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Toward a Reconceptualization of Mathematical Learning Disabilities: A Focus on Difference Rather Than Deficit

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Toward a Reconceptualization of Mathematical Learning Disabilities:
A Focus on Difference Rather Than Deficit

by

Katherine Elizabeth Lewis

A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Education in the Graduate Division of the University of California, Berkeley

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Professor Geoffrey Saxe
Professor Susan Schweik

Fall 2011
Toward a Reconceptualization of Mathematical Learning Disabilities:
A Focus on Difference Rather Than Deficit

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Katherine Elizabeth Lewis
Abstract

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Students with mathematical learning disabilities (MLDs) experience persistent challenges learning even the most elementary mathematics. While prior research on MLDs has classified students on the basis of test scores and documented performance differences between groups, this dissertation focuses on the qualitative differences in individual students with MLDs as each student attempted to learn. This study extends the understanding of MLDs by (1) focusing on the mathematical domain of fractions, (2) selecting for study students whose difficulties with mathematics are clearly due to an MLD and not other factors, (3) conducting detailed diagnostic analyses of video-taped tutoring sessions, and (4) contrasting the difficulties that the students with MLDs experience to data collected with five typically achieving fifth grade control students. The two case study students, “Lisa” (19-year-old community college student) and “Emily” (18-year-old high school student) each participated in a pretest, four weekly tutoring sessions, and a posttest focused on fraction concepts. Achievement test scores, interviews, and videotaped tutoring session data were used to establish that Lisa and Emily met classic MLD qualifications, and more stringent response-to-intervention criteria. Both students demonstrated unexplained persistent low math achievement and neither student benefited from a tutoring protocol that had been effective for typically achieving fifth grade students. The tutoring sessions were analyzed using microgenetic methods. Analysis indicated that each student relied upon a unique collection of atypical understandings, which reoccurred across the sessions, were resistant to standard instructional approaches, and proved to be highly consequential for the student’s ability to understand more complex fraction concepts. A cross case analysis revealed surprising similarities in the atypical understandings displayed by both case study students. These atypical understandings stemmed from and contributed to the student’s inability to conceptualize and manipulate representations of fractional quantity. These atypical understandings were not similarly problematic for the fifth grade control students, but did
appear in one additional student with an MLD. This suggests that there are qualitative differences in the difficulties experienced by students with MLDs and that it may be possible to design screening measures to identify these indicators of atypicality. In addition, remediation approaches should take into account and specifically target these atypicalities.
This dissertation is dedicated to the students who made this work possible, "Lisa," "Emily," and "Taylor."
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Chapter 1: Introduction

“It’s like a different language, that’s how I explain it to everyone, It’s like I’m reading Arabic, but I don’t speak Arabic.”

This is how Lisa, a 19-year-old college student, describes her experience trying to learn math. She still counts on her fingers to solve addition problems, cannot add fractions, and has just failed an arithmetic course. Despite her academic success in all non-mathematical domains, she continues to struggle with even the most elementary mathematics. No amount of tutoring or extra effort seems to address her difficulties. Her difficulties with math are localized and exceptional. In short, she has a mathematical learning disability (MLD).

Unfortunately, research on mathematical learning disabilities (MLDs) offers little insight to help understand why students like Lisa experience persistent, pervasive, and debilitating difficulties with mathematics. Even simply defining an MLD is inherently problematic, in that an MLD is understood to be “a biologically based, behaviorally defined condition for which no consensus definition currently exists” (Mazzocco, 2007 p. 44). Although the root cause of MLDs is thought to originate in biological differences in the brain, researchers have tended to identify and study these differences through behavioral approaches (e.g., performance on a written assessment). This has, in general, led to the non-specific behavioral marker of “low math achievement” being used as a proxy for MLDs. Low-achievement, although an indicator that the student might have an MLD, is not sufficient to establish that his/her difficulties stem from a cognitive origin.

Consequently, many questions about MLDs have yet to be answered. Specifically:

• How are the difficulties experienced by students with MLDs qualitatively different than those experienced by all students?
• What are the underlying causes of these persistent difficulties?
• What, if any, similarities exist in how MLDs manifest across students?
• How can researchers and practitioners best understand, identify, and remediate these difficulties?

These unanswered questions are the focus of my dissertation.

In this dissertation I conduct detailed diagnostic analyses of students with MLDs. The purpose of this research is to begin to understand and characterize the kinds of difficulties students experience and the underlying causes for those difficulties. I selectively recruited a small number of students and drew upon various data sources to ensure that their low achievement was due to persistent and debilitating difficulties with mathematics that could not be attributed to non-cognitive sources. I collected videotaped data of six individual tutoring sessions with students focused on the representationally and conceptually rich topic of fractions. The detailed diagnostic analysis of each student focused on identifying the reasons underlying the student’s learning difficulties. Each of the students with an MLD demonstrated atypical understandings, which were not evident in the fifth grade control students used to validate the tutoring protocol. These atypical understandings proved to be both persistent and detrimental to their ability to learn more complex fraction concepts. These understandings were fundamentally tied to atypical

1 Quote taken from initial interview (10/27/2008) with “Lisa,” a case study student.
ways of conceptualizing and representing fractional quantities, and provide a first step towards identifying the underlying causes of mathematical learning disabilities. For one of the case study students, a remediation was designed and implemented, which attempted to address her specific difficulties and also build upon her atypical understandings. Preliminary analysis suggests that it may be possible to design alternative instructional approaches that are more accessible for students with MLDs.

The contributions of this dissertation include: (1) an understanding of how MLDs manifest in the under-explored mathematical domain of fractions, (2) the identification of potential diagnostic indicators, which is a first step towards the development of valid diagnostic and classification tools, and (3) a new framework for the study and analysis of MLDs, which involves novel identification methods, detailed diagnostic analyses, and remediation approaches informed by the diagnostic analysis.

In this introductory chapter, I review the research on MLDs and identify four methodological challenges facing the research of MLDs. I identify the ways in which these challenges are typically addressed in MLDs research and the limitations of these approaches. I conclude this chapter by drawing upon prior work on disabilities and math cognition, to construct an alternative approach to the study of MLDs.

Prior Research on Mathematical Learning Disabilities Research

Research on MLDs is in its infancy, as compared to research on more general learning disabilities (Fletcher, Lyon, Fuchs, & Barnes, 2007; Gregoire & Desoete, 2008; Mazzocco, 2007). Much of the prior research on MLDs has focused on students’ performance on basic arithmetic calculation problems. Literature reviews synthesizing MLDs research have consistently found that the MLDs are best characterized by insufficient automaticity of arithmetic number facts (i.e., “4+5=9”; Gersten, Jordan, & Flojo, 2005; Swanson & Jerman, 2006; Swanson, 2007). Two primary hypotheses have been posited about the causes of the difficulties: domain-general versus domain-specific processing problems. On the domain-general side of the debate, insufficient working memory, is thought to be a primary contributor to the difficulties that students with MLDs experience (Geary, Hoard, Nugent, & Bryd-Craven, 2007; Geary, Bailey, Littlefield, Wood, Hoard, & Nugent, 2009; Passolunghi & Cornoldi, 2008; Swanson, Jerman & Zheng, 2008). In contrast, other researchers argue that the role of working memory has been overemphasized (Butterworth & Reigosa, 2007; Landerl, Bevan, & Butterworth, 2004; Jordan, 2007), and that a domain-specific number module in the brain is responsible for the difficulties that students with MLDs experience (Butterworth, 2005). Recently, difficulties specific to representation and manipulation of quantities, more generally referred to as “number sense,” have been gaining support in both the educational and neurological field as underlying MLDs. Although “number sense” may be one causal factor, much work remains to be done to establish the ways in which number processing difficulties would occur over time and across topic domain.

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2 Some tasks used to assess working memory (e.g., backwards digit-span) involve numerical prompts, and therefore, may not be measuring the individual’s working memory, but his/her ability to process and remember numerical information.
Currently, little is known about what causes MLDs, how to identify students with MLDs, and how to remediate the difficulties such students experience (Mazzocco, 2007; Swanson, 2007). One of the primary reasons that more progress has not been made in this field is due to the methodological challenges involved in studying a phenomenon that is as yet undefined.

**Methodological Challenges.** The study of MLDs is fraught with methodological challenges. Current challenges facing the field include: (1) how to identify students with MLDs, (2) how best to study learning disabilities in a hierarchical topic domain, like mathematics, (3) how to capture individual differences and subtypes of MLDs, and (4) how to interpret differences between students with the purpose of establishing and refining the definition of MLDs. In this section I review how prior research on MLD has dealt with these methodological challenges. Although these approaches have grown out of logical choices and a historical tradition in language-based learning disabilities research, I identify limitations of these traditional approaches. I conclude this section, by proposing an alternative approach to the study of MLDs, which provides a novel perspective on this complex topic.

**Challenge 1: Identifying students with MLDs.** Subject identification is the primary methodological issue facing researchers studying MLDs. Because there are no diagnostic assessments to accurately identify students with MLDs, low math achievement is typically used as a proxy. It is common practice to use achievement test scores below a given cut-off (typically the 25th percentile) to identify students with MLDs (Murphy, Mazzocco, Hanich, & Early, 2007; Geary, 2004). Although this approach likely does include most students with an MLD, it may include many others without MLDs. Various issues with this approach have been identified.

First, the use of “low mathematics achievement” to identify students with MLDs is problematic, because a wide range of factors (motivation, poor instruction, language fluency) might cause low scores on mathematics tests. Disentangling low achievement due to social or environmental factors from low achievement due to an MLD is a daunting task. Given the complexity of the environmental factors correlated with low-math achievement, researchers rarely attempt to control for these factors (Lewis, 2006). Hanich, Jordan, Kaplan, and Dick (2001) report that studies of MLDs have an over-representation of minority, poor, and non-native English speaking students in the MLD group. This suggests that the proxy of low achievement does not adequately address the complex social components at play in the operational definition of MLDs.

In addition, there is a lack of consistency in the cut-off score used (Francis et al., 2005). The cut off score can range from the 8th percentile to the 46th percentile (Gersten, Clarke, & Mazzocco, 2007; Swanson & Jerman, 2006). For example, recent studies on MLDs have used the 11th percentile (Geary, Hoard, Nugent, Byrd-Craven, 2008), 16th percentile (De Smedt, Holloway, & Ansari 2011), and the 26th percentile (Fuchs et al. 2010), as cut-offs to identify students with MLDs. The inconsistency in how MLDs are operationally defined has led to variability in the profile of students classified as having MLDs, suggesting that researchers are not all studying a common phenomenon (Murphy et al., 2007).

Given the lack of control within studies and the variability between studies, it can be argued that researchers are studying general characteristics of low math achievement
rather than MLDs. To make progress in defining and understanding MLDs, a subject identification approach is needed that can differentiate low-achievement due to cognitive differences from low-achievement due to other factors.

**Challenge 2: Defining the mathematical scope.** Mathematics is generally viewed as a hierarchical topic domain, in that more complex mathematics builds upon more basic levels of mathematics. In general, an individual must learn to count before he/she can add, and learn to add before he/she can multiply. Because mathematics is seen as hierarchical, researchers have logically begun investigating MLDs at the most basic level. Consequently, much of the research on MLDs has focused on elementary school students engaged in basic arithmetic calculation. Although this approach has resulted in documenting that students in the MLD group have difficulty in storing and retrieving basic number facts (Swanson, 2007; Geary, 2010, Fuchs et al., 2010), it remains unclear what difficulties would occur for students with MLDs in more advanced mathematical domains (Dowker, 2005; Geary, 2005).

The few studies that have attempted to extend the understanding of MLDs by investigating a more advanced mathematical domain and have found dramatically different results. Unlike research focusing on whole numbers, conceptual and representational issues were central findings of these studies. Hecht & Vagi (2010) found that with fractions, conceptual understanding (as measured by an open response question to an area model addition problem) was the primary factor explaining both individual and group differences for students with MLDs. They specifically note that number fact errors were not the predominant reason that students struggled with fraction problems, suggesting that the well-documented number fact difficulties were not the central factor for students with MLDs engaged in fraction problems. Similarly, Mazzocco and Devlin (2008) found representational issues were present when they analyzed students’ ordering and comparing fractions and decimals. They found that students with MLDs spent more time and made more errors on problems that involved rank ordering using an area model representation than either the typically achieving students or low achieving students. Both these studies suggest that conceptual and representational issues, not insufficient recall of basic number facts, were at the heart of the difficulties experienced by the students with MLDs. This suggests that MLDs in more complex mathematical domains may bear little resemblance to the difficulties experienced by students with basic arithmetic calculation. To provide a more complete and nuanced understanding of MLDs, mathematical topic domains that involve conceptual, procedural, and representational understandings should be investigated.

**Challenge 3: Capturing Individual differences.** Another methodological concern is how best to elicit and capture aspects of MLDs. Traditionally research on MLDs has attempted to identify what differentiates students in the MLD group from typically achieving students, the premise being that dimensions of difference between the groups are thought to provide insight into the character of MLDs. Building upon the tradition of language-based learning disabilities research, large-scale quantitative analyses have been employed to compare the groups on various experimental measures (Baglieri, Valle, Connor, & Gallagher, 2011), often documenting the student’s response time and accuracy. Although statistically significant group differences are often identified, these differences should be carefully interpreted.
Given that the group membership is determined by achievement test scores, it is unsurprising that differences are found between groups on measures like calculation error rate or response time. Students who score low on a math achievement test may have done so precisely because of inefficient and inaccurate calculation strategies. Therefore, the finding that students with MLDs have difficulty remembering and recalling basic math facts, is likely an artifact of the achievement tests initially used to classify the students, rather than a unique marker of an MLD. In interpreting group differences it is necessary to consider what might be directly correlated with the subject identification criterion.

Additionally, there is some concern that aggregated analysis of the groups might not be sensitive enough to identify the heterogeneous profile of students included in the MLD group (Mazzocco, Devlin, & McKenney, 2008). It has long been assumed that there were subtypes of MLDs with different cognitive profiles (Geary, 1994). Therefore, even if those scoring below the 25th percentile were students with MLDs, various subtypes of MLDs potentially contribute to the inability to draw consistent and reliable conclusions (Swanson, 2007). There is some evidence to suggest that the previously proposed subtypes do not well match empirical data (Mazzocco, 2009). Therefore, an aggregated analysis of group factors may be inappropriate at this stage to study the complexities of MLDs. Analytic approaches that simply document statistically significant group differences and assume those differences are defining and consequential features of MLDs are insufficient to capture individual differences, potential subtypes, and nuances of MLDs. An analytic approach, sensitive to individual differences, is necessary to begin to more deeply explore the profiles of students with MLDs.

**Challenge 4: Interpreting differences.** The final challenge has to do with how the differences found between groups are interpreted by researchers. Consistent with the historical tradition of research on language-based learning disabilities, MLDs are conceptualized as resulting from cognitive deficits (Geary, 2004). The goal of much of the research on MLDs is to document performance differences between groups and then assert that those differences were due to the deficits that characterize MLDs. Although there may be deficiencies in performance between the MLD group and the typically achieving group, it is problematic and unproductive to conclude that these differences reflect a deficit within the individual.

Reliance on the “deficit” model is fundamentally problematic because it involves defining a person in negative terms. Identifying an absence of something necessitates that researchers identify what an individual should have. In studies of MLDs, the “typically achieving” control group serves as a proxy for the “norm.” Both the presumption of a “norm” and the practice of classifying some individuals as “deficient,” have been widely criticized by disability studies scholars (e.g., Gallagher, 2004). Davis (2006) argues that the concept of “normalcy” has long been used as a way of classifying some individuals as deficient with respect to race, class, and gender, and currently it is employed to “create the ‘problem’ of the disabled person” (p. 3).

Although a deficit model has been used in the past to explain differences in performance between groups of students, it was rejected by Cole and Bruner (1971) over forty years ago, and has been systematically rejected in almost all domains of educational research (e.g., Lee, 2010; Moschkovich, 2010). Understanding differences in culture,
language, or social class, through the lens of deficits, essentializes groups of students and is ultimately unproductive. However, the deficit perspective remains the predominant theoretical lens through which researchers study learning disabilities and MLDs (Baglieri et al. 2011). To make progress in defining MLDs, we must identify what MLDs are rather than what they are not.

**Summary.** MLDs are inherently difficult to study. Prior research on MLDs following from theoretical and methodological traditions employed in language-based learning disabilities research has provided limited insight into the nature of MLDs. The traditional methodological approaches have not provided the needed traction to make headway into understanding this complex phenomenon. Subject identification methods, core to the study of MLDs, are unable to differentiate low achievement from MLDs. The focus on basic calculation provides only a narrow slice of mathematics and has neglected many conceptual, procedural, and representational aspects of what it means to do mathematics. The aggregated analyses are insensitive to individual differences and provide insight into general characteristics of low math achievement rather than fundamental characteristics of MLDs. Finally, an understanding of MLDs in terms of cognitive deficits, results in an understanding of MLDs as intractable defects within an individual, rather than a difference that should be recognized and accommodated. To address these critiques, I draw upon literature from a variety of fields including: theoretical perspectives on disability, innovative response-to-intervention learning disabilities research, and math cognition research, all which contribute to my proposed alternative theoretical and methodological approach for the study of MLDs.

**Reconceptualizing Mathematical Learning Disabilities**

In the remainder of this chapter I address the critiques of prior MLDs research by adopting an alternative theoretical and methodological approach for this dissertation. First, I propose abandoning the traditional “deficit” model of MLDs, and reconceptualizing MLDs in terms of cognitive “difference,” drawing on a Vygotskian perspective of disability. Second, to address the inherent difficulties with subject identification, I adopt Fletcher, et al.’s (2007) learning disability identification model, which attempts to differentiate low achievement from learning disabilities. Third, to extend research on MLDs beyond the study of automaticity of number facts, I explore MLDs within the mathematical domain of rational number (informally, fractions). Finally, to address the issues that arise from large-scale quantitative analyses, which are insensitive to individual differences, I propose a qualitative methodological approach that attempts to capture the nature of the student’s difficulties as they arise during a sequence of tutoring sessions.

**Alternative theoretical perspective on disability.** Instead of pathologizing human cognitive difference with respect to a non-existent idealized norm\(^3\), disability can be more productively understood in terms of natural human variation (Davis, 2006). This perspective does not deny that there may exist a biologically-based difference, but

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\(^3\)Because the terms “normal” and “non-normative” is so implicitly evaluative, I use “typical” and “atypical”, which is more neutral, to be able to contrast the difficulties experienced by students with MLDs to those typically experienced when learning math.
proposes that people are disabled not by this biological difference, but by the meaning that society makes of the difference, and the structures which limit that individual’s access (Baglieri et al., 2010). From this perspective, people who are Deaf are not disabled by their inability to hear, but by a society which understands this difference in negative terms, privileges an auditory means of communication and does not provide the individuals with access to a visual language system (Ladd, 2003; Lane, 1992; Lane, Hoffmeister, Bahan, 1996). In the case of MLDs, students may naturally process or manipulate numbers in atypical ways due to cognitive differences (supported by recent fMRI research, see Price, Holloway, Rasanen, Vesterinen, & Ansari, 2007), but the ways in which society pathologizes and labels this learning disability and our inability to provide instruction in a way which renders the mathematical content accessible, is disabling (Campione, 1989; Gallagher, 2004).

Disability scholars most often reject the biologically-based difference as the focus of analysis. Similarly, some learning disability researchers suggest that defining and delineating the student’s cognitive difference is an inappropriate and researchers should work to displace the structures that perpetuate the social construction of disability (e.g., schooling, testing, etc.; Gallagher 2004; McDermott & Varenne 1995). Unfortunately, overhauling the culturally engrained understandings of disability, while a laudable goal, is socially intractable⁴, and an unproductive way of addressing the immediate needs of individuals with learning disabilities. Instead I adopt an alternative approach, which does not pathologize the individual and which accepts the individual’s natural biological difference and attempts to better understand it for the purpose of providing access. In the same way that, for a person using a wheelchair, a ramp can provide access to an otherwise inaccessible physical location, for students with MLDs alternative instructional approaches should be designed to create conceptual access to an otherwise inaccessible mathematical terrain.

A Vygotskian perspective on disability provides a productive alternative for conceptualizing MLDs. Vygotsky claimed that a student who has a disability “is not simply a child less developed than his peers but is a child who has developed differently” (Vygotsky, 1929/1993, p. 30). The individual’s biological differences result, not in deficient development, but a different path of development. Recent MLDs research supports this idea by noting that there appear to be qualitative differences in the error patterns of students with MLDs as compared to low achieving students (Mazzocco, Devlin, & McKenney, 2008, Lewis, 2010). Differential rather than deficient development therefore, appears to be a productive way of conceptualizing MLDs.

Given the assumption that students with MLDs do not have deficient, but different paths of developments, the analytic focus shifts from documenting what students lack to what students have. This perspective shifts the focus from pathologizing student differences, to attempting to capture them. Researchers should expect students with MLDs to have developed differently, often in atypical ways, and analysis should be sensitive enough to capture these differences. Consequently, atypicality is a central theoretical construct of this dissertation, and will be discussed in greater depth with

⁴ Although scholars do attempt to address these issues, this is outside the scope and bounds of this dissertation.
respect to analytic methods. Although Vygotsky’s conceptions of disability date back almost a century, they provide a yet unleveraged theoretical alternative to the study of MLDs. This perspective provides a productive alternative to the dominant deficit-based theoretical model.

**Alternative MLD identification method.** Unlike the traditional large-scale studies, which rely solely upon achievement tests to classify students as either having MLDs or typically achieving, this dissertation leverages advances made in language-based learning disabilities research to ensure that the student’s mathematical difficulties are due to an MLD and not other factors. I applied a model of learning disabilities identification, proposed by Fletcher et al. (2007), to the study of MLDs, which requires that students demonstrate (1) low achievement (for my purposes, in mathematics), (2) no evidence of confounding factors explaining that low achievement, and (3) lack of response to an instructional intervention (Fletcher et al., 2007). The goal in subject selection was not to identify all students with MLDs, but to ensure that all students included in this study have an MLD.

Low achievement, typically established using a norm-referenced test or a comparison to class peers, was a necessary but not a sufficient criterion for an MLD classification⁵. Using the criterion of low-achievement (below the 25th percentile) not only suggested that the student may have an MLD but enabled me to maintain comparability to other studies of MLDs. Therefore, all students selectively recruited and included in this study would be categorized in the MLDs group for the majority of other studies of MLDs.

In addition, subjects classified as having MLDs for this study must not have exhibited evidence of other factors that could explain their low achievement. Those factors highly correlated with low math achievement were considered confounding factors for the purposes of this study. These included: attention or behavior problems, lack of English fluency, insufficient resources, math anxiety (Ashcraft, Krause, & Hopko, 2007; Chatterji, 2005; Diversity in Mathematics Education, 2007; Zentall, 2007)⁶. The purpose of these exclusionary criteria was to ensure that cognitive differences and not other affective factors were the cause of the student’s low achievement.

Finally, the student must not have benefited from a standard instructional intervention. In an attempt to control for the effect of prior instruction, learning disabilities “are identified only after a specific attempt is made to systematically instruct the person” (Fletcher et al., 2007, p. 65). If the student did show gains from pretest to posttest, it suggested that prior instruction might have been insufficient and an underlying cause of the student’s low achievement. If the student did not show gains from pretest to

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⁵ It is worth noting that it is possible that a student with an MLD might have sufficiently compensated for his/her difficulties. I explore this further in chapter 5, when I draw upon data collected with Taylor, a student with an MLD who has compensated so effectively that she is majoring in statistics. The low-achievement criteria is used to maintain comparability with other studies of MLDs.

⁶ I do not assume that MLD cannot co-occur with these factors, but that disentangling the effect of the confounding factor from the effect of MLD was beyond the scope of this study, and therefore these subjects were excluded from the study.
posttest, this suggested that poor instruction was not the primary cause of the student’s low achievement and indicated that the student has an MLD.

This learning disabilities identification model is longitudinal, relies on multiple data sources, and attempts to differentiate low achievement from learning disabilities, therefore addressing the fundamental subject identification issues which currently hinder MLDs research.

**MLDs in a conceptually and representationally rich mathematical domain.** Because prior work on MLDs has been almost exclusively limited to students’ calculation difficulties, I aimed to extend this understanding by exploring a more advanced mathematical domain. First, I considered the student’s use of representations, which is central to an understanding of mathematics. Second, I focused on a more authentic problem solving domain: fractions, which require conceptual, procedural, and representational knowledge.

**Importance of representations.** As noted above, prior research on MLDs has focused almost exclusively on student’s lack of mastery of basic number combinations. However, mathematics is about more than efficient and accurate calculation. Fundamental to an understanding of mathematics is an ability to understand and manipulate different mathematical representations (Ball, 1993; Kaput, 1987). Doing mathematics involves mastering a host of representations (e.g., symbols, diagrams, pictures), to procedurally and conceptually navigate a mathematical context. Prior research on MLDs has largely ignored the importance of representations, focusing primarily on the 10 numerical digits and four basic operator symbols. Representations are more than simply one aspects of mathematics, they are absolutely core to what it means to do and understand mathematics. Any investigations mathematical cognition, whether related to MLDs or not, must fundamentally engage with representations.

Although representations are central to what it means to do and understand mathematics, a representation itself does not contain meaning. Rather the meaning of the representation is a result of the interpretation of the user (Von Glasserfeld 1987; Meira, 1998). Representations are not transparent and unambiguous holders of meaning; instead the meaning must be imbued upon the representation. Because of this inherent subjectivity involved in the use of representations, how a student with an MLD perceives and operates upon a representation might be quite different than how students typically perceive and operate on that representation. For example, prior research on students understanding of basic numerical symbols, has suggested that students with MLDs may have difficulty processing both symbolic forms (e.g., “7”; De Smedt & Gilmore, 2011; Landerl & Kolle, 2009) and non-symbolic forms (e.g., “●●●●●●●●” of numerical magnitude (Piazza et al., 2010). Therefore, research considering the students

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7 Although some researchers have dealt with representations, their treatment is relatively superficial. For example, Geary, Hoard, Nugent, Byrd-Craven (2008) “teach” the number line conventions to students with one example, and then use the number line as a means of interpreting students understanding of numerical magnitude. The number line is not treated as a complex mathematical representation that requires considerable effort and time to learn (see Saxe et al., (2010) for and example of the complexities of number line representations).
use of representations should attempt to identify atypicalities with respect to students understanding of both symbolic (e.g., “3/4”) and non-symbolic (e.g., area models) representations. *Atypical understandings* of symbolic and non-symbolic mathematical representations are therefore, the central analytic construct in this dissertation and a productive focal area for the exploration of MLDs, particularly beyond basic math facts.

**Fractions.** Although a student’s use of representations could be explored in any mathematical domain, I chose to focus on the expansively researched and mathematically rich topic of fractions. A fraction, more generally referred to as a rational number, is a number that can be expressed as a ratio a/b, when a and b are integer values, and b is not equal to zero. A fraction is just one notational form used to refer to the underlying rational number. I use the term “fraction” rather than rational number because it represents the predominant notational form used during the tutoring sessions, and “fraction” is also more commonly used pedagogically when students first encounter this topic. When I refer to a fraction as a number, I am actually referring to the underlying rational number (see Lamon, 2007 for a similar treatment of the terms “rational number” and “fraction”).

For a number of reasons, fractions provide a mathematically rich terrain to explore how students with MLDs make sense of mathematical concepts. Unlike basic math facts, mastery of fractions cannot be distilled down into an automatic retrieval of the correct answer. Fractions are conceptually, procedurally, and representationally more complex than whole numbers. Students must learn that the value of the fraction is determined by the coordination of the numerator and denominator, which means that numbers that look bigger, may actually be smaller (e.g., 17/100< 1/2); that there are multiple ways to write any fraction value (e.g., 1/2, 2/4, 3/6, etc.); and that there is no next rational number. Problems with fractions often require the execution of multi-step procedures, where intermediate equivalent fraction values must be recorded as the student solves even the most basic fraction operation problem (e.g., 1/2 +1/3=). In addition, students must master a variety of representations (both symbolic and non-symbolic) used visualize, manipulate, and make sense of rational numbers (Lamon, 1996). Developing competency with fractions involves procedural, conceptual, and representational understandings, and consequently, a potentially richer and more authentic mathematical domain than basic number operations.

The selection of fractions as the mathematical domain of study was a logical choice given the extensive research base investigating the teaching and learning of fractions. Although rational numbers can take on a variety of interpretations, including: part-whole, quotient, ratio, operator, and measure (Behr, Harel, Post, & Lesh, 1992), this study was consciously limited to a part-whole understanding of fractions. The rationale behind this decision was: the part-whole understanding is the most commonly used interpretation for introducing fraction concepts, it allows for a multitude of

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8 Also note that that not all numbers written in fractional form are rational numbers (e.g., \( \frac{1}{\sqrt{2}} \)). In this study all fractions used are rational numbers, and the numerator and denominator values are almost always natural numbers.
representational forms to be explored, and it is the most thoroughly researched with respect to common student misconceptions. Commonly documented student difficulties include: confounding rational number and whole number reasoning (Mack, 1990), a tendency to rely on perceptual judgments rather than part-whole reasoning (Armstrong & Larson, 1995; Kamii & Clark, 1995), and failure to attend to representational conventions, like equal partitioning or the size of the whole when comparing fractions (Armstrong & Larson, 1995; Saxe, Taylor, McIntosh, & Gearhart, 2005). Several studies have shown that instruction can productively build on students’ informal knowledge of fair sharing to develop students’ part-whole understanding, but that rote memorization of procedures often causes difficulties (Lamon, 1996; Mack, 1990). There is extensive research into the kinds of difficulties typically achieving students experience when learning fraction concepts (see Lamon, 2007, for a review). It is therefore possible to classify the difficulties demonstrated by students with MLDs as typical or atypical. Considering MLDs in the context of fractions allows for the exploration of atypical procedural, conceptual, and representational difficulties that students with a MLD may experience in this more complex mathematical domain.

**Alternative analytic approach.** Unlike prior research on MLDs, which tend to employ quantitative evaluations of group differences, I employ an individual qualitative case study approach, focusing on the nature of the student’s atypical understandings. To provide a more nuanced understanding of MLDs, I leverage methodologies widely used in research on mathematical cognition, which provide approaches for eliciting authentic student reasoning and understanding: clinical interviews and teaching experiments.

**Clinical interview.** The clinical interview is particularly suited to revealing how a student is reasoning at a given point in time (Ginsburg 1997; Steffe, Thompson, & Von Glaserfeld, 2000). The clinical interview, inspired by Freud, Piaget, and Vygotsky, involves an interviewer asking the subject to respond to a set of predetermined tasks, with the goal of evoking the subject’s natural conception of a topic of interest. The clinical interview is more flexible than written tests in that it allows for flexible exploration of the student’s understanding of a topic based on the kinds of answers and explanations that the student gives. This is important because research has found that people have knowledge structures that do not match the canonical structure of the topic domain (Schoenfeld, Smith, & Arcavi, 1993; Smith, diSessa, & Roschelle, 1993). The clinical interview methodology is intended to elicit and explore the authentic structure of an individual’s knowledge (Clement, 2000). Consequently, a clinical interview is particularly well-suited to the study of MLDs, where the individual cognitive differences will likely result in atypical understandings and ways of thinking. This assessment technique, used for both the pretest and posttest, is sensitive to individual differences, and therefore partially addresses the issues with prior MLD research.

**Tutoring sessions.** Although clinical interviews are an excellent mechanism for determining how a student is reasoning at one point in time, they are insufficient to capture the dynamic process of learning. Building off the work on dynamic assessment (Campione, Brown, & Lidz, 1987) and teaching experiments (Saxe et al., 2010; Steffe & Thompson 2000), I explore MLDs in a one-on-one tutoring environment, as the student is in the process of learning. As with both teaching experiments and dynamic assessment, I am fundamentally concerned with engaging and documenting the student’s ways of
thinking and changes in the student’s understanding. The goal of the tutoring sessions is to capture in-the-moment how a student’s MLD manifests and is detrimental to the student’s attempts to learn. The clinical interview style pretest and posttest along with four tutoring sessions serve as the primary data source for this study.

**Microgenetic Analysis.** The goal of the analysis was to identify how MLDs occur as the student is in the process of attempting to learn. Recall that the students included in this study must not have shown gains from pretest to posttest. The purpose of the detailed analysis was to attempt to understand *why* the students did not learn. All six sessions (pretest, four tutoring sessions, and posttest) were analyzed using microgenetic analysis (Schoenfeld et al., 1993). Microgenetic analysis involves observing a subject over a period of time and documenting in fine-grained detail the qualitative changes that constitute learning. My analysis focused on the persistent difficulties that arise during the course of learning and the kinds of atypical understandings that the student displays. The atypical understandings were not the canonical understanding of the topic domain and were not common misconceptions identified in prior research on rational numbers. The analytic focus was on identifying and refining the definitions of these atypical understandings. The goal was to determine what, beyond the canonical misconceptions, resulted in the student’s learning difficulties. Much of the analytic work involved iteratively refining the definitions for various atypical understandings for each case. This kind of analytic work is well aligned with grounded theory (Glaser & Strauss, 1967), where repeated passes through the data enable a researcher to begin to iteratively develop codes and classifications.

The clinical interview, tutoring sessions, and microgenetic analysis methodologies are particularly well suited for eliciting and documenting alternative conceptions that the student may have about the topic domain. Consequently, these methodological approaches were particularly apt for the study of students with MLDs, where one would expect the student’s cognitive differences to result in atypical ways of processing or understanding mathematics.

**Conclusion**

In this dissertation I propose an alternative approach to the study of MLDs, to address the challenges facing research on MLDs. I reconceptualize MLDs in terms of cognitive differences rather than deficits, which shifts the analytic focus from documenting what students cannot do, to documenting how they understand mathematics and why they experience difficulties. The goal of this research is to begin to identify the root causes of MLDs through an individual detailed analysis of carefully recruited students, who have difficulties in mathematics that could not be explained by non-cognitive factors. This will provide a more nuanced understanding of the ways in which MLDs manifest during a student’s learning of a conceptually, procedurally, and representationally rich mathematical topic. In this dissertation I rely upon an alternative theoretical and methodological approach, essentially reconceptualizing MLDs.

**Structure of the Dissertation**

In this introductory chapter, I have provided a critique of prior MLDs research and drawn on various bodies of research to characterize my alternative approach to the
study of MLDs. In chapter 2, I expand upon these ideas and present the methods used in this dissertation research. Chapters 3-5 are analytic chapters focused on the diagnostic analyses of several students. Chapter 3 is a detailed case study of Lisa, a community college student with an MLD. Analysis indicates that she held five different atypical understandings of fraction concepts, which explained almost all her difficulties during the tutoring sessions, and which proved to be detrimental to her ability to learn more advanced fraction concepts. Chapter 4 is a detailed case study of Emily, a senior in high school with a MLD. She relied upon six atypical understandings. Like Lisa, these atypical understandings explain almost all her difficulties and were problematic in her attempts to learn. For both Emily and Lisa, their atypical understandings were fundamentally associated with their ability to represent and manipulate fractional quantities. In chapter 5 I highlight the similarities and differences between Lisa’s and Emily’s cases, contrast their atypical understandings with the fifth grade control students, and draw upon data from one other student with an MLD. These findings suggest that these atypical understandings were at the heart of the difficulties these students experience, and suggest that it may be possible to use evidence of these atypical understandings as potential diagnostic indicators of MLDs. This dissertation concludes by considering the implications of these findings for screening measures and remediation approaches. This dissertation shows that it is possible to conceptualize MLDs in terms of cognitive difference, identify the atypical and sometimes problematic understandings that the students rely upon, and design instruction that honors and builds upon those understandings.
Chapter 2: Methods

Overview

The goal of this study is to provide a detailed diagnostic analysis of students with MLDs. I sought to identify students who had persistent mathematical difficulties that could not be attributed to non-MLD causes such as general attention or behavioral issues, poor prior teaching, lack of motivation, etc. In consequence, this study employed several methodological phases to ensure accurate subject identification and comprehensive diagnostic analysis. These included: initial recruitment, data collection, student classification, and diagnostic analysis (see Figure 1). Students with potential MLDs were initially recruited either through self-identification or teacher identification. I then collected several kinds of data to attempt to determine if a student met the requirements for having an MLD. These data sources included: standardized achievement test scores, audio or video taped student interview, and video taped data of the pretest, four tutoring sessions, and posttest. These data were used to determine whether a candidate participant met the qualifications for having an MLD. Students meeting the three requirements of (1) low math achievement, (2) no confounding factors, and (3) no change from pretest to posttest, were considered to have an MLD. Those who did not meet all three the requirements were excluded from the study. For each of the students classified as having an MLD, a detailed diagnostic microgenetic analysis of all video taped sessions was conducted. In this chapter I describe the participants, the data sources, and the procedures used in this study.

![Figure 1](image-url). Schematic overview of methods used in this study.

Participants

Two students were classified as having an MLD for the purposes of this study, out of 11 students that were recruited and gave permission. These eleven students with low math achievement were recruited from a local middle school, private high school, community college, or through personal referral. Nine students were excluded from the MLD classification for one of several reasons: (1) obvious or self-reported attention or behavior problems, (2) performance at ceiling on the pretest, (3) failure to complete all

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9 At times students were excluded before all data was collected; if it was evident that they did not meet the requirements based on answers to the interview questions.
data collection sessions, or (4) substantial gains from pretest to posttest, suggesting that poor prior instruction was a possible cause of their low achievement. The remaining two students (one high school and one community college student) met the qualifications for having an MLD. Additionally a third student, a senior statistics major at a university, with a self-identified MLD was included for the purposes of corroborating data. Although she did not participate in the full complement of tutoring sessions, she was administered several of the pretest/posttest items and will be discussed in the cross-case analysis chapter. Five fifth-grade students were recruited from a local elementary school and serve as control subjects. All three subjects classified as having an MLD and all five control subjects were female\(^{10}\).

**Data Sources**

The data sources used for the purposes of classification included: standardized test scores, level of math class and performance in that class, student interview, and pretest/posttest scores from tutoring intervention. The primary data source used in the diagnostic analysis was the set of videotapes of the pretest, tutoring sessions, and posttest. The primary data source used for discussion of the remediation sessions was the set of videotapes of the remediation tutoring sessions.

**Test scores.** Standardized norm-referenced math and language STAR test scores\(^{11}\) were collected from students to establish the student’s low math achievement. If standardized assessments scores were not available, the student’s poor performance on the school’s placement test, resulting in enrollment in a remedial math class, was used to establish the student’s low math achievement as compared to peers.

**Student interview.** Before the administration of the pretest, an interview was conducted with each student focused on his/her experiences learning math. The interview questions attempted to elicit the student’s perceived level of effort and available resources (see Appendix 1 and Appendix 2 for the interview protocol).

**Pretest and posttest.** Videotaped semi-structured clinical interview pretests and posttests were administered to all subjects. The test (see Appendix 3 for the pretest/posttest protocol) was designed to cover all fraction concepts targeted in the tutoring sequence. The pretest was administered the week before the commencement of

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\(^{10}\) It is not clear why all participants in this study were female. For the control students a flyer was sent to all fifth grade parents at the elementary school. All five students whose parents responded to the flyer were included in this study. For the students with potential MLDs, the sample in was predominantly female. With the exception of 2 male participants, who both were excluded because they were at ceiling at pretest, all nine other possible candidates for having an MLD were women. Because this is a small study, were I intentionally selectively recruited students, along with the fact that I am not making generalizing from these data to make claims about any population, this over-representation of women in my study, although intriguing, is not a cause for concern.

\(^{11}\) STAR stands for Standardized Testing and Reporting, and is a California state mandated achievement test with a math and language arts section administered each spring in the state of California.
tutoring sessions and the posttest was administered the week following the completion of the tutoring sessions\(^\text{12}\). Problems included:

- **Fraction representation and interpretation problems**, which establish how the student chose to represent fractional amounts and if the student was able to interpret canonical representations of fractions.
- **Fraction comparison problems**, in which the student was asked to compare two different fraction amounts represented (1) numerically, (2) with cut-out paper shapes, or (3) with area models. The purpose of these problems was to determine the student’s fluency with fraction comparison and whether the student drew upon part-whole comparison strategy.
- **Fraction equivalence problems**, in which the student was asked to generate equivalent fractions for a given fraction.
- **Fraction operation problems**, in which the student was asked to add or subtract fractions both with like and unlike denominators.

For the majority of problems (problems 1, 2, 5, 6, and 7) there were several iterations involving isomorphic problems, but with increasingly difficult fractional values. If a student failed to answer two or more of the iterations correctly, I did not administer the rest of the iterations of that problem type.

**Tutoring sessions.** Four hour-long weekly videotaped tutoring sessions were conducted with the subjects focused on part-whole fraction concepts. These sessions were designed based on prior research on the teaching and learning of fractions and have been refined after several pilots. The questions were designed to help the student develop an increasingly refined understanding of fractions using a variety of different representations (numeric, manipulatives, area models; see Appendix 4 for a list of all problems used for the tutoring sessions with rationales). Similar to prior tutoring work (Gutstein & Mack, 1999; Mack, 1995), the goal of these questions was to help the student construct new knowledge; therefore the questions were carefully sequenced, with each attempting to build from the previous question. Each question was conceptualized as an opportunity for the student to learn and a means of assessing the student’s understanding.

The tutoring sessions had a common structure: reviewing the content covered in previous sessions, completing a series of “challenges,” playing a game, and recording notes about what the student learned. The content from the prior session was reviewed by reading and discussing the student’s previous journal entries and with “warm-up” questions. After the review, a sequence of challenges was posed to the student, which were intended to build on one another and provoke conversations about important math content. Each of the challenges had follow-up questions and/or counter suggestions. After the challenges were completed, if there was sufficient time, the student and I played a game in which the student answered questions related to the concepts learned during that session. The game served both as practice for the student in applying the concepts and as time filler if the student finished the challenges more quickly than expected. The session concluded with the student reflecting on what she had learned during the session.

\(^{12}\) Due to scheduling issues, the time between the pretest and first tutoring session sometime varied. For the students with MLDs, the posttest was administered 9 days and 14 days after the final tutoring session.
and recording in a journal, “Something that made sense” and “How would you explain it to yourself if you forgot?”

A brief description of each session will be presented, along with a description of the tutoring heuristics used to guide my interaction with the student.

**Session 1.** The goal of this session was to help the student develop an understanding of the meaning of the numerator, denominator, and whole using two different sets of foam fraction pieces (see Figure 2). The fraction pieces were labeled with the applicable unit-fraction name (e.g., “1/4”). Challenges included consideration of: how each of these fraction pieces were labeled, how fraction pieces could be used to represent non-unit fractions (e.g., 3/4), how many fraction pieces it would take to comprise the whole, a comparison of the rectangular and the circular pieces, and exploration of equivalence using fraction pieces to come up with fractions equal to one-half. The game built on the concepts explored in this session by asking students to create equivalent fractions “make 1/3 using only the 1/6 pieces.” The central concepts were (1) meaning of the denominator, (2) meaning of the numerator, (3) importance of the size of the whole, and (4) fraction equivalence for foam fraction pieces.

![Figure 2. Circular and rectangular foam fraction pieces.](image)

**Session 2.** The goal of this session was to transition from the foam fraction pieces to representing and interpreting fractional quantities using drawn area models. Challenges included: drawing a picture of a fractional quantity, developing rules for interpreting drawn pictures of fractions, comparison of equivalent fractional portions of paper which looked different, and comparison of two area models where the number of pieces, the size of the pieces, or size of the whole varied. The central concepts were (1) meaning of the denominator, (2) meaning of the numerator, and (3) importance of considering the size of the whole for area model drawings.

**Session 3.** The goal of this session was to use “fair sharing” to motivate the partitioning of unequal area models and to transition to partitioning area models to generate equivalent fractions. Challenges included: fair sharing activity for unevenly partitioned cakes and generation of equivalent fractions using area model representations with transparencies overlaid to repartition area models (see Figure 3). The central concepts were (1) importance of an evenly partitioned whole and (2) fraction equivalence using area models.

![Figure 3. Illustration of the area model squares with transparency overlays to create equivalent fractions.](image)
**Session 4.** The goal of this session was to explore fraction addition and subtraction of like and unlike denominators using the representational tools introduced in previous sessions (fraction pieces and area models). The challenges were a sequence of fraction addition problems that the student was asked to solve using the foam fraction pieces and area models. This session was structured in a way that a challenging problem was posed (e.g., \( \frac{1}{2} + \frac{1}{3} = \)) which then motivated the exploration of easier problems (e.g., \( \frac{1}{3} + \frac{1}{3} = \)) to help the student build their understanding of adding fractions with like and unlike denominators. The central concepts were (1) the importance of having a common denominator when adding and subtracting fractions and (2) using equivalent fractions as a means of generating a common denominator.

**Tutoring heuristics.** The tutoring protocol was developed to anticipate the range of student answers for a given question or prompt. In addition, general tutoring heuristics were developed to guide my actions as the tutor (see Table 1).

Table 1. General Heuristics for Tutor Responses to Student Answers.

<table>
<thead>
<tr>
<th>Student Answer</th>
<th>Tutor Response</th>
</tr>
</thead>
</table>
| Correct Answer | Ask the student to explain her answer  
                    Proceed to next question |
| “I don’t know” | Rephrase question  
                    Pose easier question |
| Incorrect      | Ask the student to explain her answer  
                    Refer back to prior work |

If the student answered a question correctly, I asked for an explanation of how the student determined that answer and then moved onto the next step of the tutoring protocol.

If a student was unsure of how to proceed or said, “I don’t know”, I rephrased the question and/or posed a simpler but related problem. For example, in the following excerpt the student was presented with the challenge: “How can you show what \( \frac{3}{4} \) would look like with fraction pieces?” the student was unsure of how to begin, so I posed two related but simpler questions: (1) how would you show what \( \frac{1}{4} \) would look like? And (2) if I had this amount (two \( \frac{1}{6} \) pieces) how much would we call that? (see Figure 4).
Figure 4. Example and illustration of the tutoring heuristic “pose easier question” which was used when the student was unable to answer the question.

After she had correctly answered easier related problems, she returned to the original challenge and correctly answered the problem. As in the example above, the student often decided when to return to the original problem with no directive prompting from me. In other cases, I specifically asked the student to consider their answer to the simpler question in relationship to the current challenge question (e.g., “if we call this two-sixths, does that help us think about how to make 3/4?”).

If the student gave an incorrect answer or explanation, I often referred back to previous problems and written work in an attempt to help the student apply previously established understandings to the current problem. For example, in the following excerpt the student believed that a one-fifth piece and a one-sixth piece together was 2/11. In my attempt to clarify, I asked her how big an eleventh fraction piece (not included in the fraction piece set) would be, and referred back to a previous problem where she had determined how big a one-eighth piece would be. She determined that you cannot create an eleventh-sized piece, because 11 is an odd number. I attempted to challenge this conception by referring back to a problem we had previously discussed, which involved fifth-sized pieces. After reminding her of the previous problem I asked her to apply that understanding to the current challenge question: whether 1/5 and 1/6 could equal 2/11 (see Figure 5).
Figure 5. Example and illustration of tutoring heuristics when a student answers a question incorrectly.

By referring back to previously established understandings, I attempted to help the student apply that understanding to the current problem. When all attempts to help the student reconcile different understandings failed, I engaged in more direct instruction.

Debrief. After the administration of the posttest, the student was encouraged to discuss outstanding questions that he/she had about the topics we had covered. These “debrief” discussions provided additional insight into the student’s understandings and were therefore included in the analysis. The student largely guided this discussion. In consequence there was no consistency in the kinds of problems addressed during this part of the sessions.
**Remediation sessions.** Remediation sessions were designed and administered to one of the case study students. These sessions were similar in structure, and included a pretest, three remediation sessions, and posttest. The remediation was designed to explicitly target the student’s difficulty and build upon her atypical understandings. These sessions will be discussed in further detail in chapter 6. Because they were based on the student’s specific difficulties, an understanding of her diagnostic analysis is necessary to contextualize this instructional approach. The goal of these sessions was (1) to design alternative representations of part-whole fraction concepts, which might be more accessible to the student and (2) to help the student construct a more stable understanding of fractional quantity.

**Procedures**

**Recruitment.** Due to the low prevalence of MLDs, I employed a range of participant recruitment approaches at several different school sites. I recruited students through (1) self-identification\(^\text{13}\), (2) teacher identification, or (3) colleague referral. Eleven students were initially recruited, 7 self-identified from a community college, 3 teacher identified (1 from a high school and 2 from a middle school), and 1 from personal referral. In additional all 5 fifth grade students whose parents responded to the electronic flyer distributed to fifth grade parents at the elementary school were included as control participants. None of these students had low achievement or identified learning problems.

