(t)_{\text{RMS}} = \sqrt{\frac{1}{T_f - T_0} \int_{T_0}^{T_f} f(\tau)^2 \, d\tau}
\]
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Table 2.2  Comparison of the experimental velocities for Scheme I at the five testing frequencies. $\Delta v = v_i - v_o$, where $v_i$ is the inner mass velocity and $v_o$ is the outer mass velocity; additionally, the subscript “no flaps” indicates testing without modulation, and the subscript “flaps” indicates testing with modulation. The superscript “RMS” indicates and average by root-mean-square, or $f(t)_{\text{RMS}} = \sqrt{\frac{1}{T_f-T_0} \int_{T_0}^{T_f} f(\tau)^2 d\tau}$

<table>
<thead>
<tr>
<th>$f$ (Hz)</th>
<th>$\Delta v_{\text{RMS no flaps}}$ (m/s)</th>
<th>$\Delta v_{\text{RMS flaps}}$ (m/s)</th>
<th>$\bar{v}_{\text{RMS no flaps}}$ (m/s)</th>
<th>$\bar{v}_{\text{RMS flaps}}$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.60</td>
<td>0.041</td>
<td>0.038</td>
<td>0.122</td>
<td>0.118</td>
</tr>
<tr>
<td>0.64</td>
<td>0.059</td>
<td>0.065</td>
<td>0.132</td>
<td>0.135</td>
</tr>
<tr>
<td>0.68</td>
<td>0.075</td>
<td>0.086</td>
<td>0.137</td>
<td>0.138</td>
</tr>
<tr>
<td>0.72</td>
<td>0.117</td>
<td>0.118</td>
<td>0.151</td>
<td>0.169</td>
</tr>
<tr>
<td>0.76</td>
<td>0.117</td>
<td>0.086</td>
<td>0.141</td>
<td>0.171</td>
</tr>
</tbody>
</table>

Table 2.3  Comparison of the average phase lag $\bar{\tau}$ of the inner mass velocity $v_i$ relative to the outer mass velocity $v_o$ and the root-mean-square (RMS) power $P$ generated by the PTO for Scheme III at the six testing frequencies. The subscript “no flaps” indicates testing without modulation, and the subscript “flaps” indicates testing with modulation. The superscript “RMS” indicates and average by root-mean-square, or $f(t)_{\text{RMS}} = \sqrt{\frac{1}{T_f-T_0} \int_{T_0}^{T_f} f(\tau)^2 d\tau}$

<table>
<thead>
<tr>
<th>$f$ (Hz)</th>
<th>$\bar{\tau}_{\text{no flaps}}$ (s)</th>
<th>$\bar{\tau}_{\text{flaps}}$ (s)</th>
<th>$P_{\text{RMS no flaps}}$ (W)</th>
<th>$P_{\text{RMS flaps}}$ (W)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.650</td>
<td>0.063</td>
<td>0.090</td>
<td>1.251</td>
<td>2.467</td>
</tr>
<tr>
<td>0.675</td>
<td>0.063</td>
<td>0.133</td>
<td>1.387</td>
<td>2.541</td>
</tr>
<tr>
<td>0.725</td>
<td>0.070</td>
<td>0.077</td>
<td>1.826</td>
<td>2.139</td>
</tr>
<tr>
<td>0.750</td>
<td>0.070</td>
<td>0.091</td>
<td>1.675</td>
<td>2.802</td>
</tr>
<tr>
<td>0.800</td>
<td>0.070</td>
<td>0.084</td>
<td>2.193</td>
<td>2.173</td>
</tr>
<tr>
<td>0.850</td>
<td>0.077</td>
<td>0.084</td>
<td>2.395</td>
<td>2.620</td>
</tr>
</tbody>
</table>
CHAPTER 2. IMPULSIVE MASS MODULATION SCHEMES FOR A CLASS OF WAVE ENERGY CONVERTERS: EXPERIMENTS, MODELS, AND EFFICACY

Firstly, and perhaps most important, one can see the effect of changing the mass modulation regime; at no testing frequency in the current experiment was the RMS relative velocity with mass modulation found to be less than that without mass modulation. Under the previous regime, this was not the case: notice that when the forcing frequency was more than 0.2 Hz away from the estimated fundamental frequency \( f_0 = 0.66 \) Hz of the WEC prototype, the RMS relative velocity is less in the mass modulated case. Notably, the RMS relative velocity without mass modulation only matched that with mass modulation at the forcing frequency of \( f = 0.800 \) Hz. It is likely that this frequency is close to a local minimum in the system’s response \( \Delta v_{\text{RMS}}(f) \) between the fundamental frequency \( f_0 \) and some harmonic \( f_n \).

Secondly, the RMS relative velocity with mass modulation is significantly larger than was the case without mass modulation at most frequencies, doubling the case without mass modulation at \( f = 0.650, 0.675 \), and \( 0.750 \) Hz. Examining the RMS inner mass velocity at the various testing frequencies reveals that, in most of the cases, the values with mass modulation are lower than those without, so it was suspected that the mass modulation was affecting a greater average phase lag of the velocity of the inner mass relative to the outer float, \( \bar{\tau} \). These values were calculated and may be seen in Table 2.3. It is evident that mass modulation results in a larger phase lag than that seen without mass modulation, so it can be deduced that this plays at least some part in overcoming the reduced RMS velocity of the inner float in the mass modulated case.

Finally, and also seen in Table 2.3, the RMS power delivered by the PTO with the resistive load \( R_L = 4.6 \) \( \Omega \) attached at both of the lead pairs was calculated. Again it can be see that, excepting the forcing frequency \( f = 0.800 \) Hz (where the power is similar in both cases), the addition of mass modulation results in greater generated power. It cannot be emphasized enough that this result was for an effective damping fundamentally dissimilar from the viscous damping assumed in the theory, and for a curve of that dissimilar damping that was selected without much evidence as to its optimality. This points towards the robustness of mass modulation’s effect on power harvesting. The frequency excepted from this trend may be explained by appealing to the RMS relative velocities in both cases: they are very close, and as such it is not at all surprising that the RMS power extracted might be slightly lower in the mass modulated case, as the process of producing an RMS mean in velocity would miss nonlinearities in the power generation that would matter to the RMS mean of power.

The above results point to validation of the incorporation of phase-dependent mass modulation on a buoy-type WEC or, more specifically, an effective modulation achieved by blocking the free-flow of water through some part of a WEC, twice a period, thus harnessing an impulsive added mass effect. Generally, these results also show that mass modulation is realizable on a buoy type WEC even in the (comparatively) unrefined prototype stage. In my opinion, it was suspected that mass modulation would be realizable on a buoy type WEC of arbitrary geometry with appropriate alternating flow-through/flow-blocking areas.

There are still some questions as to the robustness of Scheme III mass modulation. Experimental work had taken place on a scaled-down prototype of (obviously) specific inner
and outer float mass and geometry. How might the scheme perform if these parameters were
drastically different? How might hydrodynamic damping of the masses and between the
masses, particularly if taken as a function of forcing frequency, change the performance?
How might the performance change over a free selection of added mass onset and removal
in phase space? Generally, the ability to perform some sort of a numerical sensitivity anal-
ysis on a more advanced system to a wide variety of design and environmental parameters
which would be extremely difficult or impossible to change in an experimental setup was
desired. From such an analysis, the specific conclusion from the experiment - that mass
modulation is beneficial to WEC power harvesting for a wide range of incident frequencies
- could be extended from this specific prototype to a larger subset of WECs. Accordingly,
an examination of a two-degree-of-freedom model subject to such a numerical analysis is
presented below.

### 2.4 Two Degree-of-Freedom Model for the Wave Energy
Converter

To perform the aforementioned numerical analysis, a new hybrid-dynamical model is
necessitated. I model the heaving buoy WEC as a two-degree-of-freedom mass-spring-
dashpot system. The mass $M_2$ models the upper floating body while $M_1$ models the sub-
merged body (inner float) which is connected to $M_2$ by a PTO. A schematic of the oscillator
is shown in Figure 2.11 and is a more comprehensive model than the one analyzed in earlier
works [17, 54, 55, 56, 61]. By way of background, the stiffnesses $k_{1,2,3}$ and viscous damping
parameters $c_{1,2,3}$ in the model represent both mechanical and hydrodynamical forces
on the masses. Here, $c_2 = c'_2 + B$, where $B$ is the damping contribution exclusively from the
PTO. In a typical operating environment for the WEC, the six parameters $k_{1,2,3}$ and $c_{1,2,3}$
generally depend on the frequency $\omega$ of the incident forcing. As discussed in the forthcoming subsections, these parameters along with the mass $M_1$ may also depend on the mass
modulation.

#### 2.4.1 Mass modulation

The unique feature of the model is that the mass $M_1$ of the inner float varies in a state-
dependent manner. That is, if $x_1$ denotes the displacement of $M_1$, then the mass of $M_1$
depends on the value of the pair $(x_1, \dot{x}_1)$. The locations in the $x_1 - \dot{x}_1$ phase plane where the
mass changes (cf. Figure 2.12) are known as switching boundaries $S$ and the conditions un-
der which $M_1$ changes are known as switching conditions. Referring to Figure 2.12, define
two angles $\alpha$ and $\beta$ for the locations of the switching boundaries. More simply, referring
to Figure 2.12, it is convenient to define

$$\theta = \arctan \left( \frac{x_1}{\dot{x}_1} \right), \quad 0 \leq \theta < 2\pi$$

(2.4)
so that

\[
\theta \in \begin{cases} 
\overline{\Sigma}_A & \text{if } \alpha < \theta < \alpha + \beta \text{ or } \alpha < \theta - \pi < \alpha + \beta \\
\overline{\Sigma}_0 & \text{otherwise}
\end{cases}
\] (2.5)

where \(\overline{\Sigma}_A\) and \(\overline{\Sigma}_0\) are the sets of all \(\theta\) in the mass-modulated and rest mass regimes, respectively. Then, between angle \(\alpha\) in the phase plane (defined off of the positive \(x_1\)-axis) and angle \(\beta\) (defined off of the ray created by \(\alpha\)) the inner mass is equal to \(M_{1}^{\text{on}}\) (its modulated mass); this is repeated between \(\alpha + \pi\) and \(\beta + \pi\). In all other regions, the inner mass is equal to \(M_{1}^{\text{off}}\) (its unmodulated mass). The relationship between these two masses is represented by \(\mu\), a non-dimensional number indicating the increased hydrodynamic mass as a fraction of the original mass:

\[
\mu = \frac{M_{1}^{\text{on}} - M_{1}^{\text{off}}}{M_{1}^{\text{off}}} 
\] (2.6)

### 2.4.2 Impulse Momentum Considerations

The mass modulation of \(M_1\) is paired with a conservation condition at the boundary \(S\) which models the momentum transfer due to the change in mass. Generally, the change in momentum at a switching boundary is equivalent to a fluid supplied impulse \(G\) on the system at that moment:

\[
M_1^+ \dot{x}_1^+ - M_1^- \dot{x}_1^- = G
\]
The mass $M_1$ varies depending on the sign of $x_1$ and $\dot{x}_1$: $M_1 = M_{1\text{on}}^\text{On}$ (in the modulated mass region) or $M_1 = M_{1\text{off}}^\text{Off}$ (in the unmodulated mass region). The most general case is shown; the rays $S$ correspond to the locations in the state space where water is either trapped or released.

Here, $\dot{x}_1^-$ is the velocity at the instant just before the phase flow $(x_1(t), \dot{x}_1(t))$ pierces the switching boundary $S$ and $\dot{x}_1^+$ is the velocity at the instant just after the phase flow pierces the switching boundary $S$, with $M_1$ defined similarly. As $G$ is the result of fluid-body interactions that are difficult to characterize analytically, I chose to approximate $G$ as some fraction of the pre-boundary momentum:

$$G = -(1 - \varepsilon)M_1^\text{on}\dot{x}_1^-$$

such that

$$M_1^\text{on}\dot{x}_1^+ = \varepsilon M_1^\text{off}\dot{x}_1^-$$

where $(1 - \varepsilon)$ is a (constant) coefficient indicating the amount of momentum loss across $S$.

By varying $\varepsilon$, one may estimate the varying effect of the impulse $G$ without explicitly determining it. In [55, 56] and [61], the case $\varepsilon = (1 + \mu)$ was exclusively considered. This prescription for $\varepsilon$ can be questioned on physical grounds: an impulsive force of such nature is absent in the current realizations of the WEC, and I expect that without some additional external momentum impulse (that is, in addition to that which provides the effective mass change itself) $\varepsilon \leq 1$. With this in mind, one may consider two cases for the switching condition. The first pertains to when the fluid flow is blocked (the mass is modulated):

$$M_1^\text{on}\dot{x}_1^+ = \varepsilon M_1^\text{off}\dot{x}_1^- \quad (2.7)$$

When the fluid flow is allowed (the mass is unmodulated), the switching condition is

$$M_1^\text{off}\dot{x}_1^+ = \varepsilon M_1^\text{on}\dot{x}_1^- \quad (2.8)$$
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This condition pertains to the case when the phase flow passes through the switching boundary $S$ and the mass changes from $M^1_{\text{On}}$ to $M^1_{\text{Off}}$. Finally, one may note that the modulation scheme is to be designed such that it is one-directional in phase space. In other words, consulting Figure 2.12, for counter-clockwise phase flow the switching is only activated upon crossing $S$ in a counter-clockwise direction, so any clockwise jumps due to modulation across $S$ do not affect a modulation.

