Analysis for a Dislocation (Screw/Edge) Accelerating through the Shear-wave Speed

A Dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Engineering Sciences (Applied Mechanics)

by

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2007
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Chair

University of California, San Diego

2007
DEDICATION

To my son Eric, my husband Quan Gu, and my parents Zupei and Meiying.
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Chapter 4, Chapter 5 and Chapter 6, in part or in full, is being prepared to submit for publication. Prof. Xanthippi Markenscoff is the co-author of these papers.
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ABSTRACT OF THE DISSERTATION

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Supersonic motion of dislocation in solids is a topic of hot current research endeavor. The approach is mostly based on molecular dynamics and computer simulation. There is concurring evidence that dislocations cross the shear wave speed barrier, in particular under shock loading condition. Here the analytic solution is presented for a dislocation, both screw and edge, accelerating through the shear-wave speed barrier.
The analysis is based on solution for general motion of a Volterra dislocation, obtained and evaluated at the instant when the velocity of the dislocation is equal to the shear-wave speed, but acceleration is present. At this transition, the Mach wave cone is starting to form and the roots of the function that defines the interval of the dislocation motion the wavelets from which contribute to the wave-front change from complex conjugate to real, and the coefficient of the delta function of the stress at the forming Mach wave cone is logarithmic-over-square-root singular. While the step discontinuity of the displacement of a Volterra dislocation is too strong of a dislocation model for the crystal dislocation, the solution is useful because it is the kernel for a variable core model, which removes the singularity as shown here. Moreover, except in the neighborhood of the core, the analytic solution can provide comparisons for the molecular dynamics simulation solutions.
1

INTRODUCTION

Dislocation motion plays an important role in high strain rate deformation in metals under shock wave loading (e.g. Holian and Lomdahl (1998), M.A. Meyers et al (2003), E. Bringa et al (2004)), but despite a surge of recent activity (Gumbsch and Gao (1999), Li and Shi (2002), Olmsted et al (2005), Sharma and Zhang (2006), Mordehai and Kelson (2006), Marian and Caro, 2006), dislocation motion near or at supersonic speeds (exceeding the shear wave speed) is still not well understood.

The motion of a dislocation in a crystal is controlled by two physical phenomena of dissipation of energy: first, damping by scattering of elementary excitations in the lattice, which is modeled by a viscous law; and second, radiation of waves excited by the dislocation as it moves through the lattice. This thesis focuses on the latter mechanism and analyzes the radiation of waves emitted by a dislocation as it accelerates through the shear-wave speed.

Steady-state constant velocity motion of dislocations has been studied starting by Frank (1949) and Eshelby (1949, 1953, 1956, 1962, see also Markenscoff and Gupta (2006)) and it exhibits the “relativistic effect” as the shear-wave speed is reached (e.g. Hirth and Lothe (1982)), for the standard solution in an isotropic medium). However, supersonic dislocation motion (exceeding the shear-wave speed) is also a problem attacked by Eshelby (1956) who stated that supersonic motion is a “formal possibility”. Weertman (1963,1980) also provided analysis for supersonic motion and gave expression
for the energies needed to sustain such motion. Recent results by molecular dynamics simulation (Gumbsch and Gao (1999)) show that supersonic motion is possible if generated as such, while results by Olmsted et al (2005), Li and Shi (2002) show that a dislocation can accelerate through the shear-wave speed barrier, at which point it disintegrates into partials, and possibly re-integrates after that, although as stated by the authors, “the simulation results cannot be trusted after partial separation” (Olmsted et al, 2005).

From the theoretical physics point of view, the gauge field (Kadic and Edelen (1983), Edelen and Lagoudas (1988), Raifeartaigh (1997)) approach to the problem removes the singularity at the core of the moving dislocation at supersonic speed as well (Sharma and Zhang, 2006), the approach in essence being reminiscent of non-local elasticity. Evidence of supersonic motion of dislocation appears also in earthquakes (Dunham and Archuleta (2004), Xia et al (2005)), so the present analysis may have applicability to these phenomena as well.

From the lattice dynamic point of view, in a discrete lattice which is a dispersive medium in which the full spectrum of phase velocities exists, the velocity of the dislocation is always supersonic relatively to some lattice phase velocity at some particular wave-length, and subsonic relatively to others. Starting from the classical paper of Celli and Flytzanis (1970), there has been an extensive literature (see the review article by Weertman and Weertman (1980), with over 150 references), but all models involve approximations, and ab initio calculations are needed in order to give a definitive answer to the problem of supersonic motion.
In the review article of Weertman and Weertman (1980), the existence of a “Lorentz” force on the dislocation is addressed, with the conclusion from papers by Nabarro(1967), Stroh(1962) and Lothe(1960) that the analogue of the Lorentz force of electromagnetism does not exist for a moving dislocation, because the electromagnetic analogy is imperfect (Weertman and Weertman (1980)).

In this thesis the problem of a Volterra dislocation, both screw and edge, accelerating through the shear-wave speed \( c_2 \) is solved analytically. Chapter 2 and 3 contain the detailed analysis of a general accelerating motion, both subsonic and supersonic, for a screw and edge Volterra dislocation. By “supersonic” we mean also here what is referred to as “transonic” or “intersonic”, which is between the shear wave velocity \( c_2 \) and the longitudinal velocity \( c_1 \). The analysis is performed by considering the boundary-value problem of a displacement discontinuity at a half space moving non-uniformly, and applying Laplace transforms in space and time. The inversion is performed by the Cagniard-de Hoop contour, and the singularities are treated carefully, so all expressions are meaningful integrals. The motion of the dislocation jumping from rest to constant velocity, subsonic or supersonic, is treated as a special case, because the closed form solutions obtained exhibit the basic physical characteristics of the motion.

In Chapter 4 the geometry of the Mach wave fronts for a dislocation accelerating from the subsonic to the supersonic range (exceeding \( c_2 \)) is described by equations that are solved numerically. The analytical expression that relates the acceleration of the dislocation to the curvature of the wave-front is also provided. Such expression may be useful in interpreting wave-front shapes such as those of Gumbsch & Gao (1999) and
Koizumi et al (2002). The evolution of the wave-fronts in the transition from subsonic to supersonic range is provided numerically, similarly to a sonic accelerating through the sound speed in a fluid (Kaouri et al (2006), also Kaouri, Ph.D thesis, University of Oxford (2004)).

In Chapter 5, the key mathematical analysis of the formation of the Mach wave fronts in the transition from subsonic into supersonic, when there is a change of the roots of the function $f(\xi) = t - \eta(\xi) - b\sqrt{(x-\xi)^2 + z^2}$ (that defines the interval of the integration) from complex conjugates into real, is provided. The solution is then evaluated at this transition point, near a maximum $\xi^*$ of $f(\xi)$, which is now in the range of the integration. As a result, the coefficient of the delta function at the forming Mach cone is expressed in terms of the acceleration of the dislocation at the instant when the velocity of the dislocation is equal to the shear wave speed, by carrying the series expansion of the motion $\eta(\xi)$ to the next order:

$$\eta(\xi) = \eta(\xi^*) + \eta'(\xi^*)(\xi - \xi^*) + \frac{1}{2} \eta''(\xi^*)(\xi - \xi^*)^2 + o(\xi - \xi^*)^2.$$ 

The resulting coefficient of the delta function of the forming Mach wave-front at the instant of acceleration through the shear-wave speed is found to contain a logarithmic over square root singularity. We interpret this as meaning that the Volterra model (step function discontinuity in the displacement) is too strong (too singular) of a model for a crystal dislocation. Thus, a smoothing of the core is necessary in order to smooth-out the singularity, and a ramp-core model will be used and the solution for a supersonic ramp-core dislocation is provided in Chapter 6. The stress coefficient of the delta function at
the Mach wave-front now is physically acceptable since the resulting expression contains no singularity.

The motion of a dislocation decelerating from a supersonic velocity, such as \( l(t) = \frac{1}{2}at^2 \) can be studied by analogous treatment (see Freund (1972) for decelerating loads).
THE TRANSIENT MOTION OF A NONUNIFORMLY MOVING SCREW DISLOCATION

2.1 Governing equation for general motion of screw dislocation

Consider a screw dislocation parallel to the y-axis with the line direction and Burgers vector shown in Figure 2.1.

The dislocation is at rest in an infinite space in a linear elastic material until time $t = 0$ when it begins to move according to $x = l(t)$ or equivalently $t = \eta(x)$. The problem is equivalent to a boundary value problem in a half space $z > 0$. The equilibrium equation is:
\[
\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \mu \kappa^2 u_i + (\lambda + \mu) \frac{\partial}{\partial x_i} \left( \frac{\partial u_j}{\partial x_j} \right) \tag{2.1}
\]

Since there is only one displacement \( u_y \), namely \( \underline{u} = (0, u_y, 0) \) and \( u_y = u_y(x, z, t) \), the governing equation can be written as:

\[
\rho_0 \frac{\partial^2 u_y}{\partial t^2} = \mu (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}) u_y. \tag{2.2}
\]

Let \( b \) denote the shear wave slowness \( \frac{1}{c} = \sqrt{\frac{\rho_0}{\mu}} \), where \( c \) is the shear wave speed, then

\[
\left( \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2} \right) = b^2 \frac{\partial^2 u_y}{\partial t^2} \tag{2.3}
\]

For a Volterra dislocation, the boundary conditions at \( z = 0 \) are

\[
\begin{align*}
    u_y(x, 0, t) &= \frac{1}{2} \Delta u H(x) & & t < 0 \\
    u_y(x, 0, t) &= \frac{1}{2} \Delta u H(x - l(t)) & & t \geq 0 \\
    \sigma_{zz}(x, 0, t) &= 0
\end{align*} \tag{2.4}
\]

The displacements are step functions on the boundary \( z = 0 \) for Volterra dislocation. Using the above basic equation and boundary conditions, we will obtain the stress analysis for this problem.

The solution to the problem is equivalent to the superposition of the solutions of the following two problems satisfying the same differential equation and corresponding boundary conditions:

Problem I: \( u_y(x, 0, t) = \frac{1}{2} \Delta u H(x) \quad \text{for all } t \) \quad \tag{2.5}
Problem II: \( u_y(x,0,t) = \frac{1}{2} [\Delta u H(x-l(t)) - \Delta u H(x)] \) for \( t \geq 0 \). (2.6)

The solution to Problem I is known as:

\[
\frac{\partial u_y(x,z,t)}{\partial z} = \frac{\Delta u}{2\pi} \frac{x - z}{x^2 + z^2}
\] (2.7)

The solution to Problem II is obtained in the sequel.

### 2.2 Analytical solution for a screw dislocation starting from rest and moving with constant velocity

Because the solution to a dislocation starting from rest and moving with constant velocity will be needed in the treatment of the “singularities” of the general motion, it is treated first. The method of solution involves Laplace transforms in time and two-sided Laplace transform in space with the inversion carried out by means of the Cagniard-de Hoop technique.

For problem II, first applying the Laplace transform with respect to time (here \( u = u_y \))

\[
\hat{u}(x,z,s) = \int_0^\infty u(x,z,t) e^{-st} dt
\] (2.8)

The field equation and boundary conditions will become respectively:

\[
\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial z^2} = b^2 s^2 \hat{u}
\] (2.9)

B.C.: \( \hat{u}(x,0,s) = \frac{\Delta u}{2s} \left( 1 - e^{-s\sigma(x)} - H(x) \right) \)

Applying next the two-sided Laplace transform
\[ U(\lambda, z, s) = \int_{-\infty}^{\infty} \hat{u}(x, z, s)e^{-sz\lambda} \, dx \]  

(2.10)

The field equation and boundary conditions will become respectively:

\[ \frac{\partial^2 U}{\partial z^2} - (b^2 - \lambda^2) s^2 U = 0 \]  

(2.11)

B.C.: \( U(\lambda, 0, s) = \frac{\Delta u}{2} \int_{-\infty}^{\infty} \left(1 - e^{-\eta(x)} - H(x)\right) \frac{1}{s} e^{-sx\lambda} \, dx = -\frac{\Delta u}{2s} \int_{0}^{\infty} e^{-\eta(x)} e^{-sx\lambda} \, dx \)

From equation (2.11), we obtain

\[ U(\lambda, z, s) = A e^{b\lambda z} + B e^{-b\lambda z} \]  

(2.12)

Apply the corresponding boundary condition, then

\[ U(\lambda, z, s) = -\frac{\Delta u}{2s} \int_{0}^{\infty} e^{-\eta(\xi)} e^{-sx\lambda} d\xi \cdot e^{-b\lambda z} \]  

(2.13)

where \( \beta = \sqrt{b^2 - \lambda^2} \) with \( \text{Re} \, \beta > 0 \) for admissible behavior at infinity:

\[ U(\lambda, z, s) \bigg|_{z=\pm\infty} = 0. \]  

(2.14)

The inverse is performed according to the Cagniard-de Hoop technique, which modifies the path of integration in the plane to arcs of hyperbolas so that inversion in \( s-t \) is performed by inspection.

So for a dislocation starting from rest and jumping to a constant velocity \( \frac{1}{\alpha} \), namely

\[ \eta(\xi) = \alpha \xi \]  

(2.15)

Substituting equation (2.15) into equation (2.13), and taking the partial derivative with respect to \( z \), we obtain
We invert next the two-sided Laplace transform \( x - \lambda \):

\[
\frac{\partial \hat{u}}{\partial z} = \int_{Br} \frac{\partial U}{\partial z} \frac{s}{2\pi i} e^{s\lambda} d\lambda = \frac{\Delta u}{2} \frac{1}{2\pi i} \int_{Br} \frac{\beta}{(\lambda + \alpha)} e^{-s(\lambda x + \beta z)} d\lambda
\]  

(2.17)

where \( B_r \) denotes the Bronwich contour.

The Cagniard-de Hoop technique allows performing the inversion of the transform in \( s - t \) by inspection by modifying the path of integration in the \( \lambda - \)plane to arcs of hyperbolas:

Let \( \tau = -\lambda x + \beta z = \left(b^2 - \lambda^2\right)^{\frac{1}{2}} z - \lambda x \)  

(2.18)

where \( \tau \) is a real. \( \tau = \text{Re}(\lambda x + \beta z) > 0, \text{Im}(\lambda x + \beta z) = 0 \), so that

\[
\lambda_{\pm} = r_0^{-2} [-\alpha \pm iz\sqrt{\tau^2 - r_0^2 b^2}]
\]  

(2.19)

\[
\beta(\lambda_{\pm}) = r_0^{-2} [\tau \pm iz\sqrt{\tau^2 - r_0^2 b^2}]
\]  

(2.20)

with \( r_0 = \left[ x^2 + z^2 \right]^\frac{1}{2} \).

To make \( \lambda_{\pm} \) imaginary, we obtain

\[
\tau \geq r_0 b
\]  

(2.21)

\[
\text{Re} \lambda = \frac{-\alpha}{r_0^2} \leq -\frac{bx}{r_0}
\]  

(2.22)
Apply Cauchy’s theorem, since there is no singularity inside the contour ABCD, so

$$\int = \int - \int \quad \text{with} \quad R \to \infty$$

![Diagram of contour ABCD with annotations](image)

Figure 2.2 Cagniard-de Hoop technique for motion with constant velocity

So from equation (2.17), we obtain

$$\frac{\partial \hat{u}}{\partial z} = \frac{\Delta u}{2 \, 2\pi i} \left[ \int_{\frac{bx}{r_0}}^{\frac{bx}{r_0}} \beta(\lambda_+) e^{-i(\lambda_+ + \alpha \tau)} d\lambda_+ - \int_{\frac{bx}{r_0}}^{\frac{bx}{r_0}} \beta(\lambda_-) e^{-i(\lambda_- + \alpha \tau)} d\lambda_- \right]$$

$$= \frac{\Delta u}{2 \, 2\pi i} \left[ \int_{\frac{bx}{r_0}}^{\frac{bx}{r_0}} \beta(\lambda_+) \frac{\partial \lambda_+}{\partial \tau} e^{-i\tau} d\tau - \int_{\frac{bx}{r_0}}^{\frac{bx}{r_0}} \beta(\lambda_-) \frac{\partial \lambda_-}{\partial \tau} e^{-i\tau} d\tau \right]$$

(2.23)

Since $\lambda_+$ and $\lambda_-$ are conjugate to each other, So
\[
\frac{\beta(\lambda_+)}{\lambda_+ + \alpha} \frac{\partial \lambda_+}{\partial \tau} - \frac{\beta(\lambda_-)}{\lambda_- + \alpha} \frac{\partial \lambda_-}{\partial \tau} = 2i \cdot \text{Im} \left( \frac{\beta(\lambda_+)}{\lambda_+ + \alpha} \frac{\partial \lambda_+}{\partial \tau} \right)
\] (2.24)

Substitute equation (2.24) into equation (2.23), we obtain

\[
\frac{\partial \hat{u}}{\partial z}(x, z, s) = \frac{\Delta u}{2\pi} \int \text{Im} \left[ \frac{\beta(\lambda_+)}{\lambda_+ + \alpha} \frac{\partial \lambda_+}{\partial \tau} \right] e^{-is\tau} d\tau
\] (2.25)

In equation (2.25), perform inversion in $s - t$ by inspection:

\[
\frac{\partial u}{\partial z}(x, z, t) = \frac{\Delta u}{2\pi} H(t - r_0) \text{Im} \left[ \frac{\beta(\lambda_+)}{\lambda_+ + \alpha} \frac{\partial \lambda_+}{\partial \tau} \right]_{s=t}
\] (2.26)

From the expression for $\beta$ and $\lambda$ (equations (2.19) and (2.20)), we obtain

\[
\left[ \frac{\beta(\lambda_+)}{\lambda_+ + \alpha} \frac{\partial \lambda_+}{\partial \tau} \right]_{s=t} = \frac{r_0^{-2} \left( tz + ix \sqrt{t^2 - r_0^2 b^2} \right) }{\alpha + \frac{-tx + iz \sqrt{t^2 - r_0^2 b^2}}{r_0^2}} \partial t
\]

\[
= \frac{r_0^{-2} \left( iz + x \sqrt{t^2 - r_0^2 b^2} \right) }{\left( \alpha - tx r_0^{-2} \right)^2 + \frac{z^2 (t^2 - r_0^2 b^2)}{r_0^4}} \left( \alpha - tx r_0^{-2} - \frac{iz \sqrt{t^2 - r_0^2 b^2}}{r_0^2} \right) \left[ r_0^{-2} \left( -x + iz (t^2 - r_0^2 b^2)^{1/2} \right) \right]
\]

\[
= \frac{r_0^{-4} \left( -2txz + i \frac{z^2 t^2}{\sqrt{t^2 - r_0^2 b^2}} - \frac{ix^2 \sqrt{t^2 - r_0^2 b^2}}{r_0^2} \right) }{\left( \alpha - tx r_0^{-2} \right)^2 + \frac{z^2 (t^2 - r_0^2 b^2)}{r_0^4}} \left( \alpha - tx r_0^{-2} - \frac{iz \sqrt{t^2 - r_0^2 b^2}}{r_0^2} \right)
\] (2.27)

Substitute equation (2.27) into the previous expression for $\frac{\partial u}{\partial z}(x, z, t)$ (equation (2.26)), we obtain
\[ \frac{\partial u}{\partial z}(x, z, t) = \frac{\Delta u}{2\pi} H(t - r_0b) \text{Im} \left[ \frac{\beta(\lambda_+) \partial \lambda_+}{\lambda_+ + \alpha \partial t} \right] = \]

\[ \frac{\Delta u}{2\pi} H(t - r_0b) \frac{r_0^{-4} \left( \frac{z^2 t^2}{\sqrt{t^2 - r_0^2 b^2}} - x^2 \sqrt{t^2 - r_0^2 b^2} \right) (\alpha - txr_0^{-2}) + r_0^{-4} 2tz^2 x \sqrt{t^2 - r_0^2 b^2}}{r_0^2} \left( \alpha - txr_0^{-2} \right)^2 + \frac{z^2 (t^2 - r_0^2 b^2)}{r_0^4} \]

\[ = \begin{cases} 
\frac{\Delta u}{2\pi} H(t - r_0b) \frac{b^2 - t^2 x^2}{r_0^4} + \frac{z^2}{r_0^4} (t^2 - r_0^2 b^2) \left( \alpha - txr_0^{-2} \right)^2 + \frac{2tz^2 x (t^2 - r_0^2 b^2)}{r_0^6} \sqrt{t^2 - r_0^2 b^2} & (\text{for } t \geq r_0 b) \\
0 & (\text{for } t < r_0 b) 
\end{cases} \]

(2.28)

This is the solution for problem II (equation (2.6)).

Plus the solution for problem I (equation (2.5)) which is

\[ \frac{\partial u}{\partial z} = \frac{\Delta u}{2\pi} \frac{\partial}{\partial x} \left( x^2 + z^2 \right) \] (equation (2.7)), we can obtain the solution for the problem which is obtained as (Markenscoff 1980):

\[ \frac{\partial u}{\partial z}(x, z, t) = \frac{\Delta u}{2\pi} \begin{cases} 
\frac{-x}{x^2 + z^2} & (\text{for } t < rb) \\
\frac{-x}{x^2 + z^2} + \frac{b^2 - t^2 x^2}{r_0^4} + \frac{z^2}{r_0^4} (t^2 - r_0^2 b^2) \left( \alpha - txr_0^{-2} \right)^2 + \frac{2tx^2 (t^2 - r_0^2 b^2)}{r_0^6} \sqrt{t^2 - r_0^2 b^2} & (\text{for } t \geq rb) \\
\frac{-x}{x^2 + z^2} + \frac{(t^2 - r_0^2 b^2) \left( \alpha - txr_0^{-2} \right)^2 + \frac{z^2 (t^2 - r_0^2 b^2)}{r_0^4}}{x^2 + z^2} \end{cases} \]

(2.29)
Evaluate the solution for the whole problem, namely equation (2.29) at \( z = 0 \):

Since \( r^2_0 = x^2 \) at \( z = 0 \) and \( \frac{1}{x^2} = \frac{1}{|x|} \sqrt{\frac{1}{x^2}} \), so

\[
\frac{\partial u}{\partial z}(x,0,t) = \frac{\Delta u}{2\pi} \left\{ \begin{array}{ll}
-\frac{1}{x} & t < b|x| \\
\frac{\sqrt{t^2/x^2 - b^2}}{\alpha - \frac{t}{x}} \left[ 1 - \frac{1}{x} \right] & t \geq b|x|
\end{array} \right. 
\]

(2.30)

We note that at the origin, \( x \to 0 \pm \), to a first-order approximation:

\[
\lim_{x \to 0^\pm} \left\{ -\frac{\sqrt{t^2/x^2 - b^2}}{\alpha - \frac{t}{x}} \left[ 1 - \frac{1}{x} \right] \right\} = \lim_{x \to 0^\pm} \left( \frac{\sqrt{1 - b^2 x^2}}{t} \left( 1 + \frac{\alpha x}{t} \right) - 1 \right)
\]

\[
= \frac{1}{x} + \frac{\alpha}{t} - \frac{1}{x} = \frac{\alpha}{t}
\]

(2.31)

so

\[
\lim_{x \to 0^\pm} \frac{\partial u}{\partial z}(x,0,t) = \frac{\Delta u}{2\pi} \frac{\alpha}{t}.
\]

(2.32)

2.3 Analytical solution for a screw dislocation starting from rest and moving non-uniformly
We will now analyze the problem for a general dislocation with motion \( x = l(t) \) (or \( t = \eta(x) \)). From the previous results (equation (2.13)):

\[
U(\lambda, z, s) = -\frac{\Delta u}{2s} \int_0^\infty e^{-\eta(\xi)} e^{-\lambda \xi} d\xi \cdot e^{-\lambda s}
\]

and for the strain of interest:

\[
\frac{\partial U}{\partial z}(\lambda, z, s) = \frac{\Delta u}{2} \beta e^{-\lambda s} \int_0^\infty e^{-\lambda(\eta(\xi) + \beta \xi)} d\xi
\]  

(2.33)

We invert the two-sided Laplace transform \( x - \lambda \):

\[
\frac{\partial \hat{u}}{\partial z}(x, z, s) = \frac{\Delta u}{2} s \beta e^{-s(\eta(\xi) + \lambda(\xi - s))} \int_{Br} d\xi d\lambda
\]  

(2.34)

2.3.1 Solution at \( z = 0 \).

Since the solution for a non-uniformly moving dislocation at \( z = 0 \) contains singularities not present for \( z \neq 0 \) and more often the stresses on the slip plane \( z = 0 \) are of interest, the \( z = 0 \) case will be considered first.

From equation (2.15), at \( z = 0 \)

\[
\frac{\partial \hat{u}}{\partial z}(x, 0, s) = \frac{\Delta u}{2} \frac{s}{2\pi} \beta e^{-s(\eta(\xi) + \lambda(\xi - s))} \int_{Br} d\xi d\lambda
\]

\[
= \frac{\Delta u}{2} \frac{s}{2\pi} \beta e^{-s(\eta(\xi) + \lambda(\xi - x))} \int_{Br} d\lambda d\xi
\]  

(2.35)

where \( Br \) denotes the Bronwich contour.

