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STABILITY AND OSCILLATION ANALYSIS OF MAGNETIC MIRROR SYSTEMS WITH CONDUCTING END PLATES

Shalom Fisher
(Ph. D. Thesis)

November 19, 1963

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STABILITY AND OSCILLATION ANALYSIS OF MAGNETIC MIRROR SYSTEMS WITH CONDUCTING END PLATES

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STABILITY AND OSCILLATION ANALYSIS OF MAGNETIC MIRROR SYSTEMS WITH CONDUCTING END PLATES

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ABSTRACT

This thesis is an investigation of the stability of a fully ionized highly conducting plasma in static equilibrium confined by an axi-symmetric mirror field. The magnetic field interpenetrates with the plasma and is tied down by conducting end plates. These conducting end plates tend to stabilize the system, since any deformation in the plasma must be accompanied by an increase in magnetic field energy. In particular, the interchange instability is inhibited.

In Sec. II, the eigenvalue theory is applied to various energy principles, and necessary and sufficient conditions for stability are derived. In Sec. III, these stability criteria are studied for specific field configurations and particle distributions. The maximum \( \beta = 2P/B^2 \) for which these examples are stable is defined as \( \beta_c \) and computed as a function of the mirror ratio and the aspect ratio \( A = \bar{r}/a \), where \( \bar{r} \) is the maximum radial distance to the field lines, and \( 2a \) is the distance between the end plates. It is found that \( \beta_c \) decreases as the mirror ratio and the aspect ratio increase.
In Sec. IV certain geomagnetic oscillations are studied by means of a hydromagnetic model. A variational principle is used to calculate the oscillation frequency of the model; excellent agreement with the observed frequency is obtained.
I. INTRODUCTION

A poloidal magnetic field—that is, a magnetic field with axial symmetry and no component in the azimuthal direction—will trap charged particles in the regions of minimum field strength. In a magnetic-mirror configuration, the field strength is greater at the ends of the region than near the center section. The mirror machine utilizes this configuration to confine a high-temperature plasma for the release of thermo-nuclear energy. Similarly, the dipole field of the earth is a mirror field that traps the charged particles of the Van Allen Radiation belts. Mirror fields can confine single particles within the adiabatic limit; here, the gyroradius of the particles is infinitesimal compared to the characteristic distance over which the magnetic field changes, and the gyroperiod is infinitesimal compared to the time scale of the variation in magnetic field experienced by the particle. However, even in the adiabatic limit, difficulties in confinement arise as the number of particles increases.

The most important problem of plasma confinement in the mirror geometry is the question of instability. By instability we mean that there exists a perturbation which, when applied to the system, will cause it to increasingly depart from its initial state. We discuss two important categories of instability. The first of these categories consists of "micro-instabilities", which arise from gradients of the particle-distribution function in velocity space.\(^1\) By judicious scaling of the parameters of the mirror system, these instabilities can either be eliminated, or reduced to growth rates slow compared to the rate of
thermonuclear-energy production. The second category is the flute or interchange instability that arises from the interchange of neighboring flux tubes. If the magnetic lines of force enter the plasma region by means of insulators, they are free to move parallel to themselves. In this case, the interchange deformation takes place with a change of magnetic field energy that is small in comparison with the change of particle energy. If, in addition, the lines of force are concave toward the direction of higher pressure (as in the diagram), the particle energy is decreased and the system is unstable.

The interchange instability might be inhibited by the "finite" ion larmor-radius effect, which furnishes a viscous-like contribution to the pressure tensor. However, this effect can only be important in a plasma model where the adiabatic theory predicts a very slow growth rate for the interchange instability. Since this effect is therefore applicable only to a limited class of mirror-machine models, we do not consider it in this thesis. Instead, we examine the larger class of mirror-machine models where the adiabatic theory is applicable.

A method for stabilizing the mirror configuration against the interchange perturbation is to introduce the magnetic lines of force into the plasma by means of perfectly conducting endplates. By the applicable form of Ohm's law, \( \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \), (see Appendix A), any displacement,
in the plasma adjacent to these end plates would create an electric field, \( \mathbf{E} \), parallel to the interface, provided the magnetic field, \( \mathbf{B} \), has a component normal to the interface. But the perfectly conducting end plate prevents such an electric field from occurring. Hence, the lines of force are anchored or "tied" to the perfectly conducting end plates. The interchange deformation will now cause the lines of force to shear and compress, thereby increasing the magnetic-field energy. This increase tends to offset the decrease of particle energy during the deformation and thereby tends to stabilize the plasma. The system can only be unstable if the ratio of the plasma-energy to the magnetic-field-energy density is larger than a certain critical value. To be more specific, we define \( \beta \) as \( \frac{P_{\perp}}{B^2/2} \) evaluated at the mid-plane on the axis, where \( P_{\perp} \) is the pressure perpendicular to the lines of force and \( B \) is the magnetic field. Then, for instability, the \( \beta \) of the system must be larger than \( \beta_c \), where \( \beta_c \) is a critical \( \beta \).

In Sec. II we examine the magnetic-mirror configuration that has conducting end plates and an arbitrary \( \beta \) and field-line curvature. We apply the hydromagnetic-energy principle \(^7\) to obtain a necessity condition and a sufficiency condition for stability for a collision-dominated highly conducting plasma with isotropic pressure. For collisionless plasma with anisotropic pressure, we use Newcomb's energy principle \(^8\) to obtain a sufficiency condition for stability, and the Chew-Goldberger-Low energy principle \(^9\) to obtain a necessity condition for stability. These stability criteria contain integrals along a line of force and
depend upon the minimum values of certain equilibrium quantities. The variation of magnetic-field energy under the interchange perturbation, which stabilizes the system, is represented as a positive-definite term in the stability criteria. The variation of particle energy under the perturbation, which possibly destabilizes the system, furnishes terms in the stability criteria that may be negative.

In Sec. III we apply the stability criteria of Sec. II to specific equilibrium configurations. In Sec. III-A we examine the hydromagnetic-stability criteria for two different equilibrium states. The first of these is described by a finite scalar pressure and a magnetic field that includes the diamagnetism of the plasma currents; it has a small aspect ratio defined as the ratio of the midplane diameter to the distance between the end plates. A small aspect ratio implies that the field lines are curved only slightly (see Sec. III-A). The computed $\beta_c$ is found to decrease rapidly as the mirror ratio increases. The second hydromagnetic-equilibrium state we examine consists of vacuum-magnetic fields and has an arbitrary aspect ratio; that is, the field lines may be strongly curved. Numerical computation furnish $\beta_c$ as a function of the mirror ratio and the aspect ratio. It is found that for a fixed mirror ratio, $\beta_c$ increases as the aspect ratio decreases and approaches that value calculated for the equilibrium state of finite pressure and small aspect ratio. We discuss this result in detail in Sec. III-C.

In Sec. III-B we apply the stability criteria derived from the Chew-Goldberger-Low and the Newcomb energy principles to an equilibrium
state having vacuum-magnetic fields, small aspect ratio, and an anisotropic pressure tensor that vanishes at the end plates. It is convenient to use vacuum fields because the equilibrium-field equations involving the plasma currents, Eqs. (5.2) through (5.4), are nonlinear and no analytic solution is known to exist for finite curvature. The use of vacuum fields is justified in detail in Sec. III-B. We find that the sufficiency condition derived from the Newcomb energy principle furnishes a $\beta_c$ for each value of the mirror ratio very close to that furnished by the hydromagnetic sufficiency condition. This result is discussed in detail in Sec. III-B. For each value of the mirror ratio, the necessity condition of the Chew-Goldberger-Low energy principle furnishes a $\beta_c$ that is substantially less than that calculated from the hydromagnetic necessity condition. We discuss this result in Sec. III-C.

In Sec. IV of this thesis we study the oscillation of geomagnetic field lines that intersect the earth at the Auroral zones. We assume that these lines are "tied" at the ionosphere and use the density distribution of the magnetosphere furnished by Johnson. By means of the hydromagnetic energy principle we calculate the frequency of the lowest normal mode of oscillation. This calculated frequency is in good agreement with the experimental measurements of Suguira.
II. DERIVATION OF STABILITY CRITERIA

A. GENERAL EIGENVALUE THEORY

The energy principles are concerned with a fully-ionized plasma in static equilibrium; here, viscosity, electrical resistance, and all dissipative effects are considered negligible. However, the energy principles otherwise differ widely in the plasmas treated. These plasmas are discussed in detail in the Appendix for each of the energy principles dealt with in this paper. A brief summary of this discussion is given here.

In the hydromagnetic Matterhorn energy principle we consider the plasma to be collision-dominated. This requires that the matter stress tensor be isotropic not only in equilibrium, but also during the course of the perturbation, and leads to the adiabatic equation of state

\[ \frac{d}{dt} \rho \gamma = 0. \]

In the Chew-Goldberger-Low energy principle the plasma is considered to be collisionless, but it is assumed that heat flow along the lines of force vanishes for the equilibrium state and for all perturbations. These considerations leads to an anisotropic matter stress tensor, and to a double-adiabatic equation of state. The Newcomb energy principle is similar to the Kruskal-Oberman energy principle in that it is based upon the guiding center approximation, sometimes called the adiabatic-particle drift theory. However, Newcomb also allows for charge separation along the lines of force. In this treatment, heat flow along the lines of force is allowed to develop during the perturbation, and no equation of state is applicable. It is necessary to use the Vlasov equation to calculate the departure from equilibrium of the distribution function and thereby its macroscopic moments.
The energy principles are of the form $\delta W = [\xi, F(\xi)]$, where $\xi$ is an infinitesimal displacement in the plasma, and $F$ is a linear, self-adjoint (vector) operator. This $\delta W$ represents the change of potential energy of the plasma resulting from the displacement $\xi$. If $\delta W$ is less than zero for any $\xi$, the system is unstable. The procedure is to minimize $\delta W$ with respect to $\xi$, subject to a normalization condition on $\xi$. The sign of the minimum $\delta W$ then determines the stability of the system. Another way of stating this is to note that the Euler equation resulting from the minimization of $\delta W$ is of the form $F(\xi) = \lambda \xi$. The $\xi_{\text{min}}$ that minimizes $\delta W$ then has an eigenvalue $\lambda_{\text{min}}$ with respect to the operator $F$. This $\lambda_{\text{min}}$ is real and is shown below to equal the minimum value of $\delta W = \delta W_{\text{min}}$. The eigenvalue $\lambda_{\text{min}}$ is a function only of the equilibrium parameters of the system, since $F(\xi)_{\text{min}} = \lambda_{\text{min}} \xi_{\text{min}}$ and $F$ is of first order in $\xi$. The requirement that $\lambda_{\text{min}}$ be greater than zero is a stability condition. In general, it is not easy to solve for $\lambda_{\text{min}}$ exactly, but we do calculate upper bounds and lower bounds for this quantity by the techniques discussed below.

The operator $F$ is in general a second-order linear partial differential operator in three independent variables corresponding to the three spatial dimensions. The displacement $\xi$ is a vector in three dimensions for the hydromagnetic and Chew-Goldberger-Low energy principles and in two dimensions for the Newcomb and Kruskal-Oberman energy principles. However, by expressing $F$ and $\xi$ in coordinates natural
to the axisymmetric system, denoted by \((x, \psi, \theta)\), and by successively minimizing \(\delta W\) with respect to the components of the vector \(\vec{z}\), \(\delta W\) can be represented as a one-dimensional integral involving one component of the vector \(\vec{z}\). The \((x, \psi, \theta)\) coordinate system is discussed in Appendix C. The \(x\) direction is along the line of force, and the unit vector in this direction is denoted by \(e_1\). Similarly, the unit vector in the \(\psi\) direction is denoted by \(e_2\), and indicates the direction of the principal normal of the line of force. We then have \(e_3 = e_1 \times e_2\) in the \(\theta\) direction.

After minimizing \(\delta W\) with respect to the components of \(\vec{z}\) in the \(\theta\) and \(x\) directions, \(\delta W\) has the form \(\delta W = \int \delta W(\psi) \, d\psi\), where

\[
\delta W(\psi) = \int dx \left[ p(x, \psi) \left( \frac{\partial X}{\partial x}(x, \psi) \right)^2 + q(x, \psi) X^2(x, \psi) \right].
\] (1)

Here \(X(x, \psi)\) is the component of \(\vec{z}\) in the \(e_2\) direction, and the variable \(\psi\) becomes a parameter of \(\delta W\). We minimize \(\delta W(\psi)\) with respect to \(X\), subject to the normalization condition \(\int X^2(x, \psi) \, dx = 1\). This is equivalent to minimizing \(\rho = \delta W(\psi) / \int X^2(x, \psi) \, dx\). The Euler equation has the form
The associated Sturm-Liouville equation is given by

\[-\frac{\partial}{\partial x} \left[ p(x, \psi) \frac{\partial}{\partial x} X(x, \psi) \right] + q(x, \psi) X(x, \psi) = \lambda X(x, \psi). \tag{2} \]

Note that the structures of \( p(x, \psi) \) and \( q(x, \psi) \) depend upon the particular energy principle under consideration. (We derive specific Sturm-Liouville equations from the various energy principles in Secs. II-B, II-C, and II-D.)

In the following discussion we omit the parameter \( \psi \).

The integration of Eq. (1) is over a finite region, hence the eigenvalues of Eq. (3) are discrete. From a physical consideration, \( X(x) \), \( \frac{\partial X}{\partial x}(x) \), and \( \frac{\partial^2 X}{\partial x^2}(x) \) are continuous functions of \( x \). Furthermore, \( X(x) \) vanishes at the boundary because of the perfectly conducting end plates, as discussed in the introduction. The eigenfunctions \( X_n(x) \) satisfy the same properties and are orthogonal since they are eigenfunctions of a self-adjoint operator. Therefore the expansion theorem is applicable and we have \( X(x) = \sum_{n} a_n X_n(x) \). We then obtain
$SW = \sum_n \lambda_n a_n^2$.

Proof: $SW = \int \left[ p \left( \frac{\partial X}{\partial x} \right)^2 + q x^2 \right] dx = \int \left[ p \left( \frac{\partial}{\partial x} \sum_n X_n(x) \right)^2 + q \left( \sum_n X_n(x) \right)^2 \right] dx$

\[ = \int \left[ p \left( \sum_n \frac{\partial X_n}{\partial x} \right) \sum_m a_m \frac{\partial X_m}{\partial x} + q \left( \sum_n X_n(x) \right) (\sum_m a_m X_m(x)) \right] dx \]

\[ = \sum_n \sum_m a_n a_m \int \left[ p \left( \frac{\partial X_n}{\partial x} \frac{\partial X_m}{\partial x} + q X_n X_m \right) \right] dx \]

\[ = \sum_n \sum_m a_n a_m \left\{ \left( \frac{X_m}{X_n} \right) \left( \frac{\partial X_n}{\partial x} \right) \right\} \text{boundary} + X_m \left[ \frac{\partial}{\partial x} \left( \frac{\partial X_n}{\partial x} + \phi X_n \right) \right] dx \]

\[ = \sum_n \sum_m a_n a_m \int X_m \lambda_n X_n \ dx = \sum_n a_n^2 \lambda_n \], by virtue of orthogonality.

If we substitute $X_1(x) = X(x)$ into the $SW$ of Eq.(1) we obtain $SW(X_1) = \lambda_1$, where $\lambda_1$ is the smallest eigenvalue of Eq. (3).

It immediately follows from Eq. (4) that $SW_{\text{min.}} = \lambda_1$, and that $SW > \lambda_1$. Thus $\lambda_1$ is the greatest lower bound of $SW$. It also follows that for $X = X_1 + \epsilon \xi$

\[ \rho(x) = \rho(X_1 + \epsilon \xi) \geq \lambda_1, \] (5)
where $\xi$ is any arbitrary function satisfying the same continuity and boundary conditions as $X$, and $\epsilon$ is a small quantity.

Proof: \[ \rho(X_1 + \epsilon \xi) = \frac{\delta W(X_1 + \epsilon \xi)}{(X_1 + \epsilon \xi)^2} \]

\[ = \frac{\lambda_1 + 2 \epsilon \lambda_1 \int X_1 \xi \, dx + \epsilon^2 \delta W(\xi, \xi)}{\int 2 \, dx + 2 \epsilon \int X_1 \xi \, dx + \epsilon^2 \int \xi^2 \, dx} \]

\[ = \lambda_1 + \frac{\epsilon^2 \left[ \delta W(\xi) - \lambda_1 \int \xi^2 \, dx \right]}{\int X_1^2 \, dx + 2 \epsilon \int X_1 \xi \, dx + \epsilon^2 \int \xi^2 \, dx} \]

The above proof shows that if the error in the eigenfunction $X_1$ is of the order $\epsilon$, the error in the eigenvalue $\lambda_1$ is of the order $\epsilon^2$. \(^{15}\)

We now compare the eigenvalue problem involving $q(x)$ to the problem involving another function $Q(x)$, but with the same region and boundary conditions. Let $q(x) < Q(x)$ in the entire region. Let $\lambda_1$ denote the lowest eigenvalue in the problem involving $q(x)$ and let $\lambda_1'$ denote the lowest eigenvalue in the problem involving $Q(x)$. We write $\delta W(X;q)$ instead of $\delta W(X)$. We then have $\delta W(X;q) \leq \delta W(X;Q)$, which implies \(^{16}\)

\[ \lambda_1 \leq \lambda_1'. \quad (6) \]
Proof: Let \( \chi_1(x) \) denote the eigenfunction corresponding to \( \lambda_1' \), by the expansion theorem \( \chi_1 = \sum_n a_n X_n(x) \). Then
\[
\lambda_1' = \lambda_1 \int \chi_1^2(x) \, dx
\]
\[
= \lambda_1 \sum_{n=1}^{\infty} a_n^2 \leq \sum_{n=1}^{\infty} \lambda_n a_n^2 = \delta W(\chi_1; q) \leq \delta W(\chi_1; q) = \lambda_1'.
\]

Equations (5) and (6) are the basis on which the stability criteria in the following pages are constructed. As mentioned above, the sign of the lowest eigenvalue determines the stability of the system. The necessary and sufficient condition that the magnetic-mirror system be stable (under the same class of physical conditions in which \( \delta W \) is applicable) is \( \lambda_1 > 0 \). In practice it is difficult to compute \( \lambda_1 \) exactly. However, by means of Eqs. (5) and (6), it is possible to find upper and lower bounds of \( \lambda_1 \). We know that a sufficient condition for the system's stability is that the lower bound of \( \lambda_1 \) be greater than zero. In a similar fashion, a necessary condition for stability is that the upper bound of \( \lambda_1 \) be greater than zero. Two necessary and two sufficient conditions for stability can thereby be derived. The hydromagnetic energy principle furnishes a necessity and a sufficiency condition for stability. The Newcomb energy principle furnishes a sufficiency condition for stability, and the Chew-Goldberger-Low energy principle furnishes a necessity condition for stability.
B. A SUFFICIENCY CONDITION FOR STABILITY

USING THE HYDROMAGNETIC ENERGY PRINCIPLE

The hydromagnetic energy principle deals with a plasma model in static equilibrium described, in the absence of external potential, by the following equations:

\[ \mathbf{0} = \mathbf{j} \times \mathbf{B} - \nabla p, \quad (7) \]
\[ \mathbf{j} = \nabla \times \mathbf{A}, \quad (8) \]
\[ \mathbf{A} = \nabla \times \mathbf{B}, \quad (10) \]
\[ \frac{d}{dt} \gamma = 0, \quad (11) \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (12) \]
\[ \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0. \]

Here is the electric field, \( \mathbf{E} \) the magnetic field, \( \mathbf{j} \) is the electric-current density, \( \rho \) is the mass density, \( p \) is the material pressure, \( \gamma \) is the ratio of specific heats, \( e \) is the magnitude of the electronic charge, and \( \mathbf{u} \) is the mass-flow velocity. The equations are written in rationalized Gaussian units with \( c = 1 \). (The conditions for validity of the above equations are discussed in Appendix A).

The boundary conditions at the plasma-conducting surface interface are discussed in reference 7. As discussed in the introduction, Eq. (12) implies that the component of \( \mathbf{u} \) parallel to the interface vanishes if the magnetic field \( \mathbf{B} \) has a component normal to the interface. This implies that the component of \( \mathbf{A} \) parallel to the interface vanishes since \( \nabla \mathbf{A} = 0 \). We shall assume that the plasma does not pull away from the conducting
surface, which implies that the component of \( \xi \) perpendicular to the boundary is zero. Now since both the perpendicular and the parallel components of \( \xi \) vanish, we find that \( \xi \) vanishes at the boundary.

In the absence of external potentials, the hydromagnetic energy principle furnishes the following expression for \( SW(\xi) \) in the plasma region:

\[
SW = \frac{1}{2} \int d\tau \left[ Q^2 - J \cdot Q \times \xi + \gamma p (\nabla \cdot \xi)^2 + \xi \cdot \xi \cdot \nabla p \right]
\] (13)

Here \( Q = \nabla \times (\xi \times B) \), and \( d\tau \) is the volume element in space.

We next introduce the \((X, \psi, \theta)\) coordinate system. The variable \( X \) is not quite the same as the \( x \) employed in the previous section, but it is also a measure of distance along the line of force, and the transformation between \( x \) and \( X \) is given by \( dx = r^2 B^2 J \, dX \).

Here \( J \) is the Jacobian of the transformation between the \((X, \psi, \theta)\) set and cartesian coordinates. The characteristics of the \((X, \psi, \theta)\) system are discussed in Appendix C, but we note here that the lines of force lie in surfaces of constant \( \psi \). If one chooses \( \psi (0, z) = 0 \), then the magnetic flux interior to the surface, defined by \( \psi = \text{constant} \), is \( 2\pi \psi \). The gradient operator in the \((X, \psi, \theta)\) coordinate system is given by \( \nabla \equiv \varepsilon_1 \frac{1}{JB} \frac{\partial}{\partial X} + \varepsilon_2 \frac{rB}{\partial \psi} + \varepsilon_3 \frac{1}{r} \frac{\partial}{\partial \theta} \). Then the current density \( j = \nabla \times \widetilde{B} = \varepsilon_2 \frac{r}{J} \frac{\partial}{\partial \psi} JB^2 \) and \( \nabla p = \widetilde{j} \times \widetilde{B} = -\varepsilon_3 \frac{rB}{J} \frac{\partial}{\partial \psi} JB^2 \).
Thus the pressure $p$ is a function of $\psi$ alone, and $p' = dp/d\psi = j/r$ and $j/r$ is constant along a line of force. Using these results, Eq. (13) for $SW$ becomes

$$
SW = \frac{1}{2} \int J \, d\psi \, dX \, d\theta \left[ \left( \frac{r \, \partial X}{\partial \psi} \right)^2 + \left( \frac{r \, \partial Y}{\partial \psi} \right)^2 + B^2 \left( \frac{\partial X}{\partial \theta} + \frac{\partial Y}{\partial \theta} \right)^2 \right]
$$

$$
+ p' X \left( \frac{\partial X}{\partial \psi} + \frac{\partial Y}{\partial \psi} \right) + \frac{\gamma P}{J^2} \left( \frac{\partial}{\partial \psi} JX + \frac{\partial}{\partial \theta} JY + \frac{\partial Z}{\partial X} \right)^2
$$

$$
+ \frac{p' X}{J} \left( \frac{\partial}{\partial \psi} J + \frac{\partial}{\partial \theta} JY \right) + \frac{1}{J} \frac{\partial}{\partial X} \left( p' X \right]
$$

where $X(X, \psi, \theta) = rB \xi \psi, Y = \xi \psi / r$, and $Z = \xi / B$. By using the boundary conditions on $\xi$, the last term integrates to zero. Upon Fourier analysis of $\xi$ with respect to $\theta$, and integration over $\theta$, we obtain $SW = SW_0 + \sum_m SW_m$, where $m$ indicates the Fourier component of the function, and $SW_m$ is given by

$$
SW_m = \frac{\pi}{2} \int dx \, d\psi \left[ \frac{1}{rB^2} \left( \frac{\partial X_m}{\partial \psi} \right)^2 + \frac{1}{2} \left( \frac{\partial Y_m}{\partial \psi} \right)^2 + \frac{r^2}{J} + B^2 \left( \frac{\partial X_m}{\partial \psi} + \frac{\partial Y_m}{\partial \psi} \right)^2 \right]
$$

$$
+ p' X_m \left( \frac{\partial X_m}{\partial \psi} + Y_m \right) + \frac{\gamma P}{J} \left( \frac{\partial}{\partial \psi} JX_m + \frac{\partial Z_m}{\partial X} + \frac{\partial Z_m}{\partial X} \right)^2
$$

$$
+ p' X_m \left( J \frac{\partial X_m}{\partial \psi} + X_m \frac{\partial J}{\partial \psi} + J \frac{Y_m}{\partial \psi} \right] .
$$

(15)
here, $\delta W_0/2$ is obtained if we replace $Y_m$ in Eq. (15) by $\frac{m\theta}{r}$, and take the limit of $m = 0$.

Since, for each $m$, $\delta W_m$ depends only on the set $(X_m, Y_m, Z_m)$, it can be varied independently. It is clear from Eq. (15) that if $\delta W_m$ can be made negative, then $\delta W_{m+1}$ can also be made negative. This means that the minimum $\delta W_m$ is $\delta W_M$, where $M$ is the maximum value of $m$, and we need consider only $\delta W_M$ in the minimization procedure. The magnitude of $M$ is limited by the adiabatic assumptions of our model, which require $\lambda >> a_1$, where $\lambda$ is the wavelength of the perturbation and $a_1$ is the ion Larmor radius. Since we have $\lambda = \frac{r}{M}$, where $r$ is the radius of the plasma, and $\lambda$ is the wavelength in the $\theta$ direction, we must have $r/M >> a_1$ or $r/a_1 >> M$. We choose a plasma model in which this is not a severe restriction on the magnitude of $M$ and in which $M$ is so large that $1/M^2 \left( \frac{\partial Y_m}{\partial X} \right)^2 r^2/J$ is negligible in comparison to the other terms in $\delta W_M$, and after some algebraic manipulation, and suppressing the subscript $M$, $\delta W_M$ becomes

\[ \delta W = \frac{\pi}{2} \int d\psi \, dx \, J \left[ (B^2 + \gamma \rho) \left( U + \frac{X (p' + \gamma \rho \frac{\partial}{\partial \psi} \log J) + \gamma \frac{\partial Z}{\partial X}}{B^2 + \gamma \rho} \right)^2 + \frac{1}{r^2 B^2 J^2} \left( \frac{\partial X}{\partial X} \right)^2 + p'X^2 \frac{\partial}{\partial \psi} \log J - p'^2 \frac{X^2}{B^2} \right] + \frac{\gamma \rho B^2}{B^2 + \gamma \rho} \left[ X \frac{\partial}{\partial \psi} \log J + \frac{1}{J} \left( \frac{\partial Z}{\partial X} - \frac{p'X^2}{B^2} \right)^2 \right], \] (16)
where \( U = \frac{\partial X}{\partial \psi} + Y \).