2.4.3 Equations of Motion

The equations of motion for the system are derived from a balance of linear momentum on the idealization of a two-degree-of-freedom oscillator presented in Figure 2.11.

$$\begin{bmatrix} M_1 \ddot{x}_1 \\ M_2 \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \sin(\omega t + \phi) \\ F_2 \sin(\omega t) \end{bmatrix}$$

(2.9)

Here $\phi$ is the phase separation of the forcing $F_1$ on $M_1$ relative to the forcing $F_2$ on $M_2$. It is convenient to non-dimensionalize (2.9). To this end, one may choose a length scale $\ell$ and a stiffness $k_1$. In a standard fashion, these choices lead to a set of dimensionless variables and parameters:

$$\tilde{x}_\lambda = \frac{x_\lambda}{\ell}, \quad \tau = \sqrt{\frac{k_1}{M_{20}}} t, \quad \kappa_i = \frac{k_i}{k_1}, \quad \delta_i = \frac{c_i}{2\sqrt{k_1 M_{20}}} \quad (2.10)$$

$$m_1 = \frac{M_1}{M_{20}}, \quad m_2 = \frac{M_2}{M_{20}}, \quad f_\lambda = \frac{F_\lambda}{k_1 \ell}, \quad \omega = \sqrt{\frac{M_{20}}{k_1}} \omega_f \quad (2.11)$$

with $i = 1, 2, 3$ and $\lambda = 1, 2$. After some straightforward manipulations, one obtains a set of equations of motion which are equivalent to (2.9):

$$\begin{bmatrix} m_1 \dddot{x}_1 \\ m_2 \dddot{x}_2 \end{bmatrix} + \begin{bmatrix} \tilde{C} \end{bmatrix} \begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \end{bmatrix} + \begin{bmatrix} \tilde{K} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f_1 \sin(\omega \tau + \phi) \\ f_2 \sin(\omega \tau) \end{bmatrix} \quad (2.12)$$

In these dimensionless equations, I have dropped the tildes on $x_{1,2}$, the $'$ indicates $\frac{d}{dt}$ and

$$\tilde{K} = \begin{bmatrix} \kappa_1 + \kappa_2 & -\kappa_2 \\ -\kappa_2 & \kappa_2 + \kappa_3 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \delta_1 & \delta_2 \\ -\delta_2 & \delta_2 + \delta_3 \end{bmatrix} \quad (2.13)$$

The equations of motion (2.12) are supplemented by switching conditions at the boundaries $S$. As a result, the mass parameter $m_1$ will be state-dependent and one can characterize the equations of motion as an example of a hybrid dynamical system. Such hybrid systems frequently arise in many mechanical systems as outlined in [14, 15] and [31]. Within each state, the velocity and accelerations fields are smooth and only at the boundary does a discontinuity result.
2.4.4 Parameter State Dependence

Before performing simulations of the equations of motion, it is necessary to assign representative values to the parameters in (2.12). I generally allow the hydrodynamic parameters to be functions of the forcing frequency $\omega_f$ (see, e.g., [27, 64]), and the modulation regime, determined by $\theta$. Parameters $v_i$ affected by the mass-modulated state will be as follows:

$$v_i(\theta) \in \mathfrak{V} = \begin{cases} v_{i,A} \in \mathfrak{V}_A & \text{if } \theta \in \mathfrak{T}_A \\ v_{i,0} \in \mathfrak{V}_0 & \text{if } \theta \in \mathfrak{T}_0 \end{cases}$$

(2.14)

where $v_{i,A}$ is the i-th parameter in the mass-modulated regime, $v_{i,0}$ is the i-th parameter in the rest mass regime, $\mathfrak{V}_A$ is the set of all parameters in the mass-modulated regime, $\mathfrak{V}_0$ is the set of all parameters in the rest mass regime, and $\mathfrak{V}$ is the set of all parameters which depend on $\theta$.

To determine the explicit $\omega_f$ and $\theta$ dependance of hydrodynamic parameters $v_i$, the assistance of a hydrodynamic modeling software commonly used on marine structures, ANSYS AQWA, was enlisted. This software twice analyzed the simple model shown in Figure 2.11 above, via Euler's equations, subjected to an incident wave forcing of frequency $\omega_f$ for the geometry seen in 2.8. In the first analysis, the response of the masses was considered separately, and in the second, they were considered together.

The effective masses of the inner $M_1$ and outer $M_2$ floats will vary depending on the state of modulation and the forcing frequency, so they are $\in \mathfrak{V}$, and accordingly:

$$M_j = M_{j_0} + M_{j_1}^{EI}(\theta, \omega_f), \quad j = 1, 2$$

(2.15)

where $M_{j_0}$ is the rest mass of float $j$ and $M_{j_1}^{EI}(\theta, \omega_f)$ is the effective increase in mass of float $j$. From the AQWA simulation where the masses were considered separately, seen in Figure 2.13, it became evident that $M_{1_1}^{EI}(\theta, \omega_f)$ was a very weak function of $\omega_f$, so explicit dependance on the forcing frequency was removed, thereby simplifying to

$$M_{1_1}^{EI}(\theta) = \begin{cases} M_{1_A}^{EI} & \text{if } \theta \in \mathfrak{T}_A \\ M_{1_0}^{EI} & \text{if } \theta \in \mathfrak{T}_0 \end{cases}$$

(2.16)

where, after neglecting the weak $\omega_f$ dependance, $M_{1_A}^{EI}$ and $M_{1_0}^{EI}$ are constants (the mean values of $M_{1_1}^{EI}(\omega_f)$ and $M_{1_0}^{EI}(\omega_f)$ respectively). With this simplification, $\mu$ may be determined recalling (2.6):

$$\mu = \frac{M_1^{On} - M_1^{Off}}{M_1^{Off}} = \frac{(M_{1_0} + M_{1_A}^{EI}) - (M_{1_0} + M_{1_0}^{EI})}{M_{1_0} + M_{1_0}^{EI}}.$$

Hence $\mu$ represents a ratio of effective masses:

$$\mu = \frac{M_{1_A}^{EI} - M_{1_0}^{EI}}{M_{1_0} + M_{1_0}^{EI}}.$$

(2.17)
This representation of $\mu$ pertains to realizations such as the one shown in Figure 2.3(i)$^1$. That is, $\mu$ is equal to the difference in the hydrodynamic added mass of $M_1$ when the flaps are in the open state compared to the closed state. From the AQWA analysis where the floats were considered together, $M_E^I(\theta, \omega_f)$ was verified to still be a weak function of $\omega_f$. It was assumed then, for a fixed $M_{1_{10}}$, that by altering the geometry or size of the lower float the desired $\mu$ could be achieved.

Additionally, from both AQWA analyses, $M_E^I(\theta, \omega_f)$ was found to be a linearly decreasing function of $\omega_f$, as seen in Figure 2.13, with a very weak dependance on the state of mass modulation and thus weak dependance on $\theta$, thereby simplifying to:

$$M_E^I(\theta, \omega_f) = \begin{cases} M_{2A}^E(\omega_f) & \text{if } \theta \in \mathcal{A} \\ M_{20}^E(\omega_f) & \text{if } \theta \in \mathcal{O} \end{cases}$$

The regime dependence of other dynamical elements in the system was determined from the AQWA where the masses were considered together. The spring $k_1$ and dashpot $c_1$ represent the buoyancy force and hydrodynamic damping on the mass $M_1$. Although $k_1$ is a constant, $c_1$ depends on whether the flaps are open/closed and the frequency $\omega_f$

$^1$The corresponding representation of $\mu$ for realizations such as the one shown in Figure 2.3(ii) is straightforward to infer.
of the incident waves, so \( c_1 \in \mathfrak{V} \). The second spring-dashpot system represents the PTO and the coupling between the masses. In general, hydrodynamic \( k_2 \approx 0 \), \( B \) represents the damping of the PTO, and \( c'_2 \) represents hydrodynamic damping (\( \approx 0 \), removing regime and frequency dependance) and any additional mechanical damping not encompassed in the PTO, and thus \( c'_2 \notin \mathfrak{V} \). Finally, the spring \( k_3 \) and dashpot \( c_3 \) represent the buoyancy force and hydrodynamic damping on the mass \( m_2 \). I assume that \( k_3 \) is constant while \( c_3 \) is a function of \( \omega_f \). Neither parameter is assumed to depend on the configuration of the flaps.

In summary:

\[
\begin{align*}
  c_1 &= c_1 (\theta, \omega_f) \in \mathfrak{V}, \\
  c_2 &= c_2 (c'_2, B) \notin \mathfrak{V}, \\
  c_3 &= c_3 (\omega_f) \notin \mathfrak{V}.
\end{align*}
\] (2.19)

Accordingly, in performing the nondimensional numerical analyses contained below, the relevant mass, stiffness, and damping terms defined from the AQWA analysis were used, where relevant, and \( \mu \) was additionally allowed to vary as a parameter: the specific values for these mass, stiffness, and damping terms is contained in 2.8.

### 2.5 Metrics for Energy Harvesting

As in works on resonant energy harvesters (see, e.g., [17, 25, 55, 56, 66, 68]), the power generated by the WEC is taken as being proportional to the velocity across the PTO device. As such, the nondimensional average power that can be harnessed from the harvester is defined as

\[
P = \frac{B}{2T \sqrt{k_1 M_{20}}} \int_0^T \left( x'_2 - x'_1 \right)^2 d\tau
\] (2.20)

where \( T \) is a (nondimensional) period of integration which is much larger than the forcing period \( \frac{2\pi}{\omega} \). The power \( P \) is generally a function of many parameters; if one determines \( m_i, \kappa_i, \delta_i \) from the AQWA analysis, it reduces to dependence upon \( \ell \) (a suitable length scale), \( F_\alpha, \omega_f, \delta_2, \mu, \varepsilon, \phi \), and switching boundaries \( \alpha \) and \( \beta \).

For a single degree-of-freedom linear mechanical systems, one has a resonant frequency \( \omega_r \) and an associated damping \( c \) and natural frequency \( \omega_n \). If one wishes to optimally harvest energy using such a system, then it is known that one tunes \( \omega_n \) to coincide with the incident frequency and then \( B \) is chosen so that \( B = c \). For two-degree-of-freedom systems such as shown in Figure 2.11, the optimal system parameters must be determined numerically. In this case, a PTO damping \( B \) was chosen that is equivalent to the mechanical and hydrodynamic damping \( c'_2 \): hence \( c_2 = B + c'_2 = 2c'_2 \).

Selecting the optimal \( \alpha \) and \( \beta \) (both equal to \( \frac{\pi}{2} \)) as in [17] and illustrated schematically in Figure 2.3) further reduced the problem - cf. Figures 2.2, 2.14 and 2.15. In searching for optimal values for the remaining parameters in the system, I found that the effects of changing the values of \( M_{10}, M_{20}, k_1, \) and \( k_2 \) on power harvesting could be most easily interpreted using a tuning parameter \( \gamma \). This parameter, as in [68], is defined as a ratio of
frequencies:
\[
\gamma = \frac{\omega_{n,1}}{\omega_{n,2}},
\]  
(2.21)

where
\[
\omega_{n,1} = \sqrt{\frac{k_1}{M_{10}}}, \quad \omega_{n,2} = \sqrt{\frac{k_2}{M_{20}}},
\]  
(2.22)

### 2.6 Optimal Energy Harvesting

A numerical code was developed to evaluate \( P \) (cf. (2.20)) over a multi-parameter space consisting of \( \omega, \gamma, \delta, \mu, f_1, f_2, \epsilon, \phi, \alpha \) and \( \beta \); this code allowed for ranges of three parameter subsets to be investigated while all other parameters remained single valued. The resulting values of \( P \) could then be plotted, with two of the parameters of interest corresponding to the x and y axis (the z axis always corresponding to \( P \)) and the third parameter of interest encompassed in time or in multiple plots.

It should be noted that the phase separation \( \phi \) between the forcing on \( M_1 \) and \( M_2 \) was chosen to be zero in all simulations. This was dually motivated: firstly, zero phase separation is the most conservative assumption with respect to the harvested power. As this quantity is directly related to the square of the relative velocity, any phase separation increase above zero would naturally result in an increase in harvested power, and I was keen to study the effect of mass modulation independent of this tendency. Secondly, choosing a specific phase separation would be arbitrary and unmotivated, even without the first restriction, as I had no good estimate of what it would realistically be.

Here, the first two parameters were regularly chosen to be \( \omega \) and \( \mu \): \( \omega \) ranges allowed for easy comparison in the frequency space, which was almost always desirous, and \( \mu \) ranges allowed for similar comparisons in added mass space (importantly including \( \mu = 0 \), the nominal case).

Although switching boundaries \( \alpha \) and \( \beta \) were found to be optimal when set to \( \frac{\pi}{2} \) in previous work [17], it was not definitively known if this would be the case for the current model. As such, verification of these parameters’ optimality was first investigated. It was verified that the prescription \( \alpha = \frac{\pi}{2} \) and \( \beta = \frac{\pi}{2} \) was, in fact, optimal in the two mass case as well (cf. Figures 2.14 and 2.15). This was to be expected, as these parameters only effect one mass in this model as they did previously; an intuitive explanation for optimality is discussed in [17] and is not repeated here. Interestingly, while certain values of added mass onset \( \alpha \) completely negate any advantage the added mass scheme has over the nominal case, identical selections of added mass length \( \beta \) do not. Rather than resulting in higher \( P \) for increasing values of \( \mu \), these selections resulted in a shift of the dominant resonant peak at \( \omega = \omega_r \), with a \( P \) on the order of magnitude of the nominal case, for increasing values of \( \mu \): cf. Figure 2.16.

The momentum coefficient \( \epsilon \) was devised, as stated before, to investigate the effect of impulse \( G \), as one would not generally know \( G \) from first principles alone - it must be
CHAPTER 2. IMPULSIVE MASS MODULATION SCHEMES FOR A CLASS OF WAVE ENERGY CONVERTERS: EXPERIMENTS, MODELS, AND EFFICACY

\[ \alpha = 0 \]
\[ \alpha = \frac{\pi}{2} \]
\[ \alpha = \pi \]

Fig. 2.14  Plots of non-dimensional power \( P \) (cf. (2.20)) comparing the effect of changing added onset \( \alpha \) over a two-parameter space varying \( \mu \) and \( \omega \). Switching onset boundary is optimal \( (\beta = \frac{\pi}{2}) \), forcing is equivalent on both masses \( (f_1 = f_2 = 1) \), damping \( \delta_2 = 0.05 \), momentum coefficient \( \varepsilon \) is 0.8, and the tuning ratio is unity \( (\gamma = 1) \).

\[ \beta = 0 \]
\[ \beta = \frac{\pi}{2} \]
\[ \beta = \pi \]

Fig. 2.15  Plots of non-dimensional power \( P \) (cf. (2.20)) comparing the effect of changing added mass period \( \beta \) over a two-parameter space varying \( \mu \) and \( \omega \). Switching onset boundary is optimal \( (\alpha = \frac{\pi}{2}) \), forcing is equivalent on both masses \( (f_1 = f_2 = 1) \), damping \( \delta_2 = 0.05 \), momentum coefficient \( \varepsilon \) is 0.8, and the tuning ratio is unity \( (\gamma = 1) \).

determined experimentally. One would expect that larger \( \varepsilon \), or minimized momentum loss, would correspond to more robust peaks in \( P \) than smaller \( \varepsilon \), or unminimized momentum loss, would. Examining Figure 2.17, one can see that this is indeed the case; more usefully, the simulation informs one that below a certain critical value of \( \varepsilon \) (which generally varies as a function of the other parameters) the mass modulation produces a reduced \( P \) than corresponding value of \( P \) for the nominal case.

Tuning parameter \( \gamma \), the ratio of the switching mass’s natural frequency to the buoy mass’s natural frequency, is also of interest. With non-dimensionalized driving frequency \( \omega \) as given in (2.11), one would expect that altering \( \gamma \) would shift \( \omega \) in some way. Specifically how changing \( \gamma \) would affect \( P \) was not known. As can be seen in Figure 2.18, increasing \( \gamma \) does have the expected result of increasing \( \omega_r \); additionally, one can see that the peak value of \( P \) increases with \( \gamma \) up to a certain point, here at about \( \gamma = 1.5 \), after which increasing \( \gamma \) has no effect on the peak power. This suggests that there is an optimal range of \( \gamma \), above
some critical value, where $P$ is unaffected by increasing choices of $\omega_{n,1}$.

As has been mentioned previously, I narrowed my investigation of the effect of damping $\delta_2$ by selecting $B = c'_2$. With this selection, one would expect that increasing $\delta_2$ up to some value would similarly increase the maximum $P$, as it is proportional to $\delta_2$. Beyond this point, however, the potential benefit of higher PTO damping would be unrealized as the increased overall damping would limit the velocity between the masses and thus the collected power. This trend was verified and can be seen in Figure 2.19: beyond $\delta_2 = 0.05$ (in this case) the ridge of peak $P$ is broadened and reduced with increasing choices of $\delta_2$. This suggests an optimal choice of $\delta_2$ exists for any given set of parameters.