The Cagniard-de Hoop technique is used to perform the inversion of the transform in \( s - t \) by inspection by modifying the path of integration in the \( \lambda \)–plane to arcs of hyperbolas:
Let $\tau = \lambda (\xi - x)$ \hfill (2.36)

where $\tau$ is a real. $\tau = \text{Re}(\lambda (\xi - x)) > 0$, $\text{Im}(\lambda (\xi - x)) = 0$, since $(\xi - x)$ is real, so $\lambda$ has to be real. Since $\beta = \left( b^2 - \lambda^2 \right)^{\frac{1}{2}}$, so

When $\text{Re} \lambda = \lambda < b$, $\beta$ has no singularity.

When $\text{Re} \lambda = \lambda > b$, $\beta = \pm i \left( \lambda^2 - b^2 \right)^{\frac{1}{2}}$ has branch points.

Let $\lambda = \text{Re} \lambda + i \varepsilon$, we have

\[
\beta = \pm i \left( (\text{Re} \lambda + i \varepsilon)^2 - b^2 \right)^{\frac{1}{2}}
\]

\[
= \pm i \left( (\text{Re} \lambda)^2 - \varepsilon^2 + 2i \varepsilon \text{Re} \lambda - b^2 \right)^{\frac{1}{2}}
\]

Since $\text{Re} \beta > 0$, so when $\varepsilon \to 0$,

\[
\beta \to \pm i \left( (\text{Re} \lambda)^2 + 2i \varepsilon \text{Re} \lambda - b^2 \right)^{\frac{1}{2}} = \pm e^{i \varepsilon} \left( L \cdot e^{i \varepsilon} \right)^{\frac{1}{2}}
\]

\[
= \pm \left( L^{\frac{1}{2}} \cdot e^{i \frac{\varepsilon}{2} \frac{\pi}{2}} \right)
\]

\[
= \begin{cases} 
+ L^{\frac{1}{2}} \cdot e^{i \frac{\varepsilon}{2} \frac{\pi}{2}} = i (\lambda^2 - b^2)^{\frac{1}{2}} = \beta + \text{ when } \varepsilon, \varepsilon < 0 \\
- L^{\frac{1}{2}} \cdot e^{-i \frac{\varepsilon}{2} \frac{\pi}{2}} = -i (\lambda^2 - b^2)^{\frac{1}{2}} = \beta - \text{ when } \varepsilon, \varepsilon > 0
\end{cases}
\]

(2.37)

where $L = \left( (\text{Re} \lambda)^2 - b^2 \right)^{\frac{1}{2}} + 4 (\text{Re} \lambda)^2 \varepsilon^2$, $\tan \varepsilon = \frac{2 \varepsilon \text{Re} \lambda}{(\text{Re} \lambda)^2 - b^2}$. 
Apply Cauchy’s theorem, since there is no singularity inside the contour, so

\[ \int = \int_{B_r} - \int_{B_+} \text{ with } R' \to \infty, \varepsilon \to 0. \]

![Diagram of Cagniard-de Hoop technique for general motion at z = 0](image)

**Figure 2.3** Cagniard-de Hoop technique for general motion at \( z = 0 \)

The above figure only shows the \( \xi - x < 0 \) (namely \( \text{Re} \lambda < 0 \)) case. For \( \xi - x > 0 \), the result will be similar:

Since we have \( r = \sqrt{(x-\xi)^2 + z^2} = |x-\xi|, \lambda_+ = \lambda_- = \frac{\tau}{(\xi-x)} = \frac{r}{\tau} \) on \( z = 0 \). So

\[ \tau = \lambda r. \]

When \( R' \to \infty, \varepsilon \to 0 \), on \( B_+ \), \( \text{Re} \lambda = (b, +\infty), \tau = (br, +\infty) \).
\[ \text{Im} \, \lambda = +\varepsilon \rightarrow 0 , \quad \beta = -i(\lambda^2 - b^2)^{1/2} = -i \left( \frac{\tau^2}{r^2} - b^2 \right)^{1/2} \]

on \( B_+ \), \( \text{Re} \, \lambda = (b, +\infty) \), \( \tau = (br, +\infty) \).

\[ \text{Im} \, \lambda = -\varepsilon \rightarrow 0 , \quad \beta = i(\lambda^2 - b^2)^{1/2} = i \left( \frac{\tau^2}{r^2} - b^2 \right)^{1/2} . \]

The integrations on \( B_+, B_- \) can be written as:

\[ \int \beta(\lambda) e^{-i\lambda (\xi - x)} d\lambda = \int_{br}^{+\infty} \beta(\lambda) e^{-i\lambda \tau} \frac{\partial \lambda}{\partial \tau} d\tau = \left[ +\int_{br}^{+\infty} -i \left( \frac{\tau^2}{r^2} - b^2 \right)^{1/2} \right] e^{-i\tau} \frac{\partial \left( \frac{\tau}{r} \right)}{\partial \tau} d\tau \]  

(2.38)

\[ \int \beta(\lambda) e^{-i\lambda (\xi - x)} d\lambda = \int_{br}^{+\infty} \beta(\lambda) e^{-i\lambda \tau} \frac{\partial \lambda}{\partial \tau} d\tau = \left[ +\int_{br}^{+\infty} +i \left( \frac{\tau^2}{r^2} - b^2 \right)^{1/2} \right] e^{-i\tau} \frac{\partial \left( \frac{\tau}{r} \right)}{\partial \tau} d\tau \]  

(2.39)

These two integrations are conjugate, so from Cauchy’s theorem:

\[ \int = \int_{br}^{+\infty} - \int_{br}^{-\infty} \]

\[ = 2 \int_{br}^{+\infty} -i \left( \frac{\tau^2}{r^2} - b^2 \right)^{1/2} \right] e^{-i\tau} \frac{\partial \left( \frac{\tau}{r} \right)}{\partial \tau} d\tau \]

\[ = \int_{0}^{+\infty} H(\tau - br) \left( -2i \left( \frac{\tau^2}{r^2} - b^2 \right)^{1/2} \right] e^{-i\tau} \frac{1}{r} d\tau \]  

(2.40)

and from equation (2.35), we obtain

\[ \frac{\partial \hat{u}}{\partial z}(x, 0, s) = \frac{\Delta \mu}{2} \frac{s}{2\pi i} \int_{br}^{+\infty} e^{-s\eta(\xi)} \int \beta e^{-i\lambda (\xi - x)} d\lambda d\xi \]
\[
\begin{align*}
&= \frac{\Delta u}{2} s \int_{0}^{\infty} e^{-\eta(\xi)} \int_{0}^{\infty} H(\tau - br) \left[ -2i \left( \frac{\tau^2}{r^2} - b^2 \right)^{\frac{1}{2}} \right] e^{-st} \frac{1}{r} d\tau d\xi \\
&= -\frac{\Delta u}{2\pi} s \int_{0}^{\infty} e^{-\eta(\xi)} \int_{0}^{\infty} H(\tau - br) \left( \tau^2 - b^2 r^2 \right)^{\frac{1}{2}} e^{-st} \frac{1}{r^2} d\tau d\xi \\
&= \frac{\Delta u}{2\pi} \int_{0}^{\infty} e^{-\eta(\xi)} \int_{0}^{\infty} H(\tau - br) F(\tau) e^{-st} d\tau d\xi \\
&= -\frac{\Delta u}{2\pi} \int_{0}^{\infty} e^{-\eta(\xi)} \int_{0}^{\infty} H(\tau - br) F(\tau) \frac{\partial}{\partial \tau} (e^{-st}) d\tau d\xi
\end{align*}
\]

(2.41)

where \( F(\tau) = -\frac{(\tau^2 - b^2 r^2)^{\frac{1}{2}}}{r^2} \).

Since

\[
\int_{0}^{\infty} \frac{\partial}{\partial \tau} \left( H(\tau - br) F(\tau) e^{-st} \right) d\tau
\]

\[
= \int_{0}^{\infty} \frac{\partial}{\partial \tau} \left( H(\tau - br) F(\tau) e^{-st} \right) d\tau + \int_{0}^{\infty} \left( H(\tau - br) F(\tau) \right) \frac{\partial}{\partial \tau} (e^{-st}) d\tau
\]

\[
= H(\tau - br) F(\tau) e^{-st} \bigg|_{\tau=0}^{\infty}
\]

\[
= 0
\]

(2.42)

\[
\int_{0}^{\infty} \left( H(\tau - br) F(\tau) \right) \frac{\partial}{\partial \tau} (e^{-st}) d\tau = -\int_{0}^{\infty} \left( H(\tau - br) F(\tau) e^{-st} \right) d\tau
\]

(2.43)

Also
\[
\int_0^{+\infty} \left\{ \frac{\partial}{\partial \tau} \left( H(\tau - br) \right) F(\tau) \right\} e^{-\tau t} d\tau \\
= \int_0^{+\infty} \left\{ \left( \frac{\partial}{\partial \tau} (\tau - br) \right) F(\tau) \right\} e^{-\tau t} d\tau \\
= F(br) e^{-br} \\
= - \left(\frac{b^2 r^2 - b^2 r^2}{r^2}\right) e^{-br} \\
= 0
\]

(2.44)

So from equation (2.41), we obtain

\[
\frac{\partial \hat{u}}{\partial \tau} (x,0,s) = \frac{\Delta t}{2\pi} \int_0^{+\infty} e^{-s\tau(\xi)} \int_0^{+\infty} \frac{\partial}{\partial \tau} \left( H(\tau - br) F(\tau) \right) e^{-\tau t} d\tau d\xi \\
= \frac{\Delta t}{2\pi} \int_0^{+\infty} e^{-s\tau(\xi)} \int_0^{+\infty} H(\tau - br) \frac{\partial F(\tau)}{\partial \tau} e^{-\tau t} d\tau d\xi \\
= \frac{\Delta t}{2\pi} \int_0^{+\infty} e^{-s\tau(\xi)} \int_0^{+\infty} H(\tau - br) \frac{\partial F(\tau)}{\partial \tau} e^{-s(\tau + \eta(\xi))} d\tau d\xi \\
= \frac{\Delta t}{2\pi} \int_0^{+\infty} e^{-s\tau(\xi)} \int_0^{+\infty} H(\tau - \eta(\xi) - br) \frac{\partial F(\tau)}{\partial \tau} e^{-st} d\tau d\xi \\
= \frac{\Delta t}{2\pi} \int_0^{+\infty} e^{-s\tau(\xi)} \int_0^{+\infty} H(\tau - \eta(\xi)) H(\tau - \eta(\xi) - br) \left. \frac{\partial F(\tau)}{\partial \tau} \right|_{\tau = t - \eta(\xi)} e^{-st} d\tau d\xi \\
= \frac{\Delta t}{2\pi} \int_0^{+\infty} e^{-s\tau(\xi)} \int_0^{+\infty} H(\tau - \eta(\xi) - br) \left. \frac{\partial F(\tau)}{\partial \tau} \right|_{\tau = t - \eta(\xi)} e^{-st} d\tau d\xi \\
= \frac{\Delta t}{2\pi} \int_0^{+\infty} e^{-s\tau(\xi)} \int_0^{+\infty} H(\tau - \eta(\xi) - br) \left. \frac{\partial F(\tau)}{\partial \tau} \right|_{\tau = t - \eta(\xi)} e^{-st} d\tau d\xi
\]

(2.45)

Let \( t = \tau + \eta(\xi) \), then \( \tau = t - \eta(\xi) \), equation (2.45) can be written as

\[
\frac{\partial \hat{u}}{\partial \tau} (x,0,s) = \frac{\Delta t}{2\pi} \int_0^{+\infty} e^{-s(\tau + \eta(\xi))} \int_0^{+\infty} H(t - \eta(\xi) - br) \frac{\partial F(\tau)}{\partial \tau} e^{-st} d\tau d\xi
\]

(2.46)
where \( H(t - \eta(\xi))H(t - \eta(\xi) - br) = H(t - \eta(\xi) - br) \), since \( br > 0, \ t > \eta(\xi) + br > \eta(\xi) \)
is the range of integration.

Change the order of integrations in \( \xi \) and \( t \) in equation (2.46):

\[
\frac{\partial \hat{u}}{\partial z}(x,0,t) = \frac{\Delta u}{2\pi} \int_0^\infty \int_0^\infty \frac{\partial F(\tau)}{\partial \tau} \left[ H(t - \eta(\xi) - br) \right] d\xi e^{-\beta t} dt
\]  

(2.47)

In equation (2.47), carry out the inversion in \( s - t \) by inspection:

\[
\frac{\partial u}{\partial z}(x,0,t) = \frac{\Delta u}{2\pi} \int_0^\infty \int_0^\infty \frac{\partial F(\tau)}{\partial \tau} \left[ H(t - \eta(\xi) - br) \right] d\xi d\tau
\]  

(2.48)

where

\[
\frac{dF(\tau)}{d\tau} \bigg|_{\tau=t-\eta(\xi)} = \frac{d}{d\tau} \left( \frac{(r^2 - \tau^2 b^2)^{\frac{1}{2}}}{r^2} \right) \bigg|_{\tau=t-\eta(\xi)} = -\frac{(t - \eta(\xi))}{\left( (t - \eta(\xi))^2 - r^2 b^2 \right)^{\frac{1}{2}} r^2}
\]  

(2.49)

So equation (2.48) can be written as

\[
\frac{\partial u}{\partial z}(x,0,t) = -\frac{\Delta u}{2\pi} \int_0^\infty \frac{(t - \eta(\xi))H(t - \eta(\xi) - br)}{\left( (t - \eta(\xi))^2 - r^2 b^2 \right)^{\frac{1}{2}} (x - \xi)^2} d\xi
\]  

(2.50)
Consider the singularity of the integrand in equation (2.50):

\[ H(t - \eta(\xi) - rb) \neq 0 \text{ only when } t > \eta(\xi) + rb. \]

\[ (t - \eta(\xi))^2 - r^2b^2 > 0 \text{ for the integrand.} \]

\[ \therefore \text{ Only when } x = \xi, \text{ namely } (x - \xi) = 0, \text{ there is singularity for the integrand.} \]

For subsonically moving dislocation, there are three regions I, II, III that have different properties for the above integrand.

In region I:

\[ x < 0, \xi > 0. \text{ So } \xi > x, x \neq \xi. \text{ There is no singularity.} \]

In region II:

\[ t > \eta(x), \text{ and } t > \eta(\xi) + rb \text{ (necessary to make } H(t - \eta(\xi) - rb) \neq 0). \]
It is possible that $\eta(\xi) = \eta(x) \Rightarrow \xi = x$ since $\eta(\xi)$ is a monotonously increasing function.

There is a singularity in region II.

In region III:

$t < \eta(x)$, also $t > \eta(\xi) + rb$

$\eta(x) > \eta(\xi) + rb > \eta(\xi) \Rightarrow x > \xi, x \neq \xi$.

There is no singularity.

For region I, III, $x \neq \xi$, there is no singularity in the integrand, we can apply the integration in equation (2.50) as the solution for $z = 0$ directly.

For region II, it is possible that $x = \xi$ happens. We cannot directly use the integration as the solution for $z = 0$ in region II since there is a singularity $x = \xi$. We have to remove the singularity from the integrand.

For $z \neq 0$ case, since $r^2 = (x - \xi)^2 + z^2 \neq 0$ even when $x = \xi$, there will be no singularity in any of the three regions.

Removing of the singularity $x = \xi$ for $z = 0$ case in region II:

In the neighborhood of $x = \xi$, as $(\xi - x) \rightarrow 0$, use Taylor series expansion, let

$$\eta(\xi) = \eta(x) + \eta'(x)(\xi - x) + \frac{1}{2}\eta''(x)(\xi - x)^2 + ......$$  \hspace{1cm} (2.51)

Then from equation (2.35), we obtain

$$\frac{\partial \widehat{u}(x,0,s)}{\partial z} = \frac{\Delta u}{2} \frac{s}{2\pi} \int_{B_{r,0}}^{\infty} \beta e^{-s(\eta(\xi)+\lambda(\xi-x))} d\xi d\lambda$$
\[
\frac{\Delta u}{2} \frac{s}{2\pi i} \int_{Br} \beta e^{-s(\eta(x) + \lambda(\xi-x))} d\xi d\lambda - \frac{\Delta u}{2} \frac{s}{2\pi i} \int_{Br} \beta e^{-s(\eta(x)+\eta'(x)\xi-x)+\lambda(\xi-x))} d\xi d\lambda
\]

\[
+ \frac{\Delta u}{2} \frac{s}{2\pi i} \int_{Br} \beta e^{-s(\eta(x)+\eta'(x)(\xi-x)+\lambda(\xi-x))} d\xi d\lambda
\]

We can see that in equation (2.52), the following term is added and subtracted in equation (2.35):

\[
\frac{\Delta u}{2} \frac{s}{2\pi i} \int_{Br} \beta e^{-s(\eta(x)+\eta'(x)\xi-x)+\lambda(\xi-x))} d\xi d\lambda
\]

We can verify that the first two terms in equation (2.52) together has no singularity of \( x = \xi \) in region II:

\[
\frac{\Delta u}{2} \frac{s}{2\pi i} \int_{Br} \beta e^{-s(\eta(x)+\lambda(\xi-x))} d\xi d\lambda - \frac{\Delta u}{2} \frac{s}{2\pi i} \int_{Br} \beta e^{-s(\eta(x)+\eta'(x)\xi-x)+\lambda(\xi-x))} d\xi d\lambda
\]

\[
= \frac{\Delta u}{2} \frac{s}{2\pi i} \left[ \int_{Br} \beta e^{-s(\eta(x)+\lambda(\xi-x))} d\xi d\lambda \left(1 - e^{s\eta(x)} e^{-s(\eta(x)+\eta'(x)\xi-x))} \right) \right]
\]

\[
= \frac{\Delta u}{2} \frac{s}{2\pi i} \left[ \int_{Br} \beta e^{-s(\eta(x)+\lambda(\xi-x))} \left(1 - e^{s\eta'(x)(\xi-x)^2} \right) d\xi d\lambda \right]
\]

\[
= \frac{\Delta u}{2} \frac{s}{2\pi i} \left[ \int_{Br} \beta e^{-s(\eta(x)+\lambda(\xi-x))} s \eta''(x)(\xi-x)^2 d\xi d\lambda \right]
\]

In equation (2.53) there is term \((\xi-x)^2\) inside the integrand, so when we apply the same method as we use to find solution in region I, III, we can cancel the singularity \( \frac{1}{(x-\xi)^2} \) as given in the solution for region I and III.
Considering equation (2.50), it follows that the first two terms of the solution that comes from the first two terms in equation (2.52) is:

\[
\frac{\partial u}{\partial z}(x,0,t) \bigg|_{\text{partI}} = -\frac{\Delta u}{2\pi} \int_{0}^{\infty} \int_{0}^{\infty} \frac{(t-\eta(x)-\eta'(x)(\xi-x))H(t-\eta(x)-\eta'(x)(\xi-x)-br)}{\left((t-\eta(x)-\eta'(x)(\xi-x))^2-r^2b^2\right)^2 (x-\xi)^2} \ d\xi
\]

As shown in equation (2.53), the singularity is removed from the integrand to allow interchange of the order of integration while applying the Cagniard-de Hoop technique.

For the third term in equation (2.52), we can directly integrate it:

\[
\frac{\partial \hat{u}}{\partial z}(x,0,s) \bigg|_{\text{partII}} = \frac{\Delta u}{2} \int_{0}^{\infty} \int_{0}^{\infty} \beta(\lambda) e^{-\pi x} \left(\frac{\partial \lambda}{\partial \tau} \right) d\pi d\xi
\]

Apply Cauchy’s theorem in equation (2.55), so \( \int_{B_{r}} - \int_{B_{s}} \), then
\[
\frac{\partial \hat{u}}{\partial z}(x,0,s)_{\text{partII}} = \frac{\Delta u}{4\pi i} \left( \int_{B^+} e^{i(-\eta(x)+\eta'(x)x)} \frac{\beta(\lambda_+)}{(\eta(x)+\lambda_+)} e^{x\lambda_+ x} d\lambda_+ - \int_{B^-} e^{i(-\eta(x)+\eta'(x)x)} \frac{\beta(\lambda_-)}{(\eta(x)+\lambda_-)} e^{x\lambda_- x} d\lambda_- \right)
\]  

(2.56)

Apply Cagniard-de Hoop technique to carry out the inversion in \(s-t\):

In equation (2.56), let

\[
\tau = -\lambda x
\]  

(2.57)

where \(\tau\) is positive real, namely \(\tau = \text{Re}(-\lambda x) > 0\), \(\text{Im}(-\lambda x) = 0\). Since \((-x)\) is real, so \(\lambda\) has to be real. Next we will discuss both cases \(x > 0\) and \(x < 0\).

When \(x > 0, \lambda < 0\),

\[
\lambda = \lambda_+ = \lambda_- = \frac{-\tau}{x}, \quad \frac{\partial \lambda}{\partial \tau} = -\frac{1}{x}, \quad \lambda \subset (-\infty,-b), \quad \tau \subset (bx,\infty), \quad \lambda^2 - b^2 > 0.
\]

\[
\beta(\lambda_+) = i |\lambda^2 - b^2|^{\frac{1}{2}} = i \left( \frac{\tau^2}{x^2} - b^2 \right)^{\frac{1}{2}}.
\]  

(2.58)

\[
\beta(\lambda_-) = -i |\lambda^2 - b^2|^{\frac{1}{2}}.
\]  

(2.59)

From equation (2.56), we obtain

\[
\frac{\partial \hat{u}}{\partial z}(x,0,s)_{\text{partII}} = \frac{\Delta u}{4\pi i} \left( \int_{bx}^{\infty} e^{i(-\eta(x)+\eta'(x)x)} \frac{\beta(\lambda_+)}{(\eta(x)+\lambda_+)} e^{-\tau \frac{\partial \lambda_+}{\partial \tau} x} d\tau - \int_{bx}^{\infty} e^{i(-\eta(x)+\eta'(x)x)} \frac{\beta(\lambda_-)}{(\eta(x)+\lambda_-)} e^{-\tau \frac{\partial \lambda_-}{\partial \tau} x} d\tau \right)
\]

\[
= \frac{\Delta u}{2\pi} \left( \int_{bx}^{\infty} \text{Im} \left( e^{i(-\eta(x)+\eta'(x)x)} \frac{\beta(\lambda_+)}{(\eta(x)+\lambda_+)} \frac{\partial \lambda_+}{\partial \tau} e^{-\tau x} d\tau \right) \right)
\]  

(2.60)

Let \(t = \tau + \eta(x) - \eta'(x)x\),
\[ \Rightarrow \lambda_* = -\frac{\tau}{x} = -\frac{t - \eta(x) + \eta'(x)x}{x} \]  
(2.61)

\[ \Rightarrow \beta(\lambda_*) = i \left( -\frac{t - \eta(x) + \eta'(x)x}{x} \right)^2 - b^2 \right)^{\frac{1}{2}} \]  
(2.62)

Thus, equation (2.60) becomes

\[
\frac{\partial \hat{u}}{\partial z}(x,0,s) = \frac{\Delta u}{2\pi} \left. \right|_{part II} \left[ \int_0^\infty \text{Im} \left( \left( -\frac{t - \eta(x) + \eta'(x)x}{x} \right)^2 - b^2 \right)^{\frac{1}{2}} \right] \frac{-e^{-st}}{x} dt \right. 
\]

\[ = \frac{\Delta u}{2\pi} \left( \int_0^\infty H(t - bx - \eta(x) + \eta'(x)x) \left[ \left( -\frac{t - \eta(x) + \eta'(x)x}{x} \right)^2 - b^2 \right] e^{-st} dt \right) \]  
(2.63)

In equation (2.63), perform the inversion in \( s - t \) by inspection and we obtain the third term of the solution for \( x > 0 \):

\[
\frac{\partial u(x,0,t)}{\partial z} \left. \right|_{part II} = \frac{\Delta u}{2\pi} \frac{1}{t - \eta(x)} \left[ \left( -\frac{t - \eta(x) + \eta'(x)x}{x} \right)^2 - b^2 \right] H(t - bx - \eta(x) + \eta'(x)x) 
\]  
(2.64)

When \( x < 0, \lambda > 0 \),

\[ \lambda = \lambda_* = \lambda_* = -\frac{\tau}{x}, \quad \frac{\partial \lambda}{\partial \tau} = -\frac{1}{x}, \quad \lambda \subset (+b, +\infty), \quad \tau \subset (-bx, \infty), \quad \lambda^2 - b^2 > 0 \]
\[ \beta(\lambda) = -i|\lambda^2 - b^2|^\frac{1}{2} = -i\left(\frac{\tau}{x^2} - b^2\right)^\frac{1}{2} \]  

(2.65)

\[ \beta(\lambda) = +i|\lambda^2 - b^2|^\frac{1}{2} \]  

(2.66)

Thus, equation (2.60) becomes

\[
\frac{\partial \hat{u}}{\partial z}(x,0,s) \bigg|_{\text{partII}} = \frac{\Delta u}{2\pi s} \left\{ \int_{bx+\eta(x) - \eta'(x)x}^{\infty} \operatorname{Im} \left\{ -i \left( \frac{-t - \eta(x) + \eta'(x)x}{x} \right)^2 - b^2 \right\} - e^{-st} dt \right\} 
\]

\[ = -\frac{\Delta u}{2\pi} \left\{ \frac{\sqrt{-t - \eta(x) + \eta'(x)x}}{x} \sqrt{\frac{-t - \eta(x) + \eta'(x)x}{x - \eta(x)}} - b^2 \right\} e^{-st} dt \]  

(2.67)