We next minimize \( SW \) with respect to \( U \) for fixed but arbitrary \( X \) and \( Z \) by setting 

\[
U = \frac{-X(p' + \gamma p \frac{\partial}{\partial \psi} \log J) + \frac{\gamma p \partial Z}{J \frac{\partial}{\partial X}}}{B^2 + \gamma p},
\]

which reduces \( SW \) to

\[
SW = \frac{\pi}{2} \int d\psi \ dx \left[ \frac{1}{r B^2} \left( \frac{\partial X}{\partial X} \right)^2 + p' DX^2 \ J + \frac{J B^2 \gamma p}{B^2 + \gamma p} \left( XD + \frac{1}{J} \frac{\partial Z}{\partial X} \right)^2 \right]
\]

(17)

where

\[
D = \frac{\partial}{\partial \psi} \ log J - \frac{p'}{B^2} = \frac{-2}{R + B}
\]

(18)

and \( R \) is the radius of curvature of a line of force; \( R \) is negative where the line of force is concave towards the side of smaller \( \psi \) and positive where the line of force is convex.
After minimizing Eq. (18) with respect to \( z \), we obtain

\[
8W = \pi \int d\psi \delta W(\psi) \quad \text{where}
\]

\[
8W(\psi) = \int d\chi \left[ \frac{1}{2} \left( \frac{3}{2} \chi'^2 + p' J D \chi'^2 \right) + \frac{1}{2\pi} \int d\chi \chi'^2 \right] \left[ L' + \frac{V'}{\gamma_p} \right]
\]

(19)

and where

\[
f(\psi) = \frac{2\pi}{L'} \int d\chi \frac{J D \chi'}{L' + \frac{V'}{\gamma_p}} \quad \text{and} \quad L' = 2\pi \int d\chi \frac{J}{B^2} \quad \text{and} \quad V' = 2\pi \int d\chi J
\]

Equation (19) was derived in reference 1. Its derivation is repeated here for purposes of illustration. We employ the same techniques in deriving similar equations from the Chew-Goldberger-Low and the Newcomb energy principles.

The variable \( \psi \) is only a parameter in Eq. (19). The lines of force lie in surfaces of constant \( \psi \), and the coordinate \( \theta \) can be ignored. Therefore the problem is reduced to the one-dimensional consideration of the stability of a single line of force.

We are seeking a sufficient condition for stability. Since the term in \( 8W \) involving \( f(\psi) \) is positive definite, we simply set \( f(\psi) = 0 \). We shall call this new expression \( 8W' \). We now change variables by using \( dx = r^2 B^2 J d\chi \). With this substitution, \( 8W'(\psi) \) becomes,
omitting the parameter \( \psi \),

\[
8W' = \int dx \left[ \left( \frac{\partial X}{\partial x} \right)^2 + \frac{p'D}{r^2 B^2} x^2 \right].
\]  

(20)

We choose the normalization

\[
\int x^2 \, dx = 1.
\]  

(21)

Now, minimizing \( 8W' \) subject to Eq. (21), we obtain the Euler equation

\[
- \frac{\partial^2 X}{\partial x^2} + \frac{p'D}{r^2 B^2} X = \lambda X.
\]  

(22)

The associated Sturm-Liouville equation is

\[
- \frac{\partial^2 X_n}{\partial x^2} + \frac{p'D}{r^2 B^2} X_n = \lambda_n X_n.
\]  

(23)
We now use the techniques of Sec. II-A to find a lower bound for the $SW'$ of Eq. (20). As demonstrated by Eq. (24) and the discussion concerning it, we have $\lambda_1 \leq SW'$, where $\lambda_1$ is the lowest eigenvalue of Eq. (23). Hence $\lambda_1 \geq 0$ implies stability. To find a lower bound of $\lambda_1$, we notice that Eq. (23) has the same form as Eq. (3), with $p(x) = 1$ and $q(x) = \frac{p'D}{2r_B^2}$. If we substitute $q_m < q(x)$ for all $x$, the resulting $\lambda_1'$ will have the property $\lambda_1' < \lambda_1$, as we can see from Eq. (6). For $q_m$ constant, the above equation is very easy to solve, particularly with the boundary conditions that $X_n$ vanish at the end points of the domain of $x$. We have $X_n = A \sin \frac{n\pi x}{x_0}$, with

$$(\lambda_n' - q_m) = \frac{n\pi}{x_0}, \text{ or } \lambda_n' = \frac{n^2\pi^2}{x_0^2} + q_m.$$ 

Getting back to $SW$, we have

$$\lambda_1' = \frac{\pi^2}{x_0^2} + \text{Min}(p'D/r_B^2), \text{ where } x_0 = \int_0^X dx = \int_0^X d\xi r_B^2 d\xi$$

$$= \int_0^L d\xi r_B^2, \text{ where } d\xi \text{ is the unit of length along the line of force}, \text{ and } L \text{ is the total length of the line. So we have}$$

$$0 < \lambda_1' = \frac{\pi^2}{[d\xi r_B^2]^2} + \text{Min} \frac{p'D}{r_B^2} < SW \quad (24)$$

for our sufficient condition for stability.

It is to be remarked that the expression in brackets in condition
(24), \( \frac{\mathbf{D}}{\mathbf{r}^2 \mathbf{B}} \), must be evaluated on the same line of force that the integral \( \int_0^L \mathbf{d} \mathbf{r} \mathbf{^2 B} \) is evaluated. In Sec. II-A, Eq. (24) is evaluated numerically for a specific configuration, and the critical \( \beta \) is calculated as a function of the mirror ratio \( M \), and the aspect ratio \( A = \frac{r_0}{a} \), where \( r_0 \) is the radial distance to the outermost line of force in the midplane of the configuration, and \( a = L/2 \).
C. A NECESSITY CONDITION FOR STABILITY

USING THE HYDROMAGNETIC ENERGY PRINCIPLE

We begin with the hydromagnetic expression for $SW$ given by Eq. (13), and proceed exactly as in the previous section. We Fourier analyze $X, Y,$ and $Z$ with respect to $\theta$, take the limit of large $m$, and minimize $SW$ with respect to $U = Y + \frac{\partial X}{\partial \psi}$, obtaining Eq. (17):

$$SW = \frac{\pi}{2} \left[ \int d\psi \int \frac{1}{rB^2J} \left( \frac{\partial X}{\partial \psi} \right)^2 + p'DX^2J + \frac{J\gamma pB^2}{B^2 + \gamma p} \left( X^2 + \frac{1}{J} \frac{\partial Z}{\partial \psi} \right)^2 \right].$$

Now we note that we are looking for a necessary condition for stability. Therefore we do not need to minimize $SW$, but find an expression that will lie above this minimum, but as close to it as feasible. For simplicity, we set $Z = 0$. This is not the minimizing $Z$, so we obtain an expression $SW_1$ which is larger than the minimum $SW$. This yields

$$SW_1 = \frac{\pi}{2} \int d\psi SW_1(\psi)$$

where $SW_1(\psi)$ is given by

$$SW_1(\psi) = \int dX \left[ \frac{1}{rB^2J} \left( \frac{\partial X}{\partial \psi} \right)^2 + p'DX^2J + \frac{J\gamma pB^2}{B^2 + \gamma p} \right]$$

(25)
As previously, we change variables by setting \( dx = J r^2 B^2 \, dX \).

Equation (25) becomes, omitting the parameter \( \psi \),

\[
\delta W = \int dx \left[ \left( \frac{\partial X}{\partial x} \right)^2 + \frac{\partial' D X^2}{r^2 B^2} + \frac{\gamma p D^2 X^2}{r^2 B^2 \left( 1 + \frac{\gamma p}{B^2} \right)} \right].
\] (26)

We compare \( \delta W_1 \), given by Eq. (26) to \( \delta W' \), given by Eq. (20).

Note that \( \delta W' \) lies below the true minimum of \( \delta W \) and was obtained by neglecting positive definite terms. The difference between \( \delta W_1 \) and \( \delta W' \) is the term \( \frac{\gamma p D^2 X}{r^2 B^2 \left( 1 + \frac{\gamma p}{B^2} \right)} \). Since we have from Eq. (18)

\[
D^2 \sim \frac{1}{r^2 B^2 R^2} \quad \text{and} \quad \gamma p D^2 \sim \beta/r^2 R^2,
\]

we conclude that for a configuration in which the lines of force are only bent slightly (i.e., for \( R \) large), or for a configuration in which \( \beta \) is small, the errors in setting \( f(\psi) = 0 \) and \( \beta = 0 \) are small. Therefore, at this stage of development, the necessity condition should be close to the sufficiency condition, and we are near the correct minimum.

To proceed with the derivation of a necessary condition for stability, we adopt a normalization condition of the form of Eq. (21).

Let us set \( q(x) = \frac{1}{r^2 B^2} \left[ \partial' D + \frac{\gamma p D^2}{1 + \frac{\gamma p}{B^2}} \right] \). Then we have
The Euler equation resulting from minimizing $\rho$ is given by

$$
-\frac{\partial^2 X}{\partial x^2} + (\lambda - q(x)) X = 0; \quad (28)
$$

the associated Sturm-Liouville equation is

$$
-\frac{\partial^2 X_n}{\partial x^2} + (\lambda_n - q(x)) X_n = 0. \quad (29)
$$

Then, from Eq. (5) we have $\rho \geq \lambda_1$ with $\rho(X_1') = \lambda_1$. As a reminder, $X_1$ is the solution of Eq. (29) corresponding to the smallest eigenvalue $\lambda_1$. In fact for $X_1'$ different from $X_1$, we have $\rho(X_1') > \rho(X_1)$. The closer $X_1'$ is to $X_1$, the closer $\rho(X_1')$ is to $\rho(X_1)$. In fact, from the proof of Eq. (5), for $|X_1' - X_1| = \epsilon$ we have $|\rho(X_1' - X_1)| = O(\epsilon^2)$, where $O$ means order of.

In particular, let us set $q_m \leq q(x)$ for all $x$, solve the Sturm-Liouville Eq. (29) with $q_m$ substituted for $q(x)$, and use the
corresponding $X_1'$ to estimate $\rho(X_1')$. We have then $X_1' = A \sin \pi x/x_0$, with $(\lambda_1' - \alpha)^{1/2} = \frac{\pi x}{x_0}$. This function $X_1'$ has the desired boundary conditions and continuity properties and is even under reflection about the midplane. It should therefore be a reasonable approximation to $X_1$. Another way to look at this is to note that from the expansion theorem, we have $X_1' = \sum a_n X_n'$. What we are doing is keeping the first term in this expansion. We substitute $X_1' = A \sin \pi x/x_0$ into Eq. (27) and obtain:

$$
\rho(X_1') = \frac{\int \left[ \frac{\partial}{\partial x} \sin \frac{\pi x}{x_0} \right]^2 + q(x) \sin^2 \frac{\pi x}{x_0} }{\int \sin^2 \frac{\pi x}{x_0}} 
$$

$$
= \frac{\pi^2}{x_0^2} + \frac{2}{x_0} \int dx \sin^2 \frac{\pi x}{x_0} q(x)
$$

A necessity condition for stability is $\rho(X_1') > 0$. More specifically, this condition is

$$
0 < \frac{\pi^2}{x_0^2} + \frac{2}{x_0} \int \frac{dx}{r^2 B^2} \sin^2 \frac{\pi x}{x_0} \left[ p'D + \frac{B^2 y p D^2}{B^2 + y_p} \right].
$$
To simplify Condition (31), first replace $dx$ by $d\ell r^2 B$, then the above becomes

$$\frac{\pi^2}{L} \int_0^L \frac{d\ell}{d\ell r^2 B} \sin \pi \left( \frac{\ell}{L} \right) \left( \frac{p'D + \frac{B^2}{E^2} \gamma pD^2}{E^2 + \gamma_p} \right) > 0.$$  

(32)
D. SUFFICIENCY CONDITION FOR STABILITY USING THE NEWCOMB ENERGY PRINCIPLE

The Newcomb energy principle is a generalization of the Kruskal-Oberman energy principle to include the effects of charge separation along the lines of force. It deals with a class of equilibria characterized by no collisions and no dissipative effects, and it assumes the validity of the guiding-center approximation. This class of plasma models is described by the following set of equations, in which we neglect external potentials:

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{33}
\]
\[
0 = j \times \mathbf{B} - \nabla \cdot \mathbf{\Pi}, \tag{34}
\]
\[
\mathbf{J} = \nabla \times \mathbf{B}, \tag{35}
\]
\[
\nabla \cdot \mathbf{B} = 0, \tag{36}
\]
\[
\mathbf{E} + \nabla \times \mathbf{B} = 0. \tag{37}
\]

The equilibrium distribution function of a species \(a\) is determined by solving the collisionless Boltzmann equation in the guiding center approximation and is given by \(f_a = f_a(x, v_y, v_{\perp}^2)\). It is not possible to obtain a closed set of macroscopic equations unless one makes the additional assumption of zero heat flow along the lines of force for small departures from equilibrium, as in the Chew-Goldberger-Low energy principle. The above quantities have the same definitions as in the hydro-magnetic case except for \(\mathbf{\Pi}\), which is now a diagonal tensor, of the
form

\[ P = P_1 I + (P_\nu - P_L) \xi_1 \xi_1 \quad (38) \]

where

\[ P_1 = \frac{1}{2} \sum \alpha m_\alpha \int v_1^2 f_\alpha(x_1, y) \, d^3v \quad P_\nu = \sum \alpha m_\alpha \int v_\nu^2 f_\alpha(x_1, y) \, d^3v . \quad (39) \]

Here the sum is over the particle species \( \alpha \) of mass \( m_\alpha \) and distribution \( f_\alpha \). By definition, \( v_1 = (\xi_2 \cdot \chi)^2 + (\xi_3 \cdot \chi)^2 \)^{1/2} and \( v_\nu = \xi_1 \cdot \chi \).

(For further details about the assumptions of the Newcomb energy principle see Appendix A).

Newcomb develops a \( SW \) given by \( SW = SW_0 + SW_1 \), where

\[ SW_1 = \frac{1}{2} \int dt \, h^2 \left[ S \left( \frac{E^2}{m^2} \right) S \left( \frac{m}{2} \xi_{avg}^2 \right) - \left( S \left( \frac{E}{q} \xi_{avg} \right) \right)^2 \right] . \quad (40) \]

Here the operator \( S \) is defined by
\[ S \left[ F (e, \alpha, \nu) \right] = - \frac{1}{2} \pi \sum_{\alpha} B q_\alpha \, d\nu \, de \, \frac{\partial g_\alpha}{\partial e} \cdot F, \quad (41) \]

where \( q_\alpha = \left\{ 2 (e - \nu B - \frac{e_\alpha \psi}{m_\alpha}) \right\}^{1/2} = |v_\nu|, \) where \( \phi \) is the electrostatic potential along a line of force, \( g_\alpha \) is the equilibrium distribution function of the species \( \alpha, \nu = \frac{v_\perp^2}{2B} \), and \( e \) is the particle energy per unit mass. Also we have

\[ \xi = q_\alpha^2 (\varepsilon_1 \varepsilon_1 : \nabla \xi) + \nu B (\nabla \cdot \xi - \varepsilon_1 \varepsilon_1 : \nabla \xi) - \frac{e_\alpha}{m_\alpha} \hat{\phi}, \quad (42) \]

where the circumflex means the perturbed quantity, and \( \frac{1}{\hbar^2} = \frac{q^2}{4\pi S(e^2/mq^2)} \), and \( \xi \) average is defined in Appendix A. Now Newcomb shows that \( S (F^2) S (G^2) \geq (S(FG))^2 \). Actually, this is the Schwartz inequality.

By setting \( F^2 = \frac{m}{q^2} \xi_{\text{avg}}^2 \) and \( G^2 = \frac{e^2}{mq^2} \), it follows that \( 8W_1 \geq 0 \).

Accordingly, in the spirit of seeking a sufficient condition for stability, we consider only \( 8W_0 \leq 8W \).

We are left with \( 8W_0 \), where

\[
8W_0 = \frac{1}{2} \int d\tau \left\{ Q^2 + \sum_1 \xi \times Q + \sum_1 \xi \cdot \nabla P_1 \right. \\
+ \left( C + \frac{\hbar^2}{2} \right) \left( \nabla \cdot \xi - \varepsilon_1 \varepsilon_1 : \nabla \xi \right)^2 + \varepsilon_1 \varepsilon_1 : \nabla \xi \cdot \nabla \left( \nabla \cdot \xi \right) P_1 \\
+ P_\perp \left( \xi \cdot \nabla \varepsilon_1 - \varepsilon_1 \cdot \nabla \xi \right) \cdot \left( \nabla \xi \cdot \varepsilon_1 + \varepsilon_1 \cdot \nabla \xi \right) \}
\]

\[ (43) \]
where \( C = 2P_\perp - S(mv^2B^2/q^2) \), \( P_\perp = P_v - P_\perp \), \( \sigma = S(evB/q^2) \), and 
\( \mathcal{Q} = \nabla \lambda (\xi x) \). This \( \mathcal{W}_0 \) is identical to the equivalent expression derived from the Kruskal-Oberman energy principle, except for the term 
\( \sigma^2 n^2 (\nabla \cdot \xi - \varepsilon_{1} \varepsilon_{j} : \nabla \xi)^2 \) which enters from charge neutrality considerations. This term is positive-definite, hence our sufficient condition for stability is a better criterion than if the Kruskal-Oberman energy principle were employed.

We note from Appendix C that \( |\nabla \chi| = 1/jB \) and \( |\nabla \psi| = rB \). From the Serret-Frenet formulae, \( \varepsilon_{1} \cdot \nabla \varepsilon_{1} = \varepsilon_{2}/R \), where \( R \) is the radius of curvature of a field line. From Eq. (35) we write, using

\[
\frac{\partial B}{\partial \theta} = 0 \quad \text{from axial symmetry},
\]

\[
\frac{1}{\cos \theta} \cdot \nabla \times B = -\nabla \left( \frac{B^2}{2} \right) + \nabla \cdot \nabla \mathcal{Q} = \varepsilon_{2} \nabla \psi \frac{\partial B^2}{\partial \psi} - \varepsilon_{2} (\nabla \chi) \frac{\partial}{\partial \lambda} \frac{B^2}{2} \frac{\partial}{\partial \theta} R \]

\[+ \varepsilon_{1} \varepsilon_{1} \cdot \nabla (\varepsilon_{1} B) = \varepsilon_{2} \left( \frac{B^2}{R} + rB^2 \frac{\partial B}{\partial \theta} \right) \].

Similarly, we have

\[
\nabla \cdot P_\perp = \nabla \cdot (P_\perp \varepsilon_{1} \varepsilon_{1} + P_\perp I) = \varepsilon_{2} \left( \frac{\partial P_\perp}{\partial \lambda} + \frac{P_\perp}{R} \right) \]

\[+ \varepsilon_{1} \left( \frac{\partial}{\partial \lambda} P_\perp \varepsilon_{1} + \frac{1}{jB} \frac{\partial P_\perp}{\partial \lambda} \right) \].
From Eq. (34), \( j \times B = \nabla \cdot P \) we have the result

\[
\mathbf{e}_2 \left( \frac{B^2}{R} - r B^2 \frac{\partial B}{\partial \psi} \right) = \mathbf{e}_2 \left( r B \frac{\partial P}{\partial \psi} + \frac{P}{R} \right) + \mathbf{e}_1 \left( \nabla \cdot P - \mathbf{e}_1 + \frac{1}{JB} \frac{\partial P_\perp}{\partial x} \right).
\]

Equating the components of \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \), we obtain

\[
\frac{B^2}{R} - r B^2 \frac{\partial B}{\partial \psi} = r B \frac{\partial P_\perp}{\partial \psi} + \frac{P}{R} \quad (44)
\]

We have

\[
0 = \nabla \cdot P \mathbf{e}_1 + \frac{1}{JB} \frac{\partial P_\perp}{\partial x} \quad (45)
\]

or in slightly different form

\[
0 = P \frac{\partial}{\partial x} \left( \frac{1}{B} \mathbf{e}_1 + \frac{\partial P_\perp}{\partial x} \right) \quad (46)
\]
since \( \nabla \cdot \mathbf{E}_1 = \frac{1}{J} \frac{\partial}{\partial x} \mathbf{B} \). Again, from Eq. (34) we have \( j \times \mathbf{B} = \nabla \cdot \mathbf{P} \).

and by using \( j = j \mathbf{e}_3 \) (i.e., a purely azimuthal current), we solve for \( j \) and obtain

\[
j = r P_1' + \frac{P}{R_B},
\]

where the prime denotes differentiation with respect to \( \psi \). Writing

\[
\mathbf{e}_2/R = \mathbf{e}_1 \cdot \nabla \mathbf{e}_1 = - \mathbf{e}_1 \times (\nabla \times \mathbf{e}_1) = \mathbf{e}_1 \times \mathbf{e}_3 \lvert \frac{r}{J} \frac{\partial}{\partial \psi} J_B \rvert
\]

\[
= - \mathbf{e}_2 \frac{r}{J} \frac{\partial}{\partial \psi} J_B
\]

we obtain

\[
\frac{1}{R} = - \frac{r}{J} \frac{\partial}{\partial \psi} J_B.
\]
Now we are in a position to compute the integrand of Eq. (43).

We first have

\[ Q = \nabla x (\xi \times B) = -e_1 B \left( \frac{\partial X}{\partial \psi} + \frac{\partial Y}{\partial \theta} \right) + e_2 \frac{1}{r J B} \frac{\partial X}{\partial \psi} + e_3 \frac{r}{J} \frac{\partial Y}{\partial X} \]

where \( X = r B \xi \psi \), \( Y = \frac{\xi \theta}{r} \). It should be noted that the \( \xi \) of the Newcomb formalism is given by \( \tilde{\xi} = e_2 \xi \psi + e_3 \xi \theta \) and has two components as contrasted to the three component \( \xi \) of the hydromagnetic and Chew-Goldberger-Low energy principles. If we define \( U = \frac{\partial X}{\partial \psi} + \frac{\partial Y}{\partial \theta} \), then we have

\[ Q = -e_1 B U + e_2 \frac{1}{r J B} \frac{\partial X}{\partial \psi} + e_3 \frac{r}{J} \frac{\partial Y}{\partial X} \]

or

\[ Q^2 = B^2 U^2 + \frac{1}{r^2 J^2 B^2} \left( \frac{\partial X}{\partial \psi} \right)^2 + \frac{r^2}{J^2} \left( \frac{\partial Y}{\partial X} \right)^2. \quad (49) \]
The next term in the integrand can be found if we write
\[ \vec{z} = \vec{z}_2 \times \frac{X}{r} R B + \vec{z}_3 \cdot r Y, \]
and
\[ j = \vec{z}_3 \cdot j. \]
So we obtain
\[ \vec{z} \cdot \vec{z} \times \vec{z} = \frac{du}{r} \]
and we have by Eq. (47)

\[ \vec{z} \cdot \vec{z} \times \vec{z} = UX P_\perp + UX P_\perp / r R B. \]  

(50)

Now we have
\[ \nabla \cdot \vec{z} = \frac{1}{J} \left[ \frac{\partial}{\partial \psi} (r B \xi_2 J) + \frac{\xi_2 J}{r} \right] = \frac{1}{J} [U + X \frac{\partial J}{\partial \psi}] = U + X \frac{\partial J}{\partial \psi} \log J \]
and also
\[ \xi \cdot \nabla P_\perp = r B \xi_2 \frac{\partial P_\perp}{\partial \psi} = X P_\perp. \]

Therefore, we write

\[ (\nabla \cdot \xi)(\xi \cdot \nabla P_\perp) = X P_\perp \left( U + X \frac{\partial J}{\partial \psi} \right). \]  

(51)

The quantity \( \xi_1 \xi_1 : \nabla \xi \) can be written
\[ \xi_1 \cdot \nabla (\xi_2 \xi_2) + \xi_2 \xi_1 \cdot \nabla \xi_2 + \xi_1 \cdot \nabla (\xi_3 \xi_3) + \xi_3 \xi_1 \cdot \nabla \xi_3 \]
\[ = - \frac{\xi_2}{R} = - \frac{X}{r R B}, \]
where we have made use of the Serret-Frenet formulae applied to the axisymmetric geometry. Hence, we have, by using the above result for
\[ \nabla \cdot \xi, \]
\[ P_-(\nabla \cdot \xi - \varepsilon_1 \varepsilon_1 : \nabla \xi)^2 = P_- (U + X \frac{\partial}{\partial \psi} \log J + X/\text{rRB})^2. \]

(52)

Additionally we have

\[ \nabla \cdot \xi P_- = P_- U + P_- X \frac{\partial}{\partial \psi} \log J + X P_-'. \]

therefore

\[ (\varepsilon_1 \varepsilon_1 : \nabla \xi)(\nabla \cdot \xi P_-) = - \frac{X}{\text{rRB}} (P_- U + P_- X \frac{\partial}{\partial \psi} \log J + X P_-'). \]

(53)

Similarly we have

\[ \nabla \cdot \xi - \varepsilon_1 \cdot \nabla \xi = \varepsilon_1 (X/\text{rRB}) + \varepsilon_2 \left( - \frac{\partial X}{\partial \psi} / \text{rJB}^2 \right) + \varepsilon_3 \frac{\partial Y}{\partial \psi} / \text{JB} \]

and

\[ \nabla \cdot \xi + \varepsilon_1 \cdot \nabla \xi = \varepsilon_1 \left[ - 2X/\text{rRB} \right] + \varepsilon_2 \left[ - \frac{\partial X}{\partial \psi} / \text{rJB}^2 \right] + \varepsilon_3 \left[ - \frac{\partial Y}{\partial \psi} / \text{JB} \right]. \]