The parameters whose effects are the most difficult to comprehend are the forcings $f_1$ and $f_2$. Their values, both relative to each other and outright, are not well known from a purely dynamical standpoint and experimentation is necessary to specify them. Despite this, a few things may be gleaned from the numerical simulations, cf. Figures 2.20 and 2.21. The most important takeaway is that, with the other forcing fixed, increasing the direct forcing on mass one ($f_1$) has a much greater effect on $P$ than increasing that on mass two ($f_2$). This is problematic as $f_2$ is more likely to be easily determined than $f_1$ in practice. Further, $f_2$ corresponds to the forcing on the buoy mass directly excited by the incident waves - but $f_1$ seems more critical to the WEC’s operation.
Fig. 2.17  Plots of non-dimensional power $P$ (cf. (2.20)) comparing the effect of changing momentum coefficient $\varepsilon$ over a two-parameter space varying $\mu$ and $\omega$. Switching boundaries are optimal ($\alpha = \frac{x}{2}$ and $\beta = \frac{x}{2}$), forcing is equivalent on both masses ($f_1 = f_2 = 1$), damping $\delta_2 = 0.05$, and the tuning ratio is unity ($\gamma = 1$).
Fig. 2.18 Plots of non-dimensional power $P$ (cf. (2.20)) comparing the effect of changing tuning ratio $\gamma$ over a two-parameter space varying $\mu$ and $\omega$. Switching boundaries are optimal ($\alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{2}$), forcing is equivalent on both masses ($f_1 = f_2 = 1$), damping $\delta_2 = 0.05$, and the momentum coefficient $\epsilon$ is 0.8.
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Fig. 2.19 Plots of non-dimensional power $P$ (cf. (2.20)) comparing the effect of damping $\delta_2$ over a two-parameter space varying $\mu$ and $\omega$. Switching boundaries are optimal ($\alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{2}$), forcing is equivalent on both masses ($f_1 = f_2 = 1$), momentum coefficient $\varepsilon$ is 0.8, and the tuning ratio is unity ($\gamma = 1$).
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Fig. 2.20  Plots of non-dimensional power $P$ (cf. (2.20)) comparing the effect of forcing $f_1$ over a two-parameter space varying $\mu$ and $\omega$. Switching boundaries are optimal ($\alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{2}$), forcing on $M_2$ is equal to 1 ($f_2 = 1$), damping $\delta_2 = 0.05$, momentum coefficient $\varepsilon$ is 0.8, and the tuning ratio is unity ($\gamma = 1$).
Fig. 2.21  Plots of non-dimensional power $P$ (cf. (2.20)) comparing the effect of forcing $f_2$ over a two-parameter space varying $\mu$ and $\omega$. Switching boundaries are optimal ($\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{2}$), forcing on $M_1$ is equal to 1 ($f_1 = 1$), damping $\delta_2 = 0.05$, momentum coefficient $\varepsilon$ is 0.8, and the tuning ratio is unity ($\gamma = 1$).
2.7 Conclusions

Experimental work on a modified WEC specified in [17] was performed to validate the proposed optimal impulsive mass modulation contained therein. It was found that the affection of impulsive mass modulation was realizable - as was the impulsive mass modulation specified in [55, 56] and tested in [54] - and this new scheme improved the power harvested by the WEC significantly for a wide range of incident wave frequencies: moreover, it was found to only be slightly worse than the case without impulsive mass modulation at one testing frequency, which itself was significantly above the fundamental frequency of the prototype. Regardless, I devised a more robust analysis of a buoy-type WEC to affect a sensitivity analysis of the impulsive mass modulation scheme to a variety of design and environmental parameters that would be rather difficult, if not impossible, to change on a prototype WEC. Accordingly, I introduced a two-degree-of-freedom model that would more realistically reflect the dynamics of a buoy-type WEC. Using this more realistic model for the WEC I have shown the potential energy harvesting benefits of the mass modulation scheme. For example, the model shows that it is possible to increase the harvesting capabilities by 200% in some instances (cf. Figure 2.22). It should be emphasized that the effectiveness of the modulation scheme is heavily dependent on the amount of fluid momentum carried across the switching boundary. In particular, if the design of the WEC is such that \( \varepsilon \) is too small, then the mass modulation scheme does not result in significant improvements over the nominal case and may indeed be detrimental.

In closing I wish to emphasize that the results shown in this chapter verify that the impulsive mass modulation scheme is effective. Furthermore, the modulation scheme proposed here can be easily generalized - all that is required is to change the hydrodynamic
added mass (with or without trapping water) of the inner float. Broadly, one can achieve mass modulation using induced mass (virtual mass) by changing the shape of the inner float cyclically. The flaps in these experiments are but one realization of this idea. It is not too difficult to conceive of other realizations - some of which can be achieved using active control.

### 2.8 Numerically Determined Hydrodynamic Values

**Appendix**

Recalling the dimensioned mass equations (2.15), (2.16) and (2.18):

\[
M_j = M_{j0} + M^{EI}_j(\theta, \omega_f), \quad j = 1, 2
\]

\[
M^{EI}_1(\theta) = \begin{cases} 
    M^{EI}_{1A} & \text{if } \theta \in \mathcal{T}_A \\
    M^{EI}_{10} & \text{if } \theta \in \mathcal{T}_0
\end{cases}
\]

\[
M^{EI}_2(\theta, \omega_f) = \begin{cases} 
    M^{EI}_{2A}(\omega_f) & \text{if } \theta \in \mathcal{T}_A \\
    M^{EI}_{20}(\omega_f) & \text{if } \theta \in \mathcal{T}_0
\end{cases}
\]

AQWA analysis was performed on the (coupled) geometry seen in Figures 2.24a and 2.24b, where Figure 2.24a represents the system with \(M_1\) in the modulated \((M_1^{On})\) state and Figure 2.24b represents the system with \(M_1\) in the unmodulated \((M_1^{Off})\) state. It was determined using this AQWA analysis that for this example geometry, with a limited frequency range \(\omega_f \in [0.3\pi, 0.5\pi]\) Hz,

\[
M_{10} = 18247.81 \text{ kg} \\
M^{EI}_{1A}(\omega_f) = -1052.6 \text{ kg/Hz} \cdot \left(\frac{\omega_f}{2\pi}\right) + 13838 \text{ kg} \\
M^{EI}_{10}(\omega_f) = -2030 \text{ kg/Hz} \cdot \left(\frac{\omega_f}{2\pi}\right) + 4197.5 \text{ kg}
\]

Notice that, for the range of frequencies here, this means \(M^{EI}_{1A}(\omega_f)\) decreases from 13680.11 kg to 13574.85 kg. Such are variances of \(\pm 0.386\%\) from the mean value of 13627.48 kg. Similarly, for \(M^{EI}_{10}(\omega_f)\) one sees a decrease from 3893.3 kg to 3690.3 kg. Such are variances of \(\pm 2.677\%\) from the mean value of 3791.8 kg. So the simplification in (2.16) of \(M^{EI}_{1A}(\omega_f) \rightarrow M^{EI}_{1A} (= 13627.48 \text{kg, here})\) and \(M^{EI}_{10}(\omega_f) \rightarrow M^{EI}_{10} (= 3690.3\text{kg, here})\) is justified.

One may also look at the dependance of \(\mu\) on \(\omega_f\), if one had not made the simplification of (2.16) that led to (2.17), so

\[
\mu(\omega_f) = \frac{M^{EI}_{1A}(\omega_f) - M^{EI}_{10}(\omega_f)}{M_{10} + M^{EI}_{10}(\omega_f)} \rightarrow \frac{977.4 \text{ kg/Hz} \cdot \left(\frac{\omega_f}{2\pi}\right) + 9640.5 \text{ kg}}{-2030 \text{ kg/Hz} \cdot \left(\frac{\omega_f}{2\pi}\right) + 22445.31 \text{ kg}}
\]
which increases from 0.442 to 0.451, which are variances of −0.954% and 0.960% (respectively) from the mean value of 0.446. Notably, using the simplified expression for $\mu$ (2.17), one gets $\mu = 0.446$ as well.

For the values associated with the second mass, it was determined that:

\[
M_{20} = 15292.09 \text{ kg} \\
M_{2A}^E (\omega_f) = -195789 \text{ kg/Hz} \cdot \left(\frac{\omega_f}{2\pi}\right) + 97808 \text{ kg} \\
M_{20}^E (\omega_f) = -185100 \text{ kg/Hz} \cdot \left(\frac{\omega_f}{2\pi}\right) + 96225 \text{ kg}
\] (2.24)

For the damping parameters, one finds that

\[
c_1 (\theta, \omega_f) = \begin{cases} 
263.16 \text{ kg/(Hz} \cdot \text{s}) \cdot \left(\frac{\omega_f}{2\pi}\right) + 216.53 \text{ kg/s} & \text{if } \theta \in T_A \\
-480 \text{ kg/(Hz} \cdot \text{s}) \cdot \left(\frac{\omega_f}{2\pi}\right) + 391 \text{ kg/s} & \text{if } \theta \in T_0
\end{cases}
\] (2.25)

\[
c_3 (\theta, \omega_f) = \begin{cases} 
309473.6 \text{ kg/(Hz} \cdot \text{s}) \cdot \left(\frac{\omega_f}{2\pi}\right) - 18591.1 \text{ kg/s} & \text{if } \theta \in T_A \\
314700 \text{ kg/(Hz} \cdot \text{s}) \cdot \left(\frac{\omega_f}{2\pi}\right) - 21475 \text{ kg/s} & \text{if } \theta \in T_0
\end{cases}
\] (2.26)

And for the stiffnesses, one finds that

\[
k_1 = 46.7791 \text{kN/m}, \quad k_3 = 330.6602 \text{kN/m}.
\] (2.27)
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Fig. 2.23  AQWA simulation damping values.

Fig. 2.24  (a) Hydrodynamic model used to generate system parameters, with $M_1$ in modulated ($M_1^{\text{On}}$) state. (b) Hydrodynamic model used to generate system parameters, with $M_1$ in the unmodulated ($M_1^{\text{Off}}$) state.
The Roles of Impact and Inertia in the Failure of a Shoelace Knot

3.1 Introduction and Motivation

While most people have experienced accidental untying of their shoelaces, little is known and even less is documented about the physical mechanisms responsible for this ubiquitous annoyance. A popular 2005 TED Talk by Terry Moore on strategies for tying shoelaces to minimize knot failure, while helpful in suggesting a knot-tying heuristic, does not explain the source of the failure. Fortunately, some hints towards the nature of the shoelace failure mechanism can be understood anecdotally. I observed that a shoelace knot that often failed very quickly when walking (typically within 50 feet) did not fail when the leg was simply swung back and forth a similar number of cycles, that is, with no impact while sitting on a table. However, simply stomping the foot on the ground the same number of cycles also did not lead to untying. These observations suggest that the knot failure involves an interplay between the swing and stance phases (cf. Figure 3.1) of the walking motion. Stated differently, failure does not occur without both the impact and swinging experienced during walking or running.

With the goal of observing shoe lace knot failure, I examined slow-motion video footage of a runner on a treadmill whose shoes were tied with what I call the weak knot, which is based on what is commonly termed the false, granny, or granary knot. The following videos, the former slightly shorter than the latter,

https://www.youtube.com/watch?v=_-aiynIphTw
https://www.youtube.com/watch?v=fYjeUGczPNU

may assist the interested reader in re-living what I saw. The resulting images of the knot failure were striking (Figure 3.1). In particular, there appeared to be two time scales upon which untying took place: little change to the knot was observed for many strides until some untying began, after which the speed of untying was remarkable (often in less than two
CHAPTER 3. THE ROLES OF IMPACT AND INERTIA IN THE FAILURE OF A SHOELACE KNOT

Fig. 3.1 Illustration of the stages of the human gait and images from high-speed camera observation of shoelace knot failure. The three strides shown in the images come after several minutes of running on the threadmill. Images in the same column show the same stage in the stride. Each row of images corresponds to a single stride and the duration of each stride is approximately one second.

These observations informed both the hypothesized failure mechanism and experimental design.

I refer the reader to Figure 3.1 for illustrations of the phases of a walking gate and Knot Structure Appendix (a) - (g) for a comparison of the structures of weak knots and strong knots. My hypotheses for the failure of the knot are detailed in Section 3.3. I believe the failure is due to a combination of knot deformation associated with the impulsive impact of the shoe during the heel striking phase of the walking gait and the flapping motion of the shoelaces during the swinging phase of the walking gait. The deformation of the knot and the relative motion of the shoelace strands are moderated by the friction between the strands of shoelace in the knot center to create the slow-fast (or “gradually-suddenly”) time scales. To formulate the hypothesis, scientific work on knots is discussed in Section 3.3.

1 The speed of the knot failure brings to mind a Hemingway line from The Sun Also Rises describing a character’s descent into bankruptcy – which here I found to be an apropos description of knot failure – it happens “Two ways. Gradually, then suddenly.”
CHAPTER 3. THE ROLES OF IMPACT AND INERTIA IN THE FAILURE OF A
SHOELACE KNOT

3.2. After the hypothesis has been presented, an experimental validation of the proposed
failure mechanism is performed. The experiments are outlined in Section 3.4 and feature a
custom-made impact test facility. This chapter’s work also presents some challenges to the
computational mechanics community which I hope will inspire future work.

Work in this chapter was accomplished with fellow graduate student Christine E. Gregg
and the with advice and assistance of Professor Oliver O’Reilly; it resulted in the publica-
tion of [10].

![Knot Illustration](image)

**Fig. 3.2** Illustration of both (a) the false knot and (b) the square knot, upon which the weak and
strong knot are based, respectively. Each are composed of a series of stacked trefoils (a
simple crossing of lace strands), with the “first trefoil” referring to the one closest to the
shoe – it is also the first tied. The difference between the knots lies in the handedness (the
ordering of which lace crosses over which) of the “second trefoil”: that is, “left over right”
or “right over left.” In the strong knot, the handedness of the second trefoil is different
from the first.

3.2 Background

In this chapter I am primarily concerned with two versions of the common shoelace
bow-tie knot. The primary difference between the versions of this knot has to do with
their component parts: a first trefoil, or lace crossing, tied close to the lacing and a second
trefoil tied after the first (cf. Figure 3.2 for an illustration of these trefoils). The strong
version of this knot is based on the square knot: two trefoil knots of opposite handedness
are stacked with the free ends tucked into the knot center, thereby creating loops (cf. Figure
Knot Structure Appendix (h) - (i) for an illustration of the relationship between the two).
The weak version of the knot, as mentioned previously, is based on what is commonly
known as the granny or false knot. In contrast to the strong version, the two trefoils have
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the same handedness, causing the knot to twist instead of laying flat when tightened. These differences are illustrated in Knot Structure Appendix (a) - (g) and I encourage the reader to experiment with their own shoelaces. While the inferior performance of the weak version is well known in both common knot lore and surgical knot literature [37, 48, 59, 69], at the time of writing I have yet to find a proposed structural characteristic to explain the inferior behavior. Likewise, I have not found a study explaining the failure mechanism for a shoelace knot.