Correspondingly, in equation (2.67), perform the inversion in \( s - t \) by inspection and we obtain the third term of the solution for \( x < 0 \):

\[
\frac{\partial u}{\partial z}(x,0,t) \bigg|_{\text{partII}} = \frac{\Delta u}{2\pi} \left( \frac{1}{t - \eta(x)} \right) \sqrt{\frac{-t - \eta(x) + \eta'(x)x}{x}} \left( -b^2 \right) H(t - bx - \eta(x) + \eta'(x)x) 
\]

(2.68)

So for both cases \( x > 0 \) and \( x < 0 \):

\[
\frac{\partial u}{\partial z}(x,0,t) \bigg|_{\text{partII}} = \frac{\Delta u \operatorname{sgn}(x)}{2\pi} \sqrt{\frac{-t - \eta(x) + \eta'(x)x}{x}} \left( -b^2 \right) H(t - bx - \eta(x) + \eta'(x)x) 
\]

(2.69)

where \( \operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \).
Including all parts of the solution, namely solution for Problem I (which is
\(-\Delta u \frac{1}{2\pi x}\)) and solution part I (the first two terms: equation (2.54)) and solution part II (the third term: equation (2.69)) for Problem II, we can obtain the total solution for problem (2.9) in region II at \(z = 0\):

\[
\frac{\partial u}{\partial z}(x,0,t) = -\frac{\Delta u}{2\pi} \int_0^\infty \left[ \frac{(t-\eta(\xi)H(t-\eta(\xi)-br)}{(t-\eta(\xi))^2-r^2b^2}\right] (x-\xi)^2 \right] d\xi \\
- \left[ (t-\eta(x)+\eta'(x)(\xi-x))H(t-\eta(x)+\eta'(x)(\xi-x)-br) \right] (x-\xi)^2 \right] d\xi \\
+ \frac{\Delta u \text{ sgn}(x)}{2\pi t-\eta(x)} \sqrt{\left( -\frac{t-\eta(x)+\eta'(x)x}{x} \right)^2 - b^2 H(t-bx-\eta(x)+\eta'(x)x) - \frac{\Delta u}{2\pi x}}
\]

(2.70)

### 2.3.2 Solution at \(z \neq 0\).

Since when \(z \neq 0\), \(r = \sqrt{(x-\xi)^2 + z^2} > 0\), there will be no singularity in any region, we can apply the method outlined in chapter 2.2 and obtain:

\[
\frac{\partial \hat{u}}{\partial z}(x,z,s) \\
= \frac{\Delta u}{2} \frac{s}{2\pi} \int_{Br_0}^\infty \beta e^{-s(\eta(\xi)+\lambda\xi+\beta\xi)} d\xi e^{\lambda x} d\lambda \\
= \frac{\Delta u}{2} \frac{s}{2\pi} \int_{Br_0}^\infty \beta e^{-s\eta(\xi)} d\xi e^{-(\lambda\xi+\beta\xi)} d\lambda
\]

(2.71)

In equation (2.71), let

\[
\tau = \lambda\xi - \lambda x + \beta z
\]

(2.72)

where \(\tau\) is positive real.
From equation (2.72), we obtain

\[(\tau - \lambda \xi + \lambda x)^2 = (b^2 - \lambda^2)z^2\]  
(2.73)

Therefore

\[\lambda_\pm = \frac{\tau(\xi - x) \pm iz \sqrt{\tau^2 - b^2}[z^2 + (\xi - x)^2]}}{z^2 + (\xi - x)^2},\]  
(2.74)

\[
\frac{\partial \lambda_\pm}{\partial \tau} = \frac{(\xi - x) + iz \tau (r^2 - b^2 r^2)^{\frac{1}{2}}}{r^2},\]  
(2.75)

\[
\beta(\lambda_\pm) = \left[(b^2 - \lambda^2)^\frac{1}{2}\right] = \frac{\tau - \lambda \xi + \lambda x}{z}.\]  
(2.76)

From equation (2.74), we have

\[
\tau \geq b \sqrt{z^2 + (\xi - x)^2} = br\]  
(2.77)

\[
\text{Re}\lambda = \frac{\tau(\xi - x)}{z^2 + (\xi - x)^2} \geq \frac{br(\xi - x)}{r^2} = \frac{b(\xi - x)}{r}\]  
(2.78)

where \(r^2 = z^2 + (\xi - x)^2\).

Figure 2.5 Integration contour for \(z \neq 0\)
Interchange the order of integrations in equation (2.71), then apply Cauchy’s theorem and the Cagniard-de Hoop technique:

\[ \frac{\partial \hat{u}}{\partial z}(x, z, s) = \frac{\Delta u}{2} \frac{s}{\pi} \int_{0}^{\infty} e^{-s \eta(\xi)} \int_{br} \beta e^{-s(\lambda \xi \tau + \beta - \lambda k)} d\lambda d\xi \]

\[ = \frac{\Delta u}{2} \frac{s}{\pi} \int_{0}^{\infty} e^{-s \eta(\xi)} \int_{br} \operatorname{Im} \left( \beta(\lambda_x) e^{-s \lambda \tau} \frac{\partial \lambda_x}{\partial \tau} \right) d\tau d\xi \]

\[ = \frac{\Delta u}{2} \frac{s}{\pi} \int_{0}^{\infty} e^{-s \eta(\xi)} \int_{br} \operatorname{Im} \left( \frac{\tau - \lambda_x \xi + \lambda_x x e^{-s \lambda \tau} (\xi - x) + iz \tau \left( \frac{\tau^2 - b^2 r^2}{r^2} \right)^{\frac{3}{2}}}{\left( z + (\xi - x) \right)^2} \right) d\tau d\xi \]

\[ = \frac{\Delta u}{2} \frac{s}{\pi} \int_{0}^{\infty} e^{-s \eta(\xi)} \int_{br} \operatorname{Im} \left( \frac{\tau - \lambda_x \xi + \lambda_x x e^{-s \lambda \tau} (\xi - x) + iz \tau \left( \frac{\tau^2 - b^2 r^2}{r^2} \right)^{\frac{3}{2}}}{\left( z + (\xi - x) \right)^2} \right) d\tau d\xi \]

\[ = \frac{\Delta u}{2} \frac{s}{\pi} \int_{0}^{\infty} e^{-s \eta(\xi)} \int_{br} \operatorname{Im} \left( \frac{\tau^2 z^2}{r^4 \sqrt{\tau^2 - r^2 b^2}} - \frac{(x - \xi)^2 \sqrt{\tau^2 - r^2 b^2}}{r^4} \right) e^{-s \tau} d\tau d\xi \]

\[ = - \frac{\Delta u}{2} \int_{0}^{\infty} e^{-s \eta(\xi)} \int_{0}^{\infty} H(\tau - br) F(\tau, \xi) \frac{\partial (e^{-s \tau})}{\partial \tau} d\tau d\xi \] (2.79)

where \( F(\tau, \xi) = \frac{\tau^2 z^2}{r^4 \sqrt{\tau^2 - r^2 b^2}} - \frac{(x - \xi)^2 \sqrt{\tau^2 - r^2 b^2}}{r^4} \).

Since
\[\int_{0}^{\infty} \frac{\partial}{\partial \tau} \left( H(\tau - br)F(\tau, \xi)e^{-st} \right) d\tau \]

\[= \int_{0}^{\infty} \frac{\partial}{\partial \tau} (H(\tau - br)F(\tau, \xi))e^{-st} d\tau + \int_{0}^{\infty} H(\tau - br) \frac{\partial}{\partial \tau} (e^{-st}) d\tau \]

\[= H(\tau - br)F(\tau, \xi)e^{-st} \bigg|_{\tau=0} \]

\[= 0 \quad \text{(2.80)} \]

From equation (2.80), we obtain

\[\int_{0}^{\infty} H(\tau - br)F(\tau, \xi) \frac{\partial}{\partial \tau} (e^{-st}) d\tau = -\int_{0}^{\infty} \frac{\partial}{\partial \tau} (H(\tau - br)F(\tau, \xi))e^{-st} d\tau \quad \text{(2.81)} \]

Substituting equation (2.81) into equation (2.79), we can obtain

\[\frac{\partial \hat{u}}{\partial z}(x, z, s) = \frac{\Delta u}{2\pi} \int_{0}^{\infty} e^{-s\eta(\xi)} \int_{0}^{\infty} \frac{\partial}{\partial \tau} (H(\tau - br)F(\tau, \xi))e^{-st} d\tau d\xi \quad \text{(2.82)} \]

where

\[\int_{0}^{\infty} \frac{\partial}{\partial \tau} (H(\tau - br)F(\tau, \xi))e^{-st} d\tau \]

\[= \int_{0}^{\infty} \frac{\partial}{\partial \tau} (H(\tau - br)F(\tau, \xi))e^{-st} d\tau + \int_{0}^{\infty} H(\tau - br) \frac{\partial}{\partial \tau} (e^{-st}) d\tau \quad \text{(2.83)} \]

Considering

\[\int_{0}^{\infty} \left\{ \frac{\partial}{\partial \tau} (H(\tau - br))F(\tau, \xi) \right\} e^{-st} d\tau \]

\[= \int_{0}^{\infty} \left\{ \frac{\partial}{\partial \tau} (\delta(\tau - br))F(\tau, \xi) \right\} e^{-st} d\tau \]

\[= F(br, \xi)e^{-sbr} \]

\[= 0 \quad \text{(2.84)} \]
So equation (2.83) can be written as

\[ \int_0^\infty \frac{\partial}{\partial \tau} \left( H(\tau - br)F(\tau, \xi) \right) e^{-\tau} \, d\tau \]

\[ = \int_0^\infty H(\tau - br) \frac{\partial}{\partial \tau} \left( F(\tau, \xi) \right) e^{-\tau} \, d\tau \]  

(2.85)

So equation (2.82) (therefore equation (2.79)) can be written as

\[ \frac{\partial \hat{u}}{\partial z}(x, z, s) = \frac{\Delta u}{2\pi} \int_0^\infty e^{-\tau \eta(\xi)} \int_0^\infty H(\tau - br) \frac{\partial}{\partial \tau} \left( F(\tau, \xi) \right) e^{-\tau} \, d\tau \, d\xi \]

Let \( t = \eta(\xi) + \tau \), then equation (2.86) becomes

\[ \frac{\partial \hat{u}}{\partial z}(x, z, s) = \frac{\Delta u}{2\pi} \int_0^\infty e^{-\tau \eta(\xi)} \int_0^\infty H(t - \eta(\xi) - br) \frac{\partial}{\partial \tau} \left( F(\tau, \xi) \right) e^{-\tau} \, d\tau \, d\xi \]

\[ = \frac{\Delta u}{2\pi} \int_0^\infty e^{-\tau \eta(\xi)} \int_0^\infty H(t - \eta(\xi))H(t - \eta(\xi) - br) \frac{\partial}{\partial \tau} \left( F(\tau, \xi) \right) e^{-\tau} \, d\tau \, d\xi \]

\[ = \frac{\Delta u}{2\pi} \int_0^\infty e^{-\tau \eta(\xi)} \int_0^\infty H(t - \eta(\xi) - br) \frac{\partial}{\partial \tau} \left( F(\tau, \xi) \right) e^{-\tau} \, d\tau \, d\xi \]  

(2.87)

In equation (2.87), perform the inversion in \( s - t \) by inspection and obtain:

\[ \frac{\partial u}{\partial z}(x, z, t) = \frac{\Delta u}{2\pi} \int_0^\infty \left. e^{-\tau \eta(\xi)} \int_0^\infty H(t - \eta(\xi))H(t - \eta(\xi) - br) \frac{\partial}{\partial \tau} \left( F(\tau, \xi) \right) e^{-\tau} \, d\tau \, d\xi \right|_{t = \eta(\xi)} \]

(2.88)

Where

\[ \left. \frac{\partial}{\partial \tau} \left( F(\tau, \xi) \right) \right|_{t = \eta(\xi)} \]
\[
= \frac{\partial}{\partial t} \left( F(t - \eta(\xi), \xi) \right)
\]
\[
= \frac{\partial}{\partial t} \left( \frac{(t - \eta(\xi))^2 z^2}{r^4 \sqrt{(t - \eta(\xi))^2 - r^2 b^2}} - \frac{(x - \xi)^2 \sqrt{(t - \eta(\xi))^2 - r^2 b^2}}{r^4} \right)
\]
\[
= \frac{\partial}{\partial t} \left( \frac{(t - \eta(\xi))^2 z^2}{r^4 \sqrt{(t - \eta(\xi))^2 - r^2 b^2}} \right) - \frac{(t - \eta(\xi))(x - \xi)^2}{r^4 \sqrt{(t - \eta(\xi))^2 - r^2 b^2}}
\]

Considering both the solution for Problem I and II, for dislocation starting from rest and moving non-uniformly \((t = \eta(\xi))\), the solution for \(z \neq 0\) will be

\[
\frac{\partial u}{\partial z}(x, z, t)
\]
\[
= -\frac{\Delta u}{2\pi} \int_0^\infty \frac{(t - \eta(\xi))(x - \xi)^2 H(t - \eta(\xi) - rb) d\xi}{r^4 [(t - \eta(\xi))^2 - r^2 b^2]^{\frac{1}{2}}}
\]
\[
+ \frac{\Delta u}{2\pi} z^2 \frac{\partial}{\partial t} \int_0^\infty \frac{(t - \eta(\xi))^2 H(t - \eta(\xi) - rb) d\xi}{r^4 [(t - \eta(\xi))^2 - r^2 b^2]^{\frac{1}{2}}} - \frac{\Delta u}{2\pi} \frac{x}{x^2 + z^2}
\]

The solution shown above is valid for both the subsonic and supersonic cases.

### 2.4 Supersonic dislocation starting from rest and moving with constant velocity

For supersonic dislocation, the moving speed of dislocation \(\frac{1}{\alpha}\) exceeds the wave speed \(\frac{1}{b}\). That is, \(\alpha < b\).

Figure 2.6 shows the range of supersonic dislocation.
For a dislocation starting from rest and moving with supersonic constant velocity, the motion function would be: \( \eta(\xi) = \alpha \xi \), where \( \alpha < b \).

By direct formulation of the problem as a constant-velocity one, using a procedure similar to the method used in transient motion of nonuniformly moving dislocation in the previous sections, we start from equation (2.17):

\[
\frac{\partial \hat{u}}{\partial z} = \frac{\Delta u}{2} \frac{1}{2\pi i} \int_{\beta} e^{-(\lambda + \alpha)} e^{-t(\lambda x + \beta z)} d\lambda
\]

We apply the Cagniard-de Hoop technique:

Let

\[
\tau = -\lambda x + \beta z = (b^2 - \lambda^2)^{\frac{1}{2}} z - \lambda x
\]

where \( \tau \) is a real. \( \tau = \text{Re}(-\lambda x + \beta z) > 0 \), \( \text{Im}(-\lambda x + \beta z) = 0 \), so that

\[
\lambda_\pm = r_0^{-2} [-\lambda x \pm iz\sqrt{\tau^2 - r_0^2 b^2}]
\]

\[
\beta(\lambda_\pm) = r_0^{-2} [\pm x \sqrt{\tau^2 - r_0^2 b^2}]
\]
with \( r_0 = [x^2 + z^2]^{\frac{1}{2}} \).

Since \( \alpha < b \Rightarrow -\alpha > -b \) and also \(-\frac{bx}{r_0} > -b\), it is possible that \(-\alpha \geq -\frac{bx}{r_0}\). When \( \tau = r_0 b \), we have \( \lambda = -\frac{\pi x}{r_0^2} = -\frac{r_0 bx}{r_0^2} = -\frac{bx}{r_0} \). So it is possible that, namely \( \lambda + \alpha = 0 \), so that \( \lambda = -\alpha \) is a pole.

According to Cauchy’s theorem, we have

\[
\int = \int_{\gamma_1} - \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} - \int_{\gamma_5}
\]

Because there is a pole included in the integration contour, we need to introduce the integration circle \( B_3 \) (with radius \( \varepsilon \)) around the pole \( \lambda = -\alpha \).

![Integration contour for supersonic dislocation](image.png)
We can prove that $\int_{B_4} - \int_{B_5} = 0$.

On $B_1(B_4), B_3(B_5)$, we can use the same result as in the previous section.

Then we obtain the first part of the solution due to $\int_{B_3} - \int_{B_4}$:

$$\left. \frac{\partial u}{\partial z}(x,z,t) \right|_{t=2} = \left\{ \begin{array}{ll}
\frac{-x}{x^2 + z^2} & \text{for } t < rb \\
 \frac{-x}{x^2 + z^2} + \frac{(b^2 - \gamma^2 r_0^2 + z^2)(t^2 - r_0^2 b^2)}{(t^2 - r_0^2 b^2)^2 + \frac{z^2(t^2 - r_0^2 b^2)}{r_0^2}} & \text{for } t \geq rb
\end{array} \right. $$

(2.94)

On $B_3$,

$$\lambda = -\alpha + \varepsilon e^{i\theta}$$

(2.95)

where $\varepsilon > 0$, $-\pi < \theta < \pi$.

So from equation (2.95),

$$\beta = (b^2 - \lambda^2)^{1/2}$$

$$= (b^2 - \varepsilon^2 e^{2i\theta} + 2\alpha \varepsilon e^{i\theta} - \alpha^2)^{1/2}$$

(2.96)

From equation (2.95), (2.96), we obtain the limits:

$$\lim_{\varepsilon \to 0} \beta = (b^2 - \alpha^2)^{1/2}$$

(2.97)
\[
\lim_{\varepsilon \to 0} \int_{\beta_1}^\beta \frac{\beta}{(\lambda + \alpha)} e^{-s(-\lambda + \beta \varepsilon)} d\lambda \\
= \lim_{\varepsilon \to 0} \int_{-\pi}^\pi \frac{(b^2 - \alpha^2)\varepsilon}{-\alpha + \varepsilon i\theta + \alpha} e^{-s(-\alpha + \varepsilon i\theta) + \beta \varepsilon i\theta} d\theta \\
= \lim_{\varepsilon \to 0} \int_{-\pi}^\pi \frac{(b^2 - \alpha^2)\varepsilon}{\varepsilon i\theta} e^{-s(-\alpha + \varepsilon i\theta) + \beta \varepsilon i\theta} d\theta \\
= \int_{-\pi}^\pi (b^2 - \alpha^2)\varepsilon e^{-s(\alpha + (b^2 - \alpha^2)\varepsilon z)} id\theta \\
= 2\pi(i(b^2 - \alpha^2)\varepsilon e^{-s(\alpha + (b^2 - \alpha^2)\varepsilon z)}) \\
= 2\pi(b^2 - \alpha^2)\varepsilon e^{-s(\alpha + (b^2 - \alpha^2)\varepsilon z)}
\] (2.98)

Since from the definition of the Laplace transform, we have

\[
L(\delta(t - t_0)) = \int_0^\infty \delta(t - t_0)e^{-st} dt = e^{-st_0}
\] (2.99)

\[
L\left(\delta\left(t - \left(\alpha x + (b^2 - \alpha^2)\varepsilon z\right)\right)\right) = \int_0^\infty \delta\left(t - \left(\alpha x + (b^2 - \alpha^2)\varepsilon z\right)\right)e^{-st} dt = e^{-s(\alpha + (b^2 - \alpha^2)\varepsilon z)}
\] (2.100)

Therefore, the Laplace inverse of \(2\pi(b^2 - \alpha^2)\varepsilon e^{-s(\alpha + (b^2 - \alpha^2)\varepsilon z)}\) can be written as

\[
L^{-1}\left(2\pi(b^2 - \alpha^2)\varepsilon e^{-s(\alpha + (b^2 - \alpha^2)\varepsilon z)}\right) = 2\pi(b^2 - \alpha^2)\varepsilon \delta\left(t - \left(\alpha x + (b^2 - \alpha^2)\varepsilon z\right)\right) \\
= 2\pi(b^2 - \alpha^2)\varepsilon \delta\left(t - \left(\alpha \alpha + (b^2 - \alpha^2)\varepsilon z\right)\right)
\] (2.101)

And the second part of the solution that comes from \(\int_{\beta_1}\) because of the pole at \(\lambda = -\alpha\) is:

\[
\frac{\partial u}{\partial x}(x,z,t) \bigg|_3 = \frac{\Delta u}{2}(b^2 - \alpha^2)\varepsilon \delta\left(t - \alpha \alpha + (b^2 - \alpha^2)\varepsilon z\right)
\] (2.102)
Remember that in the beginning of the solving process, the general solution for equation (2.11):
\[
\frac{\partial^2 U}{\partial z^2} - (b^2 - \lambda^2)s^2 U = 0
\]
where \( U(\lambda, z, s) = \int_{-\infty}^{\infty} \hat{u}(x, z, s)e^{-s\lambda x}dx \) is
\[
U(\lambda, z, s) = Ae^{b\lambda z} + Be^{-b\lambda z} \tag{2.103}
\]
Apply the corresponding boundary condition \( U(\lambda, 0, s) = -\frac{\Delta u}{2s} \int_{0}^{\infty} e^{-s\eta(x)} e^{-b\lambda x}dx \) to equation (2.103).

In order to obtain a finite result as \(|z| \to \infty\), the solution will be:

For \( z > 0 \), the solution is
\[
U(\lambda, z, s) = -\frac{\Delta u}{2s} \int_{0}^{\infty} e^{-s\eta(x)} e^{-s\lambda x}dx \cdot e^{-b\lambda z} .
\]
This is the case we have discussed above.

For \( z < 0 \), the solution is
\[
U(\lambda, z, s) = -\frac{\Delta u}{2s} \int_{0}^{\infty} e^{-s\eta(x)} e^{-s\lambda x}dx \cdot e^{b\lambda z} .
\]
So for \( z < 0 \), \( \eta(x) = \alpha \xi \), using the same method for \( z > 0 \), we obtain
\[
\frac{\partial \hat{u}}{\partial z} = \frac{\Delta u}{2} \frac{1}{2\pi i} \int_{\beta} \frac{\beta}{(\lambda + \alpha)} e^{-s(-\lambda s - \beta)} d\lambda \tag{2.104}
\]
and
\[
\frac{\partial u}{\partial z}(x, z, t) \bigg|_{z} = \frac{\Delta u}{2} \left( b^2 - \alpha^2 \right)^{\frac{1}{2}} \delta(t - \alpha x + (b^2 - \alpha^2)^{\frac{1}{2}} z) \tag{2.105}
\]
Therefore, the final solution for the supersonic dislocation starting from rest and moving with constant supersonic velocity is:
\[
\hat{u}\left(\frac{x}{x^2 + z^2}\right) \bigg|_{b,2} = \frac{\Delta u}{2\pi} \begin{cases} 
\frac{-x}{x^2 + z^2} & \text{for } t < rb \\
\frac{-(b^2 - t^2 x^2 + z^2 (t^2 - r_0^2 b^2))(\alpha - \frac{tx}{r_0^2}) + \frac{2tx}{r_0^2} z^2 (t^2 - r_0^2 b^2)}{(t^2 - r_0^2 b^2)(\alpha - \frac{tx}{r_0^2})^2 + \frac{z^2 (t^2 - r_0^2 b^2)}{r_0^2}} \\
\frac{\Delta u}{2} \left(b^2 - \alpha^2 \right) \delta \left( t - \alpha x \mp \left(b^2 - \alpha^2 \right)^{\frac{1}{2}} z \right) & \text{"-" : } z > 0, \text{"+" : } z < 0 \end{cases}
for \ t \geq rb
\]

From the solution (equation (2.106)), we can see that the delta function is due to the fact that in the supersonic case a pole \((\lambda = -\alpha)\) is included in the integration contour.

The argument of the delta function corresponds to the shock-wave front which consists of the envelope of the wavelets that were excited by the dislocation on its way.

So the Mach wave fronts where the radiated field is of delta function strength are found for screw dislocation starting from rest and moving with constant supersonic velocity.
The equation for the shock-wave front is

\[ z = \pm \left( \frac{-\alpha x}{\sqrt{b^2 - \alpha^2}} + \frac{t}{\sqrt{b^2 - \alpha^2}} \right) \]

(2.107)

Note that for supersonic dislocation starting from rest and moving with constant velocity, the stress is zero behind the shock front and outside the circle \( t = rb \).

The wave front for nonuniformly moving supersonic dislocation will be discussed in another chapter.
3

THE TRANSIENT MOTION OF A NONUNIFORMLY MOVING EDGE DISLOCATION

3.1 Governing equation for general motion of a Volterra edge dislocation

Consider a straight infinitely-long edge dislocation aligned parallel to the y-axis and moving on the plane $z = 0$ in an infinite region as shown in Figure 3.1.

![Figure 3.1 Edge dislocation](image)

The problem of determining the elastodynamic field for this case can be reduced to an equivalent boundary value problem in a half-space $z > 0$ [see Freund(1973)]. The solution for an edge dislocation starting from rest at $t = 0$ and moving thereafter in its slip-plane $z = 0$ according to $x = l(t)$ (or equivalently $t = \eta(x)$) where $l(t)$ is any
monotonically increasing function, can be obtained as the superposition of the solutions to the following two problems:

Problem I (static problem):

The elastic fields that satisfy both the equations of elastostatics in the half-space \( z > 0 \) and the boundary conditions on \( z = 0 \)

\[
u_\Delta(x,0,t) = \frac{1}{2} \Delta u H(x) \quad \text{and} \quad \sigma_z(x,0,t) = 0
\]

are known [see HIRTH and LOTHE (1968, p.74)].