**Classification.** Students were only classified as having a math learning disability if the student (1) demonstrated low math achievement, (2) exhibited no confounding factors explaining the low achievement, (3) did not respond to the instructional intervention (as evidenced by lack of positive change in score from pretest to posttest).

**Low math achievement.** To evaluate student’s achievement in mathematics, participants were asked to provide or report their achievement test scores. Low math achievement was defined as scoring below the 25\(^{th}\) percentile (i.e., “below basic”) on the math section of the STAR test or a low score on a placement test resulting in placement in the lowest available math class (basic arithmetic for community college students or remedial math for high school students). Students were excluded from this study if their math achievement test scores were above the 25\(^{th}\) percentile, or if they placed out of the lowest available math class. Additionally, students were excluded if they reported difficulties with reading or scored below the 25\(^{th}\) percentile for English/language arts.

**Confounding factors.** Students were excluded from this study if they exhibited characteristics to which their low math achievement could be attributed. An evaluation of possible confounding factors relied on participant-reported issues from the student interview and observations of the student’s behavior during tutoring sessions. During the student interview, specific questions were intended to elicit the student’s evaluation of his/her level of effort, resources, and language fluency. During the tutoring sessions if

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\(^{13}\) A flyer was posted at a local community college, which stated: “Do you think you have a math learning disability? Do you have a difficult time learning math? Do you have trouble remembering the math you have learned?” The flyer provided contact information and an overview of the research study.
the student exhibited obvious attention and behavior problems (e.g., refusal to answer questions, off task activities like texting or doodling), the student was considered to have a confounding factor which might have caused their low math achievement. Students reporting or exhibiting potential confounding factors were excluded from the study and not classified as having a MLD.

**Response to instruction.** The student’s change from pretest to posttest was used to determine whether the student responded to the instructional sequence. A scoring rubric was developed to compare the student’s pretest and posttest performance (see Error! Reference source not found.). The total change in score from pretest to posttest was used to determine whether or not the student demonstrated evidence of learning. To qualify as having a “lack of response to instruction”, the students must have a small slope value and low posttest score. To establish what was considered to be reasonable change in score from pretest to posttest the control students’ change in score was evaluated. The average slope for the controls was 15% improvement and the average posttest score was 84%. Given that the five fifth grade controls demonstrated gains, the tutoring sessions were considered to be a reasonable learning environment. Students were excluded from this study if they had a change in score above 10% or a posttest score greater than 60%. Students who were at ceiling at the time of the pretest were also excluded from this study.

**Subject Classification.** Out of the 11 students recruited nine students failed to meet all requirements for classification as having an MLD: four students did not meet the qualifications for low math achievement (two of these students were also at ceiling at the time of the pretest), three students were excluded because of attention and behavior problems, one student reported her primary academic difficulty was with reading, and one student failed to complete the tutoring sessions (see Figure 6). Only two students, Lisa and Emily, met all the qualifications for having an MLD.

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14 This screening criteria was not intended to imply that attention/behavior issues cannot be caused by MLD, but rather for my selective recruitment, I wanted to minimize potential confounding factors.
Figure 6. Summary of classification of 11 students with potential MLD.

Lisa and Emily demonstrated low achievement, no identifiable confounding factors, and a lack of response to the tutoring intervention. Each student had posttest scores below 60% and slope values of -15% and 10% respectively (see Figure 7). For the purposes of this study, these two students were considered to have an MLD. In addition
to these two students, data from an additional student\textsuperscript{15} will be presented in the cross-case analysis chapter.

![Figure 7. Comparison of the fifth-grade control students’ average change from pretest to posttest as compared to the two case study students.](image)

**Diagnostic analytic approach.** All videotapes of the sessions (pretest, tutoring sequence, and posttest) were transcribed and all artifacts scanned. A microgenetic bottom-up analysis of the whole data corpus for each student was conducted (see Schoenfeld, Smith, & Arcavi, 1993, for an example of this kind of analysis applied to graphing). This analysis involves iterative passes through the data in an attempt to generate analytic categories that capture the nature of the student’s understanding. It is thus consistent with the kind of grounded theoretical approach discussed by Glaser & Strauss (1967) and Miles & Huberman (1984).

During the transcription of the video taped data I flagged instances that seemed to involve unusual answers or ways of understanding the mathematics. Working with printed transcripts and video I did an initial review of the transcripts for all sessions attempting to identify patterns in understanding that occurred at multiple points in time and over multiple sessions\textsuperscript{16}. In particular I attempted to understand the student’s

\textsuperscript{15} The third student, Taylor, is a fourth-year statistics major at a prestigious university. Despite her academic focus on mathematics, she has significant problems in almost all domains of mathematics.

\textsuperscript{16} Because the sessions were designed to target particular concepts, in my first analytic pass, I attempted to determine whether or not the student demonstrated evidence of those target concepts. I looked both within the session (what concepts does she draw upon in this session?) and across sessions for each target concept (how does the student use this concept over the course of the sessions?). This proved to be an unsatisfactory view of the
reasoning, particularly when she produced an incorrect answer. The goal of this was to find some level of consistency in how the student was reasoning and to be able to develop analytic categories, which capture the nature of the student’s understanding. At the end of this review I had generated a list of candidate atypical understandings with rough definitional guidelines for each. The data was then entered into excel. Each session was parsed into individual problems, which involved a question and an answer. As a general rule, a problem began with the posing of a question and ended with the student’s answer. Follow-up questions, which were not simply for the purposes of clarifying the student’s answer, were treated as distinct problems. Each session had on average about 50 problems. For each problem I coded: the question, the student’s answer and explanation, the correctness of the student’s answer, the problem type, representation type, and evidence of atypical understanding (see Table 2 for a list of all possible values for each category). The candidate atypical understandings were used during this phase of coding. In addition to flagging instances that were consistent with a given atypical understanding I also attempted to identify instances in which the student was reasoning in a way that was directly contradictory to a given atypical understanding. To be considered a non-example, the student must not only answer in a way that was in conflict with a given atypical understand, but provide an explanation that established a contradiction with a hypothesized atypical understanding. These “non-examples” had an elevated analytic importance because they provided evidence that the student was not reasoning in a way consistent with an atypical understanding, and provided potential rival hypotheses.

data. Although it highlighted some areas in which she did not have a fully developed and stable conceptual understanding, it did not provide a view of how the student was making sense of the mathematics. This top-down approach resulted in an impoverished view of the data, which seemed to be more deficit-focused than difference-focused, and was ultimately abandoned.
Table 2. Available values that could be assigned for each problem for each coding categories.

<table>
<thead>
<tr>
<th>Category</th>
<th>Possible values</th>
</tr>
</thead>
</table>
| Correctness of answer   | Right
|                         | Wrong                                           |
|                         | Unanswered                                      |
|                         | Tutor Instruction                               |
|                         | n/a                                             |
| Problem type            | Fraction construction                           |
|                         | Fraction interpretation                         |
|                         | Explaining fractions                            |
|                         | Equivalent fractions                            |
|                         | Fraction comparison                             |
|                         | Fraction operations                             |
|                         | Other (e.g., journal entry)                     |
| Representation type     | Area model                                      |
|                         | Fraction pieces                                 |
|                         | Discrete set                                    |
|                         | Number line                                     |
|                         | Numeric form (a/b)                              |
|                         | Decimal form                                    |
|                         | Paper manipulative                              |
|                         | Real life context (e.g., cake, brownies)        |
|                         | Other drawing (not otherwise classified)        |
|                         | n/a                                             |
| For each atypical       | Yes – if atypical understanding was evident      |
| understanding (e.g., “taking”) | No – if atypical understanding was not evident  |
|                         | Non-example – if the student was reasoning in a way that was in direct conflict with the atypical understanding |

For each problem, the associated transcript was included in the coding document. Therefore all lines of transcript (with the exception of off-task conversations, like discussing the markers we were using) were accounted for in this coding document. Each strand of analysis (atypical understanding) was then exported individually and iterative passes through the data were performed to further specify the definition for that atypical understanding. This involved iterative refinement of the inclusion and exclusion criteria, specifying the rationale for why each instance met the criteria for that atypical understanding, removing instances that did not meet the requirements, and returning to the larger data set to identify additional instances. As part of this work, atypical understandings sometimes merged or were differentiated.

To ensure that the analytic categories were reasonably exhaustive, I did an audit of all of the student’s incorrect answers that did not have an associated atypical understanding.

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17 The inclusion and exclusion criteria are clearly specified in the analytic chapters.
understanding flagged. I attempted to determine if other patterns in incorrect answers were evident, suggesting another kind of atypical understanding. Each atypical understanding is presented as an independent strand of analysis for each case. When first presented the atypical understanding will be referred to in quotes (e.g., “taking”) and after that quotes will used to demarcate the name of the atypical understanding only when needed for sentence clarity.

**Presentation of data**

For each of the case studies and the cross case analysis, there is a heavy reliance upon both transcripts and the drawn artifacts. Transcript conventions are detailed in Appendix 6. When transcript is presented the associated scanned artifact is generally presented immediately above the transcript for easy reference. For clarity it was sometimes necessary to include screen shots of video in conjunction with the transcript, to demonstrate how the activity unfolded. In these instances, the transcript is presented in a tabular format, with a line number, relevant lines of transcript, and associated artifact in separate columns.

**Conclusion**

These methods were employed to ensure that the students included in this dissertation had difficulties with mathematics, which were due to a cognitive rather than environmental source. The goal of the detailed diagnostic analysis was to determine what kinds of understandings appeared to be persistent and also detrimental to the student’s ability to learn mathematics. These analytic methods allowed for findings beyond mere correlations, and provided a mechanism for identifying consequential aspects of a student’s understanding, which ultimately were detrimental to their attempts to learn. As the case studies will reveal, each student did indeed demonstrate robust atypical understandings.
Chapter 3: The Case of Lisa

The first student, Lisa, had a history of unexplained low math achievement, which appeared to be due to an MLD rather than other factors. The purpose of this chapter is to provide a comprehensive view of the nature of Lisa’s MLD and how it was problematic for her attempts to learn. The analysis of the video taped data revealed five atypical understandings, which largely explain Lisa’s errors over the course of the tutoring sessions and are central to understanding Lisa’s MLD. Each of the five strands of analysis documents a robust atypical understanding, which was detrimental to her ability to learn more complex fraction concepts. In addition, a sixth analytic category, “arbitrary and ungrounded manipulation,” was identified which explained some of the remaining errors. This sixth analytic category was not an atypical understanding, in that it reflected a general orientation to mathematics rather than a specific atypical understanding. The sixth, although it completes the story of Lisa’s MLD, can best be thought of as a symptom rather than the cause of her MLD. I will refer to all six strands of analysis as “atypical patterns”. These six atypical patterns when taken together, provide an explanation for why Lisa failed to learn and therefore, are considered an apt characterization of Lisa’s MLD.

In this chapter I first present data from a variety of sources to establish that Lisa has an MLD. Second, I give an overview of the six atypical patterns and present a bird’s eye view of these patterns, which shows that the atypical patterns occurred across all sessions and accounted for nearly all of Lisa’s incorrect answers. Third, I present a detailed analysis of each of the six strands, which constitutes the majority of this chapter. Because the sixth strand is considered to be a symptom of her MLD rather than a cause of it, I will present the detailed analysis of the 5 atypical understandings first, consider the ramifications of these atypical understandings, and then present the sixth strand of analysis. The five atypical understandings along with the sixth atypical symptom, when taken as a whole, provide a reasonable explanation for why a standard instructional intervention would not have “worked” for Lisa.

Subject Classification

The first case study student, Lisa, met the MLD qualifications. Data collected during the student interview, pretest, tutoring sessions, and posttest indicated that she demonstrated (1) low math achievement, (2) no identifiable confounding factors which could explain her low achievement, and (3) lack of response to instruction as measured by her change in score from pretest to posttest.

Lisa was recruited from the community college and was considered to have low math achievement. The math placement test she took when she entered community college placed her in the basic arithmetic class. Despite completing her homework and being tutored weekly, she did not pass this class. Although standardized testing data were not available, her placement in the basic arithmetic class, and her failure to pass the class, indicate that her performance as compared to peers should be considered low.

No confounding factors, which could explain her low achievement, were identified. Lisa is a White, middle-class, native English speaker who had graduated from high school and enrolled in community college the following semester. The initial interview indicated that she had a history of failing mathematics despite adequate effort
and support. Based on the interview with Lisa and observations during the data collection sessions, lack of effort, insufficient resources, and behavior problems did not appear to be factors that could explain this student’s poor achievement in mathematics.

Lisa did not demonstrate substantial gains when comparing her pretest and posttest scores\(^\text{18}\). Lisa did worse at the time of the posttest than at the time of the pretest (see Figure 8)\(^\text{19}\). Recall that this tutoring sequence was effective for fifth grade students and consequently was judged to be an adequate instructional environment. Lisa’s lack of response to the standard intervention suggests that insufficient instruction was not the sole cause of Lisa’s low math achievement.

Lisa, therefore, meets the qualifications for having an MLD. Lisa’s lack of gains from pretest to posttest, despite my best attempts to teach her, is the entrée into the analysis. This analysis is concerned with an in depth look at why Lisa did not learn, what kinds of difficulties she experienced, and how best to understand her MLD.

![Comparison of Lisa’s Pre/Post to Controls](image)

Figure 8. Comparison of Lisa’s pretest/posttest performance as compared to the aggregated fifth grade control students.

\(^{18}\) Lisa was not administered the full version of the pretest. The pretest session with Lisa was conducted before the pre/post measure was finalized, and therefore, Lisa received a less extensive version of the pretest. She did, however, receive the full and final version of the posttest measure. For the comparison of pretest to posttest score, only items that had a corresponding pretest item were included. Lisa had a 34% accuracy rate for the complete administration of the posttest.

\(^{19}\) Lisa’s negative change in posttest score was somewhat surprising. As this dissertation will show, there is good reason, post hoc, to understand why such interventions may not “work” for students like Lisa. As explored in the remainder of the chapter, Lisa’s attempts to reconcile her atypical understandings and the standard instruction were a likely contributor to her depressed posttest score. Lisa’s negative change in posttest score makes the case that she (1) has an MLD and (2) that standard instruction will likely be ineffective for her.
Overview of Analysis

The purpose of this analysis is to explore potential causal factors that may have contributed to Lisa’s failure to benefit from the tutoring session. The goal is therefore, to capture the learning disability as it is occurring, and document the ways in which consequential aspects of her knowledge and understanding contribute to her difficulties learning.

**High-level view of Lisa’s atypical patterns** Through iterative analytic passes through the data, I identified six analytic categories. Five of these analytic categories are considered to be atypical understandings, which directly contributed to Lisa’s difficulties. These atypical understandings (depicted in Figure 9, with an example) include: (1) “taking” (2) “halving,” (3) “discrete set,” (4) “unit fraction,” and (5) “partitioning.” The sixth analytic category, “arbitrary ungrounded manipulation,” was likely a result of Lisa’s insufficient conceptual foundation, and therefore considered a consequence of the 5 atypical understandings, rather than an understanding itself.

In this section, I provide a brief description of each of the six strands of analysis along with a bird’s eye view of how instances of these analytic categories occurred over the entire data corpus. This section is intended to provide the reader with an overview of the whole before delving into the individual presentation of each strand of analysis.

**Figure 9. Overview of the six strands of analysis for Lisa.**

**Taking understanding.** “Taking” involved understanding representations of fractions in terms of an amount “taken” from the whole, which corresponded to the numerator of the fraction. For example, in the context of area models, when drawing the fraction 3/4, the 3 shaded pieces would be understood as “taken”, which often caused Lisa to shift her focus to the amount that was understood as “left” (1/4). Her interpretations of fractional amounts were therefore, sometimes dependent upon the fractional complement (non-shaded or missing pieces).

**Halving understanding.** “Halving” involved understanding the fraction 1/2 as a splitting or partitioning action, rather than the quantity one-half. In the case of area models Lisa often drew one-half by drawing the shape and splitting (or “halving”) the shape into two pieces, while omitting the shading. She often over applied this halving understanding to contexts for which she should have been drawing on a quantity understanding of one-half.

**Discrete set understanding.** “Discrete Set” was an understanding of fractions where a discrete set model was privileged. The numerator and denominator were referred to and acted upon as if they were whole numbers. Characteristic of this understanding was terminology like “three out of five” and an understanding of the fraction in terms of discrete members of a collection, rather than parts of one unified whole. Lisa often over-
applied this discrete set understanding to continuous models of fractions and fraction operations.

**Unit fraction understanding.** “Unit Fraction” was an understanding of a fractions based on the denominator value, irrespective of the numerator value. This understanding occurred most often in comparison contexts. Lisa often compared fractions as if every fraction was a unit fraction. Therefore 1/2 was considered larger than 9/10 because halves are larger than tenths. At times Lisa over-applied this unit fraction understanding to the numerator values, and would order the fractions based on the inverse size of the numerator without consideration for the denominator value (e.g., 5/8 is smaller than 3/8, because 5 is bigger than 3, and the bigger the number, the smaller the fraction).

**Partitioning understanding.** “Partitioning” was an understanding of the valid ways to partition shapes to achieve a specified number of pieces (corresponding to the denominator value). This understanding involved both the actual mechanics of partitioning and inferences made about the results of various partitioning strategies. For example, Lisa often used a repeated halving strategy to partition shapes, which resulted in Lisa inferring that it was impossible to partition a shape into an odd number of pieces. In other instances, Lisa would construct unidirectional partitions, and draw the number of partitions corresponding to the denominator value, which resulted in an additional piece.

**Arbitrary ungrounded manipulation.** “Arbitrary ungrounded manipulation” involved understanding and interacting with mathematical representations in ways that were focused on the procedural actions rather than the conceptual meaning. Lisa often manipulated mathematical representations and symbols in a way that was devoid of an understanding of the underlying meaning. Although many students learn procedural rules to solve math problems, the kinds of manipulations Lisa performed were unlike those documented in prior research (see example in Figure 9), and consequently were considered atypical. Although this sixth strand of analysis was quite different than the five atypical understandings it was an essential dimension of the kinds of difficulties that she experienced.

These six atypical patterns provide a comprehensive view of Lisa’s difficulties across the tutoring sessions. The detailed analysis of each strand will consider the ways in which these understandings were ultimately detrimental to her learning and when taken together establish a reasonable explanation for why Lisa did not benefit from the tutoring protocol. Before delving into the detailed analysis of each of these strands of analysis I present a high-level view of the occurrence of each across all video-taped data of the sessions with Lisa.

**A “bird’s-eye” view of the data corpus.** Zooming out to consider the entire data corpus (see Figure 10), high-level inferences can be made about Lisa’s overall performance. First, Lisa appeared to experience difficulty across all sessions as indicated by the frequency of instances where Lisa’s answer was coded as incorrect (red). Second, the six atypical patterns occurred in all video taped sessions. Although there were occasionally clusters of instances of one kind of understanding, in general Lisa appeared to draw upon multiple kinds of atypical understanding in a given tutoring session. This suggests that these understandings were not localized to a particular time or problem context, but instead might be more productively conceptualized as part of her repertoire...
of understandings, which she drew upon as needed to answer the questions during the sessions.

When these problems are sorted according to the correctness of the answer, different clusters of problems emerge from the data (see Figure 11). Of all the problems, 56% were coded as correct, 9% were coded as tutoring instances, 33% were coded as incorrect, and 2% were coded as “n/a” (e.g., journal entry). For the problems coded as correct, 79% did not have any of the six atypical patterns flagged, suggesting that she was either using canonical understanding or that there was insufficient data to suggest an atypical understanding was being employed. Twenty-one percent of the correct answers were given in conjunction with an atypical understanding, indicating that she was using her atypical understanding productively to produce a correct answer to the question. Similarly, in 24% of the tutoring instances, an atypical understanding surfaces. In contrast, for the incorrect answers, 83% involved her use of one or more of her atypical understandings. Only 17% (n=18) of her incorrect answers had no atypical understandings flagged. Therefore, the atypical patterns accounted for nearly all of her incorrect answers across all the tutoring sessions.

An audit was conducted on all 18 instances that were not accounted for by one of the atypical patterns. These errors were further classified to determine if there was another kind of atypical understanding at play. No predominant reason surfaced to explain these 18 errors. In many cases they can be thought of as mistakes, rather than systematic misunderstandings, and in a handful of instances Lisa eventually self-corrected her mistake. Of these 18 errors, 5 errors involved miscounting, 3 were lacking sufficient data to provide a classification beyond the incorrectness of her answer, the remaining involved unconventional problems or representations which were not a focus of analysis (improper fractions and number lines). Therefore, the six atypical patterns well capture the kinds of difficulties that Lisa was experiencing over the course of the tutoring sessions. I now turn to an in depth treatment of each of these strands of analysis.
Figure 10. Problem coding across all the data from the sessions with Lisa, with data with 6 atypical patterns flagged.
Figure 11. Problem coding across all the data from the sessions with Lisa, sorted by correctness.
Lisa’s Taking Understanding

This section presents the first of Lisa’s five atypical understandings: “taking.” Her understanding of the fractional amount as taken, caused ambiguity in whether she should attend to the amount taken or the amount left (the fractional complement). This rendered her conceptualization of fractional quantity unstable. Lisa’s “taking” understanding was robust in that it occurred across representations, persisted over time, and resisted explicit attempts to address it. This atypical understanding was detrimental to her ability to engage with more complex fraction concepts and ultimately provides one of the key pieces towards understanding why Lisa did not learn.

In this section, first I introduce the “taking” understanding and provide a prototypical example in conjunction with the operational definition. Second, I discuss, how this understanding proved to be problematic for Lisa’s understanding of more complex fraction concepts. Third, I consider the ways in which this understanding was robust and persistent across the sessions. Specifically, I consider instances of “taking” during the pretest and posttest, which suggest that this understanding originated before the tutoring sessions, and was not resolved over the course of the tutoring sessions. I consider how other representational tools, specifically fraction pieces, were also subject to her “taking” understanding. Finally, I show how this understanding resisted explicit attempts to address it. This strand of analysis illustrates that Lisa’s “taking” understanding was atypical, robust, and detrimental to her ability to learn.

**Defining and exemplifying “taking” understanding.** Lisa’s “taking” understanding involved the ways in which she understood representations of fractional quantities, specifically related to determining the value of the numerator. To highlight the atypicality of Lisa’s understanding it will be contrasted with a typical understanding of the numerator, in the most commonly used representational context: area models. Typically, when area models are used to represent fractional quantities the numerator is represented by the number of shaded pieces. To construct an area model for 3/4, an individual will shade in the number of pieces corresponding to the numerator, and when interpreting the drawn area model will count the shaded pieces to determine the numerator value (see Figure 12). The constructive and interpretive acts are, therefore, fundamentally interdependent.

A typical use of area models can be contrasted with Lisa’s “taking” understanding. Although when constructing an area model for 3/4, Lisa would shade 3 pieces, she understood the shaded region as the amount “taken.” This understanding often resulted in her attending to the non-shaded region (fractional complement) as the focal fractional quantity and referring to it as the amount “left.” A representation that she drew of 3/4, she might then interpret as 1/4 (see Figure 12). Lisa’s constructive and interpretive acts were therefore disconnected resulting in an instability in representation of fractional quantity.

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20 Although area model representations of fractions involve several conventional entailments including: representing the whole by a drawn shape, the denominator by the number of equally-sized pieces the shape is partitioned into, and the numerator by the number of shaded pieces, the focus of this exemplar is on the representation of the numerator alone.
Figure 12. Contrasting examples of typical and atypical use of area models.

**Prototypical example of taking.** Lisa’s taking understanding was evident in the following example, when instead of focusing on the 6 shaded pieces, she focused on the two *non-shaded* pieces to interpret her drawn representation of 6/8 (see Figure 13). This “taking” instance was part of Lisa’s attempts to solve the problem “7/8 - 3/4=” using area models.

Figure 13. Scanned artifact of Lisa’s area model for 6/8, which she focused on the two non-shaded pieces.

Lisa had begun solving this problem “7/8 - 3/4=” by drawing area models for both fractions. She had correctly constructed the area models for both 7/8 and 3/4 (see Figure 14), by shading the number of pieces corresponding to the numerator values.

Figure 14. Artifact from tutoring session showing Lisa’s construction of area models for the problem 7/8 – 3/4=. (This figure has been digitally recreated).

After correctly constructing area models for both 7/8 and 3/4, she determined that to subtract the quantities one would need to further partition the area model for 3/4. In the following excerpt, she correctly partitioned the area model, creating a representation of 6/8 but instead of attending to the six shaded pieces, she began attending to the 2 non-shaded pieces.
Lisa: If I were to like switch this (gestures with pen down the horizontal middle of the 3/4 area model) like that, it would be two…

Tutor: Ok, so let’s cut it in half like that.

Lisa: (draws partition line) there would be two left? Or two out of, two-sixths left. (writes 2/6) No. I’m not sure (scribbles out 2/6).

Lisa began attending to the two non-shaded pieces even before drawing the new partition line. After the partition line was drawn she restated her answer of “two”, in fractional form, “two-sixth left.” Although she herself had shaded the 3 pieces to represent 3/4, she interpreted the non-shaded pieces as indicative of the fractional quantity. Although she did reject her answer of 2/6 at the end of the excerpt, her attention to the two non-shaded pieces persisted.

In the following excerpt I reminded her that her whenever we repartition a shape the fraction name changes, and reoriented her to the representation by asserting that it was originally 3/4. Rather than attending to the shaded region, she continued to focus on the non-shaded pieces, and only partially repaired this answer by changing it from 2/6 (non-shaded/shaded) to 2/8 (non-shaded/total). She explicitly pointed to the 2 pieces that were not shaded. This excerpt ended with me asking her what the non-shaded pieces mean, and her fully correcting her answer to 6/8 (shaded/total).

Tutor: So you just cut this, right?
Lisa: Right.
Tutor: So whenever we cut it the fraction name changes. Right?
Lisa: Right.
Tutor: So, if we look at this (points to 6/8 area model), what fraction name does it have now? I mean, it used to be three-fourths, but now we are forgetting about that.
Lisa: Uh huh. Uh, two-eighths?
Tutor: And, which pieces are you counting?
Lisa: Um, I guess, 2, 4, 6, 8. (pointing to each pair of pieces in area model for 6/8 with index and middle finger) (with finger traces “/8”) There’s eight.
Tutor: There’s eight total. (*writes “/8”*)
Lisa: Yeah. And then two of which aren’t shaded in.
Tutor: Ok, if they aren’t shaded in what does that mean?
Lisa: Ok, then that wouldn’t count, so then that would be uh, four-eighths.
Tutor: Ok, and so, how –
Lisa: Oh, no, it would be six-eighths, sorry!

Although Lisa partially repaired her answer, by correctly determining the denominator value (8), she continued to attend to the non-shaded pieces to determine the numerator. It was only after I asked, “if they aren’t shaded in, what does that mean?” that she realized that the non-shaded pieces “wouldn’t count,” and changed her answer to 6/8, attending to the 6 shaded pieces. After Lisa had correctly determined the value of the equivalent fraction, she had no difficulty completing the rest of the problem and determining “7/8-6/8=” would equal 1/8. Her main difficulty with problem was therefore, her interpretation of the area model for 6/8.

This example highlights how Lisa’s taking understanding was invoked as she attempted to solve a problem. This episode was considered consistent with a “taking” understanding because it involved understanding the fractional representation in terms of the non-shaded amount, and reference to the non-shaded pieces as “left.” Although Lisa did not explicitly refer to the shaded region as an amount “taken,” this was considered a prototypical example of her “taking” understanding because after correct construction of an area model (3/4), and appropriate repartitioning of the area model (to produce 6/8), she interpreted the resulting fraction as the fractional complement. Despite her ability to represent both fractions with area models and repartition an area model to produce a common sized piece, her understanding of the shaded region as taken, and non-shaded as left, made her reinterpretation of her area model problematic. Of note is that she did not interpret the area model 7/8 as 1/8. This highlights the inconsistency with which she applied this taking understanding, which will be the focus of discussion later in this strand of analysis.

*Researcher’s operational definition of taking.* Problems were coded as indicative of a “taking” understanding if (1) Lisa used the words: “take,” “gone,” or “missing” (or any derivation) to refer to the numerator quantity, (2) if Lisa used “left” to refer to the fractional complement, (3) if Lisa was gesturing or referring to the fractional complement (represented by the non-shaded area model region or missing fraction

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21 Lisa answer of “four-eighths” was quickly corrected, and not considered to be indicative of anything other than a trivial counting mistake.
22 This problem concluded with her interpreting the area models correctly (7/8 and 6/8), me recording her interpretations by rewriting the problem, and her referring to the numeric form she answered that the solution to the problem would be “one-eighth”. Lisa had just completed this problem with fraction pieces, and determined the answer to that problem was 1/8. This activity was an attempt to use drawings to solve the same problem. Therefore, the readily apparent answer of 1/8, once the problem was rewritten as “7/8-6/8,” might be due to the previous work with the fraction pieces.
pieces) as the focal fractional quantity, or (4) if Lisa used shading to represent the removal of pieces. Non-taking\textsuperscript{23} examples included instances in which she explicitly made reference to the shaded pieces as “left” or justified her answer using the shaded pieces as indicative of the fractional amount.

There were 33 instances of Lisa’s “taking” understanding and 6 flagged non-taking examples. As seen in Figure 15, this understanding appeared across all the tutoring sessions, and was often used in conjunction with an incorrect answer.

Fig. 15. Data display of all instances of Lisa’s “taking” understanding.

**Taking as detrimental to learning.** Lisa’s taking understanding was consequential in that it hindered her ability to engage with more complex fraction concepts. In the prototypical example, her attention to the non-shaded pieces was detrimental to her ability to interpret the equivalent fraction and consequently to solve the fraction operation problem. In the next example, Lisa’s “taking” understanding fundamentally disrupted her ability to compare fractions. In this problem she was asked to compare the fractions $\frac{7}{12}$ and $\frac{1}{2}$. She correctly constructed a drawing of $\frac{7}{12}$ by partitioning the shape into 12 pieces (see Figure 16a) and shading 7 of those pieces (see Figure 16b). However, she incorrectly determined that $\frac{7}{12}$ was smaller than $\frac{1}{2}$ and when I ask her to explain she referred to the shaded pieces as “gone.”

![Fig. 16a](image1.png) ![Fig. 16b](image2.png)

Fig. 16. Reproduction of Lisa’s drawing of $\frac{7}{12}$, in which she (a) partitioned a shape into 12 pieces, and then (b) colored 7 of the pieces.

Lisa: I mean, ok, so let’s say that this is the cake (*gestures back and forth over entire shape*) and seven pieces are gone (*makes sweeping motion over the shaded pieces of the shape*).

\textsuperscript{23} Recall that non-examples were instances in which the student’s explanation for her reasoning was in direct conflict with the given atypical understanding.
Although she drew a canonical area models (using shading to represent the numerator) she subsequently interpreted the shaded region of area models as “gone.” In this example, her ability to make sense of the fraction comparison task was undermined by her atypical conception of the area model representation. During this particular problem she had no difficulty determining the equivalent fraction for 1/2 would be 6/12, but when asked to compare 6 pieces of cake (for 1/2) and 7 pieces of cake (for 7/12), she still determined that half would be more.

Tutor: So, if I have six pieces of cake, versus in this one (pointing to 7/12), I have seven pieces of cake.
Lisa: Yeah.
Tutor: Which one do I have more for?
Lisa: Um. The half.

Although in this instance I simply asked her to compare two whole numbers: 6 and 7, Lisa incorrectly determined that “the half” (6) would be more. Lisa’s “taking” understanding provides a likely rationale for her difficulty with this basic question. If she was comparing “six pieces of cake” taken, and “seven pieces of cake” taken, she could understandably come to the conclusion that there would be more cake “left” if only 6 pieces were removed. Lisa’s focus on the “taking” action, caused her to attend to the fractional complements, and rendered even this comparison of 6 and 7 problematic. This demonstrates how pervasive and potentially disruptive this focus on a “taking” action was for Lisa’s conception of fractional quantity.

Lisa’s understanding of fractions, in terms of an amount taken, appeared to disrupt her ability to build a more complete understanding of fraction equivalence, fraction comparison, and fraction addition. I now turn to an exploration of why this kind of problematic understanding persisted through the tutoring sessions and did not get resolved.

“Taking” across the sessions. This section considers the persistence and robustness of Lisa’s “taking” understanding over time. First I consider evidence of this understanding during the pretest and posttest, which suggests that Lisa was relying upon this understanding both before and after the tutoring sessions. Second, I consider the ways in which this understanding unexpectedly also appeared in conjunction with the foam fraction pieces. Third, I highlight how this understanding was at the same time persistent and also inconsistently applied across the tutoring sessions, suggesting another dimension of instability in Lisa’s conception of fractional quantity. Lastly, I present excerpts illustrating how this understanding was not easily corrected and it resisted my attempts to address it.

Persistence over time: “taking” during the pretest and posttest. Lisa’s pretest and posttest instances of “taking” were examined to determine how she was using this understanding before and after the tutoring sessions. During the pretest Lisa used shading to represent the amount taken away and referred to the numerator value as the amount “portioned off”. Similarly, at the time of the posttest she continued to use shading to represent the omission of pieces of an area model, and referred to the non-shaded pieces as “left.” Lisa’s “taking” understanding was therefore, evident both before
and after the tutoring sessions. This suggests that “taking” was part of her prior understanding and indicates that this understanding was not refined over the course of the tutoring sessions.

**Persistence across representations: “taking” with fraction pieces.** Lisa’s “taking” understanding was evident in other representational contexts, suggesting that this was not simply a misunderstanding of the shading convention used with area models. In the context of foam fraction pieces, the fractional quantity was a physical substance that could be seen, felt, and manipulated. Because the numerator was represented with physical objects, one would expect that this representation would support an understanding of the numerator as the focal fractional quantity rather than an amount taken away. However, when interpreting fraction pieces Lisa again focused on the fractional complement: the missing pieces. In the following example, Lisa was solving a fraction addition problem using the foam fraction pieces and she attended to the missing pieces (analogous to the non-shaded part of the area model) to determine the answer to the addition problem.

Lisa was solving the problem “2/5 + 3/10 =” and before this excerpt, Lisa had created both 2/5 and 3/10 using the fraction pieces (see Figure 17a), she had determined that 2/5 was the same thing as 4/10, by overlaying four purple 1/10 pieces on top of two yellow 1/5 pieces (see Figure 17b), and she accurately interpreted the equivalence and recorded it on paper (see Figure 17c).

Figure 17. Video stills of Lisa’s (a) construction of 2/5 and 3/10, (b) establishment of the equivalence of 2/5 and 4/10 and (c) record of the written equivalence (4/10=2/5).

In the following excerpt I restated what she had previously determined and asked her to consider how much we had altogether. To answer this question she attempted to find what piece would fit in the empty space.

<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript</th>
<th>Video Stills</th>
</tr>
</thead>
</table>
| 1    | Tutor: So, you said this is three-tenths. (*points to fraction pieces*)  
Lisa: Uh huh. |
| 2    | Tutor: And you said this is two-fifths. (*points to fraction pieces*)  
Lisa: Uh huh.  
Tutor: Right. But thinking about how to add them up is confusing.  
Lisa: Yeah. |
|   | Tutor: Then what you came up with was, you know look at – four-tenths are the same thing as two-fifths.  
|   | (moves four 1/10 pieces onto two 1/5 pieces) 
|   | Lisa: Uh huh. | ![Image](image1.png) |
| 4 | Tutor: So, now is it easier to tell how much we have altogether? 
|   | Lisa: Yeah. 
|   | Tutor: What fraction name would we give this whole amount that is here? 
|   | Lisa: A third? | ![Image](image2.png) |
| 5 | Tutor: And are you doing that just based on what it looks like, sort of? 
|   | Lisa: Yeah. A fourth? No. Cause, I mean how much is left? (gestures to close empty space) | ![Image](image3.png) |
| 6 | Tutor: Yeah, we are missing - 
|   | Lisa: (reaches for 1/3 fraction piece, knocks pen off table) Whoa! Sorry. 
|   | Tutor: some amount over here. 
|   | Lisa: Is a third? (moves 1/3 piece into empty space) 
|   | Yeah, a third is missing. | ![Image](image4.png) |

In this example, Lisa attended to the empty space to determine the solution to the addition problem. She initially guessed that the answer was “a third?” (Line 4). When I asked her if she got her answer based on a visual judgment, she responded affirmatively, and considered that the amount might be equal to “a fourth.” Given her gesture and reference to “how much is left?”(Line 5), Lisa was clearly referring to the missing part of the fractional amount. At the end of this excerpt she had filled the missing space with a “1/3” fraction piece and decided that “yeah, a third is missing.” Although this was a different representational system, she similarly attended to the fractional complement to determine the value of the given fraction. It’s notable that she was able to construct 2/5 by assembling two 1/5 pieces, able to construct 3/10 by assembling three 1/10 pieces, and able to interpret four 1/10 pieces as 4/10, but when asked to interpret seven 1/10 pieces she began attending to the negative space. Although she was able to eventually correct her solution to this problem\(^{24}\), the focus of this example was on how she focused and attempted to name the empty space (fractional complement) to determine the value of the assembled fractional pieces. Lisa’s “taking” understanding was sufficiently robust, in

\(^{24}\) After this excerpt, she did go on to determine that if the missing piece as one-third, that the purple pieces equaled 2/3, which required a relatively sophisticated understanding of fractional complements. I explained that I was not sure that the third piece fit and asked her if there was a way of talking about the fractional amount in terms of tenths. At this point she correctly interpreted the fractional quantity as 7/10.
that it was invoked even in the context of foam fraction pieces, and just as with area model, Lisa attended to the fractional complement, rather than the focal fractional quantity.

**Inconsistency of “taking” over time.** Although Lisa’s taking understanding recurred across all sessions and in conjunction with various representations, it was not consistently applied. There were 7 “non-taking” examples that were flagged throughout the tutoring sessions. In non-taking examples, Lisa explicitly referred to the shaded region as the focal fractional quantity and the way in which one would determine the numerator value. Recall that to qualify as a “non-taking” example, Lisa must have not only been reasoning in a way incompatible with the “taking” understanding, but also must have provided an explanation that was directly contradictory to “taking.” For example, during the second tutoring session, Lisa explained that, “whatever is colored in signifies the amount of the whole.” It is noteworthy that the non-taking examples were interspersed with instances of her using the “taking” understanding (see Figure 18). This high level view of the data helps give the sense of the inconsistency with which she understood fraction representations. At times she attended to the representation with an explicit focus on the fractional amount and at other times she attended to the fractional complement. In general almost all instances coded as non-taking examples were during problems she answered correctly.

Figure 18. Non-taking and taking instances as they occurred across the tutoring sessions.

**Robustness of Lisa’s “taking” understanding.** Lisa’s taking understanding was sufficiently well entrenched that it resisted explicit attempts to address it. During the third tutoring session, when Lisa’s tendency to attend to the non-shaded pieces became evident, I asked Lisa to imagine the area model was a picture of a cake. I asked her to think of colored frosting to help her conceptualize the shading as a quantity that was there, rather than gone. These strategies provided local correction for her interpretation of fractions, but did little to displace her tendency to understand the shading as taken. As seen in the display of “taking” understanding (see Figure 18), Lisa increasingly relied upon “taking” during the fourth tutoring session, the posttest, and debrief.

The resilient nature of this understanding was most clearly displayed during the debrief, when I was specifically attempting to address this problematic conception of shading in terms of area model comparison problems. During this 13-minute episode, Lisa continuously asserted that the shading was the amount “taken”, “taken away” or “gone”, and focused on the amount “left” to make her comparisons of the fractions.
When comparing $9/10$ and $7/12$, she focused and counted the non-shaded pieces “left” for both fractions.

Lisa: Because you are taking, more, like at the end of the day, there is only one piece left of the nine-tenths, where as there was... 1, 2, 3, 4, five. (counting the non-shaded pieces in the area model of 7/12).

Lisa’s conceptualization of the two fractions involved attending to the amount left (non-shaded pieces) rather than the amount shaded. Lisa compared the fractional complements: 1 piece left and 5 pieces left rather than the fractions 9/10 and 7/12.

To address Lisa’s focus on the pieces left, I attempted to connect the coloring of pieces to frosting on a cake. Despite this association of color to frosting, when I asked Lisa how many pieces were “gone” from the area model of 9/10, she again interpreted the nine shaded pieces as gone.

Tutor: So is the orange stuff the stuff that we are eating and getting rid of? Or is it the stuff that is still there?
Lisa: I imagined it was the stuff that we were getting rid of.
Tutor: And this is one way where I think these pictures can be really tricky, is knowing what is there and what we are getting rid of. If we think about these cakes as having orange frosting.
Lisa: Right.
Tutor: Right? So, we have a cake that is divided up into ten pieces? Right?
Lisa: Uh huh.
Tutor: How many pieces are gone from this cake?
Lisa: Um, nine?
Tutor: Ok, so, but the cake has orange frosting on it. Right, so this is sort of like a picture.
Lisa: Oh, so one. Only one.
Tutor: One piece is gone from here, how many pieces are gone from there?
Lisa: Um. Five.
Tutor: Uh huh. So one of the things that is sort of backwards in the drawing of pictures like this is that the stuff that we are shading in, is actually the stuff that is there.
Lisa: That’s actually there.
Tutor: So we are actually drawing what’s there.

Although Lisa accepted the idea that the cake had orange frosting, she still did not understand the area model representation as a picture of a fractional quantity. This episode suggests that area model representations were not a transparent representation of a fractional quantity. Instead, for Lisa area models involved an assumption of the shading as an amount removed. A simple trick for thinking of how to attend to these representations was in conflict with her intuitive conception of the meaning of shading. In the next excerpt Lisa continued to attempt to reorient her understanding to fit it into the more canonical understanding to compare 9/10 and 7/12. She correctly determined that 9/10 was larger, and in her explanation she seemed to be struggling with ascribing meaning to the shaded and non-shaded regions.

Tutor: Which cake has more there?
Lisa: Uh the this one (points to 9/10 drawing)
Tutor: Ok, and how do you know?
Lisa: Um, I guess. ok. Ok ok. So, um, if this is stuff that is there (points to shaded in the 9/10), and this is stuff that’s not there (points to non-shaded part of 7/12). Then, nine-tenths would be – wait, hold on. Then this (points to 9/10 picture) would be – this would have more.
Tutor: Ok.
Lisa: Yeah. I think.

Although Lisa was eventually able to compare 9/10 and 7/12 by focusing on the shaded regions, this reorientation to the representations appeared to be difficult for her to implement and was short-lived. On the next problem Lisa was asked to compare the fractions 1/2 and 2/3, she determined that 1/2 would have bigger pieces. She drew area models for both fractions correctly, but her comparison of the fractions was based on the amounts that were “going away.”
Tutor: Which of these has bigger pieces? *(point to 1/2 and 2/3)*
Lisa: One-half.
Tutor: Ok, so the one-half is the bigger pieces? *(writes “bigger pieces”)*
Lisa: Well, yeah. Cause if you were – yeah, I almost think I’m positive. So I don’t know. So ok *(draws square divides in two, shaded half)*. That’s like half the cake goes to someone *(draws a line coming out from the shaded part)* that’s like pretty big, whereas *(draws square divides into 3 shades 2)* No, I’m wrong. Two-thirds would be the bigger pieces.
Tutor: Ok, so what pieces are we comparing here?
Lisa: Um. This going away *(move hand over shaded area for 1/2 and move away: see Figure 19)* whereas this going away *(move hand over 2/3 shaded and away: see Figure 19)*, is more than one-half.

Figure 19. Screen snapshots with synched transcript of Lisa’s gesture when explaining that the shaded region is going away.

It is unclear from the above transcript whether Lisa was comparing the size of the pieces (halves versus thirds) or the fraction themselves (1/2 and 2/3). What is clear is that Lisa was continuing to understand the shading as the quantity taken away. To help Lisa reorient to the shaded region, I referred back to the previous problem involving the 7/12 area model, and she said, “Oh shit. I keep imagining that this *(pointing to the shaded pieces of 2/3)* is being taken away.” This suggests that Lisa’s understanding of the shaded region as being removed was robust and resistant to instruction.

**Summary and conclusion.** Lisa’s “taking” understanding resulted in an instability in the way in which she conceptualized fractional quantity. An understanding of the numerator value as “taken” led to ambiguity in how Lisa subsequently interpreted fractional representations. This understanding was robust in that it persisted throughout all the sessions and was evident across representations and problem types. This “taking” understanding was considered consequential in that it derailed her ability to make sense of more complex fraction concepts. Unlike a simple misunderstanding, “taking” was
resistant to attempts to address it, suggesting that “taking” was core and consequential to Lisa’s conceptualization of fractional quantity.

Lisa’s Halving Understanding

This section presents the second of Lisa’s five atypical understandings: “halving.” Lisa sometimes understood the fraction 1/2, not as a quantity one-half, but as the action of halving. Lisa’s over-application of a “halving” understanding resulted in difficulties, because 1/2 was used as a grounding fraction in the tutoring sessions to introduce new fraction concepts. Lisa’s halving understanding increased in frequency over the course of the tutoring sessions. Therefore, one of the primary analytic foci in this section is considering the source and cause for this increased reliance upon “halving.” This atypical understanding was detrimental to her ability to engage particularly with more complex fraction concepts and ultimately provides one of the key pieces towards understanding why Lisa did not learn.

In this section, first I introduce the “halving” understanding and provide a prototypical example in conjunction with the operational definition. Second, I discuss, how this understanding proved to be problematic for Lisa’s understanding of more complex fraction concepts. Third, I consider evidence that Lisa increasingly relied upon her halving understanding over the course of the tutoring sessions. Specifically, I consider a comparison of the pretest and posttest, in the pretest, Lisa exclusively used the quantity understanding of one-half (non-halving examples), and at the time of the posttest, she exclusively used a “halving” understanding. I then consider the origin and potential factors that may have led to the increased use of Lisa’s “halving” understanding. Although Lisa’s “halving” understanding appeared to originate before the tutoring sessions, representations used for 1/2 within the tutoring sessions may have reinforced her “halving” understanding. Finally, I explore the reasons why this halving understanding was not resolved, which was likely due, in part, to Lisa’s tendency to self-correct. This strand of analysis illustrates that Lisa’s “halving” understanding was atypical, that she increasingly relied upon “halving” when encountering more complex fraction concepts, and consequently was detrimental to her ability to learn.

Defining and exemplifying “halving” understanding. Lisa’s “halving” understanding involved the ways in which she understood and represented the fraction one-half. To highlight the atypicality of Lisa’s understanding it will be contrasted with a typical understanding of the fraction 1/2, in the most commonly used representational form: area models. Typically, when an individual uses an area model to represent the fraction 1/2, he/she draws a shape, partitions the shape into two equal pieces and shades in one of those pieces (see Figure 20). The quantity 1/2 is therefore, signified and highlighted by the shading. In contrast, Lisa sometimes represented 1/2 by drawing a shape and dividing it two pieces, omitting the shading (see Figure 20). Lisa’s drawing of one-half involved representing the halving (or splitting) action as opposed to the quantity of 1/2. This halving understanding was often over-applied to contexts in which the quantity understanding should have been used.
Figure 20. Contrast of typical and atypical use of drawn models of 1/2.

Prototypical example of halving. Lisa’s “halving” understanding was evident in the following example, in which Lisa represented the fraction one-half by partitioning several shapes in half and omitting the shading. During the posttest, Lisa was asked to draw or write one-half. She began by writing 1/2, then drawing a rectangle, dividing it in half, and then drawing a circle and dividing it in half.

Lisa provided one typical representation of 1/2, using Hindu-Arabic notation of “1/2,” but then provided two atypical representations of one-half, by drawing a shape and partitioning it in two. I followed up by asking her if there was any other way she could represent one-half. She wrote the decimal value “1.5”\(^{25}\) and then drew another rectangle and divided it in half length-wise.

\(^{25}\) Lisa’s answer of “1.5” is not a focus of analysis, because decimal notation was not a topic covered in the tutoring sessions. It is likely that her understanding of one-half as...
Tutor: Cool. Is there any other way you can think of to draw or write one-half?
Lisa: Um. Is it one point five? (writes “1.5”) I think that is one-half. Um. I could
draw it this way (draws skinny rectangle).
Tutor: So, there’s different kinds of different shapes you can use?
Lisa: Yeah.

Lisa represented one-half using three different shapes and in each case she
partitioned the shape in half, but did not shade one of the halves. Unfortunately I did not
ask her to explain her representation of one-half. Despite this, it is likely that her
representation of one-half was embodied in the partitioning of the shape into two pieces.
Other examples more explicitly provide evidence that she understood the partitioning line
itself as the representation of one-half.