Scientific work on knots can be roughly divided into three major categories: mathematical study of knot topology, rod-based models for physical knots, and experimental investigations of surgical suture knots. The first category pertains to mathematical knot theory where invariants for knots are sought and studied (see [2, 28] and references therein). The works in this field allow one to classify and distinguish types of knots and determine invariants of a particular knot topology (such as trefoil (or simple) knots, cinquefoil (or double) knots, and figure eight knots) that cannot be changed without cutting the strands of the knot. In the second category, works in the mechanics of physical knots seek to use an elastic rod model to determine the deformation of the strands of a knot [9, 47]. Due to the interest in simulating and animating strands of hair, ropes, and sutures in computer graphics, progress in this area has been rapid in the past decade (cf. [37, 72]). However, simulating the dynamics of the shoelace knot under conditions experienced during walking remains a challenging goal. An appreciation for the technical challenges can gained by examining recent works such as [9, 38] on the mechanics of rods having tangled self-contacting strands. Adding to the technical challenges, several temporal scales must be accommodated in order to examine the failure of a shoelace knot: the short duration of the impulsive impact of the shoe with the ground during the heel strike and the much longer duration of the motion of the free ends of the lace. Further, several length scales must also be considered: the small zones dominated by friction and deformation of the shoelaces and the much larger segments of flapping lace.

The challenges of numerically modeling shoelaces lead me to focus my efforts on an experimental examination of knot failure, and it is my hope that such experiments will inform future modeling work. This work was aided by the large literature on experimental characterizations of knot strength and failure in surgical sutures. Unlike knot topology, which deals with arbitrarily self-tangled curves, the suture knots that are examined are typically simple combinations of different trefoils [41, 70] that are similar to shoelace knots. Testing conducted in these works features quasi-static and cyclic tests with no inertial effects [33, 36, 67, 69]. Such tests typically lead to physical breaking rather than untying of the knot. Even if untying is accounted for, it is often secondarily commented on, and the concentration on breaking failures means that significant non-elastic effects have been experienced by the strands up until failure [36, 51]. Significant plastic deformation is not a part of the average shoelace knot’s untying, limiting the applicability of these works to the problem at hand. However, the procedures and protocols gleaned from the literature on surgical sutures were invaluable when designing the experiment discussed in this chapter.
A hypothesis for knot failure was developed based on the aforementioned high-speed observations and additional normal-speed observations of knots in both standard operation (i.e., on a shoe, walking or running) and artificial operation (i.e., swinging a shoe at the
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edge of a table in order to isolate inertial effects, or stamping of a foot in place to isolate impulsive effects). This hypothesis was further refined by data from the testing of knots mounted on the device described in Section 3.4. The hypothesized mechanism is as follows:

1. As the leg is swung forward, and then slightly backwards to impact the ground, the loops and free ends of the shoelaces (Figure 3.3 (b)) are all pulled forward (with respect to the knot center) by their inertia. The relative motion causes an opening of the knot - that is, a widening of the center space separating the two trefoils. The center space is where the free ends were pulled back through the knot to create the loops (cf. Figure Knot Structure Appendix (h) - (i)).

2. The impulsive impact force of the shoe during the striking of the heel is transmitted to the knot by the tongue of the shoe and the eyelets. As a result, the center of the knot deforms.

3. The opening of the knot, and the concomitant reduction in friction forces, facilitates relative motion of the free ends with respect to the knot center. In other words, with the center of the knot pulled apart, relative axial motion between the free end and the knot center becomes possible, and it is easier for the free end to slide out of the knot center. The tension force pulling a free end through the center is due to the imbalance of the inertia of the free end (the same inertia that helps to pull the knot apart), the inertia of the corresponding loop (to which the free end attaches through the knot), and the frictional forces at the center of the knot.

4. The repeated impacts perturb the knot such that the free end is incrementally pulled through the knot. This happens slowly at first. But as the knot is repeatedly pulled apart and more length is fed through the knot center to the free end, the inertially supplied tension forces pulling the free end through the knot are increasingly magnified, while the competing inertia forces of the loop diminish as the loop size decreases (Figure 3.3 (c)).

5. Eventually, the free end is sufficiently long that in one or two strides (impact cycles), the inertial force associated with the free end completely overpowers the loop’s inertial force, causing run-away knot failure as the free end pulls completely through the knot center. This last event signifies total failure of the second trefoil (Figure 3.3 (d)).

It should be noted that this hypothesized mechanism explains the characteristic speed/time-scales (slow-fast, as described earlier) at which catastrophic knot failure occurs. I have framed the hypothesis assuming that the inertia of the free end dominates that of the loop (an assumption that matched the vast majority of observed failures). The corresponding hypothesis when the opposite situation arises is readily formulated. In this case, the loop gets larger at the expense of the free end and the failure mechanism is similar to the case discussed in detail above. However, I believe this is rarely observed because the loops are
necessarily constrained in the orientations they can take, and do not undergo the same range of motion/acceleration as the free ends (which quite literally “whip” back and forth).

I suspect, but have not been able to prove, that the difference between the weak knot and the strong knot failure rates lies in the twist of the weak knot. The strong knot can be tightened, yet remain planar. However, to tighten the weak knot, the structure must twist completely. For the knot to fail, the deformation of the knot center noted in (ii) should contribute to the opening (rather than tightening) of the knot center, but it has proven elusive to measure the deformation of the knot center subject to a round of impulsive impact loading.

3.4 Experiments

Fig. 3.4  Illustration of the experiment used to study the effects of impact and lace dynamics on a knot. The knot is mounted to a pendulum arm (approximately 20 cm in length) that is released from rest at a prescribed angle of inclination and impacts a surface. The angle was chosen to be approximately 43°. This choice of angle, combined with a tuning of the impact surface, produced an impact deceleration of ≈ 7g.
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3.4.1 Preliminary Testing: In-Situ High Speed Failure Observation and Force Measurement

To inform controlled laboratory experiments and to hone suspected failure hypotheses, initial knot failure experiments were performed in-situ on shoes during running and walking. The first of these in-situ experiments observed knot failure using a high-speed camera (Vision Research Phantom Miro M110, 1280 x 800 pixels, 900 fps). A volunteer runner was instructed to tie her shoes using the weak version of the standard shoe knot. The volunteer then ran on a treadmill (cf. Figure 3.1 and Supplementary Video 1) until acute knot failure was observed (defined as failure of the second trefoil, more commonly understood as loop untying). The observed failure occurred rapidly and without warning, with footage confirming that full failure occurred within one to two strides of failure initiation. A single failure mode was observed where a free shoelace end pulled through the knot, thus reducing it to a single trefoil. The same volunteer was additionally instructed to tie her shoelaces using the strong version of the standard shoe knot – with approximately the same initial tightening as the weak knot – and no acute failure (that is, complete untying) was observed during the testing period.

To better understand the dynamic conditions surrounding failure, the accelerations acting on the knot during the motions of walking and running were measured. A LORD Microstrain 10g wireless accelerometer was attached to multiple types of shoes directly underneath the center of the knot (a location which varied slightly depending on the shoe). Shoes included in the initial study were typical running shoes, hiking boots, casual sneakers, and barefoot shoes. With a metronome to help her maintain a constant cadence, a volunteer was asked to both walk and run on a straight, level surface, and the accelerometer’s time history was recorded. This time history contained measurements resulting from inertial and impact sources.\(^2\) The impact sources produced accelerations which were large and impulsive. For the walking tests, the impact magnitude (or measured acceleration at the base of the knot trefoils) was typically on the order of 6-8g. Furthermore, analyses of testing results showed surprisingly similar magnitudes of acceleration regardless of shoe type (a schematic of the in-situ mounting and time history is presented in In-situ Testing Appendix). As I wished to study untying that occurs even in the least extreme cases (i.e., slow walking on a thick sole, as opposed to the extreme case of running on a thin sole), it was resolved to focus further testing on impulses corresponding to this regime. This information was used to calibrate the impact magnitude - chosen to be approximately 7g - for the controlled testing described in the following section.

3.4.2 Controlled Testing: Cyclic Impact

A experimental set-up was fabricated to isolate the effects of impact magnitude, impact direction, and free end inertial force magnitude on knot failure. Illustrated in Figure 3.4,

\(^2\) The accelerations experienced by the center of the knot are not necessarily the same as those experienced by the loops or free ends. The loops and free ends will experience additional accelerations relative to the knot center which I was unable to measure directly.
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Fig. 3.5  Representative acceleration time histories for the impact testing from the pendulum apparatus shown in Figure 3.4. The blue curve (labelled $\alpha$) represents the accelerations experienced by the knot in the impact direction. The green curve (labelled $\beta$) and red curve (labelled $\gamma$) represent off-axis accelerations in the vertical and lateral directions, respectively (cf. Figure 3.4).

An actuated pendulum was constructed to generate a primarily uni-directional impact force acting on the knot center. The pendulum was dropped from an adjustable height to control the impact magnitude, and the mounting orientation of the knot determined the effective impulse direction. An example of the time history is shown in Figure 3.5. The impact surface was tuned to produce minimal off-axis excitation and provide an impulse profile characterized by a single 7g peak similar to those observed in the accelerometer data mentioned previously (cf. In-situ Testing Appendix). The pendulum was tuned such that mean off-axis accelerations were, at most, 10% of the mean peak on-axis acceleration values. In most cases, they were less than 5%.

Except when otherwise noted, the strong knot was used for all tests. A rigorous knot tying procedure and a template were developed to ensure uniformity of knots between tests. Templates were used to manage knot geometry (size of free ends and loops) and tightening of the knots. For the latter, knot tightening was standardized by hanging weights from knot loops. Initially, the knots were tied so that the loops and the free ends were of equal length (8.26 cm). The free-end length was defined as the distance of the lace end relative to the knot center and the loop length was defined as the distance of the middle of the outstretched loop (i.e., half of its arc length/chord length) relative to the knot center (cf. Knot Structure...
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Appendix (j)). All knots studied were tied with identical store-bought black (unwaxed) dress laces with a cross-sectional diameter of 2 mm. An attempt was made to characterize the bending stiffness of the laces, but values determined via an optical and force-based method were on the order of the error of the method ($10^{-6}$ Nm$^2$). The exceptionally small value of the bending stiffness led us to conclude that it is of negligible importance relative to frictional terms.

All experiments measured the rate of knot failure. The length of the free end was measured at the beginning of the experiment and either after knot failure, if the failure occurred within a trial timeout of 15 minutes, or at the end of the trial. This was then divided by the total number of impact cycles to calculate an average slip rate (average change in the length of the free end per impact cycle).

In preliminary testing, it was observed that knot failures were almost exclusively “pull through” in nature, suggesting that the inertia of the free ends dominated that of the loops. To isolate the effect of free-end inertial force asymmetry on knot failure, masses of various sizes were added to the free ends to magnify their inertial effect. The values of these masses were chosen heuristically so that during the transition from the lowest value to the highest value a corresponding change from limited knot failure to frequent knot failure was observed. End masses of 1 gram, 2 gram, and 3 gram represented 7.4, 14.8, and 22 times the initial mass of the free end and 3.7, 7.4, and 11.1 times the initial mass of the loop, respectively, were added to the free ends of the shoelaces. The effect of different masses on the average slip rate was measured.

The effect of impact direction on failure rate was investigated by mounting the knot on the pendulum in different orientations. Two orientations were chosen: a “rear impact” where the plane in which the two trefoils lie is normal to the impacting direction and a “side impact” where that same plane is parallel to the impacting direction. One can conceptually move from a “rear impact” to a “side impact” by rotating the knot 90° about the axis of the gravitational force (cf. Figure 3.7). Additionally, the failure frequency of the strong knot was compared with the weak knot. As the weak knot does not orient itself along the same axis as the strong knot, two mounting orientations of the weak knot were tested (cf. Figure 3.8).

3.5 Results and Discussion

Two regimes of knot failure were found: gradual loosening and acute failure. To characterize these regimes, I define the average slip rate for each test as the average change in length of the free end per impact cycle (calculated by dividing the net change of the length of the free end during testing by the total number of impact cycles). Total or acute failure was defined as the complete unraveling of the second trefoil. Figure 3.9(a) shows the average slip rate for the rear impact test, with 0, 1, 2, and 3 gram free-end weights. The labels “L”, “R”, and “Average” correspond to the change in length of the left free end, right free end, and the average of the two, respectively.
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Fig. 3.6 Example of a strong knot: (a), mounted on pendulum; (b), with attached free-end weights ($\zeta$) to simulate added inertia effects; (c), partially failed (diminished loop indicated by $\nu$); and (d), completely failed (absent loop indicated by $\tilde{\nu}$). It should be noted that the knot shown in (c) has a diminished loop size and extended free ends compared to the knot displayed in (b).

Only tests conducted with 3 gram weights experienced acute knot failure, which occurred for approximately half of the 3 gram tests (53% or 8/15 tests). It is clear that tests using 3 gram free-end weights entered a regime of force asymmetry between the loop and free end that caused significantly more rapid failure. However, it was observed that acute failure was runaway in nature: the majority of the free-end length change (and thus failure) occurred within a relatively short number of impacts for all tests and amounts of free-end weight. This behavior is consistent with observations from high-speed video and is evident when examining the isolated slip rates of the 3 gram tests that did not fail (Figure 3.9(b)). If instances of runaway failure are excluded, the increase in average slip rate increases approximately linearly with free-end weight (i.e., loop/free end inertial asymmetry), whereas comparing the average slip rate of all 3 gram trials shows a more than four-fold increase in slip rate that is heavily weighted by runaway failure tests. The rear impact tests showed a significant bias towards the failure of the left lace which was likely due to the initial knot structure.

Figure 3.10(a) shows the average free-end slip rate for the side impact test, with 0,
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Fig. 3.7 Schematic of impact orientations of the knot, relative to the impulsive loading direction: (a) “rear impact” and (b) “side impact”.

Fig. 3.8 Comparison of the orientations of the weak knot on the pendulum apparatus: (a), in a horizontally mounted configuration; and (b), a vertically mounted configuration. In both figures, the scale bar has a length of 4 cm.

1, and 2 gram free-end weights. Only tests conducted with 2 gram weights experienced full knot failure. As with the rear impact tests, side impact tests exhibited runaway failure. Therefore, Figure 3.10(b) compares the average slip rates for the different free-end weights, excluding the 2 gram tests that experienced full runaway failure. Unlike the rear impact tests, the side impact slip rates for non-failed knots did not monotonically increase with increasing free-end weight (increasing inertial force asymmetry). The reason for this behavior is not fully understood and requires further investigation. Interestingly, biasing of failure between the left-free end and right-free end was not observed in the side impact
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Fig. 3.9 (a) Average rate of change of the length of the free end for rear-impacted knots with different free end masses. (b) Average rate of change of the length of the free end for rear-impacted knots with different free end masses, excluding specimens that failed completely in the 3 gram sample. For both subfigures, the label “L” refers to the left free end, the label “R” to the right free end, and the label “Average” to the mean of the left and right free ends. Error bars show standard error.

As expected, the weak knot failed much faster than the strong knot. It is commonly known in surgical knot practice and general knot lore that the disparity of load-carrying capabilities of the weak knot compared to the strong knot are significant. Figure 3.13 compares average free end slip rate for both the strong and the weak knot with added 3 gram free-end weights, impacted from the rear. The weak knot experienced a 100% failure rate in both mounting configurations and at much higher slip rates than the strong knot. Differences in slip rate between the two weak knot mounting orientations did not show
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Fig. 3.10  (a) Average free end slip rate of the free end for side-impacted knots with different free end masses. (b) Average free end slip rate for side-impacted knots with different free end masses, excluding specimens that failed completely in the 2 gram free-end weight sample. For both subfigures, the label “L” refers to the left free end, the label “R” to the right free end, and the label “Average” to the mean of the left and right free ends. Error bars show standard error.

significant differences.