Here \( \Delta u \) is a constant relative displacement in the x-direction across the plane \( z = 0 \) and \( H(x) \) is the Heaviside step function.

Problem II

We seek the dilatational potential \( \phi \) and equivoluminal potential \( \psi \) that vanish for \( t < 0 \) and satisfy the wave equations[see Fung, p187, p215]

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = a^2 \frac{\partial^2 \phi}{\partial t^2} , \quad (3.1)
\]

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = b^2 \frac{\partial^2 \psi}{\partial t^2} , \quad (3.2)
\]

where \( a \) and \( b \) are longitudinal and equivoluminal wave slowness \( a^2 = \frac{\rho}{\Lambda + 2\mu} \), \( b^2 = \frac{\rho}{\mu} \).

The solution also satisfies the following boundary conditions at \( z = 0 \):

\[
u_\Delta(x,0,t) = \frac{1}{2} \Delta u H[I(t) - x] - \frac{1}{2} \Delta u H(x) , \quad (3.2)
\]
\[ \sigma_{zx}(x,0,t) = 0, \]

for all \( t \geq 0. \)

The displacements and stresses are given in terms of the potentials \( \phi \) and \( \psi \) by:

\[
\begin{align*}
 u_x &= \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z}, \\
 u_z &= \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}, \\
 u_y &= 0, \quad (3.3)
\end{align*}
\]

\[
\begin{align*}
 \sigma_{xx} &= \Lambda \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + \mu \left( \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial z} \right), \\
 \sigma_{zz} &= \Lambda \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + \mu \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right), \\
 \sigma_{xy} &= \mu \left( \frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} \right), \quad (3.3)
\end{align*}
\]

\[ \sigma_{yz} = \sigma_{xy} = 0, \quad \sigma_{yx} = \nu(\sigma_{xx} + \sigma_{zz}). \]

where \( \Lambda \) and \( \mu \) are the Lame constants and \( \nu \) is Poisson’s ratio.

Substitution of equations (3.3) into equations (3.2) gives the boundary conditions in terms of the potentials \( \phi \) and \( \psi \)

\[
\left( \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z} \right) \bigg|_{z=0} = \frac{1}{2} \Delta u H[l(t) - x] - \frac{1}{2} \Delta u H(x), \quad (3.4)
\]

\[
\left[ \Lambda \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + \mu \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right) \right] \bigg|_{z=0} = 0,
\]

for all \( t \geq 0. \)

The solution to Problem II is obtained in the sequel.
3.2 Analytical solution for an edge dislocation starting from rest and moving non-uniformly

Here, we use the similar process previously used in obtaining solution for the nonuniformly-moving screw dislocation in Chapter 2. The solution for a non-uniformly moving dislocation at \( z = 0 \) which contains singularities not present for \( z \neq 0 \) can be treated similarly as in Section 2.3.1. In the following we only consider the \( z \neq 0 \) case.

Application first of the Laplace transforms in time,

\[
\hat{\phi}(x,z,s) = \int_0^\infty \phi(x,z,t) e^{-st} dt \quad (s > 0),
\]

\[
\hat{\psi}(x,z,s) = \int_0^\infty \psi(x,z,t) e^{-st} dt \quad (s > 0).
\]

and then two-sided Laplace transforms in space,

\[
\Phi(\lambda,z,s) = \int_{-\infty}^{\infty} \hat{\phi}(x,z,s) e^{-x\lambda} dx,
\]

\[
\Psi(\lambda,z,s) = \int_{-\infty}^{\infty} \hat{\psi}(x,z,s) e^{-x\lambda} dx,
\]

reduces the differential equations (3.1) to

\[
\frac{\partial^2 \Phi}{\partial z^2} - s^2 \alpha^2 \Phi = 0,
\]

\[
\frac{\partial^2 \Psi}{\partial z^2} - s^2 \alpha^2 \Psi = 0,
\]

and the boundary conditions (3.2) to

\[
\left( s \lambda \Phi - \frac{\partial \Psi}{\partial z} \right) \bigg|_{z=0} = \frac{-\Delta u}{2s} \int_0^\infty e^{-[q(\xi)+\lambda \xi]} d\xi,
\]
\[
\left[ \Lambda \left( s^2 \lambda^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} \right) + 2 \mu \left( \frac{\partial^2 \Phi}{\partial z^2} + s \lambda \frac{\partial \Phi}{\partial z} \right) \right]_{z=0} = 0 , \quad (3.8)
\]

where \( \alpha^2 = a^2 - \lambda^2 \) and \( \beta^2 = b^2 - \lambda^2 \).

From equations (3.7), the solution can be written as \( \Phi = Ae^{-s\alpha z}, \Psi = Be^{-s\beta z} \).

Substituting it into the boundary conditions (3.8) gives

\[
(As\lambda + Bs\beta) = -\frac{\Delta u}{2s} \int_0^\infty e^{-x [\beta(\xi) + \alpha \xi]} d\xi , \quad (3.9)_1
\]

\[
\Lambda (As^2 \lambda^2 + As^2 \alpha^2) + 2\mu (As^2 \alpha^2 - Bs^2 \lambda \beta) = 0 . \quad (3.9)_2
\]

From equations (3.9), we obtain

\[
A = -\frac{\Delta u}{b^2 s^2} \int_0^\infty e^{-x [\beta(\xi) + \alpha \xi]} d\xi , \quad (3.10)_1
\]

\[
B = -\frac{\Delta u}{2b^2 s^2} \frac{b^2 - 2\lambda^2}{\beta} \int_0^\infty e^{-x [\beta(\xi) + \alpha \xi]} d\xi . \quad (3.10)_2
\]

Therefore, the solution to (3.7) and (3.8) can be written in the form

\[
\Phi(\lambda, z, s) = -\frac{\Delta u}{b^2 s^2} \int_0^\infty e^{-x [\beta(\xi) + \alpha \xi]} e^{-s\alpha z} d\xi , \quad (3.11)_1
\]

\[
\Psi(\lambda, z, s) = -\frac{\Delta u}{2b^2 s^2} \frac{b^2 - 2\lambda^2}{\beta} \int_0^\infty e^{-x [\beta(\xi) + \alpha \xi]} e^{-s\beta z} d\xi . \quad (3.11)_2
\]

From equation (3.3), the inversion of the space transform of the stress of interest \( \sigma_{xz} \) gives

\[
\hat{\sigma}_{xz} = \int_0^\infty \sigma_{xz} e^{-st} dt = \int_0^\infty \left( \frac{2\phi(x,z) + \frac{\partial^2 \Psi}{\partial z^2} - \frac{\partial^2 \phi}{\partial x^2}}{2} \right) e^{-st} dt
\]
\[ = \mu \left( \frac{2 \partial^2 \hat{\phi}}{\partial x \partial z} + \frac{\partial^2 \hat{\psi}}{\partial x^2} - \frac{\partial^2 \hat{\psi}}{\partial z^2} \right), \]  

(3.12)

where \( \hat{\phi} = \frac{s}{2 \pi} \int_{B_r} \Phi e^{s\lambda z} d\lambda = \frac{s}{2 \pi} \int_{B_r} A e^{-s\lambda \zeta} e^{s\lambda z} d\lambda, \)  

(3.13)_1

\[ \hat{\psi} = \frac{s}{2 \pi} \int_{B_r} \Psi e^{s\lambda z} d\lambda = \frac{s}{2 \pi} \int_{B_r} B e^{-s\lambda \zeta} e^{s\lambda z} d\lambda. \]  

(3.13)_2

From equations (3.13), we can obtain

\[ \frac{\partial^2 \hat{\phi}}{\partial x \partial z} = -s^2 \frac{s}{2 \pi} \int_{B_r} A \alpha \lambda e^{-s\lambda \zeta} e^{s\lambda z} d\lambda, \]  

(3.14)_1

\[ \frac{\partial^2 \hat{\psi}}{\partial x^2} = s^2 \frac{s}{2 \pi} \int_{B_r} B \lambda^2 e^{-s\lambda \zeta} e^{s\lambda z} d\lambda, \]  

(3.14)_2

\[ \frac{\partial^2 \hat{\psi}}{\partial z^2} = s^2 \frac{s}{2 \pi} \int_{B_r} B \lambda^2 e^{-s\lambda \zeta} e^{s\lambda z} d\lambda. \]  

(3.14)_3

Substitution of equations (3.10) and (3.14) into equation (3.12) gives

\[ \hat{\sigma}_{zz} = \mu \left\{ -2 s^2 \frac{s}{2 \pi} \int_{B_r} \frac{-\Delta u \alpha}{b^2 s^2} \lambda^2 \int_{0}^{\infty} e^{-s [\eta(\xi)+\lambda \xi]} d\xi e^{-s\lambda \zeta} e^{s\lambda z} d\lambda + \right. \]

\[ s^2 \frac{s}{2 \pi} \int_{B_r} \frac{-\Delta u}{2 b^2 s^2} \lambda^2 b^2 - 2 \lambda^2 \beta \int_{0}^{\infty} e^{-s [\eta(\xi)+\lambda \xi]} d\xi e^{-s\lambda \zeta} e^{s\lambda z} d\lambda - \]

\[ s^2 \frac{s}{2 \pi} \int_{B_r} \frac{-\Delta u}{2 b^2 s^2} \beta^2 b^2 - 2 \lambda^2 \beta \int_{0}^{\infty} e^{-s [\eta(\xi)+\lambda \xi]} d\xi e^{-s\lambda \zeta} e^{s\lambda z} d\lambda \right\} \]

\[ = \mu \left\{ 2 \frac{\Delta u}{b^2} \frac{s}{2 \pi} \int_{B_r} \lambda^2 a e^{-s [\eta(\xi)+\lambda \xi]} d\xi e^{-s\lambda \zeta} e^{s\lambda z} d\lambda + \right. \]

\[ \frac{\Delta u}{2b^2} \frac{s}{2 \pi} \int_{B_r} \left( \frac{b^2 - 2 \lambda^2}{\beta} \right) e^{-s [\eta(\xi)+\lambda \xi]} e^{-s\lambda \zeta} e^{s\lambda z} d\lambda \right\} \]

(3.15)
where $Br$ is a Bromwich contour from $\lambda = -i\infty$ to $\lambda = +i\infty$ on the real side of the $\lambda$–plane.

The inversion of the time transform of equation (3.15) is completed by using the Cagniard-de Hoop technique [as in Chapter 2] to modify the path of the integration in the $\lambda$–plane to arcs of hyperbolas so that inversion from $s$ to $t$ can be performed by inspection.

In the first integral of equation (3.15), let

$$\tau = \alpha z - \lambda (x - \xi) = (a^2 - \lambda^2)^2 z - \lambda(x - \xi), \quad (3.16)$$

where $\tau$ is a real. $\tau = \text{Re}(\alpha z - \lambda(x - \xi)) > 0$, $\text{Im}(\alpha z - \lambda(x - \xi)) = 0$, so that

$$\left[\tau + \lambda(x - \xi)\right]^2 = (a^2 - \lambda^2)\xi^2,$$

$$\left[(x - \xi)^2 + z^2\right]\lambda^2 + 2(x - \xi)\tau\lambda + \tau^2 - a^2z^2 = 0,$$

$$\lambda_\pm = \frac{-2(x - \xi)\tau \pm i\sqrt{-4(x - \xi)^2 \tau^2 + 4(\tau^2 - a^2z^2)(x - \xi)^2 + z^2}}{2((x - \xi)^2 + z^2)}$$

$$= \frac{\tau(x - \xi) \pm iz\sqrt{\tau^2 - a^2r^2}}{r^2}, \quad (3.17)$$

where $r = \sqrt{(x - \xi)^2 + z^2}$. Substitution of equation (3.17) into equation (3.16) gives

$$\alpha_\pm = \frac{\tau + \lambda(x - \xi)}{z} = \frac{\tau \pm i(x - \xi)\sqrt{\tau^2 - a^2r^2}}{r^2}, \quad (3.18)$$

From the expression of equations (3.17) and (3.18), we can see that $\tau > ar$ to make $\lambda_\pm$ imaginary.
When $\tau = ar$, $\lambda_\pm = \frac{\tau (\xi - x)}{r^2} \bigg|_{\tau = ar} = \frac{a(\xi - x)}{r}$. \hfill (3.19)

Apply Cauchy’s theorem, since there is no singularity inside the contour ABCD, so

$$\int = \int - \int \text{ with } R \to \infty.$$  

From the first integral in equation (3.15), which is the dilatational component due to $\phi$, applying the Cauchy’s theorem, we obtain

$$\hat{\sigma}_{xz}^{(\phi)} = 2\mu \frac{\Delta u}{b^2} \frac{s}{2\pi i} \left\{ \int_{B_{z=0}}^{\infty} \lambda_2^2 \alpha e^{-s\eta(\xi)} d\xi e^{-s(\lambda_\xi - \lambda_x + \alpha z)} d\lambda \right\}$$

$$- \int_{B_{z=0}}^{\infty} \lambda_2^2 \alpha e^{-s\eta(\xi)} d\xi e^{-s(\lambda_\xi - \lambda_x + \alpha z)} d\lambda.$$ \hfill (3.20)

Figure 3.2 Integration contour I for edge dislocation
Interchanging in the integration in $\lambda$ and $\xi$ is permissible if $\int_\mathbb{B} (\_\_\_)d\lambda$ converges absolutely. So equation (3.20) becomes

$$\hat{\sigma}_{xx}^{(\theta)} = 2\mu \frac{s\Delta u}{\pi b^2} \int_0^\infty e^{-q(\xi)}d\xi \left[ \int_{\mathbb{B}_+} \lambda^2 \alpha_+ e^{-s(\lambda,\xi,\lambda,\xi,\alpha,\alpha,\zeta)} d\lambda - \int_{\mathbb{B}_-} \lambda^2 \alpha_- e^{-s(\lambda,\xi,\lambda,\xi,\alpha,\alpha,\zeta)} d\lambda \right]$$

$$= 2\mu \frac{s\Delta u}{\pi b^2} \int_0^\infty e^{-q(\xi)}d\xi \left[ \int_{\mathbb{B}_+} \lambda^2 \alpha_+ \frac{\partial \lambda_+}{\partial \tau} d\tau - \int_{\mathbb{B}_-} \lambda^2 \alpha_- \frac{\partial \lambda_-}{\partial \tau} d\tau \right]$$  \hspace{1cm} (3.21)

Since $\lambda_+$ and $\lambda_-$ are conjugate to each other, so

$$\lambda^2 \alpha_+ \frac{\partial \lambda_+}{\partial \tau} - \lambda^2 \alpha_- \frac{\partial \lambda_-}{\partial \tau}$$

$$= 2i \text{Im} \left( \lambda^2 \alpha_+ \frac{\partial \lambda_+}{\partial \tau} \right)$$ \hspace{1cm} (3.22)

Substitution of equation (3.22) into equation (3.21) gives

$$\hat{\sigma}_{xx}^{(\theta)} = 2\mu \frac{s\Delta u}{\pi b^2} \int_0^\infty e^{-q(\xi)}d\xi \int_{\mathbb{B}_+} \lambda^2 \alpha_+ \frac{\partial \lambda_+}{\partial \tau} e^{-s\tau} d\tau$$ \hspace{1cm} (3.23)

In equation (3.23), let $t = \tau + \eta(\xi)$, $\text{Im} \left( \lambda^2 \alpha_+ \frac{\partial \lambda_+}{\partial \tau} \right) = F_1(t, \xi) = F_1(t - \eta(\xi), \xi)$, we obtain

$$\hat{\sigma}_{xx}^{(\theta)} = 2\mu \frac{s\Delta u}{\pi b^2} \int_0^\infty F_1(t - \eta(\xi), \xi) e^{-st} dt \, d\xi$$

$$= 2\mu \frac{s\Delta u}{\pi b^2} \int_0^\infty H(t - \eta(\xi) - ar)F_1(t - \eta(\xi), \xi) e^{-st} dt \, d\xi$$ \hspace{1cm} (3.24)
Since \( e^{-st} = \frac{\partial}{\partial t} \left( e^{-st} \right)^{-1} \), using integration by parts, we obtain from equation (3.24):

\[
\hat{\sigma}_{\xi\xi}^{(s)} = -2\mu \frac{\Delta u}{\pi b^2} \int_0^\infty \int_0^\infty \frac{1}{\xi} \left[ H(t - \eta(\xi) - ar)F_i(t - \eta(\xi), \xi) \frac{\partial}{\partial t} (e^{-st}) \right] dt d\xi
\]

\[
= -2\mu \frac{\Delta u}{\pi b^2} \int_0^\infty \left[ (e^{-st} H(t - \eta(\xi) - ar)F_i(t - \eta(\xi), \xi)) \right]_{t=0}^\infty -
\]

\[
\int_0^\infty \frac{\partial}{\partial t} (H(t - \eta(\xi) - ar)F_i(t - \eta(\xi), \xi)) (e^{-st}) dt d\xi
\]

\[
= 2\mu \frac{\Delta u}{\pi b^2} \int_0^\infty \int_0^\infty \frac{\partial}{\partial t} (H(t - \eta(\xi) - ar)F_i(t - \eta(\xi), \xi)) (e^{-st}) dt d\xi
\]

(3.25)

Interchanging the integration in \( \xi \) and \( t \) in equation (3.25) gives the first integral of equation (3.15) in the form of

\[
\hat{\sigma}_{\xi\xi}^{(s)} = 2\mu \frac{\Delta u}{\pi b^2} \int_0^\infty \left[ \int_0^\infty \frac{\partial}{\partial t} (H(t - \eta(\xi) - ar)F_i(t - \eta(\xi), \xi)) dt \right] d\xi
\]

(3.26)

Performing the inversion from \( s \) to \( t \) by inspection in equation (3.26), we obtain the dilatational part of the stress \( \sigma_{\xi\xi}^{(s)} \):

\[
\sigma_{\xi\xi}^{(s)} = 2\mu \frac{\Delta u}{\pi b^2} \frac{\partial}{\partial t} \left[ \int_0^\infty \left( H(t - \eta(\xi) - ar)F_i(t - \eta(\xi), \xi) \right) dt \right] d\xi
\]

(3.27)

Similarly, in the second integral of equation (3.15),

let \( \tau = \beta \xi - \lambda (x - \xi) = \left( b^2 - \lambda^2 \right) \frac{\beta}{\lambda} z - \lambda (x - \xi) \)

(3.28)

where \( \tau \) is a real. \( \tau = \text{Re}(\beta \xi - \lambda (x - \xi)) > 0 \), \( \text{Im}(\beta \xi - \lambda (x - \xi)) = 0 \), so that

\[
[\tau + \lambda (x - \xi)]^2 = \left( b^2 - \lambda^2 \right) z^2
\]
\[
\left[(x - \xi)^2 + z^2\right] \lambda^2 + 2(x - \xi)\tau \lambda + \tau^2 - b^2 z^2 = 0,
\]

\[
\lambda_{\pm} = \frac{-2(x - \xi)\tau \pm i\sqrt{-4(x - \xi)^2 \tau^2 + 4(\tau^2 - b^2 z^2)(x - \xi)^2 + z^2}}{2((x - \xi)^2 + z^2)}
\]

\[
= \frac{\tau(x - \xi) \pm iz\sqrt{\tau^2 - b^2 r^2}}{r^2}, \quad (3.29)
\]

Substitution of equation (3.29) into equation (3.28) gives

\[
\beta_{\pm} = \frac{\tau + \lambda(x - \xi)}{z} = \frac{\tau \pm i(x - \xi)\sqrt{\tau^2 - b^2 r^2}}{r^2} \quad (3.30)
\]

Applying Cauchy’s theorem, we can obtain from the second integral of equation (3.15), which is the rotational contribution due to \(\psi\):

\[
\hat{\sigma}_{xx}^{(\psi)} = \mu \frac{\Delta u}{2b^2} \frac{s}{2\lambda} \left\{ \int_{b+}^{b+} \frac{(b^2 - 2\lambda^2)^2}{\beta_+} e^{-s\eta(x)} d\xi e^{-s(\lambda, \xi - \lambda, x + \beta, z)} d\lambda_+ \right. \\
- \int_{b-}^{b-} \frac{(b^2 - 2\lambda^2)^2}{\beta_-} e^{-s\eta(x)} d\xi e^{-s(\lambda, \xi - \lambda, x + \beta, z)} d\lambda_- \left\} \quad (3.31)
\]

Figure 3.3 Integration contour II for edge dislocation
Applying the same procedure we use for the first integral in equation (3.15), we can obtain for the second integral in equation (3.15)

\[
\dot{\sigma}^{(w)}_{xz} = \mu \frac{\Delta u}{2\pi b^2} \int_{0}^{\infty} \int_{0}^{\infty} \left( H(t - \eta(\xi) - br)F_2(t - \eta(\xi), \xi) \right) d\eta d\xi ,
\]

where \( F_2(t - \eta(\xi), \xi) = F_2(t, \xi) = \text{Im} \left( \frac{b^2 - 2\lambda^2}{\beta_\xi} \frac{\partial \lambda_x}{\partial \tau} \right) . \) (3.33)

Furthermore, from equation (3.32), we obtain

\[
\dot{\sigma}^{(w)}_{xz} = \mu \frac{\Delta u}{2\pi b^2} \int_{0}^{\infty} e^{-u} \frac{\partial}{\partial t} \int_{0}^{\infty} \left( H(t - \eta(\xi) - br)F_2(t - \eta(\xi), \xi) \right) d\eta d\xi dt \] (3.34)

Performing the inversion from \( s \) to \( t \) by inspection in equation (3.34), we obtain the rotational part of the stress \( \sigma_{xz} : \)

\[
\sigma^{(w)}_{xz} = \mu \frac{\Delta u}{2\pi b^2} \frac{\partial}{\partial \xi} \int_{0}^{\infty} \left( H(t - \eta(\xi) - ar)F_1(t - \eta(\xi), \xi) \right) d\eta d\xi ,
\]

Combining equations (3.27) and (3.35), we obtain the expression of the solution for stress \( \sigma_{xz} \) in Problem II:

\[
\sigma_{xz} = \sigma^{(w)}_{xz} + \sigma^{(w)}_{xz}
\]

\[
= 2\mu \frac{\Delta u}{\pi b^2} \frac{\partial}{\partial t} \int_{0}^{\infty} \left( H(t - \eta(\xi) - ar)F_1(t - \eta(\xi), \xi) \right) d\eta d\xi \left. \right|_{\eta=\eta(\xi)}
\]

\[
\mu \frac{\Delta u}{2\pi b^2} \frac{\partial}{\partial \xi} \int_{0}^{\infty} \left( H(t - \eta(\xi) - br)F_2(t - \eta(\xi), \xi) \right) d\eta d\xi \] (3.36)

From equations (3.17), (3.18), we can obtain

\[
F_1(t - \eta(\xi), \xi) = \text{Im} \left( \lambda_\xi^2 \lambda_x \frac{\partial \lambda_x}{\partial \tau} \right) \left. \right|_{\tau=\eta(\xi)}
\]
Let \( \tilde{x} = x - \xi \), \( T_a = \left( \tau^2 - a^2 r^2 \right)^{\frac{1}{2}} \), then equation (3.37) can be written as

\[
F_1(\tau, \xi) = \text{Im} \left[ \left( -\frac{\tilde{x} + iz T_a}{r^2} \right)^2 \left( \frac{\tau x + i z T_a}{r^2} \right) \frac{\partial}{\partial \tau} \left( \frac{\tau (\xi - x) + iz \left( \tau^2 - a^2 r^2 \right)^{\frac{1}{2}}}{r^2} \right) \right]_{x = t - \eta(\xi)}
\]

(3.37)

Let \( \tilde{x} = x - \xi \), \( T_a = \left( \tau^2 - a^2 r^2 \right)^{\frac{1}{2}} \), then equation (3.37) can be written as

\[
F_1(\tau, \xi) = \text{Im} \left[ \left( -\frac{\tilde{x} + iz T_a}{r^2} \right)^2 \left( \frac{\tau x + i z T_a}{r^2} \right) \frac{\partial}{\partial \tau} \left( \frac{\tau (\xi - x) + iz \left( \tau^2 - a^2 r^2 \right)^{\frac{1}{2}}}{r^2} \right) \right]_{x = t - \eta(\xi)}
\]

(3.37)

\[
= \text{Im} \left[ \frac{(-z^2 T_a^2 - 2iz T_a^2 \tilde{x} + \tau^2 \tilde{x}^2)}{r^8} \left( -2i \tilde{x} + i (\tau^2 z^2 T_a^{-1} - \tilde{x}^2 T_a) \right) \right]_{x = t - \eta(\xi)}
\]

\[
= \frac{(4z^2 T_a \tau^2 \tilde{x}^2 - (\tau^2 z^2 T_a - z^2 \tilde{x}^2 T_a) + (\tau^4 z^2 T_a^{-1} \tilde{x}^2 - \tau^2 \tilde{x}^4 T_a \tilde{x}^2)}{r^8}
\]

\[
= r^{-8} \left[ \frac{1}{T_a} \left( \tau^2 z^2 - T_a \tilde{x}^2 \right) \left( \tilde{x}^2 T_a^2 - T_a \tilde{x}^2 z^2 \right) + \frac{1}{T_a} \left( 4 \tau^2 T_a \tilde{x}^2 z^2 \right) \right]
\]

(3.38)

let \( \tau = \tilde{t} = t - \eta(\xi) \), we obtain

\[
F_1(t - \eta(\xi), \xi) = r^{-8} \left[ \frac{1}{T_a} \left( \tilde{t}^2 z^2 - T_a \tilde{x}^2 \right) \left( \tilde{x}^2 T_a^2 - T_a \tilde{x}^2 z^2 \right) + \frac{1}{T_a} \left( 4 \tilde{t}^2 T_a \tilde{x}^2 z^2 \right) \right]
\]

(3.39)

Similarly, from equations (3.29) and (3.30), we can obtain

\[
F_2(t - \eta(\xi), \xi) = \text{Im} \left[ \left( \frac{b^2 - 2 \lambda^2}{\beta_+} \right) \frac{\partial \lambda_+}{\partial \tau} \right]_{t = t - \eta(\xi)}
\]

\[
= \text{Im} \left[ \left( \frac{b^2 - 2 \left( -\frac{\tilde{x} + iz T_b}{r^2} \right)^2}{\frac{\tau x + iz T_b}{r^2}} \right)^2 \frac{\partial}{\partial \tau} \left( \frac{\tau (\xi - x) + iz \sqrt{\tau^2 - b^2 r^2}}{r^2} \right) \right]_{t = t - \eta(\xi)}
\]

(3.38)
\[
\sigma_{xz} = \mu \frac{2\Delta u}{\pi b^2 \gamma t} \int_{0}^{\infty} \frac{H(\gamma - \alpha r) \left( (\tilde{r}^2 z^2 - T_b^2 z^2) (\tilde{r}_a^2 z^2 - T_a^2 z^2) + \frac{1}{T_a} (\tilde{r}^2 T_a^2 z^2) \right) d\xi}{r^8 T_a} \\
+ \mu \frac{\Delta u}{2\pi b^2 \gamma t} \int_{0}^{\infty} \frac{H(\gamma - \alpha r) \left( (\tilde{r}^2 r^2 - 2(\tilde{r}_a^2 z^2 - T_a^2 z^2) + \frac{1}{T_a} (\tilde{r}^2 T_a^2 z^2) \right) d\xi}{r^8 T_b} 
\]

where \( T_b = (\tilde{r}^2 - b^2 r^2)^{1/2} \).