This prototypical example was considered consistent with a “halving”
understanding because Lisa did not represent the fractional quantity one-half. Instead she
halved each of the shapes, and accepted this as a valid representation of the fraction 1/2.
This was not simply a matter of her lack of knowledge of the shading convention,
because on the next problem she was asked to represent the fraction 3/4, and she
produced two canonical area model representations, where the shaded region
corresponded to the numerator value (see Figure 21). Therefore, Lisa’s lack of shading
in her representations of 1/2 was not a misunderstanding of representational convention,
instead it signified her understanding of the fraction 1/2 as the halving action.

Figure 21. Scanned artifact from posttest question in which Lisa was asked to draw or
write 3/4.

1.5 has to do with a literal reading of the notation 1.5 as “one”(1) and “half”(.5). Her
answer of 1.5 was also consistent with her answer on the pretest.
**Researcher’s operational definition of halving.** Problems were coded as indicative of a "halving" understanding if (1) Lisa represented 1/2 by drawing a shape, partitioning it in two, and omitted the shading, (2) if Lisa’s gestures and explanations were consistent with 1/2 as a splitting-action rather than a quantity, or (3) if Lisa justified her answer by focusing on the balance and similarity between the two quantities (part-part understanding) rather than focusing on the one part out of the total number of parts (part-whole understanding). Non-halving examples included, instances in which Lisa talked about 1/2 as being one out of 2 total parts or rejected drawings of 1/2 without shading.

There were 12 instances of Lisa’s “halving” understanding and 6 flagged non-halving examples. As seen in Figure 22, this understanding appeared towards the end of the sessions and when used, it was often in conjunction with an incorrect answer.

**Halving as detrimental to learning.** Lisa’s “halving” understanding was over-applied to contexts in which the quantity understanding of 1/2 was necessary. One example will be presented which highlights how “halving” was detrimental in the context of equivalent fractions. I then consider Lisa’s accuracy on all problems involving the fraction 1/2, to give a high level view of the ways in which her understanding of 1/2 was problematic for certain problem types. In this section I illustrate how Lisa’s “halving” understanding at times undermined her learning and generally was insufficient to build more complex fraction concepts.

In the following example, taken from the end of the third tutoring session focused on fraction equivalence, Lisa was writing notes for herself in her journal. The journaling activity was intended to be a culminating activity and opportunity to synthesize the major topics covered in the tutoring sessions. Lisa had just successfully created 14 equivalent fraction pairs, and she was attempting to record an example in her journal. She illustrated equivalent fractions by first drawing one-half with no shading, and then partitioning it into a total of six pieces. Because her initial representation of one-half omitted the shading, this led her to conclude that one-sixth was the equivalent fraction. (Note: in this session we used transparencies to help repartition area models and she referred to them during this excerpt, in Line 3)
Lisa: Ok, so like, what we were doing, if you have (draws rectangle)

like one-half (draws partition line),

and you put a transparency over it, so that's one half (writes “1/2”).

(switches pen colors)
And then like, cut it like that, (draws in purple lines) it still stays the same. It is just cut into different sections then.

Katie: So, how many sections is it cut into?
Lisa: Six. So one-sixth?

In this episode she represented 1/2 by partitioning the shape into two pieces. When drawing the vertical partition (line 2) she said “one-half”, and then re-emphasized that her drawing did in fact show the value one-half by reiterating “that’s one-half” and writing “1/2” next to her drawing (line 3). At this point she took the time to switch pen colors and then repartitioned the shape, and asserted that “it still stays the same,” (line 4) referring back to the original point of her journal entry (exemplifying equivalent fractions). When I asked her how many sections it was cut up into (line 5) she correctly determined that the shape now had 6 pieces, but incorrectly stated that the equivalent fraction would be one-sixth. Although Lisa eventually recognized that she had not shaded in her initial drawing of 1/2, this example illustrates how Lisa tended to draw upon the “halving” understanding in inappropriate contexts. Recall that this was intended to be a culmination of what she had learned, but Lisa’s “halving” understanding led her to conclude that 1/2=1/6. This example provides an illustration of how her understanding of the fraction 1/2 was not stable enough to build more complex fraction concepts.

To consider the ways in which Lisa’s understanding of one-half was potentially problematic on a more general level, I considered all occurrences of the fraction 1/2 in the tutoring sessions. On the 81 problems involving the fraction 1/2, Lisa had a 63% accuracy rate. Of particular interest was Lisa’s performance on the more advanced
problem types including: comparison, equivalent fractions, and fraction operation problems. When classified by problem type, Lisa’s accuracy on comparison and operation problems was 10% and 33% respectively (see Figure 23), supporting the hypothesis that Lisa’s understanding of one-half was not sufficient to build those fraction concepts. In contrast, her equivalent fraction performance for problems involving 1/2 was at 88% accuracy rate. An exploration of why she was so successful for fraction equivalence problems involving one-half, suggests that her “halving” strategy might have been productive. In describing equivalent fractions for one-half she said, “it [the denominator] splits itself evenly.” This suggests that Lisa’s halving understanding, an understanding that involved splitting, was productively leveraged to help her determine denominator-numerator pairings for fractions equivalent to 1/2. Although her halving understanding was potentially productive for her calculation of equivalent fractions, this “halving” understanding (as seen in the previous example) could be problematic. Therefore, Lisa’s understanding of 1/2 was not a stable foundation on which should could begin to productively explore more complex fraction concepts and Lisa’s “halving” understanding appeared at times to undermine her progress.

![Frequency counts for problems involving one-half](image)

Figure 23. Frequency counts for all problems involving 1/2 across tutoring sessions classified by problem type and correctness.

**“Halving” across the sessions.** This section considers Lisa’s increased reliance upon her “halving” understanding over time. First I contrast Lisa’s answers on the pretest to her answers on the posttest, which highlight an apparent shift in her understanding from a canonical quantity understanding of one-half to her atypical “halving” understanding. Second I consider the origin of Lisa’s “halving” understanding, Lisa difficulty with comparison problems involving one-half was also likely due to her “unit fraction” understanding, which will be discussed more extensively during that strand of analysis.

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26 Lisa difficulty with comparison problems involving one-half was also likely due to her “unit fraction” understanding, which will be discussed more extensively during that strand of analysis.
which was evident at the time of the initial screening interview and therefore, appears to have predated the pretest. Third I consider potential causes for Lisa’s increased reliance upon “halving” over the course of the tutoring sessions. In particular, I focus on Lisa’s language use when referring to the fraction one-half, and the use of one instructional tool (transparency guides) as a potential cause of her increased halving usage. Lastly, I consider why this problematic understanding was not resolved over the course of the tutoring sessions.

**Increase in use over time: “halving” during the pretest and posttest.** Lisa’s pretest and posttest instances of “halving” illustrate a shift in her understanding of one-half. Two problems will be used to demonstrate the shift in understanding: (1) a construction problem where Lisa was asked to draw and write one-half, and (2) an interpretation problem, where Lisa was asked to choose from various possible drawings of one-half, including one drawing of one-half with the shading omitted (see Figure 24).

![Figure 24](image-url)

**Figure 24.** Pretest problem administered to Lisa in which she was asked to circle all pictures that were the same thing as 1/2.

During the pretest, Lisa demonstrated an understanding of one-half as the quantity one-half, and was explicit about this both in her drawn representations and her interpretation of printed representations. When drawing one-half she used shading and explained that her drawing was “a half of a whole,” highlighting the relationship of the quantity of one-half to the specified whole. Similarly, when given the choice of an area model with no shading, she rejected it, and explained that it was not the same as one-half because, “nothing is shaded in.” During the pretest, the halving of a shape alone was insufficient to symbolize one-half. Her pretest performance was remarkably different from her answers to the same questions on the posttest.

At the time of the posttest, Lisa’s answers to both construction and interpretation problems suggested that she had shifted to relying on a “halving” understanding of the fraction 1/2. As previously discussed in the prototypical example, Lisa drew three area models for one-half all without shading. Similarly, out of all the possible representations of 1/2, she first circled the circular drawing with no shading (see Figure 25a). It was only after she looked at the canonical area model of one-half (with shading) that she returned to the non-shaded area model and added a question mark (see Figure 25b). Although she
eventually rejected this choice, she explained her initial answer with a gesture signifying a cutting or splitting action (see Figure 26).

![Figure 25. Digitally modified artifact showing the sequence of markings on this problem.](image1)

Tutor: You put a question mark by this one? *(points to non-shaded area model, see Figure 25b)*

Lisa: Yeah, I was wrong, I just automatically saw something that was *(chopping gesture with hand, see Figure 26).*

![Figure 26. Screenshots capturing Lisa’s chopping gesture.](image2)

Based on her chopping gesture, it suggests that she initially was thinking of the action of “halving” as opposed to the quantity one-half. At the time of the posttest she both drew and interpreted representations of 1/2 relying on her “halving” understanding.

During the pretest Lisa demonstrated an understanding of one-half that was focused on quantity27, whereas during the posttest she relied upon a “halving” understanding of one-half (see Figure 27). The subsequent sections examine this shift in Lisa’s “halving” understanding, specifically concerned with the origin of this understanding, potential reasons behind Lisa’s increased reliance on “halving”, and why this understanding was never resolved.

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27 The number of non-halving examples flagged in the pretest largely was an artifact of the nature of the discussions during the pretest. Recall that Lisa was administered a slightly different version of the pretest. The version that she was administered was actually more of an introduction into fraction concepts, with one-half being used as a grounding fraction. Much of our discussion involved explicit conversations about the meaning of the various fraction components, often referring to the fraction 1/2. Therefore, the prevalence of non-halving examples is inflated due to the number of questions explicitly posed about the fraction one-half.
Origin of the “halving” understanding. Given Lisa’s increased use of “halving” over the tutoring sessions, the question of the origin of this understanding is of primary import. Although the first instance of “halving” understanding did not occur until the second tutoring session, this atypical understanding appeared in Lisa’s initial screening interview, two months before the pretest was administered. I asked her to describe how she would solve the problem $\frac{1}{4} + \frac{1}{2} = \_$. To answer this problem she drew both the fraction one-fourth and one-half with no shading (see Figure 28).

Figure 28. Artifact from screening interview in which Lisa drew the one-fourth and one-half with no shading as she attempted to solve the problem $\frac{1}{4} + \frac{1}{2}$.

Lisa’s representation of $\frac{1}{2}$ during the screening interview indicated that this halving understanding predated the tutoring sessions (see Figure 29), which suggests that the tutoring sessions themselves were not the origin of this atypical understanding.

Figure 29. Data display of “halving,” highlighting the location of the first documented instance of her “halving” understanding in the screening interview.
Increasing persistence of “halving.” Although Lisa’s “halving” understanding predated the tutoring sessions, it was important to consider how the tutoring interaction may have inadvertently supported her reliance on a “halving” understanding. Ninety-two percent of all “halving” instances occurred in the second half of the sessions (see Figure 29). In this section I consider potential causes for Lisa’s increasing reliance on “halving.” Two representational forms were examined. First Lisa’s truncated linguistic reference to one-half as “half” might have unintentionally allowed for ambiguity around whether the splitting or the quantity meaning of one-half was intended. Second, the area model transparencies used to further partition area models might have supported her understanding of one-half as “halving”.

Linguistic reference to one-half as “half.” During the tutoring sessions, Lisa referred to the action of halving and the quantity one-half, using the same word: “half” (e.g., “half of a half” to mean further partitioning a representation of the quantity 1/2). Because of this potential ambiguity of meaning around the term “half”, I did frequency counts of every linguistic reference to a fraction, to determine if there was anything unusual about her way of referring to the fraction one-half. Analysis identified 5 different forms, which Lisa used to refer to fractions including: a truncated form (“half”), a standard form (“one-half”), a discrete set form (“one-out-of-two”), a separated form (e.g., 5/10: “five one-tenths”), or an unusual form (1/10 as “ten” or “one-ten” or “one-over-ten”). Frequency counts of linguistic references indicated a disparity between one-half utterances and other fraction utterances. As seen in Figure 30, Lisa most commonly referred to the fraction “one-half” using a truncated form “half.” The word form “half” can be understood and act as either a verb form (to “halve” something – often indistinguishable from “half”) or a noun form (“half” of something). The frequency of the truncated word form “half”, suggests a way in which Lisa’s understanding of one-half was ambiguously spanning a quantity and an splitting understanding.

![Lisa's Linguistic Reference to All Fractions](image)

Figure 30. Percentages for various forms of all Lisa’s fraction utterances, contrasting utterances of one-half with utterances of all other fractions.
Fraction transparencies. One potential candidate for increased occurrence of this understanding was one of the representational tools introduced during the equivalent fraction tutoring session. Lisa was asked to use transparencies to help create equivalent fractions. In this activity she selected a square area model card (see Figure 31), and then selected a transparency and overlaid the transparency on the area model. She then drew in the lines on the transparency using the markings as a guide. The resulting area model produced the equivalent fraction, which she would record. This model allowed for the removal of the partitions so the student could again “see” the original fraction.

Lisa was introduced to the transparency model during the third tutoring session (see Figure 32). One of the transparencies she used was labeled as “halves” and served the “halving” function. During this session she used the halving transparency 3 times out of 10 total transparency uses. In this way the halving action was represented by a distinct and manipulatable object. This instructional tool was intended to help support the student’s understanding of equivalent fractions; this tool, however, may have produced unintended consequences of supporting an understanding of one-half as a halving action. Given the temporal location of the introduction of the transparency model in conjunction with the documented halving instances, the transparency model is a likely candidate for supporting Lisa’s increased use of her halving understanding.

Both the transparency models and Lisa’s linguistic reference to “half” might have contributed to Lisa’s increased reliance upon her halving understanding.

Persistence of “halving.” Although this understanding appeared relatively infrequently, it persisted over the sessions. In this section, I explore the reasons that “halving” was not resolved. One of the problems with explicitly addressing this kind of
atypical understanding was the fleeting nature of its use. Lisa’s halving understanding was often self-corrected after I asked a clarifying question. Because Lisa also had access to a quantity understanding of one-half, she would often quickly correct her atypical halving understanding. For example, Lisa’s halving understanding appeared when she attempted to illustrate the solution to $1/2 + 1/4$ using area models. I asked Lisa to represent each of the parts of the problem ($1/2$ and $1/4$) and the answer separately. Lisa began constructing the shape representing one-half with no shading. She then partitioned the square into four pieces and shaded the right half of the shape. As I tried to clarify what she was doing she identified the partition line as signifying “one-half.” Midway through this explanation, she stopped herself, crossed out her representation, and drew a canonical area model for $1/2$ and for $1/4$. This episode highlights the often rapid rejection of this atypical understanding over the course of Lisa’s solution process.

<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript</th>
<th>Illustration of Artifact</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tutor: So, if we were to draw the intermediate steps.</td>
<td><img src="image1.png" alt="Illustration of Artifact" /></td>
</tr>
<tr>
<td>2</td>
<td>Lisa: um, ok. So it would be <em>(draws square)</em></td>
<td><img src="image2.png" alt="Illustration of Artifact" /></td>
</tr>
<tr>
<td>3</td>
<td>that’s one half <em>(draws vertical line partitioning)</em></td>
<td><img src="image3.png" alt="Illustration of Artifact" /></td>
</tr>
<tr>
<td>4</td>
<td>and <em>(draws horizontal line partitioning)</em></td>
<td><img src="image4.png" alt="Illustration of Artifact" /></td>
</tr>
<tr>
<td>5</td>
<td>that’s <em>(shades in right half) [inaudible]</em></td>
<td><img src="image5.png" alt="Illustration of Artifact" /></td>
</tr>
<tr>
<td>6</td>
<td>Tutor: So, you are drawing one half right now?</td>
<td><img src="image6.png" alt="Illustration of Artifact" /></td>
</tr>
<tr>
<td></td>
<td>Lisa: Yeah.</td>
<td><img src="image7.png" alt="Illustration of Artifact" /></td>
</tr>
<tr>
<td></td>
<td>Tutor: Ok.</td>
<td><img src="image8.png" alt="Illustration of Artifact" /></td>
</tr>
<tr>
<td></td>
<td>Lisa: Yeah, Well, Yeah, I guess, at the moment. This is one-half. <em>(colors in the horizontal partition line)</em></td>
<td><img src="image9.png" alt="Illustration of Artifact" /></td>
</tr>
<tr>
<td>7</td>
<td>and then to make it one-fourth <em>(draws</em></td>
<td><img src="image10.png" alt="Illustration of Artifact" /></td>
</tr>
</tbody>
</table>
In the above example, Lisa appeared to represent $1/2$ with the partition line (line 3). When I asked a clarifying question, she reemphasized that she was drawing one-half by coloring in one of the partition lines darker (line 6). With no further intervention from me, Lisa recognized her error and corrected it by crossing out the shape and drawing a canonical area model of $1/2$, which she then labeled. As in this example, in almost all cases, where she drew one-half with no shading, she corrected her own answer. Given that most of these instances were resolved with simple clarifying questions from me, I did not choose to explicitly address this halving representation of one-half\textsuperscript{28}. The interaction with the tutor, therefore, did not resolve this halving understanding, and towards the end of the data collection, this halving understanding was increasingly invoked.

**Summary and conclusion.** “Halving” involved understanding the fraction one-half to be represented by the act of splitting (represented by the partition line) as opposed to the quantity of $1/2$. Lisa’s halving understanding appeared to pre-date the tutoring.

\textsuperscript{28} Arguably, it might have been a pedagogical mistake to allow this representation of one-half to remain undiscussed, however, as the tutor, it felt like there were other issues that required more time and attention, and given that she had resolved her own inappropriate representation, it was not taken up as the topic of an extended tutor/student discussion.
sessions, and she increasingly relied on this understanding over the course of the tutoring
sessions. The representations used during the tutoring session, although not the origin of
this understanding, may have inadvertently increased her reliance on halving. This
understanding was problematic for her development of more complex fraction concepts,
and may have been particularly detrimental considering the centrality of the fraction 1/2
in the tutoring sequence. The tutoring sessions did not sufficiently address and resolve
this atypical understanding. “Halving” provides one dimension of why Lisa did not
learn.

Lisa’s Discrete Set Understanding

This section presents the third of Lisa’s five atypical understandings: “discrete
set.” This name of this understanding is derived from a discrete set model for fractions.
A discrete set model is one of way of conceptualizing fractional quantities, often related
to a ratio context (e.g., “12 out of 20 of the students are girls”). Lisa’s often understood
continuous models (e.g., area models) as if they were discrete set models, therefore
incorrectly ignoring the size of the whole or the size of the individual pieces. Lisa’s
over-application of the “discrete set” understanding to inappropriate contexts was
detrimental to her ability to engage with more complex fraction concepts and ultimately
provides one of the key pieces towards understanding why Lisa did not learn.

In this section, first I introduce the “discrete set” understanding by contrasting it
with a typical use of a discrete set model. I provide a prototypical example from the data
and specify the operational definition used to code instances of “discrete set.” Second, I
discuss, how this understanding proved to be problematic for Lisa’s understanding of
more complex fraction concepts. Third, I consider the robustness and persistence of this
understanding over the course of the tutoring sessions. Lisa relied on her discrete set
understanding during both the pretest and posttest, suggesting that this knowledge
persisted across the tutoring sessions. At times, Lisa’s discrete set understanding
appeared to serve her well, and when she was on uncertain conceptual terrain, she often
referred to fractions using terminology consistent with a discrete set understanding (“two
out of three” rather than “two-thirds”). However, her discrete set understanding was
incompatible with the primary representational tools used during the tutoring sessions
(fraction pieces and area models) and caused difficulties when she applied this
understanding to more complex fraction concepts. Lastly, I show that even a deviation
from the tutoring protocol that explicitly intended to address the differentiation of
discrete and continuous models was ineffective in refining Lisa’s understanding of when
to apply her discrete set model.

In the case of the discrete set model of fractions the numerator and denominator are
whole numbers, the denominator represents the number in the set, and the size of the
individual elements composing that set is unimportant. This can be contrasted with a
continuous model of fractions (e.g., area model), where the numerator and denominator
are not necessarily whole numbers and the size of the pieces, and the whole are crucially
important.
**Defining and exemplifying “discrete set” understanding.** Lisa’s "discrete set" understanding involved over-applying a discrete set model of fractions to inappropriate contexts (continuous models, fraction operations). For discrete set models, the size of the pieces comprising the whole set is unimportant (see Figure 33). If one out of 3 shapes is a square, it is unimportant if the square is a different size than the other shapes. However, Lisa often inappropriately applied the discrete set model to contexts where the size of the pieces was important (i.e., continuous models, like area models). This often resulted in interpretation of continuous models using inappropriate conventions (like not considering the size of the pieces or not attending to the size of the whole). This discrete set understanding can also result in inappropriate operations on fractional amounts. Although, combining a set that is 1-out-of-3 and another that is 2-out-of-5, does result in a set that is 3-out-of-8, the same is not true for continuous models, or any non-ratio fraction (see Figure 34). 1/3+2/5 does not equal 3/8, and one may not consider the one-third piece and the one-fifth pieces to be interchangeable.

Unlike the first two understandings presented (halving and taking), this understanding does appear as a common student misconception in the literature (Saxe, Taylor, MacIntosh, & Gearhart, 2005). Some students may interpret continuous models in terms of discrete set models (Saxe, Langer-Osuna, & Taylor, under review), however, this misunderstanding tends to get resolved relatively easily and does not tend to resurface as students engage in more complex fraction concepts. The atypicality of Lisa’s understanding in part has to do with the robustness of her discrete set conception, the wide domain of applicability she applies it to, and its resistance to explicit instruction intended to address the differentiation of continuous and discrete models. Although this conflation of discrete and continuous models is a common student difficulty, Lisa’s “discrete set” understanding should be thought of as atypical.

![Figure 33](image.png)

**Figure 33.** Contrast of typical (appropriate) and inappropriate use of discrete set understanding.
Prototypical example of Lisa’s discrete set understanding. Lisa’s discrete set understanding was evident in the following example, during the second tutoring session, when she was asked to draw a series of pictures of fractions. After correctly drawing an area model for $1/2$ and $1/6$, she attempted to draw an area model of $3/5$. Her representation and subsequent explanation both highlight her conceptualization of this representation in terms of a discrete set understanding.

Lisa construction of $3/5$ (see Figure 35) began by (a) drawing a rectangle and (b) partitioning the rectangle in half horizontally and then (c) partitioning the rectangle in half vertically. At this point she had created a square partitioned into 4 (relatively) equal pieces. She completed her drawing by (d) appending an extra square on the right side of the shape and (e) shading in the bottom three pieces. Her drawing was unconventional and unlike her previous two canonical area models. When I later asked Lisa how she had constructed this drawing, she explained, “I just drew 5 squares and then I shaded in 3 of them.” Both her drawing and her subsequent explanation indicated that she understood her drawing as a collection of five independent squares rather than a shape partitioned into five pieces.

This example was considered consistent with a “discrete set” understanding because Lisa’s drawing and subsequent explanation indicated that she was treating the five squares as discrete entities, rather than understanding her representation to be a whole partitioned into five pieces. Admittedly, dividing a shape into five equal pieces is a difficult partitioning task. Often times, an individual attempting to divide a shape into a number of equal pieces may end up similarly tacking an extra piece on the end (see Figure 36). However, in these instances it is implied that one is to interpret the resulting shape as the whole. Only someone who watched the construction would be able to tell...
the difference between a shape augmented with an extra piece and a shape that had been accurately partitioned. This is unlike Lisa’s treatment of her final drawing, where she explained that she “drew 5 squares” rather than dividing the shape into 5 pieces. In her case, she understood the parts as individual units (discrete set model), rather than pieces of a whole (continuous model).

![Figure 36](image)

**Figure 36. Illustration of how an individual might augment a shape to divide it into five equal pieces.**

**Researcher’s operational definition of discrete set understanding.** Problems were coded as indicative of a "discrete set" understanding if Lisa (1) ignored the intentionally unequal size of the pieces when interpreting fractional representations, (2) understood each part of a continuous model as a discrete entity (e.g., 6 squares) as opposed to a part of a whole (rectangle partitioned into 6 pieces) (3) did not account for the difference between sizes of wholes when comparing continuous models of fractions, or (4) operated on fractions as if they were represented by a discrete set (1-out-of-6 plus 1-out-of-5 equals 2-out-of-11). Non-discrete set instances involved Lisa explicitly referring to the importance of equally sized pieces.

There were 35 instances of Lisa’s “discrete set” understanding and one “non-discrete set” example. As seen in Figure 37, this understanding was often used in conjunction with an incorrect answer and persisted across all sessions.

![Figure 37](image)

**Figure 37. Data display of all instances of Lisa’s “discrete set” understanding.**

**Discrete set as detrimental to learning.** Lisa’s discrete set understanding often led to difficulties particularly in contexts where she was attempting to solve fraction operation problems. For example, Lisa was asked to solve the problem 1/3+1/3= using pictures. She used continuous area models to correctly represent the two quantities being added (1/3 and 1/3; see Figure 38), but to construct the solution to this addition problem, she partitioned the shape into 6 pieces and shaded 2 pieces. In this instance she applied a
discrete set understanding and treated both the shaded and non-shaded pieces as entities that were added together. She explained that, “it’s one-third to one-third, it would be, um, two-sixths.” Although Lisa drew consistently sized wholes for this problem, she did not attend to the difference in size of the pieces between her initial representations of 1/3 and her final representation of 2/6. Lisa’s “discrete set” understanding was often detrimental to her ability to use continuous models to represent and understand more complex fraction concepts.

Figure 38. Artifact showing Lisa’s initial attempt to represent 1/3+1/3 = using drawings and her answer of 2/6.

“Discrete set” across the tutoring sessions. This section presents evidence of the persistence and robustness of Lisa’s “discrete set” understanding over time. First, I consider the persistence of Lisa’s use of “discrete set” understanding in conjunction with continuous models during the pretest and posttest, which suggests that Lisa was relying upon this understanding both before and after the tutoring sessions. Second, I explore potential causes for this reliance upon a “discrete set” understanding. In particular I investigate the prevalence of discrete set representations in the tutoring sessions themselves, and the ways in which this understanding was at times productive for Lisa. Third, I consider the interaction of the “discrete set” understanding with the continuous models used in the tutoring sessions. Her “discrete set” understanding can be thought of as incompatible with the majority of the representations used during the tutoring sessions. Lastly, I consider the robustness of Lisa’s understanding, which resisted explicit attempts to differentiate discrete and continuous models.

Persistence over time: “discrete set” during the pretest and posttest. Lisa’s pretest and posttest instances of “discrete set” were examined to determine how she was using this understanding before and after the tutoring sessions. During both the pretest and posttest, Lisa used her “discrete set” understanding to incorrectly name the fractional quarter of paper. In this activity (see Figure 39) Lisa and I repeatedly folded a piece of paper in half, cut along the fold, then gave away one piece, eventually resulting in Lisa and I both having 1/4 of our original paper (Lisa’s paper was square and my paper was triangular, see Figure 40). When Lisa was asked to name the fraction for the piece of paper (1/4 of the whole), during both the pretest and the posttest, Lisa determined the piece was “one-third.” Her answer indicated that she was not attending to the unequal sizes of the pieces in her determination of the fractional amount. Lisa’s consistent answer of 1/3 during the pretest and posttest suggests that she persisted in understanding this continuous model of a fractional amount in terms of a discrete model of fractions.
Figure 39. Illustration of the Paper Folding/Cutting Activity.

Figure 40. Screenshot from the posttest showing the reassembled “cake”. Lisa determined the focal piece, (under right hand) was equivalent to one-third.

It should be noted that Lisa’s discrete set understanding was not simply a matter of misinterpreting unequally partitioned fractional quantities. Lisa applied a discrete set understanding to her equally partitioned area models as well. During the pretest, Lisa described her correctly drawn area model of $3/4$ using a discrete set understanding “three out of four – cause there are three different ones and there are four different squares.” This suggests that she understood her representation not as a whole that was partitioned into four pieces (continuous model) but as four independent squares (discrete model). Both Lisa’s interpretation of continuous models, and her understanding of her continuous model in discrete terms, suggested that this discrete set understanding was present both before and after the tutoring sessions.

**Potential explanations for prevalence of discrete set understanding.** Given Lisa’s reliance on the discrete set understanding, I considered (1) whether the tutoring sessions privileged a discrete set model of fractions and (2) the ways in which her discrete set understanding may have been productive for Lisa in certain contexts.

A classification of all problems based on problem type indicated that the tutoring sessions themselves did not privilege a discrete set model. Discrete set representations, including those produced by Lisa, were used in less than 5% of the problems. This can be contrasted with continuous models that were used in 75% of all problems. The remaining problems either involved numeric notation (14%), no representation (6%) or decimal notation (1%). Despite the infrequency of discrete models, Lisa often applied a discrete set understanding to continuous models.

One potential reason that the discrete set understanding was so pervasive throughout the tutoring sessions was that in many ways a discrete set understanding was productive for Lisa. Informally, I noticed while tutoring Lisa that she was the most...
accurate in her fraction naming when she was using discrete set terminology (i.e., “1 out of 3”), and that when she was struggling to make sense of challenging topics she often reverted to discrete set terminology. For example, in the following problem when she was attempting to name the equivalent fraction pair 3/4 and 6/8 (as made with area model squares and transparencies, see Figure 41), she interpreted the equivalent fraction as “two-sixths,” instead of 6/8. As she corrected her answer, she switched to using discrete set terminology to determine that the equivalent fraction was not “two-sixths” but instead “6 out of 8.”

Figure 41. Screenshot from the tutoring session showing an area model square (3/4), with a transparency partitioned in halves overlaid (creating 6/8).

In a separate analytic pass, I flagged each instance of Lisa’s use of discrete set terminology (that was not otherwise classified as indicative of discrete set understanding). On the problems classified as discrete set terminology, Lisa had a 70% accuracy rate. This figure is even more impressive if we zoom into the problems and consider her accuracy not at the problem level, but during her actual discrete set terminology utterance. Although the problem might have been coded “incorrect,” her statements when using the discrete set terminology were actually all correct (see Figure 42). Therefore, not only was Lisa’s discrete set terminology generally used in conjunction with a correct answer, but when associated with an incorrect answer, her discrete set utterance was the part of the answer that was correct. Given the incredible effectiveness of Lisa’s use of discrete set terminology, Lisa’s reliance on a discrete set understanding can be better understood. Despite the lack of discrete models used in the tutoring session, Lisa’s reference to fractions in terms of a discrete set form, suggested that this way of conceptualizing fractions (in terms of two whole numbers) was often productive for her.

Figure 42. Data display of all discrete set terminology problems across the tutoring sessions, each instance of an incorrect answer involved a discrete set terminology utterance that was correct.

_Incompatibility of discrete set understanding and continuous models._ Although Lisa used discrete set terminology effectively, particularly when she was attempting to
make sense of more complex fraction concepts, this discrete set understanding was problematic when applied to continuous models. Ninety percent of instances of Lisa’s discrete set understanding were in conjunction with continuous models. This suggests that the domain of applicability for her discrete set understanding was not appropriately constrained.

Area models. Lisa’s understanding of fractions in terms of a discrete set model was at odds with the continuous area model representations, which were a fundamental fractional representation used across the tutoring sessions. As presented in the prototypical example (1/3+1/3=2/6), Lisa often treated the pieces of the area model, both shaded and non-shaded as discrete entities that could be added. In the following example, Lisa was attempting to solve the problem 7/8-3/4=. Although she correctly constructed an area model for 7/8 and 3/4, she treated the eighths and fourths as if they were the same size. She illustrated the subtraction by treating both the shaded and non-shaded pieces as entities, and crossed out four pieces (1 non-shaded and 3 shaded) of the 7/8 to solve the problem (see Figure 43). In representing the subtraction operation, she not only treated fourths and eighths as interchangeable, but also subtracted both the non-shaded and the shaded pieces.

Figure 43. Scanned artifact from Lisa’s tutoring session showing her solution to the problem 7/8-3/4=.

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30 A discrete set model for fractions is not particularly effective for representing fraction equivalence or fraction operations, and consequently the tutoring sessions relied heavily on continuous models.
Lisa’s solution to the subtraction problem involved crossing out both the one non-shaded and three shaded pieces (corresponding to the representation of 3/4) from the area model for 7/8. She treated the non-shaded pieces of 3/4 and 7/8 as if they were discrete entities in their own right that could be subtracted. Lisa also treated the pieces in her drawing of 3/4 as if they were the same size as the pieces in her drawing of 7/8, and therefore could be directly subtracted. This suggested that she understood her representation as composed of a set of four and a set of eight interchangeable elements. Her one-to-one treatment of the eighths and fourths along with her subtraction of both the shaded and non-shaded pieces is suggestive of a discrete set understanding of these continuous area model representations.

*Fraction pieces.* Fraction pieces, another continuous model of fraction quantity, also proved to be problematic for Lisa. This example highlights both the robustness of her discrete set understanding and the incompatibility of this representational tool. Lisa’s difficulty with fraction pieces involved treating the denominator of the piece as if it represented the number or elements in the set, rather than the size of the piece. For example, she determined that a 1/5 and 1/6 piece put together would equal 2/11.

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31 Based on the video data and evident in the final artifact (see Figure 43), it is clear that Lisa only shaded half of the right piece in her model of 3/4. In her initial partitioning the right piece was significantly bigger than the other three pieces. This seemed to be a common strategy Lisa had for compensating for her inaccurate partitioning. The left piece was clearly omitted from the shading of the 3 pieces.
Lisa: I guess you could see it as two-elevenths, but you couldn’t see it as anything other than that.

In this instance she was relying upon a discrete set understanding to make sense of the value of fraction pieces. Rather than understanding the label to signify the size of the piece, she was treating it as a representation of the number in the set. This kind of reasoning was also present in her attempts to construct fractional quantities with pieces. She attempted to create something she called “one-sixteenth” by assembling a 1/10 and 1/6 piece (because 10 + 6 = 16) in one case and 1/8 and 1/8 (because 8 + 8 = 16) in the other. Lisa’s discrete set understanding was inappropriately applied to both the construction and interpretation of fraction pieces over the course of the tutoring sessions.

Lisa over-applied her discrete set understanding to continuous models, both area models and fraction pieces. These examples suggest that the continuous models used to introduce new fraction concepts was fundamentally at odds with how she naturally conceptualized fractions. In the next section I explore my explicit attempt to help Lisa differentiate continuous and discrete models, which was ultimately unsuccessful.

Robustness of “discrete set” understanding. Lisa’s discrete set understanding was sufficiently robust that it resisted my attempts to help her develop a more productive understanding of the continuous models we were relying upon in the tutoring sessions.

Fraction pieces. My first attempt to address Lisa’s difficulties with continuous models, occurred immediately following her assertion that a 1/5 and 1/6 piece together were could be 2/11. The goal of this question was to establish the conventions that should be used when working with fraction pieces. I asked Lisa to consider how big a one-eleventh piece would be and we spent three minutes discussing this. Despite the fact that she was able to determine that it would take eleven pieces labeled 1/11 to fill up a whole, she still accepted that 1/5 and 1/6 together could equal 2/11.

Lisa: So, it would take eleven one eleventh to fill up (points at 1 whole fraction piece), to make 11 pieces.

Tutor: Right, so it’s going to take 11 pieces in here. So back to the question we were sort of asking before… could this be two-elevenths? (pointing at 1/5 and 1/6 piece)

Lisa: Yeah.

Although she correctly determined how one could construct an eleventh sized piece, she did not reconcile the size of an eleventh with the size of 1/5 and 1/6, and accepted that 1/5 and 1/6 together could equal 2/11. I followed up by asking her to compare tenths
(available in the foam fraction pieces set) and an imagined eleventh size piece (not available in the foam fraction pieces set). Only once she established that an eleventh was going to be smaller than a tenth, and we constructed 2/10, did she definitively reject the possibility that 1/5 and 1/6 could be 2/11.

Tutor: How big is an eleventh going to be in comparison to a tenth?
Lisa: It’s going to be even smaller. Not by much, but it’s going to be smaller.
Tutor: So the question is, is this (points to 1/5 and 1/6 pieces) two-elevenths? (writes “2/11 = ” next to 1/5 and 1/6 fraction pieces)
Lisa: So no? (laughs)
Tutor: So, if we had two-tenths here (writes “2/10 = ”), how would we make two-tenths using these pieces?
Lisa: (moves 2 one-tenth pieces).
Tutor: And so, if we…
Lisa: Ok, so no.
Tutor: Ok, so how did you figure that out?
Lisa: Ok, because I’m almost positive that one-eleventh would be smaller than one-tenth. Like, I know that actually. So, this (point to the 1/5 and 1/6 pieces) is like blatantly larger than this (pointing to the two 1/10 pieces), so I don’t know what this (point to the 1/5 and 1/6 pieces) would be if you put it together, but it’s not that (pointing to the two 1/10 pieces).

Despite understanding that elevenths would be much smaller, it still took a side-by-side comparison of the two values for Lisa to recognize that the 1/5 and 1/6 pieces together could not equal 2/11. This extended example suggests that this discrete set interpretation of the fraction pieces appeared to be dominant, even after an explicit discussion about the size of an eleventh piece. When engaging with these pieces, she was more focused on an interpretation of the labels rather than conceptualizing these representations as a continuous quantity where size was a crucial component. As mentioned before, despite this extended discussion about the ways to use these fraction pieces, in subsequent tutoring sessions Lisa continued to treat the denominator labeled on the fraction piece as if it represented the number in the set rather than the size of the piece.

Area models. Lisa also applied her discrete set understanding to area model representations. During the third tutoring session, I explicitly deviated from the tutoring protocol to try to help differentiate between continuous and discrete fraction models, focusing on when the size of the pieces was relevant and irrelevant. Although we discussed a discrete model (chips of different sized) and contrasted it with an area model representation, this did not seem to address Lisa’s tendency to apply a discrete set
understanding to continuous models. Despite my attempts to explicitly address this over-application of a discrete set understanding by deviating from the tutoring protocol and discussing the difference between a discrete set and a continuous model, this did not result in a refinement of this understanding.

**Summary and conclusion.** Lisa’s “discrete set” understanding was far more pervasive and problematic than students typically experience when they are learning to differentiate continuous and discrete set models for fractions. Despite explicit attempts to address her over-application of her discrete set understanding, Lisa often operated on continuous models as if they were discrete models, in which the size of the whole was unimportant, the size of the pieces was unimportant, and the denominator represented the number in the set rather than the size of the piece. Lisa’s “discrete set” understanding appeared to be fundamental to her conceptualization of fractional quantities, and as such, was incompatible with the majority of continuous models used throughout the tutoring sessions. The wide applicability of her discrete set understanding was ultimately detrimental to her ability to engage with fraction equivalence and fraction operations as presented with continuous models.

Lisa’s Unit Fraction Understanding

This section discusses the fourth of Lisa’s five atypical understandings: “unit fraction.” A unit fraction is a fraction where the numerator is equal to 1 (e.g., 1/3, 1/5), and the magnitude of the fraction is inversely proportional to the denominator value (i.e., the larger the denominator value, the smaller the fraction). Lisa’s “unit fraction” understanding involved over-extending this inverse relationship and treating all fractions as if they were unit fractions. This atypical understanding indicated that Lisa was often not reconciling the numerator and denominator value together, providing further evidence that she did not have a solid understanding of fractional quantity. Lisa’s unit fraction understanding was robust and persisted across time. This atypical understanding provides a key component in understanding why Lisa ultimately did not learn.

In this section, first I introduce the “unit fraction” understanding and provide a prototypical example in conjunction with the operational definition. Second, I discuss how this understanding proved to be problematic for Lisa’s understanding of more complex fraction concepts, particularly fraction comparisons. Third, I consider the ways in which this understanding appeared to be persistent but inconsistently invoked. Specifically, I consider instances of “unit fraction” during the pretest and posttest, which suggests that this understanding was persistent in that it originated before the tutoring sessions and was not resolved by the end of the tutoring sessions. However, Lisa’s problematic over-application of her “unit fraction” understanding was inconsistent, in that during the tutoring sessions themselves, her “unit fraction” understanding was not particularly problematic or frequently invoked. I consider the ways in which this “unit fraction” understanding may have been inadvertently supported by the fraction pieces, and how Lisa’s performance on fraction comparison problems during the tutoring sessions may have over-estimated her competency. Finally, I present my attempts to address this atypical understanding, which appeared to locally correct the specific difficulty, but failed to address her tendency to over-apply this knowledge in general. This strand of analysis illustrates that Lisa’s “unit fraction” understanding was atypical,
that it reflected Lisa’s difficulty reconciling both the numerator and denominator to determine the fractional quantity, and consequently was detrimental to her ability to learn.

**Defining and exemplifying “unit fraction” understanding.** Lisa’s “unit fraction” understanding involved over-application of a unit fraction conception. To better illustrate this understanding it will be contrasted with typical use. Unit fractions can be compared simply by considering the value of the denominators. The larger the denominator, the smaller the fraction: 1/5 is smaller than 1/2 because 5 is larger than 2, or fifths are smaller than halves. Lisa’s unit fraction understanding, although it did include the canonical understanding, was insufficiently bounded. Lisa would often compare fractions based on the inverse size of the denominator, without regard for the value of the numerator. She would determine, for example, that 1/2 was bigger than 4/5 (see Figure 44), because halves are bigger than fifths. This understanding led her to conclude that one-half was the largest fraction, because 2 is the smallest denominator value. This poorly specified understanding was also sometimes applied to the numerator value (i.e., the larger the value of the numerator, the smaller the fraction). For example, she might conclude that 2/5 was bigger than 3/5 because 2 is smaller than 3. Lisa over-applied her “unit fraction” understanding to non-unit fractions, comparisons where the numerators were not the same, and occasionally to the numerator rather than denominator value. Characteristic of this kind of understanding is that the numerator and denominator values were not considered in tandem, and instead either the numerator or the denominator exclusively determined the size of the fraction.

<table>
<thead>
<tr>
<th>Typical Unit Fraction Understanding</th>
<th>Atypical Unit Fraction Understanding</th>
</tr>
</thead>
<tbody>
<tr>
<td>comparison of two unit fractions</td>
<td>comparison of non-unit fractions</td>
</tr>
</tbody>
</table>
| \[
\frac{1}{2} > \frac{1}{5}
\]

because halves are bigger than fifths.

| \[
\frac{1}{2} > \frac{4}{5}
\]

because halves are bigger than fifths and 1/2 is the largest fraction.

| \[
\frac{2}{5} > \frac{3}{5}
\]

over-application of understanding to numerator values.

because the larger the number is the smaller the fraction is. Three is larger than 2, so 2/5 is larger than 3/5.

Figure 44. Contrast of typical and atypical use of unit fraction understanding.

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32 Lisa knew that 1/1 was equivalent to 1, and consequently was not considered to be a fraction. Therefore the largest fraction possible was 1/2.
**Prototypical example of unit fraction understanding.** In the prototypical example of “unit fraction” understanding, I asked Lisa to compare two fractions for which she had drawn representations: 1/2 and 3/4. She determined that 1/2 would be bigger, and when explaining her answer she referenced the value of the denominator and the inverse relationship of the denominator’s size to the value of the fraction.

![Diagram of fractions 1/2 and 3/4]

Tutor: So if we were comparing three-fourths and one-half. Which one of those is going to be bigger?
Lisa: The half.
Tutor: Ok, and how do you know?
Lisa: Because it’s larger, because it’s closer to a whole. You have to split this up four ways to create a whole (points to drawing of 3/4). Whereas the half (points to 1/2 representation) only needs to get split into two. So, I suppose essentially, the smaller the fraction, the larger it is, and the closer it is to a whole.

Lisa determined that 1/2 was going to be larger than 3/4 because of the partitioning of a whole into 4 versus 2 pieces. This example was considered consistent with a “unit fraction” understanding because Lisa judged 1/2 to be larger than 3/4 and provided an explanation that referenced the inverse relationship of the number of partitions to the size of the fraction. This answer was also consistent with her understanding of 1/2 as the largest fraction. She justified her answer referencing the values of the denominator, and made no mention of the numerator value, or its role in determining fractional value.

Although she was referencing an unconventional representation of 1/2 in this instance (with labeling rather than shading), this kind of understanding occurred across the tutoring sessions in contexts in which the representations were better matched. This question occurred during the pretest, so this incorrect answer was not addressed at the time.

Lisa’s description of the reciprocal relationship, “the smaller the fraction, the larger it is, and the closer it is to a whole” did not explicitly refer to the denominator value. She referred to “the fraction,” rather than a term (like “denominator” or “bottom number”) that would constrain the context of applicability of this inverse relationship. Her reference to “the fraction” may partially explain why this inverse size relationship, at times, gets applied to the numerator rather than the denominator values. This will be taken up in more detail when discussing the ambiguity of Lisa’s language.
explanation suggest that she understood this fraction comparison in terms of unit fractions, comparing fourths and halves, rather than the quantity 3/4 and 1/2.

**Researcher’s operational definition of unit fraction understanding.** Problems were coded as indicative of a "unit fraction" understanding if (1) if Lisa judged the size of the fraction based solely on the denominator using an inverse relationship (e.g., 1/3 is bigger than 4/5 because thirds are bigger than fifths), (2) if Lisa judged the size of the fraction solely on the numerator using an inverse relationship (e.g., 3/5 is smaller than 2/5), (3) if Lisa asserted that 1/2 was the largest fraction, or (4) if Lisa represented the unit fraction separate from the numerator value (e.g., 5/10 represented as 5 \( \frac{1}{10} \)).

There were 42 instances of Lisa’s “unit fraction” understanding and no flagged non-unit fraction examples. It is notable that most of Lisa’s incorrect answers occurred during the pretest and posttest (see Figure 45), and that in general she used “unit fraction” correctly during the tutoring sessions.

![Figure 45. Data display of instances of Lisa’s unit fraction understanding across tutoring sessions.](image)

**Unit fraction understanding as detrimental to learning.** Although a “unit fraction” understanding can be productively applied in certain circumstances, Lisa’s underspecified “unit fraction” understanding hindered her ability to productively make sense of more complex fraction concepts, like fraction ordering and comparison. During the posttest, Lisa was asked to compare several pairs of area models to determine which one showed the larger amount. In two cases, as she was explaining her answer, she changed her correct answer and gave an explanation consistent with a unit fraction understanding.

In the first case, as Lisa was explaining why she (correctly) determined that 3/5 was larger than 2/5, she reconsidered her initial answer. She justified her new (incorrect) answer by referring to the inverse relationship of the “bigger” number (presumably the 3 in the 3/5) to the size of the fraction.

![Diagram of fraction comparison](image)

Lisa: Three-out-of-five are shaded as opposed to two-out-of-five.
Tutor: So three-out-of-five is going to be bigger than two-out-of-five?
Lisa: I mean, I guess visually, maybe not. I guess. No. Cause I think the bigger the number the smaller it would be.
Tutor: Ok.
Lisa rejected her initial (correct) answer and over-applied her unit fraction understanding to the numerator value. She then used the numerator values (2 and 3) to further exemplify this inverse relationship by explaining that 1/2 would be larger than 1/3.

Lisa: Like, like if I were to have one-half of something. That would be bigger than having one-third of something.
Tutor: Ok.
Lisa: I think.

Lisa was correct in her assertion that 1/2 is bigger than 1/3. However, Lisa appeared to be attempting to use her understanding of a 3 versus a 2 in the denominator (one-third versus one-half) to help her understand what it would mean to have a 3 versus a 2 in the numerator (three-fifths versus two-fifths). I explicitly asked Lisa if her comparison of 1/3 and 1/2 applied to the comparison of 3/5 and 2/5. She answered that she thought it did apply and by the end of her explanation she was convinced.

Tutor: Does that apply here, too?
Lisa: Yeah. So I guess visually maybe that (pointing to the picture of 3/5) would be bigger, but I guess maybe that one would be bigger (pointing to the picture of 2/5).
Tutor: Because you are saying, this is three-out-of-five and this is two-out-of-five, and so the two-out-of-five one is going to be bigger actually than the three-out-of-five one?
Lisa: I think so.
Tutor: Ok.
Lisa: I think so. I’ll put a question mark (writes “?”) Yeah, I mean yeah, I’m pretty positive actually. Cause yeah, I mean, I think.
Tutor: Ok.

Lisa ended this episode by being “pretty positive” that “two out of five” was bigger than “three out of five”. Given both her reference to the inverse relationship of the digit to the size of the fraction and her example of the comparison of 1/2 and 1/3, this example provides strong evidence that Lisa was over-applying her “unit fraction” understanding to the numerator values as well.

Similarly, in the next problem, although Lisa had (correctly) determined that 2/3 was larger than 2/4 she again changed her answer when she started explaining her choice. Lisa’s unit fraction understanding should have served her well in this circumstance (where the numerators for both fractions were 2), however, in explaining her answer, Lisa recognized that 2/4 was equivalent to 1/2, and changed her answer, explaining that one-half would be larger.
Lisa: No wait! (crosses out 2/3) This is one-half. (circles area model of 2/4)
Yeah, yeah yeah yeah yeah yeah. Yeah. This is one-half so it is definitely
the bigger one.
Tutor: Ok.
Lisa: I don’t know how I missed that.