3.6 Conclusions

High-speed video observation of in-situ shoelace knot showed failure to be a sudden and catastrophic phenomenon – that is, it occurs over two time scales: a long, slow, and gradual (and largely unnoticed) loosening followed by an extremely quick complete failure. Observations point to a failure driven by the complex interplay between impulsive impact-induced deformation of the center of the knot, dynamic swinging of the walking motion, and the inertial forces of the laces and free ends of the knot. Preliminary experimental results showed that runaway failure and loosening can be linked to a mismatch between the inertially supplied forces of the loop and the free ends that is decreasingly mediated by friction as the knot center is loosened under cyclic impact. The increasing mismatch leads to an increase in slip rate and causes an accelerating failure. The results also confirmed that the weak knot fails at higher slip rates and frequency than the strong knot. Further testing is necessary to more fully understand the effect of impact orientation on the weak knot versus the strong knot.

It was also shown that knot failure was runaway in nature, with the majority of the
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Fig. 3.11 Average free end slip rate for side-impacted knots compared to rear-impacted knots with different free end masses. The label “L” refers to the left free end, the label “R” to the right free end, and the label “Average” to the mean of the left and right free ends. Error bars show standard error.

change in length associated with complete knot failure occurring in just a few impact cycles. The direction of the impact has an effect on failure rate, with side impacts on the strong knot entering a regime of full failure at a lower threshold of inertial asymmetry (within the allotted testing time of 15 minutes) than the rear-impacted knots. This is suspected to have implications for differences of failure rate between shoe types observed anecdotally, as the location and orientation of the knot on the shoe leads to a different direction of the resultant impact force.

The work presented in this chapter on the failure of the shoelace knot is far from exhaustive. For one, the influence of the shoelace material and surface finish was not investigated. In addition, the metric used to understand knot slip rate in this chapter (measurements of free end length before and after set testing time or failure) is insufficient to elucidate finer details of knot failure progression from gradual loosening to a regime of fast failure. To do so, future tests should fail each knot, measuring free-end lengths at set intervals during testing. Such measurements will better distinguish the two regimes of knot failure I have identified (one regime where the inertial imbalance between free ends and loops is relatively stable, the other where the imbalance rapidly leads to runaway knot untying). I expect that such measurements and further testing will also lead to better understanding of the mechanical factors that cause the inferiority of the weak knot compared to the strong knot.
Fig. 3.12  Average free end slip rate for side-impacted knots compared to rear-impacted knots with different free end masses excluding specimens that failed completely. The label “L” refers to the left free end, the label “R” to the right free end, and the label “Average” to the mean of the left and right free ends. Error bars show standard error.
Fig. 3.13 Average free end slip rate for the rear-impacted weak version of the knot compared to the rear-impacted strong version of the knot, with a free-end weight of 3 grams. It should be noted that because of the inherent chirality of the weak knot, two orientations of this knot were tested. The pair of orientations can be seen and compared in Figure 3.8. The label “L” refers to the left free end, the label “R” to the right free end, and the label “Average” to the mean of the left and right free ends. Error bars show standard error.
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3.7 Knot Structure Appendix

An overhead view of the tying of the weak knot (on the left of each image panel) and the tying of the strong knot (on the right of each image panel). Notice, importantly, that the two knots only differ in the relative tying of the second trefoil to the first trefoil (cf. panels (b) and (c)). Further, after the loops are pulled out (indicated by the white arrows in panels (e) and (f)) the weak and strong knots become the prototypical false/granary/granny and square/reef knot (cf. panel (g)) respectively.
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Depictions of the (h) strong knot and (i) square knot illustrating their underlying structure. Color coding shows equivalent knot sections and demonstrates that the strong knot is equivalent to the square knot with its free ends (yellow and orange) tucked into the knot center to create loops.

Figure showing the relative dimensions of a knot tied according to the template. The free end length \( a \) is equal to half of the loop chord length \( b \).
3.8 In-situ Testing Appendix

(a) Photograph of in-situ testing of the accelerations experienced by a shoelace knot. Illustration of the recorded acceleration axes is given: the primary direction of impact lies in the plane formed by the blue arrow (labelled $\alpha$) and the green arrow (labelled $\beta$). The red arrow (labelled $\gamma$) signifies the direction of off-axis accelerations. (b) Accelerations as functions of time in three orthogonal directions for the in-situ testing of the accelerations experienced by a shoelace knot; $\alpha$, $\beta$, $\gamma$ as before.
Dynamical Analysis and Development of a Biologically Inspired SMA Caterpillar Robot

4.1 Introduction

The creation of a bio-inspired robot is, in general, a non-trivial task, with no set heuristic for the modeling and analysis of the biological inspiration that will necessarily lead to a successful and demonstrative robot. Even in the case of relatively simple source organisms, which map well to specific geometric architectures likely considered to comprise a potential robot – such as octopus tentacles, worms and elastic strings, or perhaps caterpillars and chained elastic beams [40, 44, 65]– the task of developing a minimally complex robot is still one with a daunting parameter space [39]. Take, for example, the goal of undulatory motion. In what manner should the kinematics and dynamics of the source organism be studied, particularly when the source organism has a variety of operating regimes [6]? How should the contacts seen in the source organism, often asymmetric and irregular, be translated to those of the prototype? With real-world limitations to human and numerical resources, what is the best way to reduce potentially tertiary and time-wasting diversions in the design space from those likely to affect the motion of the robot? These questions become even more intractable when the desired robot is to be soft.

In contrast to their conventional rigid counterparts, soft machines and robots are composed of deformable bodies capable of extreme changes in shape and functionality. Despite their potentially extraordinary advantages, the deformability of soft bio-inspired robots yields an infinite degree-of-freedom system that is significantly more difficult to model and control than a discrete system (e.g., piecewise rigid). Progress in the nascent field of soft robotics depends on the ability to rapidly and faithfully model the dynamical state of a soft robot and incorporate this model into a feedback control for real-time path planning and locomotion.
Here, I seek to suggest a general analysis path for the rapid prototyping of a soft robot through the specific analysis of a shape memory alloy (SMA) based, caterpillar inspired soft robot. My approach is two-pronged: analysis of the kinematics and dynamics of the caterpillar is structured through simple elastica-based models which yield estimates of forces, energetics (that would be extremely difficult to determine directly) and suggestions for open-loop motion patterns in 4.2. Simultaneously, in 4.3 simple experiments and optical tracking of SMA segments are performed to yield properties necessary for a discretized numerical simulation of potential soft robot designs. These simulation complement the work done in 4.2 by reducing the parameter space under investigation for both physical SMA prototypes and further elastica simulation.

Work in this chapter was accomplished with fellow graduate student Alyssa Novelia and with the advice and assistance of Professor Oliver O’Reilly; it is expected to be published this calendar year.
CHAPTER 4. DYNAMICAL ANALYSIS AND DEVELOPMENT OF A BIOLOGICALLY INSPIRED SMA CATERPILLAR ROBOT

Fig. 4.2 Illustration of optically tracked abdominal segments from [71]. The abdominal segments tracked are those with prolegs, numbered 3 to 7. The seventh, or terminal, proleg is abbreviated as TP. Displacements and velocities are tracked as the caterpillar crawled in positive $E_x$ direction.

4.2 Analysis of *Manduca sexta* caterpillar motion

One of the most thoroughly studied species of caterpillar is the *Manduca sexta* (tobacco hornworm), known for the ease with which it can be reared in a controlled laboratory environment and with which it can be made to perform specified crawling tasks. In order to guide the development of an SMA based, caterpillar-inspired soft robot, I sought to garner an idea of the leg movements, body deformation, and displacement of the *Manduca* from the extant literature. Of particular interest to us are studies of the kinematics of *Manduca* motion and its mechanical properties due to Trimmer et. al. [71] which were originally undertaken to understand the neural processes involved in locomotion. Experiments performed therein include optical tracking of *Manduca* abdominal segments and legs to identify swing and stance phases during crawling, in addition to determination of intersegmental deformations during crawling. Tensile tests were also performed to constitutively characterize the *Manduca* abdomen’s stress response [20, 21, 43]. Additionally, a soft-bodied caterpillar-inspired robot that is capable of performing crawling and ballistic rolling motions was constructed [44].

My interests, of course, are a few degrees abstracted from those of Trimmer et. al. Correspondingly, I resolved to perform an inverse dynamics analysis of the motion captured in [71] to determine the muscle forces required to produce the basic straight-line crawling motion, using the derived abdominal constitutive equations, assuming each segment to be subject to governing equations of rod theory. Broadly, rod theory restricts acceptable geometries to primarily axial architectures with comparably small cross sections, or combinations thereof. Admissible motion generation for an insect (in this case) or robot (my eventual goal) is similarly restricted, then, to transverse bending waves and longitudinal compressive waves – the former being most analogous to a caterpillar [19]. The *Manduca*’s motion, roughly, is composed of a wave of contraction which travels through the abdominal sections from tail to head (i.e. in the direction of desired motion), producing a characteristic traveling “hump” on a caterpillar’s body.
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4.2.1 Analysis of Manduca motion

As mentioned previously, [71] optically tracked the Manduca caterpillar to generate records of leg swing angle (relative to the horizontal), leg velocity, abdominal segment displacement and abdominal segment deformation by placing markers as shown in Figure 4.2. Study of this data revealed two key takeaways, which are assisted in explanation by Figure 4.3, which I use as a guide to my specifications of motion patterns for the soft robot. Firstly, that the swing phase (as opposed to the stance phase, in which a given segment of the caterpillar is stationary) for each segment – for example, \( t = 5 \) for the fourth segment’s proleg (A4P) – was marked by a substantial increase in the horizontal velocity of the corresponding segment leg (cf. Figure 4.3). From this one can gather that the majority of useful forward motion occurs during a segment’s swing phase, and that all other segments should be, if one was to use a simple approximation, ‘anchored’ during this period. Secondly, that the maximum compression and vertical displacement (Figure 4.3 (iii) and (iv)) of a segment roughly correspond to its peak horizontal velocity, in addition to the midpoint of the swing period for all segments aside from the rearmost one. Such implies that the time during which a given segment is maximally bent should correspond to the time when a segment is at its highest and is moving fastest – more generally, that a significant lofting motion is involved in creating the useful forward motion of the caterpillar, again while the other segments are ‘anchored’.

![Graphs of leg swing angle, leg velocity, vertical displacement and deformation](image)

**Fig. 4.3** Plots of (i) leg swing angle (relative to the horizontal) in degrees, (ii) leg velocity in cm/s, (iii) abdominal segment vertical displacement in cm and (iv) abdominal segment deformation in cm. Thanks to Professor Barry A. Trimmer for supplying the caterpillar tracking data used here.
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One of the desired outputs from analysis of the caterpillar optical data is the determination of a motion pattern, initially to guide development of a soft robot similar in specification to the caterpillar (such as in the number of segments). To that end, sinusoidal regression was performed on each segment’s deformation and compared against the source (Figure 4.4): these approximations were found to be acceptable and were compared to determine the approximate delay between the swing phase for each segment. With this analysis completed, it was endeavored to replicate *Manduca* motion with a simple generating function describing the position of material points on an idealized rod.

![Figure 4.4](image)

**Fig. 4.4** Comparison of sine wave approximations of segment deformation, achieved by regression to the source data: all deformations in cm. Fit equations and $R^2$ values are: (i) $y = 0.02\sin(3.4t + 3.5) + 0.5$ and 0.28844, (ii) $y = 0.09\sin(1.7t + 2) + 0.5$ and 0.7891, (iii) $y = 0.08\sin(1.7t + 1.6) + 0.4$ and 0.81467 and (iv) $y = 0.06\sin(1.7t + 1) + 0.4$ and 0.7371. Thanks to Professor Barry A. Trimmer for supplying the caterpillar tracking data used here.

4.2.2 Crawling motion generator: Witch of Agnesi

By observation, a segment of the centerline of the caterpillar lifts off the ground as it crawls forward in an undulating motion, while the remaining parts of the body remain attached to the ground. With that half wave shape in mind [7, 42], a symmetric, positive
CHAPTER 4. DYNAMICAL ANALYSIS AND DEVELOPMENT OF A BIOLOGICALLY INSPIRED SMA CATERPILLAR ROBOT

definite function with one local maximum and which asymptotically goes to zero at the tail ends to was needed to represent a caterpillar mostly adhering to the ground with a segment of its body forming a ‘hump’ in the air. I elect to use the so-called ‘Witch of Agnesi’ function:

\[ y(x) = \frac{8a^3}{x^2 + 4a^2} \]  \hspace{1cm} (4.1)

The curve shape is equivalent to the probability density function of the Cauchy distribution and often used to represent wave flow in geophysical sciences [4]. I modify the original function (4.1) to make a suitable motion generator by adding constant x translation (cf. Figure 4.5), in addition to making the function maxima travel at the same speed. The equations for the x- and y- positions of the material points composing the centerline \( r(\xi, t) \) are given by:

\[
\begin{align*}
\mathbf{r}(\xi, t) &= X(\xi, t)\mathbf{e}_x + Y(\xi, t)\mathbf{e}_y \\
X(\xi, t) &= \xi + ct \\
Y(\xi, t) &= \frac{8a^3}{(X(\xi, t) - \lambda(t))^2 + 4a^2} \\
\lambda(t) &= g(t) + ct
\end{align*}
\]  \hspace{1cm} (4.2-4.5)

Note that \( \lambda(t) \) (the maxima position) is composed of both a sawtooth wave \( g(t) \) and a constant speed translation \( ct \). To continue with inverse dynamics analysis of the motion, one needs four constants to be determined from the physical caterpillar’s motion: the horizontal speed \( c \), the hump height \( 2a \), and the amplitude \( A \) and period \( T \) of \( \lambda(t) \).

To calculate the value of \( c \), I perform linear regressions on the x-positions of the tracked abdominal segments as a function of time. The chosen value is the average of the slopes, which is shown in Figure 4.5 to be in agreement with the position data. The hump height \( 2a \) is similarly chosen from the y- abdominal position data. \( T \) is chosen by averaging the segment deformation period shown in Figure 4.4. As for \( A \), an ad-hoc rule to assure natural looking motion is \(|\xi_f - \xi_i| < A\) as the caterpillar body (briefly) returns back to a straight shape after a step is made. A summary of parameters used in this paper is given in Table 4.1.

Finally, note that the position function has discontinuities in the time domain at integer multiples of the crawling period \( T \) of the sawtooth wave. I choose to ignore this discontinuity and treat the time derivative of \( \lambda(t) \) as a constant \( \frac{2A}{T} \). This caveat established, corresponding partial derivatives of \( X \) and \( Y \) with respect to \( \xi \) and \( t \) may be established. Note the abbreviations \( f, \tilde{f}, f' \) and \( f'' \) indicate first and second derivatives with respect to
CHAPTER 4. DYNAMICAL ANALYSIS AND DEVELOPMENT OF A BIOLOGICALLY INSPIRED SMA CATERPILLAR ROBOT

![Plot of segment position as a function of time, for each caterpillar segment, overlaid with an approximation of the overall crawling speed $c = 0.3780 \text{ cm/s}$.