Therefore, the solution for problem II can be obtained by substituting equations (3.39) and (3.40) into equation (3.36):

\[
\sigma_{xz} = \mu \frac{2\Delta u}{\pi b^2 \gamma t} \int_{0}^{\infty} \frac{H(\gamma - \alpha r) \left( (\tilde{r}^2 z^2 - T_b^2 z^2) (\tilde{r}_a^2 z^2 - T_a^2 z^2) + \frac{1}{T_a} (\tilde{r}^2 T_a^2 z^2) \right) d\xi}{r^8 T_a} \\
+ \mu \frac{\Delta u}{2\pi b^2 \gamma t} \int_{0}^{\infty} \frac{H(\gamma - \alpha r) \left( (\tilde{r}^2 r^2 - 2(\tilde{r}_a^2 z^2 - T_a^2 z^2) + \frac{1}{T_a} (\tilde{r}^2 T_a^2 z^2) \right) d\xi}{r^8 T_b} 
\]

3.3 Analytical solution for an edge dislocation starting from rest and moving with constant transonic velocity

By direct formulation of the problem as a constant-velocity one, the solution may be obtained by a procedure similar to Section 3.2 with an additional term is introduced due to the fact that in the transonic case a pole is included in the inversion contour:

In equation (3.15), let \( \eta(\tilde{z}) = d \tilde{z} \), where \( d \) satisfies \( a < d < b \), we obtain

\[
\hat{\sigma}_{xz} = \mu \left\{ \frac{2 \Delta u}{b^2} \frac{s}{2\pi t} \int_{br}^{R} \tilde{\tilde{r}}^2 e^{-i[d \tilde{z} + \xi \lambda]} d\tilde{z} e^{-i\xi \lambda} e^{i\lambda \xi} d\lambda + \right\} \\
+ \frac{\Delta u}{2b^2} \frac{s}{2\pi t} \int_{br}^{R} \frac{\left( b^2 - 2\tilde{\tilde{r}}^2 \right) e^{-i[d \tilde{z} + \xi \lambda]} e^{-i\xi \lambda} e^{i\lambda \xi} d\tilde{z} d\lambda \right\} 
\]

In equation (3.42), making the integration in \( \tilde{z} \) directly, we can obtain

\[
\hat{\sigma}_{xz} = \mu \left\{ \frac{2 \Delta u}{b^2} \frac{s}{2\pi t} \int_{br}^{R} \frac{\tilde{\tilde{r}}^2 \alpha}{s(d + \lambda)} e^{-i\xi \lambda} e^{i\lambda \xi} d\lambda + \frac{\Delta u}{2b^2} \frac{s}{2\pi t} \int_{br}^{R} \frac{\left( b^2 - 2\tilde{\tilde{r}}^2 \right) e^{-i\xi \lambda} e^{i\lambda \xi} d\lambda \right\} 
\]

(3.43)
Applying the Cagniard-de Hoop technique, in the second term of equation (3.43), let

\[ \tau = \beta z - \lambda x = \left( b^2 - \lambda^2 \right) z - \lambda x \]  

(3.44)

where \( \tau \) is a real. \( \tau = \text{Re}(\beta z - \lambda x) > 0, \text{Im}(\beta z - \lambda x) = 0 \), so that

\[
\begin{align*}
\tau + \lambda x & = \left( b^2 - \lambda^2 \right) z^2, \\
\left[ x^2 + z^2 \right] \lambda^2 + 2x \tau \lambda + \tau^2 - b^2 z^2 & = 0, \\
\lambda_{\pm} & = \frac{-2x \tau \pm \sqrt{-4x^2 \tau^2 + 4\left( \tau^2 - b^2 z^2 \right) \left( x^2 + z^2 \right)}}{2(x^2 + z^2)} = -\tau \pm i\frac{\sqrt{\tau^2 - b^2 r^2}}{r^2}. \\
\beta_{\pm} & = \frac{\tau + \lambda x}{z} = \frac{\tau \pm i\frac{\sqrt{\tau^2 - b^2 r^2}}{r^2}}{z}.
\end{align*}
\]

(3.45)

(3.46)

From equation (3.45), we can see that \( \tau \geq rb \). Therefore \( \text{Re} \lambda = -\frac{\tau x}{r^2} \leq -\frac{bx}{r} \).

Since \( d < b \), so \( -d > -b \); also \( -b \frac{\lambda}{r} > -b \), it is possible that \( -\frac{bx}{r} = -d \). When

\[ \lambda = -\frac{bx}{r}, \]

it is possible that \( \lambda + d = 0 \). Therefore the second term in the equation (3.43) has a pole at \( \lambda = -d \).

According to Cauchy’s theorem to the second term of equation (3.43), we obtain

\[
\int_{B_r} - \int_{B_1} + \int_{B_2} - \int_{B_3} - \int_{B_4} + \int_{B_5} \]

(3.47)

Because there is a pole included in the integration contour, we need to introduce the integration circle \( B_5 \) around the pole \( \lambda = -d \).
On $B_2$ and $B_3$, we can use the same result obtained in the previous section by the Cagniard-de Hoop technique used to obtain the general solution for subsonic motion. This consists the first part of the solution.

On $B_4$, $\lambda = \text{Re} \lambda + ie'$; on $B_4$, $\lambda = \text{Re} \lambda - ie'$. We can prove that $\int_{B_1} - \int_{B_4} = 0$.

On $B_5$, $\lambda = -d + \varepsilon e^{i\theta}$, where $\varepsilon > 0$, $-\pi < \theta < \pi$. Therefore as $\varepsilon \to 0$,

$$\lambda^2 = \left(-d + \varepsilon e^{i\theta}\right)^2 = d^2 - 2d\varepsilon e^{i\theta} + \varepsilon^2 e^{2i\theta} \approx d^2,$$

$$\beta = \left(b^2 - \lambda^2\right)^\frac{1}{2} = \left(b^2 - d^2 + 2d\varepsilon e^{i\theta} - \varepsilon^2 e^{2i\theta}\right)^\frac{1}{2} \approx \left(b^2 - d^2\right)^\frac{1}{2},$$

$$\left(b^2 - 2\lambda^2\right)^\frac{1}{2} = \left(b^2 - 2d^2 + 4d\varepsilon e^{i\theta} - 2\varepsilon^2 e^{2i\theta}\right)^\frac{1}{2} \approx \left(b^2 - 2d^2\right)^\frac{1}{2},$$

$$d\lambda = \varepsilon e^{i\theta} id\theta,$$

$$\tau = \beta z - \lambda x = \left(b^2 - \lambda^2\right)z - \lambda x \approx \left(b^2 - d^2\right)z + dx.$$
Figure 3.4 Integration contours for edge dislocation starting from rest and moving with constant transonic velocity.

The second term of equation (3.43) at $B_2$ would become

$$\hat{\sigma}_{x\varepsilon}^{(v)} \bigg|_{B_2} = \frac{\Delta u}{2b^2} \int_{\theta} \frac{d^2}{d^2} e^{-s} e^{sID} d\theta$$

$$= \frac{\Delta u}{2b^2} \int_{\theta} \frac{d^2}{d^2} e^{-s} (b^2 - 2d^2) \cdot \varepsilon e^{sID} d\theta$$
Use the same technique in Chapter 2.3, the inverse of \( \sigma_{xz} \big|_{B^i} \) can be obtained from equation (3.53):

\[
\mu \frac{\Delta u}{2b^2} \left( \frac{b^2 - 2d^2}{b^2 - d^2} \right)^3 \delta(t - dx - (b^2 - d^2)^{1/2} z)
\]  

(3.54)

Here we apply the following property: \( L(\delta(t - t_0)) = e^{-st_0}. \)

Therefore we get a delta function from the existence of a pole \( \lambda = -d \). When \( \eta(\xi) = d\xi \), the final solution for \( \sigma_{xz} \) at \( z = 0 \) will be

\[
\sigma_{xz} = \frac{\mu \Delta u}{2\pi b^2} \left\{ \frac{4t^2(t^2 - a^2x^2)^{1/2} H(t - ax)}{x^3(t - ax)} - \frac{(2t^2 - b^2x^2)^2 H(t - bx)}{x^3(t^2 - b^2x^2)^{1/2}(t - dx)} \right\} \\
+ \frac{\mu \Delta u}{b^2} \left( \frac{b^2 - 2d^2}{2(b^2 - d^2)^{1/2}} \right) \delta[t - dx]
\]  

(3.55)
4.1 Mach wave front equations for generally accelerating supersonic screw Volterra dislocation

For a screw Volterra dislocation which is at rest in an infinite space until time \( t = 0 \) when it begins to move according to \( x = l(t) \) (where \( l(t) \) is a general function) or equivalently by inverting it, \( t = \eta(x) \), the wave front is seen as the envelop of the wavelets with equation:

\[
(x - l(\tau))^2 + z^2 = \frac{1}{b^2} (t - \tau)^2
\]  

(4.1)

which the dislocation emits on the way, where \( b = \frac{1}{c_2} \), \( c_2 \) being the shear wave speed. In equation (4.1), \( 0 \leq \tau \leq t \).

For subsonic motion, the wavelets emitted by the dislocation on its way do not overlap and cannot form an envelope (see figure 4.1), so in this case the wave front is the circle \( t = rb \).
Figure 4.1 For subsonic case, wavelets doesn’t overlap, no envelope is formed.

When the motion becomes supersonic, however, the wavelets emitted overlap with each other and form an envelope, which is the shock-wave front. Wave front for supersonic screw dislocation starting from rest and moving with constant velocity has been analyzed in Chapter 2.

Here, we will analyze the wave front for supersonic motion with acceleration or deceleration and the forming of the wave front will be illustrated later in this chapter.

These wavelets emitted by a dislocation during its motion \( x = l(\tau) \) are at time \( t \) on the circle:

\[
F(\tau) = (x - l(\tau))^2 + z^2 - c^2(t - \tau)^2 = 0 \tag{4.2}_1
\]

and in terms of \( \xi \):

\[
f(\xi) = t - \eta(\xi) - \sqrt{(x - \xi)^2 + z^2} b = 0 \tag{4.2}_2
\]

The envelope of these circles is the line satisfying equations (4.2) as well as

\[F'(\tau) = 0 \text{ or } f'(\xi) = 0, \text{ i.e.} \]

\[
\frac{d}{d\tau} F(\tau) = -l'(\tau)(x - l(\tau)) + c^2(t - \tau) = 0, \tag{4.3}_1
\]
and in terms of \( \xi \):

\[
\frac{d}{d\xi} f(\xi) = -\eta'(\xi) + \frac{(x - \xi)b}{\sqrt{(x - \xi)^2 + z^2}} = 0
\]  

Equations (4.2) and (4.4) yield the parametric form of the envelope (the wave front for the moving dislocation), namely

\[
x \equiv g(\tau) = l(\tau) + \frac{c^2}{l'(\tau)}(t - \tau)
\]  

\[
z \equiv h(\tau) = \pm c(t + \tau) \left\{ 1 - \frac{c^2}{l'(\tau)^2} \right\}^{\frac{1}{2}}
\]

Moreover, from the above equations, we have

\[
g'(\tau) = l''(\tau) \left( 1 - \frac{c^2}{l'(\tau)^2} \right) - \frac{c^2 l''(\tau)}{l'(\tau)^2} (t - \tau)
\]  

\[
\pm h'(\tau) = -c \left\{ 1 - \frac{c^2}{l'(\tau)^2} \right\}^{\frac{3}{2}} + c^3 (t - \tau) l''(\tau) \left\{ 1 - \frac{c^2}{l'(\tau)^2} \right\}^{\frac{1}{2}}
\]

So that the slope of the layout to the envelop is also given parametrically. From equations (4.5) it is seen that the slope of the tangent depends linearly on the acceleration of the dislocation \( l''(\tau) \).

Thus, we see that a dislocation starting from rest moves subsonically and at time \( t \) such that \( V_d \equiv at = \frac{1}{b} \) or \( t = \frac{1}{ab} \) the motion becomes supersonic until that time the wavelets do not have envelopes, since they do not intersect and are inside the circle \( t = rb \). For the first wavelet to get out of the circle \( t = rb \), we must have
\[
\frac{1}{2} a \tau^2 + c (t - \tau) = ct, \text{ or } \tau^0 \left( \frac{1}{2} a \tau^0 - c \right) = 0, \text{ or } \tau^0 = \frac{2}{ab}. \]

The wavelets that are now emitted form an envelope that intersects the circle \( t = rb \) at the point that satisfies

\[
ct \cos \theta = x = l(\tau^0) + \frac{c^2}{l'(\tau^0)^2} (t - \tau^0) \quad (4.6)_1
\]

\[
ct \sin \theta = z = \pm c (t - \tau^0) \left( 1 - \frac{c^2}{l'(\tau^0)^2} \right)^{\frac{1}{3}} \quad (4.6)_2
\]

So that

\[
\frac{1}{c^2 t^2} \left( l(\tau^0) + \frac{c^2}{l'(\tau^0)^2} (t - \tau^0) \right)^2 + \left( \frac{t - \tau^0}{t^2} \right)^2 \left( 1 - \frac{c^2}{l'(\tau^0)^2} \right) = 1 \quad (4.7)
\]

The roots of which \( \tau^0 = \tau^0 (t) \) define the intersection of the envelope (Mach front) with the circle \( t = rb \), which is the subsonic wave-front.

### 4.2 Evolution of Mach wave fronts for constant acceleration motion and transition from subsonic to supersonic motion

The following figures show the wavelets and their envelopes as the motion starts at \( \tau = 0 \) is subsonic \( 0 < \tau < \frac{1}{ab} < t \) in which case the wavelets do not form envelopes, becomes supersonic after that, when envelopes form inside the circle \( t = rb \), which subsequently crosses the \( t = rb \) and the Mach fronts are outside.

We take \( a = 4m/s^2 \), \( c = 1m/s \), \( b = 1s/m \) (we take \( c = 1m/s \) to normalize this problem) so that \( l(\tau) = 2\tau^2 \), draw the wavelets for different \( \tau \) and study their
relationship and how they form envelope. The evolution of the Mach wave fronts is shown in the Figures 4.2-4.6.

These following figures at different times illustrate how the transition from subsonic into supersonic motion for a dislocation starting from rest at $t=0$ and how the wave front is formed by the wavelets.

When $\tau < 0.25$, the motion of the dislocation is subsonic. So when $\tau = 0.1$, there is no envelop (Figure 4.2). The outside circle is the wavelet let off at $\tau = 0$ (namely $r = t = 0.1$).

![Diagram showing wavelet formation](image)

Figure 4.2 when $t = 0.1s$, the motion is subsonic and the wavelets do not overlap. There is no envelope. $c$ is normalized to $1 m/s$.
When \( t = 0.25 \), the velocity of the dislocation equals the wave velocity (Figure 4.3). This is the transition time when wavelets start to cross with each other and form envelope.

![Figure 4.3](image)

Figure 4.3 when \( t = 0.25 \) s, the velocity equals to the wave speed.

The envelope starts to form. \( c \) is normalized to \( 1 m/s \)

In order to make it supersonic, it is needed that \( t \geq 0.25 \). So \( t \geq \tau \geq 0.25 \).

Draw the envelops from the envelop equation for different time \( t \geq 0.25 \).

When \( t > 0.25 \), the motion is supersonic. The wavelets overlap each other. They start to form the envelop inside the circle \( r = t \).

For \( t = 0.3 \), the envelope is inside the circle \( r = t = 0.3 \) (Figure 4.4).

For \( t = 0.4 \), the envelope is inside the circle \( r = t = 0.4 \) (Figure 4.5).
Figure 4.4 when $t = 0.3s$, the motion is supersonic and the envelope is still inside the circle $r = 0.3m$. $c$ is normalized to $1 m/s$. 
Figure 4.5 when \( t = 0.4s \), the motion is supersonic and the envelope is still inside the circle \( r = 0.4m \). \( c \) is normalized to \( 1m/s \)

At time \( t = 0.5s \), the wavelets emitted by the dislocation on its way at time, \( 0.5 \geq \tau \geq 0.25 \) form the envelope, which reaches the circle \( r = 0.5 \) (which is the first wavelet let off at \( \tau = 0 \), namely the subsonic wave-front) (Figure 4.6).
Figure 4.6 when $t = 0.5s$, the motion is supersonic and the envelope touches the circle $r = 0.5m$. $c$ is normalized to $1m/s$.

So, wavelets emitted at time starting from $\tau \geq 0.25$ (the condition to be supersonic) will form the envelop at $t = 1$ (see Figure 4.7).
Figure 4.7 when $t = 1s$, the motion is supersonic and the envelope grows out of the circle $r = 1m$. $c$ is normalized to $1m/s$.

In the next part, we will discuss the curvature of the wave front.

### 4.3 Effect of acceleration on curvature of Mach wave-front

For different motions (namely, different $l(t)$) of dislocation, we study the shape of the envelopes. Geometrically, the envelope consists of two mirror curves jointed at the position of the dislocation. One of the important properties of shape of a curve is its curvature.

The curvature of a parametric plane curve is
\[ \kappa = \frac{\ddot{g}(\tau)h(\tau) - \dot{g}(\tau)\ddot{h}(\tau)}{\left(\dot{g}(\tau)^2 + \dot{h}(\tau)^2\right)^{\frac{3}{2}}} \]  \hspace{1cm} (4.8)

where \( x = g(\tau), z = h(\tau) \). By using the parametric form of the envelope (equations (4.4)), we can find the curvature of the envelope (the Mach wave front).

For dislocation starting from rest and moving with constant velocity, i.e., \( l(\tau) = v_d \tau \), where \( v_d \) is the velocity of the moving dislocation, the envelope can be written in parametric form as:

\[ x \equiv g(\tau) = v_d \tau + \frac{c^2}{v_d} (t - \tau) \]  \hspace{1cm} (4.9)_1

\[ z \equiv h(\tau) = \pm \left( t + \tau \right) \left\{1 - \frac{c^2}{v_d^2}\right\}^{\frac{1}{2}} \]  \hspace{1cm} (4.9)_2

From the above equations we can see that \( v_d \geq c \) is the condition for the envelope to start to form.

The corresponding curvature for the dislocation starting from rest and moving with constant velocity is \( \kappa = 0 \). So the envelop for dislocation starting from rest and moving with constant supersonic velocity are two straight lines. Please see Figure 4.8, where \( v_d = 2m/s, c = 1m/s \) (normalized).
Figure 4.8 At time $t = 2s$, Mach wave-front for a dislocation with constant supersonic velocity $(v_d \geq c)$ (thus zero acceleration), $v_d = 2m/s$,

c is normalized to $1m/s$.

For dislocation starting from rest and moving with an acceleration, the envelope will be curved. See Figure 4.9, 4.10, 4.11, where the curved envelopes are formed by moving dislocations that all have the same velocity $2m/s$ at time $t = 1s$. 

Figure 4.9  At $t = 1s$, effect of acceleration on curvature of Mach wave front for a dislocation starting from rest and moving according to $l(\tau) = \frac{4}{3} \tau^3$, $c$ is normalized to $1m/s$. 
Figure 4.10  At $t = 1s$, effect of acceleration on curvature of Mach wave front for a dislocation starting from rest and moving according to

$$l(\tau) = \frac{4}{e} e^\tau,$$  
$c$ is normalized to $1m/s$. 
Figure 4.11 At $t = 2s$, effect of acceleration on curvature of Mach wave front for a dislocation starting from rest and moving according to

$$l(\tau) = \frac{4}{e^\tau}, \quad c \text{ is normalized to } 1m/s.$$  

The curvature can be calculated for the corresponding dislocation motion at time $t$:

For quadratic motion $l(\tau) = 2\tau^2$: curvature $\kappa = \pm \frac{1}{16\tau^5 - t}$. 

For cubic equation \( l(\tau) = \frac{4}{3} \tau^3 \): curvature \( \kappa = \pm \frac{2}{16\tau^5 + \tau - 2t} \).

For exponential motion \( l(\tau) = \frac{4}{e^\tau} \): curvature \( \kappa = \pm \frac{1}{16 e^{2\tau} - t + \tau - 1} \).

Where \( \tau \) corresponds to the wavelet emitted at time \( \tau \) and forming part of the envelope. The intersection point of the wavelet and the envelope can be found using equation (4.4).

In the following, we compare the curvatures for different motions in the same figure. Solid line ‘''' denotes cubic motion; dotted line ‘:' denotes quadratic motion; dash dot line ‘-.’ denotes exponential motion.

Figure 4.12 At \( t = 1s \), curvatures versus \( \tau \), the time when the wavelet is emitted, for different accelerating motions. \( c \) is normalized to \( 1m/s \).
An infinity in curvature signifies a cusp. We can see from Figure 4.13 that a dislocation moving according to the exponential function \( l(\tau) = \frac{4}{e^\tau} \) will not form any cusp at time \( t = 1s \). From its curvature expression \( \kappa = \pm \frac{1}{16 \frac{\tau}{e^\tau} - t + \tau - 1} \), we can see that only when \( \frac{16}{e^\tau} e^{2\tau} - t + \tau - 1 = 0 \), there will have an infinity. Since \( 0 \leq \tau \leq t \), so the condition for it to be satisfied is \( t \geq \frac{16}{e^2} - 1 = 1.1654 \). So when \( t=1s \), there is no infinity in
curvature for dislocation motion \( l(\tau) = \frac{4}{e^\tau} \). From Figure 4.13, we can see that at \( t = 3s \), there will be infinity for its curvature. For the other two dislocation motions, their curvatures will have infinity and therefore there will be cusp on the wave-front right from the start \( t = 0s \).

We compare the wave-fronts for the above three different dislocation motion in Figure 4.14, 4.15.

Solid line ‘-’ denotes cubic motion; dotted line ‘:’ denotes quadratic motion; dash dot line ‘-.’ denotes exponential motion.

\[
- l(\tau) = \frac{4}{3} \tau^3 \\
\cdots \quad l(\tau) = 2 \tau^2 \\
- \cdot l(\tau) = \frac{4}{e^\tau}
\]

Figure 4.14  \( t = 1s \), wave fronts for different accelerating motions. \( c \) is normalized to \( 1m/s \).
(a) the full graph

Figure 4.15 at t=3s, wave fronts for different accelerating motions. $c$ is normalized to $1m/s$.
Figure 4.15 at t=3s, wave fronts for different accelerating motions. $c$ is normalized to 1 m/s.

From Figure 4.14, 4.15, we can see that there is a difference in shape between exponential motion and the other two motions. From $t = \tau = 0$, the exponential motion $l(\tau) = \frac{4}{e} e^\tau$ has a velocity $\frac{4}{e}$ which is greater than the wave velocity $c = 1$ m/s. So the exponential motion is supersonic right from start. While the other two motions starts with zero velocity.

The infinity of curvature corresponds to the cusp of the envelope. From Figure 4.14, we can see that the wave-front for exponential motion $l(\tau) = \frac{4}{e} e^\tau$ has no cusp since
it has no infinity with its curvature at \( t = 1s \). At \( t = 3s \), we can see from the Figure 4.15(b) that there is a cusp on the wave-front for the exponential motion. While for the two other motions there are cusps at both \( t = 1s \) and \( t = 3s \) since they have singularities with their curvatures right from \( t=0 \).