Although I pressed Lisa to provide more of an explanation for why one-half was bigger
than two-thirds, she simply re-asserted that two-fourths was the same thing as one-half
and therefore, the larger of the two fractions.

These examples suggest that Lisa’s over-application of her unit fraction
understanding to the numerator and her mistaken assumption that 1/2 is the largest
fraction, was detrimental to her understanding of fraction comparison.

“Unit fraction” understanding across the sessions. This section considers the
persistence and robustness of Lisa’s “unit fraction” understanding over time. First I
consider a card sort problem from the pretest and posttest, which suggests that Lisa was
relying on a unit fraction understanding both before and after the tutoring sessions.
Second, I consider why this understanding was so problematic during the pretest and
posttest while it was infrequently invoked during the tutoring sessions themselves and
when it was it almost always led to a correct answer. In this analysis I consider potential
reasons for this inconsistency in application of her understanding. Specifically, I explore
whether the questions during the tutoring sessions were fundamentally different than
those posed on the pre/posttests. Analysis indicates that Lisa demonstrated apparent
competency during the tutoring sessions, particularly on fraction comparison problems,
that may have over-estimated her understanding of the topic. Third, I consider the
potential contributory factors to her persistent over-application of her unit fraction
understanding, including her numerical representation of the fraction pieces and the
ambiguity around her description of fraction comparisons. Finally, I consider my
attempts to address this atypical understanding, which appeared to locally correct the
specific difficulty, but failed to address her tendency to over-apply this knowledge in
general.

Persistence over time: “unit fraction” during the pretest and posttest. Lisa’s
pretest and posttest instances of “unit fraction” were examined to determine how she was
using this understanding before and after the tutoring sessions. Lisa’s pretest and posttest
involved a high incidence of “unit fraction” understanding\(^{35}\) (see Figure 46), I will focus
on the problems from a card sort, in which Lisa ordered 11 cards from least to greatest.

\(^{35}\) The high incidence of “unit fraction” understanding is likely due to the fact that Lisa’s
administration of the pretest and posttest involved a card sort where she was asked to
order fractions from least to greatest. Because this sorting task involved multiple
embedded comparison problems, it was a context where this understanding was often
cued.
She produced an *identical incorrect* sort during both the pretest and the posttest (see Figure 47). The card sort activity demonstrated a surprising consistency in her application of a unit fraction understanding and suggests that this was part of her prior knowledge, and was not resolved over the course of the tutoring sessions.

Figure 46. Lisa’s unit fraction understanding over the course of the tutoring sessions, with the problems involved in the card sort highlighted.

Figure 47. Illustration of Lisa’s ordering of fraction values from smallest (left) to largest (right) during the pretest.

Lisa’s incorrect ordering of cards can be almost entirely attributed to her unit fraction understanding. Because Lisa’s explanations and justifications were entirely consistent from pretest to posttest, and in the interest of space, the clearest excerpts will be presented. The two unit fractions, 1/100 and 1/2 were justified using canonical unit fraction understanding. She explained that, 1/100 would be the smallest value, “cause this is the largest number. *(pointing to 100)*” and 1/2 “*(pointing at 1/2 card)* would be the closest thing to one.” Although she correctly applied her unit fraction understanding to the two unit fractions in the sort, she then over-applied it to the placement of the remaining fractions: 9/10, 7/12, 3/2, 2/4, and 2/3. She explained, “I was just looking at the numerator. And it was larger, so it [the fraction] would be smaller.” In the case when the numerators were the same (2/4, 2/3), she explained that she reverted back to ordering by the denominator, still applying the inverse ordering of values.

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36 During both sorts during the pretest and posttest she also correctly established the equivalence of 8/8 and 1. She also correctly placed 1 2/5 as larger than 1. These fractional values were not problematic for Lisa, did not appear to involve a unit fraction conception, and therefore are not a focus of this analysis.
In both the pretest and the posttest she explained her fraction ordering based on the inverse ordering of numerical values, sometimes the denominator values and sometimes the numerator values. Characteristic of all her ordering of fractions involved attending to only one of the two values comprising the fraction, she either sorted based on the numerator value or based on the denominator value, always presuming an inverse relationship of the value of the digit to the magnitude of the fraction. The surprising similarity both in Lisa’s incorrect sort order and the explanations she gave for her answer suggests persistence in her over-application of her unit fraction understanding.

**Inconsistent use of “unit fraction” understanding.** Although there was a high occurrence of problematic “unit fraction” understanding during the pretest and posttest, the instances during the tutoring sessions themselves were relatively infrequent and when they occurred it was in conjunction with a correct answer (see Figure 48).

![Figure 48](image)

Figure 48. Data display of all instances of Lisa’s unit fraction understanding.

Several analytic passes were made through the data to attempt to determine whether there was some inconsistency in the kinds of problems posed during the pretest/posttest and the kinds of problems posed during the tutoring sessions. Lisa’s unit fraction understanding occurred primarily during comparison problems (76% of all unit fraction instances). Comparison problems accounted for 11% of the problem types during the tutoring sessions (see Figure 49). This was not a disproportionally small percentage. Therefore, comparison problems were sufficiently represented during the tutoring sessions. Lisa also accurately solved comparison problems that were isomorphic to the comparison problems she struggled with on the pre/posttest.

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37 The card sort was the last activity Lisa completed, and therefore, was not the cause of Lisa’s over-application of a unit fraction understanding to the numerator values in the previously presented area model comparison problems.

38 The frequency counts did reveal that there were no comparison problems administered during the third or fourth tutoring sessions, this may partially explain the low frequency of unit fraction instances during those sessions.
For example, at the end of the second tutoring session, Lisa was able to compare area models of fractions with different denominators and different numerators (see Figure 50). The only comparison problems she answered incorrectly involved area model comparisons where the sizes of the whole were not the same. This was dramatically different than her performance on the posttest. At the time of the posttest her unit fraction understanding seemed to have corrupted her ability to compare even simple area models of fractions that during the tutoring session she had no difficulty comparing. Therefore, the competency that Lisa displayed during the tutoring sessions might have over-estimated her understanding of fraction comparisons.

Potential contributors to the persistence of “unit fraction.” One central aspect of Lisa’s “unit fraction” understanding was that she did not reconcile the denominator and numerator values in tandem. Analysis indicated that the use of fraction pieces might have unintentionally supported the independent consideration of denominator and numerator values. In addition, her underspecified unit fraction explanations (“the bigger the number the smaller the fraction”) might have contributed to her over-application of her unit fraction understanding to the numerator values.

*Fraction pieces.* When working with the fraction pieces, Lisa tended to record the label on the fraction piece independently from the number of pieces comprising the
fraction. For example, when Lisa was asked to write down fractional amounts equal to 1/2, she produced several different equivalence relations, but when she recorded these fractions, she wrote the numerator separate from denominator, which was written as the unit fraction. Rather than writing the fraction 5/10, she recorded it as “5 \( \frac{1}{10} \)” (Figure 51).³⁹

![Figure 51. Scanned artifact showing Lisa’s representation of fractional amounts equal to 1/2.](image)

The fraction pieces appeared to suggest this kind of recording of the values, where the number of pieces was represented separately from label on the pieces. Lisa did specify that she did not mean to write a mixed number, “not like five and a tenth, but like five one-tenths makes up one-half.” As the problem continued she recorded three other fractions in this form. Although we discussed this notation, and she replaced it with a more canonical notation, at the end of the session, Lisa chose to record the numerator and denominator separately when creating her journal entry (see Figure 52). This representation of the denominator as an independent unit fraction value occurred in subsequent sessions and was likely due to this journal entry. Both the fraction pieces and the resulting journal entry may have contributed to her understanding of the numerator and denominator values as independent rather than integrated components of the fractional value. Although it is possible that this was simply a notational issue stemming from the use of fraction pieces, it may be that this specific representation well reflected her natural conception of numerator and denominator as separate quantities.

³⁹ Lisa almost seemed to be writing this equivalence relationship as a sentence, note the apostrophes, which could be read as “five one-tenth’s equal one-half.” It is unclear if the “’s” was needed because 1/10 by itself is singular and read as “one-tenth” not “one-tenths”.
Figure 52. Scanned artifacts of Lisa’s journal entry written after the first tutoring session.

**Ambiguous domain of applicability.** Although Lisa’s “unit fraction” understanding was based on a correct and canonical understanding, the domain of applicability of this unit fraction understanding was insufficiently specified. Even explanations Lisa provided for correct unit fraction comparisons reflected a potentially problematic lack of the specification. For example, when explaining her answer that 1/6 was bigger than 1/8, she alluded to the inverse relationship of the numerical value to the size, by giving the example of how magnitude works for negative numbers.

Lisa: It’s kinda like with negatives, like the bigger the negative is, the smaller it is. Like the farther it is away from zero. It’s kinda, I think, like the same thing, where, like, the larger the number the smaller it is.

Lisa captured her “unit fraction” understanding quite succinctly, by stating “the larger the number the smaller it is.” In her explanation, she did not constrain this inverse relationship to the denominator value and made no reference to the importance of the numerator values being 1 (or the same). This explanation hints at potential ambiguity that could result in over-application of this understanding. Previous examples have shown that this ambiguity played out in detrimental ways because Lisa often did not considering the value of the numerator in conjunction with the denominator, and did not constrain this inverse relationship of numeral to the size of the fraction to the denominator value. Even when appropriately applying this understanding, she was unable to sufficiently specify the conditions of the problem that allowed for accurate comparison. Lisa’s “unit fraction” understanding, even when appropriately applied, was explained in ambiguous language, which allowed for misapplication of this understanding.

**Robustness of “unit fraction” understanding.** Because Lisa’s unit fraction understanding did not occur frequently during the tutoring sessions, there was limited data with respect to how effective explicit instruction would have been for helping her resolve this problematic conception. However, at the end of the first tutoring session, we had a discussion about the comparison of 9/10 and 1/2 using the fraction pieces as a support. Although Lisa was able to explain how to construct 9/10 with fraction pieces, she still initially judged that 9/10 would be less than 1/2. I explicitly drew her attention to the meaning of the numerator, and she corrected her answer.
Tutor: So, if we were thinking about it in terms of pieces that we had, If we were thinking about what nine-tenths would mean, how many 1/10 pieces would we have?
Lisa: Ten.
Tutor: So, ten would fit in the whole, right?
Lisa: Yeah.
Tutor: How many for 9/10?
Lisa: Nine.
Tutor: Yeah, so we would have nine of these. Would that be more or less than a half?
Lisa: Less?
Tutor: If we put nine of these together?
Lisa: More.
Tutor: Yeah, cause we are going to almost fill up the whole. (puts 1/10 piece on the blue 1 whole piece, see Figure 53) Right?
Lisa: Ok. So like the 7/10 and the 9/10 would be like way the heck over here. (gesturing to the right)
Tutor: Yeah, so they would be over 1/2.
Lisa: Ok. That didn’t make sense. I mean, it makes sense now. That’s something that I wouldn’t have known before. Once you introduced everything it made more sense.

Figure 53. Screenshot of the tutor’s placement of a 1/10 piece on the blue one whole piece.

Lisa ended this episode asserting that this kind of fraction comparison now made sense. Although I did not anticipate an immediate and permanent change in her understanding, this interaction was encouraging. In addition to addressing this understanding explicitly, during the tutoring sessions Lisa and I spent a great deal of time discussed basic fraction concepts, including the meaning of the numerator and denominator. Given her apparent competency with these ideas and her lack of difficulty when comparing area models of fractions during the tutoring sessions, it was surprising that her unit fraction understanding was problematic in the same ways at the time of the posttest.

40 At the start of this problem, Lisa had used the physical space in front of her to talk about fraction ordering, putting “0” on her left and “1½” on her right and asked about the placement of 9/10. Based on this initial set-up to the problem, it is probable that when she was gesturing to her right, she was suggesting that the fraction was larger. I confirmed this by saying that it would be “over 1/2”, by which I meant greater than 1/2.
Summary and conclusion. Lisa over-applied a “unit fraction” understanding to non-unit fractions and occasionally to the numerator value. The high incidence of this understanding during the posttest indicates that this understanding, although not problematic during the tutoring sessions, was still fundamental to her understanding of fraction comparisons. The fraction pieces may have unintentionally supported her treatment of the numerator and denominator as separate quantities. Although primarily occurring with fraction comparisons problems, her “unit fraction” understanding reflects that Lisa’s conception of fractional quantities did not involve a unified understanding of the numerator and denominator. This atypical understanding proved to be problematic for her, and ultimately contributed to her difficulties learning.

Lisa’s Partitioning Understanding

This section describes the fifth of Lisa’s five atypical understandings: “partitioning.” Lisa had difficulty partitioning shapes to produce the desired number of pieces. Her partitioning difficulties led her to inappropriate conclusions (e.g., you can’t partition into an odd number of pieces) and resulted in unequally partitioned representations. Although Lisa’s partitioning understanding underwent a transition from a less sophisticated halving strategy, to a more sophisticated unidirectional partitioning strategy, both approaches led to difficulties. Her partitioning understanding often obscured the fraction concepts the representations were intended to highlight and consequently contributed to her learning difficulties.

In this section, first I introduce the “partitioning” understanding and provide a prototypical example in conjunction with the operational definition. Second, I discuss how this understanding obscured the meaning of fraction representations and was problematic particularly for Lisa’s understanding of more complex fraction concepts. Third, I consider how even though Lisa’s partitioning approaches eventually shifted from a halving strategy to a unidirectional partitioning strategy both of these strategies were somewhat problematic. This strand of analysis illustrates that Lisa’s “partitioning” understand was atypical, robust, and both her initial halving and subsequent unidirectional partitioning understandings were detrimental to her ability to learn.

Defining and exemplifying “partitioning” understanding. Lisa’s partitioning understanding was primarily involved with the mechanics and strategies for partitioning shapes to represent fractions. Typically, an individual partitioning a shape into 5 pieces will attempt to visually estimate the size of one piece, and draw 4 lines to partition the shape into 5 relatively equally sized pieces (see Figure 54). In contrast, Lisa would often use a halving partitioning strategy to attempt to construct fifths, which either led to a non-normative representation involving an appended piece, or her assertion that it was impossible to partition into an odd number of pieces. In the instances when she did employ a unidirectional partitioning strategy, she would often draw the number of lines corresponding to the denominator, (i.e., drawing 5 partition lines), which resulted in a shape that was partitioned into 6ths.
Figure 54. Contrast of typical and atypical partitioning of a shape into fifths.

Prototypical example of partitioning. Lisa’s partitioning understanding was evident in the following example, in which she attempted to partition a shape into five pieces. Both her tendency to use a halving strategy and her strategy of drawing the number of partition lines corresponding to the denominator were evident at various points during this example. In this posttest problem, Lisa was asked to draw a picture of 3/5. After quickly abandoning her circular area model, she drew a rectangle and 5 partition lines.

<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript</th>
<th>Recreated artifact</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tutor: If you were to draw three-fifths, how would you do that? Lisa: (starts drawing a circle, then stops scribbles it out) Oh my god, I always get really tripped up with the fifths. (writes 3/5)</td>
<td><img src="image1.png" alt="Artifact 1" /></td>
</tr>
<tr>
<td>2</td>
<td>Lisa: Like drawing them. (draws rectangle, divides into 5 pieces)</td>
<td><img src="image2.png" alt="Artifact 2" /></td>
</tr>
</tbody>
</table>
3  Lisa: *(splits last piece, creating six pieces, points to each piece from right to left)*

4  Lisa: Forget this exists *(crosses out the left piece)*
That’s not a part of it.

5  Lisa: *(shades in three right pieces).*

As Lisa was attempting to draw a shape partitioned into fifths she commented on her difficulty with fifths. She drew 5 partition lines, which resulted in a shape that was partitioned into sixths. After drawing the final partition line she pointed to each of her pieces with her pen (presumably silently counting). After pointing to the sixth and left-most piece, she repaired her answer by crossing that piece out.

I then asked the standard follow-up question and asked if there was any other way of drawing or writing three-fifths. She attempted to construct a model of three-fifths using a circle, and partitioned it using a halving approach. She made two different attempts, and then explicitly asked how it was possible and talked about “break[ing] off” a part to create fifths.
Tutor: Is there any other way you can think of drawing or writing three-fifths?
Lisa: (Draws circle, divides vertically in half) How would I do that? How would I break off – like, this isn’t even like math related, (draws circle and then partitions horizontally) but just like drawing, like sectioning it into fifths?

In this instance Lisa demonstrated that one of her approaches to partitioning into fifths involved partitioning-by-halving. Even though she quickly abandoned her initial attempts to halve a shape to attain fifths, she suggested that “breaking off” a piece was necessary to attain an odd denominator. Admittedly circles are difficult to divide up into fifth-sized pieces, but Lisa’s difficulties go beyond the mere mechanics of equal partitioning. This problem highlights her tendency to understand partitioning in terms of successive halving and the creation of an odd number of pieces by dividing into an even number and removing the extraneous piece.

This example was considered to be characteristic of partitioning for several reasons. First, Lisa drew the number of partition lines corresponding to the denominator value in her first attempt to draw 3/5 and her drawing had six parts. Second, in her attempts to use a circle to represent 3/5, she used a halving partitioning strategy. Her halving approach and her suggestion that she would need to “break off” a piece suggests that Lisa recognized that a halving strategy would not result in an odd number. This example highlights both the correspondence between denominator value and number of drawn partitions and her tendency to employ an inappropriate partitioning-by-halving strategy.

**Researcher’s operational definition of partitioning.** Problems were coded as indicative of a "partitioning" understanding (1) if Lisa attempted to partition a shape into an odd number with a partitioning-by-halving strategy (2) if Lisa asserted that it was impossible to partition a shape into an odd number of pieces, (3) if Lisa drew the number of partitions corresponding to the denominator value (resulting in an extra piece), or (4) if Lisa’s final representation involved a non-normative partitioning of a shape.

Although there were no instances of this understanding during the pretest, there were 23 instances of Lisa’s “partitioning” understanding spanning the remaining sessions (see Figure 55). When this understanding was invoked it was often occurred in conjunction with an incorrect answer.
Figure 55. Data display of all instances of Lisa’s “partitioning” understanding

“Partitioning” as detrimental to learning. Lisa’s partitioning difficulties were particularly problematic in the context of fraction equivalence and fraction operations. When dealing with these problem contexts, she often had to partition twice, once to create the original area model and once to produce the equivalent fraction representation. Consequently, in these contexts, her partitioning issues were exacerbated. In the following example Lisa was asked to solve the problem “1/4 + 1/3 =”. Although Lisa was able to navigate many of the challenging conceptual issues in solving this problem, her partitioning issues were salient and ultimately distracted from the meaning of the representations.

Before the start of the first focal excerpt, Lisa had drawn an area model for 1/4 (using a partitioning-by-halving strategy) and an area model for 1/3 (see Figure 56). I asked her to redraw her representation of 1/4 to more closely match the area model squares that we had been using, so that the equivalence would be more obvious. She was able to redraw 1/4, determined that twelfths would be the common denominator, partitioned the 1/3 into 12 pieces and determined the equivalent fraction was 4/12.

Figure 56. Digitally recreated artifact from student work on problem 1/4 + 1/3 =.

The focus of this episode is on what follows, and specifically on how Lisa partitioned the 1/4 area model into twelfths. In the first excerpt she initially repartitioned the area model for 1/4 correctly, producing an area model of 3/12. However, after quick consideration, she split the first column in half, creating an additional partition and 16 total pieces, rather than the desired 12.

<table>
<thead>
<tr>
<th>Line Number</th>
<th>Transcript</th>
<th>Digitally recreated artifacts</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tutor: So we need to make these pieces (points to 1/4) the same size as these pieces (points to 1/3 picture). Lisa: Ok. Tutor: How can we do that?</td>
<td><img src="image-url" alt="Digitally recreated artifact" /></td>
</tr>
</tbody>
</table>
In Lisa’s first attempt to partition the shape (Line 2), she correctly drew 2 lines to partition the one-fourth into 12 pieces. She pointed to the left column, potentially counting by groups\(^{41}\), and then decided to add another partition line by splitting the first column in half. At this point the number of partition lines (3) was the same as the desired number of columns (3). When she counted up the number of pieces she determined that she had inappropriately partitioned the area model for 1/4. Her repartitioning of 1/4 had both created unequal pieces and the incorrect number of pieces.

In the next segment, I suggested redrawing 1/4, and during her reconstruction of 1/4 she was cautious to count between each addition of a partition line. When repartitioning 1/4, rather than replicating her initially correct answer, she split the shape in half, and then renamed the fraction 2/8, and then split one of the halves and renamed the fraction 3/12.

<table>
<thead>
<tr>
<th>Line Number</th>
<th>Transcript</th>
<th>Digitally recreated artifacts</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Lisa: (divides twice, points at each piece in right column)</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>3</td>
<td>Lisa: (draws another vertical line, dividing left third)</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td></td>
<td>Lisa: There.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Tutor: How many pieces do we have total?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Lisa: 1, 2, 3, 4.  (counting) Wait… AH! Ok, that doesn’t work. That was sixteen, we want twelve.</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Tutor: Let’s just redraw our one-fourth and try to partition it again.</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td></td>
<td>Lisa: (redraws rectangle, draws top two partition lines, counts pieces)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Lisa: (draws third line, counts bottom up)</td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>6</td>
<td>Lisa: Ok, so right now we are at one-fourth (shading piece).</td>
<td><img src="image5" alt="Diagram" /></td>
</tr>
</tbody>
</table>

\(^{41}\) It is possible that Lisa, having drawn two partition lines, was counting by twos as she pointed to each of the pieces in the left column. Counting by groups by pointing to one column or row was a commonly used strategy for her throughout the sessions when she was trying to determine the total number of pieces in a shape. The fact that she added an additional partition line suggests that she was not counting by threes.
Lisa’s strategy involved successive halving until the desired denominator (12) was attained. By employing this strategy she was able to determine the numerical value of the equivalent fraction, but her resulting representation of 3/12 involved uneven partitions, therefore obscuring the transformation of 1/4 to 3/12 that the area models were intended to afford.

Also of note in this excerpt is that Lisa experienced difficulty in partitioning her final construction of 1/4. When redrawing the area model for 1/4, Lisa drew 2 partition lines (Line 4), counted the pieces and then drew the last line (Line 5), she then recounted the number of pieces to confirm that there were four. This counting and recounting was typical of Lisa’s partitioning activities, particularly those that involved unidirectional partitioning. When drawing in partition lines, it seemed that she did not understand what the addition of an extra partition line would mean for the total number of partitions, so her constructions involved repeated counting, partitioning, and recounting. This is important to consider simply because it appeared that the actual act of partitioning required considerable attention and monitoring, and was potentially distracting from other aspects of the given problem.

During this problem, Lisa’s partitioning issues rendered opaque the fundamental conceptual entailments using area models were intended to highlight. Her final representations of the equivalent fraction for 1/4 (both 4/16 and 3/12) involved unequal pieces (see Figure 57. Scanned artifact from Lisa’s solution to the problem 1/4+1/3=).

Lisa’s partitioning activity absorbed a great deal of attention, and potentially distracted from the purpose of this question, which was to explore the use of fractional equivalence with area models to help solve fraction operation problems. Lisa’s partitioning understanding was therefore detrimental, particularly to problems involving fraction equivalence in which representations were repartitioned.
“Partitioning” across the sessions. This section considers the robustness of Lisa’s “partitioning” understanding, as well as shifts in the nature of this understanding over time. First I consider both the first instances of her partitioning understanding and instances from the posttest, which suggests that this atypical partitioning understanding was part of her prior understanding and also not sufficiently refined over the course of the tutoring sessions. Second, I consider how her initial conceptualization of partitioning-by-halving was not easily refined, and resisted my attempts to address it. Third, I explore Lisa’s eventual but incomplete refinement in understanding which was evident in her shift in partitioning strategy. Over the course of the tutoring sessions she shifted from partitioning-by-halving to a unidirectional partitioning strategy. As previously noted however, both of these strategies were problematic and both partitioning understandings were evident at the time of the posttest. This suggests that this understanding was not sufficiently refined, and would likely continue to be problematic for her.

Persistent over time: “partitioning” during pretest and posttest. Although there were no instances of partitioning understanding during the pretest, Lisa’s partitioning understanding was evident during the first instance in which Lisa was asked to reason about the size of a piece resulting from an odd numbered denominator. During the first tutoring session Lisa rejected the idea that it was possible to create 1/11, because it was “uneven.” Lisa’s understanding of partitioning involved repeated halving, and therefore she assumed that it was only possible to make even numbers of pieces. Given that Lisa’s partitioning understanding surfaced during the first instance in which she was asked to reason about an odd denominator, suggests that it was likely part of her prior understanding, rather than an idea introduced by the tutoring sessions.

By the time of the posttest, although she recognized that shapes could be partitioning into an odd number of pieces, her partitioning strategy still lacked sophistication. As previously discussed in the prototypical example, Lisa partitioned into fifths initially by drawing 5 partitioning lines. She then attempted to use a circular shape to represent fifths, but her halving strategy would not accommodate and odd numbered denominator. This initial instance of her partitioning understanding, and the instances during the posttest, suggests that Lisa’s “partitioning” understanding was part of her prior understanding during the pretest was likely due to the limited number of fractions that Lisa represented during the pretest. Lisa made 6 drawings of fractional amounts and in four cases these were for the fraction 1/2 and in another 2 cases it was for the fraction 3/4. Lisa’s commonly used strategy of partitioning-by-halving shapes works well for all of the fractions drawn during the pretest (i.e., denominators that were powers of 2).
understanding and that this understanding, although somewhat revised, was not sufficiently refined over the course of the tutoring sessions.

Robustness of Lisa’s “partitioning” understanding. In the first tutoring session I explicitly attempted to address Lisa’s assumption that odd numbered\textsuperscript{43} denominators were not possible, by referring back to several previously completed challenges. As previously discussed, Lisa asserted that eleventh sized pieces were not possible to make because 11 was “uneven”.

Tutor: Ok, so how big would an eleventh be?
Lisa: I don’t think it could be evenly distributed.
Tutor: Ok.
Lisa: Like with all these (gestures to area that has eighths, tenths, and fourths fraction pieces, see Figure 58) it’s like an equal amount and that’s like important, to make up a whole. But if you were to have an uneven amount. Then it couldn’t be evenly proportioned with the other one.

Figure 58. Screenshot of video showing the placement of the fraction pieces that included eighths (pink), tenths (purple) and fourths (green).

Lisa explained that it was impossible to make an eleventh-sized piece and referred to the fact that the pieces could not be “evenly distributed” and that with an eleventh you would have “an uneven amount.” She seemed to reject the possibility that a whole could be divided into an odd amount.

To address this assertion, I referred back to a previously completed challenge, in which she had partitioned the whole to create eighth sized pieces. She determined it was not possible to create elevenths in the same way because you would end up with 12 pieces instead of 11.

Tutor: Ok, so if we were to go back to the problem before and instead of making eighths here, if you were to try to make 11ths, is it possible to make 11ths?
Lisa: I don’t – no. No. Cause you would make 12 and have an extra piece.

In this instance, it is clear that she believed it was only possible to create even numbers of pieces\textsuperscript{44}. I again referred to a previous challenge in which we established that the

\textsuperscript{43} Although Lisa at times asserted that an odd number of parts were impossible to create, this seeming impossibility did not appear to apply to her understanding of thirds, which she successfully partitioned in all instances.

\textsuperscript{44} It is worth noting that Lisa’s halving strategy would likely only produce powers of 2, rather than all even numbered denominators.
denominator corresponded to the number of pieces it would take to fill a whole, and specifically referenced the 1/5 pieces.

Tutor: Part of it has to do with how you are dividing it up. Right?
Lisa: Yeah.
Tutor: And circular pieces are sort of hard to think of in this way. Um, but back when we were thinking about – how many one-fifths would it take to equal a whole?
Lisa: 5. Oh!
Tutor: Cause, fifths are sort of not as easy to divide up either.
Lisa: But, nonetheless it’s possible…. So yeah, I guess, I guess you would just have. I guess it is possible then… to do elevenths, I think.
Tutor: Cause it was going to take 5 one-fifths.
Lisa: So, it would take eleven one-elevenths to fill up… to make 11 pieces.

Lisa appeared to have an “aha!” moment when she realized that the fraction set contained 1/5 pieces and therefore, it was possible to make fifth sized pieces. At the end of this episode Lisa stated that it would take 11 elevenths to make up a whole and was able to reason about the relative size of an eleventh with respect to a tenth. Despite this seeming progress in debunking the idea that odd denominators were impossible, later in that same session, when working with the rectangular fraction pieces, Lisa referred back to the situation with the elevenths to explain why the fifths could not possibly be used to construct the whole, because it was missing one piece. Lisa had created a rectangular one-whole piece using various fraction pieces, when I asked her to do the same thing with the 1/5 pieces, she referred back to the impossibility of an odd number of pieces.45

Tutor: Cool, and what about the one-fifth pieces?
Lisa: (puts 2 rows of 1/5 on top of one another, see Figure 59) Um…I don’t know, what to do.
Tutor: So right now you have five-fifths, should this be the same thing as the six-sixths (points to 6/6) and the one-third – or the three-thirds (points to 3/3), and the ten-tenths (points to 10/10)?
Lisa: I guess no, because this is an odd number. Um, It’s kinda like the 11 thing, we need another thing right here (draws a line up from the third 1/5 in the last row).
Tutor: Uh huh.
Lisa: Uh, so it’s not going to go in evenly.

45 Lisa’s assertion that an odd number of parts were impossible to create did not appear to apply to her understanding of thirds.
Figure 59. Screenshot from tutoring session, in which Lisa was trying to fit 5/5 (yellow) into the traced whole representation. Also shown are the 3/3 (green), 6/6 (orange) and collection of 10/10 (purple).

Although Lisa was guided through a local correction, which established that odd numbered denominators were in fact possible, she reverted back to her understanding of the impossibility of an odd numbered denominator when faced with fifth sized pieces later in the session. This understanding surfaced again during the posttest when she talked about breaking off a piece to create fifths using her halving strategy. This suggests that partitioning-by-halving was a robust understanding which was not sufficiently resolved over the course of the tutoring sessions.

**Shift in “partitioning” understanding from halving to unidirectional partitioning.** Lisa did eventually transition to a unidirectional partitioning strategy and an understanding that odd numbered denominators were possible. To capture this shift in partitioning approach, I compiled all instances in which Lisa partitioned an area model (n=67) and classified those with respect to whether she partitioned using a halving strategy and/or a unidirectional strategy. I removed all instances in which she was partitioning into halves or thirds, because in 100% of the cases she correctly partitioned, and halves necessarily involve a unidirectional halving strategy, and thirds always involved a unidirectional partitioning strategy. The resulting data set involved 43 instances of area model partitioning. From the graphic representation (see Figure 60) we can see a transition from instances in which she is using a partitioning-by-halving strategy to one in which she was using a unidirectional partitioning strategy. In the pretest, and the first and second tutoring sessions, Lisa exclusively used a halving strategy to partition fractions. It was not until the fourth tutoring session, after the introduction of the transparency model, that she first drew unidirectional partition lines. During this session she used both halving and unidirectional partitioning strategies and all instances or her errors involved unidirectional partitioning. This was also the first instance that the denominator value was inappropriately associated with the number of drawn partitions, and this accounts for all of her errors during that session. During the posttest Lisa relied on both halving and unidirectional partitioning strategies, and both these strategies led to errors.
Figure 60. Display of all instances of Lisa’s partitioning of area models across tutoring sessions (excluding instances of halves or thirds).

Lisa shifted in the predominant strategy she used for partitioning shapes from a halving strategy to one with unidirectional partitions. A potential factor contributing to the shift is the introduction of the area model squares with transparency overlays (see Figure 61 for an illustrated example of how these were used during the tutoring sessions). The area model squares and the transparency overlays both have unidirectional partitions. To solve problems during the third tutoring session Lisa was primarily using these representational tools. In the fourth tutoring session, Lisa was required to begin to represent these kinds of equivalent fraction relationships for the purposes of adding fractions. Two things should be noted. First, Lisa’s exposure to the area model squares and transparencies likely increased her tendency to partition shapes using unidirectional partitions. The introduction of the area model squares with transparencies occurred prior to her shift in partitioning strategy mentioned above (see Figure 60). Second, the area model squares with transparencies, offloaded the portioning work into the representations themselves. The area model squares were pre-partitioned as were the transparencies. Therefore although during the third session Lisa had exposure to unidirectionally partitioned shapes, she was not responsible for doing the partitioning herself. This means that she did not have practice with coordinating the number of partition lines with the number of pieces, potentially resulting in her inappropriate strategy of drawing the number of partition lines corresponding to the denominator, which occurred starting in the fourth tutoring session.

<table>
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<td>Shift from partitioning-by-halving to unidirectional partitioning</td>
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<td>Predominant unidirectional</td>
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Key
- Correct partitioning
- Incorrect partitioning
- Other partitioning strategy, not halving or unidirectional

Figure 61. Screen shots demonstrating the transparency use with fraction 3/5 and transparency labeled “fourths”.

selects area model card 3/5
selects and overlays transparency (fourths)
draws in lines on transparency
evaluates resulting equivalent fraction
allows for removal of partitions
Summary and conclusion. Lisa’s “partitioning” understanding was problematic in every session with the exception of the pretest. Lisa’s partitioning understanding underwent a shift over the course of the tutoring sessions from one that exclusively involved a partitioning-by-halving to one that involved a unidirectional partitioning approach. Both of these strategies proved to be problematic in different ways. When employing the halving strategy, Lisa asserted that it was impossible to divide wholes into odd numbers of pieces. When employing a unidirectional partitioning strategy Lisa often drew the number of partition lines corresponding to the denominator value. This partitioning understanding was ineffective for supporting her understanding of more complex fraction concepts, particularly fraction equivalence. Although Lisa’s partitioning understanding underwent a transition from a less sophisticated halving strategy, to a more sophisticated unidirectional partitioning strategy, at the time of the posttest both of these strategies led to difficulties in her partitioning of a shape into fifths. Her partitioning understanding often obscured the fraction concepts the representations were intended to highlight and consequently contributed to her difficulties learning.

Implications of the 5 Atypical Understandings: Arbitrary Ungrounded Manipulation

Lisa demonstrated five kinds of atypical understanding that explain almost all incorrect answers given during the data collection. There is a sixth category, which accounts for some of the remaining errors, but which is better conceptualized as a symptom rather than a cause of her MLD. Although this strand is presented last, it was not an afterthought in the analysis of the student’s difficulties. This kind of arbitrary and ungrounded manipulation of representations has been the most consistent characteristic of my experience tutoring students with MLDs. To appropriately contextualize this sixth strand of analysis, it is necessary to briefly consider the implications of the five atypical understandings as a whole.

There are both local and global ramifications for these 5 atypical understandings. When considered individually, these atypical understandings each appeared to be persistent, robust, and problematic for Lisa’s development of more complex fraction conceptions. When taken together, these 5 atypical understandings suggest that Lisa’s conceptualization of even the most basic understanding of fraction quantity was unconnected and disjointed. The five atypical understandings are each concerned with different fractional components (numerator, denominator, and/or complement), suggesting that each atypical understanding had a different domain of applicability. In addition, these 5 atypical understandings supported a vacillating fractional quantity, rather than a stable fractional quantity. I argue that Lisa’s understanding of fractional quantity was fundamentally unconnected and unstable; this leads to the sixth strand of analysis: her arbitrary ungrounded manipulation of mathematical representations.

Domain of applicability. Lisa’s atypical understandings appeared to be largely unconnected. This may be partially due to the domain of applicability for each of these kinds of understandings. Lisa’s conceptualization of fractions can be thought to involve three different quantities: the denominator value, the numerator value, and the fractional complement. For the fraction 3/5, the numerator value is 3, denominator value is 5, and the complement is 2. Each of Lisa’s five atypical understanding mapped onto different
fractional values (see Figure 62). Taking involved either the numerator value (3 taken) or the complement (2 left), but not both. Halving involved both the numerator value and the fractional complement, with a focus on the balance between the two values. For example, the fraction 3/6 might be considered to represent 1/2 because of the balance between the numerator and the fraction complement (e.g., three parts were shaded and parts were not shaded), whereas, the fraction 3/5 would not be considered the same as 1/2, because the numerator (3) does not equal the fraction complement (2). Discrete set involved both the numerator and denominator values, but these values were treated as whole numbers and not reconciled into an understanding of a single value for fractions. Unit fraction involved either judging the value of the fraction based on the inverse size of the denominator or the numerator value, but never both. Partitioning involved the denominator value alone, and was concerned with the mechanics of dividing a shape into the specified number of pieces. Therefore, each of the five atypical understandings had a unique domain of applicability. This suggests that her atypical understandings, upon which she heavily relied, supported a disconnected understanding of the values composing the fraction (numerator and denominator), and consequently had implications for her understanding of fractional quantity.

Figure 62. Illustration of the mapping of each of Lisa’s atypical understandings onto the various salient fractional values: numerator, denominator, and complement.

46 Although both the unit fraction and partitioning understanding at times exclusively applied to the denominator value, unit fraction was primarily concerned with the interpretation of the fraction, while partitioning was primarily concerned with the construction of the fraction.
Atypical understandings and fractional quantity. The most basic fractional concept is an understanding of a fractional quantity as an amount less than 1. Lisa’s atypical understandings can be seen as resulting from and indicative of a lack of understanding of fractional quantity. Instead of understanding a fractional quantity as a fixed amount, her atypical understandings allowed for a shifting fractional value.

Several of Lisa’s atypical understandings allowed for an inconsistent evaluation of fractional quantity. For example, Lisa’s “taking” understanding could result in her drawing a picture of 3/4, but subsequently interpreting it as 1/4. Lisa’s “unit fraction” understanding also allowed for shifting judgments of quantities based on how the fraction was represented. After determining 1/2 was the same as 2/4, she might judge that 2/3 could be both larger than 2/4 (because thirds are larger than fourths), and smaller than 1/2 (because thirds are smaller than halves). Her “unit fraction” allowed the same fractional value to be both bigger and smaller than another fraction. Finally Lisa’s “halving” understanding allowed her to shift her focus from the fractional quantity 1/2, to the act of splitting. Therefore, the most common and most well understood fractional quantity for most students was a site of potential ambiguity for Lisa.

The remaining two atypical understandings, although not supporting a shifting understanding of fractional quantity, also reflect an insufficiently developed understanding of a fractional value. Her discrete set understanding did sometimes involve attending to the numerator and denominator value simultaneously, however, Lisa conceptualization involved two disparate whole numbers that were not reconciled to give meaning to the fractional quantity. Lisa treated both the numerator and denominators as countable amounts. In addition, partitioning suggested that odd denominator fraction quantities could not exist, further suggesting that her underlying conceptualization of the nature of a fractional quantity was insufficiently developed.

Each of the five atypical understandings suggests that her understanding of fractional quantity was unstable and insufficient. Rather than fractional quantities being a fixed amount upon which Lisa could reason about and manipulate, Lisa’s conceptualization of fractional quantities appeared to allow for shifting quantities. This fundamental difficulty with conceptualizing fractional quantities may be more generally thought of as poor fraction sense, an idea that I will return to during the discussion section.

These five atypical understandings suggest that Lisa did not have a well-integrated or stable conceptualization of fractional quantity, the most basic and fundamental fraction concept. With this in mind, it was unsurprising that at times, Lisa demonstrated misremembered, misapplied, and seemingly random manipulation of representations and symbols in her attempts to deal with different fraction problems. The sixth strand of analysis deals explicitly with what appears to be arbitrary and ungrounded manipulation of mathematical representations, which can be thought of as a consequence of her lack of conceptual foundation.

Lisa’s Arbitrary Ungrounded Manipulation

This section presents the sixth strand of analysis: “arbitrary ungrounded manipulation.” At times, Lisa’s manipulation of representation or symbols was decontextualized from the underlying meaning. This can be thought of as a focus on the
(often abstracted) procedural actions while ignoring the context of the problem. Characteristic of this approach was a focus on the procedural steps of manipulation rather than on the meaning of the symbols and representations. A useful analogy is thinking of Lisa attempting to navigate the mathematical terrain using a series of written directions (like googlemap directions) rather than using a map. She more or less knew the sequence of turns to make along the way, but was not able to find her way back to the path if she accidentally strayed and aspects of the journey that seemed familiar could cue a sequence of directions that was inappropriate at that moment.

As in the other strands of analysis, I contrast Lisa’s “arbitrary ungrounded manipulation” with typical kinds of errors that students’ commonly experience in this mathematical domain and provide an operational definition for how these instances were identified. Second, I provide a prototypical example of this kind of manipulation, which illustrates how this kind of manipulation was atypically employed. Third, I consider how her procedural focus was problematic and potentially indicative of a more general orientation towards mathematical manipulations. This strand of analysis illustrates that Lisa’s arbitrary ungrounded manipulation was atypical and likely an outcome of her lack of sufficient conceptual grounding.

**Defining and exemplifying “arbitrary ungrounded manipulation.”** Lisa’s “arbitrary ungrounded manipulation” involved manipulating representations in inappropriate ways. Although, many students when unsure of how to answer a question might manipulate symbols without meaning (Hiebert, 1984; Hiebert & Lefevre, 1986; Hiebert & Wearne, 1986), these answers are generally within standard bounds of typical student misconceptions and misunderstandings. For example, a student facing the problem $\frac{1}{2}+\frac{1}{4}=\frac{2}{6}$ (see Figure 63). In contrast, Lisa might approach this problem by changing the denominator of $\frac{1}{4}$ to a “2” so that the problem had common denominators. This is both not a common student error, and also reflects a lack of understanding of the fractional quantity $\frac{1}{4}$, which cannot simply be rewritten as $\frac{1}{2}$. Characteristic of these errors was that they were unusual and surprising errors and indicate that Lisa was operating with a relatively superficial understanding of the mathematical representations and symbols.

![Figure 63. Contrast of typical and atypical errors.](image)

Prototypical example of “arbitrary ungrounded manipulation.” Lisa’s arbitrary and ungrounded manipulation of was evident in the following example; she changed both
the operator and the value of the fraction. During the posttest Lisa was asked to solve the problem “1/2+1/4=." She attempted to work through this problem, first by changing the addition sign to a multiplication sign and then changing the denominator of the 1/4 to a 2.

Lisa: I don’t remember which one, you have to like just multiply across for (changes addition sign to multiplication sign), I think that’s if you have like a division problem. But I know you have to somehow find the common denominator between the two. I guess that could be two, right? (writes 2)

Tutor: Ok.

Lisa: I think. So then that would be two-over-two (writes 2/2), but then that wouldn’t make sense, because it’s equal to a whole. I’m not sure.

Lisa changed both the operation sign and the denominator of the second fraction in the process of solving this problem. Lisa’s solution of 2/2, suggests that despite her modification to the operator, that she was adding these fractions rather than multiplying the fractions. Although, Lisa quickly asserted that her answer of “2/2” did not make sense, she failed to recognize that her rewriting of 1/4 as 1/2, resulted in a problem that was “1/2+1/2=”, where the answer of 2/2 did make sense. In this excerpt, there were hints of correct procedural knowledge. She seemed to know that (1) multiplication was part of the solution for adding fractions with different denominators, (2) she needed to find a common denominator, and (3) multiplication was also involved in the division of fractions. She was however unable to reconstruct for herself the complete procedure for adding two fractions with different denominators. This is somewhat surprising, given that on the previous problem she had listed seven equivalent fractions for 1/2, including 2/4. Lisa manipulated the symbols in this problem in a way that suggests that her understanding of them was devoid of the underlying meaning.

This example was considered consistent with an arbitrary ungrounded manipulation because Lisa’s manipulation of the symbols in the problem was lacking a connection to the underlying meaning of those symbols. Multiplication is not the same as addition of fractions, and 1/4 cannot simply be transformed into 1/2. Both of these manipulations suggest that Lisa was attempting to accomplish a specific procedural goal (i.e., make the denominators the same) without reference to the underlying meaning of the symbols.

**Researcher’s operational definition of arbitrary ungrounded manipulation.** Problems were coded as indicative of a "arbitrary ungrounded manipulation" if (1) Lisa manipulated symbols and representations based on superficial aspects without respect to their underlying meaning, (2) Lisa performed mathematical computations on values without regard to the meaning or relationship of those values (“2/6 equals 1/12 because 2 times 6 is 12”), or (3) Lisa interpreted representations by attending to superficial aspects
of the form, (e.g., “one-half” can be written as “1.5” because it is “1” (“one”) and “.5” (“half”)).

There were 10 instances coded as arbitrary ungrounded manipulation, and almost all instances were associated with an incorrect answer (see Figure 64).

![Figure 64](image)

**Key**

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Figure 64. Data display of all instances of Lisa’s “arbitrary ungrounded manipulation” understanding.

**Arbitrary ungrounded manipulation as detrimental to learning.** Lisa’s arbitrary ungrounded manipulation of representations was detrimental to her ability to make sense of more complex fraction concepts, particularly fraction addition that required the generation of equivalent fractions. In the following example, Lisa’s understanding of the abstracted procedure for repartitioning area models led her to conclude that the fraction 1/4 could be repartitioned and then renamed 1/3.

Prior to this episode, Lisa had solved the problem “2/3+1/4=” using the transparency squares, and created twelfths by partitioning the 1/4 square into 3 pieces (see Figure 65).

![Figure 65](image)

Figure 65. Screen shot of Lisa’s construction of the equivalent fraction for 1/4 (3/12) using area model squares with overlaid transparencies.

In this problem she attempted to employ the same procedure for solving the problem “1/4+1/3=” but her area model of 1/4 involved both horizontal and vertical partitions (see Figure 66). She gestured over her area model of 1/4 as if she was going to partition it into thirds vertically. She did not recognize that drawing two vertical partition lines would not result in equally sized pieces, and she asserted that the resulting fraction would be 1/3.
Figure 66. Digitally recreated artifact showing Lisa’s solution to the problem $1/4+1/3$ at the time of the excerpt.

Lisa: So then. If I was to do this. Should I use a different color? (picks up different colored pen) This (gestures over area model of $1/4$: see dotted red lines) To make it one-third, right?

Lisa attempted to apply a procedure for creating equivalent fractions, which was inappropriate for her area model of $1/4$. She applied a decontextualized partitioning rule to solve this addition problem, which indicates that she did not understand the conceptual entailments of the repartitioning of the area model representation. Lisa’s abstracted manipulation was inappropriately applied in this context, and it led to her assertion that $1/4$ could transform into $1/3$.

**Persistence of ungrounded symbolic manipulation.** Lisa seemed to have abstracted “rules” for solving certain kinds of problems that appeared to be based on superficial aspects of the representations. I have presented several examples that suggest that at times when she was solving problems she was not connecting the representations and symbols to their underlying meaning. I believe that these few instances of arbitrary ungrounded manipulation might be overt manifestations of a more widely applied approach. Suppose for example, that in the previous excerpt, she had partitioned the $1/4$ using horizontal partitions only. Her modification of the area model would have appeared to be entirely appropriate and I likely would have overestimated her understanding of the mathematical content. These few instances of arbitrary ungrounded manipulation may therefore be symptomatic of a wider issue, which often served Lisa to productively determine the answer to problems. Although I have identified 10 instances in the data, it is likely that she may have been using a similar kind of arbitrary ungrounded manipulation more successfully in other problems, leading at times to a correct answer. These specific problems hint at a larger and more systematic issue with how it is she understands the mathematical concepts and procedures.

**Conclusion.** At times Lisa manipulates symbols and representations without regard for the underlying meaning of those symbols or representations. It is likely that
this kind of arbitrary manipulation stemmed from her lack of sufficient conceptual grounding of basic fraction quantity. Specific instances of Lisa’s arbitrary ungrounded manipulation may suggest a wider prevalence of this kind of superficial and procedurally-oriented approach to mathematics. This strand of analysis, although essential to an understanding of Lisa’s difficulties with learning, is considered symptomatic rather than causal in an understanding of the nature of her MLD.

Discussion of Lisa’s Atypical Understandings

Lisa demonstrated five atypical understandings that explain almost all incorrect answers given during the data collection. These five understandings along with Lisa’s arbitrary and ungrounded manipulation provide an explanation for why Lisa did not learn over the course of the tutoring sessions. These understandings did not provide the necessary footholds for the development of more complex concepts like fraction equivalence or fraction operations. In this section I consider the implications of the suite of atypical understandings. First I consider the atypicality of these understandings. In particular, I address how even when similar kinds of understandings are displayed in younger students that they are transitory and refined relatively easily during the course of standard instruction. Second, I consider how Lisa’s instability in understanding of fractional quantity may best be conceptualized as an insufficiently developed fraction sense. Lastly, I address how Lisa’s understandings can be thought of as incompatible with the representational systems used within the tutoring sessions. This incompatibility renders these representations inaccessible and provides a likely explanation for why this standard instructional approach was ineffective for Lisa.