Hence we can write

\[ P_-[\varepsilon_1 \varepsilon_1 : \nabla \xi + (\varepsilon_1 \cdot \nabla \varepsilon_1 - \varepsilon_1 \cdot \nabla \xi) \cdot (\nabla \xi \cdot \varepsilon_1 + \varepsilon_1 \cdot \nabla \xi)] = P_[- - X^2/(\text{rRB}^2) - \left( \frac{\partial X}{\partial \psi} \right)^2 / \text{rJB}^2 - r^2 \left( \frac{\partial Y}{\partial \psi} \right)^2 / \text{JB}^2]. \]

(54)
Let us define $I$ as the integrand of Eq. (44); then, from Eqs. (49) through (54), $I$ is given by

$$I = B^2 U^2 + \left( \frac{\partial X}{\partial X} \right)^2 / r^2 J^2 B^2 + \left( \frac{\partial Y}{\partial X} \right)^2 r^2 / J^2 + U X P_{-} + U X P_{-} + U X P_{-} / r \ R \ B$$

$$+ U X P' + X^2 P_{-} \frac{\partial}{\partial \psi} \log J + (C + \sigma h^2)$$

$$(W + X \frac{\partial}{\partial \psi} \log J + X/r RB)^2$$

$$- (X/r RB)(P_{-} W + P_{-} X \frac{\partial}{\partial \psi} \log J + X P_{-})$$

$$+ P_{-} (- X^2 / r^2 R B^2 - \left( \frac{\partial X}{\partial X} \right)^2 / r^2 J^2 B^2 - r^2 \left( \frac{\partial Y}{\partial X} \right)^2 / J^2 B^2)$$

Thus we have for $Sw_0$, by collecting terms and setting $\sigma_\psi = P_{-} / B^2$,

$$Sw_0 = \frac{1}{2} \int J dX dY d \psi \left[ \left( 1 - \sigma_\psi \right) \left( \frac{\partial X}{\partial X} \right)^2 \right] \left[ \frac{1}{r^2 J^2 B^2} + (1 - \sigma_\psi) \frac{r^2}{J^2} \left( \frac{\partial Y}{\partial X} \right)^2 \right]$$

$$+ (B^2 + \sigma^2 h^2 + C) \left( W + X \left( \frac{P_{-}}{B^2} + (C + \sigma^2 h^2) \frac{\partial}{\partial \psi} \log J + \frac{1}{r RB} \right) \right)^2$$

$$+ X^2 \left( P_{-} \frac{\partial}{\partial \psi} \log J + (C + \sigma^2 h^2) \frac{\partial}{\partial \psi} \log J + \frac{1}{r RB} \right)$$

(Eq. (56) cont.)
We Fourier analyze with respect to $\theta$, and obtain, as in the hydromagnetic result, $\delta W = \delta W_0 + \sum_{m=1}^{\infty} \delta W_m$. Since we have $\delta W_{m+1} \leq \delta W_m$ for all $m$, we take the limit of infinite $m$. This yields, by using also the Eq.(48) and suppressing the subscript $\infty$,

$$\delta W = \frac{\pi}{2} \int J \, d\chi \, d\psi \left[ (1 - \sigma_-) \left( \frac{\partial \chi}{\partial \chi} \right)^2 \frac{1}{r^2 R^2} \delta W \right. + (B^2 + \sigma^2 \eta^2 + C) \left( U + \frac{X \left( P_{-} + \left( C + \sigma^2 \eta^2 \right) \left( \frac{1}{B} \frac{\partial B}{\partial \psi} \right) \right)}{B^2 + \sigma^2 \eta^2 + C} \right)^2 \right]$$

$$+ \left( \frac{P_{-} + 2P_{+}}{rRB} \right) + \frac{P_{+}^2}{B^2} + \frac{\sigma_-}{(rR)^2} (1 - \sigma_-)$$

$$+ (C + \sigma^2 \eta^2) \left( \frac{1}{B} \frac{\partial B}{\partial \psi} \right)^2 \left( C + \sigma^2 \eta^2 \right)^2 / B^2 + \sigma^2 \eta^2 + C$$

\( (57) \)

We minimize $\delta W$ with respect to $U$ for fixed, but arbitrary $X$, collect terms in $X^2$ and use Eq. (45) to obtain $\delta W = \frac{\pi}{2} \int d\psi \, \delta W(\psi)$, where, after we suppress the parameter $\psi$, $\delta W(\psi)$ is given by
\[ \delta W = \int J \, dx \left[ (1 - \sigma_-) \left( \frac{dx}{\partial x} \right)^2 + \frac{1}{r^2} \frac{\partial^2}{\partial \rho^2} \right] X^2 \left( 1 - \frac{1}{r R_B} \left( P_+ - 2 P_+ \right) \right) \]

\[ + \frac{X^2}{c + \frac{2}{h^2} + \frac{B^2}{\sigma_-}} \left( \frac{1}{r R_B} \right) (1 - \sigma_-) (C + \sigma_- h^2 + B^2 \sigma_-) \]

(58)

It should be mentioned that requiring \( 1 - \sigma_- \geq 0 \) is the firehose stability criterion.\(^\text{17}\) It should also be noticed that the leading term in \( 1/R \) in the coefficient of \( X^2 \) becomes identical to the equivalent term in the hydromagnetic energy principle, Eq.(17), for the case when \( P_+ = P_\perp \), which implies that \( P_- = \sigma_- = 0 \). Hence, for a configuration in which the lines of force are bent only slightly and in which the pressure is isotropic, the hydromagnetic approximation is more nearly valid.

By setting \( dx = dX r B J \sqrt{1 - \sigma_-} \) and \( q(x) = \left( \frac{1}{r R} \right) \left( \frac{1 - \sigma_- (C + \sigma_- h^2 + \sigma_- B^2 \sigma_-)}{C + \sigma_- \frac{2 h^2}{h^2} + \frac{B^2}{\sigma_-}} \right) \),

we obtain \( \delta W = \int dx \left[ \left( \frac{dx}{\partial x} \right)^2 + q(x) X^2 \right] \). With \( \delta W \) in this familiar form, we follow the same procedure as in Sec. II-A. First we adopt the normalization condition \( \int X^2 \, dx = 1 \), and then obtain the Euler equation

\[ - \frac{\partial^2 X}{\partial x^2} + \left( \lambda - q(x) \right) X = 0 \]
and the Sturm-Liouville equation

\[- \frac{\partial^2 X_n}{\partial x^2} + (\lambda_n - q(x)) X_n = 0. \tag{60}\]

We substitute \( q_m \leq q(x) \) for all \( x \) into the above equation in order to obtain the lowest eigenvalue \( \lambda'_1 \). From Eq. (6) we have \( \lambda'_1 \leq \lambda_1 \), where \( \lambda_1 \) is the lowest eigenvalue of Eq. (60). Since \( X_n' \) must vanish at the boundary, we obtain as before \( X_n' = A \sin \frac{n\pi x}{x_0} \), where \( X_n' \) is the solution of Eq. (60) with \( q_m \) substituted for \( q(x) \). The corresponding eigenvalue, \( \lambda_n' \) is given by

\[ \lambda_n' = \frac{n^2 \pi^2}{x_0^2} + q_m. \]

The condition for stability becomes

\[ 0 < \lambda'_1 = \frac{n^2 \pi^2}{x_0^2} + \min \{ q(x) \} \leq \lambda_1 \leq 8 \pi, \tag{61} \]

or more explicitly, by setting \( d\ell = JB \, dX \) we obtain the sufficiency condition for stability

\[ 0 < \frac{\pi^2}{I} \cdot \text{Min} \left[ \frac{1 - \sigma}{r^2 B^2} \left( \frac{P'_{r} + 2P'_{\ell}}{rRB} \right) + \frac{1 - \sigma}{(rR)^2} \frac{C + \sigma h^2 + B^2 \sigma}{(C + \sigma h^2 + B^2 \sigma)} \right] . \tag{62} \]
where $I_1 = \int_0^L \frac{\hat{x} I^2 \hat{r}}{(1 - \sigma)}$, and $L$ is the length of a line of force. It should again be stated that the quantity in the brackets must be evaluated on the same line of force, characterized by the parameter $\psi$, that $I_1$ is computed. If there exists any line of force for which the above criterion is violated, the system may be unstable.

In Sec. III, the right hand side of the Inequality (62) is evaluated for certain specific configurations. The critical $\beta$ is computed as a function of the mirror ratio $M$. 
E. NECESSITY CONDITION FOR STABILITY

USING THE CHEW-GOLDBERGER-LOW ENERGY PRINCIPLE

The Chew-Goldberger-Low energy principle is useful when calculating a necessary condition for stability because of the inequality
\[ \delta W_{\text{Newcomb}} \leq \delta W_{\text{CGL}} \]. Because of this inequality, the minimum of \( \delta W_{\text{CGL}} \) will always lie above the minimum of the more exact \( \delta W_{\text{Newcomb}} \). The reason for this inequality is that the Chew-Goldberger-Low treatment places an additional constraint on the displacement \( \xi \); i.e., there must be no heat flow along the lines of force in the perturbed state. Consequently, the domain of allowable displacements is restricted and the second-order variation in potential energy \( \delta W \) must always lie above the actual \( \delta W \) given by the Newcomb treatment.

The Chew-Goldberger-Low energy principle deals with a plasma model described by the following system of equations, with no external potential:

\[ 0 = j \times B - \nabla \cdot P, \quad \text{(63)} \]
\[ j = \nabla \times B, \quad \text{(64)} \]
\[ 0 = \nabla \cdot B, \quad \text{(65)} \]
\[ \frac{d}{dt} \frac{P_L}{\rho B} = 0, \quad \text{(66)} \]
\[ \frac{d}{dt} \frac{P_E^2}{\rho^3} = 0, \quad \text{(67)} \]
\[ \nabla \times E = -\frac{\partial B}{\partial t}, \quad \text{(68)} \]
and \[ E + \mathbf{u} \times B = 0. \quad \text{(69)} \]
Here, as before, \( \mathbf{E} \) is the electric field, \( \mathbf{B} \) is the magnetic field, \( \mathbf{j} \) is the electric current density, \( \mathbf{y} \) is the mass flow velocity, and \( \rho \) is the mass density. Also we have \( \mathbf{P} = \mathbf{P}_1 + (\mathbf{P}_h - \mathbf{P}_1) \mathbf{e}_1 \mathbf{e}_1 \), as in Eq. (38), and \( \mathbf{P}_1 \) and \( \mathbf{P}_h \) are defined in Eq. (39). Eqs. (66) and (67) are applicable as a result of the constraint mentioned above, that is, no heat flow along the lines of force under the perturbation described by the displacement \( \mathbf{\hat{x}} \). Further details about the assumptions of the Chew-Goldberger-Low energy principle are furnished in Appendix A.

The Chew-Goldberger-Low energy principle gives the following expression for \( \delta W \):

\[
\delta W = \frac{1}{2} \int \alpha d\tau \left( \mathbf{Q}^2 - \mathbf{j} \cdot \mathbf{Q} + \mathbf{\nabla} \cdot \mathbf{Q} + \mathbf{\nabla} \cdot \mathbf{Q} \right) + \left( \mathbf{\nabla} \cdot \mathbf{\hat{x}} \right) + \mathbf{\nabla} \cdot \mathbf{\hat{x}} + \mathbf{\nabla} \cdot \mathbf{\hat{x}}
\]

\[
+ \frac{1}{2} \mathbf{P}_1 \left[ \mathbf{j} \cdot \mathbf{e}_1 \mathbf{e}_1 : \mathbf{\nabla} \mathbf{\hat{x}} \right]^2 + \left( \mathbf{\nabla} \cdot \mathbf{\hat{x}} \right) \mathbf{\nabla} \cdot \mathbf{\hat{x}} \cdot \mathbf{P}_1
\]

\[
+ \left[ \mathbf{h} \left( \mathbf{\nabla} \cdot \mathbf{\hat{x}} \right) + \left( \mathbf{\nabla} \cdot \mathbf{\hat{x}} \right) \cdot \mathbf{\nabla} \cdot \mathbf{\hat{x}} \right]
\]

Here, as before, we have \( \mathbf{Q} = \mathbf{\nabla} \times (\mathbf{\hat{x}} \times \mathbf{B}) \), and \( \mathbf{\hat{x}} \) has three components as in the hydromagnetic case.
This expression for $8W$ is treated in the same manner as is the Newcombe $8W$ in the previous section, although the algebra is much harder because of the additional component of \( \tilde{z} \) along the lines of force. We express Eq. (70) in the \((X, \psi, \theta)\) coordinate system, and after much algebraic manipulation and after integrating by parts we obtain:

\[
8W = \frac{1}{2} \int J dX d\psi d\theta \left[ (1 - \sigma_\nu) \left( \frac{\partial Y}{\partial X} \right)^2 \frac{r^2}{J^2} \right.
\]

\[
+ (1 + 2K) \left( BU + B \left[ \frac{K}{1 + 2K} \left( \frac{1}{JB} \frac{\partial ZB}{\partial X} - \frac{X}{rRB} \right) + X \frac{\partial K}{\partial \psi} + \frac{Z \partial K}{J \partial X} \right] \right)^2
\]

\[
+ \left( \frac{3P''}{B} - \frac{K^2 B}{1 + 2K} \right) \left( \frac{1}{JB} \frac{\partial ZB}{\partial X} - \frac{X}{rRB} \right)
\]

\[
- \frac{K \left( \frac{X \frac{\partial B}{\partial \psi} + \frac{Z \partial B}{\partial X} + \frac{B X \frac{\partial K}{\partial \psi} + \frac{Z \partial B}{\partial X}}{1 + 2K} }{3P''} \right)^2}{B - \frac{K^2 B}{1 + 2K}}
\]

\[
+ z^2 \left( - \frac{B}{J^2} \left( \frac{\partial B}{\partial X} + \frac{B}{1 + 2K} \frac{\partial K}{\partial X} \right) \frac{2}{3P''} \frac{B K^2}{B^2 - \frac{K^2 B}{1 + 2K}} \right)
\]

(Eq. (71) cont.)
Here we use \( K = \frac{P_1}{B^2} \).

For the derivation of Eq. (71) see Appendix D.

Equation (71) was derived by Vuillemin\(^{18}\) who used the Chew-Goldberger-Low energy principle to obtain a sufficient condition for stability. His sufficient condition is applicable when heat flow is absent along the lines of force during the perturbation, and when

\[ P_1 B^2 / \rho^2 \] and \( P_1 / \rho B \) are constant along the lines of force. We then Fourier analyze with respect to \( \theta \), take the limit of \( m = M \), where \( M \) is the maximum value of \( m \) and is so large that \( \left(1/M^2\right)(1 - \sigma) \)

\[ (r^2/J) \left| \frac{\partial \psi}{\partial X} \right|^2 \] is negligible in comparison to the other terms, and suppress the subscript \( M \). We also minimize with respect to \( U \) for fixed but arbitrary \( X \) and \( Z \) by setting the quantity in the brackets involving \( U \) equal to zero. Since we are seeking a necessity condition for stability, we follow the same procedure as in Sec. II-C with the hydro-magnetic energy principle by setting \( Z = 0 \), which is not the minimizing
\[ \begin{align*}
\text{Z, but which simplifies the calculation. The resulting expression,} \\
\delta W_1 & \text{ is greater than the minimum } \delta W. \text{ This } \delta W_1 \text{ is given by} \\
\delta W_1 & = \frac{\pi}{2} \int d\psi \delta W_1 (\psi) \text{ where } \delta W_1 (\psi) \text{ is given by} \\
\delta W_1 (\psi) & = \int dX \left[ (1 - \sigma_\bot) \left( \frac{\partial X}{\partial \psi} \right)^2 - \frac{1}{r^2} \frac{\partial^2}{\partial \psi^2} + X^2 \left\{ - \frac{1}{rRB} \frac{\partial P}{\partial \psi} + \frac{\partial P}{\partial \psi} \right\} \right] \\
& \quad + \frac{X^2}{r^2} \left( \frac{1}{1 + 2K} \right) \left[ 7K + 5K^2 + 2K\sigma_\bot + 4\sigma_\bot^2 \right] \tag{72}
\end{align*} \]

Note that for small curvature, when the lines of force are bent only slightly, this expression reduces to Eq. (58), which is the equivalent equation in the Newcomb energy principle. This is not surprising, because, as the lines of force become straight, we tend toward a uniform situation in which heat flow is unimportant.

To derive a necessary condition for stability, we substitute
\[ \delta X = dX \frac{r^2 B^2}{\delta_\bot - \sigma_\bot} \]
and adopt the normalization \( \int X^2 dX = 1 \). Then by minimizing \( \rho = \delta W/\int X^2 dX \) we obtain the usual Euler equation
\[ \frac{\partial^2 X}{\partial X^2} + (\lambda - q(x)) X = 0 , \tag{73} \]
and the associated Sturm-Liouville equation

\[ \frac{d^2 X_n}{dx^2} + (\lambda_n - q(x)) X_n = 0, \quad (74) \]

where \( q(x) \) is the coefficient of \( X^2 \) in the SW of Eq. (72) after Eq. (72) is transformed to the \( x \) variable. Then from Eq. (5), we have \( \rho \geq \lambda_1 \) with \( \rho(X_1) = \lambda_1 \), where \( \lambda_1 \) is the smallest eigenvalue of Eq. (74). Then, to find an approximation to \( X_1 \), the corresponding eigenfunction, we substitute \( q_m \leq q(x) \) for all \( x \) into the above Sturm-Liouville equation. This furnishes the function

\[ X_n' = A \sin \left( \frac{n \pi x}{x_0} \right), \quad \lambda_n' = \frac{n^2 \pi^2}{x_0^2} + q_m. \]

From the expansion theorem, we obtain

\[ X_1 = \sum_{n=1}^{\infty} a_n X_n'. \]

Substituting the first term of this expansion into \( \rho \), we have, from Eq. (5), \( \rho(X_1') \geq \lambda_1 \).

This substitution yields

\[ \rho(X_1') = \frac{\pi^2}{x_0^2} + \frac{2}{x_0} \int q(x) \sin^2 \frac{n \pi x}{x_0} \, dx. \quad (75) \]

with the necessity condition for stability, \( 0 < SW_{\text{min}} < \rho \).

This stability criterion can be written in the form.
\[ 0 < \frac{\pi^2}{I_1} + 2 \int_0^L \frac{dl}{B} q(\ell) \sin^2 \left( \frac{\pi}{I_1} \int_0^L dl' \frac{r^2(\ell')B(\ell')}{[1 - \sigma(\ell')]} \right), \]

where \[ I_1 = \int_0^L \frac{dl}{1 - \sigma} \frac{r^2 B}{r}. \]

and \[ q(\ell) = -\frac{(P_1' + P_2')}{rRB} + \frac{1}{|rR|} \left( \frac{2 \left( 7K + 5K + 2K \sigma - \sigma^2 + 4\sigma \right)}{1 + 2K} \right). \]
F. SUMMARY OF SECTION II

We have investigated the stability of the axisymmetric plasma configuration. We have derived stability criteria from various energy principles, by the techniques of eigenvalue theory. These stability criteria are quite general. They are applicable to adiabatic plasmas that are axisymmetric, and which end in conducting plates. Hence they can be applied to mirror machines with arbitrary curvature and $\beta$.

Essentially, the sufficient conditions depend upon the minimum value of $(P_\|' + P_\perp')/R$, evaluated along a line of force. Here $R$ is the radius of curvature of the line of force. (The sign of $R$ is indicated in the sketch.) The prime denotes partial differentiation with respect to the stream function $\psi$. If there is any region along the line of force where $(P_\|' + P_\perp')/R$ is large in magnitude and negative in sign, the sufficient conditions will be very restrictive, even if at other regions of the line of force this function is large and positive. The sufficient conditions become so restrictive because they pick out the minimum values of the function. The necessary conditions depend upon the integral over a line of force of $(P_\|' + P_\perp')/R$ times a weighting factor, that is the square of an approximation to the minimizing $\hat{\xi}$. Their
validity depends upon how well we have approximated this minimizing $\xi$. In the following sections we calculate the stability criteria for certain specific configurations.
III. COMPUTATION OF THE STABILITY CRITERIA
OF SECTION II

A. COMPUTATION OF THE STABILITY CRITERIA
DERIVED FROM THE HYDRONAGNETIC ENERGY PRINCIPLE

In what follows we obtain specific equilibrium models and study
them by means of the stability criteria of Sec. II. We obtain hydro-
magnetic equilibrium models and apply to them the stability criteria of
Secs. II-B and II-C.

The equilibrium values of the quantity $j$, $B$, and $p$ in the
hydromagnetic model must satisfy Eqs. (7) through (9). Combining equations
(9) and (8), we write

$$0 = (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p$$

(77)

It is convenient to express Eq. (77) in terms of the stream function
$\psi(r,z)$, defined by $\psi = A_\theta r$, where $A_\theta$ is the $\theta$ component of the
vector potential $A$. Since $\beta$ is poloidal, $A$ has only the component
$A_\theta$. In terms of $\psi$, Eq. (77) becomes

$$r \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial \psi}{\partial r} \right] + \frac{\partial^2 \psi}{\partial z^2} = - r \frac{\partial p}{\partial \psi}.$$  

(78)
The solution of Eq. (78) is not completely determined since there are two unknown functions, \( \frac{dp}{d\psi} \), and \( \psi \). For simplicity, we choose

\[
p = C(\overline{\psi} - \psi),
\]

(79)

where \( C \) and \( \overline{\psi} \) are constants to be determined from the physical parameters. With this substitution, Eq. (8.2) becomes

\[
\frac{r}{\partial z} \frac{\partial \Psi}{\partial r} + \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2} = C r^2.
\]

(80)

The solution of Eq. (8) can be written as \( \psi = \Psi_I + \Psi_P \), where \( \Psi_I \) is a solution of the homogeneous equation and \( \Psi_P \) is a particular solution. A solution \( \Psi_I \) which is symmetric about the midplane defined by \( z = 0 \), is

\[
\Psi_I = \sum_{n=1}^{\infty} b_n r \cos \frac{n\pi z}{a} I_1 \left( \frac{n\pi r}{a} \right) + \frac{B r^2}{2}
\]

(81)

and a solution \( \Psi_P \) is

\[
\Psi_P = \frac{C r^4}{8},
\]

(82)
where $B_0$ and $b_n$ are constants to be determined by the physical parameters of the system and $a$ is the half length of the system. For simplicity, we choose the first term in the sum of Eq. (81) and adopt a solution for $\psi$ suggested by Harold Grad:\footnote{19}$

$$\psi = - \frac{akr}{\pi} \left[ \cos \frac{\pi z}{a} \right]_1 I_1 \frac{\pi r^2}{a} + B_0 \frac{r^2}{2} + \frac{\lambda r^4}{4},$$

where $C = 2\lambda B_0$, and $b_1 = - \frac{akr}{\pi}$. We then have

$$B_z = \frac{1}{r} \frac{\partial \psi}{\partial r} = B_0 \left( 1 + \lambda r^2 \right) - k \left[ \cos \frac{\pi z}{a} \right]_0 \frac{\pi r^2}{a},$$

and

$$B_r = - \frac{1}{r} \frac{\partial \psi}{\partial z} = - k \left[ \sin \frac{\pi z}{a} \right]_1 \frac{\pi r^2}{a}.$$ 

We define the physical parameters as $\beta = \frac{2p(z = r = 0)}{B^2(z = r = 0)}$, mirror ratio $M = B(z = \bar{r} a)B(z = 0)$ and aspect ration $A = \bar{r} / a (z = 0)$, where $\bar{r}$ is $r$ evaluated at the boundary of the plasma region. For
convenience, we substitute \( \alpha = k/B^0 \) and \( N = \psi/B^0 \) into Eqs. (82) through (85) and obtain

\[
N = -\frac{a\alpha}{\pi} \cos \frac{\pi x}{a} r I_1 \left( \frac{\pi r}{a} \right) + \frac{r^2}{2} + \frac{\lambda r^4}{4} ,
\]  

(86)

\[
\frac{B_z}{B^0} = 1 + \lambda r^2 - \alpha \cos \frac{\pi x}{a} r I_1 \left( \frac{\pi r}{a} \right) ,
\]  

(87)

and

\[
\frac{E_r}{B^0} = -\alpha \sin \frac{\pi x}{a} r I_1 \left( \frac{\pi r}{a} \right) .
\]  

(88)

From Eq. (77) we have \( \nabla p = r B \frac{\partial}{\partial \psi} \hat{\psi} = -\hat{B} \times (\nabla \times \hat{B}) \)

\[
= -\frac{\nabla B^2}{2} + \hat{B} \cdot \nabla \hat{B} = \hat{B} \left( -r B \frac{\partial}{\partial \psi} \left( \frac{B^2}{2} + \frac{B^2}{R} \right) \right) ,
\]

or we can write

\[
\frac{\partial}{\partial \psi} \left( \frac{p + B^2}{2} \right) = \frac{B^2}{R} .
\]

Integrating this equation across the boundary \( \psi = \bar{\psi} \), we discover that \( (B^2/2 + p) \) must be continuous across the boundary. However, from Eq. (79), we have \( p = C(\bar{\psi} - \psi) \); because \( p \) is zero at the boundary, we see that \( B^2 \) must be continuous across the boundary, which implies that \( \psi \) is continuous, since \( B \) is related to the gradient of \( \psi \). We therefore must choose a medium outside the plasma in which solutions to \( \psi \) can be found that will match the Solution (83) at \( \psi = \bar{\psi} \). We can satisfy the boundary conditions by surrounding the plasma with a perfect conductor with its surface everywhere tangent.
to the plasma and therefore tangent to the stream line \( \psi = \bar{\psi} \).

We place on this conductor a surface current \( \mathbf{K} \) such that the magnetic field inside the conductor is zero. From \( \nabla \times \mathbf{B} = \mathbf{J} \)

we have \( \mathbf{n} \times [[\mathbf{B}]] = \mathbf{K} \) which implies that \( + |\mathbf{K}| = \mathbf{B} (\psi = \bar{\psi}) \),

where \( \mathbf{K} = - \mathbf{\varepsilon}_3 |\mathbf{K}| \). (The double bracket \([[]]\) means the change of the bracketed quantity across the interface.) Since we know that \( \mathbf{B} = 0 \) in the conductor, the flux \( \psi \) is everywhere constant.

To summarize, our model consists of a plasma surrounded by a perfect conductor in which the magnetic field vanishes. At the plasma-conductor boundary there is a surface current on the conductor.
1. EVALUATION OF HYDROMAGNETIC SUFFICIENCY CONDITION FOR $A \ll 1$

Here, we evaluate Eq. (24) using Eqs. (86) and (87). We consider terms of order $A$, but neglect terms of order $A^2$. Since the curvature $1/R$ is of order $A$, this procedure is valid when the lines of force are only slightly bent; that is, when the curvature is small but not infinitesimal.