Part of the material in this chapter is being prepared to submit for publication. Prof. Xanthippi Markenscoff is the co-author for this paper.
5 STRESS FIELD ANALYSIS OF SINGULARITY FOR VOLTERRA SCREW AND EDGE DISLOCATION ACCELERATING THROUGH THE SHEAR WAVE SPEED BARRIER

5.1 Wave front equations for a Volterra Screw Dislocation Accelerating through the Shear Wave Speed Barrier

For a Volterra Screw dislocation starting from rest and moving non-uniformly \((t = \eta(\xi))\), the solution has been obtained in equation (2.89):

\[
\frac{\partial u}{\partial z}(x, z, t) = -\frac{\Delta u}{2\pi} \int_0^\infty (t - \eta(\xi))(x - \xi)^2 H(t - \eta(\xi) - rb) \frac{d\xi}{r^4[(t - \eta(\xi))^2 - r^2b^2]^2} \\
+ \frac{\Delta u}{2\pi} \int_0^\infty (t - \eta(\xi))^2 H(t - \eta(\xi) - rb) \frac{d\xi}{r^4[(t - \eta(\xi))^2 - r^2b^2]^2} \frac{\partial}{\partial t} - \frac{\Delta u x}{2\pi x^2 + z^2}
\]

where \(r^2 = (x - \xi)^2 + z^2\).

The solution shown above is valid for both the subsonic and supersonic case, but the nature of the argument of the step function \(f(\xi) = t - \eta(\xi) - rb\) is different for the subsonic and supersonic case.

The wave front of a dislocation not only separate deformed region and undeformed region, it also has a concentration in the stress at the wave front.
We need to evaluate the stress at the wave front at the time that the shear wave speed barrier is crossed.

5.1.1 Analysis of the function $f(\xi) = t - \eta(\xi) - rb$ for constant velocity case

In order to study the wave-front behavior of the solution given above, we have to look closer at the behavior of the integral, and first, at the form of the function $f(\xi) = t - \eta(\xi) - rb$ which defines the intervals of the integration. Only when $f(\xi) = t - \eta(\xi) - rb \geq 0$, the step function $H(t - \eta(\xi) - rb) \neq 0$, there is contribution to the solution (2.89).

For the constant-velocity case, we discuss the range of the integration:

Let $\eta(\xi) = \alpha \xi$, $f(\xi) = t - \alpha \xi - rb = t - \alpha \xi - b\sqrt{(x - \xi)^2 + z^2} = 0$, we obtain

\[ t^2 - 2\alpha t \xi + \alpha^2 \xi^2 = b^2(x - \xi)^2 + b^2z^2, \]

\[ \xi_{\pm} = \frac{\alpha - b^2x \pm b\sqrt{(t - \alpha \xi)^2 - z^2(b^2 - \alpha^2)}}{\alpha^2 - b^2}, \]

\[ (\xi - x)_{\pm} = \frac{\alpha(t - \alpha \xi) \pm b\sqrt{(t - \alpha \xi)^2 - z^2(b^2 - \alpha^2)}}{\alpha^2 - b^2} \] (5.1)

When $z = 0$, $f(\xi) = t - \alpha \xi - b|x - \xi| = 0$, we obtain

For $x > \xi$, $\xi_0 = \frac{bx - t}{b - \alpha}$; for $x < \xi$, $\xi_1 = \frac{bx + t}{\alpha + b}$. (5.2)

Therefore, $f(\xi)$ can be plotted in Figure 5.1(a) and Figure 5.1(b) for supersonic and subsonic motion, respectively.
Figure 5.1 $f(\xi) = t - \eta(\xi) - rb$ for subsonic and supersonic dislocation
For subsonic motion, the lower limit of integration is always zero, while for supersonic motion, the interval of the integration may not contain the origin and \( f(\xi) \) exhibits a maximum in the range of integration. For a particular combination of \( t, x, z \), the range of integration vanishes (namely \( f(\xi) = 0 \)). It may be easily seen that this particular combination corresponds to the wave-front.

In the following, for a general motion, we will study the particular combination of conditions that corresponds to the wave-front.

Suppose \( \xi_0, \xi_1 \) are roots of the function \( f(\xi) = t - \eta(\xi) - rb \).

Let the range \([\xi_0, \xi_1]\) of integration tends to zero, i.e. \( \xi_0, \xi_1 \to \xi^* \)(Please see Figure 5.2).

If \( f(\xi^*) = t - \eta(\xi^*) - rb \) has a maximum in this range, the wave front equation can be defined by

\[
f(\xi^*) = 0 \tag{5.3}
\]

\[
\frac{df(\xi^*)}{d\xi} = 0 \tag{5.4}
\]

namely

\[
f(\xi^*) = t - \eta(\xi^*) - \sqrt{(x - \xi^*)^2 + z^2b} = 0 \tag{5.5}
\]

\[
\frac{d}{d\xi} (t - \eta(\xi^*) - \sqrt{(x - \xi^*)^2 + z^2b}) = 0 \tag{5.6}
\]
Since the curve \( f(\xi^*) = t - \eta(\xi^*) - rb = 0 \) separates the zero and nonzero range of integration, therefore separate the deformed region and the un-deformed region, the solution for equation (5.3) and (5.4) obviously should correspond to the wave-front.

By analyzing the wave front equations, we can find the property of the wave front itself and the property of stress field on the wave front. In the following, we will study two particular cases where we can obtain explicit results: dislocation starting from rest and moving with constant velocity; dislocation starting from rest and moving with constant acceleration.

For dislocation starting from rest and moving with constant velocity, i.e., \( l(\tau) = \frac{\tau}{\alpha} \), or inversely \( \eta(\xi) = \alpha \xi \), the wave front can be written as

\[
f(\xi) = t - \alpha \xi - \sqrt{(x - \xi)^2 + z^2} b = 0
\]

(5.7)

\[
\frac{d}{d\xi} f(\xi) = -\alpha + \frac{(x - \xi)b}{\sqrt{(x - \xi)^2 + z^2}} = 0
\]

(5.8)
From equation (5.7),

\[ f(\xi) = (t - \alpha x + \alpha - \alpha \xi) - \sqrt{(x - \xi)^2 + z^2} b = 0 \]

\[ (t - \alpha x + \alpha - \alpha \xi)^2 = \left( (x - \xi)^2 + z^2 \right) b^2 \]

So

\[ (\xi - x) = \frac{\alpha(t - \alpha x) \pm \sqrt{\alpha^2(t - \alpha x)^2 - (\alpha^2 - b^2)(t - \alpha x)^2 - b^2 z^2}}{\alpha^2 - b^2} \]

(5.9)

When \( \alpha^2(t - \alpha x)^2 - (\alpha^2 - b^2)(t - \alpha x)^2 - b^2 z^2 = 0 \), namely

\[ z = \pm \frac{t - \alpha x}{(b^2 - \alpha^2)^{1/2}} \]

(5.10)

where we need to have \( b > \alpha \), namely the velocity of the moving dislocation must be bigger than the shear wave speed and it is moving supersonically.

We will get two same real roots

\[ (\xi - x) = (\xi - x) = \frac{\alpha(t - \alpha x)}{\alpha^2 - b^2} \]

(5.11)

So we can conclude that \( z = \pm \frac{t - \alpha x}{(b^2 - \alpha^2)^{1/2}} \) is the wave front which in this case is two straight lines (see Figure 2.9). On the wave front, the equations for wave front will have two equal real roots.

For \( b < \alpha \), there is no real root for the equations. Therefore, there is no shock-wave front for subsonic case.

For \( b > \alpha \):

When \( |z| > \frac{t - \alpha x}{(b^2 - \alpha^2)^{1/2}} \), there is no real root for the equations.
When \( |z| < \frac{t - \alpha x}{(b^2 - \alpha^2)^{1/2}} \), there will be two real unequal roots.

When \( z = \pm \frac{t - \alpha x}{(b^2 - \alpha^2)^{1/2}} \), there will be two real equal roots and this case corresponds to the wave-front.

5.1.2 Analysis of the front for constant acceleration and transition from subsonic to supersonic

Let us examine in more detail the motion with constant acceleration, i.e.,

\[ t(t) = \frac{1}{2} a t^2 \] or inversely \( \eta(\xi) = \sqrt{\frac{2\xi}{a}} \) where \( a \) is the acceleration. Thus, the wave front can be written as:

\[ f(\xi^*) = t^* - \sqrt{\frac{2\xi^*}{a}} - \sqrt{(x - \xi^*)^2 + z^2} b = 0 \]  \hfill (5.12)

\[ \frac{d}{d\xi} f(\xi^*) = -\sqrt{\frac{2}{a}} \frac{1}{\sqrt{\xi^*}} + \frac{(x - \xi^*)b}{\sqrt{(x - \xi^*)^2 + z^2}} = 0 \]  \hfill (5.13)
Figure 5.3  \( f(\xi) = t - \sqrt{\frac{2\xi}{a}} - \sqrt{(x-\xi)^2 + z^2} \) \( \xi \) with \( a = 4 \) \( m/s^2 \), \( b = 1 \) \( s/m \),

\( x = 0.5 \) \( m \), \( z = 0.1 \) \( m \). \( (t=0.15s,0.20s,0.25s,0.30s,0.35s,0.40s) \)

Since the Mach fronts form when there are real roots \( \xi^* \) of the equations (5.5) and (5.6), we study the roots of the equations (5.12) and (5.13) for motion of constant acceleration.

First, the function \( f(\xi) = t - \sqrt{\frac{2\xi}{a}} - \sqrt{(x-\xi)^2 + z^2} \) \( \xi \) is illustrated for different time t’s in Figure 5.3, with \( a = 4 \), \( b = 1 \), \( x = 0.5 \), \( z = 0.1 \). On Figure 5.3 are shown curves I and II. When \( f(\xi) \) is curve I, there are no real roots (intersection of \( f(\xi) \) with
the $\xi$-axis) for $f(\xi) = 0$ therefore $H(f(\xi)) = 0$ in the equation (2.89) and the field point is in region where the wave has not yet reached (no motion). When $f(\xi)$ is curve II (as time passes), the field point is in region where there are contributions from the subsonic motion, coming from the interval $(0, \xi_2)$ (see Figure 5.3), and contributions from the supersonic motion, coming from the interval $(\xi_0, \xi_1)$.

On the Mach wave-front, the minimum of the function $f(\xi)$ corresponds to the front BC (see Figure 4.4 ~4.6), while the maximum to the front AB, and B is a cusp point with a singularity that we do not analyze here. In order to study the nature of the Figure 5.3 when the function $f(\xi) = t - \frac{2\xi}{a} - \sqrt{(x - \xi)^2 + z^2}b$ not only equals zero but also is a maximum, we do some further study into the two wave-front equations (5.12) and (5.13). We analyze the transition of the roots of $f(\xi)$ from complex conjugates roots to real ones, i.e. the transition from region outside the Mach wave-front to region inside the Mach wave-front when the two complex conjugates roots become one double real root and split into two real roots (see also figure 5.2)

From (5.13), we can see that $x > \xi > 0$. Eliminating $z$ in equations (5.12) and (5.13), we have:

$$\theta^3 - \frac{(1 + ab^2x)}{ab^2} \theta + \frac{at}{ab^2} = 0$$

(5.14)

where $\theta = \sqrt[3]{\xi}$.

Equation (5.14) is a cubic equation obtained from equations (5.12) and (5.13), the roots of which determine the nature of the wave-front. We analyze it to study the wave
front itself. Since for a cubic equation \( \theta^3 + a_2 \theta^2 + a_1 \theta + a_0 = 0 \), \( D = Q^3 + R^2 \) determines the property of its roots (For cubic equation, see http://mathworld.wolfram.com/CubicEquation.html).

where \( Q \equiv \frac{3a_1 - a_2^2}{9} = -\frac{1}{3} \left( x + \frac{1}{ab^2} \right) \), \( R \equiv \frac{9a_2a_1 - 27a_0 - 2a_2^3}{54} = -\frac{1}{2\sqrt{2a}} \frac{1}{b^2} t. \)

Therefore,

\[
D = Q^3 + R^2 = -\frac{1}{27} \left( x^3 + 3x^2 \frac{1}{ab^2} + 3x \frac{1}{a^2 b^4} + \frac{1}{a^3 b^6} \right) + \frac{1}{8a} t^2. \tag{5.15}
\]

When \( D = 0 \), the cubic equation has one double real root which corresponds to the transition point for the cubic equation from containing two complex conjugate roots to real ones in which case there are contribution from supersonic motion.

For any \( x \), \( D = 0 \) gives the time \( t \) at which the Mach front crosses field point \((x, z)\), where \( z \) can be obtained from substituting \( x \) and \( t \) into equation (5.12) and (5.13).

Now, we want to study expression (5.15) at the time \( t \) when the dislocation crosses the shear wave speed barrier \( c_2 \). Then, for motion of constant acceleration, namely for \( l(t) = \frac{1}{2} a t^2 \), \( \dot{l}(t) = at = c_2 \), so that \( t = \frac{c_2}{a} \) is the time when the dislocation velocity equals the shear wave speed \( c_2 \) while the dislocation is at \( l(t) = \frac{1}{2ab^2} \) and for \( \left( x = \frac{1}{2ab^2}, t = \frac{c_2}{a} \right) \), \( D \equiv 0 \). This confirms that this position \( x \) is at the Mach wave-front at the time \( t \) when the dislocation becomes supersonic.
In the next section we focus at the time $t$ when the dislocation crosses the shear wave speed, that is when the dislocation velocity $\eta'(\xi) = b$, but in the presence of acceleration. This means that the analysis of the wave-front equations (5.5) and (5.6) will be applied for a time when $\eta'(\xi^*) = b$.

5.2 Stress analysis for a Volterra screw dislocation accelerating through the shear wave speed barrier and singularity analysis at the wave front

The stress for transient motion of a Volterra screw dislocation is given by equation (2.89). We are interested in evaluating the stress $\sigma_{xz}$ at the Mach wave-front at the moment when the dislocation becomes supersonic. On the other hand we are interested in the front AB (see Figure 4.4 ~4.6) passing through the field point $x$, which is when the cubic equation (5.14) has a double real root $\xi^*$ at the maximum of $f(\xi)$. At a time $\Delta t$ later this splits into two real roots $\xi_0^*, \xi_1^*$ (Figure 5.2). The minimum gives no delta function contribution as shown by Callias and Markenscoff (1980), so the stress field is continuous as we approach the front BC.

Before we analyze the stress at the instance of the dislocation becoming supersonic, we look at the case when the dislocation velocity is supersonic but not near the shear wave speed $c_2$. So we study equations (5.5) and (5.6) for any given $(x,z)$, which represents the time when the Mach wave-front passes through any field point.

Each of the two integrals appearing in the general solution for the stress (equation (2.89)) can be studied asymptotically as we approach the wave-front $(t^*, \xi^*)$ in a systematic way. Since we are near the wave-front we can expand the integrands in $t - t^*$. 
On the other hand, since in the limit the range of the integration \([\xi_0, \xi_1]\) becomes small approaching the wave-front, we can expand the integrands in \(\xi - \xi^*\).

In order to study the property of the integrands at the wave-front, we need to study the property of the function \((t - \eta(\xi))^2 - r^2b^2\) near the wave front.

Expanding asymptotically \((t - \eta(\xi))^2 - r^2b^2\) near the wave front \((t^*, \xi^*)\), we let

\[
t = t - t^* + t^*
\]

\[
r^2 = (x - \xi)^2 + z^2
\]

\[
= (x - \xi^* + \xi^* - \xi)^2 + z^2
\]

\[
= (x - \xi^*)^2 + (\xi - \xi^*)^2 + 2(x - \xi^*)(\xi^* - \xi) + z^2
\]

Using Taylor series to the first order expansion, we have

\[
\eta(\xi) = \eta(\xi^*) + \eta'(\xi^*)(\xi - \xi^*) + o(\xi - \xi^*)
\]

From (5.16), (5.17), (5.18), we will get

\[
(t - \eta(\xi))^2 - r^2b^2
\]

\[
= [t - t^* + t^* - \eta(\xi^*) - \eta'(\xi^*)(\xi - \xi^*) + o(\xi - \xi^*)]^2
\]

\[
- (x - \xi^*)^2 + (\xi - \xi^*)^2 + 2(x - \xi^*)(\xi^* - \xi) + z^2
\]

\[
= (t - t^*)^2 + (t^* - \eta(\xi^*))^2 + \eta'(\xi^*)(\xi - \xi^*)^2 + 2(t - t^*)(t^* - \eta(\xi^*))
\]

\[
- 2(t^* - \eta(\xi^*))\eta'(\xi^*)(\xi - \xi^*) - 2\eta'(\xi^*)(\xi - \xi^*)(t - t^*) - (x - \xi^*)^2 b^2
\]

\[
- (\xi - \xi^*)^2 b^2 - 2(x - \xi^*)(\xi^* - \xi)b^2 - z^2b^2
\]

Since from equation (5.14), we have

\[
(t^* - \eta(\xi^*))^2 - (x - \xi^*)^2 b^2 - z^2b^2 = 0
\]

So equation (5.19) becomes
\[(t - \eta(\xi))^2 - r^2b^2\]
\[= (t - t^*)^2 + [(\eta' (\xi^*))^2 - b^2](\xi - \xi^*)^2 + 2(t - \eta(\xi^*))(t - t^*)
- [2\eta' (\xi^*)(t - \eta(\xi^*)) - 2b^2(x - \xi^*)](\xi - \xi^*) - 2\eta' (\xi^*)(t - t^*)(\xi - \xi^*)
+ o((\xi - \xi^*), (t - t^*))\]

(5.21)

The roots of the expression (5.21) give \(\xi_0, \xi_1\) in the asymptotic limit, so we can write (5.21) as

\[-((\eta' (\xi^*))^2 - b^2)(\xi - \xi_0)(\xi_1 - \xi)\]

(5.22)

Thus to the leading order the term in the general solution (equation (2.89)) that gives rise to the \(\delta\) – function, i.e. the second term can be written near the wave front \((t^*, \xi^*)\) as

\[
\frac{\Delta u}{2\pi} z^2 \frac{\partial}{\partial t} \int_{\xi_0}^{\xi_1} \frac{(t - \eta(\xi^*))^2 H(t - t^*)d\xi}{[(x - \xi^*)^2 + z^2] (b^2 - \eta' (\xi^*)^2) \frac{1}{2} \sqrt{(\xi - \xi_0)(\xi_1 - \xi)}}
\]

(5.23)

where the step function \(H(t - t^*)\) indicates that the integral vanishes for \(t < t^*\) (outside the wave-front).

To the leading order in \((t - t^*)\), the above expression (5.23) obviously gives the stress at the Mach wave-front:

\[A \delta(t - t^*)\]

(5.24)

where

\[A = \lim_{\xi_0, \xi_1 \to \xi^*} \frac{\Delta u}{2\pi} z^2 \frac{(t - \eta(\xi^*))^2}{[(x - \xi^*)^2 + z^2] (b^2 - \eta' (\xi^*)^2) \frac{1}{2} \sqrt{(\xi - \xi_0)(\xi_1 - \xi)}} \int_{\xi_0}^{\xi_1} \frac{d\xi}{\sqrt{(\xi - \xi_0)(\xi_1 - \xi)}} \]

(5.25)

Since
\[
\lim_{\xi_0,\xi \to \xi^*} \int_{\xi_0}^{\xi^*} \frac{d\xi}{\sqrt{(\xi - \xi_0)(\xi_1 - \xi)}} = \pi, \tag{5.26}
\]

From equation (5.25), we have

\[
A = \frac{\Delta u \cdot z^2}{2} \cdot \frac{(\xi^* - \eta(\xi^*))^2}{\left[(x - \xi^*)^2 + z^2\right] \cdot \left(b^2 - \eta'(\xi^*)^2\right)^{\frac{1}{2}}} \tag{5.27}
\]

Looking at the denominator of equation (5.27), we can see that the coefficient \(A\) for the \(\delta\)–function will be finite when \(\eta'(\xi^*) < b\), the dislocation being moving supersonically. When \(\eta'(\xi^*) = b\), namely when the speed of the moving dislocation equals the shear wave speed, the coefficient \(A\) for the \(\delta\)–function will be infinite which is meaningless. The velocity of the moving dislocation \(\eta'(\xi^*) = b\) represents the transition point for the motion of dislocation from subsonic to supersonic. If we can obtain finite stress and therefore finite energy, on the wave front at this transition point, we can conclude that the transition from subsonic to supersonic motion is possible. We will next focus on this transition.

Since the expression (5.27) of the coefficient \(A\) for the \(\delta\)–function contains \(\eta'(\xi^*) - b\) which is zero at this transition point, in the first order of the Taylor series expansion. The terms in equation (5.18) which depend on the acceleration are no longer ignored. We need to use the Taylor series expansion to the next order which involves the higher derivatives of \(\eta'(\xi^*)\) at this point, and neglect the higher order terms.

Using the next order of the Taylor series expansion, we will obtain
\[
\eta(\xi) = \eta(\xi^*) + \eta'(\xi^*)(\xi - \xi^*) + \frac{1}{2}\eta''(\xi^*)(\xi - \xi^*)^2 + o(\xi - \xi^*)^2 \tag{5.28}
\]

Applying expression (5.16), (5.17), (5.28), we can expand asymptotically \((t - \eta(\xi))^2 - r^2b^2\) near the wave front \((t^*, \xi^*)\) to the next order, which includes the effect of the acceleration \(\eta''(\xi^*)\):

\[
I \equiv (t - \eta(\xi))^2 - r^2b^2
\]

\[
= [t - t^* + t^* - \eta(\xi^*) - \eta'(\xi^*)(\xi - \xi^*) - \frac{1}{2}\eta''(\xi^*)(\xi - \xi^*)^2 + o(\xi - \xi^*)^2]^2 - r^2b^2
\]

\[
= [[(\eta'(\xi^*))^2 - b^2](\xi - \xi^*)^2 + 2(t^* - \eta(\xi^*))(t - t^*) + (t - t^*)^2 - 2(t - t^*)\eta'(\xi^*)(\xi - \xi^*) - [2\eta^* (t^* - \eta(\xi^*)) - b^2(x - \xi^*)](\xi - \xi^*) + [(t^* - \eta(\xi^*))^2 - (x - \xi^*)^2b^2 - z^2b] + \frac{1}{4}(\eta''(\xi^*))^2(\xi - \xi^*)^4 + \eta'(\xi^*)\eta''(\xi^*)(\xi - \xi^*)^3 - [t - t^*]\eta'(\xi^*)(\xi - \xi^*)^2 - [t^* - \eta(\xi^*)]\eta''(\xi^*)(\xi - \xi^*)^2 + o(\xi - \xi^*)^2[t - \eta(\xi^*) - \eta'(\xi^*)(\xi - \xi^*) - \frac{1}{2}\eta''(\xi^*)(\xi - \xi^*)^2]^2 \tag{5.29}
\]

We note that part of the expression (5.29)

\[
II \equiv [((\eta'(\xi^*))^2 - b^2](\xi - \xi^*)^2 + 2(t^* - \eta(\xi^*))(t - t^*) + (t - t^*)^2 - 2(t - t^*)\eta'(\xi^*)(\xi - \xi^*) - [2\eta^*(t - \eta(\xi^*)) - b^2(x - \xi^*)](\xi - \xi^*) \tag{5.30}
\]

actually is the first order asymptotic expansion of \((t - \eta(\xi))^2 - r^2b^2\), and can be written as

\[
-((\eta'(\xi^*))^2 - b^2)(\xi - \xi^*)(\xi - \xi^*) \tag{5.31}
\]

where \(\xi_1\) and \(\xi_0\) are roots of expression (5.30). Suppose that \(\xi_1 > \xi_0\).