Typicality of atypical understandings. The five atypical understandings identified for Lisa are not prevalent or persistent in most students. Although characteristics of these atypical understandings may occur in students who are first grappling with fraction concepts, these misconceptions are generally quickly dispensed with and do not cause pervasive difficulties. For example, one of the fifth-grade control students (Mary) had momentary confusion about the meaning of the shading. To draw 2/3 of a cake using the whole paper, she initially shaded just one piece and although she corrected her drawing to have 2 shaded pieces, she was not initially convinced that this made sense. Mary explained, “usually when someone says, and then I took away two-thirds of the cake, they shade two thirds so that’s the part that they are taking or that, that is not there.” In this statement she associated the shaded region with the amount removed from the whole. This statement bears a striking resemblance to Lisa’s “taking” understanding. However, Mary immediately reconsidered this evaluation and continued by saying, “but in a way it could be because then if you have two-thirds and you added another third, you would adding onto it and not subtracting from it. So I think this makes sense.” In this second excerpt she reconsidered the meaning of shading in an invented context of the problem 2/3+1/3= and recognized that shading would be the amount that was added rather than removed. In the span of less than 30 seconds, with no intervention from me, Mary went from understanding the shaded region as taken away to rejecting that interpretation and understanding the shaded region as the amount cake that was there. After this one instance this understanding did not resurface for Mary and remained unproblematic for the rest of her sessions. This will be further elaborated in the cross
case comparison chapter. Although traces of these understandings may appear as students learn both the fraction concepts and the conventions of the representations, for most students these misunderstandings are often quite easily refined and do not cause persistent and pervasive issues as they did for Lisa.

**Fraction sense.** Lisa’s atypical understandings and her arbitrary manipulation indicate that she did not have a stable understanding of fractional quantity, more generally referred to as number sense. Lisa’s atypical understanding of “taking” and “halving” are both fundamentally concerned with conceptualizations of fractional quantity. In both of these atypical understandings, Lisa focuses on an action (“taking” and “halving”) rather than a quantity. Additionally, for the “discrete set” and “unit fraction” understandings, Lisa was conceptualizing fraction values in terms of whole numbers rather than on the relationship between the numerator and denominator value. Lisa’s final atypical understanding, partitioning, may potentially be considered the most problematic of all atypical understandings. Many researchers argue that experiences partitioning are core to the development of an understanding of rational number quantities. “The ability to divide an object or a set of objects into equal parts appears critical to the logical development of part-part and part-whole relationships and notations of equality and inequality.” (Lamon, 1996, p. 170). Lisa’s partitioning understanding, which did not allow for odd numbered denominators, was detrimental, rather than supportive, of her understanding of fractional quantity. Finally, Lisa’s lack of understanding of fractional quantity was reflected in her often arbitrary and ungrounded manipulations of fractional quantities, for example, thinking that 1/4 can become 1/3, or simply changing a denominator value. Together, these 5 atypical understandings and evidence of her arbitrary manipulation, suggest that Lisa did not have a stable conceptualization of fractional quantity. Similar to other studies of mathematical learning disabilities that point to student’s poor number sense as the causal factor in the student’s difficulties (Berch, 2005; Butterworth & Reigosa, 2007; Gersten, Jordan, Flojo, 2005; Piazza et al., 2010), Lisa’s unstable fraction sense seemed to be at the heart of her difficulties.

**Representations as viewed through Lisa’s atypical understandings.** Lisa’s difficulties with fraction sense were left largely unresolved, because the representations intended to support her understanding of fractional quantity were largely inaccessible and problematic for Lisa. Consideration of the three main representational tools used (area models, fraction pieces, and area model squares with transparencies), suggests that not only were the representations incompatible with her atypical understandings, but her use of these representations sometimes led to unintended consequences. Figure 67 illustrates both occurrence of a given atypical understanding in conjunction with each representation, and also incompatibilities between the representation and the atypical understandings.
Area models were ineffective instructional tools because of the incompatibility with Lisa’s taking, halving, discrete set, and partitioning atypical understandings. Lisa’s taking understanding resulted in a disconnection between her constructive and interpretive acts. Therefore, the creation of an area model itself allowed for the mutation of a fractional quantity (e.g., 3/4 may “become” 1/4). Lisa’s halving understanding involved a non-standard area model representation, where the shape was partitioned, but not shaded. Lisa’s discrete set understanding involved ignoring the size of the pieces of the model, and treating each piece (both shaded and non-shaded) as if it were uniform in meaning and size. Lisa’s partitioning understanding, involved difficulties partitioning shapes into a given number of pieces, often leading Lisa to believe that area models of odd number denominators could not be created. These four atypical understandings rendered area models problematic for the representation of fractional quantity. For Lisa, area models were not a transparent representation of fractional quantity; instead there was ambiguity with respect to how the parts and the shading in area models should be understood.

Fraction pieces were similarly problematic for Lisa. In particular, the fraction pieces were incompatible with her discrete set and unit fraction understandings. Lisa’s understanding of fractions in terms of a discrete set model (where the denominator specifies the number in the set) allowed for erroneous interpretations of the fraction pieces. In addition, the use of the fraction pieces may have reinforced her unit fraction understanding, by allowing for separate representation of the numerator and denominator. Despite multiple attempts to make the labels and the pieces more meaningful for Lisa, this representational system continued to be problematic and ultimately made the fraction pieces an inaccessible representational tool for her to use to build more complex fraction concepts like equivalence and fraction operations.

The area model with transparency overlays, although in many respects similar to the standard drawn area models, involved additional problematic components. The

Figure 67. Illustration of incompatibilities across representations and atypical understandings.
transparency grids potentially increased Lisa’s reliance on her halving understanding. The transparencies, although allowing Lisa to shift her partitioning strategy from partitioning-by-halving to unidirectional partitioning, off-loaded the responsibility for repartitioning the shapes, and subsequently may have led to some of her issues partitioning and repartitioning later in the tutoring sessions.

The three primary representational tools (area models, fraction pieces, and area model squares with transparencies) had unintended consequences, and were often at odds with Lisa’s atypical understandings. Considering these representations in conjunction with Lisa’s atypical understandings begins to sketch some of the difficulties facing Lisa’s development of a stable understanding of fractional quantity.

**Conclusion**

Lisa’s five atypical understandings contributed uniquely and in tandem to Lisa’s difficulties learning. She did not have a stable understanding of fractional quantity and the primary representational tools were often incompatible with Lisa’s understandings. These atypical understandings present a portrait of why Lisa’s attempts to learn, and my attempts to teach, were doomed to failure. Teaching a student with an MLD requires understanding how they atypically approach mathematics, understanding that standard representations may be inaccessible for these students, and devising instruction that both builds upon their firmly held atypical understandings and provides alternative representations which are compatible with their understandings.

In the next chapter I will present a similar detailed diagnostic analysis of Emily. She demonstrates some similar and some unique atypical understandings. The subsequent chapter will consider a cross-case analysis considering the similarities between Lisa and Emily. These atypical understandings were not evident in the data from the fifth grade control students, but were evident in one additional student with an MLD. These analyses suggest that atypical understandings will remain central for the students with MLD and teaching students with MLDs should involve creating representations that are more compatible with their understandings and aim to provide accessibility to the mathematic concepts. In this final chapter, I return to Lisa and present a brief overview of my attempts to do just that.
Chapter 4: The Case of Emily

The second case study student, Emily, had a history of unexplained low math achievement, which appeared to be due to an MLD rather than other factors. As in the previous chapter, the purpose of this chapter is to provide a comprehensive view of the nature of Emily’s MLD and how it was problematic for her attempts to learn. The analysis of the video taped data revealed six atypical understandings, which largely explained Emily’s difficulties over the course of the tutoring sessions and are central to understanding Emily’s MLD.

In this chapter I first present data from a variety of sources to establish that Emily has an MLD. Second, I give an overview of the six atypical understandings with a bird’s eye view of these understandings, which shows that the atypical understandings occurred across all sessions and accounted for nearly all of Emily’s incorrect answers. Third, I present a detailed analysis of each of the six strands, which constitutes the majority of this chapter. These six atypical understandings when taken together, provide an explanation for why Emily did not learn from a standard instructional intervention and therefore, are considered an apt characterization of Emily’s MLD.

Subject Classification

The second case study student, Emily, meets the MLD qualifications. Data collected during the student interview, pretest, posttest, and tutoring sessions indicate that she demonstrated (1) low math achievement, (2) no identifiable confounding factors which could explain her low achievement, and (3) lack of response to instruction as measured by her change in score from pretest to posttest.

Emily’s standardized test scores from sixth through tenth grade years show that she had a persistent low-achievement in mathematics. Emily scored at the lowest achievement levels (“far below basic” and “below basic”) across all five years. Her English language-arts scores were consistently higher than her math scores. Percentile rankings were available only for her 6th and 7th grade STAR test and were at the 23rd percentile and 11th percentile respectively, placing her below the standard 25th percentile cut-off. Emily completed the tutoring sessions at the end of her senior year of high school. Although percentile ranks were not provided for her eighth through tenth grade test administrations, she continued to score at comparably low levels\(^47\). Therefore, Emily’s history of low math achievement was considered adequate to establish her comparability to other studies of MLDs.

No confounding factors, which could explain her low achievement, were identified. Emily is a White, middle-class, native English speaker who had graduated from high

\(^{47}\) Emily scored in the “below basic” or “far below basic” categories on all math achievements test from her sixth grade to tenth grade years. Emily’s scores were classified as “below basic” for both her sixth and seventh grade years, when she scored in the 23rd and 11th percentile respectively. Therefore, scores of “below basic” and “far below basic” are considered to be consistent with scoring below the 25th percentile. The STAR test has 5 classification levels, including: far below basic, below basic, basic, proficient, and advanced.
school and was enrolled in a four-year college the following year. The initial interview indicated that she had a history of failing mathematics despite adequate effort and support. Based on the interview with Emily and observations during the data collection sessions and during Emily’s math class, lack of effort, insufficient resources, and behavior problems did not appear to be factors that could explain this student’s poor achievement in mathematics.

Emily did not demonstrate substantial gains when comparing her pretest and posttest scores. Emily’s pretest score was 49% and posttest score was 54% (see Figure 68). Emily’s lack of response to the standard intervention, which was successful for fifth-grade students without MLDs, suggests that insufficient instruction was not the sole cause of Emily’s low math achievement.

Emily, therefore, meets the qualifications for having an MLD. This analysis is concerned with an in depth look at why Emily did not learn, what kinds of difficulties she experienced, and how best to understand her MLD.

Figure 68. Comparison of Emily’s pretest/posttest performance as compared to the aggregated fifth grade control students.

**Overview of Analysis**

The purpose of this analysis is to explore potential causal factors that may have contributed to Emily’s lack of positive gains from pretest to posttest.

**High-level view of Emily’s six atypical understandings.** Through iterative analytic passes through the data, I identified six atypical understandings, which directly contributed to Emily’s difficulties. These atypical understandings (depicted in Figure 69, with an example) include: (1) “smaller part” (2) “halving,” (3) “part-part,” (4) “more
pieces,” (5) “quarters” and (6) “rule-based navigation.” Although there is some overlap between Emily’s and Lisa’s atypical understandings, I will address the comparison of their understandings for the cross-case comparison chapter (see Chapter 5).

In this section, I provide a brief description of each of the six strands of analysis along with a high level view of how instances of these atypical understandings occurred over the entire data corpus.

**Figure 69. Overview of the six strands of analysis for Emily.**

**Smaller part.** “Smaller part” involved Emily’s tendency to understand that the numerator value was the part comprised of the fewest number of pieces. Although she would interpret an area model of 1/4 correctly, she might interpret an area model of 3/4 also as 1/4, because she was attending to the one non-shaded piece, as opposed to the 3 shaded pieces. Her interpretations of fractional amounts, particularly for fractions greater than 1/2, were often dependent upon the fractional complement (non-shaded or missing pieces).

**Halving.** “Halving” involved understanding the fraction 1/2 as a splitting or partitioning action, rather than the quantity one-half. In the case of area models, Emily often drew one-half by drawing the shape and splitting (or “halving”) the shape into two pieces, while omitting the shading. She sometimes over applied this halving understanding to other unit fractions and understood the fraction to be represented by the number of partitions (e.g., 1/4 is represented by splitting a shape into 4 pieces).

**Part-part.** “Part-part” was an understanding that the focal fractional quantities were comprised of two parts, the numerator and the fractional complement. This can be contrasted with a part-whole understanding, where the fraction is understood to be comprised of a designated part of the whole. Although this understanding was sometimes productively applied in the context of the fraction 1/2 (e.g., 3/6 equals 1/2 because 3 are shaded and 3 are not shaded), it led to one-half specific strategies for understanding fractions, which could not be extended to other fractional values.

**More pieces.** “More pieces” was an understanding of fractional magnitude in terms of the number of pieces rather than the size of the pieces. For example, Emily might incorrectly conclude that 1/4 was larger than 1/3 because 1/4 was comprised of more pieces. This understanding was most often applied to the denominator value, but sometimes was overextended to the numerator value or equivalent fractions.

**Quarters.** “Quarters” was an understanding that fourths were a kind of special fractional value. Quarters, similar to halves, had a unique status, and were often referred to as “quarters” rather than “fourths” and were associated with the value 25, rather than the value 4. Figural aspects of various representations appeared to cue her quarters understanding, even when it was not appropriate.
Rule-based navigation. “Rule-based navigation” involved understanding and interacting with mathematical representations in ways that were focused on the procedural actions rather than the conceptual meaning. Emily’s abstraction of procedural rules often treated nonessential aspects as central to the procedure.

These six atypical understandings provide a comprehensive view of Emily’s difficulties across the tutoring sessions. The detailed analysis of each strand will consider the ways in which these understandings were ultimately detrimental to her learning and when taken together establish a reasonable explanation for why Emily did not learn. Before delving into the detailed analysis of each of these strands of analysis I present a high-level view of the occurrence of each atypical understanding across all video-taped data of the sessions with Emily.

A “bird’s-eye” view of the data corpus. Zooming out to consider the entire data corpus (see Figure 70), high-level inferences can be made about Emily’s overall performance. First, Emily appeared to experience difficulty across all sessions, but particularly during tutoring session #2 (area models) and tutoring session #4 (fraction operations) as indicated by the frequency of instances where Emily’s answer was coded as incorrect (red) or unanswered/tutoring (yellow). Second, Emily’s atypical understandings appear to be more clustered than in Lisa’s case, suggesting that Emily might have drawn upon particular atypical understandings to answer particular kinds of problems. Despite this clustering, all six of the atypical understandings persisted across the sessions suggesting that these understandings were central to her understanding of fractions.
Figure 70. Problem coding across all the data from the sessions with Emily, with data with 6 atypical understandings flagged.
When these problems were sorted according to the correctness of the answer, different clusters of problems emerged from the data (see Figure 71). Of all the problems, 67% were coded as correct, 10% were coded as tutoring instances, 19% were coded as incorrect, and 3% were coded as “n/a” (e.g., journal entry). For the problems coded as correct 11% were associated with an atypical understanding. For the problems coded as tutoring or unanswered 41% were associated with an atypical understanding. Although atypical understandings were at times evident during correct or unanswered questions, this can be contrasted with the prevalence of atypical understandings associated with her incorrect answers. For Emily’s incorrect answers, 84% involved her use of one or more of her atypical understandings. Only 16% (n=12) of her incorrect answers had no atypical understandings flagged. This suggests that the atypical understandings accounted for nearly all of her incorrect answers across all the tutoring sessions.

An audit was conducted on all 12 instances that were not accounted for by one of the atypical understandings. These errors were further classified to determine if there was another kind of atypical understanding at play. Three errors were due to a calculation mistake, which Emily self-corrected. Four errors were due to Emily not considering the difference in the size of the wholes. Although this may represent a distinct kind of understanding, this did not warrant a separate analytic category for several reasons. First Emily did not apply this understanding with any consistency, and while this understanding was evident in these 4 errors, there were many more instances in which she did correctly consider and evaluate the size of the wholes. Second, this is also a common difficulty experienced by students learning fraction concepts. Therefore, although there was similarity in the underlying reason for the error, this was not considered an atypical understanding. For the remaining five errors, there was not sufficient data to provide a classification. Therefore, the six atypical understandings well capture the kinds of difficulties that Emily experienced over the course of the tutoring sessions. I now turn to an in depth treatment of each of these strands of analysis.

\[48\] In two of these instances, it was likely that she was drawing upon a smaller part understanding, but Emily’s explanation was insufficiently elaborated.
Figure 71. Problem coding across all the data from the sessions with Emily, sorted by correctness.
Emily’s Smaller Part Understanding

This section presents the first of Emily’s six atypical understandings: “smaller part.” Her interpretation of fractional representations often involved assuming that the numerator was represented by the smaller of the two parts. For example, Emily might interpret an area model of 4/5 as 1/5, because there were fewer non-shaded than shaded pieces. This understanding often involved her interpreting fractional representations by attending to the part that was missing (fractional complement) rather than the part designating the fractional quantity (e.g., shaded pieces). Although similar in some ways to Lisa’s “taking” understanding, Emily’s “smaller part” understanding occurred only when she was interpreting fractional representations and tended to occur for fractions that were greater than one-half. Emily’s “smaller part” understanding was robust in that it occurred across representations, persisted over time, and resisted explicit attempts to address it. This atypical understanding was detrimental to her ability to engage with more complex fraction concepts and ultimately provides one of the key pieces towards understanding why Emily did not learn.

In this section, first I introduce the “smaller part” understanding and provide a prototypical example in conjunction with the operational definition. Second, I discuss, how this understanding proved to be problematic for Emily’s understanding of more complex fraction concepts. Third, I consider the ways in which this understanding occurred over time. Specifically, I consider the persistence of this understanding across the tutoring sessions, how this understanding occurred in the context of various representations, and the ways in which this understanding led to ambiguity in her interpretation of fractional representations. Finally, I show how this understanding resisted explicit attempts to address it. This strand of analysis illustrates that Emily’s “smaller part” understanding was atypical, robust, and detrimental to her ability to learn.

**Defining and exemplifying “smaller part” understanding.** Emily’s “smaller part” understanding involved the ways in which she understood representations of fractional quantities, specifically related to determining the value of the numerator. To highlight the atypicality of Emily’s understanding it will be contrasted with a typical understanding of the numerator, in the most commonly used representational context: area models. Typically, when area models are used to represent fractional quantities the numerator is represented by the number of shaded pieces. Given area models partitioned into 4 parts, an area model with 1 piece shaded will be interpreted as 1/4 and an area model with 3 pieces shaded will be interpreted as 3/4 (see Figure 72). The shaded pieces are understood to designate the focal fractional quantity.

A typical use of area models can be contrasted with Emily’s “smaller part” understanding. She often understood the numerator to be represented by the part composed of fewer pieces. Given area models partitioned into 4 parts, although she would correctly interpret an area model with 1 shaded piece as 1/4, she might also interpret an area model with 3 shaded pieces as 1/4 (see Figure 72). Emily’s tendency to attend to the smaller part was so central to her understanding of fractions that at times she incorrectly interpreted her own drawings. For example, immediately after drawing a correct area model for 3/4, she might interpret her own drawing as 1/4. Emily’s “smaller part” understanding led to instability in her interpretations of fraction.
Prototypical example of “smaller part”. Emily’s “smaller part” understanding was evident in the following example. She had just correctly drawn a picture of the fraction 1/5 in which she shaded one piece. However, when I asked her to interpret a drawn representation of the fraction 5/6, she attended to the non-shaded piece and interpreted the fraction as 1/6.

Despite having drawn a correct area model for 1/5, she interpreted the area model that I drew with respect to the non-shaded piece, which comprised the smaller part. When I asked Emily to explain her answer, she said, “there’s like, all, except one shaded.” For Emily, the fractional quantity was not consistently determined by the shaded pieces or the non-shaded pieces. Instead, her understanding of the fractional quantity was dependent upon the part with fewer pieces (1 non-shaded as opposed to 5 shaded). Her correct drawing of 1/5 and her incorrect interpretation of 5/6 as 1/6, suggests that Emily’s understanding allowed for ambiguity in how area models were interpreted.

This episode was considered consistent with a “smaller part” understanding because it involved understanding the fractional representation in terms of the part with fewer pieces, in this case the non-shaded part. I will return to this example when I discuss the robustness of this understanding.
Researcher’s operational definition of smaller part. Problems were coded as indicative of a “smaller part” understanding if Emily interpreted a fractional representation based on the part composed of fewer pieces. In addition, because her attention to smaller part created ambiguity with respect to how shading should be interpreted, problems were also coded as “smaller part” if she interpreted the fraction as the fractional complement (corresponding to the non-shaded or missing pieces), irrespective if it was the smaller of the two quantities.  

Non-smaller part examples included instances in which she explicitly made reference to the shaded pieces as signifying the fractional quantity. There were 26 instances of Emily’s “smaller part” understanding and 5 flagged non-smaller part examples. As seen in Figure 73, this understanding appeared across almost all the sessions, and was often used in conjunction with an incorrect answer.

Figure 73. Data display of all instances of Emily’s “smaller part” understanding.

Smaller part as detrimental to learning. Emily’s smaller part understanding hindered her ability to engage with more complex fraction concepts, including fraction comparison, fraction equivalence, and fraction operations. In the following example, a simple fraction comparison was problematic for Emily because of her tendency to focus on the smaller part of her own drawn representations.

In this problem Emily was asked to compare the fractions 2/8 and 5/8. She correctly drew a representation of both fractions, using shading to represent the numerators (see Figure 74). However, in her interpretation of her drawings she shifted

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49 As with all these atypical understandings, the name is simply a gloss to try to capture the essence of the student’s understanding. The original name for this understanding was “missing part”, because she sometimes attended to the non-shaded pieces or missing fraction pieces when interpreting fractional representations. Her descriptions of how she determined whether she should attend to the shaded or non-shaded pieces, suggested that at the heart of this understanding was a focus on the smaller part. This understanding was therefore renamed “smaller part” to better reflect Emily’s understanding, but the few instances in which she attended to the non-shaded pieces, even when it was not the smallest part, were considered to be consistent with this understanding.

50 Recall that non-examples were instances in which the student’s explanation for her reasoning was in direct conflict with the given atypical understanding.
from attending to the shaded pieces to attending to the non-shaded pieces. She interpreted her drawing of 5/8 as 3/8 and pointed to each of the non-shaded pieces in her drawing of 2/8. Although she ultimately said that did not know how to answer this question, her interactions with her drawn representations suggests that she was attending to the fractional complements rather than the fractional quantity. Given the complexity of the gestures in this excerpt, the transcript is presented in conjunction with screen shots and scanned artifacts (below).

Figure 74. Scanned artifact of Emily’s drawing of the fractions 2/8 and 5/8.

<table>
<thead>
<tr>
<th>Line#</th>
<th>Transcript</th>
<th>Scanned Artifacts / Video Stills with description</th>
</tr>
</thead>
</table>
| 1     | Tutor: What if we had the problem two-eighths and five-eighths, which one would be bigger there?  
Draws 8 rectangles, shades in 5) | Correct drawing of 2/8 and 5/8                                                                 |
<table>
<thead>
<tr>
<th></th>
<th>Emily: So, this is. <em>(points to each of the 5 shaded pieces.)</em></th>
<th>Emily points to each of shaded pieces of 5/8 with her finger</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Emily: <em>(points to each of the non-shaded pieces with her pen, and traces a “3” in the air with her pen)</em></td>
<td>Emily points to each of the non-shaded pieces of 5/8 with her pen.</td>
</tr>
<tr>
<td>4</td>
<td>Emily: <em>(Points to each of the non-shaded pieces in the drawing 2/8 with her pen.)</em></td>
<td>Emily points to each of the non-shaded pieces of 2/8 with her pen.</td>
</tr>
</tbody>
</table>
Although Emily was able to correctly represent both fractions, these drawings actually hindered her ability to compare the fractional amounts. Once drawn, Emily shifted from attending to the shaded pieces to attending to the non-shaded pieces for 5/8. She interpreted 5/8 as 3/8, and appeared to start to reinterpret her drawing of 2/8 in terms of the non-shaded pieces. It was only after pointing to each of the non-shaded pieces in the fraction representation for 2/8 that she rejected her interpretation of 5/8 as 3/8. It is possible that counting the non-shaded pieces, which composed the *larger* part of 2/8, helped her recognize, in this particular situation that counting the non-shaded pieces was inappropriate. This example demonstrates how Emily’s tendency to focus on the smaller part led to ambiguity with respect to how to interpret fraction representations, and caused her to be unable to answer this relatively simple comparison problem. As in this example, Emily’s smaller part understanding of fractions appeared to disrupt her ability to build a more complete understanding of fraction comparison, fraction equivalence, and fraction operations. I now turn to an exploration of why this kind of problematic understanding persisted through the tutoring sessions and did not get resolved.

**“Smaller part” across the sessions.** This section considers the persistence and robustness of Emily’s “smaller part” understanding over time. First I consider evidence of this understanding during the pretest and posttest, which suggests that Emily was relying upon this understanding both before and after the tutoring sessions. Second, I consider the generality of this understanding both in terms of the fraction values which evoked this understanding, and how this understanding might have influenced all of her interpretations across the tutoring sessions. Third, I consider the ways in which this understanding also appeared in conjunction with the foam fraction pieces. Fourth, I highlight how this understanding was at the same time persistent and also inconsistently applied across the tutoring sessions, suggesting another dimension of instability in Emily’s conception of fractional quantity. Lastly, I present excerpts illustrating how this understanding was not easily corrected and it resisted my attempts to address it.

**Persistence over time: “smaller part” during the pretest and posttest.** Emily’s pretest and posttest instances of “smaller part” were examined to determine how she was using this understanding before and after the tutoring sessions. Both during the pretest and posttest Emily displayed ambiguity with respect to whether she should attend to the shaded or non-shaded pieces. During the pretest Emily incorrectly interpreted 7 fraction...
representations in terms of the fractional complement, attending to the non-shaded pieces, in most cases the smaller part. Similarly, during the posttest, Emily interpreted two fractions with respect to the non-shaded pieces and explicitly asked if she was supposed to judge the fraction based on the shaded or non-shaded part. Emily’s “smaller part” understanding was therefore, evident both before and after the tutoring sessions. This suggests that “smaller part” was part of her prior understanding and indicates that this understanding was not refined over the course of the tutoring sessions.

**Generality of “smaller part” understanding.** Because Emily’s smallest part understanding was primarily used in the context of fractions greater than one-half\(^5\), I conducted a separate analytic pass of all fraction interpretation problems\(^2\) to determine if Emily had difficulty interpreting fractions greater than 1/2 in general. For fractions less than or equal to one-half, Emily interpreted 96% correctly, whereas for fractions greater than one-half, she only interpreted 65% correctly. Indeed, for all of her incorrect interpretations, 89% of those problems involved a fraction greater than 1/2. This separate analytic pass, which included all Emily’s fraction interpretations showed that Emily’s had much greater difficulty interpreting fractions greater than 1/2. This suggests that her “smaller part” understanding potentially influenced her fraction interpretations and provides additional support for the centrality of Emily’s “smaller part” understanding.

**Persistence across representations: “smaller part” with fraction pieces.** Emily’s “smaller part” understanding was also evident when she used the foam fraction pieces, suggesting that her understanding was not simply a misunderstanding of area model conventions. When interpreting fraction pieces, particularly those greater than 1/2, Emily focused on the fractional complement: the missing pieces. In Emily’s first interaction with the fraction pieces, I asked her to explain how the fraction pieces were labeled. She used ten 1/10 pieces to construct a whole and identified as such. She continued her explanation, removing one of the 1/10 pieces and defining the fractional quantity in terms of the piece that was missing (i.e., the empty space).

\(^5\) In 19 out of the 24 instances (79%) her smaller part understanding was used in conjunction with a fraction greater than 1/2. In only 1 case was her smaller part understanding used in conjunction with a fraction less than 1/2. The remaining 5 instances were impossible to classify because it was ambiguous which fractional part she was attending to, or the problem did not involve a fraction interpretation. Emily’s smaller part understanding was used almost exclusively in the context of fractions greater than 1/2.

\(^2\) I excluded interpretation problems that involved number lines, unequal partitions, or interpretations where she simply stated that the fraction was not (e.g., that is not 2/3) without stating the fractional value.
<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript</th>
<th>Video Stills</th>
</tr>
</thead>
</table>
| 1    | Tutor: So your first challenge is, how would you explain to someone else how they labeled each of these fraction pieces?  
Emily: Oh! Ok. Oh. First is this one (pulls out and lines up 1/10 pieces, creates a whole with ten 1/10 pieces) I don’t know. Is this right? Ok. (points to each piece of the whole) So there are 10 of the same size pieces, and those all make up a whole.  
Tutor: Ok.                                                                                                                          | ![Image](image1.png) |
| 2    | Emily: So. Yeah. So, this (takes piece out and holds it in her fist).                                                                                                                                              | ![Image](image2.png) |
| 3    | Emily: If you take this out (moves fist away from pieces, continues to look at nine 1/10 pieces on table), then that is one-tenth.                                                                                | ![Image](image3.png) |
| 4    | Tutor: Ok.  
Emily: And yeah.  
Tutor: Ok, so if we take this out then this is one-tenth? (makes circular pointing gesture above nine 1/10 pieces)                                                                 | ![Image](image4.png) |
| 5    | Emily: This (pointing to empty space) is one-tenth.                                                                                                                                                               | ![Image](image5.png) |
Emily began by constructing a ten 1/10 pieces and identifying her construction as a whole, by saying, “so there are 10 of the same size pieces, those all make up a whole” (line 1). In this statement, she was identifying the pieces comprising the whole as both what she was focusing her attention on, and also how she was determining the quantity. She shifted her attention as soon as she removed a piece. Rather than removing 1 piece and renaming the quantity 9/10, she renamed the quantity 1/10 and began attending to the piece that was missing. Emily’s interpretation of the nine 1/10 pieces involved attending to the smaller part (i.e., the missing 1/10 piece, represented by empty space) and naming the fractional quantity with respect to that amount.

Although this was partially resolved over the course of the next two minutes, in which Emily toggled between attending to the empty space and attending to the fraction pieces, her focus on the empty space, in the context of fraction pieces, persisted across the remaining tutoring sessions. Her attention to the empty space (i.e., smaller part) was particularly problematic for her when she attempted to represent fraction operations with the fraction pieces. She often understood the addition of two fraction pieces in terms of the smaller part, and attempted to determine what would fit in the empty space. For example, she attempted to solve the problem 1/2+1/3= by putting a 1/5 piece in the empty space (see Figure 75). This piece did not fit and she was unable to solve the problem. Similarly, when solving the problem 1/2+1/4= she determined that a 1/4 piece fit in the empty space (see Figure 76), and said the answer to the problem “1/2+1/4=” was “one-fourth.” Emily’s “smaller part” understanding was sufficiently robust, in that it was invoked even in the context of foam fraction pieces, and just as with area models, Emily attended to the smaller part, which was often the fractional complement rather than the focal fractional quantity.

Figure 75. Screen shots of Emily’s attempt to solve the problem 1/2+1/3= by placing a 1/5 piece in the empty space.

Figure 76. Screen shot of Emily’s attempt to solve the problem 1/2+1/4= by placing a 1/4 piece in the empty space.

**Ambiguity resulting from a “smaller part” understanding.** Emily’s smaller part understanding was problematic because it allowed for ambiguity in how fractional representations were interpreted. As seen in the prototypical example, Emily could shift from attending to the shaded pieces (e.g., 1/5) to attending to the non-shaded pieces (e.g., seeing 5/6 as 1/6). Emily’s consistency in applying her “smaller part” understanding, and attending to the part comprised of the fewest pieces, may actually have increased the ambiguity of fractional representations for her. For example, in one problem during the pretest Emily was attempting to identify fraction representations equal to 4/5. She first
interpreted a discrete set representation of 8/10 incorrectly focusing on the 2 non-shaded pieces, then interpreted 4/9 correctly focusing on the 4 shaded pieces, then interpreted an area model of 4/5 again focusing on the 1 non-shaded piece (see Figure 77). In each case her determination of the numerator was based on the part with the fewest pieces, irrespective of whether that part was shaded or not shaded and consequently she interpreted two of these representations incorrectly. Although at times she may consistently be reasoning with a smaller part understanding, this suggested a larger inconsistency in how she interpreted area model representations. Because her smaller part allowed her to understand either the shaded or non-shaded pieces as the focal fractional quantity, this created an ambiguity in how to interpret representations.

![Figure 77. Emily’s interpretation of several fraction representations consistently applying her smaller part understanding, but inconsistently attending to shading.](image)

This ambiguity can be seen in the following example, in which, Emily attended to the non-shaded pieces, even though it was not the smaller part. When Emily was asked to solve the problem “1/2+1/4=” she asked if she could draw pictures to help her solve the problem. She constructed two correct area models\(^{53}\) (see Figure 78). After constructing these area models she began to attend to the non-shaded pieces.

![Figure 78. Scanned artifact showing Emily’s drawn area models for the problem “1/2+1/4=”.](image)

\(^{53}\) When Emily originally drew her representation of 1/2, she did not include shading. This will be addressed in the second strand of analysis “halving”. Before the start of this excerpt she had corrected her drawing and shaded in one of the two pieces.
<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript</th>
<th>Video screen shots with descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Emily: Um. So, one-half <em>(points to non-shaded part of the 1/2 area model with pen)</em></td>
<td>Points with pen to non-shaded part of 1/2 area model.</td>
</tr>
<tr>
<td>2</td>
<td>Emily: plus one-fourth <em>(points to area model of 1/4)</em>.</td>
<td>Points with pen to area model of 1/4.</td>
</tr>
<tr>
<td>3</td>
<td>Emily: Um. So then. * (pause: 10 seconds) * Um. This would leave. Three-quarters – three-fourths <em>(gestures over the non-shaded pieces in area model of 1/4 - gesture repeated 3 times)</em> And then <em>(pause: 8 seconds)</em> I don’t know. Tutor: So this one is confusing? Emily: Yeah.</td>
<td>Gestures with pen over the 3 non-shaded pieces of the picture of 3/4. She repeated the gesture depicted here 3 times.</td>
</tr>
</tbody>
</table>

This example highlights three aspects of inconsistency. First, in this example, Emily attended to the non-shaded pieces even though they did not represent the smaller part in either fraction. Second, Emily herself drew the area models for 1/2 and 1/4. Although she drew these correctly, using shading to designate the numerator value, she immediately shifted to interpreting these in terms of the non-shaded pieces. Third, Emily interpreted the area model for 1/4 first as 1/4, and then shifted to focusing on the 3 non-shaded pieces. There was considerable inconsistency in how she interpreted fractional representations, at times she attended to the focal fractional quantity and at times she attended to the fractional complement. For Emily, these drawn representations did not represent a stable fractional quantity; instead the same drawing could represent both 1/4 and 3/4. This suggests that these kinds of representations should be considered inaccessible for Emily, and likely ineffective for building more complex fraction concepts.

**Robustness of Emily’s “smaller part” understanding.** Emily’s smaller part understanding was sufficiently well entrenched that it resisted explicit attempts to address it through standard instruction. In the second tutoring session Emily had iteratively
refined a list of rules for interpreting area models. Her journal entry read: “rules: put the shaded # of pieces on the top. Put the number of all pieces including shaded pieces on the bottom”. We reviewed her journal at the start of the third tutoring session, she explained her journal entry by reading out the rules and then she gave an example of the application of her rule to the area model 1/5 that she had drawn. Despite her seeming focus on the shaded region to determine the fractional quantity, as seen in the prototypical example, when I asked her to interpret an area model of 5/6, she switched to attending to the smaller part (the non-shaded pieces). After incorrectly determining that the answer was 1/6, she asked if she should be applying the rules from her journal entry. Although she was able to determine that the rules would specify the fraction value was 5/6, she was unsure of which was the correct answer.

Emily: That’s um. *(pause 15 seconds)* That’s one-sixth.
Tutor: Ok, so this one *(pointing to area model of 5/6)* is going to be one-sixth?
Emily: Wait, are we doing that? *(points to rules)*
Tutor: Ok, if we were following your rules, would we end up with one-sixth?
Emily: No.
Tutor: What would we get if we were following your rules?
Emily: Uh, we would get. Uh, five *(pause)*, five-sixths.
Tutor: So, how do we know whether this is a picture of five-sixths or this is a picture of one-sixth?
Emily: I don’t know.

Emily’s own method for interpreting fractional representations was clearly at odds with the written rules we had generated in the earlier session. Her smaller part understanding resulted in an answer of 1/6, while the written rules resulted in an answer of 5/6. The rules that she had generated and recorded were insufficient to help her correct her answer. Instead she was unsure of which answer was correct.

In the following excerpt I asked her to interpret her drawing of 1/5. She correctly determined that it was 1/5, and acknowledged that she was attending to the shaded piece. When I asked her to consider what was different about the drawing of 5/6 she focused on the smaller part, the one non-shaded piece, as the reason for the discrepancy.

Tutor: So. If we were to cover this one up *(covers 5/6 picture)* and we were to look at this one. Let’s imagine you couldn’t see the fraction there. What would you say this is a picture of?
Emily: One-fifth.
Tutor: One-fifth. Ok, so you were using the shaded ones here?
Emily: Uh huh.
Tutor: What makes it different down here?
Emily: Like all, there’s like, all except one shaded.

Despite having written rules specifying how she should interpret drawn area models, Emily still was struggling with ambiguity about how to determine the numerator value. As the discussion continued, she eventually shifted to attending to the shaded region, and correctly interpreted the area model for 5/6. She explained: “Well, I know that this is five-sixths, because it’s not one-sixth because, there’s like six-sixths is a whole, and that one would be, (gestures as if shading the last piece) shaded in. It’s just five-sixths, because only one isn’t shaded in. So it’s five-sixths.” She used her understanding of the whole to make sense of the part of the representation she should attend to. Although she had constructed a relatively articulate explanation, leveraging her understanding of a whole, this did not create a lasting change in her interpretation of fractional representations greater than 1/2. Despite reviewing her journal at the start of the fourth tutoring session and immediately before the posttest, Emily’s “smaller part” understanding continued to persist and led to errors. This suggests that Emily’s understanding of the fraction as defined by the smaller part was robust and resistant to instruction.

**Summary and conclusion.** Emily’s “smaller part” understanding resulted in instability in the way in which she conceptualized fractional quantity. She often interpreted the fractional quantity in terms of what part was comprised of fewer pieces. This understanding was robust in that it persisted throughout all the sessions and was evident across representations and problem types. This “smaller part” understanding was considered consequential in that it derailed her ability to make sense of more complex fraction concepts. Unlike a simple misunderstanding, Emily’s “smaller part” was resistant to attempts to address it, suggesting that “smaller part” was core and consequential to Emily’s conceptualization of fractional quantity.

**Emily’s Halving Understanding**

This section presents the second of Emily’s six atypical understandings: “halving.” Emily’s halving understanding was quite similar to Lisa’s halving understanding. Emily understood and represented the fraction 1/2 as the action of halving, or splitting in two, instead of the quantity one-half. However, unlike Lisa, Emily occasionally extended this halving understanding to other unit fractions as well. At times she understood partitioning into four pieces to be a representation of the fraction 1/4. Emily’s halving understanding sometimes caused difficulties for her in the context of more complex fraction concepts and provides another dimension to understanding why Emily did not learn.

In this section, first I introduce Emily’s “halving” understanding and provide a prototypical example in conjunction with the operational definition. Second, I discuss, how this understanding proved to be problematic for Emily’s understanding of more complex fraction concepts. Third, I consider the occurrence of Emily’s halving over time. Specifically, I consider the persistence of this understanding, the ways in which the instruction may have reinforced or extended this understanding, and why this understanding was not resolved. This strand of analysis illustrates that Emily’s “halving”
understand was atypical, that she appeared to extend this halving understanding to other unit fractions particularly in the context of more complex fraction problems, and consequently was detrimental to her ability to learn.

**Defining and exemplifying Emily’s “halving” understanding.** Emily’s “halving” understanding involved the ways in which she understood and represented the fraction one-half. Just like Lisa, Emily atypically represented 1/2 by drawing a shape and dividing it two pieces, omitting the shading (see Figure 79). Emily’s drawing of one-half involved representing the halving action rather than emphasizing the quantity 1/2 with shading. She occasionally extended this halving understanding to other unit fractions and understood the fraction to be represented by the creation of partitions (e.g., representing 1/4 by partitioning in 4 parts).

![Figure 79](image)

**Figure 79. Contrast of typical and atypical use of drawn models of 1/2.**

**Prototypical example of halving.** Emily’s “halving” understanding was evident in the following example, in which Emily represented the fraction one-half by partitioning a circle in half and omitting the shading. During the pretest Emily was asked to draw or write the fraction 1/2. She drew a circle partitioned it in half and wrote the numeric notation for 1/2. When asked to explain her drawn representation she identified the “one” as the whole circle, and the “half” as the partition line. She then focused on the importance of the equivalence of the two parts.

![Diagram](image)

Tutor: Can you explain to me how these two things that you have written out are one-half?

Emily: Well, this is one (traces with pen around the circle). And it’s cut in half, and there is two of the exact same – I mean, they are not the exact same, but they are supposed to be the exact same size.
Emily explained her non-shaded representation of one-half, by mapping the words onto aspects of the representation. “Half” was signified by the act of partitioning into two pieces, and “one,” instead of representing one of the two resulting pieces, was understood to be represented by the entire circle. In this excerpt, both the partitioning act and the balance between the resulting two parts were central to her representation of one-half.

This prototypical example was considered consistent with a “halving” understanding because Emily did not represent the fractional quantity one-half. Instead she halved a circle, and identified the partition line as signifying “half.” As with Lisa, Emily’s lack of shading was not simply a matter of her lack of knowledge of the shading convention, because on a subsequent problem she was asked to represent the fraction 3/5, and she produced a canonical area model representation, where the shaded region corresponded to the numerator value (see Figure 80). Although Emily knew of the shading convention, she used partitioning rather than shading to represent the quantity 1/2.

Figure 80. Scanned artifact from pretest question in which Emily was asked to draw or write 3/5.

**Researcher’s operational definition of halving.** Problems were coded as indicative of a “halving” understanding if (1) Emily represented 1/2 by drawing a shape, partitioning it in two, and omitted the shading, (2) Emily’s gestures and explanations were consistent with 1/2 as a splitting-action rather than a quantity, or (3) Emily understood the partitioning of a shape to represent that unit fraction quantity (e.g., partitioning into 2 is a representation of 1/2, partitioning into 3 is a representation of 1/3, etc.). Non-halving examples included, instances in which Emily explained 1/2 as being comprised of one out of 2 total parts or rejected drawings of 1/2 without shading.

There were 11 instances of Emily’s “halving” understanding and 2 flagged non-halving examples. As seen in Figure 81, although this understanding was not evident in all tutoring sessions, this understanding was often used in conjunction with an incorrect answer.

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54 Admittedly, equally partitioning is an important area model convention, however, Emily appeared to understanding the fraction 1/2 in terms of the equal partitions. At no point during the episode did she signify that one-half was one of the two pieces.

55 Instances in which Emily focused on the balance between the two parts, although coded as “halving” for Lisa, were not coded as “halving” for Emily. Emily’s part-part understanding represents a distinct understanding for her, and so will be addressed in the next strand of analysis.
Figure 81. Data display of all instances of Emily’s “halving” understanding.

**Halving as detrimental to learning.** Emily’s halving understanding was particularly problematic when it was extended to other unit fractions. In the following example, Emily applied her halving understanding to the fraction 1/4. Emily was in the process of solving the problem “2/3 + 1/4 =”. After writing down the problem, she drew 2/3 with a green pen. She then repartitioned the area model into 4 horizontal rows with a black pen (see Figure 82).

![Figure 82](image)

Figure 82. Scanned and digitally modified recreation of Emily’s written work for her journal entry, which shows the problem 2/3 + 1/4 and her drawn representation.

Emily was able to correctly represent 2/3 and repartition it appropriately, but she was unsure of how to complete the problem. In this excerpt, I asked if she had drawn a picture of one-fourth. Although she was initially unsure, when I suggested that she draw a picture of 1/4, she decided that she already had, and referenced the area model for 2/3 that she had repartitioned into four horizontal rows.
Tutor: So what did you do?
Emily: I did, two-thirds, in green, and then one-fourth.
Tutor: So did you represent – did you show what one-fourth is? *(points at 1/4 written on whiteboard)*
Emily: I don’t know. Um.
Tutor: What if we were to draw a picture of one-fourth?
Emily: Well, actually, yeah I did.
Tutor: Ok.
Emily: Yeah cause.
Tutor: Can you say something more about that?
Emily: I don’t know. It’s like divided evenly into four.

In this episode Emily indicated that she thought she represented both the fraction 2/3 and the fraction 1/4 in her drawing. She specified that the green marker was her representation of 2/3, and that she drew a picture of 1/4 because it was “divided evenly into four.” Emily did not mistakenly think that the drawing of 2/3 was a representation of 1/4, instead, she thought that by repartitioning the area model, she had in fact represented the quantity 1/4. In this way she was allowing the partitions themselves to be the representation of the fraction. This example highlights the ambiguity around Emily’s understanding of fractional quantity. Instead of adding the quantity 1/4 to the fraction 2/3, she operated on her representation of 2/3, by repartitioning it into 4 horizontal rows and understood that to be the answer to the problem. This example highlights how Emily’s halving understanding – in particular her understanding of quantity in terms of partitions – was problematic for the development of more complex fraction concepts.

“Halving” across the sessions. This section considers Emily’s halving understanding over the course of the sessions. First I consider evidence of this understanding during the pretest and posttest, which suggests that Emily was relying upon this understanding both before and after the tutoring sessions. Second, I consider the ways in which the transparency overlays might have unintentionally supported Emily’s extension of her halving understanding to other unit fractions. Third, I present how explicit rules for understanding the use of shading with area models were insufficient to refine Emily’s halving understanding.

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56 Also, of note is that Emily was solving this problem as part of a journaling activity at the end of the tutoring session. She had solved this problem correctly only 7 minutes before and her written work from her previous solution was in clear view throughout this entire episode.
Persistence over time: “halving” during the pretest and posttest. Emily’s pretest and posttest instances of “halving” show similarity in how she represented the fraction 1/2. During both the pretest and the posttest she represented the fraction 1/2 by partitioning a shape in two pieces and omitting the shading (see Figure 83). Her explanations for her drawing were nearly identical during the pretest and posttest. During the pretest she stated, “Well, this is one (traces around the circle). And it’s cut in half, and there is two of the exact same [size]” and during the posttest she said, “so, this whole thing is one (traces with pen around exterior of horizontally partitioned rectangle) and there’s two.” In both cases she understood the numerator value to be represented by the one whole as opposed to one of the two pieces.

There was however a slight difference in her pretest and posttest performance when she was asked to determine whether a drawing of one-half without was a valid representation of 1/2 (see Figure 83). During the pretest she accepted the drawing of 1/2 with no shading, and said, “it is a half, because it is divided evenly.” This explanation was consistent with her justification given for her answer to the 1/2 construction problem. However, at the time of the posttest, despite having drawn shapes that were similarly unshaded on the first problem, she rejected the drawing of 1/2 with no shading. She explained that she did not select it, “because one is not shaded.” Although she did not select the non-shaded area model at the time of the posttest, her four drawings of 1/2 without shading suggests that her halving understanding was still present at the time of the posttest. Therefore, Emily’s halving understanding was evident during both the pretest and the posttest, suggesting that halving was part of her prior knowledge and was not resolved over the course of the tutoring sessions.

Potential reinforcement of “halving”. Over the course of the tutoring sessions Emily extended this halving understanding to other unit fractions. As in Lisa’s case, it is

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57 It is possible that her rejection of the non-shaded drawing had more to do with the other options that were presented during the interpretation task (including a canonical area model representation of 1/2, which included shading). Emily’s drawings of 1/2, therefore are considered to be a more unbiased reflection of how she was thinking about the fraction 1/2 at the time of the posttest.
possible that Emily’s halving understanding might have been unintentionally reinforced by the transparency overlays. Recall that the transparency overlays were used to help the students create equivalent fraction pairs (see Figure 84). Data from the tutoring sessions suggests that the transparency overlays might have supported her understanding of one-half as halving and provide a plausible origin of her extension of halving to other fractional amounts.