<table>
<thead>
<tr>
<th>Kinematic</th>
<th>Value</th>
<th>Material</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_f$</td>
<td>1.93 cm</td>
<td>$E$</td>
<td>0.9333 MPa</td>
</tr>
<tr>
<td>$c$</td>
<td>0.3780 cm/s</td>
<td>$\rho_0$</td>
<td>0.0398 kg/m</td>
</tr>
<tr>
<td>$a$</td>
<td>0.075 cm</td>
<td>$R$</td>
<td>3.45 mm</td>
</tr>
<tr>
<td>$T$</td>
<td>3.63 s</td>
<td>$t_m$</td>
<td>0.079 mm</td>
</tr>
<tr>
<td>$A$</td>
<td>2.2 cm</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1 List of kinematical and material parameters used by the simulation in this paper.

time and material coordinate, respectively. Below are the first two partial derivatives:

\[
\begin{align*}
\dot{X} &= c, \quad \ddot{X} = 0, \quad X' = 1, \quad X'' = 0 \\
\dot{Y} &= 16a^3 \tilde{g}(t) \frac{\xi - g(t)}{((\xi - g(t))^2 + 4a^2)^2} \\
\ddot{Y} &= 16a^3 \tilde{g}(t)^2 \frac{3(\xi - g(t))^2 - 4a^2}{((\xi - g(t))^2 + 4a^2)^3} \\
Y' &= -16a^3 \frac{\xi - g(t)}{((\xi - g(t))^2 + 4a^2)^2} \\
Y'' &= 16a^3 \frac{3(\xi - g(t))^2 - 4a^2}{((\xi - g(t))^2 + 4a^2)^3}
\end{align*}
\]
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As evidenced, \( Y'' = \frac{1}{g'(t)^2} \ddot{Y} \) where \( g(t) \) is constant and \( Y(\xi, t) \) is the solution of D’Alembert’s wave equation and is thus periodic, as is required.

Functions of higher order derivatives of \( X \) and \( Y \) are featured in the kinetics of the centerline. Among those required are the direction of the unit tangent vector \( e_t \) along the centerline \( \theta \), stretch \( \mu \), and curvature \( \kappa \):

\[
\theta = \arctan \left( \frac{Y'}{X'} \right) \quad (4.6)
\]
\[
\mu = \sqrt{(X')^2 + (Y')^2} \quad (4.7)
\]
\[
\kappa = \frac{1}{\mu} \theta' \quad (4.8)
\]

### 4.2.3 Constitutive equations for the estimation of caterpillar forces and energetics

I will now lay down the equations governing the elastic properties of the caterpillar, which I am modeling as an elastic string or as an Euler beam (elastica) that allows resulting forces and energetics to be determined. The chosen geometry is a variation on the modeling work done in \([45]\) where the caterpillar is assumed to be a hydraulic skeleton composed of two concentric cylindrical shells of ventral internal lateral muscle and cuticle. I assume that the beam consists solely of the muscle part while keeping the value of the mass of the entire caterpillar fixed (about 2 g). The muscle takes the shape of a thin cylindrical shell with radius \( R \) and thickness \( t_m \), from which one can calculate the surface area \( A \) and second moment of area \( I \).

For the elastica model, I choose to model the caterpillar body as an extensible, linearly elastic material with a constant Young’s modulus \( E \) – this value determined from the slope of the secant line of preconditioned muscle’s loading-unloading curve. Although the muscle behaves as a nonlinear, pseudo-elastic, transversely isotropic material with strain rate dependence \([20, 21]\), such modeling would appear to be unnecessarily sophisticated. The list of material constants used are listed in Table 4.1.

#### String approximation

The constitutive equation for strain energy is:

\[
\rho_0 \Psi(\mu, \xi) = \frac{EA}{2} (\mu - 1)^2
\]

(4.9)

where \( \rho_0 \) is mass density (per unit length), \( \Psi \) is strain energy density function, \( E \) is Young’s modulus, \( A \) is the cross-sectional area kept planar during bending and \( \mu \) is the axial strain of the caterpillar segment. The strain energy density function is independent of the material
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points \( \xi \) which comprise the string. From here, one can invoke the energy balance in order to solve for a prescription for the contact force:

\[
\mathbf{n} = EA (\mu - 1) \mathbf{e}_t, \quad \mathbf{e}_t = \frac{\mathbf{r}'}{||\mathbf{r}'||}
\]  

(4.10)

Note that, despite its name, the contact force \( \mathbf{n} \) will generally vary across the string/rod (as seen in (4.10)) and, moreover, it is generally non-zero in areas that are not in contact with any external surface. It may take on a (single) value of a (boundary prescribed) contact force in a few simple cases, hence the nomenclature. I am merely following the naming convention devised by Naghdi [34, 35, 52, 57] to describe this stress over cross sectional area-like quantity. Note that the contact force only exists in the direction tangent to the material curve, as a string is only able to support tension along its length.

These definitions presented, a balance of linear momentum (per unit length – consult [57] for details) is all that is needed to determine the body forces per unit length supplied by the idealized caterpillar elastica:

\[
\rho_0 \ddot{\mathbf{r}} = \rho_0 \mathbf{f} + \mathbf{n}'
\]

(4.11)

which can be decomposed into \( \mathbf{E}_x \) and \( \mathbf{E}_y \) constituent equations:

\[
\rho_0 \mathbf{f} \cdot \mathbf{E}_x = \rho_0 \ddot{\mathbf{x}} - \mathbf{n}' \cdot \mathbf{E}_x
\]

(4.12)

\[
\rho_0 \mathbf{f} \cdot \mathbf{E}_y = \rho_0 \ddot{\mathbf{y}} - \mathbf{n}' \cdot \mathbf{E}_y
\]

(4.13)

where \( \mathbf{n}' \) is given by:

\[
\mathbf{n}' = EA \left( \frac{\mu'}{\mu^2} X' \right) \mathbf{E}_x + EA \left( \frac{\mu'}{\mu^2} Y' + \frac{\mu - 1}{\mu} Y'' \right) \mathbf{E}_y
\]

(4.14)

Elastica approximation

For the string’s extensible elastica counterpart, I opt for the constitutive equation:

\[
\rho_0 \Psi(\Delta \kappa, \mu, \xi) = \frac{1}{2} EI (\kappa - \kappa_0)^2 + \frac{1}{2} EA (\mu - 1)^2
\]

(4.15)

Note the dependence of the strain energy function on curvature \( \kappa \) (relative to the intrinsic curvature \( \kappa_0 \)), and the second moment of area \( I \) in the direction of bending. This time, the energy balance gives a prescription for the bending moment throughout the rod \( \mathbf{m} \) and the contact force component in the tangential direction \( \mathbf{n} \cdot \mathbf{e}_t \):

\[
\mathbf{m} \cdot \mathbf{E}_z = \frac{1}{\mu} EI (\kappa - \kappa_0)
\]

(4.16)

\[
\mathbf{n} \cdot \mathbf{e}_t = EA (\mu - 1) - \frac{1}{\mu} \kappa EI (\kappa - \kappa_0)
\]

(4.17)
There are still four more unknowns to determine: \( \rho_0 f \cdot \mathbf{E}_x, \rho_0 f \cdot \mathbf{E}_y, \mathbf{n} \cdot \mathbf{e}_n, \) and \( \mathbf{m}_a, \) but with only linear momentum and angular momentum balance laws remaining, I have to make an engineering approximation for the system to be solvable. Under the assumption that applied moment per unit length \( \mathbf{m}_a \) is equal to zero, the contact force component in the normal direction is:

\[
\mathbf{n} \cdot \mathbf{e}_n = \frac{\rho_0 I}{\mu} \theta - \frac{1}{\mu^2} EI (\kappa' - \kappa'_0) - \frac{\mu'}{\mu^3} EI (\kappa - \kappa_0)
\]

That leaves the applied force to be solved using the balance of linear momentum as in Equations (4.12) and (4.13).

**Comparison and analysis**

![Comparison and analysis](attachment:image.png)

**Fig. 4.6** Applied force per unit length \( \rho_0 f(\xi^* = 0, t) \) required to actuate a material point \( \xi^* = 0 \) which belongs to a thin rod according to the prescribed motion generator for one cycle of crawling.

A plot of the applied force per unit length for a chosen material point in one cycle of crawling is shown in Figure 4.7. In both string and elastica models, one can see that the
Fig. 4.7 Surface representation of the applied force per unit length $\rho_0 f$ required for any given time $t$ and material point $\xi$.

values move away from zero as the representative material point lifts off the ground, around $t = 1$. However, one also finds that the values differ by two orders of magnitude.

Recall that the applied force per unit length is influenced by the material derivative of the contact force, which is largely influenced by curvature (and its derivatives) in the elastica model. This causes the applied force per unit length to assume a large value at an instant when the material point is at maximum $y$-value, as the signed curvature of the ‘Witch of Agnesi’ plunges to a minimum. This explains the large difference in the force per unit length values for the string and elastica model, since the string model is unaffected by curvature.

Another point of interest is highlighted when the forces in the $y$-direction assume a negative value, briefly, before and after $t = 1$ as the material point takes flight and lands back to the ground. For readers assuming that the vertical applied force per unit length corresponds to reaction force applied by the ground, this may seem like an oddity. However, since the applied force is a sum of ground reaction force, gravitational body force, and the force supplied by the caterpillar muscle, it is possible for the total applied force to have a negative value, especially during the time the material point is off the ground. Therefore, it is safe to say that the sole contributor to the applied force at that instant is the force due to the muscle, which points in negative $y$-direction (the gravitational force being rather small in comparison).

Variation of the applied force per unit length as a function of material point and time is plotted in Figure 4.7. Here, I superpose the applied force per unit length over one cycle of crawling for five material points along the length of the curve. It is evidenced that each material point experiences the exact amount of force as any other while performing the same motion later in the cycle. This is because the horizontal motion of material points is restricted to steady axial motion $\xi + ct$. 
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The power consumed by the entire caterpillar during one cycle of crawling can be calculated using the following expression:

\[ P(t) = \int_{\xi_i}^{\xi_f} \rho_0 f(\xi, t) \cdot \dot{r}(\xi, t) d\xi \quad (4.19) \]

From the graph of power versus time shown in Figure 4.8, one can see that energy addition and extraction occurs when the material curves transition from a straight line to the ‘Witch of Agnesi’ and back during one crawling cycle. It is interesting to note that in the string model, energy is gained as the material curve is deformed from the straight configuration, and lost as it returns to a straight line. Meanwhile, the opposite phenomena is observed in the elastica model. The material curve loses energy when it deforms from straight to ‘Witch of Agnesi’, and gains energy to return to a straight line. Note that the energy change is close to zero at any other time, even during the translation of the peak of the ‘Witch of Agnesi’ curve.

The high value of peak force observed in the material points during deformation is balanced by the low velocity at which they are traveling, which results in low power values. Additionally, one can say that net energy change is small but positive at the end of one crawling cycle since the material points move in the positive \( x \)-direction with constant velocity due to positive work done by the external forces. The energy expenditure here still ends up being much larger than approximated in [46] by particle rectilinear motion, which does not take into account the effects stretching and recoil.

Cumulative integration of the power with respect to time gives us the work done by the external forces over time, or in other words the amount of energy going in and out of the system at any given time. While previously I learned that energy is lost and gained within a crawling cycle through the graph of power \( P(t) \), it is more apparent in Figure 4.9 that the net energy change over a cycle is positive, which means that the external forces do positive work towards the caterpillar body since it ends up crawling forward in the positive \( x \)-direction. This fact is better illustrated by observing the overall increasing trend of the work done for four crawling cycles.

4.3 SMA characterization and numerical modeling

Mapping SMA segments onto rod-based elastica hierarchies requires the determination of properties which are seldom directly available to a robot developer. Specifically, for one of the most simple constructions – a inextensible planar rod – at a minimum the density, intrinsic curvature, flexural rigidity and contact characteristics (such as contact stiffness and frictional properties) in both the activated and deactivated regimes must be determined. The first property is trivially within reach and the last would likely be extremely difficult to determine precisely or cheaply, which suggested a novel approach to the authors: gain an approximation of the intrinsic curvature and flexural rigidity of an SMA segment through a simple experiment and determine the correct contact characteristics through comparison of simulated motion via a numerical method with that of the inspiration source.
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4.3.1 Optically-based SMA characterization

I developed my characterization of an initial design for a SMA limb on an optical method first presented in [12, 13], in those instances to characterize a pneu-net architecture soft robot, with substantive modifications. In line with those works, multiple (n=5) photographs (at slightly different positions) are taken of multiple (n=4) SMA segments in actuated (that is, above a threshold power input of approximately 11 W) and unactuated regimes, at view-angles nearly normal to the SMA side; note that the SMA was positioned slightly offset from a regularly spaced, 5 mm grid for proper scale. In each photograph, the SMA was clamped at one end and free to deform along the rest of the length of its body (cf. Figure 4.10). Departing from the previously developed method, the actuation direction of the SMA was selected to be in the plane made by gravity and the horizontal: comparisons were made between this arrangement and that in [13] (that is, such that the actuation direction is about the axis of gravity) and no significant variations in SMA deflection were noted. The convenience and repeatability of taking photographs in this arrangement and a desire to avoid significant out-of-plane deformation (even accounting for the inherently stiffer nature of the SMA in comparison to the pneu-net arm) solidified this choice.

Significant enhancement of the aforementioned method was achieved through the enlargement of the number of SMA centerline identification points. This can be briefly ex-

Fig. 4.8 Power over time $P(t)$ during one cycle of prescribed crawling motion.

$\times 10^{-3}$ String Power

<table>
<thead>
<tr>
<th>time (s)</th>
<th>J/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>1.5</td>
<td>1.5</td>
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<tr>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td>3.5</td>
<td>3.5</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>time (s)</th>
<th>J/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
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<td>2.5</td>
<td>1</td>
</tr>
<tr>
<td>3.5</td>
<td>0</td>
</tr>
</tbody>
</table>

$\times 10^{-3}$ Elastica Power
explained after a small foray into curvature: if one defines the position of any material point on the SMA centerline by \( \mathbf{r}(\xi) = x(\xi)\mathbf{E}_x + y(\xi)\mathbf{E}_y \), then the curvature of the line at any point \( \xi \) may be established:

\[
\kappa = \frac{x'y'' - x''y'}{[(x')^2 + (y')^2]^{3/2}}
\]  

(4.20)

Therefore, it is necessary to twice numerically differentiate position values obtained optically, which will result in a reduction of optically recognized sites on the centerline by two. Moreover, for optically recognized sites which are closely spaced together (as they were in [12], and would be on this SMA) small errors in site placement or recognition will be greatly magnified when taking first and second derivatives with respect to the length parameter \( \xi \), which was the case previously and correspondingly necessitated significant averaging that diluted the accuracy of any curvatures derived thusly. This SMA was also significantly shorter than the pneu-net arm, which combined with the desire to ensure that all derivatives with respect to \( \xi \) were sufficiently continuous, motivated my modification.

Instead of a denoting the centerline of the SMA by a finite number of dots, to be optically parsed in the black and white spectrum, I masked and coated the SMA centerline (in the cross-section) with a continuous, optically distinct cyan marker fluid that did not alter the deformation of the SMA. This cyan fluid, although not uniquely necessary to the success of this method, was found to be easily identifiable when processing the images (in
Fig. 4.10  Illustration of characterization setup, with sample cross-sections of clamped ($\gamma$) SMA limbs with (i) no end forces (ii) spring dynamometer ($\alpha$) supplied end forces – used during characterization of initial SMA architecture (iii) weight ($\beta$) supplied end forces – used during characterization of new SMA architecture. Spacing of the grid $\nu$ is 5 mm.