To first order in $r/a \ll A$, Eq. (86) for $N$ becomes:

$$N = -\frac{2\pi}{r^2} \cos \frac{\pi z}{a} \left[ \frac{r^2}{2} + \frac{\lambda r^4}{4} \right] = \frac{r^2}{2} \left[ 1 - \alpha \cos \frac{\pi z}{a} \right] + \frac{\lambda r^4}{4}$$

In order to solve for $r = r(z)$ on a particular line of force, we utilize the constancy of $N$ on a line of force. Solving Eq. (89) for $r = r(N, z)$ by holding $N$ fixed we obtain

$$\lambda r^2 = \left[ 1 - \alpha \cos \frac{\pi z}{a} \right] \left[ 1 + \frac{4 \lambda N}{\lambda r^2} \right]^{1/2}.$$  

(90)

Now $N$ is some fraction of $\bar{N} = N$ at the boundary of the plasma. Hence, we set $N = \epsilon \bar{N}$, where $\epsilon$ is a parameter characterizing the location of the line of force. By substituting this into the above expression for $\lambda r^2$, we obtain
\[ \lambda r^2 = -(1 - \alpha \cos \frac{\pi z}{a}) + [(1 - \alpha \cos \frac{\pi z}{a})^2 + 4 \epsilon \lambda \bar{N}]^{1/2} \]

(91)

In a similar fashion we have

\[ B_z = B^0[(1 - \alpha \cos \frac{\pi z}{a}) + \lambda r^2] = B^0[(1 - \alpha \cos \frac{\pi z}{a})^2 + 4 \epsilon \lambda \bar{N}]^{1/2} \]

(92)

and

\[ B_r = -B^0 \frac{\alpha \pi \sin \frac{\pi z}{a}}{2a} \sin \frac{\pi z}{a} \]

Then we have

\[ \beta = 2 \rho (z = r = 0) /B^0 (z = r = 0) = 4 \lambda B^0 \frac{\pi}{B^0} \bar{\Omega}^2 (1 - \alpha)^2 \]

\[ = 4 \lambda \bar{N} / (1 - \alpha)^2 \]

which implies
\[ 4 \lambda \bar{N} = \beta (1 - \alpha)^2 . \]  

(93)

With this relationship, Eq. (92) becomes

\[ B_z = B^0[(1 - \alpha \cos \frac{\pi z}{a})^2 + \varepsilon \beta (1 - \alpha)^2]^{1/2}. \]  

(94)

To determine \( a \), we use

\[ M = B(z = \pm a)B(z = 0) = [(1 + \alpha)^2 + \varepsilon \beta (1 - \alpha)^2]^{1/2} / (1 - \alpha)(1 + \epsilon \beta)^{1/2}. \]  

(95)

Solving for \( a \) we obtain

\[ \alpha = \frac{[M^2(1 + \epsilon \beta) - \epsilon \beta]^{1/2} - 1}{(M^2 - 1)(1 + \epsilon \beta)}. \]  

(96)

For \( \epsilon \) or \( \beta = 0 \), \( a \) reduces to \( a = (M - 1)/(M + 1) \).

We are now in a position to determine the functions involved in Eq. (24). First we calculate the curvature \( 1/R \) given by

\[ \frac{d}{dz} \left( \frac{dr}{dz} \right) / \left( 1 + \left( \frac{dr}{dz} \right)^2 \right)^{3/2}. \]  

The sign of \( R \) is indicated in the figure.

We get \( \frac{dr}{dz} |_\psi = \frac{B_r}{B_z} \).
hence \( 1/R = \frac{\frac{d}{dz} B}{d(B^2)} \left( \frac{1 + \frac{B_r^2}{B_z^2}}{B_z^{3/2}} \right)^{3/2} = \frac{B_z^{-3/2} \frac{d}{dz} B}{d(B^2)} \).

However, in the approximation \( A \ll 1 \), we have

\[
B_r^2 = \frac{B_0^2 \alpha \pi^2}{4} \sin^2 \frac{\pi z}{a}
\]

\( = \mathcal{O}(A^2) \). Thus we see that \( B \approx B_z \), which implies that \( 1/R = \frac{\frac{d}{dz} B}{d(B^2)} \); by substituting Eq. (94) for \( B_z \) and Eq. (92) for \( B_r \) we obtain

\[
\frac{1}{R} = \frac{\alpha \pi^2 B_0^2}{2a B^z} \left[ - \cos \frac{\pi z}{a} + \frac{\alpha}{2} \frac{B_0^2}{B_z^2} \sin \frac{\pi z}{a} + \frac{\alpha B_0^2}{B_z^2} \sin \frac{\pi z}{a} \left[ 1 - \alpha \cos \frac{\pi z}{a} \right] \right].
\]

(97)

Thus we see that \( 1/R \approx \frac{r}{a} = \mathcal{O}(A) \). It is also apparent that the minimum value of \( 1/R \) is at the midplane where \( z = 0 \). We also note from Eqs. (91), (93), (94) and (97) that \( \frac{\frac{d}{dz} \frac{1}{R}}{r^3 B^2} = \sin \frac{\pi z}{a} f(z, \alpha, \beta) = 0 \) for \( z = 0, \pm a \). This implies that

\[
p' \frac{p}{r^2 B^2} = -2 \frac{p'}{r B} \frac{r^2 B^2}{r^3 B^2} = 2 \frac{|p'|}{r^3 B^3}
\]

has its minimum value at the midplane defined by \( z = 0 \). Now \( B(z = 0) \) is from Eq. (94) equal to
\[ B(z = 0) = B^0[(1 - \alpha)^2 + \epsilon \beta(1 - \alpha)^2]^{1/2} = (1 - \alpha)(1 + \epsilon \beta)^{1/2}. \] (98)

Also, from Eq. (97), \(1/rR\) at the midplane becomes

\[
\frac{\alpha R^2}{2a^2} \frac{B^0}{B_z} \left[ -1 \right] = \frac{1}{rR} \text{ at midplane.} \tag{99}
\]

From Eqs. (91) and (93) we have

\[ r^2(z = 0) = (1 - \alpha)[(1 + \epsilon \beta)^{1/2} - 1]/\lambda. \tag{100} \]

From Eqs. (80) and (81) we get \(p' = -2 \lambda B^0\). Combining the above, we then obtain

\[
\begin{align*}
\text{Min}\{p' D/r^2 B^2\} &= -\alpha R^2 \left( -2 \lambda B^0 \right)/( -2 ) / 2 a^2 B_z r^2 \\
&= -2 \alpha \lambda^2 \pi^2 / B^0 a^2 (1 - \alpha)^{1/2} (1 + \epsilon \beta)^2 \\
&\quad \times [(1 + \epsilon \beta)^{1/2} - 1] (1 - \alpha) \\
&= -2 \alpha \lambda^2 \pi^2 / B^0 a^2 (1 - \alpha)^5 (1 + \epsilon \beta)^2 [(1 + \epsilon \beta)^{1/2} - 1] \\
&\quad \times (101)
\end{align*}
\]

We also have

\[
\int_0^L dl \ r^2 B \approx \int_{-a}^{+a} dz \ r^2 (z, N)B_z = \int_{-1}^{+1} dy \ G(y) \frac{aB^0}{\lambda}, \tag{102}
\]
where \( G(y) = \left[ (1 - \alpha \cos \pi y)^2 + \epsilon \beta (1 - \alpha)^2 \right]^{1/2} \left[ (1 - \alpha \cos \pi y) + \epsilon \beta (1 - \alpha)^2 \right]^{1/2} \).

Therefore Condition (24) becomes

\[
0 < \frac{\pi^2 \lambda^2}{a^2 B^0 (\int_{-1}^{+1} G(y) \, dy)^2} - \frac{2 \alpha \lambda^2 \pi^2}{a^2 B^0 (1 - \alpha)^5 (1 + \epsilon \beta)^2 [(1 + \epsilon \beta)^{1/2} - 1]}
\]

or

\[
0 < \frac{1}{(\int_{-1}^{+1} G(y) \, dy)^2} - \frac{2\alpha}{(1 - \alpha)^5 (1 + \epsilon \beta)^2 [(1 + \epsilon \beta)^{1/2} - 1]} = F(\alpha, \beta) .
\]

Expression (103) is a condition on \( \beta \), and \( \alpha = \alpha(M) \). From physical considerations we have \( \beta \geq 0 \) and \( \alpha > 0 \). For certain allowable values of \( \alpha \) and \( \beta \), we have \( F(\alpha, \beta) > 0 \), which implies stability.

For \( \beta = 0 \), we have \( F(\alpha, \beta) > 0 \) for all allowable values of \( \alpha \).

As \( \beta \) increases from zero, by holding \( \alpha \) fixed, \( F(\alpha, \beta) \) decreases. For a certain value of \( \beta \), which we have defined as \( \beta_c \), \( F(\alpha, \beta_c) = 0 \).

For \( \beta > \beta_c \), we have \( F(\alpha, \beta) < 0 \) and the sufficiency criterion is violated. Now, actually, there is a third parameter, \( \epsilon \), which is always a multiplier of \( \beta \). Therefore, instead of calculating \( \beta_c \) such
that $F(\alpha, \beta_c) = 0$, we calculate $\epsilon \beta_c$ such that $F(\alpha, \epsilon \beta_c) = 0$.

Thus for each $\alpha$ there is a continuum of values of $\beta_c$ corresponding to the range of values that $\epsilon$ can assume. Since $N = \epsilon \bar{N}$, we have $\epsilon_{\text{max}} = 1$. Therefore, the minimum value of $\beta_c$ for each $\alpha$ occurs for $\epsilon = 1$, which implies $N = \bar{N}$. The least stable line of force is at the boundary of the plasma, where $N = \bar{N}$. The equation $F[\alpha(M), \beta_c] = 0$ has been solved numerically to determine $\beta_c$ for various values of $M$. (The results are depicted in Table I and Figure 1.)
TABLE I.

Evaluation of the hydromagnetic sufficiency stability criterion.

A. Maximum $\beta = \beta_c$ given as a function of $M$ for real fields, $A \ll 1$

<table>
<thead>
<tr>
<th>$\beta_c$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.781</td>
<td>2</td>
</tr>
<tr>
<td>0.321</td>
<td>3</td>
</tr>
<tr>
<td>0.199</td>
<td>4</td>
</tr>
<tr>
<td>0.143</td>
<td>5</td>
</tr>
<tr>
<td>0.06</td>
<td>10</td>
</tr>
</tbody>
</table>

B. Maximum $\beta = \beta_c$ given as a function of $M$ and $A$ for vacuum fields, arbitrary $A$

<table>
<thead>
<tr>
<th>$\beta_c$</th>
<th>$M$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.580</td>
<td>2</td>
<td>0.1</td>
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<tr>
<td>0.46</td>
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<td>0.298</td>
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<td>0.5</td>
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<tr>
<td>0.277</td>
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<tr>
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<td>0.25</td>
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<tr>
<td>0.002</td>
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<td>0.5</td>
</tr>
</tbody>
</table>
2. EVALUATION OF SUFFICIENCY CONDITION
FOR STABILITY FROM THE HYDROMAGNETIC ENERGY PRINCIPLE
FOR ARBITRARY $A$

The starting point for this calculation is again obtained from Eqs. (85) through (87). We are interested in examining the influence of a large aspect ratio $A$ upon the stability of the mirror system. In accordance with this idea, for simplicity we choose Eqs. (85) through (87) in their exact form, but with $\lambda = 0$. This means that we are using vacuum fields in the calculation instead of real fields. However, the error made by this approximation should not be large, particularly for large $M$, in view of the small values for $\beta$ calculated in the preceding section. This error will be of order $\beta^2$ in calculating $\min \{p'D/r^2B^2\}$, since the expression itself is of order $\beta$, and will be small in calculating $\int_0^L \tilde{\psi} d\ell r^2 B^2$. In fact, returning to the case for $A \ll 1$, we can write for $\beta \ll 1$,

$$B_z/B^0 \approx \left[ (1 - \alpha \cos \frac{\pi z}{a})^2 + \beta(1 - \alpha)^2 \right]^{1/2} \approx (1 - \alpha \cos \frac{\pi z}{a})$$

$$+ \frac{\beta}{2} \frac{(1 - \alpha)^2}{(1 - \alpha \cos \frac{\pi z}{a})} ; \quad r^2 \approx \frac{1}{\lambda} \left[ -(1 - \alpha \cos \frac{\pi z}{a})^2 \right]$$

$$+ \left[ (1 - \alpha \cos \frac{\pi z}{a})^2 + \beta(1 - \alpha)^2 \right]^{1/2}$$

$$\approx 2N \left[ 1 - \frac{\beta}{16} \frac{(1 - \alpha)^2}{(1 - \alpha \cos \pi z)} \right] / (1 - \alpha \cos \pi z) ,$$
hence

\[
\frac{r^2 B_z}{B^0} \approx 2N \left(1 + \frac{7}{16} \frac{\beta (1 - \alpha)^2}{(1 - \alpha \cos \frac{\pi z}{a})^2}\right).
\]

Defining \( I = \int_0^L d\ell \, r^2 B \), we have then, for \( A \ll 1, \beta \ll 1 \);

\[
I = \int_{-\alpha}^{\alpha} dz \, r^2 B_z \approx 2N B^0 \left[2a \, \frac{7}{16} \beta (1 - \alpha)^2 \int_{-\alpha}^{\alpha} dz / (1 - \alpha \cos \frac{\pi z}{a})^2\right]
\]

\[
= 4a \, N \, B^0 \left[1 + \frac{7}{16} \frac{\beta (1 - \alpha)^{1/2}}{(1 + \alpha)^{3/2}}\right].
\]

Setting \( \alpha = |M - 1|/M + 1 \), from Eq. (91) for \( \beta = 0 \), we obtain

\[
I = I^0 (1 + I') = I^0 (1 + \frac{7}{32} \beta \frac{M + 1}{M^{3/2}}).
\] (104)

Thus, the first order (in \( \beta \)) correction to \( I \) is a small quantity.

For example, let us set \( M = 2 \), then from Fig. 1, \( \beta < 3/4 \), and

\( I = I^0 (1.16) \). This gives a maximum error of 16\% in \( B \) and the error decreases rapidly as \( M \) increases. For example, for \( M = 4 \), the error is 3\%. We are on relatively safe ground in neglecting this
correction in order to investigate the effects of a finite aspect ratio.

In this approximation we have for \( N \)

\[
N = - \frac{\alpha a}{\pi} r I_1 \left( \frac{\pi r}{a} \right) \cos \frac{\pi z}{a} + \frac{r^2}{2} = a^2 \left[ - \frac{\alpha a}{\pi} I_1 \left( \pi A \right) + \frac{A^2}{2} \right],
\]

(105)

where we have set \( r(z = 0) = \bar{r} \), in accordance with examining the stability of the outermost line of force. Similarly, we have

\[
\frac{B_z}{B_0} = 1 - \alpha I_0 \left( \frac{\pi r}{a} \right) \cos \frac{\pi z}{a}
\]

and

\[
\frac{B_z}{B_0} = - \alpha \sin \frac{\pi z}{a} I_1 \left( \frac{\pi r}{a} \right).
\]

(106)

Now the mirror ratio \( M = B(z = \pm a)/B(z = 0) \)

\[
= \frac{[1 + \alpha I_0 \left( \frac{\pi r}{a} \right) (z = 0)]}{1 - \alpha I_0 (\pi A)}.
\]

Since the lines of force converge at the mirrors, we make the approximation
\[ r(z = \frac{1}{2} a)/a \ll 1 \] and obtain for \( M \)

\[
M = \frac{(1 + \alpha)/(1 - \alpha I_0(\pi a))}{(1 + \alpha)/(1 - \alpha I_0(\pi a) + 1)} \tag{107}
\]

Solving for \( \alpha \) we obtain

\[
\alpha = \frac{(M - 1)/(M I_0(\pi a) + 1)}{\pi a} \tag{108}
\]

From \( p = C(\Psi - \psi) \) and \( \beta = 2p(z = r = 0)/B^2(z = r = 0) \)

\[
= 2C \frac{\Psi}{B^2(1 - \alpha)^2} = 2 C \frac{N/B^0(1 - \alpha)^2}{2} \; \text{we find} \; C = \beta B^0(1 - \alpha)^2/2N .
\]

Now the integral \( \int_0^L \int dz r^2 B_z = \int_{z=-a}^{z=a} \int dz r^2 B_z + \int_{z=-a}^{z=a} drr B_r . \)

Using \( \frac{dr}{dz} = \frac{r}{B_z} \),

\[
\int dz r^2 B_z = \int_{z=-a}^a dr r^2 B^0 \frac{(1 - \alpha \cos \frac{\pi z}{a} I_0(\frac{\pi z}{a}))}{-\alpha \sin \frac{\pi z}{a} I_1(\frac{\pi z}{a})} \int_{z=-a}^a dx \left( (1 - \alpha I_0(\pi x) \cos \frac{\pi z}{a})^2 / \alpha \sin \frac{\pi z}{a} I_1(\pi x) \right),
\]

where \( x = r/a \). Now, from Eq. (98), \( \cos \frac{\pi z}{a} = \left( \frac{x^2}{2} - \frac{N}{2a^2} \right) / \frac{\pi x}{a} I_1(\pi x) \)

and \( \sin \frac{\pi z}{a} = -(1 - \cos^2 \frac{\pi z}{a})^{1/2} \) over the range which we are considering.

Thus we have

\[
\int_{z=-a}^a dz r^2 B_z = 2a^3 \int_{z=-a}^0 dx x^2 f^2(x)/\alpha g(x), \tag{109}
\]

where
\[ f(x) = 1 - \pi I_0 \left( \frac{x^2}{2} - \frac{N}{a^2} \right) / x I_1 (\pi x), \text{ and} \]

\[ g(x) / I_1 (\pi x) = -\sin \frac{\pi z}{a} = \left(1 - \cos^2 \frac{\pi z}{a}\right)^{1/2} \]

\[ = \left(1 - \frac{\pi^2}{a^2} \left(\frac{x^2}{2} - \frac{N}{a^2}\right) / x^2 I_1^2 (\pi x)\right)^{1/2}. \]

In the computation the limits on the integral are set up to be the zeros of \( g(x) \), or where \( \sin \pi z / a = 0 \). Similarly, we set

\[ 2 \int dr r^2 B_r = 2a^3 \int dx x^2 B^0 \alpha g(x). \]

Finally we have

\[ \int_0^L dl r^2 B = 2 \int dr r^2 B_r + \int dr r^2 \frac{B_z}{B_r} \]

\[ = 2a^3 B^0 \int dx x^2 \left( \frac{r^2}{\alpha g} + \alpha g \right). \]

Now

\[ p'B^2 / r^2 B^2 = -2p' / r^3 B R = \beta B^0 (1 - \alpha)^2 / NR r^3 B^3. \]

From the preceding section, we have

\[ 1/R = \frac{B_z^3}{B^3} \frac{d}{dz} \frac{R_z}{B_z}. \]
At the midplane \( B_r = 0 \) and \( B = B_z \), hence we get

\[
\frac{1}{B_0} = \frac{1}{B_0} \frac{dB}{dz} = -\frac{\alpha \pi I_1(\pi A)}{4} \frac{1}{1 - \alpha I_0(\pi A)},
\]

where the subscript 0 indicates evaluation of the function at the midplane. From Eq. (99) \( B_0 = B_0^0(1 - \alpha I_0(\pi A)) \); therefore, we have

\[
p^2 \frac{D}{r^2 B^2} = -\beta(1 - \alpha)^2 \frac{\alpha \pi I_1(\pi A)}{N a} r_0^3 B_0^0(1 - \alpha I_0(\pi A))^2
\]

\[
= \frac{1}{B_0^2} a^6 \left[ \frac{\beta(1 - \alpha)^2 \alpha \pi I_1(\pi A)}{N^2} \right] \frac{1}{A^3 (1 - \alpha I_0(\pi A))^4}
\]

(110)

The problem therefore consists of examining the inequality

\[
0 < \frac{\pi^2}{12} \beta(1 - \alpha)^2 \frac{\alpha \pi I_1(\pi A)}{a^2} \frac{N}{A^3} (1 - \alpha I_0(\pi A))^4
\]

(111)

The maximum \( \beta \) for which this inequality holds is now a function of \( M \) and \( A \). These results were calculated by numerical methods and are tabu-
lated in Table I, and plotted in Fig. 1. These results agree well with expectations because for fixed $M$, $\beta$ decreases as the aspect ratio is increased. Thus we have once again the result that as the lines of force get more bent and the curvature increases, the system becomes less stable. It should be emphasized that the $M$ used in this section is a function of $\psi$, and is different for each line of force. Since $M$ increases as $\psi$ increases, the average mirror ratio will be less than $M$. 
3. EVALUATION OF THE NECESSITY CONDITION FOR STABILITY

DERIVED FROM THE HYDROMAGNETIC ENERGY PRINCIPLE

In this section we evaluate the stability criterion of Eq. (32) using the same equilibrium configuration as in part A of this section. That is, we are keeping only terms linear in A for simplicity, and Eqs. (89) through (93) are applicable. Since it has been shown that 1/R is of order A, and since D is of the order of 1/R, the inequality of Eq. (32) simplifies to

$$0 < \frac{\pi^2}{L} \int_0^L \frac{d\ell}{r^2 B} + 2 \int_0^L \frac{d\ell}{B} (p'D) \sin \lambda \int_0^L \frac{d\ell'}{r^2 B}$$

Equation (112) is also applicable without neglecting terms in $D^2$ if we consider the line of force at the boundary of the plasma, since the pressure goes to zero there.

The integral $\int_0^L \frac{d\ell}{r^2 B}$ is the same integral as in the sufficient condition and is given in this approximation by Eq. (102),

$$\int_0^L \frac{d\ell}{r^2 B} = \int_{-\lambda}^0 \frac{aB_0}{\lambda} dy G(y), \text{ where } G(y) \text{ is given by Eq. (103)}.$$  

However, we now have another integral to deal with. The plasma pressure is constant along a line of force and therefore $p$ can be taken
out of the integration. We substitute Eq. (97) for $\frac{1}{R}$ and $B_0^0[(1 - \alpha \cos \frac{\pi z}{a})^2 + \epsilon \beta (1 - \alpha)^2]^{1/2}$ for $B_z$ to obtain

$$D/B = -2/rR \frac{B_z^2}{a} = -2 \alpha \frac{\pi^2 B_0}{a^2 B_z^3} \left[ -\cos \frac{\pi z}{a} + \frac{\alpha B_0}{2 B_z} \sin^2 \frac{\pi z}{a} ight]$$

$$+ \frac{\alpha B_0^2}{B_z^2} \sin^2 \frac{\pi z}{a} (1 - \alpha \cos \frac{\pi z}{a}).$$

Using the relation $p' = -2 \lambda B_0^0$, we then write the inequality (112) as

$$0 < \frac{\lambda \pi^2}{a B_0^0} \int_{-1}^1 dy G(y) + 4 \frac{\lambda \pi^2 \alpha}{a B_0^0} \int_{-1}^1 dy M(y), \quad (113)$$

or

$$0 < \frac{1}{\int_{-1}^1 dy G(y)} + 4 \alpha \frac{1}{\int_{-1}^1 dy M(y)} \int_{-1}^1 dy M(y)$$

where

$$M(y) = P(\hat{y})[(1 - \alpha \cos \pi y)^2 + \epsilon \beta (1 - \alpha)^2]^{-1}$$
\[
\sin^2(\pi \int_{-1}^{1} G(y')/ \int_{-1}^{1} G(y) \ dy) \] 

and 

\[
P(y) = \left[ (1 - \alpha \cos \pi y)^2 + \epsilon \beta (1 - \alpha)^2 \right]^{-1/2} 
\]

\[
= \left[ - \cos \pi y + \frac{\alpha}{2} \sin^2 \pi y((1 - \alpha \cos \pi y)^2 + \epsilon \beta (1 - \alpha)^2)^{-1/2} 
+ \alpha \sin^2 \pi y(1 - \alpha \cos \pi y)((1 - \alpha \cos \pi y)^2 + \epsilon \beta (1 - \alpha)^2)^{-1} 
+ \epsilon \beta (1 - \alpha)^2 \right]. 
\]

We have also changed variables by noting that \( L = 2a \) and \( \ell = z - a \). As in the preceding treatment, \( \beta \) occurs multiplied by \( \epsilon \) and therefore the minimum \( \beta \) is developed at the boundary where \( \epsilon = 1 \). We are thus further justified in neglecting the terms in \( D^2 \) since they are positive definite and reduce to zero at the boundary.

The inequality (113) is evaluated by numerical means. The results are tabulated in Table II and plotted in Fig. 3. It is found that Inequality (113) is satisfied for all \( \beta \) for \( M = 2 \), and only for large \( M \) does \( \beta_c \) decrease to a reasonable value. The reason for this anomalous result lies in the hydromagnetic approximation which requires that
the pressure be constant along a line of force. This means that significant contributions to the integral are furnished in the regions of positive curvature, which in a real plasma contain only a small amount of material. In the following sections, where the plasma is assumed to vanish at the mirrors, more reasonable values for $\beta_c$ are determined.
TABLE II

Evaluation of the hydromagnetic necessary stability criterion.

<table>
<thead>
<tr>
<th>β_c</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>all values</td>
<td>2</td>
</tr>
<tr>
<td>all values</td>
<td>3</td>
</tr>
<tr>
<td>all values</td>
<td>4</td>
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<tr>
<td>1.47</td>
<td>5</td>
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<tr>
<td>0.723</td>
<td>10</td>
</tr>
</tbody>
</table>
B. COMPUTATION OF THE STABILITY CRITERION DERIVED FROM THE CHEW-GOLDBERGER-LOW AND NEWCOMB ENERGY PRINCIPLES

In this section we obtain an equilibrium plasma model which we study by means of the stability criteria of Secs. II-D and II-E.

Equations (33) through (35) must be satisfied by this equilibrium state. We combine Eqs. (33) and (34) and obtain

\[(\nabla \times B) \times B = \nabla \cdot P\]

Since the magnetic field is poloidal, we utilize a vector potential \( A = \zeta \theta A_\theta \), where \( B = \nabla \times A \). This \( B \) will satisfy Eq. (35), \( \nabla \cdot B = 0 \). Now we have three unknowns, \( A_\theta \), \( P_\perp \) and \( P_\parallel \). However, we have only two equations relating these quantities since the vector \( \nabla \cdot P \) in Eq. (114) has only two components. Hence we must make an additional assumption to mathematically determine \( P_\perp \), \( P_\parallel \) and \( A_\theta \).