So when \(\eta'(\xi^*) = b\), we will obtain for expression (5.30):
II=0. \quad (5.32)

Also, we have from (5.20):

\[
(t^* - \eta(\xi^*))^2 = (x - \xi^*)^2 + z^2 b^2.
\]

Therefore applying (5.32) and (5.20), when the velocity of the motion of the dislocation equals to shear wave speed, namely \( \eta'(\xi^*) = b \), we will obtain from the expression (5.29):

\[
I = (t - \eta(\xi))^2 - r^2 b^2 = \frac{1}{4} \left(\eta''(\xi^*)\right)^2 (\xi - \xi^*)^4 + \eta'(\xi^*) \eta''(\xi^*)(\xi - \xi^*)^3 - \left[t - \eta'(\xi^*)\eta''(\xi^*)(\xi - \xi^*)\right]^2 + \frac{1}{2} \eta''(\xi^*)(\xi - \xi^*)^2 \]

\[
= (\xi - \xi^*)^2 \left[-(t - \eta(\xi^*) \eta''(\xi^*)) + o(\xi - \xi^*)^2 \right]. \quad (5.33)
\]

Thus, the second term in the general solution (equation (2.89)) for stress can be written near the wave front as

\[
\frac{\Delta u}{2\pi} z^2 \frac{\partial}{\partial t} \int_0^\infty \frac{(t - \eta(\xi))^2 H(t - \eta(\xi) - rb)}{r^4 [(t - \eta(\xi))^2 - r^2 b^2]^{1/2}} d\xi
\]

\[
\approx \frac{\Delta u}{2\pi} z^2 .
\]

\[
\frac{\partial}{\partial t} \int_0^\infty \left[ \frac{(t - t^* + t^* - \eta(t^*) - \eta'(t^*) (t^* - \xi^*))^2}{((x - \xi^*)^2 + z^2)^2 (t - \eta(\xi^*))^2 (t - \eta(\xi^*))^2} \right] d\xi
\]

\[
= \frac{\Delta u}{2\pi} z^2 .
\]

\[
\frac{\partial}{\partial t} \int_0^\infty \left[ \frac{(t - t^* + t^* - \eta(t^*) - \eta'(t^*) (t^* - \xi^*))^2}{((x - \xi^*)^2 + z^2)^2 (t - \eta(\xi^*))^2 (t - \eta(\xi^*))^2} \right] d\xi
\]
\[ H\left(t - t^* + t^* - \eta(\xi^*) - \eta'(\xi^*)(\xi - \xi^*) - b\sqrt{(x - \xi)^2 + z^2}\right) \] (5.34)

Next we need to evaluate the expression (5.34) in the neighborhood of the double real root at which there is also the maximum of function \( f(\xi) = t - \eta(\xi) - rb \):

On the wave front, we have

\[ t^* - \eta(\xi^*) - \sqrt{(x - \xi^*)^2 + z^2} b = 0 \] (5.35)

and near the wave front, we have

\[ t - t^*, \xi - \xi^* \to 0 \] (5.36)

So near the wave front, we can write in leading order,

\[ t - \eta(\xi) - rb \]

\[ \approx t - t^* + t^* - \eta(\xi^*) - \eta'(\xi^*)(\xi - \xi^*) - b\sqrt{(x - \xi)^2 + z^2} \]

\[ = t - t^* + b\sqrt{(x - \xi^*)^2 + z^2} - \eta'(\xi^*)(\xi - \xi^*) - b\sqrt{(x - \xi)^2 + z^2} \]

\[ \to t - t^* \]

\[ (t - \eta(\xi))^2 \]

\[ \approx (t - t^* + t^* - \eta(\xi^*) - \eta'(\xi^*)(\xi - \xi^*))^2 \]

\[ = (t - \eta(\xi^*))^2 + (t - t^*)^2 + (\eta'(\xi^*)(\xi - \xi^*))^2 + 2(t - t^*)(t^* - \eta(\xi^*)) + 2(t - t^*)(\eta'(\xi^*)(\xi - \xi^*)) \]

(5.37)

\[ r^4 = \left((x - \xi)^2 + z^2\right)^2 \]

\[ = \left((x - \xi^* + \xi^* - \xi)^2 + z^2\right)^2 \]

\[ = \left((x - \xi^*)^2 + (\xi^* - \xi)^2 + 2(x - \xi^*)(\xi^* - \xi) + z^2\right)^2 \]
\[
\rightarrow \left( (x - \xi)^2 + z^2 \right)^2 \tag{5.39}
\]

Since \( \xi_1 \) and \( \xi_0 \) are roots of expression (5.30) which is the first order asymptotic expansion of \( (t - \eta(\xi))^2 - r^2 b^2 \), and

\[
(t - \eta(\xi))^2 - r^2 b^2 = (t - \eta(\xi)) - rb \left( t - \eta(\xi) + rb \right) \tag{5.40}
\]

If \( \eta(\xi) \) increases monotonically and remains positive through the whole motion, then \( (t - \eta(\xi)) - rb < (t - \eta(\xi)) + rb \), the two roots of \( t - \eta(\xi) - rb \) are inside the two roots \( \xi_1' \) and \( \xi_0' \) of \( (t - \eta(\xi)) + rb \). So the roots of \( (t - \eta(\xi))^2 - r^2 b^2 \) and \( t - \eta(\xi) - rb \) will be the same, namely \( \xi_1 \) and \( \xi_0 \). The range of the integration obtained from \( H(t - \eta(\xi) - rb) \neq 0 \) will be \( (\xi_0, \xi_1) \).

So the second term of the general solution can be written to leading order near the wave front as,

\[
\frac{\Delta u}{2\pi} z^2 \frac{\partial}{\partial t} \int_{\xi_0}^{\xi_1} \frac{(t^* - \eta(\xi^*))^2 H(t - t^*)d\xi}{[z^2 + z^2 \eta''(\xi^*)^2(t - \eta(\xi^*))^2]} \tag{5.41}
\]

where the step function \( H(t - t^*) \) indicates that the integral vanishes for \( t < t^* \) (outside the wave-front).

To leading order in \( t - t^* \), after taking the derivative with respect to \( t \), the above expression obviously gives
\[ A \delta(t-t^*) \] (5.42)

with

\[
A = \lim_{\xi_0 \to \xi^*} \frac{\Delta u}{2\pi} \frac{b^2}{\left(1 + m^2\right) \left(\eta(\xi^*)(t - \eta(\xi^*))\right)^\frac{1}{2}} \int_{\xi_0}^{\xi} \frac{d\xi}{\left(\xi - \xi^*\right)}
\] (5.43)

For the time \( t \) near the time \( t^* \) when the dislocation crosses the shear wave speed, the Mach wave-front itself is near the position of the dislocation (as observed also from figure 4.3). So that \( t \to t^*, \ x \to \xi^* \) and \( \eta(\xi^*) = b \). We are interested in studying the stress field near the current position of the dislocation as it crosses the shear wave speed barrier. So the field point \((x,z)\) is near the dislocation. We choose to approach the dislocation at an angle so that \( \frac{z}{x - \xi^*} = m \), where \( m \) is a constant. Also when \( t \to t^* \),

\[
\left(t - \eta(\xi^*)\right) \to \left(t^* - \eta(\xi^*)\right) \to b\sqrt{1 + m^2} (x - \xi^*)
\]

The field point is \( O(\varepsilon) \) from the dislocation and \( O(\varepsilon) \) from the last wavelet that was emitted from the path of the dislocation and had the time to reach the field point (Callias and Markenscoff, 1988). So \((x - \xi^*) = O(\xi - \xi^*)\).

Thus, equation (5.43) becomes

\[
A = \lim_{t \to t^*} \lim_{\xi_0 \to \xi^*} \frac{\Delta u}{2\pi} \frac{m^2 b^2}{\left(1 + m^2\right) \left(\eta(\xi^*)(t - \eta(\xi^*))\right)^\frac{1}{2}} \int_{\xi_0}^{\xi} \frac{d\xi}{\left(\xi - \xi^*\right)}
\]

\[
= \lim_{\xi_0 \to \xi^*} \frac{\Delta u}{2\pi} \frac{m^2 b^2}{\left(1 + m^2\right) \left(\eta(\xi^*)b\sqrt{1 + m^2} (x - \xi^*)\right)^\frac{1}{2}} \ln \left(\xi - \xi^*\right)_{\xi_0}
\]
\[
\lim_{\xi, \xi_0, \xi_1 \to \xi^*} O\left( \frac{\ln|(\xi - \xi^*)|^{\frac{1}{2}}}{\left|(\xi - \xi^*)\right|^2} \right)
\]

(5.44)

where the limit \(\xi, \xi_0, \xi_1 \to \xi^*\) is taken as the maximum of \(f(\xi)\) is approached, and the two real roots coincide to a double real one. This again corresponds to the Mach front AB crossing the field point near the current position of the dislocation, i.e. \(x \to \xi^*\).

The expression (5.44) is singular since it \(\to \infty\) as \(\xi, \xi_0, \xi_1 \to \xi^*\).

Since from expression (5.44) the coefficient \(A\) to the delta function in (5.42) is infinite, we will get infinite stress on the wave front, which implies that a Volterra dislocation with Heaviside step function in the displacement is too strong of a discontinuity to model the crystal dislocation. In order to remove this singularity we will use a ramp-core model in the next chapter instead of the Volterra one. The Volterra solution is useful as the kernel to the ramp-core solution.

5.3 Stress analysis for a Volterra edge dislocation accelerating through the shear wave speed barrier and singularity analysis at the wave front

For a Volterra edge dislocation starting from rest and moving non-uniformly \((t = \eta(\xi))\), the solution has been obtained as:

\[
\sigma_{\xi\xi} = \sigma^{(\theta)}_{\xi\xi} + \sigma^{(\nu)}_{\xi\xi}
\]

\[
= \frac{2\Delta u}{\rho b^2} \int_0^\infty \frac{H(t - \eta(\xi) - ar)}{\sqrt{(t - \eta(\xi))^2 - a^2 r^2}} \left[ \left( t - \eta(\xi) \right)^2 z^2 - \left( t - \eta(\xi) \right)^2 - a^2 r^2 \right] x - \xi \right] dx - \left[ \left( t - \eta(\xi) \right)^2 (x - \xi)^2 - \frac{4(t - \eta(\xi))^2}{(t - \eta(\xi))^2 - a^2 r^2} (x - \xi)^2 z^2 \right] d\xi
\]
The solution given above is for both subsonic and supersonic motion.

We want to evaluate the stress at the wave front at the time that the shear wave speed barrier \((1/b)\) is crossed.

From the solution we can see, that when we expand asymptotically the given solution near the wave front \((t^*, \xi^*)\) and also let \(\eta'(\xi^*) = b\), only the second term will produce a singularity.

Expanding asymptotically the second term near the wave front, and letting \(\eta'(\xi^*) = b\), we have

\[
t - \eta(\xi) - rb \\
\approx t - t^* + \xi - \eta'\xi^* - \xi^*) - \eta'(\xi^*) - b\sqrt{(x - \xi^*)^2 + z^2} \\
= t - t^* + b\sqrt{(x - \xi^*)^2 + z^2 - \eta'(\xi^*) - b\sqrt{(x - \xi^*)^2 + z^2}} \\
\rightarrow t - t^* \tag{5.46}
\]

\[
r^8 = \left( (x - \xi^*)^2 + z^2 \right)^4 \\
= \left( (x - \xi^* + \xi^* - \xi^* + z^2) \right)^4 \\
= \left( (x - \xi^*)^2 + (\xi^* - \xi^*) + 2(x - \xi^*) + z^2 \right)^4 \\
\rightarrow \left( (x - \xi^*)^2 + z^2 \right)^4 \tag{5.47}
\]
\[(t - \eta(\xi))^2 - r^2 b^2 \]

\[
\approx -((\eta'(\xi^*))^2 - b^2)(\xi - \xi_0)(\xi_1 - \xi) + \frac{1}{4}(\eta''(\xi^*))(\xi - \xi^*)^2 + \frac{1}{2}(\eta''(\xi^*)\eta''(\xi^*)\eta^*\eta^*(\xi - \xi^*)]^3
\]

\[- [t - \eta(\xi)]^2 (\xi - \xi^*)^2
\]

\[
+ o(\xi - \xi^*)^2 [t - \eta(\xi^*) - \eta^*(\xi^*)](\xi - \xi^*)(\xi - \xi^*) - \frac{1}{2} \eta''(\xi^*)(\xi - \xi^*)^2 ]
\]

\[
\rightarrow (\xi - \xi^*)^2 [- (t - \eta(\xi^*))\eta^* (\xi^*)] \quad (5.48)
\]

\[(t - \eta(\xi))^2 \]

\[
\approx \left( t - t^* + t^* - \eta(\xi^*) - \eta^*(\xi^*)(\xi - \xi^*) \right)^2
\]

\[
= \left( t^* - \eta(\xi^*) \right)^2 + \left( t - t^* \right)^2 + \eta^* (\xi^*)(\xi - \xi^*)^2 + 2(t - t^*)^2(t^* - \eta(\xi^*))
\]

\[- 2(t^* - \eta(\xi^*)^2(t^* - \eta(\xi^*)) - 2(t - t^*)^2(t^* - \eta(\xi^*))
\]

\[
\rightarrow (t^* - \eta(\xi^*))^2 \quad (5.49)
\]

Also

\[(x - \xi)^2 \]

\[
\approx (x - \xi^* + \xi^* - \xi)^2
\]

\[
= (x - \xi^*)^2 + (\xi - \xi^*)^2 + 2(\xi^* - \xi)(x - \xi^*)
\]

\[
\rightarrow (x - \xi^*)^2 \quad (5.50)
\]

So from the previous results, when \( t \rightarrow t^*, \xi \rightarrow \xi^* \), also letting \( \eta'(\xi^*) = b \), we can obtain that

\[
\left[ b^2 r^4 - 2(t - \eta(\xi))^2 (x - \xi)^2 - (t - \eta(\xi))^2 - b^2 r^2 \right] z^2
\]

\[- 16(t - \eta(\xi))^2 (t - \eta(\xi))^2 (x - \xi)^2 z^2
\]

\[
\approx \left[ b^2 \left( (x - \xi^*)^2 + z^2 \right)^2 - 2(t^* - \eta(\xi^*)^2 (x - \xi^*)^2 - (x - \xi^*)^2 z^2 (t - \eta(\xi^*))\eta^* (\xi^*)] \right]^2
\]
\[-16(t^* - \eta(\xi^*))\bigg[ - (\xi - \xi^*)^2 z^2 (t - \eta(\xi^*))\eta''(\xi^*) \bigg] (x - \xi^*)^2 z^2 \]

\[
= \left[ b^2 \left( (x - \xi^*)^2 + z^2 \right)^2 \right]^2 \left( (t - \eta(\xi^*))\right)^2 (x - \xi^*)^2 z^2 \quad (5.51)
\]

Using the wave front equation (5.3), we obtain

\[(t^* - \eta(\xi^*))^2 = ((x - \xi^*)^2 + z^2) b^2 \]

So equation (5.50) may be written as

\[
 \left[ b^2 r^4 - 2((t - \eta(\xi))^2 (x - \xi^*)^2 - ((t - \eta(\xi))^2 - b^2 r^2)) z^2 \right]
\]

\[-16(t - \eta(\xi))^2 \left( (t - \eta(\xi))^2 - b^2 r^2 \right) (x - \xi^*)^2 z^2 \approx \left[ b^2 \left( (x - \xi^*)^2 + z^2 \right)^2 \right]^2 \left( (t - \eta(\xi^*))\right)^2 (x - \xi^*)^2 z^2 \]

\[
= \left[ b^2 \left( (x - \xi^*)^2 + z^2 \right)^2 \left( z^2 - \left( x - \xi^* \right)^2 \right) \right]^2 \]

\[
\rightarrow b^4 \left( z^4 - \left( x - \xi^* \right)^4 \right)^2 \text{ or } (t^* - \eta(\xi^*))\left( z^2 - \left( x - \xi^* \right)^2 \right)^2 \quad (5.52)
\]

Thus the asymptotic expansion near the wave-front \((t^*, \xi^*)\), of the second term in the general solution for an edge dislocation crossing through shear wave speed barrier (let \(\eta'(\xi^*) = b\)), will result in

\[
- \frac{\Delta u}{2 \pi b^2} \frac{\partial}{\partial t} \int_{\xi^*}^{\xi} \left( z^2 - (x - \xi^*)^2 \right)^2 \left( t^* - \eta(\xi^*) \right)^4 H(t - t^*) d\xi \quad (5.53)
\]

or in the form

\[
- \frac{\Delta u}{2 \pi} b^2 \frac{\partial}{\partial t} \int_{\xi^*}^{\xi} \left( z^2 - (x - \xi^*)^2 \right)^2 \left( t^* - \eta(\xi^*) \right)^4 H(t - t^*) d\xi \quad (5.53)
\]
where $\xi_0, \xi_1$ are the roots of the expression for the first order asymptotic expansion of 
\[(t - \eta(\xi))^2 - r^2b^2.\]

To the leading order in $\left(t - t^*\right)$, after taking the derivative with respect to $t$, the above expression obviously gives

\[B\delta\left(t - t^*\right)\]  
(5.54)

with

\[B = \lim_{\xi_0, \xi_1 \to \xi^*} \frac{-\Delta u}{2\pi b^2} \frac{\left(z^2 - (x - \xi^*)^2\right)^2 (t^* - \eta(\xi^*))^4}{\left[(x - \xi^*)^2 + z^2\right] \left[\eta''(\xi^*) (t^* - \eta(\xi^*))\right]^2} \int_{\xi_0}^{\xi_1} d\xi \left(\frac{\xi - \xi^*}{|\xi - \xi^*|}\right)^2 \]  
(5.55)

where the limit $\xi_0, \xi_1 \to \xi^*$ is taken as the maximum of $f(\xi)$ is approached, and the two real roots coincide to a double real one. This again corresponds to the Mach front AB crossing the field point near the current position of the dislocation, i.e. $x \to \xi^*$.

Performing the similar analysis as for screw dislocation, we can obtain the same form of results as equation (5.44):

\[B = \lim_{\xi_0, \xi_1 \to \xi^*} O\left(\frac{\ln\left(\xi - \xi^*\right)}{\xi - \xi^*}\right)\]  
(5.56)

It is singular since \[\left(\frac{\ln\left(\xi - \xi^*\right)}{\xi - \xi^*}\right)\] as $\xi, \xi_0, \xi_1 \to \xi^*$.

Since from expression (5.56) the coefficient $B$ of the delta function in (5.54) is infinite, we will get infinite stress on the wave front using the Volterra edge dislocation with Heaviside step function displacement discontinuity.
In order to remove this singularity, smoothed-out core model (ramp core model) instead of the Volterra dislocation should be used to describe the dislocation.

Part of the material in this chapter is being prepared to submit for publication. Prof. Xanthippi Markenscoff is the co-author for this paper.
THE REMOVAL OF SINGULARITY OF STRESS FIELD FOR SCREW DISLOCATION PASSING THROUGH SHEAR WAVE SPEED BARRIER WITH RAMP CORE MODEL

6.1 Ramp-core model as a convolution of the Volterra dislocation

The Volterra dislocation uses a Heaviside step function to simulate the core of the dislocation and predicts infinite stress on the wave front. Here a more realistic core model is introduced to analyze this problem. We use the ramp-like core model including arctan displacement functions with time-dependent or constant width.

Using a ramp-core model, the problem can be described as:

$$\left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2} \right) = b^2 \frac{\partial^2 u_y}{\partial t^2} \quad (6.1)$$

with boundary condition on the slip-plane $z = 0$:

$$u_x(x,0,t) = \begin{cases} \frac{B}{2} f_0(x), & t < 0 \\ \frac{B}{2} f_1(x,t), & t \geq 0 \end{cases} \quad (6.2)$$

where $b$ is the shear wave slowness with $b = \sqrt{\frac{\rho_0}{\mu}}$, $B$ denotes the Burgers vector, and $f_0(x)$ and $f_1(x,t)$ (with $f_1(x,0) = f_0(x)$) denote the ramp functions.
The above problem can be decomposed into the superposition of the following
two problems satisfying the same governing differential equation (6.1) and corresponding
boundary conditions:

Problem I:

\[
 u_s(x,0,t) = \frac{B}{2} f_0(x) \quad \text{for all } t
\]  
(6.3)

Problem II:

\[
 u_s(x,0,t) = \begin{cases} 
 0, & t < 0 \\
 \frac{B}{2} \left[ f_1(x,t) - f_0(x,t) \right], & t \geq 0.
\end{cases}
\]  
(6.4)

Problem I is a static problem, the solution of which is obtained by convolution as

\[
 u(x,z) = \frac{B}{2\pi} \frac{z}{x^2 + z^2} * f_0(x)
\]  
(6.5)

where * denotes the convolution integral defined by

\[
 f(x) * g(x) = \int_{-\infty}^{\infty} f(x-\xi) g(\xi) d\xi
\]  
(6.6)

while the strain \( \frac{\partial u}{\partial z} (x,z) \) is given by

\[
 \frac{\partial u}{\partial z} (x,z) = \frac{B}{2\pi} \frac{x^2 - z^2}{(x^2 + z^2)^2} * f_0(x).
\]  
(6.7)

The solution of Problem II is sought by means of Laplace transform in time and
two-sided Laplace transform in space:

For Problem II, apply first the Laplace transform in time \( t \)

\[
 \hat{u}(x,z,s) = \int_0^\infty u(x,z,t)e^{-st} dt
\]  
(6.8)

to the governing equation (6.1) and the boundary condition (6.4) and yield:
\[ \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial z^2} = b^2 s^2 \hat{u} \]  \hspace{1cm} (6.9) \\

with boundary condition:

\[ \hat{u}(x,0,s) = \int_0^\infty B e^{-st}(f_1(x,t) - f_0(x)) dt \]  \hspace{1cm} (6.10) \\

Define

\[ f_2(x,t) = f_1(x,t) - f_0(x). \]  \hspace{1cm} (6.11) \\

Obviously, we have

\[ f_2(x,0) = 0 \]  \hspace{1cm} (6.12) \\

Assume that \( \frac{\partial f_2}{\partial t} \) and its one-sided Laplace transform in time exists, also define

\[ f_3(x,t) = \frac{\partial}{\partial t} f_2(x,t) = \frac{\partial}{\partial t} (f_1(x,t) - f_0(x)). \]  \hspace{1cm} (6.13) \\

Integrating equation (6.10) by parts and applying (6.12), we obtain from the boundary condition

\[ \hat{u}(x,0,s) = \frac{B}{2s} \int_0^\infty e^{-st} f_3(x,t) dt. \]  \hspace{1cm} (6.14) \\

Apply next the two-sided Laplace transform in x

\[ U(\lambda, z, s) = \int_{-\infty}^\infty \hat{u}(x,z,s)e^{-s \lambda x} dx. \]  \hspace{1cm} (6.15) \\

and equation (6.9) yields

\[ \frac{\partial^2 U(\lambda, z, s)}{\partial z^2} - (b^2 - \lambda^2) s^2 U(\lambda, z, s) = 0 \]  \hspace{1cm} (6.16) \\

where
\[ \beta \equiv \left[ b^2 - \lambda^2 \right], \quad \text{Re}\beta \geq 0. \tag{6.17} \]

So that the solution of equation (6.16) is given by

\[ U(\lambda, z, s) = U(\lambda, 0, s)e^{-s\beta z}, \tag{6.18} \]

where

\[ U(\lambda, 0, s) = \int_{-\infty}^{\infty} e^{-s\lambda x} \hat{u}(x, 0, s)dx = \frac{B}{2s} \int_{-\infty}^{\infty} e^{-s\lambda x} \left[ \int_{0}^{\infty} e^{-s\beta t} f_{3}(x, t)dt \right]dx \tag{6.19} \]

and for the strain of interest \( \frac{\partial u}{\partial z} \), we have:

\[ \frac{\partial U(\lambda, z, s)}{\partial z} = -\frac{B}{2} \lambda e^{-s\beta z} \int_{-\infty}^{\infty} e^{-s\lambda x} \left[ \int_{0}^{\infty} e^{-s\beta t} f_{3}(x, t)dt \right]dx \tag{6.20} \]

In (6.20), inverting the two-sided Laplace transform with respect to \( x \) and using the convolution theorem follows:

\[ \frac{\partial \hat{u}(x, z, s)}{\partial z} = \int_{br} \frac{\partial U}{\partial z} \frac{s}{2\pi i} e^{s\lambda x} d\lambda \]

\[ = -\frac{B}{2} \int_{br} \frac{s}{2\pi i} e^{s\lambda x} d\lambda \int_{-\infty}^{\infty} \beta e^{-s\beta z} e^{s\lambda(x-\xi)} d\lambda \left[ \int_{0}^{\infty} e^{-s\beta t} f_{3}(\xi, t)dt \right]d\xi \]

\[ = -\frac{B}{2} \int_{br} \frac{s}{2\pi i} e^{s\lambda x} d\lambda \left[ \int_{0}^{\infty} e^{-s\beta t} f_{3}(x, t)dt \right] \tag{6.21} \]

Inverting next the one-sided Laplace transform with respect to \( t \) we obtain from (6.21):

\[ \frac{\partial u(x, z, t)}{\partial z} = -\frac{B}{2} F^{-1} \left\{ \int_{br} \frac{s}{2\pi i} e^{s\lambda(x-\xi)} d\lambda \left[ \int_{0}^{\infty} e^{-s\beta t} f_{3}(\xi, t)dt \right]d\xi \right\}, \tag{6.22} \]

where \( F^{-1}(\cdot) \) symbolizes the inverse one-sided Laplace transform:
\[ F^{-1}[\hat{g}(s)] = \frac{1}{2\pi i} \int_{Br} \hat{g}(s)e^{st} \, ds. \] (6.23)

Equation (6.22) is further written as

\[
\frac{\partial u(x,z,t)}{\partial z} = -\frac{B}{2} \frac{\partial}{\partial t} \left\{ F^{-1} \left[ e^{-st} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \beta e^{-s\lambda} e^{i\lambda(x-z)} d\lambda \left( \int_{0}^{\infty} e^{-sw} f_3(\xi,w) \, dw \right)d\xi \right] \right\} 
\] (6.24)

where the following standard Laplace transform property is used:

\[
\frac{\partial}{\partial t} F^{-1}[\hat{g}(s)] = F^{-1}[s\hat{g}(s)]. \] (6.25)

In equation (6.24) we interchange the inverse Laplace transform \( F^{-1} \) and the integration in \( \xi \) and obtain

\[
\frac{\partial u(x,z,t)}{\partial z} = -\frac{B}{2} \frac{\partial}{\partial t} \left\{ \int_{-\infty}^{\infty} F^{-1} \left[ \left( \int_{-\infty}^{\infty} \beta e^{-s\lambda} e^{i\lambda(x-z)} d\lambda \right) \left( \int_{0}^{\infty} e^{-sw} f_3(\xi,w) \, dw \right) \right] d\xi \right\} 
\] (6.26)

In equation (6.26), we evaluate the inverse Laplace transform in time by the convolution theorem:

\[
F^{-1} \left[ \left( \int_{-\infty}^{\infty} \beta e^{-s\lambda} e^{i\lambda(x-z)} d\lambda \right) \left( \int_{0}^{\infty} e^{-sw} f_3(\xi,w) \, dw \right) \right] 
\]

\[
= \frac{1}{2\pi i} \int_{Br} \left[ \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \beta e^{-s\lambda} e^{i\lambda(x-z)} d\lambda \right) \left( \int_{0}^{\infty} e^{-sw} f_3(\xi,w) \, dw \right) e^{st} \right] ds 
\]

\[
= \frac{1}{2\pi i} \int_{0}^{\infty} \left[ \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \beta e^{-s\lambda} e^{i\lambda(x-z)} d\lambda \right) e^{s(t-w)} \right] f_3(\xi,w) \, dw 
\]

\[
= \frac{1}{2\pi i} \left[ \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \beta e^{-s\lambda} e^{i\lambda(x-z)} d\lambda \right) e^{st} \right] * f_3(\xi,t) 
\]

\[
\left( F^{-1} \left[ \int_{-\infty}^{\infty} \beta e^{-s\lambda} e^{i\lambda(x-z)} d\lambda \right] \right) * f_3(\xi,t) \] (6.27)
so equation (6.26) gives

\[
\frac{\partial u(x,z,t)}{\partial z} = -\frac{B}{2} \frac{\partial}{\partial t} \left\{ \int_{-\infty}^{\infty} \left[ \text{F}^{-1} \left( \frac{1}{2m_{br}} \left[ \beta e^{s\beta} e^{s\lambda(x-\xi)} d\lambda \right] \right) \right] \ast f_3(\xi,t) \right\} d\xi \tag{6.28}
\]

where \( \ast \) denotes the convolution with respect to \( t \) defined by

\[
h_1(t) \ast h_2(t) = \int_0^\infty h_1(t-w)h_2(w)dw \tag{6.29}
\]

Thus, the problem has been reduced to quadratures once the inverse transform

\[
\text{F}^{-1} \left( \frac{1}{2m_{br}} \int \beta e^{s\beta} e^{s\lambda(x-\xi)} d\lambda \right) \tag{6.30}
\]

is evaluated by complex analysis. This was obtained by Markenscoff (1980) on the basis of the Cagniard-de Hoop technique:

Let \( \tau = \beta z - \lambda(x-\xi) \), so \( \lambda = -\tau(x-\xi)z + \sqrt{\tau^2 - \hat{\rho}^2 b^2} \frac{\rho}{\hat{\rho}} \), then the inverse transform can be evaluated as

\[
\text{F}^{-1} \left( \frac{1}{2m_{br}} \int \beta e^{s\beta} e^{s\lambda(x-\xi)} d\lambda \right) = \frac{H(t-\hat{r}b)}{\pi} \int_{-\infty}^{\infty} \text{Im} \left( \beta(\lambda) \frac{\partial \lambda_+}{\partial \tau} e^{-\tau \delta} d\tau \right)
\]

\[
= H(t-\hat{r}b) \left[ \frac{\hat{r}^2 z^2}{\rho^4 \sqrt{\tau^2 - \hat{r}^2 b^2}} - \frac{(x-\xi)^2 \sqrt{\tau^2 - \hat{r}^2 b^2}}{\hat{r}^4} \right] \equiv H(t-\hat{r}b) \frac{F(t,x-\xi)}{\pi}. \tag{6.31}
\]
where \( \hat{r} = \left[ (x - \xi)^2 + z^2 \right]^{\frac{1}{2}} \), \( \lambda = \frac{-\tau (x - \xi) + iz \sqrt{\tau^2 - \hat{r}^2 b^2}}{\hat{r}^2} \) and \( H(\cdot) \) denotes the Heaviside step function.