MATLAB) through a CMYK (cyan, magenta, yellow, key) color space, particularly after removal of the key (black) channel. An example of correctly identified points may be seen in Figure 4.11. Once identified, an extremely dense ($n \propto$ order of identified points) curve was least-squares fitted to the data, and through an iterative process was reduced, twice differentiated with respect to $\xi$ and re-integrated for comparison with the original data. The maximally reduced curve with the highest fidelity (in a least-squares sense) to the original data was chosen as the proper centerline, thus avoiding the data sparsity and averaging problems experienced in [12] and ensuring sufficient curvature smoothness, even for an arc as short as that on this SMA. Curves so identified were compared for each photograph of a given SMA sample to obtain a single curve, from which curvature as a function of $\xi$ was determined, in both the actuated and unactuated regimes (cf. Figure 4.12).

To determine the flexural rigidity of the SMA, only a slight modification of the aforementioned method was necessary. A force dynamometer was connected to the free end of the SMA via a small string, such that the initial force measured, when unactuated, was 0 N. The SMA was then actuated, causing it to deform some amount less than without the dynamometer, and the force registered by the dynamometer was recorded. Following along as before, I could feed photographs of the SMA in this position through MATLAB and determine its curvature with the applied end force. This curvature could then be compared to the intrinsic curvature obtained previously for the same sample, allowing us to determine the flexural rigidity $EI$ of the SMA in the actuated regime. Specifically, if one assumes the
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Fig. 4.11  Simplified schematic of optical SMA characterization of the initial architecture: (i) orthonormal photograph of SMA segment, with centerline enhancing cyan coating, in actuated position. (ii) high-density identification of centerline enhancing coating. (iii) iterative determination of oriented, ordered centerline arc. Note that the dimensions of the initial SMA architecture are (approximately, accounting for variation in production) $L = 3 \text{ cm} \times W = 3 \text{ cm} \times H = 0.3 \text{ cm}$.

Fig. 4.12  Comparison of centerline arcs of actuated SMA: the dashed line is with the spring dynamometer applied force resisting full SMA deflection, whereas the dash-dot line is the natural actuated position of the SMA. Curvatures are derived from these centerline arcs.
Fig. 4.13 Comparison of centerline arcs of the revised SMA: (i) in the unactuated state, (ii) in the actuated state, and (iii) in the actuated state with the applied 1g end force – notice the additional deflection and corresponding change in curvature. Note that the dimensions of the revised SMA architecture are (approximately, accounting for variation in production) $L = 5.5 \text{ cm} \times W = 2.4 \text{ cm} \times H = 0.1 \text{ cm}$.

### Flexural Rigidity $EI\left(\text{N}\cdot\text{m}^2 \times 10^{-4}\right)$  
### Intrinsic Curvature $\kappa\left(\text{m}^{-1}\right)$

<table>
<thead>
<tr>
<th></th>
<th>Actuated</th>
<th>Un-actuated</th>
<th>Actuated</th>
<th>Un-actuated</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.681</td>
<td>51.647</td>
<td>0.563</td>
<td>6.646</td>
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<tr>
<td>$\sigma$</td>
<td>0.201</td>
<td>7.202</td>
<td>0.409</td>
<td>5.183</td>
</tr>
</tbody>
</table>

*Table 4.2* Values of flexural rigidity $EI$ and intrinsic curvature $\kappa$ obtained for the SMA segments, as described in Section 4.3.1, for both the actuated and unactuated states. For both, $\eta$ indicates the mean of the quantity and $\sigma$ the standard deviation. The unactuated intrinsic curvature does not have these quantities as per the method in which it was determined.
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SMA can be modeled as an inextensible linearly elastic rod, one has:

\[
M(\xi) = EI (\theta'(\xi) - \kappa_0(\xi)) 
\]

(4.21)

\[
0 = n' + \rho f 
\]

(4.22)

\[
M' = (\sin(\theta)E_x - \cos(\theta)E_y) \cdot n 
\]

(4.23)

and the boundary condition:

\[
n(\xi = l) = F 
\]

(4.24)

where \(\kappa_0\) is the intrinsic curvature of the SMA, \(n(\xi)\) is the contact force at position \(\xi\), \(f\) is a body force (per unit length), and \(F\) is the dynamometer-measured applied force at the SMA tip. The contact force is simply determined using (4.22) and the boundary condition, which enables integration of (4.23) for the value of the left-hand-side of the constitutive relation (4.21). Once accomplished, the constitutive equation may easily be rearranged to solve for the flexural rigidity \(EI\), which may not be single valued. For my purposes, an average (representative) value of the flexural rigidity was determined – variation, either due to prototype manufacturing tolerances, deviation between the SMA behavior and the linear behavior assumed, or numerical error, was not of interest for my initial, simple characterization.

As this technique could not be leveraged in the unactuated regime, a solution from the literature was found. My initial SMA segment design was that of a Martensite/Austenite transfer SMA wire surrounded by a liquid metal embedded elastomer (LMEE). It was reasoned that the ratio between the actuated and unactuated flexural rigidity should be the same as the ratio between Martensite and Austenite Young’s moduli of the SMA wire by itself: that is, that the SMA wire was the controlling stiffness, as far as bending was concerned. This ratio was easily determined from the literature [74]. I additionally had to consult the literature to find the load/unload contact characteristics of the SMA, or more specifically, the LMEE surrounding the SMA wire: this was supplied from collaborators [5].

A totally redesigned second architecture, incorporating three pre-strain levels, was also characterized in a similar manner – generally speaking, each new SMA limb was thinner, more malleable, and was both less curved in the unactuated and actuated states. Compared to previously characterized limbs, the new limbs (at all pre-strain levels) have actuation positions that are nearly flat (that is, with nearly zero curvature), which changed the feasibility of the previous measurement direction. Moreover, measured dynamometer forces were an order of magnitude smaller than those measured previously, or on the order of the error of the spring-dynamometer. Correspondingly, end-load forces were created from known sub-gram weights at three levels (0.27, 0.49, and 1 g), from which SMA cross sectional positions were pulled and \(EI\) calculated (cf. Figure 4.13). The 1 g level was eventually used, as it provoked the most significant displacement from the reference case, creating the most consistent and error free results.
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Parameter values determined using the novel method are presented in Table 4.2. I note that the large values for the standard deviation in both quantities are likely due to the small sample size (n=4), prototype manufacturing variation – particularly with regards to torsional deflection – and the limits of the method (or any other) as mean \( \kappa \) goes to zero (or as an SMA becomes more flat). In such a case, small variations about the mean centerline can cause large variation of determined values. I believe that with a large population size composed of non-prototype members, these numbers would be significantly reduced and a more precise mean obtained. Irrespective of this, these numbers provide considerable agreement with those determined via a discrete elastic rods based method by my collaborators (private correspondence) and give us the parameter order of magnitude necessary to proceed with numerical motion prototyping – in this paper, I restrict myself to the parameters gathered for the initial SMA architecture.

4.3.2 MADYMO based rapid numerical prototyping

My goals for numerical-simulation based prototyping of an SMA caterpillar were relatively modest. Using a binary simplification of the motion data extracted from the bio-inspiration in 4.2 (cf. Figure 4.14), simulations were to be ran that (i) studied variations in ground-robot frictional contact, (ii) studied variations in segment intrinsic curvature and inter-segment connections and (iii) studied the effect of changes of the relative timing of this motion pattern. Through these, informed by the timing and energetic results from 4.2, an investigation of the design parameter space could be undertaken that would illuminate a physical SMA caterpillar robot’s form. In addition, results could also be used to reduce the necessary parameter space under investigation in future elastica models (for example, in determination of proper frictional regimes) that would lessen research time and improve results. Force and energetic quantities could also be simultaneously output from these simulations (often of essentially unmeasurable quantities, a la 4.2.3) and compared to the bio-inspiration, additionally guiding the design process. Note that the binary pattern, whilst generated for the subject caterpillar’s legs, is used for the actuation of segments in the numerical model. Two main approximations were incorporated into modeling here to make the simulations numerically expedient and to reduce complexity.

First, the SMA continua, heretofore modeled as a beam or rod like object, was discretized into an \( n \)-body system connected by rotational springs and dampers (cf. Figure 4.15), whose action would approximate the function of the flexural rigidity \( EI \). Specifically, I compare the energy contained within a clamped-free beam which is subject to an end force \( P \) and an \( n \)-body approximation subject to the same force:

\[
U_b = \frac{P^2 L^3}{6EI}, \quad U_n = \sum_{i=1}^{n-1} \frac{k}{2} (\theta_{i+1} - \theta_i)^2
\]

\[
\theta_i = \frac{-Px_i}{2EI} (x_i - 2L), \quad x_i \in [0, L]
\]
where one can compare from $U_b = U_n$ and rearrange to solve for the equivalent rotational spring stiffness $k$ for this $n$-body system, in both the actuated and unactuated regime:

$$k = \frac{4EIL^3}{3} \sum_{i=1}^{n-1} \left\{ \left( x_{i+1}^2 - x_i^2 \right) - 2L(x_{i+1} - x_i) \right\}^2$$  \hspace{1cm} (4.25)

and the value is appropriately agnostic of the end load applied, as one would expect. The average intrinsic curvature $\kappa$ in both regimes was used in determining the offset (or null point) of these rotational springs. Rotational damping $c$ was determined heuristically within the numerical simulation software (described below) by specifying a damping that reduced oscillations to zero on the order of the time of the actuation of the SMA ($\approx 0.25$ seconds).

Second, the switch from the actuated to unactuated regimes was approximated as a jump condition. This was motivated by the magnitude and variation of the SMA’s switching time.
Fig. 4.15  Illustration of $n$-body discretization ($\beta$) of SMA limb ($\alpha$), with expanded view of rotational spring and damper ($\gamma$). Additionally, example three-limb caterpillar in MADYMO, with $n = (i)$ 5, (ii) 10, (iii) 20, and (iv) 30. Beyond $n = 15$ no appreciable difference in robot steady-state or dynamic motion was discernible, whilst computation time increased dramatically.

– although typically on the order of tenths of seconds, the actual value varied significantly, particularly after long duty cycles, and could not be uniquely determined. Moreover, any sort of more gradual transition would require characterization of an intermediate stage of the SMA, which I did not have.

Initial work was, for expedience, performed on commercial software. Naturally, there are many commercially available softwares in this field, but I needed one that incorporated both finite-element (FE) and rigid-body (RB) elements, allowed for on-the-fly parameter switching, and accommodated a non-trivial friction model. A simulation suite called MAtematical DYnamic MOdel (MADYMO R7.6 – TASS International, Helmond, The Netherlands) was chosen for its sophistication in all the aforementioned areas. Originally developed for the study of vehicle occupant interactions during car crashes, and used primarily in the collision and automotive development fields, its corresponding emphasis on dynamic simulation (as opposed to static or quasi-static) was particularly useful to us. Numerical methods employed by MADYMO – alternatively a Runge-Kutta or modified Euler method, as individual simulations required – are not limited to automotive simulations and there are no scaling limitations contained in the program.

In many respects it is difficult to investigate, much less discuss, how changes to a single parameter or small group of parameters effect the success of undulatory motion as, natu-
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rally, values set for the rest of the parameter space will consistently modify this success – no one parameter choice is truly independent of all others. That being said, numerical simulations in MADYMO tended to reveal that a specific choice of a study parameter gave rise to the most successful motion for a range of values of the non-study parameters, which advanced the numerical prototyping in the direction defined below. Entirely alternative designs, such as an SMA caterpillar which utilizes rolling motion, or an SMA-based snake robot, could be similarly investigated with quite distinct parameter fields, all without having to build a physical prototype until the design space was significantly reduced.

Contact Analysis

A determination of the minimally complex ground-robot contact interaction was necessary, which primarily became a question of contact geometry and friction. The robot under study was limited to contact the ground at set points at the end of each segment: this was both numerically expedient (in that it limited the number of interactions the solver needed to check for) and it was motivated by the motion of the bio-inspiration (which, in its normal operating mode, touches the ground in a few set places). Each of these contact points was connected to abutting SMA segments via a rotational spring which emulates bending resistance and a translational spring which emulates compression/tension resistance – this will be elaborated on with some detail in 4.3.2. While a stress/strain hysteresis curve for the SMA was determined from the literature,[5] this was somewhat a secondary concern – it was expected that the contact point’s resistance to lateral motion would be the decisive factor in the success of undulatory motion.

Fig. 4.16 Comparison of modified Coulombic ($\alpha$), pure stick-slip ($\beta$) and modified stick-slip ($\gamma$) friction prescriptions.

At the most basic level, determination of contact and generation of corresponding forces is the most efficient when using RB elements (as opposed to FE elements), as the number
of intersections to inspect is reduced. With this modeling, MADYMO only has allowances
for a rather simplistic (modified) Coulombic friction – that is, friction forces generated are
a function of the normal forces at the contact point alone, augmented by a 0 to 1 ramp
function at low velocities. Specifically

\[ F_f = C \left( |v_{surf}| \right) \cdot \hat{\mu} \cdot |N| \quad (4.26) \]

where \( C \left( |v_{surf}| \right) \) is the ramp as a function of surface velocity \( v_{surf} \) and \( \mu (|N|) \) is the friction
coefficient generated as a function of the normal force \( N \). Such a prescription disallows
explicit stick-slip friction, which I anticipated might be key to accurate simulations of a
robot’s motion. If I use the FE elements, however, friction can be formulated as a function
of \( v_{surf} \):

\[ F_f \leq \hat{\mu}_s \cdot |N|, \quad |v_{surf}| = 0 \]
\[ F_f = \hat{\mu}_d \cdot |N|, \quad |v_{surf}| > 0 \]

where \( \hat{\mu}_s \) is the static friction coefficient and \( \hat{\mu}_d \) is the dynamic friction coefficient: ex-
plicit stick-slip is accommodated. In addition to the added computation time required for
determination of contact in the FE model, such a prescription introduces small forces and
corresponding vibrations as the contact oscillates between stick and slip. This is problem-
atic from a computational perspective, as such an effect pushes the necessary integration
timestep down and the overall computation time up. For actual simulation, the friction
prescription is relaxed somewhat to help reduce this computation time:

\[ F_f = \tilde{\mu} \left( |v_{surf}| \right) \cdot |N| \quad (4.27) \]

with

\[ F_f \leq \tilde{\mu}_s \cdot |N|, \quad |v_{surf}| \leq v^* - \varepsilon \]
\[ F_f = \tilde{\mu}_d \cdot |N|, \quad |v_{surf}| > v^* + \varepsilon \]

where \( v^* \ll 1 \) and \( \varepsilon \ll v^* \) (cf. Figure 4.16). While the relaxation of the jump discontinuity
reduces the numerical load of strict stick-slip, it does so only slightly. Correspondingly, I
compared each of these friction models to determine if stick-slip was necessary. A sim-
ulation was run with a modest lateral end force (on the order of the gravitational force of
one segment) applied to a 4-segment SMA in MADYMO. This test is visualized in Figure
4.17. The simulation utilizing the Coulombic friction was determined to have qualitatively
different behavior than the stick-slip friction simulation – not insignificant rigid-body mo-
tion suggests that the former friction model cannot fully replicate the ‘anchoring’ function
of the contacts which is known to occur (cf. 4.2.1).