This assumption is a prescription on the form of \( P_\perp \) on the midplane in such a way that the pressure goes to zero on the boundary. As in the hydromagnetic equilibrium, the ends of the region are considered to be at the mirrors.

Using Eq. (114) and setting \( A_\perp = \psi / r \), where \( \psi \) is the stream function characterizing the lines of force, we obtain the two pressure balance Eqs. (44) and (46) in the form

\[-\frac{1}{r^2} \left[ r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial \psi^2} \right] = \frac{P_\perp}{rRB} + \frac{\partial P_\parallel}{\partial \psi} \]  

(115)
\begin{equation}
0 = P - \frac{\partial}{\partial \ell} \frac{1}{B} \frac{\partial P}{\partial \ell} + \frac{1}{B} \frac{\partial P}{\partial \ell}
\end{equation}

where \( d\ell \) is the unit of length along the line of force. Now Eq. (115) is a second-order nonlinear partial differential equation, since \( rB = |\nabla \psi| \). Therefore, for simplicity, we employ an expansion in \( \beta \) by setting \( \psi = \psi^0 + \psi^1 \), where \( \psi^0 \) satisfies the homogeneous equation, and \( \psi^1 \) is order \( \beta \). In this calculation, however, only \( \psi^0 \) is utilized.

It is shown below that this is a reasonable approximation. The solution \( \psi^0 \) is given by Eq. (66),

\[
\psi^0 = \sum_{n=1}^{\infty} b_n r (\cos \frac{n \pi z}{a}) I_1 \left( -\frac{n \pi z}{a} \right) + B^0 \frac{r^2}{2}.
\]

As before, we keep only the first term in summation and obtain

\[
\psi^0 = B^0 \left[ \frac{r^2}{2} - \frac{a \alpha}{\pi} \cos \frac{\pi z}{a} r I_1 \left( \frac{\pi r}{a} \right) \right],
\]

(117)

where the constants \( B^0 \) and \( a \) are determined by the physical parameters of the system. In the limit that we consider, \( \frac{r}{a} < A \ll 1 \), we obtain
\[ \psi^0 = B^0 \frac{\pi^2}{2} (1 - \alpha \cos \frac{\pi z}{a}) \]  

(118)

with

\[ \frac{B^0_z}{B^0} = 1 - \alpha \cos \frac{\pi z}{a} \quad \text{and} \quad \frac{B^0_r}{B^0} = \frac{\alpha \pi}{2a} \sin \frac{\pi z}{a}. \]  

(119)

Henceforth, we omit the superscript \( o \) on the \( B_z \) and \( B_r \). We next select a distribution function \( f \), in order to evaluate the various moments involved in the stability criteria. The only requirement on \( f \) is that it obey the guiding-center approximation; i.e. \( f = f(\nu, \epsilon, \psi) \) as discussed in Appendix I. For analytic simplicity, we choose

\[ f(\nu, \epsilon, \psi) = g(\epsilon, \psi) \left( \frac{\nu B_m}{\epsilon - 1} \right)^n, \quad \text{for } \epsilon/B_m < \nu < \epsilon/B, \]  

(120)

otherwise it equals zero. Here we have \( n \geq 0 \) and \( B_m = B_{\text{max}} \).

This distribution function will vanish for those values of \( \epsilon \) and \( \nu \) which permit the particles to escape from the mirrors. We thus exclude those particles in the loss cone, defined by \( \theta < \theta_c \), where \( \cos^2 \theta_c = B/B_m \) and \( \theta \) is the pitch angle of the velocity vector of the particle defined by \( \cos \theta = \nu \cdot B / |\nu| B \). We assume that \( T_e \ll T_i \), where \( T_e \)
is the electron temperature and \( T_1 \) is the ion temperature. We also assume that there is no electrostatic potential in the equilibrium state. In Appendix I, it is shown that for \( T_e \ll T_1 \) there is negligible electrostatic potential in the perturbed state. Also the quantity \( \sigma = S(e v B/q^2) \) is negligibly small. As a consequence, the stability criteria we consider are the same as those we would have derived from the Kruskal-Oberman energy principle, which neglects charge separation effects. (For further discussion of these points, see Appendix A.)

The moments of the distribution function that we will require are \( P_\perp \) and \( P_\parallel \), defined by Eq. (39). Because of the assumption \( T_e \ll T_1 \), we need consider only the ionic contribution to these moments. Now, in cylindrical coordinates, the volume element in velocity space is

\[
\mathrm{d}^3v = 2\pi v_\perp \mathrm{d}v_\perp \mathrm{d}v_\parallel = 4\pi v_\perp \mathrm{d}v_\perp \left|\mathrm{d}v_\parallel\right|.
\]

Using \( v_\perp^2/2B = v \) and \( \epsilon = \frac{v_\parallel^2}{2} + v B \), which implies \( \frac{\partial \epsilon}{\partial |v_\parallel|} = \frac{\partial \epsilon}{\partial |v_\parallel|} = |v| \), we obtain

\[
\frac{\hbar \pi B \mathrm{d}v \mathrm{d}\epsilon}{|2(\epsilon - vB)|^{1/2}}.
\]

We therefore have

\[
P_\parallel = M \int v_\parallel^2 f \, \mathrm{d}^3v = 4\pi M B \int f \, \mathrm{d}v \, \mathrm{d}\epsilon \sqrt{2(\epsilon - vB)}
\]

\[
= 4\pi M B \int \mathrm{d}v \mathrm{d}\epsilon g(\psi, \epsilon) \left( \frac{v B}{\epsilon} - 1 \right)^n \sqrt{2(\epsilon - vB)},
\]

(121)
where the limits on $v$ are from $\epsilon/B_m$ to $\epsilon/B$ and $\epsilon$ varies from 0 to $\infty$. Setting $x = v B_m / B$, we obtain

$$P'' = 4\pi \sqrt{2} M \frac{B}{B_m} \int_0^\infty dc \frac{c^{3/2} g(\psi, c)}{1} \int \left( x - 1 \right)^n \left( 1 - \frac{B}{B_m} x \right)^{1/2} dx;$$

setting

$$y = \left( 1 - x B / B \right) / \left( 1 - B / B_m \right),$$

we obtain

$$P'' = 4\pi \sqrt{2} M \frac{B}{B_m} \left[ 1 - \frac{B}{B_m} \right]^{n+3/2} \int_0^\infty G(\epsilon) \epsilon^{3/2} d\epsilon \int_{1-y}^1 (1-y)^{n+1/2} dy;$$

and finally we get

$$P'' = A(\psi) \left( \frac{B_m}{B} \right)^n \left[ 1 - \frac{B}{B_m} \right]^{n+3/2}, \quad (122)$$

where
Similarly, we obtain

\[ P_{\perp} = \frac{M}{2} \int v_{\perp}^2 f d^3v = A(\psi) \left( \frac{B_{m}}{B} \right)^n \left( 1 - \frac{B}{B_{m}} \right)^{\frac{n+\frac{1}{2}}{2}} \left( n + 1 + \frac{B}{2B_{m}} \right). \]

These expressions are similar to those of Oppenheim, who also chooses a distribution function of the same form as Eq. (120). However, Oppenheim does not make the assumption that \( T_e \ll T_i \), hence he considers the electronic contribution to the pressure as well as the ionic contribution. In addition, his \( A(\psi) \) differs from the above by a factor of 2, and because his distribution function was slightly different from Eq. (120), he obtains a factor of \( \epsilon^{n+3/2} \) in the integration over \( \epsilon \) instead of \( \epsilon^{3/2} \). It is convenient to express \( P_u \) and \( P_{\perp} \) in terms of \( P_{\perp 0} \), the value of \( P_{\perp} \) at the midplane. We call \( M = M(\psi) = B_{m}(\psi)/B_{0}(\psi) \), where \( B_{0} \) = value of \( B \) at midplane and \( B_{m} = B \) at the mirrors. Then by Eq. (123), we have

\[ A(\psi) = \frac{P_{\perp 0}(\psi)}{M^n} \left( 1 - \frac{1}{M} \right)^{-\frac{n+\frac{1}{2}}{2}} \left( n + 1 + \frac{1}{2M} \right)^{-1}. \]
From Eq. (124) we obtain

\[ \frac{P_L}{P_H} = \frac{(n + 1) \frac{B_m}{B} + \frac{1}{2}}{\frac{B_m}{B} - 1} \]  \hspace{1cm} (125) 

We now choose \( P_{L0} \) in a similar manner to that of \( P \) in the hydromagnetic case by setting \( P_{L0} = C(\overline{\psi} - \psi) \), where \( C \) and \( \overline{\psi} \) are constants to be determined by the physical conditions.

Let us return to the pressure-balance condition of Eqs. (115) and (116). Since distribution function obeys the guiding-center approximation by being constant along a line of force, Eq. (116) is satisfied.
Proof: First, we restate Eq. (116):

$$( P_{\parallel} - P_{\perp} ) \frac{\partial}{\partial \xi} \frac{1}{B} + \frac{1}{B} \frac{\partial}{\partial \xi} \frac{\partial}{\partial t} P_{\parallel} = 0,$$

where

$$P_{\parallel} = 4 \pi MB \int d\nu d\varepsilon f(\psi, \varepsilon, \nu) \sqrt{2(\varepsilon - \nu B)}$$

and

$$P_{\perp} = 4 \pi MB^2 \int d\nu d\varepsilon f(\psi, \varepsilon, \nu) \sqrt{\frac{\varepsilon}{2} \nu B}.$$

Then we have

$$\frac{\partial}{\partial \xi} P_{\parallel} = 4 \pi MB \frac{\partial B}{\partial \xi} \int d\nu d\varepsilon f(\psi, \varepsilon, \nu) \sqrt{2(\varepsilon - \nu B)}$$

$$= \frac{P_{\parallel} \frac{\partial B}{\partial \xi}}{B} - \frac{P_{\perp} \frac{\partial B}{\partial \xi}}{B} = (P_{\parallel} - P_{\perp}) \frac{1}{B} \frac{\partial B}{\partial \xi}$$

and

$$(P_{\parallel} - P_{\perp}) \frac{\partial}{\partial \xi} \frac{1}{B} = -(P_{\parallel} - P_{\perp}) \frac{1}{B^2} \frac{\partial B}{\partial \xi}. $$
Consequently, we get

\[ \frac{1}{B} \frac{\partial P_\psi}{\partial \xi} + (P_\psi - P_\lambda) \frac{\partial}{\partial \xi} \frac{1}{B} = 0. \]
1. EVALUATION OF SUFFICIENCY CONDITION FOR STABILITY

DERIVED FROM THE NEWCOMB ENERGY PRINCIPLE

We are now in a position to examine the stability criterion of Eq. (62). As in the hydromagnetic treatment, for simplicity we consider the case when $A \ll 1$. Since $1/R$ is of order $A$, we neglect the terms of order $1/R^2$ and write for Eq. (62)

$$0 < \frac{\pi^2}{I^2} + \min \left[ \left( \frac{1 - \sigma_+}{r^2B^2} \right) \right] - \left( \frac{P_{-}'' + 2P_{-}'}{rRB} \right),$$

(126)

where $I = \int_0^L d\ell r^2 B/(1 - \sigma_+)$ and $\sigma_+ = P_+/B^2$. As before, we consider only the line of force on the boundary. Since the pressure is zero there, we set $\sigma_+ = 0$. The minimum of the bracketed expression, for the fields we consider, is at the midplane. To determine $1/R$ and $B$, we choose the zero-order solution of Eq. (115) given by Eqs. (118) and (119). Since $\beta$ turns out to be small, and since by Eq. (104)

$I \approx I^0 \left[ 1 + \frac{7}{32} \frac{\beta(M + 1)}{M^{3/2}} \right]$, we can justify this approximation. We note that Eq. (104) was derived using the hydromagnetic approximation in which the pressure was constant along a line of force. In this calculation the density is a maximum in the midplane and goes to zero at the mirrors, and so we expect the plasma currents to give an even smaller contribution to the magnetic field and thus to $I$ than is suggested by Eq. (104). Accordingly, we now proceed to evaluate the Inequality (126) by using
Eq. (119) for the stream function and magnetic field, and by using Eqs. (122) and (123) for the plasma pressure. Now, the power \( n \) is an arbitrary parameter. To observe the effects of increasing \( n \), we study two cases, \( n = 0 \) and \( n = 5 \).

a. Case when \( n = 0 \). In this example, we obtain for Eq. (125)

\[
P_{\|} / P_{\perp} = \left[ \frac{B_m}{B} + \frac{1}{2} \right] \left[ \frac{B}{B_m} - 1 \right],
\]

which implies

\[
P_{\|0} = \frac{2P_{\perp0}(B_m - B_0)^{3/2}}{(2B_m + B_0)(B_m - B_0)^{1/2}} = \frac{2P_{\perp0} (M - 1)^{3/2}}{(2M + 1)(M - 1)}.\]

Hence, we get

\[
P_{\perp0} = 2P_{\perp0} (M - 1)/(2M + 1).
\]

We also have \( M = B_m/B_0 = (1 + \alpha)/(1 - \alpha) \) which implies \( \alpha = \frac{M - 1}{M + 1} \).

In general, \( M = M(\psi) \), but the simplified configuration we are describing has \( M \) constant, since \( \alpha \) is a constant. Another quantity we require is

\[
\frac{1}{R} = \frac{d}{dz} \left( \frac{B_r}{B_z} \right) = \frac{\alpha r \pi^2}{2a} \left[ - \cos \frac{\pi z}{a} + \frac{3}{2} \alpha \sin^2 \frac{\pi z}{a} / (1 - \alpha \cos \frac{\pi z}{a}) \right] / (1 - \alpha \cos \frac{\pi z}{a}).
\]

(128)

at the midplane this becomes
\[
\frac{1}{R_0} = \frac{-\alpha r_0 \pi^2}{2a^2(1 - \alpha)}.
\] 

(129)

From Eq. (127) we can write

\[
P_\perp + 2P_\parallel = P_\perp + P_\parallel = P_{\perp 0}(\psi)(B_m - B)^{1/2}
\]

\[
= \frac{(4B_m - B)/(B_m - B_0)^{1/2}(2B_m + B_0)}{2B_m + B_0} ;
\] 

(130)

therefore, we have

\[
P_{\parallel 0} + P_{\perp 0} = P_{\perp 0}(4M - 1)/(2M + 1) = C(\bar{\psi} - \psi)(4M - 1)/(2M + 1).
\] 

(131)

Since our choice of \( M \) is not a function of \( \psi \), we have

\[
P_{\parallel 0} + P_{\perp 0} = C(4M - 1)/(2M + 1).
\]

To find \( C \), use

\[
\beta = 2P_\perp(z = r = 0)/B^2(z = r = 0) = \frac{2C\bar{\psi}}{B_0^2}/(1 - \alpha)^2
\]

(Eq. (132) cont.)
where

\[ \overline{N} = \frac{\psi}{B^0}. \]

This implies \( C = \beta B^0 (1 - \alpha)^2 / 2\overline{N} \) and, therefore,

\[ P_{10} + P_{\#0} = - \beta B^0 (1 - \alpha)^2 (4M - 1) / 2\overline{N}(2M + 1). \quad (132) \]

Because of the constancy of \( \psi \), and hence \( N \), along a line of force we write

\[ N = \frac{r^2}{2} (1 - \alpha \cos \frac{\pi z}{a}) = \frac{r_0^2}{2} (1 - \alpha), \]

where \( r_0 \) is evaluated at the midplane, \( z = 0 \). Using this relation, and from Eqs. (129) and (131), we obtain

\[
\frac{P_{10} + P_{\#0}}{r_0^3 B_0^3 R_0} = \beta B^0 (4M - 1) \alpha \pi^2 / 8 \overline{N} a^2 B^0 (2M + 1)(1 - \alpha) \\
= \beta (4M - 1)(M - 1) \pi^2 / 16 N a^2 B^0 (2M + 1); \\
\]

from \( \alpha = \frac{M - 1}{M + 1} \).

\( \quad (133) \)
We can immediately write

\[ I = \int d\ell \, r^2 B = \int_{-a}^{+a} dz \, r^2 B_z = \int_{-a}^{+a} dz \, \frac{2N B^0 (1 - \alpha \cos \frac{\pi z}{a})}{(1 - \alpha \cos \frac{\pi z}{a})} = 4 a B^0 \tilde{N}; \]

thus:

\[ 0 < \frac{\pi^2}{I^2} + \text{Min} \left[ \frac{(2P^i + P^i)}{r^2 B^0 R} \right] = \frac{\pi^2}{16N^2 B^2 a^2} - \frac{\pi^2 \beta (4M - 1)(M - 1)}{16N^2 B^0 a^2 (2M + 1)} \]

or

\[ 0 < 1 - \frac{\beta (4M - 1)(M - 1)}{2M + 1} \]

and

\[ \beta < \frac{(2M + 1)/(M - 1)(4M - 1)}{.} \quad (134) \]

The \( \beta_c \) obtained from Eq. (134) is plotted in Fig. 2 and tabulated in Table III. The interesting result of this calculation is that the \( \beta_c \) obtained is almost identical to the result obtained from the hydro-magnetic sufficient condition, Table I and Fig. 1. (In Fig. 4 we com-
pare the two results.) The reason for this again depends on Eq. (104), which establishes the validity of using vacuum fields, and Eq. (127), which shows that for large \( M \), \( P_{\parallel 0} \) approximates \( P_{\perp 0} \). Therefore, we have \( P_{\parallel 0} + P_{\| 0} = 2 P_{\perp 0} \), and our choices for \( P_{\perp 0} = C(\bar{\psi} - \psi) \) and the hydromagnetic pressure \( P = C(\bar{\psi} - \psi) \) have the same form.

Therefore the quantity in brackets of the hydromagnetic energy principle, \( p'D/r^2B^2 = -2p'/r^3B^3 \), becomes identical at the midplane to the corresponding quantity of Eq. (133).

b. Case when \( n = 5 \). We use Eq. (125) to write

\[
P_{\|}/P = \left[ (n+1)m - \frac{m}{2} \right]/\left[ -\frac{m}{2} + 1 \right] = (12 m + B)/(2(B_m - B))
\]

for \( n = 5 \). Thus we have

\[
P_{\|} = 2P_{\perp}(B_m - B)/(12B_m + B)
\]

and

\[
P_{\| 0} = 2P_{\perp 0}(M - 1)/(12 M + 1)
\]

We therefore have

\[
P_{\| 0} + P_{\perp 0} = P_{\perp 0}(14M - 1)/(12M + 1);
\]

(135)
by using the results of case a, we obtain

\[
\frac{P_{10} + P_{n0}'}{R_0 r \frac{3}{2} B_0} = \frac{\beta \pi^2 (M - 1)}{16 N B^2 a^2} \frac{14 M - 1}{12 M + 1}.
\]

We therefore write Eq. (126) as

\[
0 < \frac{\pi^2}{12} + \min \left[ - \frac{(P_{n} + P_{L}')}{{r^2 R B^3}} \right] = \frac{\pi^2}{16 N^2 B^2 a^2} \frac{14 M - 1}{12 M + 1}, \tag{136}
\]

or

\[
0 < 1 - \beta (M - 1) \left[ \frac{14 M - 1}{12 M + 1} \right],
\]

which implies

\[
\beta_c = \frac{12 M + 1}{(M - 1)(14 M - 1)}.
\]

The results of this calculation are plotted in Fig. 2 and tabulated in Table III. It appears that the effect of increasing n is to obtain a more stable configuration. However, this is not the case. By increasing n, we decrease the ratio of \(P_n/P_L\), as shown from
Eq. (125). Therefore, we decrease the destabilizing effect of \( P_{\parallel 0} \), as determined from Eq. (126). We define \( \beta \) in terms of \( P_{\perp 0} \), and so it appears that a larger \( \beta \) is needed to decrease the bracketed expression above. But this does not necessarily mean that we can tolerate a larger density. If we choose a \( \beta \) that included the contribution of \( P_{\parallel} \), then it would remain unchanged as we increased \( P_{\parallel} \) at the expense of \( P_{\perp} \). In fact, an increasing \( n \) should reduce the stability of the plasma, since from Eq. (122), \( P_{\perp} \sim (B_m - B)^{n+1/2} \). As \( n \) increases, the pressure drops off more and more steeply in the stabilizing mirror regions where \( B = B_m \).
TABLE III.

Evaluation of sufficiency condition derived from the Newcomb stability condition.

A. Maximum $\beta = \beta_c$ given as a function of $M$; vacuum fields;

$$A \ll 1; f = g(\epsilon, \psi) \left( \nu \frac{B - 1}{m} \right)^0$$

<table>
<thead>
<tr>
<th>$\beta_c$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.714</td>
<td>2</td>
</tr>
<tr>
<td>0.318</td>
<td>3</td>
</tr>
<tr>
<td>0.200</td>
<td>4</td>
</tr>
<tr>
<td>0.145</td>
<td>5</td>
</tr>
<tr>
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<td>10</td>
</tr>
</tbody>
</table>

B. Vacuum fields; $A \ll 1$; $f(\epsilon, \nu, \psi) = g(\epsilon, \psi) \left( \frac{\nu B}{m} \right)^5$ for $\frac{\epsilon}{B_m} < \nu < \frac{\epsilon}{B}$.

<table>
<thead>
<tr>
<th>$\beta_c$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.926</td>
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</tr>
<tr>
<td>0.451</td>
<td>3</td>
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<tr>
<td>0.300</td>
<td>4</td>
</tr>
<tr>
<td>0.221</td>
<td>5</td>
</tr>
<tr>
<td>0.10</td>
<td>10</td>
</tr>
</tbody>
</table>
2. EVALUATION OF NECESSARY CONDITION FOR STABILITY

DERIVED FROM THE CHEW-GOLDBERGER-LOW ENERGY PRINCIPLE

In this section we evaluate the stability criterion of Eq. (76) by using the equilibria defined by Eqs. (118), (119) and (124). As discussed above, this implies that we are using the vacuum fields with \( A \ll 1 \), and therefore we neglect terms in \( A^2 \). For simplicity, we take the case when \( \mu = 0 \) and consider the stability of the line of force at the boundary. In this approximation, Eq. (76) becomes

\[
0 < \frac{a^2}{I} + 2 \int_0^L \frac{d\ell}{B} \left[ - \frac{P'_1 + 2P'_{1,1}}{r \rho B} \right] \sin^2 \pi \frac{0}{I} \int_0^\ell d\ell' r^2 B,
\]

where \( I = \int_0^L d\ell \ r^2 B \). We show below that

\[
\int_0^\ell r^2 B d\ell'/I = \ell/L,
\]

which implies that

\[
\sin \frac{0}{I} = \sin \frac{\pi \ell}{L} = \cos \frac{\pi z}{2a}.
\]

Let us call
\[ D(z) \equiv \frac{1}{B(z)} \left[ \cos^2 \frac{\pi z}{2a} \left( \frac{P_\perp' + 2 P_\perp}{rRB} \right) \right]. \]

For \( n = 0 \), Eq. (114) is valid, and

\[ 2P_\perp + P_- = P_\parallel + P_\perp = P_\perp^0(\psi) \frac{(B_m - B)^{1/2}(4B_m - B)}{(B_m - B_0)^{1/2}(2B_m + B_0)}. \]

Since we are examining the case when \( \psi = \overline{\psi} \), we can write

\[ P_\parallel' + P_\perp' = -C \frac{(B_m - B)^{1/2}(4B_m - B)}{(B_m - B_0)^{1/2}(2B_m + B_0)}. \]

Now from

\[ B = B^0(1 - \alpha \cos \frac{\pi z}{a}) \]

we find
Similarly, we find

$$\sqrt{B_m - B} = B^0 \frac{1}{2} \left[ 1 + \alpha - (1 - \alpha \cos \frac{\pi z}{a}) \right]$$

$$= B^0 \frac{1}{2} \left[ \alpha (1 + \cos \frac{\pi z}{a}) \right] = \left| 2 B^0 \frac{1}{2} \right| \cos \frac{\pi z}{2a}.$$

(138)

By using $M = B_m / B_0$, we then write

$$4 B_m - B = B^0 (4 + 4 \alpha - 1 + \alpha \cos \frac{\pi z}{a}) = B^0 \left( 3 + 4 \alpha + \alpha \cos \frac{\pi z}{a} \right).$$

Using Eq. (128) for $1/R$, we then write
\[ \cos^2 \frac{\pi z}{2a} (P_{\mu'} + P_L') = -\frac{c}{B^2 \sqrt{2\alpha}} \pi^2 \alpha f(z) P(z) \cos^2 \frac{\pi z}{2a} \]

\[ \frac{1}{(1 - \alpha)^{3/2} (M - 1)^{1/2} (2M + 1) 2a^2 (1 - \alpha \cos \frac{\pi z}{a})^2}; \]

(140)

Here

\[ P(z) = -\cos \frac{\pi z}{a} (1 - \alpha \cos \frac{\pi z}{a})^{-1} + \frac{3}{2} \alpha \sin \frac{\pi z}{a} (1 - \alpha \cos \frac{\pi z}{a})^{-2}. \]

By using

\[ \int_{0}^{L} \int_{0}^a \frac{r^2}{B} = \int_{-a}^{a} \int_{0}^{z} 2N B^0 = 4a \frac{N}{a} B^0, \]

which implies

\[ \frac{\ell}{\int_{0}^{L} \int_{0}^a \frac{r^2}{B}} = \frac{\ell}{L} = \frac{z - a}{2a}. \]
and since
\[ C = B^0(1 - \alpha)^2/2N , \]
the Inequality (137) becomes

\[
0 < \pi^2 \frac{1}{4aN^0} + \frac{\sqrt{2\alpha} \pi^2 \alpha B^0(1 - \alpha)^2\beta}{N^0B^0(1 - \alpha)^{3/2}(M - 1)^{1/2}(2M + 1)2a^2} \]

\[
\int_{-\alpha}^{\alpha} \frac{f(z)}{\cos \frac{\pi z}{a}} \cos \frac{\pi z}{a} \frac{2a}{\cos \frac{\pi z}{a}} \, dz ; \quad (141)
\]

or, we must evaluate

\[
0 < 1 + \frac{(2\alpha)^{3/2}(1 - \alpha)^{1/2}\beta I'}{(M - 1)^{1/2}(2M + 1)} = 1 + \frac{4I'(M - 1)\beta}{(M + 1)^2(2M + 1)} , \quad (142)
\]

where

\[
I' = \int_{-1}^{1} \frac{f(z) P(z)}{(1 - \alpha \cos \pi z)^2} \, dz
\]
after we substitute $z$ for $z/a$ and $M - 1/M + 1$ for $\alpha$. In Appendix C, we evaluate $I'$ analytically. Its value is

$$I' = -\frac{(M + 1)^2}{(M - 1)^2} \frac{1}{8\pi} \left[ 6M^2 - 8M + 30 + (M - 1)^{-1/2}ight.
\left. (6M^3 - 13M^2 - 4M - 16) + \tan^{-1}(M - 1)^{1/2} \right] .$$

Since $I'$ is less than zero, the stability criterion (142) becomes

$$0 \leq 1 - 4\beta(M - 1) | I' | / (M + 1)^2 (2M + 1)$$

or

$$\beta \leq (M + 1)^2 (2M + 1) / 4(M - 1) | I' |$$

which implies

$$\beta_c = (M + 1)^2 (2M + 1) / 4(M - 1) | I' |$$

The maximum $\beta = \beta_c$ that obeys (144) is tabulated in Table IV and plotted in Fig. 3.
It can be seen that the approximation of vacuum fields is not really justified except for large $M$, since even for $M = 10$, we have $\beta_c = 0.465$. However, we do have a qualitative description of the behavior of $\beta_c$, if not a quantitative one. The effect of having a pressure distribution which vanishes at the mirrors can be seen by comparing the result of this section with the corresponding results of the previous section, using hydromagnetic theory. The values of $\beta_c$ found here are much more realistic, and lie closer to those at the sufficiency criteria. This is because there is less plasma in the stabilizing regions near the mirrors. In fact, as mentioned in the preceding section, the larger the value of $n$, and consequently the more the plasma is concentrated at the midplane, the less stable the system; as $n$ increases, the necessity condition should approach the sufficiency condition since the sufficiency condition is based upon the value of $P_{\perp}/R$ at the midplane. This limit is indicated in the previous section. Consider Eq. (136), which implies

$$\beta_c = \frac{1}{M - 1} \left[ \frac{12M + 1}{14M - 1} \right].$$

As $n$ increases, the bracketed expression approaches 1, and both the necessity condition and the sufficiency condition approach $\beta_c = 1/(M - 1)$. 
TABLE IV.