One episode in particular suggests that Emily treated the halves transparency as a representation of the fraction 1/2. In this challenge Emily was asked to use the transparency overlays to create 5 different fractions all equal to 1/2. After recording her first equivalent fraction pair (1/2 = 2/4), which was comprised of the area model square of 1/2 and the halving transparency (see Figure 85) she attempted to use the transparency—rather than the area model square to create another equivalent fraction to 1/2. She placed the “halves” transparency over the area model for the fraction 2/5 (see Figure 86). Instead of focusing on the quantity one-half, and generating 5 equivalent fractions for that quantity, Emily focused on the halving action and attempted to use the halving transparency on another fractional value (2/5). This suggests that her attention was focused on the transparency overlays, not as a tool of partitioning area models, but as focal quantities in and of themselves.

In addition to treating the halves transparency as the focal fractional quantity, it is possible that the transparency overlays might have contributed to Emily’s extension of
her halving understanding to other unit fractions. Just as there were transparency overlays labeled “halves”, there were also transparency overlays labeled “thirds” and “fourths.” It is possible that Emily’s experience with these transparencies reinforced her understanding of partitioning as a representation of quantity. Indeed, Emily’s extension of “halving” to other unit fractions all occurred after the introduction of the transparency overlays, and all in the context of fraction operations. Given the temporal location (see Figure 87) of the introduction of the transparency model, the transparency model was a likely candidate for supporting Emily’s over-extension use of her halving understanding to other unit fractions.

Therefore, the transparency overlays might have both unintentionally supported her understanding of the fraction 1/2 as halving and caused Emily to extend her halving understanding to other fractional amounts.

![Figure 87](image)

**Figure 87. Data display with introduction of the transparency model highlighted.**

**Robustness of “halving.”** Although Emily’s halving understanding occurred relatively infrequently, it was still the predominant way that she chose to represent 1/2 at the time of the posttest. There was no opportunity to directly address Emily’s representation of 1/2 with no shading during the tutoring sessions, because all seven instances occurred during the pretest or posttest. The 5 times Emily drew a representation of 1/2 during the tutoring session, it was a canonical area model representation that included shading. Although her lack of shading was never addressed directly, during the course of the tutoring sessions, Emily and I had spent a considerable amount of time discussing the representational conventions used with area models, both when drawing them and when interpreting them. These were captured as “rules” in her journal, which she reviewed just prior to the administration of the posttest. The rules written in her journal read: “Put the shaded # of piece on the top. Put the number of all pieces including the shaded pieces on the bottom” (see Figure 88).
In the following excerpt Emily attempted to explain her drawings of 1/2, and although she referenced the rules she was unable to reconcile those rules with her drawings of 1/2 with no shading.

Emily: So, this (pointing to the horizontally partitioned rectangle) is two groups. I mean, So, this whole thing is one (traces with pen around exterior of shape) and there’s two. So, uh, if we follow the rules to that thing then, well, I don’t know. So, this is one, and there’s two, so you put the… (gesturing up and down with pen) I forgot.

Emily’s was not able to connect her recorded rules with her drawn representations of 1/2. This was understandable in some respects, because the rules highlighted the importance of shading, and Emily’s representations did not contain shading. Her reference to the rules, which she understood and could implement in other contexts, was not enough to cue her to shade one of the halves. Therefore, despite considerable discussion around the
meaning of shading in drawings of area models, the rules were insufficient to displace Emily’s natural inclination to think of the fraction 1/2 as halving.

**Summary and conclusion.** “Halving” involved understanding the fraction one-half to be represented by the act of splitting (represented by the partition line) as opposed to the quantity of 1/2. When Emily’s halving understanding was extended to other fractions, it became problematic. The tutoring sessions did not sufficiently address nor resolve this atypical understanding and at the time of the posttest, representing one-half as halving was the predominant way she understood 1/2. One-half was a central fraction to these tutoring sessions, and Emily’s “halving” understanding, along with the next strand of analysis “part-part”, indicates that Emily had an atypical understanding of the quantity 1/2, which although productive in many contexts did not provide a sufficient base on which to build an understanding of other fractional quantities. Emily’s “halving” understanding therefore provides another dimension of why Emily did not learn.

**Emily’s Part-part Understanding**

This section presents the third of Emily’s six atypical understandings: “part-part.” Emily’s part-part understanding involved interpreting representations of fractions with respect to 2 quantities: the numerator and the fractional complement. For example, she might refer to a drawing of 2/3 as ‘2 shaded and 1 not shaded’. Characteristic of a part-part understanding was that quantification of the whole was not considered. Unlike most other atypical understandings, this understanding was quite productive for her during the sessions particularly when used with fractions equal to 1/2. However, her tendency to evaluate fractional representations in terms of part-part rather than part-whole, allowed for an ambiguity that was occasionally problematic and might ultimately lead to more significant issues.

In this section, first I introduce Emily’s “part-part” understanding and provide a prototypical example in conjunction with the operational definition. Second, I discuss, how this understanding proved to be problematic for more complex fraction interpretations, specifically those that involved equivalent fractions. Third, I consider the occurrence of Emily’s part-part understanding over time. Specifically, I consider what kinds of problem types and representations evoked this understanding and how part-part might have introduced ambiguity in her interpretations of fraction representations. This strand of analysis illustrates that Emily’s “part-part” understanding was atypical and potentially detrimental to her ability to use 1/2 as a grounding fraction to build an understanding of other fractions.

**Defining and exemplifying Emily’s “part-part” understanding.** Emily’s “part-part” understanding involved the ways in which she interpreted and parsed representations of fractional quantity. To highlight the atypicality of Emily’s part-part understanding it will be contrasted with a typical part-whole understanding. Typically, when an individual interprets a drawn representation of a fraction, he/she identifies two quantities: (1) the numerator, typically represented with shading, and (2) the number of parts comprising the whole (see Figure 89). The fractional quantity is understood to be the part of the whole that is signified, typically with shading. In contrast, Emily sometimes interpreted representations of fractions in terms of two separate and distinct quantities: the shaded and the non-shaded part (see Figure 89). Although she might be
able to correctly determine that the representation of 3/6 was equal to 1/2, her justification would focus on the balance between the two parts (i.e., the equivalence of 3 shaded and 3 non-shaded) versus, the relationship of the fraction part (3) to the whole (6).

**Typical part-whole interpretation**  
The fraction is understood as part of the whole collection.

![Typical part-whole interpretation](image1)

**Atypical part-part interpretation**  
The fraction is understood as comprised of 2 distinct groups.

![Atypical part-part interpretation](image2)

Figure 89. Contrast of typical part-whole and atypical part-part interpretation of a fractional representation for 3/6.

**Prototypical example of part-part.** Emily’s “part-part” understanding was evident in the following example, in which Emily was judging whether or not a collection of representations were valid ways of representing 1/3. During the pretest she incorrectly rejected the discrete set model for 2/6, and identified each of the parts (2 and 4), rather than the part and the whole (2 out of 6).

Tutor: Ok, and then you said this one wasn’t one-third *(points to 2/6).*
Emily: No, because there are 2 shaded and there are 4 not shaded.

Emily interpreted this representation as two distinct groups, a group of shaded pieces and a group of non-shaded pieces. The focal quantities in this representation were therefore 2 and 4, rather than 2 and 6. Given that she was parsing the representation in this way it was unsurprising that she was unable to determine that this representation was equivalent to 1/3. This can be contrasted with her performance on the same item on the posttest, in which she applied a part-whole interpretation. As she evaluated the representation she said, “Ok. *(pointing to each of the circles)* 1, 2, 3, 4, 5, 6. So, how I know is that it’s six. And then it’s two out of six. So that’s one-third. *(circles discrete set model 2/6).*” I argue that her change from pretest to posttest had less to do with her understanding of equivalent fractions and had more to do with how she was parsing and interpreting the representation in each case. The multiplicative relationship of the numerator to denominator was clear when she focused on the values of 2 and 6, but was obscured when she focused on the values 2 and 4.

The prototypical example was considered consistent with a part-part understanding, because she interpreted the representation with respect to two distinct parts, the shaded amount and the non-shaded amount.
Researcher’s operational definition of part-part. Problems were coded as indicative of a "part-part" understanding if (1) Emily referred to a fractional amount in terms of the numerator and the complement rather than the numerator and the whole, or (2) Emily focused on the balance between the two parts comprising the whole. Instances in which she identified the fraction in part-part terms, but continued her explanation to identify the whole were excluded from this understanding, because in addition to identifying the two parts, she was also attending to the amount comprising the whole, and is therefore more suggestive of a part-whole understanding.

There were 14 instances of Emily’s “part-part” understanding, which occurred only during the pretest and posttest (see Figure 90). Although this understanding occasionally was associated with an incorrect answer, it was often incredibly productive for Emily. Given how productive this understanding often was, non-part-part examples were not identified for this understanding.⁵⁸

There were 14 instances of Emily’s “part-part” understanding, which occurred during the pretest and posttest (see Figure 90). Although this understanding occasionally was associated with an incorrect answer, it was often incredibly productive for Emily. Given how productive this understanding often was, non-part-part examples were not identified for this understanding.⁵⁸

Figure 90. Data display of all instances of Emily’s “part-part” understanding.

Part-part as detrimental to learning. Although Emily’s part-part understanding was in many cases incredibly productive, there were limits to the applicability of this understanding. The instances in which this understanding was employed in the context of

⁵⁸ Non-part-part examples were not identified for several reasons. First, the purpose of the non-examples is often to show how the student was occasionally reasoning in more “typical” ways, which were in conflict with the atypical understanding. Because Emily’s part-part understanding was often incredibly productive it was therefore more difficult to contrast with non-part-part examples. Second, almost all Emily’s fraction interpretations did involve identifying the part and the whole. Therefore, in most cases she was answering interpretation problems typically. Identifying non-part-part (i.e., part-whole) instances was therefore considered to be an unproductive analytic path. Part-part was nevertheless included as a strand of analysis because of the prevalence of the instances during the pretest and posttest.
more complex fraction interpretation problems, highlighted the potential for this understanding to become more widely problematic. In one of the more complex fraction interpretation problems, Emily was asked to judge if various representations were valid representations of 4/5. When considering a discrete set drawing of 8/10, she incorrectly determined that it was not equal to 4/5, and although she did count all the pieces, she named the fraction using a part-part understanding.

Tutor: Why didn’t you pick this one (points to picture of 8/10)?
Emily: Because it didn’t seem right, there are two, um, wait, let me see. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. And then there is eight out of two. I mean there is two out of eight. So, yeah.

Emily interpreted the representation of 8/10 first as “eight out of two” (shaded/non-shaded) and then as “two out of eight” (non-shaded/shaded). Both of these interpretations she identified the two parts, the shaded and non-shaded, as the focal amounts, rather than the 10 that comprised the whole. Emily shifted her answer from 8/2 to 2/8, likely because she was more accustomed to having the numerator be the smaller of the two values. Not only did she not attend to the whole in this instance, but the values for the two parts, suggested that the non-shaded should be represented as the numerator value. Her part-part understanding, in addition to causing her difficulties in the context of more complex fraction interpretation problems, also may have contributed to her focus on the non-shaded pieces.

“Part-part” across the sessions. In this section I consider the occurrence of Emily’s part-part understanding over time. First I consider why this understanding only surfaced during the pretest and posttest, particularly focused on the nature of the problem types during which this understanding was evoked. Second, I consider how this understanding occurred in the context of several different representational forms. Third, I consider the ramifications for this kind of understanding, both with respect to the ambiguity that it introduced with respect to interpretation of fraction representations, and the ramifications for her understanding of the fraction 1/2.

Persistence over time: “part-part” during the pretest and posttest. Unlike the other atypical understandings, Emily’s part-part understanding occurred only during the pretest and posttest sessions and always in conjunction with a fraction interpretation problem. The reason for the localization of this understanding in these two sessions was likely due to the unique problem types. Although Emily did interpret various representations of fractions during the tutoring sessions, during the pretest and posttest she was asked to interpret representations with respect to a target fraction (e.g. “is this a drawing of 1/2?”). In this way, this interpretive activity was unlike any of the problems given during the tutoring sessions. As Emily justified why she selected or did not select a given representation, a part-part understanding often was evoked in a way it was not during simple interpretation problems. Emily’s use of the part-part understanding at the
time of the pretest and posttest suggests that the tutoring sessions were unsuccessful in changing her orientation to focus on the part of the whole.

**Pervasiveness of understanding: part-part across multiple representations.**

Emily relied upon her part-part understanding in the context of several representation types. As previously mentioned, this understanding occurred primarily in the context of problems in which she determined whether a representation matched a target fractional quantity (see Figure 91). Part-part was used in conjunction with all representation types included in the problem: area models, number lines, and discrete set drawings. One example will be given for each type of representation in which she judged the representation to be equal to $1/2$.

**Figure 91.** Example interpretation problem given during the pretest/posttest sessions, in which students are asked to identify all fraction representations matching the target fraction.

In the first example, Emily explained why she selected an area model of $1/2$. To justify her selection, she identified each of the parts, and focused on the similarity in size between the two parts.

Emily: This one is shaded (points to shaded piece) and this one is not (points to non-shaded piece) so it’s - same size.

Rather than focusing on the relationship of the one shaded part to the whole (two pieces), she focused on and identified each of the parts separately, and then focused on the balance between the two parts.

Similarly, in the next example, Emily focuses on the balance between the two parts. She again justified her answer based on the enumeration of both of the parts.
Emily: And then there is (points to picture of 3/6) three not shaded and three shaded.

Again, in justifying why she selected this representation she identified the two parts (3 and 3) rather than the relationship of the shaded part to the whole. Also of note in this example, was that she identified the non-shaded pieces first and then the shaded. This suggests that the part that was non-shaded was as central to her interpretation of this representation as the shaded pieces. Her understanding of this representation involved two quantities, which were differentiated by color.

In the final example, Emily applied a part-part understanding to a number line representation. She determined that a number line from 0 to 4 with an arrow pointing to the 2 was equivalent to 1/2. In this excerpt she focused on the balance in the representation and pointed to each of the elements on the left and right side of the “2” as a way of justifying that the representation was equal to 1/2.

Emily: Two is like a half in this one, because it’s like 0, 1, and 3, 4 (pointing to each number as saying it). I guess. I don’t know. Cause it’s like equal numbers (points to both sides of the 2).

Emily focused on the equality between the two identified parts. The first part was composed of two elements (0 and 1) and the second part was also composed of two elements (3 and 4). As she reiterated her explanation, again focusing on the equality of the two distinct parts, she reconsidered her answer, because she was unsure if “0” would “count.”

Emily: It’s just like it’s an equal amount of spaces to get to 2 (points to 0 and 1). And then if you like, 4, 3, 2, and then, I mean 4, 3, 2, (pointing to each number as saying it) and then 0, 1, 2 (pointing to each number as saying it). But, I don’t actually I don’t know because zero doesn’t count.
This excerpt provides further evidence that Emily was reasoning about the number line with a part-part understanding. Emily pointed to each of the elements composing the set, working in a symmetrical fashion, starting from the outside most points and working towards the halfway point (2). The symmetrical motion suggests that she was focusing on the number of elements rather than the inherent value of those elements. By the end of the excerpt she was unsure whether this representation was equal to 1/2 because “zero doesn’t count.” The balance of her two parts was contingent upon zero being one of the elements, if zero didn’t “count” the balance would not be maintained. Emily did not judge this representation to be equal to 1/2 in part-whole terms (because 2 is half of four), but instead identified the equality of two distinct parts (i.e., two numbers to the right of the arrow and two numbers to the left of the arrow).

Emily relied upon her part-part understanding to evaluate area models, number lines, and discrete set representations. These examples have illustrated how she parsed these representations in two distinct parts, rather than focusing on the part and the whole.

**Implications of part-part understanding.** Unlike many other atypical understandings, Emily’s part-part understanding was more often associated with a correct answer (62% correct). Her part-part understanding can be thought of as a productive understanding for her in certain contexts, particularly with respect to the fraction 1/2. In this section, I consider the ramifications of Emily’s part-part understanding with respect to (1) the potential ambiguity this understanding could introduce in the context of area model representations and (2) the implications for understanding the fraction 1/2, as a balance between two quantities.

**Ambiguity of non-shaded.** One of the main difficulties in using a part-part understanding, particularly in conjunction with area models, is the tendency to see both parts as composed of a quantity or an amount. Whereas with a part-whole understanding, the shaded region is understood to be the focal fractional quantity, with the total number of pieces (shaded and non-shaded combined) representing the whole. In contrast, with a part-part understanding both the shaded and non-shaded take on independent significance. Rather than the non-shaded pieces representing emptiness or an incompleteness of the whole, Emily treated the non-shaded pieces as quantities in their own right. As seen in the example with 8/10, which she interpreted as 8/2 and then 2/8, she treated the parts as relatively interchangeable. There was not one focal fractional part in her interpretation of the fraction, but instead, two parts that were treated as interchangeable. For example, rather than referring to the non-shaded pieces as “unshaded” or “not shaded” Emily sometimes referred to those as “white pieces”. This terminology reflected her tendency to see these non-shaded pieces as entities with a color, rather than as empty placeholder pieces. To better illustrate this, consider the colored representation in Figure 92, there is no sense of emptiness. The red and the orange pieces each constitute a separate group. I argue that at times Emily understood standard area model representations not as shaded (filled) and non-shaded (empty) but as two different groups, each having its own color (see Figure 93).
Implications of part-part understanding of $\frac{1}{2}$. In addition to the ambiguity around the interpretation of area models, Emily’s part-part understanding also had specific ramifications with respect to her understanding of the fraction $\frac{1}{2}$. Emily’s understanding of the fraction $\frac{1}{2}$ in terms of a balance between parts was often productive for her. However, this understanding of $\frac{1}{2}$ did not allow for the fraction $\frac{1}{2}$ to serve the grounding function that it normally does in instruction. In the tutoring sessions, new concepts were often introduced first with the fraction $\frac{1}{2}$ and then extended to other fractional values. Because Emily had one-half-specific strategies, $\frac{1}{2}$ did not serve as a stable foundation upon which to build an understanding of more complex fractional values or more complex fraction concepts. For example, although she was able to judge that $\frac{4}{8}$ was the same as one-half because there were “four of these shaded, and another four not shaded”, she was unable to extend her part-part understanding to recognize the equivalence of $\frac{2}{6}$ to $\frac{1}{3}$. The ramifications of a specific one-half-strategy should not be underestimated. As in the tutoring sessions, instruction often positions $\frac{1}{2}$ as an intuitive way to introduce and explore the entailments of rational number in an accessible context. However, the strategies that Emily developed around her use and
understanding of the fraction 1/2, were specific only to that fraction, and therefore, could not provide the necessary footholds for exploration of other fractional quantities.

**Summary and conclusion.** Emily’s “part-part” understanding involved focusing on the fractional part and fractional complement rather than understanding the fraction in terms of the part and whole. This understanding appeared primarily when Emily was in the process of judging whether a given representation matched a target fraction, and consequently only appeared during the pretest and posttest sessions. This understanding occurred in conjunction with area models, number lines, and discrete set model representations, and most often in conjunction with the fraction 1/2. Although often productive in the context of 1/2, this understanding was problematic when applied to other fractional values.

Emily’s part-part understanding can be thought of as the interpretive counterpart to her “halving” understanding. While “halving” often occurred in conjunction with a construction of a representation, “part-part” always occurred in the context of an interpretation of a representation. Emily’s tendency to understand the fraction 1/2 in terms of a splitting (i.e. halving) or a balance (part-part) were strategies that were specific to the unique value of 1/2 and could not be productively extended to other fractional values. Consequently, Emily’s “part-part” understanding provides one important dimension of why Emily did not benefit from the standard instruction.

**Emily’s More Pieces Understanding**

This section presents the fourth of Emily’s six atypical understandings: “more pieces.” This understanding involved judging the fractional quantity, often in the context of area models, based on one parameter alone: the number of pieces. Characteristic of this understanding was that Emily did not attend to the size of the pieces. “More pieces” was often understood to be indicative of the larger fraction, and would lead Emily to incorrectly conclude, for example, that 3/5 was larger than 3/4, because 3/5 had more pieces. Emily’s “more pieces” understanding was robust in that it resisted attempts at instruction and it persisted over time. This atypical understanding was detrimental to her ability to engage with more complex fraction concepts and ultimately provides one of the key pieces towards understanding why Emily did not learn.

In this section, first I introduce Emily’s “more pieces” understanding and provide a prototypical example in conjunction with the operational definition. Second, I discuss how this understanding proved to be problematic for Emily’s understanding of more complex fraction concepts particularly fraction comparison and fraction equivalence. Third, I consider the ways in which this understanding occurred over time. Specifically I consider the persistence and robustness of her “more pieces” understanding and how attempts to refine this understanding led to additional issues. This strand of analysis illustrates that Emily’s “more pieces” understanding was atypical, robust, and detrimental to her ability to learn.

**Defining and exemplifying “more pieces” understanding.** Emily’s “more pieces” understanding involved the ways in which she understood fractional quantities, particularly related to judging the magnitude of the fraction. To highlight the atypicality of Emily’s understanding it will be contrasted with a typical understanding of the fraction comparison. Consider the comparison of area models of 1/3 and 1/4. Because there is
one piece shaded in each case, determining which fraction is larger requires determining which of the two shaded pieces is larger. Typically, the focus of the comparison is therefore on the size of the piece (see Figure 94). Determining the number of pieces comprising each representation is a valid strategy for determining which of the two pieces is larger (e.g., the greater the number of pieces, the smaller each individual piece is). Therefore, an area model partitioned into three, rather than four, pieces will have larger pieces and consequently, 1/3 is larger than 1/4.

In contrast, Emily focused exclusively on the number of pieces comprising the whole. She inferred that the area model with “more pieces” was the larger fractional value. Because she focused exclusively on the quantification of the parts rather than the size of the parts, she might conclude that 1/4 was larger than 1/3 because 1/4 had more pieces\(^59\) (see Figure 94). Unlike a typical understanding, she did not count the pieces to determine the size of the pieces; she counted the pieces to determine the value of the fraction. In this way, size, a crucial aspect of continuous models, like area models, was not salient for her.

![Figure 94. Contrasting examples of typical and atypical comparison of area models.](image)

Prototypical example of “more pieces”. Emily’s “more pieces” understanding was evident in the following example in which she judged 3/5 to be bigger than 3/4. This example occurred at the end of the second tutoring session as we were playing a game which involved selecting the larger of two area models given on the game card (tasks adapted from Armstrong & Larson 1995). Each problem involved a comparison of two area models that were partitioned in different directions. The size of the whole, numerator value, or denominator value varied for each. When Emily was presented with the comparison of 3/4 (orange) and 3/5 (green), she first asserted that the wholes were the

\(^{59}\) Although conflating whole number knowledge and fraction knowledge is a common issue experienced by students (Mack, 1993), I argue that Emily’s “more pieces” was different than a simple over-application of whole number concepts. Emily’s “more pieces” understanding was fundamentally characterized by an attention to the number of pieces rather than the size of the pieces. For her, size did not appear to be a relevant dimension.
same size, and then proceeded to count the number of total pieces in each. She eventually determined that \( \frac{3}{5} \) had more pieces and selected \( \frac{3}{5} \) as the larger fraction.

Emily: They are both the same size, and, um, but this one (points to orange area model), 1, 2… (pointing to all 4 pieces of the orange \( \frac{3}{4} \) area model) it’s in fourths. And then this one (points to green \( \frac{3}{5} \) area model) is out of 1, 2, 3, 4, 5 (as points to each of the pieces of green \( \frac{3}{5} \) area model). Um. So. Um. I pick. Um. I pick… I don’t know. This one is hard.

Tutor: Ok, so do we want to talk through this? So we have - So how many pieces in each cake\(^{60}\)?

Emily: There’s four and then there is 1, 2, 3, 4, 5. So that one (points to green \( \frac{3}{5} \) area model) has more pieces.

Tutor: So this one is divided into four and this one is divided into five.

Emily: Uh huh. So it’s this one. (points to green \( \frac{3}{5} \))

Emily selected \( \frac{3}{5} \) as the larger of the two fractions. Her interpretation and comparison of the two quantities did involve comparing the relevant dimension: the denominator values. However, although she correctly determined that \( \frac{3}{5} \) had “more pieces”, she incorrectly inferred that it was the larger fractional amount. Emily’s only reference to size in this excerpt was in her assertion that both of the area models had the same size wholes. This was characteristic of the way in which she attended to size across many of the comparison problems. Of note is that these problems were designed by Armstrong and Larson (1995) to evaluate how students were reasoning about various fractional values, without allowing for direct comparison of the sizes of the pieces. By partitioning the shapes in different directions, the student’s understanding of the relationship of the denominator value to the size of the pieces was evoked. Emily’s answer indicated that she understood that a greater number of pieces signified a greater fractional value, and made no reference to the size of the individual pieces.

This episode was considered consistent with the “more pieces” understanding because Emily’s evaluation of the fractional comparison was focused on which area model was comprised of “more pieces.” In this example, she focused exclusively on the number of pieces rather than the size of the pieces.

**Researcher’s operational definition of more pieces.** Problems were coded as indicative of a “more pieces” understanding if (1) Emily focused on the number of pieces in the whole, rather than on the size of the pieces, particularly when size was the relevant

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\(^{60}\) I asked “so how many pieces in each cake, are shaded?” but Emily talked over the “are shaded” utterance, and answered this question as if I had asked how many pieces each cake was divided into. The “are shaded” was omitted from the transcript presented here for clarity and because it appeared that Emily did not hear this part of the question.
dimension, (2) Emily asserted that the larger denominator had the larger sized pieces, (3) Emily asserted that the fraction with the larger denominator was the larger fractional value\textsuperscript{61}, or (4) Emily focused on the number of pieces without referencing the size of the pieces in the context of equivalent fractions. In addition, because over the course of the tutoring sessions, “more pieces” was refined to be associated with the smaller fraction, problems in which Emily asserted that the fraction with the larger numerator was the smaller fraction (e.g. 3/5 is smaller than 2/5 because 3 is larger than 2) were also included. Non-more pieces examples were defined as instances in which she explained that the larger denominator was associated with a smaller sized piece.

There were 16 instances of Emily’s “more pieces” understanding and no identified non-more pieces instances\textsuperscript{62}. As seen in Figure 95, this understanding appeared across almost all the sessions, and was often used in conjunction with an incorrect answer.

![Figure 95: Data display of all instances of Emily’s “more pieces” understanding.](image)

**More pieces as detrimental to learning.** Emily’s “more pieces” understanding was problematic primarily because it highlighted the number of pieces and ignored the size of those pieces. Two examples will be given which demonstrate how this was

\textsuperscript{61} Unlike Lisa, Emily did not simply ignore the value of the numerator when using the denominator to determine the magnitude of the fraction, instead, she compared denominators only when the numerator values were the same.

\textsuperscript{62} Recall that to constitute a non-example, Emily must not only be reasoning in a way in conflict with the understanding, but must provide a justification for her answer that is in conflict with the understanding. An additional analytic pass will be discussed in the subsequent section that analyzes all instances in which Emily made reference to the size.
problematic for Emily. In the first example, Emily discounted differences in the size of the pieces and relied upon the area model with more pieces to determine the larger fraction. In the second example, Emily’s focus on “more pieces” without reference to size obscured the conceptual entailments of using area models to represent equivalent fractions. Ultimately, Emily’s “more pieces” understanding was detrimental to her ability to reason about fractional magnitude and equivalent fractions.

As seen in the prototypical example, Emily focused on the number of pieces comprising each whole, and inferred that more pieces signified the larger fraction. Her attention to the number rather than size of the pieces was so pervasive that it occasionally caused her difficulties even with unit fraction comparisons. For example, during the posttest, Emily was asked to compare the fraction $1/6$ and $1/8$. She drew pictures of both fractions and despite recognizing that $1/8$ had smaller pieces, she determined that $1/8$ was bigger because it had more pieces.

Emily: (draws $1/6$, draws $1/8$) One-eighth is bigger.
Tutor: Ok, and how do you know?
Emily: Because there are more pieces.
Tutor: Ok, and so what about the fraction -
Emily: Well... Yeah, never mind. I don’t know. They’re, (pointing to $1/8$ area model) like the pieces are smaller.
Tutor: Ok. So, which is going to be bigger?
Emily: One-eighth.

In this instance, even though Emily referenced the idea of smaller pieces, her final determination of fractional magnitude was based on the number of pieces rather than on the size of the pieces.\textsuperscript{63} This excerpt suggests that Emily’s understanding of fractional quantity in the context of area models was potentially problematic. Even in this rare instance when she did attend to the size of the piece, the size was seen as less important to determining fractional quantity than the number of pieces. Emily judged fraction magnitude primarily based on the number of pieces in the area model.

In the second example Emily again focused on the number of pieces and did not attend to the size of the pieces. In this problem Emily had drawn an area model of $2/3$.

\textsuperscript{63} Although Emily did refer to $1/8$ as having smaller pieces, this was not classified as a “non-more pieces” instance. Because she ultimately determined that the $1/8$ was bigger because it had more pieces, this entire problem was classified as a “more pieces” instance rather than a non-more pieces instance. Although problems can be associated with more than one atypical understanding, a problem cannot be associated with both the atypical understanding and the non-example (non-more pieces).
and then transformed it to 4/6 by partitioning it down the center. Emily was asked to
determine what aspects of the representation had changed and what had stayed the same.
She identified that there were now more pieces, but did not identify any other change to
the representation.

Tutor: Great. Ok. So the last question is, what’s changed and what’s stayed the
same about this cake?
Emily: It’s cut into more pieces.
Tutor: Ok, so there’s more pieces, do we want to write more pieces down?
Emily: (writes “more pieces”)
Tutor: Is there anything else that has changed?
Emily: Nope.

Emily’s evaluation of the changes to the area model, as it was transformed from 2/3 to
4/6, involved attending solely to the number of pieces. It is unclear if Emily was
referring to the change in the number of shaded pieces (from 2 to 4) or the change in the
total number of pieces (from 3 to 6), or the change in both (2 to 4 and 3 to 6). It is clear
however, that Emily did not consider the change in the size of the pieces to be a relevant
aspect of the transformation. Although when asked Emily was eventually able to
determine that the size of the pieces had changed, this was not something that she
attended to or volunteered without explicit questioning. Using area models to support an
understanding of fractional equivalence necessitates understanding the inverse
relationship of the number of pieces and the size of the pieces. Despite repeated exposure
to these kinds of problems, Emily did not coordinate the change in the number of pieces
and the size of the pieces.

These examples have illustrated how Emily’s “more pieces” understanding was
problematic in the context of fraction comparisons and fraction equivalence. Emily used
“more pieces” to identify the larger of two fractions, and also used “more pieces” in a
context in which the fractions were equivalent. Emily’s use of “more pieces” to justify
both inequality and equality of fractions suggests that this understanding was ripe for
misapplication. These examples suggest that Emily’s understanding of fractional value,
particularly in the context of area models was insufficient.

“More pieces” across the sessions. This section considers the persistence and
robustness of Emily’s “more pieces” understanding over time. First, I consider evidence
of this understanding during the pretest and posttest, which suggests that Emily was
relying upon this understanding both before and after the tutoring sessions. Second, I
present data from a separate analytic pass, which indicates that Emily did attend to the
property of size when working with continuous models, however she was primarily
focused on the equality of the partitioning or equality of the wholes being compared
rather than the size of the individual pieces. Third, I illustrate how attempts to refine Emily’s “more pieces” understanding were counterproductive. Instead of shifting her focus to the size of the pieces, “more pieces” became inappropriately associated with the smaller fractional value, which caused significant problems when she used her understanding of “more pieces” with reference to the numerator value.

**Persistence over time: “more pieces” during the pretest and posttest.** Emily had difficulty with fraction comparison problems during both the pretest and posttest. In each instance she privileged and attended to the number of pieces rather than the size of the pieces. At the time of the pretest she was unable to determine which fraction was larger 1/6 or 1/8. She drew pictures for both fractions, but focused on the number of pieces comprising the whole rather than on the size. Similarly, as previously mentioned, at the time of the posttest she determined that 1/8 was larger than 1/6, because 1/8 had more pieces. During both the pretest and posttest Emily drew area models, but her understanding of fractional magnitude, was based on the number of pieces rather than on the size of the pieces. This suggests that “more pieces” was part of her prior understanding and indicates that this understanding was not refined over the course of the tutoring sessions.

**References to “size” across the tutoring sessions.** Because Emily’s “more pieces” understanding specifically excluded an evaluation of the importance of the size of the pieces, I did a separate analytic pass of the entire data corpus attempting to identify instances in which she referenced size with respect to the denominator. Out of the 44 instances in which she referred to size, less than 7% of those were in reference to the individual pieces comprising the whole (e.g., the denominator). Emily’s references to size typically involved an assertion of equality or inequality between two different wholes or between the individual pieces of the area models. The tutoring sessions included non-standard questions in which the size of the wholes varied, or the wholes were unevenly partitioned. For these non-standard kinds of problems, one must attend to the size of the wholes or the equality of the individual pieces. It is possible that her attention to these non-standard size variations may have distracted from evaluations of size as related to the denominator.

**Robustness of Emily’s “more pieces” understanding.** Emily’s more pieces understanding was sufficiently well entrenched that despite instruction, she continued to attend exclusively to the number of pieces while ignoring the size of the pieces. In addition, the refinements that Emily did make to her “more pieces” understanding over the course of the tutoring sessions led to additional issues. Emily began to associate “more pieces” with the smaller, rather than larger, fraction. Therefore, her refinement of “more pieces” only involved the inference she made about the fraction with more pieces, rather than including a consideration of the size of the pieces. Several excerpts will be presented to highlight both how this understanding was robust and how the refinement of her “more pieces” understanding actually led to further issues.

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64 This separate analytic pass involved reading through all the transcripts and flagging anytime she used the word “size” or made reference to size with words like “same”, “smaller”, or “bigger”.
This sequence of examples was taken from the area model comparison game previously referred to in the prototypical example. During this game, the student drew a card, circled the larger of the two area models, and then checked her answer by looking at the game key (see Figure 96). If the student answered correctly she moved her token forward the designated number of spaces. Because there were various isomorphic problems, Emily compared the fractions 3/4 and 3/5 several times. In the span of 15 minutes, Emily answered this comparison question incorrectly 3 separate times. Emily’s persistent incorrect answer despite my attempts to help her focus on the size of the pieces, suggests that this understanding was relatively robust.

![Key]

Figure 96. Example of area model comparison game card and the associated answer key.

In the first instance, Emily was unsure how to answer the question, I asked her to focus on one piece and determine which was larger. She incorrectly determined that the 3/5 had the larger pieces.

Emily: This one’s (points to orange 3/5) out of five, and this one is out of four (points to green 3/4). And um. I don’t know.

Tutor: So, in thinking about the size of the pieces, am I going to have bigger sized pieces if I was to just look at one of the pieces here and one of the pieces there, am I going to have bigger sized pieces in the green one or the orange one?

Emily: The orange one?

Tutor: Ok, so these (points to orange 3/5) are going to be bigger sized?

Emily: Uh huh.
I attempted to address her incorrect assumption that the fifths would be larger than the fourths by talking through a fair sharing context. When I talked about sharing with four people, versus sharing with five people, Emily corrected her answer.

Tutor: Cause what I did here (points to orange 3/5), was I took a whole cake, and I divided it up into five pieces, and here (points to green 3/4) I took a whole cake and divided it up into four pieces.
Emily: (nods)
Tutor: Am I going to get more cake…?
Emily: You are going to get more cake in the four.

The fair sharing context helped Emily determine that, “you are going to get more cake in the four.” However, Emily’s momentary focus on size in the fair sharing context was fleeting and not applied four minutes later to the prototypical example. As previously presented, Emily again focused on the area model with more pieces.

Emily: There’s four (pointing to the orange 3/4) and then there is 1, 2, 3, 4, 5 (pointing to the green 3/5). So that one (pointing to green 3/5) has more pieces.
Tutor: So this one is divided into four and this one is divided into five.
Emily: Uh huh. So it’s this one. (points to green 3/5)

Emily again determined that 3/5 was the larger fraction. To address this error, I asked her to focus on the size of the piece and reminded her of fair sharing to help further establish the relationship of the denominator to the size of the piece. Although she changed her answer and said that 3/4 was larger, again, this did not appear to result in any lasting change.

Less than 10 minutes later, faced with the same problem, Emily again said that 3/5 was larger than 3/4.

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65 Fair sharing is a pedagogical strategy for helping students reason about fractional values by building upon their informal understanding of partitioning and sharing (usually food) among people (Ball, 1993; Mack, 1990).
Emily: So, that’s three-fourths (points to orange 3/4).
Tutor: (writes 3/4)
Emily: And then that’s three-fifths (points to green 3/5).
Tutor: (writes 3/5)
Emily: So um. Green.

Again Emily associated the larger denominator with the bigger fraction. After using the game’s answer key and determining that her answer was incorrect, I attempted to use the foam fraction pieces to highlight the size of the pieces. I identified the area models as composed of fourths and fifths, and then I pulled out a 1/4 and a 1/5 piece and placed them next to the area models.

<table>
<thead>
<tr>
<th>Line#</th>
<th>Transcript</th>
<th>Scanned Artifacts / Video Stills with description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tutor: So here we have fourth sized pieces (pointing to green area model of 3/4) here we have fifth sized pieces (pointing to green area model of 3/5).</td>
<td>Scanned playing card</td>
</tr>
<tr>
<td>2</td>
<td>Tutor: So, here we have fifth sized pieces (pulls out yellow 1/5 piece)</td>
<td>Screen shot of yellow 1/5 piece with respect to problem card.</td>
</tr>
<tr>
<td>3</td>
<td>Tutor: and here we have fourth sized pieces (puts 1/4 fraction piece). Does it help to think about them with respect to these? Emily: Yeah. Yeah. I get it, but it’s just kinda confusing.</td>
<td>Screen shot of green 1/4 piece with respect to problem card.</td>
</tr>
</tbody>
</table>
Although I continued to make reference to the foam fraction pieces during subsequent comparison problems, Emily did not use the fractions pieces or a fair sharing context in any of her explanations. When Emily was asked to compare 3/4 and 3/5 the fourth time, she did correctly determine that 3/4 was larger, but when asked to explain her reasoning only said “because”, and provided no further explanation.

By the end of the game, Emily was able to answer the 3/4 and 3/5 area model comparison problem correctly. However, her explanation gave no indication that she had refined her understanding to consider the size of the pieces. The next example suggests that rather than appropriately refining her “more pieces” understanding to reason about the size of the pieces, she had simply associated “more pieces” with the smaller fractional value. Less than a minute after correctly answering that 3/5 was smaller than 3/4, she incorrectly determined that 2/5 was larger than 3/5.

Emily: Number 5. Um. So 1, 2, 3, 4, 5 (pointing to each of the pieces in the orange 3/5 area model). Uh 1, 2, 3, 4, 5 (point to each of the pieces in the green 2/5 area model). So. Three-fifths (points to area model of 3/5) and two-fifths (points to area model of 2/5). Um. (pause 4 seconds). I am so confused.

Tutor: I’m sorry.
Emily: Like two-fifths? I think. It’s green. (circles 2/5)

Emily determined that 2/5 was larger than 3/5. Because the denominators were the same in this case, it is reasonable to assume that Emily was comparing the numerator values 3 and 2. She determined that 2/5 was the larger fraction, which is consistent with understanding that the fraction with “more pieces” is smaller. There is further evidence that “more pieces” was associated with the smaller fractional value. During the posttest when solving an analogous problem comparing area models of 3/5 and 2/5 (see Figure 97), Emily explained that, “three is bigger” and chose 2/5 as the larger fraction. Emily appeared to associate “more pieces” with the smaller fractional value.

Figure 97. Scanned artifact of the 3/5 and 2/5 area model comparison problem from the posttest.
The comparisons of 3/5 and 2/5 suggest two things about Emily’s refinement of her “more pieces” understanding. First, it suggests that she modified her understanding of “more pieces” to be associated with the smaller rather than larger fraction. Second, that it was possible to apply her “more pieces” understanding to the numerator value. Emily’s refinement of her more pieces understanding did not involve a consideration of the size of the pieces, and it appeared to result in a more problematic understanding of fractional magnitude.

Emily’s “more pieces” understanding was robust, in that she experienced repeated difficulty with the same comparison problem. Explicit instruction using both fair sharing concepts and fraction pieces manipulatives were ultimately unsuccessful in changing Emily’s orientation to area models. Refinements to her “more pieces” understanding, rather than incorporating an understanding of the size of the piece, led to more difficulties when she applied it to the numerator. Emily’s “more pieces” understanding suggests that she was not connecting symbols to the referents. This will be explored in more detail in the final strand of analysis.

Summary and conclusion. Emily’s “more pieces” understanding persisted across the tutoring sessions. Her tendency to focus on the number of pieces rather than the size of the pieces was detrimental to her understanding of fraction comparisons and fraction equivalence. Her attempts to refine her “more pieces” understanding involved only a relatively superficial correction to the inference she made about the fraction with more pieces and did not sufficiently incorporate a consideration of the size of the pieces. Emily’s “more pieces” understanding was robust and problematic and fundamentally limited her ability to reason about fractional magnitude.

Emily’s Quarters Understanding

This section presents the fifth of Emily’s six atypical understandings: “quarters.” For Emily, fourths seemed to have a special status as a fraction, and were referred to as “quarters.” Just as students often treat one-half as a special fraction, Emily seemed attuned to perceptual elements of representations that reminded her of quarters. Although this was often a productive and seemingly intuitive way for Emily to make sense of fourths, this understanding was occasionally cued in unproductive contexts. The terms and strategies that she developed around “quarters” were particular to fourths, and consequently might have contributed to her difficulties extrapolating more general fraction principles. Although this was not the most common atypical understanding, it provides another dimension to Emily’s understanding of fractional values.

In this section, first I introduce Emily’s “quarters” understanding and provide a prototypical example in conjunction with the operational definition. Second, I discuss, how this understanding proved to be problematic for Emily’s understanding of more complex fraction concepts. Third, I consider the persistence of Emily’s quarters understanding and over time and across representational form. This strand of analysis illustrates that Emily’s “quarters” understand was atypical and was potentially detrimental to her ability to abstract fractional concepts using the quantity 1/4.

Defining and exemplifying Emily’s “quarters” understanding. Emily’s “quarters” understanding involved the ways in which she understood and referred to fourths. To highlight the atypicality of Emily’s understanding it will be contrasted with a
typical understanding of the fraction 1/4. Typically, an individual treats the fraction 1/4 in the same way that he/she would treat any other fraction. When interpreting an area model, the shaded and total number of pieces map onto the terminology and numeric form used to name the fraction (e.g., 1 shaded piece out of 4 total pieces, equals “one-fourth” or “1/4”) (see Figure 98). In contrast, Emily often understood fourths in terms of “quarters.” Although she was able to correctly interpret area models using the “quarters” terminology, and was often able to successfully use the terminology to produce the fractional form of “1/4”, “quarters” were also associated with the numeral 25. Emily treated quarters as a special kind of fraction, similar to how many students treat 1/2.

Typical understanding of fourths
Fourths are interpreted just like any other fraction, the linguistic term matches the representation and the fractional form.

Atypical understanding of fourths
Fourths are interpreted as quarters and associated with the number 25.

![Diagram of typical and atypical understanding of fourths](image)

Figure 98. Contrast of typical and atypical understanding of fourths.

**Prototypical example of quarters.** Emily’s “quarters” understanding was evident in the following example, in which Emily referred to and represented her understanding of the fraction 3/4 using “quarters” and the numeral 25. During the pretest Emily was asked to draw or write the fraction 3/4. She first wrote down 75% and then drew a picture to illustrate 3/4, which was a circle partitioned into 4 pieces with “25” written in three of the parts. In her explanation she used “quarters” terminology and only when pushed did she refer to the fraction as “three-fourths.”
Emily: *(draws circle, divides into 4 and writes “25” in three of the parts)*

Tutor: Can you explain to me what that is a drawing of?

Emily: Well, it’s in quarters. So. It’s three-out-of-four?

Tutor: Is that how we say this fraction? Three-out-of-four?

Emily: Three-quarters.

Tutor: Three-quarters…

Emily: Three-out-of-four.

Tutor: Three-out-of-four, three-quarters, is there any other way of saying that?

Emily: Three-fourths.

Emily’s quarters understanding was central in this example. In describing her picture, she explained that the shape was “in quarters” rather than fourths or 4 pieces. This was similar to how someone might describe a shape as cut “in half”. Emily only produced the standard “three-fourths” terminology after repeated questioning. In addition, Emily’s drawing of 3/4, involved representing each of the three fourths in terms of the numeral 25. Although this might have been an attempt to link the fraction 3/4 to her percentage representation of 75%, it suggests that she understood fourths in this circumstance primarily in terms of quarters.

This prototypical example was considered consistent with a “quarters” understanding because Emily referred to the partitioning of her shape as “in quarters” and initially referred to the fraction 3/4 as “three-quarters” rather than the more canonical “three-fourths.” Additionally, she represented each of the fourths with the numeral “25”, further suggesting a quarters understanding of 3/4. There is nothing about this answer that was incorrect, it merely establishes Emily’s tendency to understand and represent fourths as quarters.

**Researcher’s operational definition of quarters.** Problems were coded as indicative of a “quarters” understanding if (1) Emily referred to “fourths” in terms of quarters, (2) Emily used the numeral 25 to represent fourths, or (3) Emily attended to perceptual and figural cues (like perceived right angles) to judge the fractional value as equivalent to 1/4.
There were 13 instances of Emily’s “quarters” understanding. As seen in Figure 99, this understanding was evident in almost all sessions, and this understanding was often used in conjunction with an incorrect answer.

Figure 99. Data display of all instances of Emily’s “quarters” understanding.

Quarters as detrimental to learning. Emily’s quarters understanding, although sometimes used productively, was occasionally evoked in inappropriate contexts, often by perceptual similarity to the fraction 1/4. The following example highlights Emily’s tendency to “see” quarters in representations, particularly those with horizontal and vertical halving partitions. In this problem, Emily was asked to identify representations that were equivalent to “four-fifths.” Emily drew upon her quarters understanding in this context, presumably because of the numerator value (4), and the similarity of an area model to one partitioned in quarters.
Tutor: What if we had four-fifths?
Emily: Four-fifths… um… I think this one is four-fifths (circling unequally partitioned area model)
Tutor: Ok.
Emily: Um. (counting discrete set model of 8/10) Um. Well this one (pointing to unequally partitioned area model) I can kinda relate to kind of. Um.
Cause – there four squares and then like quarters.

Emily’s choice, in conjunction with her explanation highlights how Emily’s quarters understanding could potentially be detrimental. She explained the unequal area model as 4 squares (see Figure 100), and pointed to the similarity of the representation to “quarters”. Despite the fact that Emily initially called the fraction “four-fifths” she did not identify the area model of 4/5 as a valid answer. It is likely that the 4 in the numerator of 4/5, and the perceptual similarity of the unequally partitioned area model to a shape partitioned in quarters cued Emily’s quarters understanding. Emily’s attention to a representation that she saw as quartered disrupted her ability to conceptualize or identify even relatively straightforward representations of 4/5. This example highlights how Emily’s quarters understanding was cued, sometimes in inappropriate contexts.

Figure 100. Representation of how Emily was likely parsing the unequally partitioned area model into 4 squares, in which she “saw” quarters.

“Quarters” across the sessions. This section considers Emily’s quarters understanding over the course of the sessions. First I consider evidence of this understanding during the pretest and posttest, which suggests that Emily was relying upon this understanding both before and after the tutoring sessions. Second, I consider
the ways in which this “quarters” understanding occurred across several different representations. Third, I consider Emily’s linguistic usage across the tutoring sessions, which suggested that she shifted from referring to fourths as “quarters”, to referring to them by the more conventional “fourths” terminology. Emily’s quarters understanding appeared to be perceptually cued and although often productive, may have contributed to her developing local strategies for fourths that could not be extended to other fractional values.

**Persistence over time: “quarters” during the pretest and posttest.** Emily drew upon her quarters understanding in similar ways both during the pretest and posttest. Two problems in particular evoked Emily’s quarters understanding. In the first problem, as seen in the prototypical example, Emily used the numeral 25 as part of her representation for 3/4. Although she represented 3/4 correctly during the pretest, with 75% and then labeled each of the quarters, at the time of the posttest she simply determined that 3/4 was equivalent to 25% 66. In both these instances, quarters, particularly in the context of the numeral “25” seemed central to Emily’s understanding of representing fourths. In the second problem, Emily was asked to name the fractional portion of a piece of paper from the paper folding and cutting activity previously described (see Figure 101). In both cases, Emily named the 1/4 of the piece of a paper “a quarter.” During the pretest, she said that the “quarter” could also be referred to as the fraction “one-third”, whereas at the time of the posttest, she renamed the quarter “one-fourth.”