A subsequent simulation was also run with a significantly (4x) larger applied force, as
I expected transient impulsive forces experienced during undulatory motion to be poten-
tially quite large. This test is visualized in Figure 4.18, and it similarly revealed substan-
tially different behavior under load between the friction models. The simulation utilizing
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Fig. 4.17 Comparison of (a) rigid-body/Coulombic friction and (b) finite-element/stick-slip friction models of an SMA soft robot upon application of a modest lateral end force $F$ at time $t_a = 0$ and $t_b = 0.298$ s. Notice that the former experiences deflection at both the front ($\alpha$) and end ($\beta$) segments, which is indicative of rigid body motion; in the latter, the front deflection ($\dot{\alpha}$) is greatly reduced and rear deflection is nonexistent.

Coulombic friction experienced a complete collapse of the standing position – whereas the stick-slip simulation stayed planted. These observations, combined with those from the modest force simulation, suggested that the more numerically expedient Coulombic friction model was inappropriate to use in these simulations. More generally, it also suggests that any simulation attempting to model the undulatory motion of a robot where there are impulsive loads on a small subset of contact points needs to carefully consider the accurate modeling of friction for full verisimilitude.

Fig. 4.18 Comparison of (a) rigid-body/Coulombic friction and (b) finite-element/stick-slip friction models of an SMA soft robot upon application of a substantial lateral end force $F$ at time $t_a = 0$, $t_b = 0.150$ and $t_c = 0.298$ s.
Intrinsic curvature and inter-segment connection analysis

An iterative process was used to determine an initial size and orientation of the contact segments. It was quickly determined that smooth variations in contact curvature gave rise to the most controlled transitions between actuated and unactuated steady-state contact position, so all geometries considered were ellipsoidal in nature. Two orientations were seriously considered: ellipsoids with their semi-major axis positioned vertically (\(\alpha\)), inter-segment contact ellipsoids with the semi-major axis oriented horizontally (\(\hat{\alpha}\)), SMA segments with standard unactuated intrinsic curvature (\(\kappa_{UA}\)) and the ‘anchoring’ segment with reduced unactuated intrinsic curvature (\(\tilde{\kappa}_{UA}\)).

To help achieve the ‘anchoring’ effect that was repeatedly seen to be a component of successfully simulated motion, it was determined that the last segment needed to be more parallel to the ground when unactuated than the other segments. Specifically, \(\kappa_{UA}\) was brought closer to the value of \(\kappa_A\) via reduction in its value by 47% (cf. Figure 4.19). Modulation of the leading segment’s intrinsic curvature was correspondingly believed to be beneficial to the undulation of the SMA robot but the exact manner that is most beneficial has yet to be determined.

It was also discovered that the SMA segment-contact were most effective in enabling undulatory motion when they were much less (rotationally) stiff than the flexural rigidity of the SMA in the unactuated state: specifically, for the ‘anchor’ segment, a reduction of 50%, and for the front segment a further order of magnitude reduction from the ‘anchor’
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or an overall reduction of 95%. This has the effect of making both of the contact sections relatively easy to rotate and maximizes their contact area with the ground; for the front segment, this is particularly key, as a ‘mis-step’ can derail any subsequent motion. Notice, in Figure 4.20, that after one cycle (≈ 3.5 seconds) the SMA robot with the lower rotational stiffness at end contact points is in a better position to begin the next step than that with rotational stiffness equal to that of the SMAs themselves; by the third cycle (≈ 10 seconds) the benefit of this greater adherence has manifested itself in greater net motion of the former robot. For the intra-segment contact areas, it was not necessary to reduce the rotational stiffness from the unactuated value.

Motion pattern timing analysis

Initially, the motion pattern used by default when investigating the parameter field described in 4.3.2 and 4.3.2 was a scaled-down approximation of the binary simplification of the motion data extracted from the bio-inspiration seen in Figure 4.14; the scaling (by a factor of 10) was performed to reduce the time necessary to run each simulation whilst still utilizing the more sophisticated friction model. Once the motion pattern itself was to be investigated, this could no longer be the case, and the more numerically taxing full-scale pattern (and variations thereof) had to be implemented. A fairly direct adaptation of the bio-inspired motion and two variations were considered: (i) an approximation of the inspiration pattern contained in Figure 4.14, (ii) a relaxation of this pattern where the third segment stays actuated as the fourth did previously and where the fourth segment remains at rest in a modified unactuated position and (iii) a variation on this relaxation where the third segment’s lengthened actuation is dropped (cf. Figure 4.21 (i), (ii) and (iii)). Note that I have kept the modified rear-segment intrinsic curvature $\tilde{\kappa}_{UA}$ to make the patterns more

![Fig. 4.20](image.png)

(i) After one cycle (≈ 3.5 seconds) the SMA robot with the lower rotational stiffness at end contact points ($\alpha$) is in a better position to begin the next step than that ($\beta$) with rotational stiffness equal to that of the SMAs themselves: (ii) by the third cycle (≈ 10 seconds) the benefit of this greater adherence has manifested itself in greater net motion of $\alpha$. 

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Fig. 4.21 Illustration of actuation patterns for a four-segment SMA robot in MADYMO, with (i) an approximation of the inspiration pattern contained in Figure 4.14, (ii) a relaxation of this pattern where the third segment stays actuated as the fourth did previously and where the fourth segment remains at rest in a modified unactuated position and (iii) a variation on this relaxation where the third segment’s lengthened actuation is dropped.

directly comparable.

Fig. 4.22 Comparison of schemes (i), (ii) and (iii) after three cycles.

These particular alternate patterns were considered for a few reasons. First, it was readily apparent from the motion of the bio-inspiration (cf. Figure 4.14 - I note that the on/off pattern, whilst generated for the subject caterpillar’s legs, is used for the actuation of segments in the numerical model.) that the rearmost leg (A7) stays in the ‘actuated’ position for most of its duty cycle. Therefore, an approximation of the segment as continuously actuated would not be large change from that suggested by the physical caterpillar. Second, during the investigations in 4.3.2, it was determined that a reduction of the rearmost
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Fig. 4.23  Frictional forces applied to the robot contact points during MADYMO runs as a function of time for schemes (i), (ii) and (iii). A moving average of 256 ms has been applied to each force signature to allow easy comparison, as large and narrow impulsive peaks might otherwise hinder analysis. The units of the x axes are milliseconds (ms) and the y axes are newtons (N). Note that inter-segment 1 has particularly low values for these forces as it experiences significant lift off during undulation, which is a function of the intrinsic curvatures and rotational stiffnesses discussed above.

The three motion patterns considered here, in addition to many others that are not presented for the sake of brevity, do not result in undulation that is terribly smooth. Rather, motion occurs over a series of leaps and bounds, with large and spiky loadings at the contact areas (both normally – that is, elastically – and frictionally) evidenced by significant jumps in the force and energy signatures, some of which are recorded here. The force terms, in particular (cf. Figure 4.23) were somewhat difficult to interpret without some sort of averaging filter (here a 256 ms smoothing filter was used) to distinguish between the schemes. The rapid onset of the actuation, resultant forces and ensuing motion of the robot’s constituent parts attests to the essentially impulsive nature of the underlying undu-
CHAPTER 4. DYNAMICAL ANALYSIS AND DEVELOPMENT OF A BIOLOGICALLY INSPIRED SMA CATERPILLAR ROBOT

![Graph](image)

Fig. 4.24 Work applied to overall robot by external loading during MADYMO runs as a function of time for schemes (i), (ii) and (iii). The units of the x axes are milliseconds (ms) and the y axes are joules (J).

In terms of the distance progressed in one cycle of undulatory motion, the pattern heretofore called (i) – that is, the pattern most directly adapted from the *Manduca* motion that retains its rearmost segment actuation – was the least successful (cf. Figure 4.22). Energetically, it was very similar to scheme (iii) (cf. Figure 4.24), with only a 1% penalty over three cycles for the additional actuation of segment 4, although I reiterate this is so low partially due to the retention of \( \tilde{k}_{UA} \). Differences between schemes (ii) and (iii) were more nuanced, in terms of forces and motion, and less nuanced in terms of energetics. Consulting Figure 4.22, it is apparent that scheme (iii) has a slight advantage over (ii) of approximately (depending on which point of the robot is used as the baseline) 0.5 cm, which is about 25% of the overall motion accomplished during three actuation cycles of scheme (iii) (which was \( \sim 2 \) cm over three cycles – about half the value of the caterpillar, but I note that the actuation can easily be sped up). The most differentiating feature of the frictional force signatures at the robot’s contact points (cf. Figure 4.23) are present at the rearmost contact End 2. I note that both schemes (ii) and (iii) experience significantly (\( \sim 200\% \)) higher forces at this point, buttressing the importance of the ‘anchoring’ role discussed earlier. Schemes (ii) and (iii) are also slightly different from each other here, as scheme (iii) reaches lower minimum values (\( \sim 130\% \)) than (ii), which indicates that it is better at anchoring in both directions than (ii). Such, I hypothesize, is how it undergoes the most motion per cycle. Energetically, (ii) is worse than (iii) – by a factor of \( \sim 14\% \) after three cycles – and this can be squarely laid at the feet of the nearly doubled actuation period per cycle of segment 3, which can be seen through examination of Figure 4.25. The same figure also reveals the penalty paid from scheme (i)’s modulation of segment 4 is, as predicted, rather low, again
due to the reduction of $\kappa_{UA}$ to $\tilde{\kappa}_{UA}$ – such would likely not be the case if it was the same as the other segments in the unactuated regime.

**Fig. 4.25** Work applied to SMA segments by external loading during MADYMO runs as a function of time for schemes (i), (ii) and (iii). The units of the x axis are milliseconds (ms) and the y axis are joules (J).

I note that, broadly comparing Figures 4.24 and 4.25 to Figure 4.9, the power required to provoke undulatory motion is similar in magnitude and trend for both the SMA robot considered here and the elastica. Although the time history from the discretized simulation is notably notchier (due to the incorporation of frictional contact and the binary nature of the actuation) than the elastica, it suggests that the continua approximation, even lacking ground interactions, can lead to ballpark of energetics required to move sequentially organized crawlers, a surprising result.

### 4.4 Conclusions

I have sought to suggest a general analysis path for the rapid prototyping of a soft robot through the specific analysis of a shape-memory-alloy (SMA) based, caterpillar inspired soft robot. I have introduced a theoretical framework for analyzing a caterpillar’s crawling motion based on the mechanics of a rod, where the specific models of the string and elastica have been utilized. The parameters of the mechanical model were adjusted to match the crawling motion recorded in the works of extant biological studies, and resultant kinematical data were also used as a baseline for the design of a bio-inspired soft robot made out of multiple connected SMA actuators. A simple experiment was formulated to gain an approximation of the intrinsic curvature and flexural rigidity of the SMA actuator.
sample through optical characterization during its on and off state. Additionally, a rapid-prototyping numerical model of the connected soft robot design was constructed by means of a collection of rigid bodies connected by torsional springs, whose stiffness is defined from characterization of the SMA sample. Calculations were performed in order to determine the correct contact characteristics that enable the locomotion, using the locomotion pattern converted from the leg velocity data from the recorded caterpillar motion. The energy cost of actuating the connected, individually actuated soft robot design as a whole is roughly equivalent to the energy cost of the theoretical approximation which is an undulating single rod made out of elastic material. These results seem like a convincing start to expand the elastic rod framework to allow modeling and study of individual SMA actuators and their collective undulatory behavior in the future, in tandem with the numerical approach I have described here.
Chapter 5

Closing Comments

In this dissertation I have presented in-depth research was undertaken on three dynamical systems, which I term ‘case-studies’, whose most interesting dynamics were largely precipitated by abbreviated periods where impulsive loading dominated.

In the first case-study I concerned myself with an impulsively-loaded wave energy converter (WEC) for which mass modulation schemes have been proposed; the goal of the schemes is to improve the energy harvesting capabilities of these devices by taking advantage of the ambient water. In this case-study, experimental results for a pair of passively and impulsively initiated schemes were presented and one of them was shown to be effective in increasing the energy harvesting potential of a WEC; numerical analysis of the model also showed the potential benefits of the mass-modulation scheme and, moreover, validated the benefits of harnessing impulsively applied fluid pressures which are often neglected in the design of a WEC. I wish to emphasize that the results shown in this case-study verified that the impulsive mass modulation scheme is effective; furthermore, the modulation scheme proposed can be easily generalized - all that is required is to change the hydrodynamic added mass (with or without trapping water) of the inner float. Broadly, one can achieve mass modulation using induced mass (virtual mass) by changing the shape of the inner float cyclically. The flaps in these experiments are but one realization of this idea. It is not too difficult to conceive of other realizations - some of which can be achieved using active control.

In the second case-study, I examined the accidental untying of a shoelace while walking, which occurs often and typically with minimal prior warning. In this case-study, I discussed the series of events that lead to a shoelace knot becoming untied. As demonstrated using slow-motion video footage and a series of experiments, the failure of the knot happens in a matter of seconds, often without warning, and is catastrophic. The controlled experiments showed that increasing inertial effects of the swinging laces leads to increased rate of knot untying, that the directions of the impact and swing influence the rate of failure, and that the knot structure has a profound influence on a knot’s tendency to untie under cyclic impact loading. The work presented is far from exhaustive. For one, the influence of the shoelace material and surface finish was not investigated. In addition, the metric used
CHAPTER 5. CLOSING COMMENTS

to understand knot slip rate in this chapter (measurements of free end length before and after set testing time or failure) is insufficient to elucidate finer details of knot failure progression from gradual loosening to a regime of fast failure. To do so, future tests should fail each knot, measuring free-end lengths at set intervals during testing. Such measurements will better distinguish the two regimes of knot failure I have identified. I expect that such measurements and further testing will also lead to better understanding of the mechanical factors that cause the inferiority of the weak knot compared to the strong knot.

The final case-study concerned the ground-up development of prototyping techniques for a soft-robot whose motion was modeled after that of the common caterpillar. Like the other two case-studies, impulsive loading (here experienced during brief periods of robot actuation) is the primary motivator – in this case, of undulatory motion. Through analysis of a specific soft-robot, I sought to suggest an analysis path for rapid prototyping of a SMA based, caterpillar inspired soft-robot to undergo undulatory motion. My approach was two-pronged: analysis of the kinematics and dynamics of the caterpillar are structured through simple, string and elastica based models which yield estimates of forces and energetics that would be extremely difficult to determine directly, in addition to suggestions for open-loop motion patterns. Simultaneously, simple experiments and optical tracking of SMA segments were performed to yield properties input directly into a robust numerical solver used to simulate a prototype soft-robot’s undulatory motion and reveal facets leading to the success or failure of its undulatory motion. The motion patterns considered therein did not result in undulation that is terribly smooth. Rather, motion occurred over a series of leaps and bounds, with large and spiky loadings at the contact areas evidenced by significant jumps in the force and energy signatures. The rapid onset of the actuation, resultant forces and ensuing motion of the robot’s constituent parts attests to the essentially impulsive nature of the underlying undulation here, and the degree to which small changes in the timing of said impulses greatly effect the chance of undulation’s success.
References