Evaluation of the necessity condition derived from the Chew-Goldberger-Low energy principle.

Critical $\beta = \beta_c$ given as a function of $M$ for vacuum fields;

$$f(\varepsilon, \nu, \psi) = g(\varepsilon, \psi); \frac{\varepsilon}{B_m} < \nu < \frac{\varepsilon}{B}.$$

| $\beta_c$ | $M$ | $\beta_c = \frac{(M + 1)^2(2M + 1)}{(M - 1)|I'|}$ | $|I'| = \frac{|M + 1|}{M - 1}$ |
|-----------|-----|---------------------------------|-----------------|
| 1.96      | 2   | $\frac{1}{5} \left[ 6M^2 - 6M + 50 + (M - 1)^{-1/2} \right]$ | $(6M^3 - 13M^2 - 4M - 16) \tan^{-1} \sqrt{M - 1}$ |
| 1.22      | 3   |                                              |                  |
| 0.89      | 5   |                                              |                  |
| 0.465     | 10  |                                              |                  |
C. SUMMARY OF SECTION III

In this part of the thesis we applied the stability criteria derived in Sec. II to specific equilibrium configurations modelled after the physical states that exist in real mirror machines. The purpose of these applications is to indicate the range of parameters for which a real mirror machine will be stable. We defined a critical $\beta = \beta_c$, which is the maximum $\beta$ allowable for stability. (The parameter $\beta$ is defined as equal to $2 \frac{P}{B^2}$ evaluated on the axis, at the midplane.)

We calculated $\beta_c$ as a function of the mirror ratio $M$ and the aspect ratio $A$. The results are given in Tables I to Iv and are plotted in Figs. 1, 2, 3 and 4. These results show that $\beta_c$ decreases as the mirror ratio and the aspect ratio increase.

They also exhibit an interesting consistency in the $\beta_c$ determined from the various models. In Sec. III-B we discuss the correspondence between the $\beta_c$ determined from applying the sufficiency condition derived from the Newcomb energy principle to a guiding-center equilibrium state and the $\beta_c$ determined from applying the hydromagnetic sufficiency condition to a hydromagnetic model. This correspondence is illustrated in Fig. 4.

Another noteworthy feature is in the similarity of the shape of the curves depicting the variation of $\beta_c$ with the mirror ratio and the aspect ratio for the two hydromagnetic equilibrium states examined. One of these equilibrium states, which we shall call state 1, included the effect of the plasma currents on the magnetic field, and had a small aspect ratio in the sense that $A^2 \ll 1$. The other equilibrium state, which
we shall call state 2, had a vacuum magnetic field, and an arbitrary aspect ratio. States 1 and 2 are in a sense complementary simplifications to a more realistic model of a mirror machine in which the aspect ratio is finite and in which the magnetic field includes the diamagnetism of the plasma currents. States 1 and 2 are discussed in detail in Sec. II-A. We are interested in determining the effect of neglecting the plasma diamagnetism in state 2. This information is provided in Fig. 1, in which the values of $\beta_c$ for small $A$ as determined from state 1 are compared to those values of $\beta_c$ for finite $A$ as determined from state 2. From this figure it can be seen that no anomalous results are obtained and that each set of values of $\beta_c$ appear as continuations of the other set. For example, in Fig. 1b the values of $\beta_c$ for each $M$ determined from state 1 are plotted on the $A = 0$ axis and lie on continuations of the curves of $\beta_c$ versus $A$ which have been determined from state 2. Similarly, in Fig. 1a, the curve of $\beta_c$ versus $M$ for small $A$, determined from state 1, lies parallel to and somewhat above the curves of $\beta_c$ versus $M$ for $A = 0.1$ and $A = 0.5$, as determined from state 2. We can conclude that, for the purposes of this calculation, our hypothetical model of finite aspect ratio and diamagnetic plasma currents can be well approximated by state 2, in which the diamagnetism of the plasma currents is neglected.

The necessity condition derived from the Chew-Goldberger-Low energy principle was applied to an equilibrium configuration characterized by infinitesimal aspect ratio, vacuum magnetic fields, and a distribution function which is constant along a line of force. Complete details about
this equilibrium configuration are given in Sec. III-B. We also applied the necessity condition derived from hydromagnetic theory to state 1. In Fig. 3, we compare these two calculations in a plot of $\beta_c$ versus $M$. The two curves have almost the same shape, indicating a similar dependence of $\beta_c$ on $M$. However, the hydromagnetic theory gives a much larger value of $\beta_c$ for each value of $M$ than does the guiding-center theory. The reason for this is that in the hydromagnetic theory the pressure is constant along a line of force, while in the guiding-center theory the pressure decreases away from the midplane. Hence, for a given $\beta$ defined at the midplane, the hydromagnetic equilibrium has a much greater pressure at the stabilizing mirror regions than does the guiding-center equilibrium, and it will tend to be more stable.
D. CONCLUSION TO SECTIONS II AND III

In order to explain our results it is important to discuss the meaning of the concepts of sufficiency and necessity. Let us assume that we are studying a model of an equilibrium state by applying to it a sufficiency condition for stability and a necessity condition for stability. Each of these stability criteria will furnish a maximum allowable $\beta = \beta_c$ as a function of the mirror ratio and the aspect ratio. Let us denote the $\beta_c$ furnished by the sufficiency condition as $\beta_c^s$ and the $\beta_c$ furnished by the necessity condition as $\beta_c^n$. Since the sufficiency condition is obtained from a lower bound on $\delta W$, while the necessity condition is obtained from an upper bound on $\delta W$, $\beta_c^s$ will always be less than $\beta_c^n$. If the $\beta$ of the model equilibrium state is less than $\beta_c^s$ it will be stable under the interchange perturbation. If $\beta$ is greater than $\beta_c^n$, the equilibrium state will be unstable. If the inequality $\beta_c^n > \beta > \beta_c^s$ holds, the equilibrium state may or may not be unstable. There does exist an exact $\beta_c$, which we shall denote by $\beta_c^e$, such that the system will always be unstable for $\beta > \beta_c^e$. The values of $\beta_c^s$ and $\beta_c^n$ are lower and upper bounds of $\beta_c^e$. To determine $\beta_c^e$, one must furnish a stability condition that is both a sufficiency condition and a necessity condition.

In Sec. III we applied the hydromagnetic sufficiency condition and the hydromagnetic necessity condition to a hydromagnetic model of an equilibrium state consisting of magnetic fields which include the diamagnetism of the plasma currents; the equilibrium state is characterized by a small aspect ratio. This is state 1 discussed the previous section.
State 1 is not a very good representation of a real plasma for several reasons. First of all, the hydromagnetic assumptions are not applicable because in a real mirror machine the collision time is long compared to the characteristic time for development of the interchange instability. Secondly, in the hydromagnetic model the pressure is constant along a line of force, which means that the plasma is in contact with the end plates, which we have placed at the mirrors. In a real mirror machine the plasma pressure vanishes at the mirrors. Additionally, in our hydromagnetic model the plasma is bounded by a perfect conductor that is parallel to the outermost line of force in the plasma. The reason for this conducting boundary is discussed in Sec. III-B. In a real mirror machine the plasma is bounded by vacuum. In Fig. 5, we plot the $\beta^s_c$ and $\beta^n_c$ versus the mirror ratio. The range between these values indicates the location of $\beta^e_c$. Unfortunately, for each value of the mirror ratio $M$ the difference between $\beta^s_c$ and $\beta^n_c$ is large, in fact infinite for $4 > M$. A better set of values of $\beta^s_c$ and $\beta^n_c$ is provided by a guiding-center model of a plasma.

In Sec. III-B we applied the sufficiency condition derived from the Newcomb energy principle and the necessity condition derived from the Chew-Goldberger-Low energy principle to a guiding-center model of a plasma. In this model, we used vacuum-magnetic fields and a small aspect ratio. As discussed in Sec. III-B the effect of neglecting the diamagnetism of the plasma currents is not large. In this model the pressure vanishes at the mirrors and therefore at the end plates, the collision time is assumed to be long compared to the characteristic time of the interchange...
mode, and the plasma is surrounded by vacuum, except for the end plates. Hence this model is a much better representation of a real plasma than the hydromagnetic model described above. In Fig. 6 we plot $\beta_c^s$ and $\beta_c^n$ versus $M$. The range between these values, which indicates the size of $\beta_c^e$, is much smaller than in the hydromagnetic case. The reason for this is that $\beta_c^n$ determined from the hydromagnetic model is much greater for each value of $M$ than is the $\beta_c^n$ determined from the guiding-center model. We discuss these results in Sec. X.

The $\beta_c$ calculated in Sec. III are favorable for thermonuclear confinement. From power-loss considerations in a mirror machine utilizing cryogenic or superconducting magnets, Post has estimated that an optimum $\beta$ would be about 0.1 with mirror ratios between 3.3 and 10 and infinitesimal aspect ratios. Comparing these values of $\beta$ to our calculated values of $\beta_c^s$ varying between 0.3 and 0.06 for mirror ratios between 3.3 and 10 and infinitesimal aspect ratio, we conclude that the thermonuclear mirror machines postulated by Post would probably be stabilized against the interchange mode by the addition of conducting end plates.

The $\beta$ given by Post is defined in a different manner than the $\beta$ used throughout this paper. His $\beta$ is defined as $\beta = P_l/B^2/2$, where $B$ is evaluated at the boundary of the plasma region and $P_l$ is evaluated on the axis, at the midplane. To illustrate the difference in the definition of $\beta$, if Post's $\beta = 0.1$, the $\beta$ in this paper would be given approximately by $\frac{0.1}{1 - 0.1}$. 
IV. APPLICATION OF A VARIATIONAL PRINCIPLE

A. COMPARISON OF GEOMAGNETIC OSCILLATIONS WITH A HYDROMAGNETIC MODEL

Sugiura observed oscillations in the geomagnetic field at College, Alaska, a station located at about 65 degrees latitude. These oscillations had a period of approximately 6 minutes and had an amplitude of 100 \( \gamma \), where \( \gamma = 10^{-5} \) gauss. A possible excitation mechanism for these oscillations may be the viscous-like interaction between the solar wind and the magnetosphere, proposed by Axford and Hines. This viscous-like interaction would lead to convective motion of the geomagnetic field lines of high latitude and to turbulence in the boundary between the magnetosphere and the solar wind. The turbulence could easily generate the oscillations in the geomagnetic field observed by Sugiura.

In this section we study these oscillations using a hydromagnetic plasma as a model for the magnetosphere. The conditions for the applicability of the hydromagnetic assumption are discussed in Appendix A. We can assume that for these oscillations the plasma is strictly tied to the magnetic lines of force since the diffusion time of the magnetic field is much greater than the period of oscillation. Additionally, we can assume that the neutral particles do not participate in the oscillation since the neutral-ion collision time is also much greater than the period of oscillation. We further assume that the ionosphere provides a rigid boundary through which the geomagnetic field enters the ionized region. According to this assumption, the magnetic lines of force are tied down at the end points of the plasma. We also assume that the magnetic field is of perfect dipolar shape. The distribution of charged particles above 600 km is taken to be the hydrostatic distribution suggested by Johnson.
1. CALCULATION

To calculate the frequency of the oscillations in our model, we use the hydromagnetic-energy principle of Bernstein, Frieman, Kruskal and Kulsrud. We estimate, by a variational approach, the frequency of oscillation \( \omega \) generated by an infinitesimal displacement \( \xi \); we use the equation

\[
\omega^2 = \frac{1}{2} \int \rho \xi^2 \, d\tau.
\]  \tag{145}

Here, \( \delta W \) is the change of potential energy under the infinitesimal displacement \( \xi \), and \( \rho \) is the density. For the parameters of this problem the ratio of plasma pressure to magnetic-field pressure, \( \beta \), is very much less than one. Accordingly, we can neglect all terms in \( \delta W \), except \( \int \frac{Q^2}{\partial \tau} \, d\tau \), where \( Q = \nabla \times (B \times B) \). Then \( \delta W \) becomes in the \((X, \psi, \theta)\) coordinate system described in Appendix D,

\[
\delta W = \frac{1}{8\pi} \int J d\psi \, d\theta \left[ \int \frac{1}{rJB} \frac{\partial}{\partial X} rB \xi \psi \right]^2 + \left| \frac{r}{J} \frac{\partial}{\partial X} \frac{\xi \theta}{r} \right|^2 \tag{146}
\]

\[
+ B^2 \left| \frac{\partial}{\partial \psi} rB \xi \psi + \frac{\partial}{\partial \theta} \frac{\xi \theta}{r} \right|^2
\]

Setting...
\[ \xi_{\psi} = \sum_{m=0}^{\infty} \frac{X_m(X, \psi)}{r B} \begin{bmatrix} \cos \\ \sin \end{bmatrix} m \theta \]

and

\[ \xi_{\theta} = \sum_{m=1}^{\infty} \frac{r Y_m(X, \psi)}{m} \begin{bmatrix} \cos \\ \sin \end{bmatrix} m \theta + \xi_{\theta}^{(0)}(X, \psi) \]

we have \( \delta W = \delta W_0 + \sum_{n=1}^{\infty} \delta W_m \), where \( \delta W_m \) is given by

\[
\delta W_m = \frac{\pi}{4\pi} \int d\chi d\psi \left[ \frac{1}{2B^2 J} \left( \frac{\partial X_m}{\partial \chi} \right)^2 + \frac{r^2}{2 J} \left( \frac{\partial Y_m}{\partial \psi} \right)^2 \right]
+ B^2 J \left| Y_m + \frac{\partial X_m}{\partial \psi} \right|^2 \tag{147}
\]

For simplicity, we assume \( m \) is large and we set \( Y_m + \frac{\partial X_m}{\partial \psi} = 0 \), and obtain for \( \delta W_m \)

\[
\delta W_m = \frac{\pi}{4\pi} \int d\chi d\psi \frac{1}{2B^2 J} \left( \frac{\partial X_m}{\partial \chi} \right)^2 \tag{148}
\]
In a similar manner, we have

\[ \int \rho \xi^2 \, d\tau = \int J \, d\chi \, d\psi \, d\Theta \rho (\xi_{\psi}^2 + \xi_{\Theta}^2) \]

\[ = 2\pi \sum_{m=1}^{\infty} \int \rho \, J \, d\psi \, d\chi [\left(\frac{X_m}{r_B}\right)^2 + \frac{r^2}{m^2} \frac{Y_m}{m^2}] \]

\[ \approx 2\pi \sum_{m} \int \rho \, J \, d\psi \, d\chi \frac{X_m^2}{r_B^2} , \]

where we have assumed that only modes of large \( m \) appear. Then we have for the oscillation frequency

\[ \omega^2 \approx \frac{1}{4\pi} \sum_{m} \int \rho \, d\chi \, d\psi \, \frac{1}{r_B^2 J} \left( \frac{\partial X_m}{\partial \chi} \right)^2 \]

\[ \times \int \rho \, J \, d\psi \left( \frac{X_m}{r_B} \right)^2 \]

(149)

Now if we assume that all of the \( X_m \) have the same \( \chi \) and \( \psi \) dependence, we can write \( \omega^2 \) as

\[ \omega^2 = \frac{1}{4\pi} \left[ \int d\chi \, d\psi \, \frac{1}{r_B^2 J} \left( \frac{\partial X}{\partial \chi} \right)^2 \right] / \int \rho \, J \, d\chi \frac{X}{r_B^2} \, d\psi \ , \quad (150) \]

where we have suppressed the subscript \( m \). We now change variables by writing \( d\chi \, r_B^2 \frac{J}{J} = dx \), and obtain
\[ \omega^2 = \frac{1}{4\pi} \int dx \frac{\partial x}{\partial x}^2 \, d\psi \]

\[ \int p \frac{\chi^2}{r^4 B^4} \, dx \, d\psi \]

we choose \( \chi^2 = \left[ \sin^2 \frac{\pi x}{x_0} \right] \delta(\psi - \psi_0) \), where \( x_0 = \int_0^\infty r^2 B^2 \, dx = \frac{L}{\int_0^\infty r^2 B^2 \, dl} \), where \( dl \) is the unit of length along the line of force characterized by \( \psi_0 \) and \( L \) is the length of the line of force; we write

\[ \omega^2 = \frac{1}{4\pi} \int dx \frac{\partial x}{\partial x}^2 \, d\psi / \int p \, dx \, d\psi \frac{x^2}{r^2 B^2} = \frac{\pi x_0}{r^2 B^2} / \int \frac{dx}{4 B^4} \sin^2 \frac{\pi x}{x_0} \]

(151)

and finally we have

\[ \omega^2 = \pi/8 \left[ \int_0^L dl \, r^2 B \left[ \int \frac{p dl}{r^2 B^3} \sin^2 \left( \frac{\pi}{x_0} \int_0^l dl \, r^2 B \right) \right] \right) = \frac{\pi}{8 B} \]

(152)

where

\[ \int_0^L r^2 B \, dl = J \]

and
It should be emphasized that $I$ and $J$ are evaluated along a line of force. The evaluation of $J$ is quite straightforward. First, we note from Fig. 7 that $r$ is the distance from the axis to a point on the line of force and not the distance from the center of the earth to that point, which we denote by $R$. In fact, $r = R \sin \theta$, where $\theta$ is the colatitude. Now we have for $R$

$$R = R_0 \sin^2 \theta,$$  \hspace{1cm} (153)

where $R_0$ is shown on Fig. 7 and is the maximum distance from the line of force to the center of the earth. The field strength $B$ is given by

$$B = B_0 \sqrt{1 + 3 \cos^2 \theta} / \sin^6 \theta,$$ \hspace{1cm} (154)

where $B_0 = B(R_0, \theta = \pi/2)$. So we have $d\ell = Rd \theta \left[1 + \frac{1}{R^2} \frac{dR^2}{d\theta}\right]^{1/2}$

$$= R_0 \sin \theta d\theta \left[1 + 3 \cos^2 \theta\right]^{1/2}.$$
Then we obtain

\[ J = 2 \int_{\theta=\theta_{\text{initial}}}^{\pi/2} r^2 B \, dl = 2B_0 R_0^3 \int_{\theta=\theta_{\text{initial}}}^{\pi/2} \sin \theta \, d\theta \left( 1 + 3 \cos^2 \theta \right). \]

Taking \( \theta_{\text{initial}} = 25 \frac{1}{2} \) degrees corresponding to a line of force entering the earth at 65 degrees latitude, we obtain

\[ J = 2B_0 R_0^3 (1.62). \]  \hspace{1cm} (155)

The second integral \( I \) is much harder to evaluate, because it is necessary to know the value of the density at points along the line of force. These values have been estimated by Johnson,\textsuperscript{10} and can be summarized as follows:

(a) Between 300 km and 1300 km altitude, the primary ionic constituent of the atmosphere is \( O^+ \), which exponentially decreases with a scale height of about 300 km.

(b) Between 1300 and 1800 km altitude, there is a transition zone in which ionized oxygen is mixed with an equal density of ionized hydrogen.

(c) Above 1800 km, the primary ionic constituent is hydrogen in thermal equilibrium, distributed according to the hydrostatic law

\[ N = N_0 \exp \left( -\phi / kT \right), \] where
\[ \phi = -\frac{g m M}{R} - \frac{1}{2} m \Omega^2 \cos^2 \alpha = \frac{g m R_0^2}{R} - \frac{1}{2} m \Omega^2 \cos \alpha. \]

Here \( G \) is the gravitational constant, \( g \) is the acceleration of gravity at the earth's surface, \( m \) is \( 1/2 \) the proton mass, \( M \) is the mass of the earth, \( R \) is the distance above the earth's center, \( R_e \) is the radius of the earth, \( \Omega \) is the rotational velocity of the earth, \( \alpha \) is the latitude as shown in Fig. 7, \( N \) is the density, and \( N_0 \) is taken to be \( 3 \times 10^3 \) particles/cm\(^3\) at 600 km altitude.

(The factor of \( 1/2 \) for the proton mass develops because of charge separation between ions and electrons.) The temperature is assumed to be constant at \( 1250^0K \). The protons of the Van Allen belts are not included in this hydrostatic distribution since there are relatively few of them in the region of the exosphere under consideration. Johnson claims that data from Whistler observations verifies the hydrostatic distribution.

The integral \( I \) is divided into three segments corresponding to the three regions discussed above. We show that the first two regions give an insignificant contribution to the integral \( I \).

For region 1, \( R \) varies from \( 300 \text{ km} + R_e = R_1 \) to \( 1500 \text{ km} + R_e = R_2 \), where \( R_e \) is taken to be \( 6400 \text{ km} \). The change in the angle \( \theta \) between these limits can be found from Eq. (153) and is of the order of \( 1/25 \) radians. We
have then

\[ \sin^2 \frac{\pi}{x_0} \int_0^{\frac{\pi}{x}} d\phi \frac{r^2 B}{r^2 B} = \sin^2 \frac{\pi}{3.64} \left[ 1.62 - \cos \left( \theta_1 + \frac{1}{25} \right) \right] \]

\[ - \cos^3 \left( \theta_1 + \frac{1}{25} \right) \approx 9 \times 10^{-4} \]

where \( \theta_1 \) is the angle corresponding to \( R_1 \) and is 0.495 radians \( = 25 \frac{1}{2} \) degrees. The charged particles in region 1 are mostly oxygen ions having a maximum density of \( 10^6 \) particles/cm\(^3\), and decreasing with a scale height \( h \) of 300 km. Therefore we can write for \( I_1 = I \) evaluated over region 1

\[ I_1 \leq \frac{9 \times 10^{-4}}{B(R_2)} \frac{\rho(R) dR}{R^2} \leq \frac{9 \times 10^{-4}}{B(R_2)} \int \frac{\rho(R) dR}{R^2} \leq \frac{9 \times 10^{-4} h \rho(R)}{B(R_2)^3} \frac{R}{R_1^2} \]

\[ < \frac{1.4 \times 10^{-5}}{N_a B_e^3}, \quad (156) \]

where \( N_a \) is Avogadro's number, and \( B_e \) is the field strength at the earth's surface.

In region 2, we set \( R_1 = R_e + 1500 \) km and \( R_2 = R_e + 2500 \) km, where \( R_1 \) and \( R_2 \) are the lower and upper boundaries. The massive charged particles consist of oxygen and hydrogen ions. The oxygen-ion density at 1500 km altitude is about \( 7 \times 10^3 \) particles/cm\(^3\), decreasing exponentially with a scale height \( h \) of about 100 km. The hydrogen-ion density is
essentially constant in region 2 with a density of about $3 \times 10^3$ particles/cm$^3$. We can therefore write, using Eqs. (155) and (154), where $I_2$ is $I$ evaluated over region 2,

\[
I_2 < \frac{\pi \times 10^{-2}}{\sin^2 \theta, R_e^2 B^3(R_2)} \left[ \int \rho_0^- \, dr + \int \rho_H^+ \, dr \right]
\]

\[
< \frac{\pi \times 10^{-2}}{\sin^2 \theta, R_1^2 B^3(R_2)} \left[ h \rho_0^- + 1000 \rho_H^+ \right]
\]

\[
< \frac{1.5 \times 10^{-7}}{B_e^3 N_a} = 10^{-2} \times I_1,
\]

where $h = 100$ km, $\rho_0^+(R_1) = 7 \times 10^3 \times 16/N_a$ gm/cm$^3$, and $\rho_H^+ = 3 \times 10^3/N_a$ gm/cm$^3$.

In region 3, $R$ ranges from $R_e + 2500$ km to $5.6 R_e$, corresponding to a range in $\theta$ from 0.5236 radians (= 30$^\circ$) to 1.5708 radians (= 90$^\circ$). Using Eqs. (153) and (154) and the relation

\[
dl = R_0 \sin \theta \, d\theta (1 + 3 \cos^2 \theta)^{1/2}
\]

we write

\[
dl/r^2 B^3 = \sin^{13} \theta \, d\theta /R_0 B_0^3 (1 + 3 \cos^2 \theta).
\]

By employing the techniques used in deriving Eq. (155) we can write
We express the hydrostatic-density distribution in terms of the angle \( \theta \) corresponding to a point on the line of force a distance \( R \) from the earth's center. We use as a reference the density of \( 3 \times 10^3 \) particles/cm\(^3\) at an altitude of 600 km, and substitute the numerical values of \( g, R_e, \Omega, m, kT \) for \( T = 1250^0 K \), and \( R_0 \) to obtain

\[
N = 3 \times 10^3 \exp \left( 0.5390 / \sin^2 \theta + 0.1634 \sin^4 \theta - 2.77 \right)
\]  

(160)

The values of \( N = N(\theta) \) computed from Eq. (159) are tabulated in Table V for 13 values of \( \theta \) between 0.5236 radians and 1.5708 radians.