![Figure 101: Illustration of the paper folding and cutting activity, in which student is asked to name the fractional amount of paper she had.](image)

At the time of the pretest and posttest, Emily drew upon her quarters understanding. During both the pretest and posttest her quarters understanding was productive to answer one problem, but it was inappropriately applied in the other problem. These instances reflect persistence in this type of understanding, and also the potential ways that it may be misapplied.

**“Quarters” understanding occurring across representations.** As seen in the previously presented examples, Emily’s quarters understanding was used in conjunction

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66 Emily eventually corrected this error and changed her answer to 75%, but then immediately rejected that option as well.
with drawn area models and paper manipulatives. This understanding was also used in conjunction with the circular foam fraction pieces. In the context of the circular fraction pieces, the perceptual entailments of this quarters understanding appeared most clearly. Emily appeared to “see” some fraction pieces as quarters because of a perceived similarity to a quarter sized piece. For example, when asked to identify fraction pieces that were face down, Emily identified the 1/6 piece as the 1/4 (see Figure 102). Although she corrected her answer, it appeared that Emily initially identified something about the shape of the piece that cued her understanding of quarters. It is likely that she perceived the 1/6 to be quarter shaped, and did not recognize that the shape did not have a full right angle.

![Figure 102](image)

Figure 102. Scanned artifact of Emily’s written work to identify fraction pieces in which she initially identified the 1/6 as 1/4.

Similarly, in the next example, Emily appeared to allow the 1/5 piece’s perceived perceptual similarity to quarters to disrupt her understanding of the relationship of the denominator to the whole. Prior to this excerpt, Emily had determined that it would take eight 1/8 pieces to make a whole, six 1/6 pieces, four 1/4 pieces, and ten 1/10 pieces. However, when I ask this question in the context of the 1/5 piece, she determined that it would take four 1/5 pieces to make a whole.

Tutor: Excellent. Ok. How many one-fifths \( (\text{pulls out 1/5 piece and points to piece}) \) would it take to fill up a whole?

Emily: Um, oh, um, four.

To follow up on Emily’s assertion that it would take four 1/5 pieces to make a whole, I asked her to test out her answer. As she started to test her answer, she referred back to the eight 1/8 pieces she had already assembled on top of the 1 whole piece. She seemed
to reference the eighths in pairs, pointing 4 times, once to each pair of 1/8 pieces. She then returned to assembling the 1/5 pieces on top of the 1/8 pieces. It was only after she assembled three 1/5 pieces on top of the eighths that she reconsidered her answer.

<table>
<thead>
<tr>
<th>L#</th>
<th>Transcript</th>
<th>Scanned Artifacts / Video Stills with description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tutor: Ok. You want to test it? Emily: Yeah <em>(picks up 1/5 piece)</em></td>
<td><img src="image1.png" alt="Image" /></td>
</tr>
<tr>
<td>2</td>
<td>Emily: Wait. <em>(points 4 times to each of the pair of 1/8 pieces)</em></td>
<td><img src="image2.png" alt="Image" /></td>
</tr>
<tr>
<td>3</td>
<td>Emily: <em>(puts three 1/5 pieces on top of 1/8 pieces)</em> No. Not four. <em>(puts piece on)</em> Five <em>(puts fifth piece on)</em></td>
<td><img src="image3.png" alt="Image" /></td>
</tr>
<tr>
<td>4</td>
<td>Tutor: Ok. Emily: Ok, yeah, duh. I know. I knew that, anyway. Tutor: Ok. So – it would take… Emily: It would take five to be yeah.</td>
<td><img src="image4.png" alt="Image" /></td>
</tr>
</tbody>
</table>

Emily’s quarters understanding may have been evoked by the size and shape of the 1/5 piece, or may have been cued by the fact that eighths had already been assembled on top of the one-whole circular fraction piece. Regardless of the original trigger, Emily was unable to extrapolate the meaning of the denominator in the case of the 1/5 pieces. Rather than following the pattern that she had previously established with respect to the meaning of the denominator, she incorrectly determined that it would take four 1/5 pieces to make a whole. This example highlights how Emily’s quarters understanding, while in some respects quite productive, had the tendency to produce quarter-specific strategies. Although she did not refer to “quarters” in this excerpt, this was classified as an instance of her quarters understanding, because she appeared to understand the whole in terms of quarters, both when she pointed 4 times to the 1/8 fraction pieces and when she determined that it would take four 1/5 to make a whole.
Linguistic use of “quarter.” A separate analytic pass was conducted to determine the prevalence of the linguistic form: “quarter.” Frequency counts of Emily’s reference to the word “quarter” and “fourth” was conducted on all transcripts. (see Table 3). During the pretest Emily’s use of “quarter” outnumbered her use of the term “fourth.” Over the course of the tutoring sessions, as I exclusively used the term “fourth”, Emily shifted I her terminology usage. Although she began using “fourths” more consistently, it did not eliminate her occasional reference to “quarters”. By the posttest, Emily continued to use the term “quarter”, this along with instances of her quarters understanding suggests that this was a productive way for her to understand fractional quantities.

Table 3. Frequency counts of Emily’s usage of the terms “quarter” and “fourth” across all sessions.

<table>
<thead>
<tr>
<th></th>
<th>Frequency counts of instances of “quarter”</th>
<th>“fourth”</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>Tutoring Session #1</td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>Tutoring Session #2</td>
<td>1</td>
<td>18</td>
</tr>
<tr>
<td>Tutoring Session #3</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>Tutoring Session #4</td>
<td>0</td>
<td>11</td>
</tr>
<tr>
<td>Posttest</td>
<td>1</td>
<td>9</td>
</tr>
</tbody>
</table>

Summary and conclusion. Emily’s “quarters” understanding involved understanding fourths as a special kind of fraction. Emily appeared to attend to perceptual aspects of various representations that sometimes productively and sometimes unproductively caused her to focus on quarters. This understanding although not that problematic during the tutoring sessions, had the possibility of undermining her ability to generate general properties of fractions. Just as 1/2 was a central fraction in these tutoring sessions, 1/4, 2/4, and 3/4 were also central and frequently used fractions. Emily’s “quarters” understanding might not have provided a sufficient base on which to abstract fraction concepts.

Emily’s Rule-based Navigation

This section presents the sixth strand of analysis: “rule-based navigation.” It appeared that Emily navigated the mathematical terrain largely based upon abstracted and memorized procedures, which were largely disconnected from the underlying mathematical concepts. As evident with Emily’s “more pieces” understanding, Emily’s understanding of fraction symbols and representations was not grounded in the underlying referents. Therefore, her primary way of navigating the mathematical terrain was devoid of the often self-corrective and constraining functions a connection to the underlying referents would afford. Although Emily’s “rule-based navigation” had some similarities to Lisa’s “arbitrary ungrounded manipulation,” this kind of understanding was more pervasive for Emily, and appeared to influence how she understood mathematical representations. Again, a helpful analogy is thinking of Emily navigating
using a sequence of written directions rather than a map. She could easily get lost and had few resources to identify where she strayed off course. Characteristic of this kind of understanding was that repeated exposure to a particular kind of problem often resulted in a set of abstracted rules, which relied upon superficial aspects of the problem, rather than the underlying meaning. Emily’s rule-based orientation to the mathematics left her susceptible to misremembering and misapplying rules, and consequently, provides the final aspect explaining why Emily did not benefit from the tutoring instruction.

In this strand of analysis, I explore Emily’s rule-based navigation in a variety of contexts in which it arose. First, I begin with a simple arithmetic rule that Emily used relatively consistently for generating equivalent fractions. This rule, although often productive, was sometimes misapplied. Second, I give an example of a rule as it was being created in the context of a novel kind of problem. Emily’s rule appeared to incorporate irrelevant aspects involved in solving the problem, suggesting that she was abstracting procedural steps while not attending to the underlying concepts. Third, I will discuss how this rule-based orientation heavily influenced her understanding and use of mathematical representations. Emily treated mathematical representations as answers in and of themselves rather than holders of meaning. This strand of analysis is different than the rest, because it reflects a general orientation to mathematics rather than a specific kind of understanding. However, an appreciation for this rule-based orientation is absolutely critical to contextualize the kinds of difficulties that Emily experienced across the tutoring sessions.

**Defining and exemplifying “rule-based navigation.”** Emily’s “rule-based navigation” involved solving problems by following a sequence of abstracted procedures without connection to underlying concepts. This orientation to mathematics was most apparent when inconsequential aspects of a procedure were treated as central. For example, Emily may have abstracted a procedure for creating equivalent fractions that involved multiplying both the numerator and denominator by the denominator value (see Figure 103). She might not recognize that it is possible to multiply the numerator and denominator by *any* value (see Figure 103). In this way, Emily took a nonessential part of the algorithm and made it central.
**Typical abstraction of rules**
You can create equivalent fractions by multiplying both the numerator and denominator by **any number**.

\[
\frac{2}{5} \times \frac{2}{2} = \frac{4}{10} \\
\frac{2}{5} \times \frac{3}{3} = \frac{6}{15} \\
\vdots
\]

\[
\frac{3}{4} \times \frac{2}{2} = \frac{6}{8} \\
\frac{3}{4} \times \frac{3}{3} = \frac{9}{12} \\
\vdots
\]

**Atypical abstraction of rules**
You can create equivalent fractions by multiplying both the numerator and denominator by the **denominator value**.

\[
\frac{2}{5} \times \frac{5}{5} = \frac{10}{25} \\
\frac{3}{4} \times \frac{4}{4} = \frac{12}{16}
\]

Figure 103. Contrast of typical and atypical abstraction of rules.

**Prototypical example of “rule-based navigation.”** Emily’s rule-based navigation often was most evident across multiple problems, rather than within one problem. The prototypical example presented here was a rule that Emily appeared to apply across various problems: equivalent fractions can be made by multiplying both the numerator and denominator by the denominator value. This rule, although a valid way of producing some equivalent fractions, led to an overly constrained understanding of equivalent fractions. During the pretest, Emily produced six equivalent fractions in which she multiplied by the denominator of the original fraction (see Table 4).
Table 4. Emily’s generation of equivalent fractions during the pretest that were consistent with the rule-based approach to multiplying both the numerator and denominator by the denominator value.

<table>
<thead>
<tr>
<th>Original Fraction</th>
<th>Equivalent fraction(s)</th>
<th>Scanned artifact</th>
</tr>
</thead>
<tbody>
<tr>
<td>5/4</td>
<td>20/16</td>
<td><img src="https://example.com/image1.png" alt="Image" /></td>
</tr>
<tr>
<td>3/4</td>
<td>12/16</td>
<td><img src="https://example.com/image2.png" alt="Image" /></td>
</tr>
<tr>
<td>1/3</td>
<td>3/9; 9/27</td>
<td><img src="https://example.com/image3.png" alt="Image" /></td>
</tr>
<tr>
<td>2/5</td>
<td>10/25; 50/125</td>
<td><img src="https://example.com/image4.png" alt="Image" /></td>
</tr>
</tbody>
</table>

Although Emily never said that it was necessary to multiply by the denominator value of the original fraction, her answers in Table 4 were consistent with this rule. To explain the equivalence of 5/4 and 20/16, she said she multiplied 5/4 “times four over four.” Not only was this consistent with the “rule” but when I asked her if there was another way she could write or draw 5/4, she answered that she did not know. Therefore, her understanding of this procedure appeared to be overly constrained to allow multiplication by the denominator value alone. Although Emily was able to execute her procedure to produce one or sometimes two equivalent fractions, this did not reflect an understanding of how to create equivalent fractions in general.

There was further evidence that Emily’s understanding of the rule was primarily procedural, because she misapplied this rule during the posttest. It has been well documented that when students memorize procedures without connection to the underlying concepts, they are often misremembered or misapplied (Hiebert, 1984; Hiebert & Lefevre, 1986; Hiebert & Wearne, 1986). Emily misapplied this rule in her attempt to create equivalent fractions for 1/3. As in the previous example, she used the denominator value, but in this case she relied primarily on an additive rather than multiplicative relationship. She determined that 1/3 was equal to 3/6, presumably by
Emily applied her rule of using the denominator value to create equivalent fractions, and incorrectly determined that 1/3 was equal to both 3/6 (i.e., 1/2) and 6/9 (i.e., 2/3). Although she continued to use the denominator value to produce the equivalent fractions she misremembered her rule and added rather than multiplied the denominator value. This episode highlights both that at the time of the posttest she still had an overly constrained understanding of creating equivalent fractions and her rule-based approach to equivalent fractions was susceptible to being misapplied. Emily’s assertion that 1/3 was equal to 3/6 was particularly notable because on the previous problem written on the same page, Emily had determined that 1/2 was equivalent to 3/6 (see Figure 104).

Figure 104. Scanned artifact of Emily’s written equivalent fractions, in which she determined 3/6 was an equivalent fraction for 1/2 and for 1/3.

These examples highlight Emily’s “rule-based navigation” because they illustrated how she consistently applied a rule that relied on nonessential aspects of the procedure and that this rule was susceptible to being misapplied. Emily’s rule-based

Note that Emily’s approach for generating equivalent fractions equal to 1/2 did not follow her rule-based approach for multiplying by the denominator. As previously discussed, Emily treated 1/2 as a special kind of fraction and had generated one-half-specific strategies. Emily appeared to rely upon the relationship of the numerator and denominator to produce fractions equivalent to 1/2, and therefore, did not apply her rule-based understanding of equivalent fractions to this context.
navigation although often productive for her in allowing her to produce equivalent fractions, appeared to be abstracted away from the underlying meaning.

**Researcher’s operational definition of rule-based navigation.** Problems were coded as indicative of "rule-based navigation" if (1) Emily’s answers and explanations were consistent with a non-standard “rule” (e.g., multiplying the numerator and denominator by the denominator value to create equivalent fractions), (2) Emily abstracted a set of rules for solving a particular kind of problem that suggests a focus on the procedures devoid of conceptual meaning, (3) Emily attempted to apply a rule in an inappropriate context, (4) Emily treated a representation as the answer to a question rather than a holder of meaning, or (5) Emily used associations of numbers to produce answers devoid of the meaning of the symbols and representations (e.g., 1/4 and 1/6 make 1/10).

There were 39 instances coded as rule-based navigation, and almost all instances were associated with an incorrect answer (see Figure 105). Instances associated with a correct answer, were almost always for problems on which Emily applied a non-standard rule to correctly answer a question (as seen in Table 4).

![Figure 105. Data display of all instances of Emily’s “rule-based navigation”](image)

**Creation of a rule.** Emily’s rule-based navigation was often difficult to identify and assess. It was only in cases where she abstracted irrelevant features of the procedure that it became apparent how she was procedurally orienting to the mathematical task. In the following example, Emily appeared to extrapolate a rule for solving a problem in which she was required to come up with a fraction that was close to, but not equal to, one-half for a given denominator. She solved several problems correctly, determining that 4/10, 6/10, 10/22, 12/22, 51/100, and 4/6 were all close to, but not equal to, one-half. Her rule can be loosely stated as: determine what fraction would equal 1/2, and then add or subtract 1 to the numerator value. The fact that she was artificially constrained to the procedures rather than the general principles was evident when she tried to find a fraction close to but not equal to 1/2 with the denominator of 7. I asked her what value would make the fraction exactly equal to 1/2, and she correctly determined that 3.5 would be exactly 1/2. Rather than answering that 3/7 or 4/7 was close to one-half, she determined that the numerator value should be 2.5.

Tutor: So what would be close to three point five that you could use for the numerator?
Emily: Uh. (pause) um.
Tutor: Sorry I don’t know if my question is clear, if we want to come up with something that is close to, but not equal to a half. You just found what was equal to a half for seven, right? It’s three point five over seven, is going to be exactly equal to a half. What if we just want to be close?
Emily: Uh. Two point five.

Emily seemed to have created an algorithm for creating a fraction close to one-half that involved determining the value for 1/2 and then adding or subtracting 1. This is an example of how Emily appeared to attend to irrelevant aspects of the solution process and those features became central to her created rule or procedure.

**Representations as answers.** Emily’s procedurally-based orientation to mathematics extended to her understanding and use of representations. She appeared to treat representations in terms of procedural steps rather than as mathematical objects with meaning. Her treatment of representations suggests that she operated with rules for creating representations, which produced answers. To highlight the atypicality of her orientation to representations, consider a situation in which a student is asked to draw a picture of 5/8. For most students if you immediately ask them what their drawing shows, they will likely answer “five-eighths” without needing to count the pieces. They don’t need to count the pieces, because their own drawing holds meaning for them, and their constructive act of drawing 5/8, results in something that understood to be 5/8 (see Figure 106). However, this was not the case for Emily. Recall the example presented in the “smaller part” strand of analysis in which, immediately after drawing 5/8, Emily began interpreting this as 3/8 (see Figure 107). Emily treated the construction of 5/8 as an answer, which was not connected to the subsequent interpretation of the representation (see Figure 106). Her constructive and interpretive acts, rather than being natural inverses were separate activities entirely.

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68 Although this example was also consistent with Emily’s “smaller part” understanding the focus here is on how her constructive and interpretive activities were disconnected.
Figure 106. Contrast of typical and atypical understanding and use of representations.

Figure 107. Scanned and digitally recreated artifact which highlights Emily’s drawing of 5/8 and reinterpretation of 5/8 as 3/8.

There is further evidence of Emily’s difficulty coordinating both construction and interpretation of representations. During the second tutoring session, Emily was asked to produce a sequence of rules that I would follow to interpret her drawn representations. Emily would choose a card with a fraction value, write down that fraction value, draw a picture of the fraction, and then fold the paper over so I couldn’t see the original fraction value. I would use the rules she wrote on the whiteboard to interpret her fractional drawing and “guess” her fraction (see Figure 108). This activity was intended to involve several iterative steps, in which I would intentionally misinterpret her rules to help her further clarify the conventions used for interpreting area models. Although she correctly constructed all representations of the given fraction values, she had an incredibly difficult time in generating rules to interpret her drawings. Her modifications to the interpretive rules consistently involved changing aspects of the construction of the representation rather than the interpretation. For example, when trying to modify her rule “write down the number of pieces,” she focused on constructive rather than interpretive procedures. In
four of her modifications of the interpretive rules she wanted to add a direction for me to shade pieces, despite the fact that she had already drawn the picture and I was simply writing down the fraction value. Similarly, when I intentionally inverted the fraction and guessed 8/3 rather than 3/8, she wanted to correct the interpretive rules by drawing an arrow to the shaded region of the picture of 3/8, rather than writing “put the shaded number on top.”

Despite the fact that the instruction in this segment specifically attempted to differentiate the act of construction from the act of interpretation, Emily continued to focus on her constructive acts when attempting to write interpretive rules. Rather than understanding the interpretation of a representation as the natural analog to the construction of a representation, Emily appeared to have difficulty relating these two activities. For Emily, the drawing was not the natural pivot point in shifting from construction to interpretive acts.

Figure 108. Scanned artifacts of Emily’s fraction drawings and the final set of interpretive rules she wrote on the whiteboard.

**Conclusion.** Emily appeared to have a general orientation to mathematics that involved a rule-based navigation of the content. This resulted in her abstracting procedural rules away from the underlying mathematical meaning. At times these procedures contained nonessential aspects, and at times they were misapplied. This rule-based orientation to mathematics appeared to incorporate her understanding of mathematical representations, and gave representations the status of an outcome or an answer to a question rather than a representation of a fractional quantity. This may partially explain why mathematical representations like area models and fraction pieces were ineffective in helping Emily develop a more refined understanding of fractional concepts.

**Discussion of Emily’s Atypical Understandings**

Emily demonstrated six atypical understandings that explain almost all incorrect answers given during the data collection. These six atypical understandings therefore, can be understood to be responsible for the difficulties Emily had during the tutoring sessions and provide an explanation for why Emily did not benefit from the instruction.
There are both local and global ramifications for these 6 atypical understandings. When considered individually, these atypical understandings each appeared to be persistent, robust, and problematic for Emily’s development of more complex fraction conceptions. When taken together, these 6 atypical understandings suggest that Emily’s conceptualization fraction quantity was unstable. In this section I consider the implications of the suite of atypical understandings. First I consider how “halving”, “part-part”, and “quarters” all involve fraction-specific strategies, which may not provide the necessary footholds for other fraction values. Second, I explore how “smaller part” and “more pieces” allowed for a shifting understanding of fractional quantity. Lastly I consider how Emily’s procedural orientation to representations may have exacerbated her issues with understanding fractional quantity. Together, all six of these atypical understandings suggest that Emily had an insufficient understanding of fractional quantity, or fraction sense.

**Fraction-specific strategies.** Emily’s “halving”, “part-part”, and “quarters” understandings were all associated with specific fraction values. When these understandings were used productively, both halving and part-part were primarily associated with the fraction value 1/2 and quarters was primarily associated with the fraction 1/4. This suggests that Emily had developed fraction-specific strategies for dealing with some common fraction values (see Figure 109). In this section I consider the ramifications of fraction-specific strategies in general and then with respect to Emily’s atypical understandings.

![Figure 109. Illustration of Emily’s fraction-specific atypical understandings.](image)

**Typicality of fraction-specific strategies.** Fraction-specific strategies may not be inherently problematic. Students may often develop fraction-specific strategies for commonly used fractions. For example, children generally understand the fraction 1/2 at a very early age based on their informal experience sharing, but their understanding may not reflect their fluency with other fractional values (Hunting & Davis, 1991). These fraction specific strategies are not unique to young children. Even adults retain and rely upon fraction-specific understandings and strategies. Smith, diSessa, & Roschelle (1993) illustrated how both novices and experts rely upon a range of strategies for solving fraction problems, which are often dependent upon the fraction values provided. The use of fraction-specific strategies should be considered quite common. Although all students may have some fraction-specific strategies, for most students these strategies are used as a matter of efficiency. These fraction-specific strategies are often not central to their
understanding of fractions and do not cause issues in learning more complex fraction concepts.

**Emily’s fraction-specific strategies.** Although fraction-specific strategies may be common in most students, the ramifications for Emily were problematic. First, Emily’s understanding of general properties of fractions appeared *not* to apply to these special fraction values. Second, Emily’s fraction specific strategies did not provide a sufficient foundation upon which to build more complex fraction concepts. Finally, Emily’s attempts to extend these fraction-specific strategies were problematic. In this section, I discuss each of these three issues in turn.

**Applicability of general principles to fraction-specific understandings.** Emily’s understanding of general fraction concepts appeared *not* to apply to her understanding of $\frac{1}{2}$. Emily appeared to treat the fraction $\frac{1}{2}$, as a special case, not subject to the general rules for fractions. For example, recall that when Emily attempted to apply her written rules for interpreting fractions to her drawing of $\frac{1}{2}$, she was unable to reconcile her drawing and the rules she had recorded in her journal (see Figure 110). Emily’s drawing of $\frac{1}{2}$ without shading did not conform to the general properties she relied upon for constructing and interpreting other fractional amounts. Therefore, the properties that applied to fractions in general were *not* applied to this special fraction value.

![Figure 110](image)

Figure 110. Illustration of how Emily’s understanding of $\frac{1}{2}$ did not appear to be subject to the general rules she had for interpreting fractions, which were generated by Emily and recorded in her journal.

**Limitations of fraction-specific understandings.** Emily’s reliance upon her fraction-specific strategies suggested she might not be able to draw upon her familiarity with common fractions like $\frac{1}{2}$ and $\frac{1}{4}$ to bootstrap more complex fraction concepts. Emily applied her atypical understandings, often productively, in the context of $\frac{1}{2}$ (or $\frac{1}{4}$), but this had limited applicability. Consider how Emily might apply each of her atypical understandings in the context of fraction equivalence (see Figure 111). While Emily’s “halving” understanding allowed her to create fraction equivalent to $\frac{1}{2}$ by splitting an even denominator in 2 to determine the numerator, it was not possible to
extend this understanding to non-unit fraction values. Emily could not rely upon an easily accessible multiplicative relationship between the numerator and denominator for a non-unit fraction like 2/3. Similarly, although Emily was able to use her part-part understanding to correctly assert that 4/8 was equivalent to 1/2 because there were four pieces shaded and four pieces not shaded, this could not extend to other fraction values. When Emily extended her part-part understanding to the fraction 1/3, she determined that 2/6 was not equivalent to 1/3 because 2 pieces were shaded and 4 were not. Lastly, although this did not occur during the tutoring sessions, Emily might have leveraged her understanding of quarters to help her determine that 2/8 was equivalent to 1/4 because of her perceptual familiarity with quarters. Again, this strategy cannot be applied to other fractional values because the perceptual entailments were specific to the value 1/4.

Although these fraction-specific strategies might be productive for understanding fraction equivalence for 1/2 (or 1/4 in the case of quarters), these strategies could not be extended to all other fractional values (see Figure 111). Emily’s halving, part-part, and quarters understanding were a potentially unstable foundation for the abstraction of equivalent fraction understandings to other values.

Figure 111. Illustration of Emily’s halving, part-part, and quarters understandings in the context of generating equivalent fractions for 1/2 and lack of applicability of these understandings to other fraction values.

Problematic extension of fraction-specific understandings. In the instances when Emily did attempt to build upon her “halving”, “part-part”, and “quarters” understanding to extend it to other fractions, it was fundamentally problematic. Emily’s extension of

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69 Emily was able to partially extend her halving strategy to other unit fraction values by using the multiplicative relationship of the numerator and denominator. In the same way that she might determine that 5/10 was equal to 1/2 because 5 times 2 is 10, Emily determined that 3/9 was equal to 1/3, because 3 times 3 was 9. However, this strategy for creating equivalent fractions would not extend to any non-unit fraction values.
her atypical understandings to other fraction is one main area in which she experience difficulties (see Figure 112). For example, when Emily extended her halving understanding to the fraction 1/4, she thought that she had represented the quantity 1/4 by creating four partitions. Similarly, when extending her part-part understanding to 1/3, she was unable to recognize that 2/6 was equivalent to 1/3, because she attended to the two non-shaded pieces and the four shaded pieces, rather than understanding the representation as 2 out of 6. Lastly, when Emily extended her quarters understanding to the fraction pieces labeled 1/5, she incorrectly determined that it would take four 1/5 pieces to create one whole. In each of these instances Emily’s extension of her atypical understanding to other contexts was problematic.

![Figure 112. Illustration of Emily’s extension of halving, part-part, and quarters to other fractional values.](image)

Although most students do have fraction specific strategies, Emily’s fraction-specific strategies were problematic for her ability to engage with more complex fraction concepts. The instruction was designed to explore novel concepts by starting with more common, and more readily understandable fraction values, and then abstracting to more complex fraction values. This instructional approach might have not been productive for Emily because of her tendency to rely upon fraction-specific strategies for dealing with these common fractions.

**Shifting fractional quantity.** Emily’s “smaller part” and “more pieces” understanding appeared to allow for a shifting evaluation of fractional quantity. Emily’s “smaller part” understanding allowed for a representation of a fractional quantity to change in value. A representation that Emily drew of 3/4 could be immediately interpreted at 1/4 (see Figure 113). The lack of consistency in how Emily interpreted fractional representations led to an understanding of fractional quantity as a changeable rather than stable amount.

![Figure 113. Illustration of instability in Emily’s understanding of fractional quantity.](image)
More pieces also involved a shifting evaluation of fractional quantity, but in a more subtle way. The inferences that were made about the magnitude of the fraction (often with respect to another fraction) were dependent upon how “more pieces” was being applied. Emily applied “more pieces” to (1) the number of total pieces, corresponding to the denominator, (2) the number of shaded pieces, corresponding to the numerator, or (3) the number of pieces in an equivalent fraction, often not indicating whether she was referring to the numerator, denominator, or both. This application of “more pieces” to varying fractional components was particularly problematic for her understanding of a stable fractional quantity because of the direct inferences about the magnitude of the fraction based on which had more pieces. The inferences made about “more pieces” are dependent upon the context in which it is applied. When all other factors are held constant, more total pieces implies a smaller amount, more shaded pieces implies a larger amount, and more pieces in the context of equivalent fractions implies equal amounts (see Figure 114). Emily’s undifferentiated understanding of “more pieces” resulted in seeming instability in the implications of more pieces. Recall that during the tutoring sessions, Emily refined her understanding of “more pieces” to be associated with the smaller fraction and then inappropriately applied that understanding to a context in which “more pieces” was applied to the numerator value. Emily’s more pieces understanding was detrimental to her understanding of a stable fractional quantity, because more pieces was used to refer to representations that should be judged as larger, smaller, and equivalent to another representation with fewer pieces.
Various Applications of "more pieces"

"more pieces" implies:

- smaller amount
- larger amount
- equal amounts

Figure 114. Illustration of more pieces, with related inferences, when applied to the denominator, numerator, or in the context of equivalent fractions.

**Fraction sense.** The five atypical understandings discussed thus far, have suggested that Emily did not have a stable understanding of fractional quantity. Representations intended to support her understanding of fractional quantities were at best ineffective and at worst detrimental to her development of an understanding of fractional quantity. Emily’s difficulties with understanding, representing, and manipulating fractional quantities were exacerbated by the sixth atypical understanding: rule-based navigation. Emily oriented to representations specifically, and mathematics in general, in terms of procedural steps which she enacted to arrive at an answer. Because she treated representations as a kind of problem to be solved, rather than a mathematical object to use to reason, there was not a solid foundation on which she could build. Together, these 6 atypical understandings suggest that Emily did not have a stable conceptualization of fractional quantity. Similar to other studies of mathematical learning disabilities that point to student’s poor number sense as the causal factor in the student’s difficulties ((Berch, 2005; Butterworth & Reigosa, 2007; Gersten, Jordan, Flojo, 2005; Piazza et al., 2010), Emily’s unstable fraction sense seemed to be at the heart of her difficulties.
Conclusion

Emily’s six atypical understandings contributed to her difficulties learning. Emily’s atypical understandings suggest that she did not have a stable understanding of fractional quantity on which she could build. Some of her atypical understandings led to fraction-specific strategies that were exclusively effective for dealing with common fraction values, like 1/2 and 1/4, and detrimental when applied to other fraction values. Some of Emily’s other atypical understandings led to an understanding of fractional quantity and fractional magnitude as a changeable and shifting value. The representations that were central to this instructional sequence were at least partially inaccessible for Emily because of her rule-based orientation to these mathematical objects. These six atypical understandings provide a comprehensive view of the difficulties that Emily experienced during the tutoring sessions and explain why standard instruction was ineffective. Therefore Emily’s atypical understandings provide an apt characterization of her MLD.

In the next chapter I present a cross-case analysis where I consider the similarities and differences between Lisa’s and Emily’s atypical understandings. I contrast Lisa’s and Emily’s atypical understandings with the data collected with the fifth grade control students and draw upon data from one other student with an MLD. The cross case comparisons provide further insight into the origin and prevalence of these kinds of atypical understandings.
Chapter 5: Cross-case Comparison

The case studies of Lisa and Emily attempted to understand why neither student benefited from the tutoring sessions. The case study analyses revealed that a collection of atypical understandings largely accounted for the difficulties that the student’s experienced. Although each student had a unique collection of atypical understandings, there were some surprising similarities in the atypical understandings they demonstrated. In this chapter, in addition to comparing and contrasting Lisa’s and Emily’s atypical understandings I draw out the commonalities of some of the atypical understandings to establish a higher-order category of “indicators of atypicality.” The ultimate purpose of this chapter is to use these indicators of atypicality as an analytic frame to explore the data from both the fifth grade control students and one other student with an MLD.

In this chapter, first I compare the case of Lisa and the case of Emily, specifically pairing several of their atypical understandings and abstracting commonalities of each pairing. The similarity of several atypical understandings was striking. Three primary indicators of atypicality common to both become the analytic lens used for the remainder of the chapter. Second I consider the evidence of these three indicators of atypicality in the data collected with the fifth grade controls. Although traces of one or more of these indicators did occur for some of the students, the atypical understandings were resolved relatively quickly. This suggests that these atypical understandings are likely to be naturally occurring, but are persistent and problematic only for students with MLDs. Third, I consider interview data from a student with an MLD who has managed to compensate so effectively for her learning difficulties that she is completing her B.S. in statistics. Despite her ability to pass upper division math classes, she too demonstrated all three indicators of atypicality. This suggests that unlike the case with the fifth grade students who demonstrated only fleeting traces of the indicators, these atypical understandings continue to persist and remain central for students with MLDs. The analysis presented here suggests that there is some commonality across students with MLDs in their reliance and persistence of the three primary atypical understandings. The implications of these findings for screening measures and remediation approaches will be addressed in the final chapter.

Comparison of Lisa and Emily

Lisa and Emily both displayed six atypical understandings, which largely explained their difficulties during the tutoring sessions. The purpose of this section is to reconcile the two analytically distinct case studies by comparing the atypical understandings displayed by each student. The commonalities identified between the cases are used to abstract a superordinate analytic categorization: indicators of atypicality.

Although both Lisa and Emily each displayed a unique profile of atypical understandings, there were natural pairings to consider for the comparison of cases. (see Figure 115). Lisa’s taking and Emily’s smaller part, both involved a focus on the fractional complement. Lisa’s halving and Emily’s halving and part-part involved understanding the fraction 1/2, not as a quantity, but as a splitting or a balance between parts. Lisa’s unit fraction and Emily’s more pieces understandings both were used
primarily in the context of comparison problems and involved a failure to coordinate the numerator and denominator values. For each of these paired atypical understandings I abstract up to a level of commonality. The commonalities serve as the basis for the three indicators of atypicality: fractional complement, halving, and uncoordinated fractional values. In addition, both Lisa and Emily tended to operate on fraction symbols and representations in a procedural way devoid of meaning. However, this procedural orientation will not be a focus of analysis. I flag this similarity because it may be quite common among students with MLDs, however, it represents more of an orientation to mathematics rather than a specific identifiable understanding. The remaining atypical understandings had no natural analogs. Emily did not demonstrate a discrete set understanding or partitioning difficulties. Similarly, Lisa did not appear to attend to quarters in the same way that they were perceptually salient to Emily.

Pairings of Atypical Understandings

![Diagram]

Figure 115. Illustration of the pairing of Lisa’s and Emily’s atypical understandings, with the associated indicator of atypicality.

In this section each of the pairings will be discussed in turn, followed by a brief exploration of the unpaired atypical understandings. For each atypical understanding pairing I provide a side-by-side comparison of Lisa’s and Emily’s understandings, focusing on similarities and differences. I rely heavily upon examples previously presented in the individual cases and juxtaposition these images for easy comparison.
The quotes presented here are often extracted from the transcript to highlight the relevant part of the student’s explanations. Ultimately the goal is to identify commonalities which appear to cut across both Emily’s and Lisa’s cases. This section concludes by using the commonalities identified to operationally define the indicators of atypicality. The indicators of atypicality are then used as the primary analytic frame for the remainder of the chapter.

**Taking and smaller part.** Central to Lisa’s taking and Emily’s smaller part understandings was a problematic focus on the fractional complement. Lisa’s “taking” understanding involved conceptualizing the shaded region of area models as removed, which resulted in her attending to the fractional complement. Emily’s “smaller part” understanding involved determining the numerator value by the part comprised of the fewest pieces. Despite the difference underlying their explanations, both of these atypical understandings involved attending to the fractional complement rather than the target fractional quantity (see Figure 116). In this section I highlight both the similarities and differences between these atypical understandings and consider how although distinct, these two atypical understandings can be thought of as belonging to a higher-order category: focus on the fractional complement.

![Diagram](image.png)

**Figure 116.** An illustration and comparison of taking and smaller part atypical understandings.

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70 The goal is clarity and brevity to highlight the similarities and differences between Lisa’s and Emily’s understandings rather than presenting how the explanation unfolded. The majority of these quotes are taken from episodes presented in the individual cases, where the quotes were given in context.
**Similarities of taking and smaller part understandings.** Both Lisa’s taking and Emily’s smaller part understandings resulted in a focus on the fractional complement and common kinds of errors. For example, in the context of area models, both Lisa and Emily attended to the non-shaded pieces. In the same way that Lisa interpreted 6/8 as 2/8 by focusing on the non-shaded pieces, Emily interpreted 5/6 as 1/6 (see Figure 117). Both Lisa and Emily understood the non-shaded pieces to be the focal fractional quantity and incorrectly interpreted the area models as the fractional complement.

<table>
<thead>
<tr>
<th>Question: Interpret the drawn fraction representation</th>
</tr>
</thead>
</table>
| **Artifact** | ![Artifact](image)
| Lisa | Emily |
| **Answer** | 2/8 (incorrect) | 1/6 (incorrect) |
| **Similarity** | Focus on non-shaded pieces (fractional complement) | Focus on non-shaded pieces (fractional complement) |
| **Quotes** | Two-eighths. ... There’s eight. ... And then two of which aren’t shaded in. | That’s one-sixth. ... Like all, there’s like, all except one shaded. |

Figure 117. Illustration of the similarity of Lisa’s taking and Emily’s smaller part understanding in the context of area models.

Similarly, in the context of fraction pieces, they both focused on the empty space and attempted to define the fractional amount by determining what piece would fit in the missing space (see Figure 118). When Lisa was presented with seven 1/10 pieces, she cupped her hand around the empty space and determined that “a third” would fit in the space. When Emily was asked to interpret nine 1/10 pieces, she too focused on the empty space and named the fractional amount as the fractional complement: 1/10.

Although 1/3 is not the fractional complement for 7/10, it is clear that Lisa was focused on the empty space in her attempt to name the fraction amount.
**Question:** What is this amount?

<table>
<thead>
<tr>
<th></th>
<th>Lisa</th>
<th>Emily</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Answer</strong></td>
<td><img src="image1.png" alt="Image" /> <em>(gestures with hand around empty space)</em></td>
<td><img src="image2.png" alt="Image" /> <em>(pointing to empty space)</em> is one-tenth.</td>
</tr>
<tr>
<td><strong>Similarity</strong></td>
<td>Focus on empty space (fractional complement)</td>
<td>Focus on empty space (fractional complement)</td>
</tr>
<tr>
<td><strong>Lisa’s response</strong></td>
<td>How much is left? Is a third? Yeah, a third is missing.</td>
<td>7/10 interpreted as 1/3 (incorrect)</td>
</tr>
<tr>
<td><strong>Emily’s response</strong></td>
<td>This is one-tenth.</td>
<td>9/10 interpreted as 1/10 (incorrect)</td>
</tr>
</tbody>
</table>

Figure 118. Illustrations with screenshots and selected quotes of Lisa’s and Emily’s focus on the empty space when working with fraction pieces.

Both Lisa and Emily attended to the empty space or non-shaded pieces when interpreting fractional representations. For both Lisa and Emily, this focus on the fractional complement occurred across multiple problem types and continued to be problematic at the time of the posttest.

**Differences between taking and smaller part understandings.** Despite these marked similarities, “taking” and “smaller part” were treated as analytically distinct atypical understandings; although both may result in interpreting an area model of 3/4 as 1/4, the underlying rationale for the incorrect interpretation was different. For Lisa, the shading clearly represented the pieces that were taken and this understanding occurred in conjunction with her constructive act. As she drew 3/4, she understood that the shading was a removal of pieces. Even after an extensive conversation about interpreting area models, she said, “I keep imagining that this *(pointing to shaded part)* is being taken away” (see Figure 119). In contrast, “taking” was not part of Emily’s understanding of area models. During the debrief when I asked Emily to think about why she focused on the non-shaded pieces and interpreted 8/10 as 2/10, she said, “Because if a lot of them are shaded, it just seems kinda weird.” In her explanation for why she attended to the non-shaded pieces, the proportion of shaded pieces was a factor. When a majority of the pieces were shaded she tended to focus on the non-shaded pieces, corresponding to the smaller part. Because I had completed a partial analysis of the Lisa case, I asked her explicitly whether she thought of the shaded pieces as “taken”. She rejected this idea. “I don’t really think I’m removing.” She later confirmed that the number of pieces was crucial to her interpretation and she tended to focus on the part comprised of fewer pieces. Taking and smaller part understandings were different in so far as the underlying explanations for the answers seemed to involve taking for Lisa but not for Emily.
Because of the difference in how they interpreted shading in the area model, these atypical understandings were treated as distinct for the individual case studies. While this difference is important to appreciate, particularly when considering remedial approaches, the overt manifestations of these two different atypical understandings were remarkably similar.

| **Problem:** What does the shading mean? |
|---|---|
| **Lisa** | **Emily** |
| Artifact |  |
| Quote | “I keep imagining that this (pointing to shaded part) is taken.” | “I don’t really think I’m removing” |
| Difference | Shaded pieces understood to be taken | Rejection of shaded pieces as “taken” |

Figure 119. Illustration of the difference between Lisa’s and Emily’s understanding of the shaded region.

**Conclusion.** Emily and Lisa both focused on the non-shaded pieces or empty space to determine the fractional value. Although the underlying understandings appeared to be different in nature, the similarity of the errors arising from these atypical understandings is striking. Central to both was a tendency to focus on the fractional complement and consequently these atypical understandings are classified as part of a higher-order category: focusing on the fractional complement.

**Halving and part-part.** Lisa’s and Emily’s halving understandings were the most similar of all their atypical understandings. Both Lisa and Emily represented one-half by halving a shape and omitting the shading. They also both focused on the balance between the two parts when interpreting representations of 1/2 (see Figure 120). Part-part was treated as a separate analytic category for Emily because it captured a prevalent type of understanding, whereas for Lisa part-part was not as apparent. In the discussion of the similarities between their understandings, Emily’s halving and part-part understanding will both be discussed in relationship to Lisa’s halving understanding. In this section I identify similarities and differences of these understandings and I argue that Lisa’s halving and Emily’s halving and part-part all should be thought of as belonging to the higher-order category of “halving”.

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Just as Emily sometimes identified the two parts (shaded and non-shaded) to name the fraction, Lisa also sometimes similarly named fractions using part-part terms. In the two instances in which Lisa used a part-part understanding to name the fraction, she identified the non-shaded region as the numerator, so these two instances were classified as “taking” and did not constitute a separate analytic category.
Figure 120. Illustration and comparison of Lisa’s halving understanding and Emily’s halving and part-part understandings.

**Similarities of Lisa’s halving and Emily’s halving and part-part understandings.** Lisa’s and Emily’s halving understandings were incredibly similar. This can be best illustrated by their answers to the posttest questions where they were asked to draw or write one-half (see Figure 121). Both drew several different shapes that were partitioned in half and omitted the standard shading of one of the two pieces.
**Question:** Draw 1/2

<table>
<thead>
<tr>
<th></th>
<th>Lisa</th>
<th>Emily</th>
</tr>
</thead>
<tbody>
<tr>
<td>Posttest Answer</td>
<td><img src="image1" alt="Lisa's drawing" /></td>
<td><img src="image2" alt="Emily's drawing" /></td>
</tr>
<tr>
<td>Similarity</td>
<td>Non-standard representation of 1/2</td>
<td>Non-standard representation of 1/2</td>
</tr>
<tr>
<td></td>
<td>Drew 1/2 without shading</td>
<td>Drew 1/2 without shading</td>
</tr>
</tbody>
</table>

Figure 121. Illustration of the similarity of Lisa’s and Emily’s representations of 1/2 at the time of the posttest.

In addition, when evaluating whether a representation was equal to 1/2, they both relied upon explanations pointing to the equivalence of the parts rather than the relationship of the part to the whole (see Figure 122). Lisa counted both the shaded and non-shaded pieces and made a balancing gesture with her hands. Similarly, Emily identified the non-shaded first and then the shaded. Not only did they justify their answers using a balancing strategy that would not extend beyond the fraction one-half, but their parsing of the representation into shaded pieces and non-shaded pieces highlighted the numerator and fractional complement, rather than relationship of the numerator to the whole. These understandings were considered synonymous because both Lisa and Emily represented 1/2 by splitting a shape and understood 1/2 as a balance of parts.

**Question:** Is this 1/2?

<table>
<thead>
<tr>
<th></th>
<th>Lisa</th>
<th>Emily</th>
</tr>
</thead>
<tbody>
<tr>
<td>Answer</td>
<td><img src="image3" alt="Lisa's answer" /></td>
<td><img src="image4" alt="Emily's answer" /></td>
</tr>
<tr>
<td></td>
<td>There is four of the shaded ones and four of the non-shaded ones, then you can see that they are one-half, even though it doesn’t look like one-half. <em>(making balancing gesture with her hands)</em></td>
<td>And then, three aren’t shaded and three are shaded.</td>
</tr>
<tr>
<td>Similarity</td>
<td>Drawing is 1/2 (correct) – part-part</td>
<td>Drawing is 1/2 (correct) – part-part</td>
</tr>
</tbody>
</table>

Figure 122. Illustration of the similarity of Lisa’s and Emily’s correct identification of a drawing of 1/2, focusing on the balance of the two parts.
Differences between Lisa’s halving and Emily’s halving and part-part understandings. The primary difference between Lisa’s halving and Emily’s halving and part-part was the fraction values that these understandings were applied to. Lisa’s halving understanding applied only to the fraction 1/2. In contrast, Emily extended her halving understanding to other fraction values. Just as she represented 1/2 by partitioning a shape into 2 pieces, she believed she had represented 1/4 by partitioning a shape into 4 pieces. Emily’s part-part understanding was also extended beyond a simple focus on the balance between parts for fractions equal to 1/2, and was applied to other fractional values. This application to other fraction values was one of the primary reasons that Emily’s part-part understanding was treated as analytically distinct from her halving understanding. However, for the purposes of this cross-case comparison, this distinction of halving and part-part is of less import. Although Lisa did not similarly extend her halving understanding to other fraction values, this reflects a difference in application of the understanding rather than a fundamental difference between Lisa’s and Emily’s understandings. The common core to both Lisa’s and Emily’s atypical understandings were a representation of 1/2 as a splitting and a focus on the balance of parts.

Conclusion. Lisa and Emily both constructed representations of 1/2 by partitioning a shape into two pieces, and interpreted representations of 1/2 by focusing on the balance between parts. Although these understandings were occasionally evoked in conjunction with different problems, these understandings were persistent and sometimes problematic for both of them. The primary overt manifestation of these atypical understanding for both Emily and Lisa was the same: drawing 1/2 with no shading. These understandings are classified as part of a higher-order category also called “halving.”

Unit fraction and more pieces. Although on the surface Lisa’s unit fraction understanding and Emily’s more pieces understanding seem quite different, both understandings were primarily employed in the context of fraction comparisons, and can best be conceptualized as two sides of the same coin. To accurately compare fractions, with a part-whole understanding, one must coordinate both the size of the pieces and the number of pieces. Lisa’s “unit fraction” understanding involved attending exclusively to the size of the pieces and not the number of pieces. In contrast, Emily’s “more pieces” understanding involved attending exclusively to the number of pieces and ignoring the size of the pieces. Both of these atypical understandings resulted in errors in fraction comparison because of a lack of coordination of both number of pieces and the size of the pieces (see Figure 123). Given that these two understandings are fundamentally different, in this section I highlight how they can be considered complementary and argue that they can both be classified as belonging to the higher-order category of: uncoordinated fractional values.
Atypical Comparison of Fractions

Lisa’s Unit Fraction Understanding

Focus on the **size** of pieces
ignoring **number** of pieces

\[
\frac{1}{2} > \frac{4}{5}
\]

because halves are bigger than fifths and 1/2 is the largest fraction

Emily’s More Pieces Understanding

Focus on the **number** of pieces
ignoring **size** of pieces

more pieces

larger fraction

Commonality: Not coordinating **size** and **number** of pieces

smaller number is larger fraction  **or**  larger number is larger fraction

Figure 123. Illustration and comparison of Lisa’s unit fraction understanding and Emily’s more pieces understanding.

**Complementary nature of unit fraction and more pieces understandings.** Both Lisa’s unit fraction understanding and Emily’s more pieces understanding involved attending to only one dimension of the fractional value. When comparing fractions, Lisa tended to focus exclusively on size and Emily tended to focus exclusively on the number of pieces. When size was the relevant dimension, Lisa would answer correctly and when number of pieces was the relevant dimension, Emily would answer correctly. For example, during the posttest Lisa and Emily were asked to compare 1/6 and 1/8. Lisa, focusing on size, correctly answered that 1/6 was bigger, while Emily, focusing on the numerical value, determined that 1/8 was bigger, because it was comprised of more pieces (see Figure 124).
**Question:** Which is bigger 1/6 or 1/8?

<table>
<thead>
<tr>
<th>Artifacts</th>
<th>Lisa</th>
<th>Emily</th>
</tr>
</thead>
</table>
| ![Image](1/6) ![Image](1/8) | One-sixth is bigger.  
...  
The larger the number the smaller it is. | One eighth is bigger.  
...  
Because there are more pieces. |

<table>
<thead>
<tr>
<th>Difference</th>
<th>Lisa</th>
<th>Emily</th>
</tr>
</thead>
</table>
| Focus on size of parts | 1/6 is bigger (correct) | 1/8 is bigger (incorrect)  
Focus on number of parts |

Figure 124. Contrast of Lisa’s correct and Emily’s incorrect answer to the comparison of 1/6 and 1/8.