In equation (6.31), we have defined

\[
F(t, x - \xi) = \frac{t^2 z^2}{\hat{r}^4 \sqrt{t^2 - \hat{r}^2 b^2}} - \left( \frac{x - \xi}{\hat{r}^4} \right) \frac{\sqrt{t^2 - \hat{r}^2 b^2}}{\hat{r}^4}
\]  

(6.32)

Thus, by inserting equation (6.31) into (6.28), the solution is obtained for a general ramp function \( f_3(\xi, t) \), i.e. for a ramp function that generally changes during the motion, we obtain

\[
\frac{\partial u(x, z, t)}{\partial z} = -\frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{-\infty}^{\infty} d\xi \left[ H(t - \hat{r}b)F(t, x - \xi) \right]^*, f_3(\xi, t) \right\}
\]

\[
= -\frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{-\infty}^{\infty} d\xi \int_{0}^{\infty} dw \left[ H(t - w - \hat{r}b)F(t - w, x - \xi) \right] f_3(\xi, w) \right\}.
\]  

(6.33)

Next some special forms of \( f_3(\xi, w) \) are specified and integral (6.33) are evaluated. First we retrieve the classical case for Volterra dislocation moving nonuniformly:

\section{The Volterra dislocation as the limit of the ramp-core dislocation}

The Volterra dislocation is obtained from the general ramp function by setting

\[
f_3(x) = H(x), \quad f_3(x, t) = H(x - l(t)) \text{ for } t \geq 0.
\]  

(6.34)
where \( x = l(t) \), or equivalently \( t = \eta(x) \), defines the position of the center of the core of the dislocation.

By substituting equation (6.34) into equation (6.7), we retrieve the static solution of the Volterra screw dislocation:

\[
\frac{\partial u}{\partial z}(x, z) = \frac{B}{2\pi} \frac{x^2 - z^2}{(x^2 + z^2)^2} \ast f_0(x)
\]

\[
= \frac{B}{2\pi} \frac{x^2 - z^2}{(x^2 + z^2)^2} \ast H(x)
\]

\[
= \int_{-\infty}^{\infty} \frac{B}{2\pi} \frac{(x - \xi)^2 - z^2}{(x - \xi)^2 + z^2} H(\xi) d\xi
\]

\[
= -\frac{B}{2\pi} \frac{x}{x^2 + z^2}
\]

(6.35)

While assuming \( l(t) \) (so \( \eta(x) \))is monotonous and \( \eta(x) \) is \( >0 \) \( t<0 \), substituting equation (6.34) into equation (6.13) gives,

\[
f_1(x,t) = \frac{\partial}{\partial t} \left( f_1(x,t) - f_0(x) \right) = \frac{\partial}{\partial t} \left( H(\eta(x) - t) - H(x) \right)
\]

\[
= \begin{cases} 
0, & t < 0, \\
-\delta(\eta(x) - t), & t > 0.
\end{cases}
\]

(6.36)

For \( f_1(x,t) \) given by equation (6.36) the general solution (equation (6.33)) yields

\[
\frac{\partial u(x,z,t)}{\partial z} = \frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{-\infty}^{\infty} d\xi \int_{0}^{\infty} dw \left[ H(t - w - \hat{b}) F(t - w, x - \xi) \delta(\eta(\xi) - w) \right] \right\}
\]

\[
= \frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{-\infty}^{\infty} d\xi H(t - \eta(\xi) - \hat{b}) F(t - \eta(\xi), x - \xi) \right\}
\]
which coincides with equation (3.2) of Markenscoff (1980).

Thus, the solution for a Volterra dislocation moving nonuniformly is obtained in equation (6.37) as a special case of the ramp-core dislocation model.

### 6.1.2 Nonuniformly moving ramp-core dislocation with delta sequence (arctan displacement) core

In this case, we consider a dislocation with \( \arctan(\varepsilon x) \) displacement function on the slip-plane \( z = 0 \). The dislocation is at rest until time \( t = 0 \), when it starts moving nonuniformly, with the center of the core moving as \( x = l(t) \). Two cases are considered:

1. **Constant width:** \( \varepsilon = \text{constant} \)
2. **Variable width:** \( \varepsilon = \varepsilon(t) \).

For the ramp function, we have

\[
f_0(x) = H_\varepsilon(x), \quad f_1(x, t) = H_\varepsilon(x - l(t))
\]

where

\[
H_\varepsilon(x) = H(x) \frac{\varepsilon}{\pi \varepsilon^2 + x^2}
\]
\[
\int_{-\infty}^{\infty} \frac{1}{\pi} \frac{d}{1 + \left(\frac{\xi}{\epsilon}\right)^2} = \left[ \frac{1}{\pi} \tan^{-1}\left(\frac{\xi}{\epsilon}\right) \right]_{\xi=-\infty}^{\infty} = \left(\frac{1}{\pi}\right) \arctan\left(\frac{x}{\epsilon}\right) + \frac{1}{2}
\]

So the problem with ramp function in (6.38) is:

The governing equation

\[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2} = b^2 \frac{\partial^2 u_x}{\partial t^2}\]  \hspace{1cm} (6.40)

with boundary condition

\[u_x(x,0,t) = \begin{cases} 
\frac{B}{2} H_x(x), & t < 0 \\
\frac{B}{2} H_x(x-l(t)), & t \geq 0 
\end{cases}\]  \hspace{1cm} (6.41)

which is decomposed into the superposition of the following two problems:

Problem I:

\[\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2} = \frac{P_0}{\mu} \frac{\partial^2 u_y}{\partial t^2}\]

\[u_y(x,0,t) = \frac{B}{2} H_x(x) \quad \text{for all } t \]  \hspace{1cm} (6.42)

Problem II:

\[\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2} = \frac{P_0}{\mu} \frac{\partial^2 u_y}{\partial t^2}\]

\[u_y(x,0,t) = \begin{cases} 
0, & t < 0 \\
\frac{B}{2} [H_x(x-l(t)) - H_x(x)], & t \geq 0
\end{cases}\]  \hspace{1cm} (6.43)
The solution to the static problem--Problem I is

\[ u(x,z) = \frac{B}{2\pi} \frac{z}{x^2 + z^2} * H_\varepsilon(x) \]  

\[ = \frac{B}{2\pi} \frac{z}{x^2 + z^2} \left[ H(x) \frac{1}{\pi \varepsilon^2 + x^2} \right] \]

\[ = \frac{B}{2\pi} \arctan \left( \frac{x}{\varepsilon + z} \right) + \frac{B}{4} \]

We also obtain the stress from equation (6.44):

\[ \frac{\partial u}{\partial z}(x,z) = \frac{B}{2\pi} \frac{x^2 - z^2}{(x^2 + z^2)^2} * H_\varepsilon(x) = \frac{B}{2\pi} \frac{x^2 - z^2}{(x^2 + z^2)^2} * \left[ H(x) \frac{1}{\pi \varepsilon^2 + x^2} \right] \]

\[ = \left[ \frac{B}{2\pi} \frac{x^2 - z^2}{(x^2 + z^2)^2} * H(x) \right] \frac{1}{\pi \varepsilon^2 + x^2} \]

\[ = \left[ \int_{-\infty}^{\infty} \frac{B}{2\pi} \frac{(x - \xi)^2 - z^2}{((x - \xi)^2 + z^2)^2} H(\xi) d\xi \right] \frac{1}{\pi \varepsilon^2 + x^2} \]

\[ = \left[ \int_{0}^{\infty} \frac{B}{2\pi} \frac{2(x - \xi)^2 - \left((x - \xi)^2 + z^2 \right)}{(x - \xi)^2 + z^2} d\xi \right] \frac{1}{\pi \varepsilon^2 + x^2} \]

\[ = \left[ \frac{B}{2\pi} \frac{(x - \xi)}{(x - \xi)^2 + z^2} \right]_{\xi=0}^{\infty} \frac{1}{\pi \varepsilon^2 + x^2} = \frac{B}{2\pi} \frac{x}{x^2 + z^2} \]

\[ = -\frac{B}{2\pi^2} \int_{-\infty}^{\infty} \frac{(x - \xi)}{(x - \xi)^2 + z^2} \frac{\varepsilon}{\varepsilon^2 + \xi^2} d\xi \]

\[ = -\frac{B}{2\pi} \frac{x}{x^2 + (z + \varepsilon)^2} \]  

\[ (6.45) \]
For the solution of Problem II, two cases are discussed: \( \varepsilon = \text{constant} \) and \( \varepsilon = \varepsilon(t) \), that is, constant or time-dependent core width, and solution for each of them is obtained in the following:

1. \( \varepsilon = \text{constant} \):

From equations (6.13), (6.38), (6.39), we have when \( \varepsilon = \text{constant} \):

\[
f_1(x,t) = -\frac{1}{\pi} \frac{\varepsilon l'(t)}{\varepsilon^2 + (x-l(t))^2}
\]

which, when inserted into equation (6.33), gives the solution of equation (6.43) in the form

\[
\frac{\partial u(x,z,t)}{\partial z} = \frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{-\infty}^{\infty} d\xi \int_{0}^{\infty} dw [H(t-w-\hat{r}b)F(t-w,x-\xi)] \frac{1}{\pi} \frac{\varepsilon l'(w)}{\varepsilon^2 + (\xi-l(w))^2} \right\}
\]

(6.47)

which is simplified by changing the variable of integration \( w \) to the variable \( \zeta = l(w) \) (also \( w = \eta(\zeta) \)). So that \( d\zeta = l'(w)dw \), and we obtain

\[
\frac{\partial u(x,z,t)}{\partial z} = \frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{-\infty}^{\infty} d\xi \int_{0}^{\infty} d\zeta \left[ \frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + (\xi-\zeta)^2} H(t-\eta(\zeta)-\hat{r}b)F(t-\eta(\zeta),x-\zeta) \right] \right\}
\]

(6.48)

Change variables in equation (6.48) from \( (\xi,\zeta) \) to \( (h,\zeta) \) with \( h = \xi - \zeta \), so that

\[ d\xi = dh \quad \zeta = h + \xi \]

and equation (6.48) takes the form

\[
\frac{\partial u(x,z,t)}{\partial z} = \frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{-\infty}^{\infty} dh \int_{0}^{\infty} d\zeta \left[ \frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + h^2} H(t-\eta(\zeta)-\hat{r}b)F(t-\eta(\zeta),x-h-\zeta) \right] \right\}
\]

\[
= \frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{0}^{\infty} d\zeta \left[ H(t-\eta(\zeta)-\hat{r}b)F(t-\eta(\zeta),x-\zeta) \right] \frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + h^2} \right\}
\]
We can see that solution (6.49) is the convolution of the RHS of equation (6.37) with \( \frac{1}{\pi \varepsilon^2 + x^2} \), which is expected since the two problems satisfy boundary conditions (6.34) and (6.38) that are related to each other by equation (6.39).

2. \( \varepsilon = \varepsilon(t) \):

For \( \varepsilon = \varepsilon(t) \), from equations (6.38), (6.39), we obtain

\[
f_0(x) = \frac{1}{\pi} \arctan\left(\frac{x}{\varepsilon(0)}\right) + \frac{1}{2},
\]

\[
f_1(x,t) = \frac{1}{\pi} \arctan\left(\frac{x - l(t)}{\varepsilon(t)}\right) + \frac{1}{2}
\]

and from equation (6.13), we obtain

\[
f_3(x,t,\varepsilon) = \frac{\partial}{\partial t} f_1(x,t) = -\varepsilon(t) \left[ \frac{(x - l(t))\varepsilon'(t) + \varepsilon(t)\varepsilon''(t)}{\pi \varepsilon^2(t) + (x - l(t))^2} \right]
\]

which, when inserted into equation (6.33), gives the solution for the nonuniformly moving ramp core dislocation with time-dependent width \( \varepsilon = \varepsilon(t) \) in the form

\[
\frac{\partial u(x,z,t)}{\partial z} = \frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{\infty}^{\infty} d\xi d\omega \left[ H(t - w - \hat{v}b) F(t - w, x - \hat{\xi}) \frac{(\xi - l(w))\varepsilon'(w) + \varepsilon(w)\varepsilon''(w)}{\pi \varepsilon^2(w) + (\xi - l(w))^2} \right] \right\}
\]

which can be explicitly evaluated if \( l(t) \) and \( \varepsilon(t) \) are known functions.

### 6.2 Stress analysis at the Mach wave front for the ramp-core dislocation accelerating through the shear-wave speed barrier
For transient screw dislocation jumping from rest and moving nonuniformly, the ramp-core model (with delta sequence arctan displacement core) has been introduced to analyze this problem. The solution to this problem using ramp-core model with constant width \((\varepsilon = \text{constant})\) is equation (6.49):

\[
\frac{\partial u(x,z,t)}{\partial z} = \frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{0}^{\infty} d\xi \left[ H(t - \eta(\xi)) - \hat{\rho} b \right] F(t - \eta(\xi), x - \xi) \right\} \frac{1}{\pi \varepsilon^2 + x^2}
\]

where

\[
F(t - \eta(\xi), x - \xi) = \frac{(t - \eta(\xi))^2 z^2}{\hat{\rho}^4 \sqrt{(t - \eta(\xi))^2 - \hat{\rho}^2 b^2}} - \frac{(x - \xi)^2 \sqrt{(t - \eta(\xi))^2 - \hat{\rho}^2 b^2}}{\hat{\rho}^4}
\]

\(\hat{\rho} = \left[ (x - \xi)^2 + z^2 \right]^{\frac{1}{2}}.\)

Inserting equation (6.54) into equation (6.49), we obtain:

\[
\frac{\partial u(x,z,t)}{\partial z} = \frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{0}^{\infty} d\xi \left[ H(t - \eta(\xi)) - \hat{\rho} b \right] \frac{(t - \eta(\xi))^2 z^2}{\hat{\rho}^4 \sqrt{(t - \eta(\xi))^2 - \hat{\rho}^2 b^2}} \right\}
\]

\[
\times \frac{1}{\pi \varepsilon^2 + x^2}
\]

\[
= \frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{0}^{\infty} d\xi H(t - \eta(\xi)) \left[ \frac{(t - \eta(\xi))^2 z^2}{(x - \xi - \eta)^2 + z^2} \sqrt{(t - \eta(\xi))^2 - b^2 [(x - \xi - \eta)^2 + z^2]} \right]
\]

\[
- \frac{(x - \xi - \eta)^2 \sqrt{(t - \eta(\xi))^2 - b^2 [(x - \xi - \eta)^2 + z^2]} \left[ \frac{1}{\pi \varepsilon^2 + \xi^2} d\xi \right] \frac{1}{\pi \varepsilon^2 + x^2}
\]

(6.55)
At the Mach wave front \((t^*, \zeta^*)\), we have

\[
[t^* - \eta(\zeta^*)] = (x - \zeta^*)^2 + z^2 b^2
\]  

(6.56)

From equation (6.55), we can see that in order to find the property of solution at the Mach wave front, we have to analyze the property of expression

\[
I = (t - \eta(\zeta))^2 - b^2 \left[ (x - \xi - \zeta)^2 + z^2 \right]
\]  

(6.57)

near the Mach wave front \((t^*, \zeta^*)\).

Expand RHS of equation (6.57) asymptotically near \((t^*, \zeta^*)\):

\[
I = (t - \eta(\zeta))^2 - b^2 \left[ (x - \xi - \zeta)^2 + z^2 \right]
\]

\[
= \left[ t - t^* + t^* - \eta(\zeta^*) - \eta'(\zeta^*) (\zeta - \zeta^*) - \frac{1}{2} \eta''(\zeta^*) (\zeta - \zeta^*)^2 + o((\zeta - \zeta^*)^3) \right]^2
\]

\[
- b^2 \left[ (x - \xi - \zeta + \zeta^* - \zeta^*)^2 + z^2 \right]
\]  

(6.58)

Rearrange the RHS of equation (6.58), we obtain

\[
I = (t - \eta(\zeta))^2 - b^2 \left[ (x - \xi - \zeta)^2 + z^2 \right]
\]

\[
= (t - t^*)^2 + (t^* - \eta(\zeta^*))^2 + \eta'(\zeta^*)^2 (\zeta - \zeta^*)^2 + \frac{1}{4} \eta''(\zeta^*)^2 (\zeta - \zeta^*)^4 + 2(t^* - \eta(\zeta^*)) (t - t^*) (t^* - \eta(\zeta^*))
\]

\[
- 2(t - t^*) \eta'(\zeta^*) (\zeta - \zeta^*) - (t - t^*) \eta''(\zeta^*) (\zeta - \zeta^*)^2 - 2(t^* - \eta(\zeta^*)) \eta''(\zeta^*) (\zeta - \zeta^*)
\]

\[
- (t^* - \eta(\zeta^*)) \eta'''(\zeta^*) (\zeta - \zeta^*)^3 + \eta''(\zeta^*) \eta''(\zeta^*) (\zeta - \zeta^*)^3
\]

\[
- b^2 \left[ (x - \xi - \zeta)^2 + (\zeta^* - \zeta)^2 + z^2 - 2(x - \zeta^*) \xi + 2(x - \zeta^*) (\zeta^* - \zeta) - 2(\zeta^* - \zeta) \xi + z^2 \right]
\]

\[
+ o((\zeta - \zeta^*)^2)
\]  

(6.59)

Part of expression (6.59)

\[
II = (\zeta - \zeta^*)^2 \left[ \eta''(\zeta^*) - b^2 \right] + 2(t^* - \eta(\zeta^*)) (t - t^*) - 2 \eta'(\zeta^*) (t - t^*) (\zeta - \zeta^*)
\]
\[ -2\left[(t^* - \eta(\xi^*)\eta'(\xi^*) + b^2 \xi - b^2 (x - \xi^*)\eta(\xi^*) + (t - t^*)^2 \right] \]  

(6.60)

can be written as

\[ II = -\left[\eta'(\xi^*)^2 - b^2 \right] (\xi - \zeta_0)(\zeta_1 - \zeta) \]  

(6.61)

where \( \zeta_0, \zeta_1 \) are roots of RHS of equation (6.60).

When \( \eta'(\xi^*) = b \), namely the velocity of the moving dislocation equals the wave speed, we obtain from equation (6.61) that \( II = 0 \), so that the expression of \( I \), namely equation (6.59) gives

\[ I = \left((t - \eta(\xi))^2 - b^2\right) \left[(x - \xi - \xi)^2 + z^2\right] \]

\[ = \frac{1}{4} \eta''(\xi^*) (\xi - \xi^*)^2 - (t - t^*) \eta''(\xi^*) (\xi - \xi^*)^2 - (t - \eta(\xi^*) \eta''(\xi^*) (\xi - \xi^*)^2 \]

\[ + \eta''(\xi^*) \eta'(\xi^*) (\xi - \xi^*)^2 - b^2 [\xi^2 - 2(x - \xi^*) \xi] + o(\xi - \xi^*)^2 \]

\[ = -(t - t^*) \eta''(\xi^*) (\xi - \xi^*)^2 - (t - \eta(\xi^*) \eta''(\xi^*) (\xi - \xi^*)^2 \]

\[ - b^2 \xi^2 + 2b^2 (x - \xi^*) \xi + o(\xi - \xi^*)^2 \]  

(6.62)

Rearrange equation (6.62), we obtain

\[ I = \left((t - \eta(\xi))^2 - b^2 \right) \left[(x - \xi - \xi)^2 + z^2\right] \]

\[ = -b^2 \xi^2 + 2b^2 (x - \xi^*) \xi - (t - \eta(\xi^*) \eta''(\xi^*) (\xi - \xi^*)^2 + o(\xi - \xi^*)^2 \]  

(6.63)

Substitute equation (6.63) into equation (6.55). So when \( \eta'(\xi^*) = b \), namely the velocity of the moving dislocation equals the wave speed, near Mach wave front \((t^*, \xi^*)\), the stress for dislocation using ramp core model starting from rest and moving nonuniformly can be written as
\[
\frac{\partial u(x, z, t)}{\partial z} = \frac{B}{2\pi} \frac{\partial}{\partial t} \left\{ \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\xi} d\zeta H\left(t - \eta(\zeta) - \sqrt{(x - \xi - \zeta^*)^2 + z^2 b}\right) \right. \\
\left. \left( (t^* - \eta(\zeta^*)) \frac{z^2}{(x - \xi - \zeta^*)^2 + z^2} \right) \sqrt{-b^2 \xi^2 + 2b^2 (x - \xi - \zeta^*) \xi - (t - \eta(\zeta^*)) \eta''(\zeta^*)(\zeta - \zeta^*)^2} \right\} \frac{1}{\pi \frac{\varepsilon}{\varepsilon^2 + \frac{\varepsilon^2}{\xi^2}}} d\xi \\
(6.64)
\]

From equation (6.64), we can see that the existence of the convolution makes the solution not longer singular as \( \zeta_0, \zeta_1 \rightarrow \zeta^* \). So the use of ramp-core model will remove the singularity on the wave front obtained from Volterra dislocation model that uses the Heaviside step function to describe the core.

Part of the material in this chapter is being prepared for publication. Prof. Xanthippi Markenscoff is the co-author for this paper.
CONCLUSIONS

In this thesis,

• The analytic solution for general dislocation motion has been obtained by means of Laplace transforms—with particular emphasis on the nature of the wave-front equations for subsonic and supersonic motion.

• The relation between dislocation acceleration and curvature of the Mach-wave front for supersonic motion has been obtained.

• The function determining the Mach front as the dislocation accelerates through the shear wave speed has been studied by analyzing the general solution at the instant that the dislocation velocity is equal to \( c_2 \).

• The stress singularity at the forming Mach front has been analyzed at the instant of acceleration through the shear wave speed both for screw and edge dislocations. The coefficient of the delta function for the stress at the wave front is found to be a logarithmic over square root singularity, which implies that the Volterra dislocation is too strong of a dislocation model for the crystal dislocation.

• The analytic solution for the stress of a ramp-core dislocation accelerating through the shear-wave speed has been obtained, which removes the singularity.
REFERENCES