Summing up Eqs. (158) through (160), we write for \( I_3 \), since \( \rho = Nm \),

\[
I_3 = \frac{\int_0^{1.5708} \theta \, d\ell}{\theta = 0.5236 \int_0^{1.5708} \frac{d\ell}{r^2 B^3} \rho \sin^2 \left( \frac{\pi}{2} \right) \int_0^\ell \frac{d\ell'}{r^2 B} \}
\]

\[
= \frac{3 \times 10^3}{N \, R_e B_0} \int_0^{1.5708} \frac{\theta = 1.5708}{\theta = 0.5236} f(\theta) \, d\theta ,
\]  

(161)

where \( f(\theta) \) is given by
\[ f(\theta) = \frac{\sin^{13} \theta}{(1 + 3 \cos^{2} \theta)} \left( \sin^{2} \frac{\pi}{3.64} (1.62 - \cos \theta - \cos^{3} \theta) \right) \]

\[ \left( \exp \left( \frac{.5390}{\sin^{2} \theta} + .1634 \sin^{4} \theta - 2.77 \right) \right) \].

The integral \( I_3 \) was divided into 13 equal segments and evaluated numerically by Simpson's rule. The values of \( f(\theta) \) for each of 13 equally-spaced points are given in Table V. We find that \( I_3 \) is given by

\[ I_3 = 1.056 \times 10^2 / N_a R_o B_0^3 \approx 1 / N_a B_e^3 \quad (163) \]

Comparing Eq. (163) to Eqs. (156) and (157), we see that \( I_3 \) is larger than \( I_1 \) and \( I_2 \) by a factor of \( 10^5 \). Therefore, we can neglect \( I_1 \) and \( I_2 \) and set \( I = 2I_3 \) where the factor of 2 occurs because we have only integrated to \( \theta = \pi/2 \). From symmetry considerations the remainder of the integration is equal to \( I_3 \).

From Eq. (149), we can now calculate the oscillation frequency. Using the values of \( I \) and \( J \) derived above we write

\[ \omega^2 = \frac{\pi}{8I J} = \frac{\pi}{8B_0 R_0^3 \times 3.24 \times \frac{1.056 \times 10^2 \times 2}{R_0 B_0^3 N_a}} = \frac{\pi R_0^2 N_a}{1.372 R_0^2 \times 10^3 \times 8} \quad (164) \]
We can now write,

\[ \frac{\omega}{2\pi} = \nu = 1.449 \times 10^{-3} \text{ cycles/sec} \]  \hspace{1cm} (165)

The period \( \tau \) is \( \nu^{-1} \) and is given by

\[ \tau = \frac{1}{\nu} = \frac{10}{1.449} = 690 \text{ sec} \approx 11\frac{1}{2} \text{ minutes} \]  \hspace{1cm} (166)
2. DISCUSSION

The calculated period of oscillation, 11 1/2 minutes, is within an order of magnitude of the observed period of 6 minutes. Although this agreement may be accidental, it at least implies that we chose a trial function, \( \xi \), and a magnetosphere model which are good representations of reality. Since the frequency of oscillation is very sensitive to the charged-particle density, the hydrostatic-density distribution seems to be a good approximation to the real density distribution. Similarly, the assumption that the geomagnetic lines of force are rigidly tied at the ionosphere seems to be valid. If this constraint were lifted somewhat, that is, if the feet of the lines of force were able to move in a restricted manner, the frequency of oscillation would be lowered and the period would be raised. Of course if the feet could move freely, the magnetosphere would be unstable and would dissipate. The physical mechanisms for tying the lines of force is the high conductivity of the ionosphere, which in turn is coupled to the lower portions of the atmosphere by viscosity.

It therefore seems likely that the oscillation observed by Sugiura consists of an adiabatic motion of lines of force oscillating decoupled from one another.
### TABLE V.

Numerical Data to Evaluate $I$ in Eq.(161).

<table>
<thead>
<tr>
<th>Angle (deg.)</th>
<th>Density $N = N(\theta)$</th>
<th>$R/R_p$</th>
<th>$f(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$16.20 \times 10^2$ particles (cm$^3$)</td>
<td>1.40</td>
<td>$7.58 \times 10^{-4}$</td>
</tr>
<tr>
<td>35</td>
<td>9.74</td>
<td>1.84</td>
<td>$1.37 \times 10^{-2}$</td>
</tr>
<tr>
<td>40</td>
<td>7.02</td>
<td>2.31</td>
<td>$1.11 \times 10^{-1}$</td>
</tr>
<tr>
<td>45</td>
<td>5.65</td>
<td>2.80</td>
<td>$6.62 \times 10^{-1}$</td>
</tr>
<tr>
<td>50</td>
<td>4.86</td>
<td>3.29</td>
<td>2.82</td>
</tr>
<tr>
<td>55</td>
<td>4.40</td>
<td>3.76</td>
<td>9.06</td>
</tr>
<tr>
<td>60</td>
<td>4.13</td>
<td>4.20</td>
<td>$2.45 \times 10^1$</td>
</tr>
<tr>
<td>65</td>
<td>3.96</td>
<td>4.61</td>
<td>$5.66 \times 10^1$</td>
</tr>
<tr>
<td>70</td>
<td>3.87</td>
<td>4.94</td>
<td>$1.11 \times 10^2$</td>
</tr>
<tr>
<td>75</td>
<td>3.82</td>
<td>5.22</td>
<td>$1.88 \times 10^2$</td>
</tr>
<tr>
<td>80</td>
<td>3.81</td>
<td>5.43</td>
<td>$2.78 \times 10^2$</td>
</tr>
<tr>
<td>85</td>
<td>3.79</td>
<td>5.56</td>
<td>3.49</td>
</tr>
<tr>
<td>90</td>
<td>3.79</td>
<td>5.6</td>
<td>3.79</td>
</tr>
</tbody>
</table>
ACKNOWLEDGMENTS

The author wishes to express his utmost thanks to Dr. Allan N. Kaufman, who suggested the problems considered in this thesis, and who gave invaluable advice during all stages of its development. The author wishes to express particular thanks to his wife, Rae, whose patience and encouragement were a necessary ingredient in this work. The author appreciates the assistance given by Dr. William A. Newcomb and Dr. Theodore G. Northrop. He also appreciates the encouragement given by Dr. Wulf B. Kunkel.

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An energy principle is an expression for the second-order variation of potential energy of a well-defined equilibrium configuration under an infinitesimal displacement $\xi$ in the plasma. Since the plasma is assumed to be instatic equilibrium, the first-order variation of the potential energy is zero. There are, in general, two different types of energy principles: those based upon the macroscopic moments of the distribution function, and those based upon individual particle motion and the collisionless Boltzmann equation. The former include the hydromagnetic or Matterhorn energy principle and the Chew-Goldberger-Low energy principle. The latter include the Kruskal-Oberman energy principle and the Newcomb energy principle. A brief discussion of the requirements of the various equilibria treated by these energy principles is given below.

All of these energy principles can be written in the form

$$\delta W = \frac{1}{2} \int \xi \cdot F(\xi) \, dt,$$

where $F(\xi)$ is a self-adjoint operator. Because of the self-adjointness of $F$, it has real eigenvalues, hence $\delta W$ is real. As a consequence the phenomenon of overstability (growing oscillations) does not appear in this theory. The system is either stable or unstable with respect to a particular displacement $\xi_p$ depending upon the sign of the eigenvalue $\lambda_p$ where $F(\xi_p) = \lambda_p \xi_p$. If we choose a normalization
\[ \frac{1}{2} \int \xi_p^2 \, dt = 1, \text{ then we have } \delta W(\xi_p) = \frac{1}{2} \int [\xi_p \cdot F(\xi_p)] = \lambda_p. \]

What we do in this paper is to try to find that particular \( \xi_p \) which minimizes \( \delta W \). The sign of the corresponding \( \lambda_p \) then determines whether the system is stable. This \( \lambda_p \) is a function of the equilibrium parameters of the system. The requirement that \( \lambda_p \) be greater than zero is a stability criterion; i.e., it is a condition that the equilibrium parameters must satisfy if the configuration is to be stable.

The Matterhorn or hydromagnetic-energy principle considers a plasma in which collisions are frequent enough to maintain an isotropic pressure, and zero heat flow. In this approximation, the adiabatic law

\[ \frac{d}{dt}(p - \gamma) = 0 \]

is applicable. Additional assumptions are:

(a) Quasi-neutrality; i.e., \(|n_i - n_e| \ll n_i\), where \(n_i\) and \(n_e\) are the electron and ion densities, respectively. This assumption is valid when the Debye radius \( \lambda_D \) is infinitesimal in comparison to the characteristic length of change of macroscopic quantities, \( L \).

(b) Quadratic terms in \( v \) and \( j \) are negligible. This is valid when the macroscopic speed \( v \) is small compared to sound speed \( C_s = \sqrt{\gamma p/\rho} \) or to hydromagnetic speed \( C_K = \frac{B}{\sqrt{\rho}} \).

(c) The ratio of the electron mass \( m \) to the ion mass \( M \) is negligible in comparison with unity.

(d) The displacement current is negligible. This holds if \( v \) is small compared to the speed of light.

(e) Ohm's Law in the form of Eq. (12) is valid. Spitzer \(^{23}\) gives the complete generalized Ohm's Law which may be written in the form
The electron-inertia term \( \frac{m}{e^2} \frac{\partial j}{\partial t} \) is negligible when the characteristic frequency \( v_c \) is infinitesimal compared to the plasma frequency \( \omega_p = (ne^2/m)^{1/2} \). The ion-inertia term \( \frac{M}{e} \frac{\partial v_i}{\partial t} \) is negligible when \( v_i \) is infinitesimal compared to the ion Larmor frequency \( eB/M \). The electrical-resistance term \( \eta \frac{j}{\partial} \) is negligible when the time characteristic of relative diffusion of matter and magnetic flux is long compared to \( (v_c)^{-1} \). The term involving the ion-pressure gradient \( \frac{\nabla P_i}{ne} \) is negligible when \( r_L c_s/L \) \( u \ll 1 \), where \( r_L \) is the ion Larmor radius.

The Chew-Goldberger-Low energy principle considers a collisionless plasma. In this case the mean free path \( l \) goes to infinity, but \( r_L \) takes its place as a parameter, where \( r_L \) is the ion Larmor radius.

Thus it is assumed that

\[
r_L/L \ll 1
\]

and also

\[
\omega/\omega_i \text{ and } \omega/\omega_p \ll 1, \quad (A-1)
\]
where $\omega_i$ is the ion-cyclotron frequency, $\omega_p$ is the plasma frequency, $L$ is the characteristic length, and $\omega$ is the inverse-time scales.

The above statements are the defining equations of the guiding-center approximation. We also assume quasi-neutrality and that the ratio of electric field along the lines of force to the electric field perpendicular to the lines of force is of order $m/M \ll 1$. With these approximations it has been shown that the pressure tensor $\mathbf{P}$ can be written in the form

$$\mathbf{P} = \mathbf{P}_l + (\mathbf{P}_n - \mathbf{P}_l) \mathbf{e}_l \mathbf{e}_l,$$

where

$$\mathbf{P}_l = \sum_{\alpha} \frac{M_{\alpha}}{2} \int v_+^2 f d^3v$$

and

$$\mathbf{P}_n = \sum_{\alpha} m_{\alpha} \int v_+^2 f d^3v,$$

$I$ is the unit dyadic, and $\mathbf{e}_l$ is the unit vector along the lines of force. Making the additional restriction that no heat flow occurs in the course of the displacement, Chew, Goldberger and Low arrive at the adiabatic equations of motion,
\[
\frac{d}{dt} \left( \frac{P \cdot B^2}{\rho^3} \right) = 0
\]

and

\[
\frac{d}{dt} \left( \frac{P \cdot B}{\rho B} \right) = 0.
\]

Using these equations of motion, Maxwell's equations neglecting displacement current, and the force-law equation

\[
\frac{\rho}{\partial t} \frac{\partial \mathbf{v}}{\partial \mathbf{t}} = \mathbf{\nabla} \cdot \mathbf{F} + \mathbf{J} \times \mathbf{B},
\]

Chew, Goldberger and Low derive the expression for \( SW \) given by Eq. (70).

The Kruskal-Oberman and Newcomb energy principles deal with equilibria in which the guiding-center approximations of Eq. (A-1) are applicable. In these equilibria, the Debye length \( \lambda_D = \left( \frac{4\pi n e^2}{K T} \right)^{1/2} \) is very much less than \( L \) and also \( E_y/E_L \) is of order \( r_L/L \), and displacement current is neglected. As a consequence of the guiding center approximation, the motion of a particle is such that the drifts are of order \( r_L/L \); hence, the motion of the particle consists of a rapid gyration about the guiding center, and the guiding center is held to a given line of force. The \( \psi \) coordinate of the particle is
therefore a constant of the motion, where $\psi$ is the stream function defined in Appendix C. Also, the energy $\epsilon$ and magnetic moment $\nu$ are constants of the motion and the distribution function $f$ is independent of the azimuthal angle $\theta$ in coordinate space. From Liouville's theorem, the distribution function $f$ is constant along a particle trajectory. Therefore, $f$ is a function of the constants of the motion and we write $f = f(\epsilon, \nu, \psi)$. There is no heat flow in the equilibrium state, but there is for small displacements from equilibrium. To eliminate multistream types of instabilities, we assume that $\delta f/\delta \nu < 0$. This assumption implies zero heat flow in the equilibrium state, but not for small displacements from equilibrium.

These assumptions do not yield a closed set of equations governing the changes of the various moments of $f$. Instead, one must go back to the collisionless Boltzman equation and compute the variation of $f$ under a displacement $\delta f$. This variation in $f$ then furnishes the information to compute the variation in the matter-stress terms and the moments of $f$. As a result, the expression for $8W$ involves integrations over velocity spaces, as well as volume integrations. To make matters worse, averages defined by $F_{\text{avg}} = \int \frac{df}{q} F / \int \frac{df}{q}$, where $q = |v_\parallel|$, are also involved. These averages occur as a result of certain constraints on the system. It is the presence of these averages in $8W$ which made it desirable to use the approximation techniques discussed in Sec. II-C.

The difference between the Kruskal-Oberman treatment and the Newcomb energy principle is the concept of charge separation along a line of force. Newcomb considers the effect of a longitudinal electric
field of order $r_L/L$, derivable from a scalar potential $\phi$. The concept of charge neutrality, $0 = \sum e_\alpha \int_{\alpha} d^3v$, is considered to be a constraint on the distribution function.

This charge-neutrality constraint leads to an integral equation for the perturbed electrostatic potential $\hat{\phi}$ given by

$$\hat{\phi} = \hbar^2 S(e^2 \zeta_{\text{avg}}/q^2) - \hbar^2 S(e \lambda/q^2) ,$$  \hspace{2cm} (A-2)

where

$$\lambda = q^2 (\vec{e}_1 \cdot \vec{e}_1 : \nabla \zeta) + v_B (\nabla \cdot \zeta - \vec{e}_1 \cdot \nabla \zeta)$$

and

$$\zeta = \lambda - \frac{e}{\hbar} \hat{\phi} \quad \text{and} \quad 1/\hbar^2 = S(e^2/mq^2)$$

where the operator $S$ is defined in Eq. (5.9). For the case

$$f_\alpha (v, \epsilon, \psi) = G_\alpha (\epsilon, \psi) (vB_m/\epsilon - 1)^n$$

and

$$G_\alpha (\epsilon, \psi) = \text{const.} e^{-m\alpha \epsilon/\alpha},$$

we obtain
\[ \frac{1}{\hbar^2} = \sum_{\alpha} \frac{e^2}{m} \int B \, dv \, d\varepsilon \left/ q^2 \frac{\partial f}{\partial \varepsilon} \right| \ln \pi = \sum_{\alpha} n_\alpha \frac{e^2}{\theta_\alpha}, \]

(A-3)

where

\[ \theta_\alpha = \frac{KT}{\alpha} \]

and

\[ n_\alpha = 4\pi \int B \, \frac{dv \, d\varepsilon}{q_\alpha} f_\alpha. \]

Since we have

\[ n_e = n_1, \]

then we get

\[ \frac{1}{\hbar^2} = ne^2 \left| \frac{1}{\theta_i} + \frac{1}{\theta_e} \right|. \]

Then for

\[ \theta_i \gg \theta_e, \]
we have

$$\frac{1}{2} \frac{1}{\hbar^2} = \frac{ne^2}{\theta_e}$$

or

$$\hbar^2 = \frac{\theta_e}{ne^2}$$

For cold electrons therefore, we can neglect $\hbar^2$ and consequently $\hat{\phi}$. This justifies the statement made in Sec. III-B. Similarly, we get

$$\sigma = \mathcal{S}(e \nu B/q^2) = - \frac{4\pi}{\alpha} \sum_{\alpha} e_{\alpha} \int B \frac{dv}{q} \nu B \frac{d\epsilon}{d\epsilon}$$

$$= - \sum_{\alpha} e_{\alpha} \int d\epsilon \frac{d\nu}{d\epsilon} \frac{v}{\epsilon} B \int \frac{d\nu}{d\epsilon} \left( \frac{v B_m}{\epsilon} - 1 \right)^n$$

$$= F\left(\frac{B}{B_m}\right) \sum_{\alpha} e_{\alpha} \frac{\theta_{\alpha}}{m_{\alpha}} \frac{3/2}{m_{\alpha}}$$

where

$$F\left(\frac{B}{B_m}\right) = \frac{4\pi}{2 \Gamma(n + 3/2)} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2}\right) \left(1 - \frac{B_m}{B}\right)^{(n-1)} \left(1 - \frac{B_m}{B}\right)^{n+1/2}$$

Hence, for
\[ \frac{\theta_1}{m_1} \approx \frac{\theta_e}{m_e}, \]

\( \sigma \) is very small and we can neglect \( \sigma^2 n^2 \) in comparison with \( C \).

The contribution to \( 8W \) from the terms involving charge separation is positive definite. Hence the Newcomb \( 8W \) is greater than or equal to the Kruskal-Oberman \( 8W \). The reason for this is that Newcomb has an additional constraint on the distribution function, reducing the class of admissible perturbations of the system. In fact, Newcomb proves the following inequalities:

\[ 8W_{\text{hydromagnetic}} \leq 8W_{\text{KO}} \leq 8W_{\text{Newcomb}} \leq 8W_{\text{CGL}}, \]

where the inequality on the left is valid only for isotropy. Thus, if we are looking for a necessary condition for stability, it is natural to consider \( 8W_{\text{CGL}} \), while for a sufficient condition, a lower limit to \( 8W_{\text{KO}} \) or \( 8W_{\text{Newcomb}} \) is sought.
B. EXAMINATION OF EQUATION (142)

Here we evaluate

\[ I' = \int_0^L \frac{P_1' + P_1'}{r RB} \sin^2 \frac{\pi l}{L} \, dl = \text{const. } I' \]

where

\[ I' = \int_{-a}^a \frac{f(z) P(z)}{\left| 1 - a \cos \frac{\pi z}{a} \right|^2} \, dz ; \quad f(z) = \cos \frac{\pi z}{2a} \left( 3 + 4\alpha + \alpha \cos \frac{\pi z}{a} \right) \]

and

\[ p(z) = \left[ - \cos \frac{\pi z}{a} \left| 1 - \alpha \cos \frac{\pi z}{a} \right|^{-1} + \frac{3}{2} \alpha \sin^2 \frac{\pi z}{a} \left| 1 - \alpha \cos \frac{\pi z}{a} \right|^{-2} \right]. \]

Then we have

\[ I' = - (3 + 4\alpha)I_1 = \alpha I_2 + \frac{3}{2} \alpha(3 + 4\alpha)I_3 + \frac{3}{2} \alpha^2 I_4, \]

where

\[ I_1 = \int_{-1}^1 dz \frac{\cos^3 \frac{\pi z}{2} \cos \frac{\pi z}{2}}{\left| 1 - \alpha \cos \frac{\pi z}{a} \right|^3}, \]
We first consider $I_1$. From symmetry about $z = 0$, and by transforming variables, we obtain

$$I_1 = 2 \int_0^1 dz \frac{\cos^3 \pi z \cos \pi z}{(1 - \alpha \cos \pi z)^3} = \frac{2}{\pi} \int_0^\pi dy \frac{\cos^3 y \cos y}{(1 - \alpha \cos y)^3}$$

$$= \frac{4}{\pi} \int_0^{\pi/2} dx \frac{\cos^3 x \cos x}{(1 - \alpha \cos 2x)^3}$$

$$= \frac{4}{\pi} \frac{\pi/2}{(2\alpha)^3} \int_0^{\pi/2} dx \cos x \frac{(1 - \sin^2 x)(1 - 2 \sin^2 x)}{(1 - \alpha^2 + \sin^2 x)}$$
By setting $u = \sin x$, and using $\alpha = \frac{M - 1}{M + 1}$, which implies

$$\frac{1 - \alpha}{2 \alpha} = \frac{1}{M - 1},$$

we get

$$I_1 = \frac{1}{2\pi} \left( \frac{M + 1}{M - 1} \right)^3 \int_0^{\frac{1}{M - 1}} \frac{(1 - u^2)(1 - 2u^2)}{(1 + u^2)^2} du \left( 1 - u \right) \left( 1 - 2u \right)$$

$$= \frac{1}{2\pi} \left( \frac{M + 1}{M - 1} \right)^3 \left( J_1 - 3 J_2 + 2 J_3 \right).$$

Similarly, we get

$$I_2 = \frac{1}{2\pi} \left( \frac{M + 1}{M - 1} \right)^3 \left( J_1 - 5 J_2 + 8 J_3 - 4 J_4 \right),$$

$$I_3 = \frac{1}{\pi} \left( \frac{M + 1}{M - 1} \right)^4 \left( K_2 - 2 K_3 + K_4 \right),$$

and

$$I_4 = \frac{1}{\pi} \left( \frac{M + 1}{M - 1} \right)^4 \left( K_2 - 4 K_3 + 5 K_4 - 2 K_5 \right).$$
where:

\[ J_1 = \int_0^1 \frac{du}{\left( \frac{1}{M-1} + u^2 \right)^3} \]

\[ = \frac{(M-1)^3}{8M^2} (3M+2) + \frac{3}{8} (M-1)^{5/2} \tan^{-1} (M-1)^{1/2} , \]

\[ J_2 = \int_0^1 \frac{du u^2}{\left( \frac{1}{M-1} + u^2 \right)^3} \]

\[ = \frac{(M-1)^2}{8M^2} (M-2) + \frac{1}{8} (M-1)^{3/2} \tan^{-1} \sqrt{M-1} , \]

\[ J_3 = \int_0^1 \frac{du u^4}{\left( \frac{1}{M-1} + u^2 \right)^3} \]

\[ J_4 = \int_0^1 \frac{du u^6}{\left( \frac{1}{M-1} + u^2 \right)^3} \]

\[ = \frac{1}{8M^2} (8M^2 + 9M - 2) - \frac{15}{8} (M-1)^{-1/2} \tan^{-1} \sqrt{M-1} , \]

\[ K_2 = \int_0^1 \frac{du u^2}{\left( \frac{1}{M-1} + u^4 \right)^3} \]
\[
\frac{(M - 1)^3}{48M^3} (3M^2 + 2M - 8) + \frac{(M - 1)^{5/2}}{16} \tan^{-1}\sqrt{M - 1},
\]

\[
K_3 = \int_0^1 \frac{d u u^4}{(\frac{1}{M - 1} + u^2)^4}
\]

\[
= \frac{(M - 1)^2}{48M^3} (3M^2 - 14M + 8) + \frac{(M - 1)^{3/2}}{16} \tan^{-1}\sqrt{M - 1},
\]

and

\[
K_5 = \int_0^1 \frac{d u u^8}{(\frac{1}{M - 1} + u^2)^4}
\]

\[
= \frac{1}{48M^3} (48M^2 + 87M^2 - 38M + 8) - \frac{35}{16} (M - 1)^{-1/2} \tan^{-1}\sqrt{M - 1}.
\]

with these substitutions, we can write

\[
(3 + 4\alpha)I_1 + \alpha I_2 = 3 + 4 \frac{(M - 1)}{M + 1} I_1 + \frac{M - 1}{M + 1} I_2
\]
\begin{align*}
\frac{1}{M+1} \left[(7M-1) I_1 + (M-1) I_2\right] &= \frac{1}{2\pi (M-1)^2} (3M^2 - 8M - 7) \\
&\quad + \frac{1}{2\pi} \left(\frac{M+1}{M-1}\right)^{5/2} (3M^2 - 10M + 17M + 2) \tan^{-1} \sqrt{M-1}.
\end{align*}

Similarly, we get
\begin{align*}
I_3 &= \left(\frac{M+1}{M-1}\right)^{1/2} \frac{1}{\pi} \frac{M-1}{4\sqrt{M}} (3M^2 - 10M - 8) \\
&\quad + \frac{1}{16} \left(\frac{M+1}{M-1}\right)^{1/2} (M^2 - 4M + 8) \tan^{-1} \sqrt{M-1}
\end{align*}

and
\begin{align*}
I_4 &= \frac{M+1}{M-1} \frac{1}{\pi} \frac{1}{4\sqrt{M}} (3M^2 - 19M^2 - 186M - 8) \\
&\quad + \frac{1}{16} (M-1)^{-1/2} (M^3 - 7M^2 + 36M + 40) \tan^{-1} \sqrt{M-1}.
\end{align*}
Then, we have

\[
\frac{3}{2} \alpha^2 I_4 + \frac{3}{2} \alpha(3 + 4\alpha) I_3 = \left[ \frac{M + 1}{M - 1} \right] \frac{1}{8\pi} (6M^2 - 24M - 58)
\]

\[
+ \frac{3}{8\pi} \frac{(M + 1)^2}{(M - 1)^{5/2}} (2M^3 - 9M^2 + 24M + 8) \tan^{-1}\sqrt{M - 1}.
\]

Finally we have

\[
I' = -(3 + 4\alpha)I_1 - \alpha I_2 + \frac{3}{2} \alpha(3 + 4\alpha) I_3
\]

\[
= - \left[ \frac{M + 1}{M - 1} \right] \frac{1}{8\pi} (6M^2 - 8M + 30) + \frac{(M + 1)^2}{(M - 1)^{5/2}}
\]

\[
(6M^3 - 3M^2 - 4M - 16) \tan^{-1}\sqrt{M - 1}
\]

This is the expression substituted into Eq. (142).
C. DISCUSSION OF THE \((x, \psi, \theta)\) AND THE \((x, \psi, \theta)\) COORDINATE SYSTEMS

We have defined \(\psi = r A_\theta(r, z)\), where the poloidal magnetic field \(B\) is given by \(\nabla \times (\mathbf{A}_\theta A_\theta) = -\frac{1}{r} \mathbf{A}_\theta \times \nabla \psi\). It follows from this definition that \(B \cdot \nabla \psi = 0\). Thus, the lines of force lie in the surface \(\psi = \text{constant}\) and in the planes \(\theta = \text{constant}\). The magnetic flux \(\phi\) interior to the surface \(\psi = \text{constant}\) is given by

\[
\phi = 2\pi \int_{r'=0}^{r'=r} B_z(r', z) r' \, dr'.
\]

From \(B = -\frac{1}{r} \mathbf{A}_\theta \times \nabla \psi\), it follows that \(B_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}\) and \(B_z = \frac{1}{r} \frac{\partial \psi}{\partial r}\). Hence we have

\[
\phi = 2\pi \int \frac{\partial \psi}{\partial r'} \, dr' = 2\pi \left[\psi(r, z) - \psi(0, z)\right].
\]

Choosing the constant of integration as \(\psi(0, z) = 0\), we find that the flux interior to the surface \(\psi = \text{constant}\) is given by \(2\pi \psi\).

We use \(\psi\) and \(\theta\) as coordinates and introduce a function \(X(r, z)\) whose level surfaces are perpendicular to the surfaces \(\psi = \text{constant}\) and \(\theta = \text{constant}\). We choose \(X\) so that the set \((X, \psi, \theta)\) forms a right-handed coordinate system. The volume element in this coordinate system is \(d\tau = J \, dx \, d\psi \, d\theta\), where \(J = 1 |\nabla X| = \nabla \psi \cdot \nabla \theta \times \nabla \psi\).
The gradient operator in the \((X, \psi, \theta)\) coordinate system is given by

\[
\nabla = |\nabla X| e_X \frac{\partial}{\partial X} + |\nabla \psi| e_\psi \frac{\partial}{\partial \psi} + |\nabla \theta| e_\theta \frac{\partial}{\partial \theta}
\]

\[
= \frac{1}{JB} e_X \frac{\partial}{\partial X} + rB e_\psi \frac{\partial}{\partial \psi} + \frac{1}{r} e_\theta \frac{\partial}{\partial \theta}
\]

where

\[
\xi = \frac{\nabla X}{|\nabla X|} , \quad \xi_\psi = \frac{\nabla \psi}{|\nabla \psi|} , \quad \xi_\theta = \frac{\nabla \theta}{|\nabla \theta|} .
\]

We often use the notation \(\xi_1, \xi_2\), and \(\xi_3\) in place of \(\xi_X, \xi_\psi\), and \(\xi_\theta\). We often use the coordinate \(x\) in place of the coordinate \(X\). The coordinate \(x\) in the hydromagnetic model is not the same as in the collisionless models. In the hydromagnetic model, \(x\) is defined by the equation \(dx = dX J \rho_\perp B^2\). In the collisionless models, \(x\) is defined by \(dx = \frac{dX J \rho_\perp B^2}{1 - \sigma_-}\), where \(\sigma_- = \frac{P_n - P_L}{B^2}\). We should emphasize that \(1 - \sigma_-\) is always greater than zero in our plasma models, since \(0 < 1 - \sigma_-\) is the condition that the plasma be stable against the "firehose" mode of displacement.
We start with Eq. (70) for $8W$:

$$8W = \frac{1}{2} \int d\tau \left[ q^2 - \nabla \cdot \xi \times \tilde{q} + \frac{5}{2} P_1 (\nabla \cdot \tilde{\xi})^2 
+ (\tilde{\xi} \cdot \nabla P_1) \nabla \cdot \tilde{\xi} + \frac{1}{2} P_1 (\nabla \cdot \tilde{\xi} - \tilde{z} \cdot \tilde{e}_1 \cdot \nabla \tilde{\xi})^2 
+ (\tilde{e}_1 \cdot \nabla \tilde{\xi}) (\nabla \cdot \tilde{\xi} P_1) 
+ P_1 [4 (\tilde{e}_1 \cdot \nabla \tilde{\xi})^2 
+ (\tilde{\xi} \cdot \nabla \tilde{e}_1 - \tilde{e}_1 \cdot \nabla \tilde{\xi}) \cdot (\nabla \tilde{\xi} \cdot \tilde{e}_1 + \tilde{e}_1 \cdot \nabla \tilde{\xi})] \right].$$

(D-1)

We then write, using the $(X, \psi, \theta)$ coordinate system:

$$\nabla \cdot \nabla = \frac{1}{J} \frac{\partial}{\partial x} \frac{V_1}{B} + \frac{\partial}{\partial \psi} (V_2 r JB) + \frac{\partial}{\partial \theta} \frac{V_3}{r} \quad (D-2)$$

$$\nabla \times \nabla = \varepsilon_1 \left[ B \frac{\partial}{\partial \psi} r \frac{V_3}{rB} - \frac{\partial}{\partial \theta} \frac{V_2}{rB} \right] + \varepsilon_2 \left[ \frac{1}{rJB} \frac{\partial}{\partial \theta} V_1 JB - \frac{\partial}{\partial x} r \frac{V_3}{r} \right]$$

$$+ \varepsilon_3 \left[ \frac{r}{J} \frac{\partial}{\partial x} \frac{V_2}{rB} - \frac{\partial}{\partial \psi} V_1 JB \right] \quad (D-3)$$

where $V_1 \equiv V_x$, $V_2 \equiv V_\psi$, $V_3 \equiv V_\theta$. 
We now have, using Eq. (D-2) and Eq. (D-3)

\[ Q = \nabla \times (e_1 \times B) = -e_1 \left[ \frac{B}{\psi} r B e_2 + \frac{\partial}{\partial \psi} \frac{e_3}{r} \right] + e_2 \left[ \frac{1}{r B} \frac{\partial}{\partial x} r B e_2 \right] \\
+ e_3 \left[ \frac{r}{J} \frac{\partial}{\partial x} \frac{e_3}{r} \right], \quad (D-4) \]

\[ \nabla \cdot \xi = \frac{1}{J} \left[ \frac{\partial}{\partial x} \frac{e_1}{B} + \frac{\partial}{\partial \psi} r B \frac{e_2}{J} + \frac{\partial}{\partial \psi} \frac{e_3}{r} \right], \quad (D-5) \]

and

\[ e_1 e_1 : \nabla \xi = (e_1 \cdot \nabla \xi) \cdot e_1 = -\frac{e_2}{R} + \frac{1}{J B} \frac{\partial}{\partial x} e_1 \cdot \xi. \quad (D-6) \]

Noting that \( J = J e_3 \), we write

\[ \xi \times B = J B e_3 \times e_2 = J B e_2 = \nabla \cdot \xi \]

\[ = \nabla \cdot P + \nabla \cdot (P e_1 e_1) = e_2 \left( r B \frac{\partial P}{\partial \psi} + \frac{P}{R} \right) \]

\[ + e_1 \left[ \frac{1}{J} \frac{\partial}{\partial x} \frac{P}{B} + \frac{1}{J B} \frac{\partial P}{\partial X} \right]. \]

We therefore have
\[ j = r P_1' + \frac{P_1'}{RB}, \]  

(D-7)

where the prime denotes \( \frac{\partial}{\partial \psi} \),

\[ 0 = \frac{1}{B} \frac{\partial P_1}{\partial x} + \frac{\partial}{\partial x} \frac{P_1}{B} \]  

(D-8)

We can also write

\[ \xi \times \zeta : j = \xi \times \zeta \cdot \varepsilon_3 j = (\xi_1 Q_2 - Q_1 \xi_2) j \]

= \( \left( r P' + \frac{P_1'}{RB} \left[ \frac{\xi_1}{rb} \frac{\partial r b}{\partial x} \xi_2 + b \frac{\partial r b}{\partial \psi} \xi_2 + \frac{\partial \xi_2}{r} \right] \right) \)  

(D-9)

Similarly, we write

\[ P_1 (\xi \cdot \nabla \xi_1 - \xi_1 \cdot \nabla \xi) \cdot (\nabla \xi \cdot \xi_1 + \nabla \xi) \]

= \( P_1 \left[ - 2 \left( \frac{\xi_2}{r} - \frac{1}{JB} \frac{\partial}{\partial x} \xi_1 \right)^2 - \left( \frac{r}{r b} \frac{\partial \xi_2}{\partial x} \right)^2 - \left( \frac{1}{r b^2} \frac{\partial r b}{\partial x} \right)^2 \right] \)
Now we consider the last term in the right side of the above equation. The integral of this term becomes after an integration by parts,

\[- \frac{\xi_1}{R} \frac{1}{rJB^2} \frac{\partial}{\partial x} rB \xi_2 - \frac{1}{JB} \frac{\partial}{\partial \psi} \xi_1 \frac{\partial}{\partial x} rB \xi_2 \]

\[- \left[ \frac{\partial}{\partial \theta} \xi_1 \right] \frac{1}{JB} \frac{\partial}{\partial x} \frac{\xi_3}{r} \right]. \quad (D-10)\]

Also, the next to last term becomes, after an integration by parts,

\[- \int J d\psi d\theta \xi_1 \frac{1}{rJB^2} \frac{\partial}{\partial \psi} \frac{\xi_3}{r} \]

\[- \int d\chi d\psi d\theta \frac{P - \frac{P}{B}}{\psi} \frac{\partial}{\partial x} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial x} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \psi} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \theta} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \psi} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \theta} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \psi} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \theta} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \psi} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \theta} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \psi} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \theta} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \psi} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \theta} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \psi} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \theta} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \psi} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \theta} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \psi} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \theta} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \psi} \frac{\xi_3}{r} \]

\[- \int d\theta d\chi d\psi \xi_1 \frac{P}{B} \frac{\partial}{\partial \theta} \frac{\xi_3}{r} \]
By defining a new function

\[ U = \frac{\partial}{\partial \psi} rB \xi_2 + \frac{\partial}{\partial \psi} \frac{\xi_2}{r} , \]

and by using the above results, Eq. (D-10) becomes

\[ \int J \, d\mathbf{x} \, d\mathbf{v} \, d\theta \, P_2(\mathbf{v} \cdot \mathbf{e}_1 - \mathbf{e}_1 \cdot \nabla \xi) \cdot (\nabla \cdot \mathbf{e}_1 + \mathbf{e}_1 \cdot \nabla \xi) \]

\[ = \int J \, d\mathbf{x} \, d\mathbf{v} \, d\theta \, P_2 \left[ - \left( \frac{1}{rJB} \frac{\partial}{\partial \mathbf{x}} rB \xi_2 \right)^2 - \left( \frac{r}{JB} \frac{\partial}{\partial \mathbf{x}} \frac{\xi_2}{r} \right)^2 \right] \]

\[ - \frac{\xi_1}{R} \left( \frac{1}{rJB} \frac{\partial}{\partial \mathbf{x}} rB \xi_2 \right) \]

\[ - 2 \left[ \frac{\xi_2}{R} - \frac{1}{JB} \frac{\partial}{\partial \mathbf{x}} \frac{\xi_1}{r} \right]^2 \]

\[ + \frac{\xi_1}{J} \left( \frac{\partial}{\partial \mathbf{x}} rB \xi_2 \right) \frac{\partial}{\partial \psi} \frac{P_{-}}{B} - \frac{U}{J} \frac{\partial}{\partial \mathbf{x}} \left[ \frac{\xi_1}{r} \frac{P_{-}}{B} \right] \]

\[ (D-11) \]

The integral

\[ \int \mathbf{d}x \, d\psi \, d\theta \, J \left( \mathbf{e}_1 \cdot \mathbf{e}_1 : \nabla \xi \right) \left( \nabla \cdot \xi \right) \cdot P_{-} \]

becomes, by using
\[
\frac{1}{R} = - \frac{r}{J} \frac{\partial}{\partial y} JB
\]

and Eqs. (D-5) and (D-6)

\[
\int d\xi d\psi \frac{d}{d\theta} J \left( I_{s2} \xi_1 : \nabla \xi \right) (\nabla \cdot \nabla P_\perp)
\]

\[
= \int d\psi d\xi d\theta J \left[ -\xi_2 \frac{P_\perp}{r} \frac{\partial}{\partial y} rB \xi_2 - \xi_2^2 rB \left( -\frac{P_\perp}{rRB} + B \frac{\partial}{\partial y} \frac{P_\perp}{B} \right) \right]
\]

\[
+ \int d\psi d\xi d\theta J \left[ -\frac{\xi_2}{r} \frac{\partial}{\partial y} \frac{\xi_1^2 P_\perp}{B} - \frac{\xi_2^2 P_\perp}{R} \right]
\]

\[
\left( \frac{\partial}{\partial y} \frac{\xi_2}{r} \right) + \frac{UP_\perp}{JB} \frac{\partial}{\partial x} \xi_1 + \left[ rB \xi_2 \frac{\partial}{\partial x} \xi_2 \right]
\]

\[
\left( -\frac{P_\perp}{rJRB^2} + \frac{1}{J} \frac{\partial}{\partial y} \frac{P_\perp}{B} \right) + \left[ \frac{1}{J} \frac{\partial}{\partial x} \xi_1 \left( \frac{\partial}{\partial x} \frac{\xi_1 P_\perp}{B} \right) \right]
\]

\( (D-12) \)

Similarly, we can also write

\[
\int J d\xi d\psi d\theta \left( \nabla \cdot \xi_1 \right) P_\perp \nabla \cdot \xi = \int J d\xi d\psi d\theta \left[ -\frac{U_\perp \xi_1}{J} \frac{\partial}{\partial x} \frac{P_\perp}{B} \right]
\]

\[
+ rB \xi_2 U_\perp P_\perp + rB \frac{\xi_2 P_\perp}{J} \frac{\partial}{\partial x} \xi_1
\]

\( (Eq. (D-13) cont.) \)
Adding Eqs. (D-9), (D-11), (D-12) and (D-13) we obtain

\[
\int d\mathbf{\psi} d\mathbf{\theta} J \left( \varepsilon_1 \varepsilon_1 : \nabla \cdot \varepsilon_1 \right) + \left( \varepsilon_1 \cdot \nabla \varepsilon_1 - \varepsilon_1 \cdot \nabla \varepsilon_1 \right) + \nabla \cdot \varepsilon_1 \nabla \cdot \varepsilon_1 \right)
\]

\[
= \int d\mathbf{\psi} d\mathbf{\theta} J \left[ -2 U \left( \frac{\partial}{\partial \mathbf{\psi}} \frac{\mathbf{P}}{\mathbf{B}} \right) + \left( r \mathbf{B} \varepsilon_2 \right)^2 \left( \mathbf{P}_B \cdot \frac{\partial}{\partial \mathbf{\psi}} \log J \right) \frac{1}{\mathbf{B}} \left( \frac{\partial}{\partial \mathbf{\psi}} \frac{\mathbf{P}}{\mathbf{B}} \right) \right]
\]

\[
- \frac{\varepsilon_1^2}{J^2} \left( \frac{\partial}{\partial \mathbf{\psi}} \frac{\mathbf{P}}{\mathbf{B}} \right) \left( \frac{\partial}{\partial \mathbf{\psi}} \frac{\mathbf{P}}{\mathbf{B}} \right) - \mathbf{P}_B \left( \frac{1}{r \mathbf{B}^2} \frac{\partial}{\partial \mathbf{\psi}} \frac{\mathbf{B}}{\mathbf{B}} \varepsilon_2 \right)^2
\]

\[
- \mathbf{P}_B \left( \frac{\varepsilon_2}{R} - \frac{1}{R \mathbf{B} \frac{\partial}{\partial \mathbf{\psi}}} \right) + \left( 2r \mathbf{B} \varepsilon_2 \mathbf{P}_B \cdot \mathbf{U} \right)
\]

\[
- \frac{\varepsilon_1}{R \mathbf{J}} \frac{\partial}{\partial \mathbf{\psi}} \frac{\mathbf{P}}{\mathbf{B}} + \frac{1}{J} \frac{\partial}{\partial \mathbf{\psi}} \left( r \mathbf{B} \varepsilon_2 \frac{\varepsilon_1}{\mathbf{B}} \right) + \left( r \frac{\varepsilon_2}{J} \frac{\partial}{\partial \mathbf{\psi}} \log J \right) \frac{1}{\mathbf{B}} \left( \frac{\partial}{\partial \mathbf{\psi}} \frac{\mathbf{P}}{\mathbf{B}} \right)
\]

Equation (D-14)
\[ \frac{1}{R} = -r \frac{\partial}{\partial \psi} J B \]

and Eq. (D-8), and by integrating by parts, the last four terms in Eq. (D-14) become

\[
\int J \, d\alpha \, d\psi \, d\theta \left[ \frac{x_1}{R} \frac{x_2}{B} \frac{\partial}{\partial \psi} \frac{P}{B} + \left( \frac{\partial}{\partial \psi} rB \right) \frac{x_1}{x_2} \frac{\partial}{\partial \psi} \frac{P}{B} \right.
\]

\[
+ \frac{P}{x_1} \frac{\partial}{\partial \psi} \left( \frac{x_2}{B} \right) \left( \frac{x_1}{x_2} \right) \frac{\partial}{\partial \psi} \log J \left( \frac{\partial}{\partial \psi} \frac{P}{B} \right) \]

\[
= \int J \, d\alpha \, d\psi \, d\theta \left[ 2x_1 x_2 \left( \frac{\partial}{\partial \psi} \frac{P}{B} \right) \frac{\partial}{\partial \psi} \right]. \quad \text{(D-15)}
\]

Finally, using Eqs. (D-14), (D-15) and (D-6), we have

\[
\int \, d\alpha \, d\psi \, d\theta \, J \left[ \xi_1 \xi_1 : \nabla \xi \cdot (\xi \cdot \frac{\partial}{\partial \psi} P) + (\xi \cdot \nabla \xi_1 - \xi_1 \cdot \nabla \xi) \right.
\]

\[
\cdot (\nabla \xi \cdot \xi_1 + \xi_1 \cdot \nabla \xi) P_1 + \frac{1}{2} \cdot \xi \times \xi
\]

\[
+ (\xi \cdot \nabla P_1) \nabla \cdot \xi + 4 P_1 (\xi_1 \xi_1 : \nabla \xi)^2 \right].
\]
\[
\int d\mathbf{x} d\psi d\theta \left[ -2 U \frac{\xi_1}{J} \frac{\partial P}{\partial \mathbf{x}} - \frac{\partial^2 P}{\partial \mathbf{x}^2} + 2 |U \mathbf{rB} \xi_2| P_1' \right] \\
- \left( \frac{\xi_1^2}{J^2} \frac{\partial P}{\partial \mathbf{x}} \right) \frac{\partial}{\partial \mathbf{x}} \frac{1}{B} - (rB \xi_2)^2 \left( P_1' \frac{\partial}{\partial \psi} \log J \right) \\
- \frac{1}{rR} \frac{\partial P}{\partial \psi} - P_1' \left( \frac{1}{rJ B^2} \frac{\partial}{\partial \mathbf{x}} \frac{rB \xi_2}{\mathbf{rB}} \right)^2 \\
- P_1' \left( \frac{r}{J B} \frac{\partial \xi_3}{\partial \mathbf{r}} \right)^2 + 3P_1' \left( \frac{\xi_2}{R} \frac{\partial}{\partial \mathbf{x}} \xi_1 \right) \\
+ \frac{2}{J} \left( \frac{\xi_1}{B} \frac{rB \xi_2}{\mathbf{rB}} \right) \left( \frac{\partial P}{\partial \mathbf{x}} \frac{\partial}{\partial \psi} \right). 
\]

We similarly obtain, by using Eqs. (D-5) and (D-6)

\[
\int d\mathbf{x} d\psi d\theta \left[ \frac{P_1}{3} \left( \nabla \cdot \xi - 3 \xi_1 \xi_2 : \nabla \xi \right)^2 + \frac{5}{3} P_1 (\nabla \cdot \xi)^2 \right] \\
= \int d\mathbf{x} d\psi d\theta \left[ 2 P_1 \left[ U + rB \xi_2 \frac{\partial}{\partial \psi} \log J \right] + \frac{\xi_2 U}{R} + \frac{\xi_2}{R} \left( \frac{\partial}{\partial \psi} \log J \right) \\
+ 2 U \frac{\xi_1}{J} \frac{\partial}{\partial \mathbf{x}} \frac{1}{B} + 2 \left[ rB \xi_2 \right] \frac{\partial}{\partial \mathbf{x}} \log J \left( \frac{\xi_1}{J} \frac{\partial}{\partial \mathbf{x}} \frac{1}{B} \right) + U \frac{\partial}{\partial \mathbf{x}} \xi_1 \\
+ rB \xi_2 \frac{\partial}{\partial \psi} \log J \left( \frac{\partial}{\partial \mathbf{x}} \xi_1 \right) + \frac{\xi_1 \xi_2}{R J} \frac{\partial}{\partial \mathbf{x}} \frac{1}{B} + \frac{\xi_2}{R J B} \left( \frac{\partial}{\partial \mathbf{x}} \xi_1 \right) 
\] 

( Eq.(D-17)cont. )
Adding Eqs. (D-16) and (D-17), and using Eq. (D-4), we obtain for $SW$

\[
SW = \frac{1}{2} \int d\theta d\psi \, dx \, J \left[ \left( 1 - \frac{P}{B^2} \right) \frac{1}{r B} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \left( \frac{r B}{P} \right) \frac{P}{B^2} \frac{r B}{P} \frac{\partial}{\partial x} \right]^2
\]

\[
+ (B^2 + 2P_1) U^2 + 2UB^2 \left[ \frac{r B}{P} \frac{\partial}{\partial x} \left( \frac{r B}{P} \right) \frac{P}{B^2} - \frac{r B}{P} \frac{\partial}{\partial x} \left( \frac{r B}{P} \right) \frac{P}{B^2} \right]
\]

\[
+ \frac{\xi_1}{B} \frac{\partial}{\partial x} \left( \frac{P}{B^2} \right) + \frac{P_1}{B} \frac{\partial}{\partial x} \left( \frac{\xi_1}{B} \right) + \left| \frac{r B}{P} \frac{\partial}{\partial x} \left( \frac{r B}{P} \right) \frac{P}{B^2} \right|^2 \left[ 2P_1 \frac{\partial}{\partial x} \log J \right]^2
\]

\[
+ \frac{2P_1}{r B} \frac{\partial}{\partial x} \log J + P_1 \frac{\partial}{\partial x} \log J - \frac{1}{r B} \frac{\partial}{\partial x} \frac{P}{B}
\]

\[
+ \frac{3P_1}{r^2 R B^2} \] + \[ 3P_1 \left( - \frac{2r B \frac{\xi_2}{B} \frac{\partial}{\partial x} \frac{\xi_1}{B} \frac{\partial}{\partial x}}{r R B^2 J} + \frac{2P_1}{r B} \frac{\xi_1}{B} \frac{\partial}{\partial x} \frac{\xi_1}{B} \frac{\frac{\partial}{\partial x}}{B} \right) ^2
\]

\[
+ 3P_1 \left( \frac{1}{B} \frac{\partial}{\partial x} \frac{\xi_1}{B} \right) ^2 + \left( \frac{\xi_1}{B} \frac{\partial}{\partial x} \frac{P_1}{B} \right) \left( \frac{\partial}{\partial x} \frac{1}{B} \right)
\]

We consider now the terms involving $U$. We first set $X = rB \frac{\xi_2}{B}$ and $Z = \frac{\xi_1}{B}$ and by completing the square in $U$, we obtain
\[(B^2 + 2P) U^2 + 2 UB^2 \left( x \frac{\partial P}{\partial \psi B^2} - \frac{X P}{r RB^3} + \frac{Z}{J} \frac{\partial P}{B^2} + \frac{P}{J B^3} \frac{\partial P}{\partial X B^2} \right) \]

\[= \left[ B^2 + 2P \right] \left[ U + \frac{B^2}{B^2 + 2P} \left( x \frac{\partial P}{\partial \psi B^2} - \frac{X P}{r RB^3} + \frac{Z}{J} \frac{\partial P}{B^2} + \frac{P}{J B^3} \frac{\partial P}{\partial X B^2} \right) \right]^2 \]

\[\frac{B^2}{B^2 + 2P} \left( x \frac{\partial P}{\partial \psi B^2} - \frac{X P}{r RB^3} + \frac{Z}{J} \frac{\partial P}{B^2} + \frac{P}{J B^3} \frac{\partial P}{\partial X B^2} \right)^2 \]

\[\text{(D-19)}\]

We employ Eq. (D-19), and collect terms to transform Eq. (D-18) for \(SW\) into Eq. (71).
REFERENCES


FIGURE CAPTIONS

1. Plot of $B_c$ versus $M$ and $A$ for hydromagnetic sufficiency condition.
2. Plot of $B_c$ versus $M$ for Newcomb sufficiency conditions, $A$ infinitesimal, vacuum fields.
3. Plot of $B_c$ versus $M$ for necessity conditions; $A$ infinitesimal.
4. Comparison of $B_c$ versus $M$ for hydromagnetic and Newcomb sufficiency conditions.
5. Hydromagnetic model of a plasma.
7. Dipole field line.
Fig. 2.

\[ f(\nu, \epsilon, \psi) = g(\epsilon, \psi)(\nu \frac{B_m}{\epsilon} - 1)^5 \]

\[ f(\nu, \epsilon, \psi) = g(\epsilon, \psi) \]
Hydromagnetic necessity condition; $\nabla \times B = \nabla p$

Chew-Goldberger-Low necessity condition; $\nabla \times B = 0$
Hydromagnetic equilibrium; $A$ infinitesimal, $\nabla \times B = \nabla p$

Guiding-center equilibrium; $A$ infinitesimal, $\nabla \times B = 0$
Fig. 5.

$\beta_c^n$ from necessity condition

$\beta_c^s$ from sufficiency condition
Fig. 6.

$\beta_c^n$ from necessity condition

$\beta_c^s$ from sufficiency condition

$\beta_c$ vs $M$
Fig. 7.
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