In contrast, when the number of pieces was the relevant dimension in the comparison of 2/8 and 5/8, Lisa answered incorrectly and Emily answered correctly. In Lisa’s answer she explained that 2/8 was bigger than 5/8 by referencing previous comparison problems (1/6 and 1/8; 2/5 and 2/7) in which she had established the inverse relationship of the size of numeral to the size of the fraction. In contrast, Emily had no difficulty with this comparison problem where number of pieces was the relevant dimension.
<table>
<thead>
<tr>
<th>Artifacts</th>
<th>Lisa</th>
<th>Emily</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image.png" alt="Image" /></td>
<td>L: Two-eighths would be the larger one. … Because it applies the same way <em>points to previous comparison of 2/5 and 2/7</em> … Tutor: So, two is the smaller number here, so two-eighths is going to be bigger than five-eighths? L: Yeah, exactly.</td>
<td>None</td>
</tr>
</tbody>
</table>

| Difference | Focus on size of parts | Focus on number of parts |

| Answer | 2/8 is bigger (incorrect) | 5/8 is bigger (correct) |

Figure 125. Contrast of Lisa’s incorrect and Emily’s correct answer to the comparison of 2/8 and 5/8.

These understandings can be thought of as complementary, because they both involved exclusive focus on one dimension alone and did not coordinate the two essential components of a part-whole understanding. Consequently, these understandings led to *opposite* patterns of errors on the posttest (see Figure 126). Lisa’s answers were consistent with choosing the smaller numerical value as the larger fraction, related to her unit fraction understanding. In contrast, Emily’s answers were generally consistent with choosing the larger numerical value to represent the larger fraction, consistent with her more pieces understanding.
Conclusion. Both Lisa’s unit fraction understanding and Emily’s more pieces understanding resulted in errors in judgment of fractional magnitude. Lisa focused exclusively on the size of the pieces, associating a smaller numerical value with the larger fraction. Emily focused exclusively on the number of pieces. These two understandings both involve failing to coordinate the two necessary components for a part-whole comparison: size and number. These understandings can be considered part of a higher-order category: uncoordinated fractional values.

Arbitrary and rule-based navigation. Both Lisa’s arbitrary manipulation and Emily’s rule-based navigation involved operating procedurally on symbols and representations without connection to the underlying meaning or referents. However, it was not possible to abstract a higher-order category given the range of ways in which this could manifest. For both students this procedural operation was more of a general orientation to mathematics, representations and symbols, rather than a specific and identifiable atypical understanding. Therefore, despite the similarity, this was not pursued as a higher-order category.

Unique atypical understandings. Although Emily and Lisa displayed many similar kinds of understandings, there were several understandings that had no analogs in the other case. Although quarters were perceptually salient for Emily, they were not for Lisa. Lisa used the word “quarter” to describe 3/4 on only one occasion during the pretest, she never used this term again. Unlike Lisa, Emily did not experience difficulties partitioning shapes into odd numbered denominators and did not rely upon a discrete set understanding of fractions. In fact, Emily was very concerned with the evenness of the wholes and the evenness of each of the pieces comprising the whole over all the sessions. Given the lack of commonality for Lisa and Emily with respect to these three atypical understandings (quarters, partitioning, and discrete set understanding) it was not possible
to extrapolate a higher-order category. While these remain essential to the understanding of the individual cases, they are not pursued in the remainder of this chapter.

**Indicators of atypicality.** The commonalities between Lisa’s and Emily’s atypical understandings allowed for an abstraction of superordinate categories: indicators of atypicality (see Figure 127). In this section I use the similarities observed in Lisa’s and Emily’s cases to operationally define these indicators of atypicality.

*Fractional complement* is operationally defined as attending to the fractional complement of a given representation (either the missing or the non-shaded pieces). This indicator of atypicality includes both referring to the shaded amount as “taken” and the non-shaded amount as “left,” or understanding the numerator of the fraction to be comprised of the part with the fewest pieces.

*Halving* is operationally defined as drawing or interpreting the fraction 1/2 without shading, or justifying the equivalence of a representation to one-half by focusing on a balance of similar parts.

*Uncoordinated fractional values* is operationally defined as comparing fractions based on one value alone. The larger or smaller value might be judged as signifying the larger fraction, but essential to this indicator of atypicality is that both of the values are not reconciled with each other.

Fractional complement, halving, and uncoordinated fractional values were considered the three primary indicators of atypicality. These indicators were used as an analytic tool to allow for a top-down analysis of the data from the fifth grade control students and one other student with an MLD. The premise being, when an indicator is identified it may signal an underlying atypical understanding. These indicators will serve as the primary analytic lens throughout the remainder of the chapter.

![Figure 127](image.png)

Figure 127. Depiction of the relationship of the atypical understandings to the superordinate indicators of atypicality.
**Comparison of Students With MLDs to Fifth-grade Controls**

The purpose of this section is to determine if the control students displayed similar atypical understandings and if so, explore how they occurred for those students. The indicators of atypicality were used to analyze the data and flag any potential instances that were consistent with fractional complement, halving, or uncoordinated fractional values. Those flagged indicators were the starting point of analysis to determine if the data from the student suggested an underlying atypical understanding, and if so, how did that atypical understanding occur over the course of the tutoring sessions. This section addresses the analytic question: Was the occurrence of atypical understandings qualitatively different for the control students than it was for the students with MLDs? These data suggest that although the indicators of atypicality sometimes occurred in the sessions with the fifth graders, in general they did not reflect an underlying atypical understanding. In the few cases where an atypical understanding was evident it was not detrimental and easily resolved. At the time of the posttest, the fifth grade students demonstrated almost no indicators of atypicality, further suggesting that these atypical understandings were not problematic for the control students.

In this section I present the analytic methods used to analyze the data from the fifth grade students, and then present a high level view of the five fifth grade students as seen through indicators of atypicality. For each of the indicators of atypicality I explore how it occurred for each of the fifth grade students, to determine if the student demonstrated an atypical understanding and the ways in which it was problematic and/or persistent. I conclude by refining the definitions of these indicators based on the differences between the students with MLDs and the controls.

**Analytic methods.** The sessions with the fifth grade students were content logged and any potential indicator of atypicality was flagged and the relevant problem was transcribed. As needed related problems were also transcribed to further sketch the bounds of the atypicalities identified (e.g., if the pretest showed indicators of atypicality the same question would be transcribed for the posttest). For any student displaying an indicator of atypicality, frequency was determined by the number of answers or explanations given which were consistent with that indicator. The flagged instances were analyzed to determine if this understanding was indicative of an underlying atypical understanding. If so, a more comprehensive analysis of the sessions were performed to determine the pervasiveness of this atypical understanding. The analysis was not a systematic analysis of the learning experiences for these students; instead it is a view of these fifth graders through the pre-established lenses of these indicators of atypicality.

**Overview of fifth graders.** Although several students demonstrated indicators of atypicality, the prevalence of these indicators was quite low (as seen by the frequency counts presented in Figure 128). Fractional complement was the most common indicator identified for these students, however, the indicators were only reflective of an underlying atypical understanding for Mary. Only two students demonstrated halving and only one student demonstrated uncoordinated fractional values. In each section I explore the instances flagged for each indicator of atypicality to determine the ways in which it occurred and the ways in which it persisted or was resolved.
Incidence and frequency of each of the indicators of atypicality for each fifth grade control students.

**Fractional complement indicator.** Fractional complement was the most prevalent indicator of atypicality present for the fifth grade controls (see Figure 129). For one student, Mary, analysis of the instances of indicators of atypicality suggested that she *did* in fact have an underlying atypical understanding similar to Lisa’s “taking” understanding. For the remaining 3 students, nearly all flagged instances of this indicator were in conjunction with a correct answer, or an answer that was subsequently self-corrected. In this section I first present details from Mary’s data and discuss how her atypical understanding occurred and was then resolved. I then briefly describe the flagged instances for the three other students and explain why they did not constitute evidence of an underlying atypical understanding.

Figure 129. Data display of the flagged indicators of fractional complement for each of the fifth grade control students.

**Evidence of Mary’s atypical understanding.** The indicators of fractional complement flagged for Mary revealed that she sometimes represented a fraction by shading the fractional complement and sometimes understood the shaded region as taken. During the pretest she was asked to draw the fraction $3/4$. She drew a shape, divided it into 4 pieces, but shaded only one of the pieces (see Figure 130). Mary therefore represented the fraction by shading the fractional complement, and the fraction was represented by the non-shaded pieces.
During the third tutoring session this understanding surfaced again, in a slightly different manner. Although she used shading canonically both to draw 1/4 and interpret 5/6, she described the shaded region as “taken.”

Mary: There’s cake here (points to non-shaded piece of 5/6), but there’s not cake here (points to each of the shaded pieces of 5/6) because somebody ate it.

Tutor: Ok. So, the pink is going to show us where there is not cake?

Mary: Yes. What’s taken.

Tutor: Ok, is that the same with your picture here? (points to 1/4)

Mary: Uh yes. It is.

Tutor: So one-fourth is going to be taken from there?

Mary: Yes.

Mary understood the shaded region of both 5/6 and 1/4 as “taken” or gone “because somebody ate it.” These examples were considered consistent with a “taking” atypical understanding. Mary sometimes understood the non-shaded region to be the focal fractional quantity (as in the case of 3/4) and she understood the shaded region as “taken” away.

Despite clear evidence that Mary had a “taking” understanding, this understanding was inconsistently applied. For example, immediately after drawing the representation of 3/4 by shading 1 piece, she drew a canonical representation of 2/5 shading 2 of the pieces. During the second tutoring session, she drew and interpreted area model representations correctly and was incredibly adept at writing rules for interpreting drawings of fractions. Her rule specified that, “the shaded part is the

73 These two indicators of fractional complement were coded as correct because Mary did not attend to the fractional complements (1/6 or 3/4), but instead drew a correct representation of 1/4 and interpreted 5/6 correctly.
numerator. And the total number of pieces is the denominator.” Although this “taking” understanding was evident in her answers, she also applied it inconsistently.

**Resolution of Mary’s atypical understanding.** The last two indicators of fractional complement flagged for Mary involve her again representing a fraction by shading the fractional complement, and then resolving this ambiguity for herself. During the third tutoring session I asked her to use a whole sheet of paper to draw a picture of 2/3. She divided the paper into three pieces and then shaded only one of the pieces. She reflected on her representation in which the fractional complement (1/3), not the desired fractional quantity (2/3) was shaded. She referenced her understanding of the shaded portion as the amount taken away, which resulted in 2/3 left. However, she then reconsidered her answer and shaded an additional piece.

<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript</th>
<th>Screenshot</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Mary: <em>shades in one of the pieces of cake, caps pen – pause 8 seconds</em> I think this might be if you wanted to take it away, and then hey look, two-thirds is left. <em>(covers the one-third)</em></td>
<td><img src="image1.png" alt="Screenshot" /></td>
</tr>
<tr>
<td></td>
<td></td>
<td><img src="image2.png" alt="Screenshot" /></td>
</tr>
<tr>
<td></td>
<td>Mary: Or you could shade two pieces instead and say oh look, two-thirds are shaded. Tutor: Uh huh. Is it helpful to think of this as - Mary: <em>shades second piece</em></td>
<td><img src="image3.png" alt="Screenshot" /></td>
</tr>
</tbody>
</table>

After some consideration Mary modified her drawing of 2/3 so that two pieces rather than just one piece was shaded. Her final answer involved a canonical representation of 2/3.
Because Mary had noticeably changed her answer, I asked her to reflect upon whether it made sense that her drawing was 2/3. She began by arguing that it did not make sense and referred again to taking.

Tutor: Ok. Is that a way that it makes sense to you?
Mary: No… Well, because. Usually when someone says, and then I took away two-thirds of the cake, they shade two thirds so that’s the part that they are taking or that, that is not there.

Although Mary had shaded the second piece in on her representation of 2/3, she still applied a “taking” understanding and argued that the shading showed the pieces that were “not there.” However, immediately after finishing this statement she considered a context in which 1/3 was added to 2/3, and she concluded that shading two pieces did in fact make more sense.

Mary: But in a way it could be because then if you have two-thirds and you added another third, you would adding onto it and not subtracting from it. So I think this makes sense. (points to two shaded pieces)

In this second excerpt she reconsidered the meaning of shading in an invented context of the problem “2/3+1/3=” and recognized that shading would be the amount that was added rather than removed. In the span of less than 30 seconds, with no intervention from me, Mary went from understanding the shaded region as taken away to rejecting that interpretation and understanding the shaded region as the amount of cake that was there.
Mary continued her explanation of the shading by posing the problem “2/4+1/4=” and enacting this problem using an area model, emphasizing that the shaded pieces were considered the pieces that were there and the non-shaded piece was not there.

<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript</th>
<th>Screenshot</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Mary: Like, if you had like two-fourths for example and you added a fourth, you’d get three fourths. So (draws rectangle, divides into 4 pieces)</td>
<td><img src="image1.png" alt="Screenshot" /></td>
</tr>
<tr>
<td>2</td>
<td>if you had two fourths (shades in 2 pieces).</td>
<td><img src="image2.png" alt="Screenshot" /></td>
</tr>
<tr>
<td>3</td>
<td>And this (points to shaded pieces) this was the cake. This was the cake and this is what you had,</td>
<td><img src="image3.png" alt="Screenshot" /></td>
</tr>
<tr>
<td>4</td>
<td>and then from somebody else’s cake, you brought it another fourth (shades third piece). Then you had three-fourths, and it would be positive. and this one (points to non-shaded), you wouldn’t have this.</td>
<td><img src="image4.png" alt="Screenshot" /></td>
</tr>
</tbody>
</table>

Mary’s explanation of the representation of the problem $2/4+1/4=$ used shading consistently to indicate the amount that was there. After these episodes in which Mary appeared to reconcile the representational ambiguity of the shading, this understanding did not resurface for Mary and remained unproblematic for the rest of her sessions.

**Comparison of Mary and the students with MLDs.** Although the indicators of atypicality did uncover an atypical “taking” understanding, the nature of Mary’s understanding was qualitatively different than it was for Lisa and Emily. First, Mary identified the problematic ambiguity in the meaning of the shading for herself and then resolved this understanding. Second, when this atypical understanding did occur it was not similarly problematic. Although Mary was inconsistent across problems and allowed the shading to be representative of the amount there or the amount taken away, she was internally consistent within problems. After drawing her representation of $3/4$ using non-shaded pieces, she did not then interpret it as the fractional complement. Similarly, in her understanding of the shaded region as taken for $5/6$ and $1/4$, she did not focus on the
fractional complements (1/6 or 3/4). Despite the fact that she allowed for flexibility in the meaning of the shading across problems, she never attended to the fractional complement. Mary’s “taking” understanding is a lovely example of how a similar atypical understanding occurred but was not similarly problematic nor persistent. Mary’s taking understanding is therefore considered to be qualitatively different than Lisa’s and Emily’s tendency to attend to the fractional complement.

**Zora’s, Parish’s, and Kara’s indicators of fractional complement.** Three other students also demonstrated indicators of atypicality (see Figure 131), but unlike Mary’s case, these indicators did not appear to suggest and underlying atypical understanding.

Kara’s only instance was during the posttest, in which she considered an area model of 2/3 to be a valid representation of 1/3 (see Figure 132) because “it was divided into three”. Because this was the only indicator of fractional complement, and because her focus was on the partitioning of the shape rather than the non-shaded piece, this was not considered indicative of an underlying atypical understanding.

Parish also accepted the area model of 2/3 as 1/3 on the pretest. Her remaining 2 indicators of fractional complement she self-corrected as she explained her answer. Again, this was not considered to reflect an underlying atypical understanding.

Zora also accepted the area model of 2/3 as 1/3 during the pretest, but her remaining instances, all associated with correct answers, primarily involved her commenting on the representational ambiguity inherent in representations of fractions. She, quite correctly, argued that the shading could represent the amount there or the amount taken away depending on the problem context, and specified that to know the meaning of the shading “you are going to need to know by what they wrote or what they say.” Again, this was not considered to reflect an underlying atypical understanding, but a sophisticated understanding of the ambiguity in conventions used for creating mathematical representations.

The one common problem that all three students answered incorrectly was the interpretation of 2/3 as 1/3 (see Figure 132). Interestingly, neither Lisa nor Emily, despite their tendency to focus on the fractional complement, selected this answer on the pretest or posttest. This suggests that the answer to this question may not be the best indicator of fractional complement. I return to this when discussing the refinement of the operational definition for the indicators of atypicality.
Conclusion. Although fractional complement was the most common indicator of atypicality, it was only reflective of an underlying atypical understanding in Mary’s case. The data from the fifth grade controls reflected a qualitatively different experience with the fractional complement. The instances were either fleeting or resolved relatively easily, whereas for the students with MLD, the atypical understandings were persistent and problematic. The incidence of fractional complement for the fifth grade students was therefore quite dissimilar to how it occurred for Lisa and Emily.

Halving indicator. Two of the control students displayed indicators of halving, Zora and Kara (see Figure 133). Both of Zora’s indicators occurred during the pretest and it appeared that she underwent a shift from a halving understanding to a quantity understanding of 1/2. In contrast, Kara demonstrated an indicator of halving during both the pretest and the posttest. However, further examination of Kara’s data suggests that this was not her primary understanding of 1/2. In this section I present evidence of Zora’s change from pretest to posttest, and evidence that Kara’s primary understanding of 1/2 did not involve halving.

Zora’s indicators of halving. Both Zora’s construction and interpretation of the fraction 1/2 appeared to undergo a shift from pretest to posttest. Zora’s indicators of halving at the time of the pretest suggested an underlying “halving” understanding. When asked to draw a representation of 1/2, she drew an area model representation partitioned in half with no shading (see Figure 134)\textsuperscript{74}.

\textsuperscript{74} Zora also drew several other shapes partitioned in half with no shading, but these shapes and partitions were less standard (e.g., triangle with a lightening bolt shaped partition line) so were excluded from the discussion here for the sake of succinctness.
Zora’s construction and interpretation of an area model of $\frac{1}{2}$ without shading were suggestive of an atypical halving understand. Despite this initial indication that Zora relied upon an atypical halving understanding, there were no indicators of halving during the tutoring sessions or at the time of posttest. Indeed, for these same problems on the posttest, her answers highlighted a canonical quantity understanding of $\frac{1}{2}$. When asked to draw $\frac{1}{2}$, she drew a canonical area model of $\frac{1}{2}$ (see Figure 136) and said, “I would think of that, one-half is shaded.”

Zora: Let’s say someone circled this (pointing to the area model with no shading), then I’d automatically know that they’d be wrong. It’s either one whole or
none. But let’s say that this part was shaded (shades in one piece), if this was shaded like that, then they would be right (circles area model).

At the time of the posttest Zora did not accept a shape partitioned in half as a valid representation of 1/2. The contrast of Zora’s answers on the pretest and posttest suggest that she shifted her understanding from a halving understanding to a quantity understanding of 1/2. Although it is unclear how this understanding was resolved, this atypical understanding was not problematic in the context of more complex fraction concepts, and was not apparent at the time of the posttest.

**Kara’s indicators of halving.** Unlike Zora, Kara did not appear to undergo a shift in understanding from pretest to posttest. During both the pretest and the posttest she selected the non-shaded area model of 1/2 as a valid representation of 1/2. Her explanations were relatively consistent. At the time of the pretest she said, “there’s a line down the middle… well, it’s divided into two halves,” and at the time of the posttest she said, “because the line went straight down the middle and it’s half of a circle.” These answers were classified as an indicator of halving. However, on both 1/2 interpretation problems, she also circled the area model of 1/2 with shading, a discrete set representation of 1/2 and 3/6, an area model of 4/8, and a number line representation of 2/4 (see Figure 137). While she might have accepted one-half without shading as one of the many possible way of representing 1/2, her own drawn constructions 1/2 are thought to be a better indication of how she understood 1/2.

![Scanned artifacts of Kara’s pretest and posttest answers for the problem “circle all the pictures you think are the same as 1/2.”](image)

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Zora had spent considerable amount of time explicitly discussing the shading conventions, and although she allowed for the shading to represent the amount that was there or gone flexibly given the problem context, she never toggled to attending to the fractional complement. In this example, she talked about the two possible interpretations of the drawing, either as a whole (white pieces as the focal fractional quantity) or as none (if shaded pieces were the focal fractional quantity).
During both the pretest and the posttest, her own drawings of 1/2 highlight the shading. At the time of the pretest when asked to describe and then draw what she thought of when she thought of 1/2 she said, “Fifty percent or like half of a circle shaded in,” and then drew a circle with one-half shaded (see Figure 138). Similarly at the time of the posttest, she said, “a box and one-half shaded in”, and then drew several valid area model representations of 1/2, all using shading appropriately (see Figure 139). In all her drawings she shaded the quantity corresponding to one-half, and in both her explanations she explicitly highlighted shading as a relevant feature of her understanding.

Figure 138. Scanned artifact of Kara’s pretest answer to the question: draw 1/2.

Although Kara did allow 1/2 to be represented with no shading during the interpretation problems, her own drawn representations of 1/2, along with her explanations, suggest that shading was a central component of her understanding. Kara was therefore focused on one of the two pieces in her drawing that was consistent with a quantity understanding of 1/2 rather than an atypical halving understanding.

Conclusion. Although both Zora and Kara demonstrated indicators of halving, it was clear that it was not problematic in either instance. Zora refined her understanding of 1/2 between the pretest and the posttest, and Kara’s representation of 1/2 suggested that she favored a quantity understanding rather than a halving understanding. These indicators of halving were therefore considered to be qualitatively different than in Lisa’s and Emily’s cases.

Uncoordinated fractional values. Only one control student, Parish, demonstrated an indicator of uncoordinated fractional values in her answers to the
fraction comparison problems (see Figure 140). Parish answered a sequence of comparison problems in a way that demonstrated her lack of coordination between the numerator and denominator values.

![Figure 140](image)

At the time of the pretest Parish’s errors on comparison questions were consistent with Emily’s more pieces understanding. She judged that 1/8 was larger than 1/6, 2/7 was larger than 2/5, and 5/8 was larger than 2/8. In each case she identified the fraction comprised of the largest numeral as the larger fraction. When I asked her to compare the fractions where both the numerator and denominator varied, in the case of 3/6 and 5/10, she said, “I don’t think I can handle that.” At the time of the pretest, Parish was unable to coordinate the numerator and denominator values, and her answers were consistent with Emily’s answers on the pretest.

Although her answers on the pretest suggest a potential underlying atypical understanding, this understanding did not surface again. After working with the fraction pieces during the first tutoring session, Parish was asked to order fractions from largest to smallest. She correctly sorted three sets of cards with no assistance: (1) 1/100, 1/10, 1/3; (2) 2/16, 2/8, 2/5; and (3) 3/5, 6/5, 18/5. In each case she made correct magnitude determinations based on whether the larger number was in the numerator or the denominator. At the time of the posttest in addition to correctly answering the comparison problems she had gotten incorrect on the pretest, she also answered much more complex comparison problems (see Table 5). Although the indicator of uncoordinated fractional values occurred for Parish during the pretest, it appeared that her interaction with the fraction pieces quickly resolved this issue, and it never surfaced again for her.

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76 This indicator is slightly different than the others in that it is only evident across multiple problems. There were therefore, not three separate instances of “lack of coordination,” instead all three instances in combination were used to flag an indicator of lack of coordination of numerator and denominator.
Table 5. Overview of Parish’s answers on fraction comparison problems during the pretest and posttest.

<table>
<thead>
<tr>
<th>Comparison (which is bigger?)</th>
<th>Pretest Answer</th>
<th>Posttest Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/6 and 1/8</td>
<td>Incorrect (1/8)</td>
<td>Correct (1/6)</td>
</tr>
<tr>
<td>2/7 and 2/5</td>
<td>Incorrect (2/7)</td>
<td>Correct (2/5)</td>
</tr>
<tr>
<td>2/8 and 5/8</td>
<td>Correct (5/8)</td>
<td>Correct (5/8)</td>
</tr>
<tr>
<td>3/6 and 5/10</td>
<td>Not answered</td>
<td>Correct (equal)</td>
</tr>
<tr>
<td>4/5 and 2/3</td>
<td>Not administered</td>
<td>Correct (4/5)</td>
</tr>
<tr>
<td>3/7 and 2/3</td>
<td>Not administered</td>
<td>Correct (2/3)</td>
</tr>
</tbody>
</table>

**Conclusion.** The indicators of uncoordinated fractional values at the time of the pretest were consistent with Emily’s “more pieces” atypical understanding. However, after working with the fraction pieces, Parish was able to focus on size of the piece as a relevant dimension. Unlike Lisa and Emily, this atypical understanding was not persistent, and during the first tutoring sessions she correctly ordered fractions and there was no further trace of this understanding at the time of the posttest.

**Contrasting controls and students with MLDs.** Although there were traces of these atypical understanding for the fifth grade students, these atypical understandings were often refined quite easily and were not detrimental to the students understanding of fractions. For the most part, these atypical understandings were not evident at the time of the posttest. Figure 141 illustrates the difference between the students with MLD and the control students at the time of the posttest. Lisa and Emily had all three indicators of atypicality at the time of the posttest. In contrast, for the fifth grade controls there were almost no indicators of atypicality. As previously mentioned, Kara’s answers on the posttest that were classified as indicators included her accepting a non-shaded halved circle as a valid representation of 1/2 and a representation of 2/3 as a valid representation of 1/3. Neither of these answers was considered to be indicative of an underlying atypical understanding.
Figure 141. Contrast of students with MLDs and control students indicators of atypicality on the posttest.

The indicators of atypicality were operationally defined based on the commonalities of Lisa’s and Emily’s cases. With the addition of the fifth grade controls, it is possible to refine these operational definitions. The goal of this refinement is to better capture the indicators that are reflective of an underlying atypical understanding. For example, drawing a fractional representation of 1/2 by halving a shape and omitting the shading appears to differentiate the students with MLD from the fifth grade students (see Figure 142) more effectively than their answers on the non-shaded one-half interpretation problem. Both Lisa and Emily drew 1/2 with no shading at the time of the posttest, but both students rejected the drawing of 1/2 with no shading. In contrast, although none of the fifth graders drew 1/2 with no shading at the time of the posttest, one did allow for a drawing 1/2 with no shading to be a valid representation of 1/2. The best indicator of halving therefore appears to be the student’s construction of a representation of 1/2, rather than whether they accept a non-shaded area model.

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77 Although both Mary and Erica had a drawing of 1/2 that did not involve shading, they both labeled the focal fractional quantity 1/2, so this answer was not considered indicative of halving.
In addition the interpretation problem where students are asked to judge if an area model of 2/3 is a valid representation of 1/3 did not appear to effectively signal that the student may have issues with attending to the fractional complement. Several of the fifth-grade students selected (or qualified their selection) on either the pretest and/or posttest, whereas neither Lisa nor Emily ever selected this representation, despite their tendency to attend to the fractional complement. This specific problem is therefore not a good indicator of an underlying atypical understanding. By ruling out these two boundary cases as poor indicators of the underlying atypical understanding, we see a much cleaner contrast of the indicators for the students with and without MLDs (see Figure 143).

Figure 143. Refined indicators of atypicality at the time of the posttest for both the students with MLDs and the fifth grade control students.
**Discussion.** The data from the fifth grade students suggests that the indicators of atypicality and indeed some of the underlying atypical understandings are in some ways quite typical. These atypical understandings have origins in common misinterpretations or common misunderstandings. They are not atypical in the sense that they never occur for students. What makes these understandings atypical for the students with MLDs is the persistence and ways in which they become reified and detrimental to the students learning. The fifth graders who had some of these same atypical understandings were able to recognize and often repair those understandings over the course of the tutoring sessions. As in Mary’s case when she resolved her “taking” understanding, it was her understanding of fractional quantity itself that helped her sort out the representational ambiguity that exists when using area models. She grounded her understanding of the shading in two contexts. She explored the meaning of the shading in the context of adding $1/3$ to $2/3$ and for the problem $2/4+1/4=\cdot$. She was able to deduce from her enactments of these problems that shading must represent the “amount there.” In contrast, the students with MLDs did not appear to have a grounded understanding of fractional quantity. Hence, it is possible their understanding of fractional quantity could not serve the corrective function that it did for Mary. This presents a bit of a catch-22. The students with MLD could not repair their understanding of representations of fractions because they did not have a firm conceptual grounding of fractional quantities, and the primary way of helping students understand fractional quantity is through those very representations. For the students with MLDs, this potentially produced a self-perpetuating cycle where their understanding of fractional quantity was not grounding for the representations and the representations were not grounding for their understanding of fractional quantity.

**Conclusion.** Although several of the fifth grade students demonstrated indicators of atypicality, in only a few cases were these indicative of an underlying atypical understanding. The students who had an atypical understanding resolved it over the course of the tutoring sessions and it was neither problematic for their learning nor evident at the time of the posttest. These atypical understandings likely have roots in common misunderstandings or misinterpretations. For students without MLDs, these misunderstandings are quickly dispensed during the process of learning. For students with MLDs, like Lisa and Emily, these misunderstandings become reified and detrimental to their understanding of fractions. In the next section I use these indicators again to explore the case of another student with an MLD. Despite her ability to compensate for her MLD, these atypical understandings remained central to her conception of fractions.

**Taylor’s Insights into the Atypical Understandings**

Taylor is a unique case of a student with an MLD who was compensating so effectively that she was completing her B.S. in statistics. Despite her ability to pass upper division math classes, even basic arithmetic remains a challenge for her. As part of a separate study we explored and documented her difficulties and ways of compensating across a variety of mathematical topics. During one interview I asked her to answer and reflect upon her understandings for a small subset of the problems from this study. Two fascinating things happened. First, she demonstrated all three of the indicators of
atypicality. Second, although she relied upon the atypical understandings, they were used in productive ways. The purpose of this section is to establish Taylor’s comparability to Lisa and Emily through the indicators of atypicality and then explore Taylor’s reflections on those understandings and identify her ways of compensating.

In this section I describe the methods from the separate study as well as how a subset of data were analyzed for this study. I present an overview of her indicators of atypicality followed by an in depth exploration of each indicator. For each indicator, I establish the similarity of Taylor’s understanding to Lisa’s and Emily’s atypical understandings drawing upon examples from Taylor’s interviews. Taylor’s reflections and rationale for her understandings provide insight into why these atypical understandings might be persistent. In particular I focus on the ways in which Taylor navigates her atypical understandings so they are not ultimately problematic.

Methods. These data were taken from a separate study focused on documenting Taylor’s difficulties and compensatory strategies. We met weekly for two hours over several months for a total of 16 hours. All sessions were video and audio taped. As part of this project she completed the Dyscalculia Screener (Butterworth, 2003) and a computerized number fact program (http://www.cs.berkeley.edu/~colleenl/kelewis/mathFacts.swf). The structure of these sessions was co-negotiated, and generally involved Taylor solving a range of problems and reflecting upon her process, difficulties, and strategies for compensating.

Taylor did not complete the tutoring sessions; instead, she was asked to solve a subset of problems from the pretest/posttest and tutoring sessions and asked to reflect upon her understandings. Several follow-up questions were posed a week later to clarify her understandings. I transcribed the interviews and did a first analytic pass to flag indicators of atypicality. Unlike the fifth graders, frequency and correctness were not useful dimensions to capture. Frequency counts were not relevant because Taylor was not administered the same protocol and because the explicit purpose of the interviews was to explore her reflections upon these kinds of atypical understandings. Similarly, a classification of problems as correct or incorrect was unproductive as Taylor answered every question correctly during the interviews.78 Instead, indicators of atypicality were only used as a first parsing of the transcribed data and to establish existence of the indicators in Taylor’s data. The transcribed interview data was then analyzed to determine whether or not Taylor was relying upon atypical understandings. An atypical understanding was identified only if Taylor reasoned in a consistent way over the course of the interviews. A final analytic pass involved identifying how these atypical understandings were not problematic for Taylor and identifying her ways of compensating.

Classification of Taylor as a student with an MLD. Taylor was a White, native English speaker, and a senior at a prestigious university who was majoring in statistics and passing upper division math classes. Although Taylor would not meet the requirement of low achievement, a component of the operational definition of MLD for this study, she was nevertheless considered to have an MLD. Essential to my

78 Taylor answered every question correct during the interviews. The only possible exception to that was her representation of 1/2 with no shading.
conceptualization of MLDs is that it is possible for students to compensate for these cognitive differences. Taylor is a case of a student who was compensating so effectively that she would not be classified as low achieving. However, her reported experiences along with observations of her doing math provide compelling evidence that she (1) has an MLD and (2) has an exceptional ability to compensate.

The interviews revealed that her difficulties with mathematics were not localized, and permeated all mathematics domains discussed. Even solving the most basic math problem was often an arduous process. It took her over 10 seconds to solve the problem “8x3=.” She later explained her calculation process, “two eights is sixteen, and then I’m adding [another] eight. Sixteen plus what equals 20? Sixteen plus four. Now the eight, minus the four, and there is four left over. Twenty plus four, twenty-four.” Rather than retrieving this solution from memory, she was solving four independent calculation problems each with an intermediate sum. Her difficulties processing number permeated all aspects of her life, she reported that she could not read an analog clock, she often forgot her pin number, and she had difficulty remembering her house number in her street address79. On a computer assessment specifically designed to evaluate student’s ability to process numerical information (Dyscalculia screener; Butterworth, 2003), she was classified as having “dyscalculiac tendencies with compensatory aspects”. It is only through her exceptional ability to develop and employ compensatory processes that she was able to succeed in mathematics. Taylor defied standard operational definitions for MLD, and yet, her exceptional difficulties with even the most basic arithmetic suggests that she has a cognitive difference. For the purposes of this study she was considered to have an MLD.

**Overview of Taylor’s indicators of atypicality.** Although Taylor was only administered a subset of questions, Taylor demonstrated all three indicators of atypicality. Taylor talked about shading as “pulling away” (similar to Lisa’s taking), represented 1/2 without shading (similar to both Lisa’s and Emily’s halving), and was unable to consider the size of the piece in a comparison of 1/6 and 1/8 (similar to Emily). In short, all three indicators of atypicality that Lisa and Emily displayed were evident in Taylor’s interview (see Figure 144). Unlike Lisa and Emily, Taylor was often able to navigate and use these atypical understandings in productive ways.

In the next section I present excerpts from Taylor’s interview that illustrates each of these atypical understandings along with her reflections on these understandings. For each of the indicators flagged, there did appear to be an underlying atypical understanding. Excerpts presented here are for the purposes of (1) establishing the similarity of her atypical understandings to Lisa and Emily, (2) highlighting the rationale she provided for her atypical understandings, and (3) addressing how she compensated for these atypical understandings.

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79 To remember her house number she wrote out the number over and over again, so that she remembered the feeling of writing it. When asked her street address she could write it down and then read it.
Figure 144. Illustration of Taylor’s similarity to Lisa and Emily for the three focal indicators of atypicality.

**Fractional complement.** Taylor attended to the fractional complement in ways similar to Lisa and Emily. In this section I begin by presenting two examples that establish the similarity of Taylor’s answers to the atypical understandings displayed by Lisa and Emily. I then explore Taylor’s reflections on this understanding of fractions, which appeared to be grounded in the establishment of the whole. I conclude by attempting to identify how she compensated for this atypical understanding so it was not ultimately problematic.

**Taylor’s indicators of fractional complement.** Taylor’s understanding of fractional quantity was similar to Lisa’s and Emily’s and led to two common types of answers. First, in talking about her construction of fraction representations she talked about ‘pulling an amount away’, which was remarkably similar to Lisa’s taking understanding. Taylor’s representation of a fractional quantity involved a removal of an amount from the whole. She understood 2/5 as “five objects, and then I’d like pull away two.” Not only did she talk about pulling away the numerator value (2), but her gesture (see Figure 145) was remarkably consistent with Lisa’s “taking” understanding. Over the course of the interviews Taylor used the terms “pull away,” “get rid of”, and “taken” to refer to the fractional quantity. Just like Lisa, the fractional amount was understood as removed, potentially highlighting the fractional complement.

<table>
<thead>
<tr>
<th>Indicator</th>
<th>Lisa</th>
<th>Emily</th>
<th>Taylor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fractional complement</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Halving</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Uncoordinated fractional values</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

**Key**
- + Indicator of atypicality evident
- No indicator of atypicality evident
Second, just like both Lisa and Emily, Taylor filled in the empty space when working with fraction pieces. In this excerpt Taylor was asked to identify the fractional amount for a collection of eight 1/10 pieces. To solve this interpretation problem she first filled in the empty space with two 1/10 pieces. She then pulled away the eight 1/10 pieces, and counted the 8 pieces to interpret the fractional value.

<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript</th>
<th>Descriptions and Screenshots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tutor: So if we pulled a bunch of tenths together. (assembles eight 1/10 pieces) How would you think about determining how much total we had here?</td>
<td>Eight 1/10 pieces</td>
</tr>
<tr>
<td>2</td>
<td>Taylor: I’d look for the other ones until it was a whole circle. (puts two 1/10 pieces in empty space)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>And then I could, pull them away from the whole. (moves 8/10 to the left)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>And then count them. Um, yeah. So, 1, 2, 3, 4, 5, 6, 7, 8. (pointing to each of the 1/10 pieces)</td>
<td></td>
</tr>
</tbody>
</table>

Just like Lisa and Emily, the empty space (i.e., fractional complement) was a central component of her interpretation of the fraction pieces. Only after she filled the empty space was she able to count the pieces to interpret the fractional amount.

These two examples illustrate how Taylor’s understanding and representations of fractions similarly highlighted the fractional complement. Like Lisa, Taylor understood the focal fractional quantity as being pulled away, taken, or removed from the whole. Like Lisa and Emily, Taylor attended to the empty space when working with fraction
pieces. Her answers and explanations during the interviews were consistent with Lisa’s “taking” understanding and the indicators of atypicality were considered to reflect an underlying atypical understanding.

**Taylor’s reflections.** As Taylor reflected on how she understood fractions, the centrality of the whole emerged as fundamental to her understanding. She said “I want to start with the original [whole], and then take away the part that is modified or of interest.” This was clear both in her constructions of fractional representations and in her interpretation of 8/10. Recall that to interpret 8/10 she began by constructing the whole. She explained that the labels on the pieces were meaningless to her and she needed first be able to see the whole to be able to interpret the fractional amount. “I don’t take it as face value that that’s a tenth until I actually see ten pieces all together, fitting into a whole.” The centrality of the whole for Taylor appeared to be grounding for her. However, by starting with the whole, it resulted in a construction of fractions that involved both the fractional quantity and the fractional complement. The preference of grounding the fraction by starting with a whole and the inaccessibility of the labels on the fraction pieces might partially explain why both Lisa and Emily similarly filled the empty spaces when interpreting fractional quantities.

**Taylor’s compensations.** Although the fractional complement was highlighted in Taylor’s construction and interpretation of fractional quantities, she never incorrectly attended to the fractional complement as the focal quantity. Taylor appeared to explicitly maintain focus on the focal fractional quantity using either imagined or physical separation of pieces.

The spatial location she established as she conceptualized fractional quantities seemed central to her ability to maintain focus on the relevant part. She identified that the focal fractional quantity was mentally held on her left side. “Yeah, stuff that I’m getting rid of goes to my left side, and things that I’m holding stay on my right… like I’m not disregarding it, it’s actually of more interest because it’s being moved away to my left.” Although the fractional complement was highlighted in Taylor’s description (i.e., “things that I’m holding”), she was consistently able to maintain focus on what she described as being “of interest,” which was mentally located on her left side. Consider one subtle difference in Lisa’s and Taylor’s gestures when referring to the removal of the numerator quantity. Lisa pushed the pieces away from her body with an open hand, whereas Taylor’s motion moved from one side of her body to the other and ended in a fist (see Figure 146). This difference in gesture was suggestive of a different status for the amount removed. Whereas for Taylor this became the amount “of interest” for Lisa this amount was pushed away and out of consideration.
Question: What is this amount?

<table>
<thead>
<tr>
<th></th>
<th>Taylor</th>
<th>Lisa</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Hand gesture" /></td>
<td><img src="image2" alt="Hand gesture" /></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Similarity</th>
<th>“Pull away” (removal from whole)</th>
<th>“Going away” (removal from whole)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Difference</td>
<td>Amount held in fist</td>
<td>Amount pushed away from body</td>
</tr>
</tbody>
</table>

Figure 146. Illustration of the similarities and differences of Taylor’s and Lisa’s gestures with reference to the numerator of the fraction.

Similarly when solving the 8/10 interpretation question, Taylor was explicit about maintaining a physical separation between the focal fractional quantity and the extra pieces. When interpreting eight 1/10 pieces she explained that she would complete the whole, but maintain a physical separation so she did not confuse herself.

<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript</th>
<th>Descriptions and Screenshots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Taylor: Like if I was presented this, and I was asked how much is this? (gestures around eight 1/10 pieces)</td>
<td><img src="image3" alt="Screenshot 1" /></td>
</tr>
<tr>
<td>2</td>
<td>Then I’m going to look for the other two pieces. (moves two 1/10 pieces)</td>
<td><img src="image4" alt="Screenshot 2" /></td>
</tr>
<tr>
<td>3</td>
<td>I probably won’t put them all the way in, otherwise I’ll confuse myself, and then I’ll see that two would fit kinda in there, and then I’d count the number of pieces that I was originally presented with.</td>
<td><img src="image5" alt="Screenshot 3" /></td>
</tr>
</tbody>
</table>
The physical space between the pieces served to allow her to designate the fractional quantity even after she had assembled the fractional complement. This was unlike Lisa and Emily, who filled the space and then would lose track of the original fractional quantity (see Figure 147).

<table>
<thead>
<tr>
<th>Question: What is this amount?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taylor</td>
</tr>
<tr>
<td>Problem:</td>
</tr>
<tr>
<td>Answer</td>
</tr>
<tr>
<td>Correct: 8/10</td>
</tr>
<tr>
<td>Similarity</td>
</tr>
<tr>
<td>Difference</td>
</tr>
</tbody>
</table>

Figure 147. Contrast of Taylor’s answer, which maintained physical separation of the focal fractional quantity, and Lisa’s and Emily’s answers which involved filling in the empty spot without maintaining a separation.

**Conclusions.** Although Taylor clearly relied upon the fractional complement and had an atypical understanding similar to Lisa’s taking, she did not similarly attend to the fractional complement. This ability to maintain focus on the focal fractional quantity differentiated her from Lisa and Emily. Taylor appeared to rely primarily upon physical or imagined spatial separation of the fractional quantity from the fractional complement. Her ability to attend to the fractional quantity, often identified on her left side, allowed her to productively rely upon her atypical understanding.

**Halving.** Taylor displayed a halving understanding which was similar to Lisa’s and Emily’s halving understanding. In this section, I begin by presenting one example that established Taylor’s similarity to Lisa and Emily. I then briefly discuss Taylor’s reflections on her understanding, which provided additional insight into potential reasons why the non-shaded area model might have been her predominant understanding of 1/2. I conclude by considering potential reasons why this understanding might not have been problematic for Taylor in the same way that it was for Lisa and Emily.

**Taylor’s indicators of halving.** Taylor’s representation and explanation for her drawing of 1/2 were consistent with Lisa’s and Emily’s halving understanding (see Figure 148).
### Question: Draw 1/2

<table>
<thead>
<tr>
<th></th>
<th>Taylor</th>
<th>Lisa</th>
<th>Emily</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="1" alt="Image" /></td>
<td><img src="2" alt="Image" /></td>
<td><img src="3" alt="Image" /></td>
<td><img src="4" alt="Image" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Similarity</th>
<th>Non-standard representation of 1/2</th>
<th>Non-standard representation of 1/2</th>
<th>Non-standard representation of 1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drew 1/2</td>
<td>Drew 1/2 without shading</td>
<td>Drew 1/2 without shading</td>
<td></td>
</tr>
</tbody>
</table>

Figure 148. Comparison of Taylor, Lisa, and Emily’s answers to the problem “draw one-half”.

When I asked Taylor to describe and draw 1/2, she talked about cutting something in half and then drew a shape and partitioned it into two parts.

Tutor: If you were to just close your eyes and imagine the fraction one-half, what would you sort of naturally think of?
Taylor: I think of cutting something in half.
Tutor: Ok.
Taylor: Two pieces of a whole, it’s a nice one to think about because it’s a pair. Um, which are easy to work with.
Tutor: Uh huh. So you were saying that you think of cutting something in half, how would you think about representing that if you were asked to draw something or show…
Taylor: So, (draws circle divides in two pieces).

Taylor’s understanding of 1/2 involved cutting something in half, which produced “two pieces of a whole.” Similar to Lisa and Emily, she focused on the splitting action and never identified one of the two pieces as the representation of 1/2. Taylor’s answers and explanations were considered to be consistent with an atypical “halving” understanding.

**Taylor’s reflections.** During the follow-up interview I explicitly asked Taylor to reflect on the similarities and differences between an area model of 1/2 and an area model of 1/2 with no shading. When comparing side-by-side representations, she thought that
the non-shaded area model was a better representation of 1/2. She explained that the splitting action itself created the fraction 1/2, and that the shading was extraneous.

Taylor: It’s like this physical process of like cutting a whole (points with pen down division line of non-shaded area model: see Figure 149) into two parts, you don’t need to shade in one side, to say that that’s like a half, you just know that you’ve cut this whole into two parts, and they’re halves.

Figure 149. Screen shots of Taylor describing the cutting of the shape into two parts.

They physical act of creating the partition embodied the meaning of 1/2 for Taylor. She explicitly rejected the idea that one of the two pieces needed to be designated. Not only was the shading considered extraneous to Taylor, but she explained that it was almost problematic, because it resulted in two different things (a shaded piece and a non-shaded piece) rather than two similar things.

Taylor: Again, it’s like two parts of a whole, they have to be equal they have to be balanced (palms up and makes a balancing scale movement with hands: see Figure 150) this (points to shaded area model) like it’s two different things, seeing them shaded, which is kinda weird to me. Like they’re two things that make up the same thing, they’re not two different things.

Figure 150. Taylor’s gesture showing the balance between the two parts, moving her hands up and down at alternating times.
Taylor argued that shading one of the pieces makes one of the two things different, which was not consistent with her understanding of 1/2 as two of the same things. For Taylor cutting the shape in half was how 1/2 was best represented and this sense of 1/2 was strengthened by the similarity of the two parts. Both Taylor’s initial drawing of 1/2 and her subsequent reflections on the shaded and non-shaded area models of 1/2 indicated that she was operating primarily with a halving understanding of the fraction 1/2.

**Taylor’s compensations.** Despite the centrality of halving to her understanding of the fraction 1/2, this understanding was never problematic for her during the interviews and (aside from her atypical representation of 1/2) never led to an incorrect answer. It is likely that this understanding was not problematic for Taylor primarily because she did not draw upon it when solving problems involving the fraction 1/2.

When asked to solve problems involving one-half, rather than relying upon one-half-specific strategies like Lisa and Emily, she treated 1/2 just like every other fraction. For example, when she was solving a card sort (fraction values: 0, 1/100, 1/2, 2/4, 2/3, 7/12, 9/10, 1, 8/8, 1 2/5, and 3/2), she did not identify 1/2 as a special quantity. Instead she said that she wanted to start immediately by looking for a common base (i.e. denominator). She identified 3/2 and 1/2 as the only two fractions with a common base. When I asked her if she wanted paper to help her solve this problem, she said, “what I’d do on paper is automatically make these all the same base.” By treating all fractions the same way, regardless of the specific fraction value, Taylor did not draw upon her halving knowledge during her attempt to complete the card sort. It is possible that her compartmentalizing of her problem solving techniques rendered any atypical halving understanding non-detrimental.

Not only did she not draw on her halving understand when solving problems, but it appeared that she did not ever connect her halving understanding to the fractional notation. She explained that her understanding of cutting a shape into two pieces never connected to the fractional notation: 1/2.

Taylor: I have this huge disconnect between the notation and what it is. So, this is something I never grasped, and my dad always found really frustrating because you’re just cutting a pie into slices, and this is the same thing as writing one over two. There is no way that those two things ever joined together which is really strange because I usually take pictures and associate them with numbers, but not with fractions.⁸⁰

It may be precisely because she did not integrate her understanding of one-half with the fractional notation that this understanding was not cued and could not become detrimental to her ability to solve problems involving 1/2.

**Conclusion.** Taylor demonstrated a surprising similarity to Lisa’s and Emily’s halving understanding. Although she preferred a halving understanding when

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⁸⁰ It is possible that her disconnection between the cutting scenario she described and the written notation “1/2” had to do with the fact that she never explicitly identified one of the two pieces as being the representation of 1/2, and instead thought of 1/2 as two pieces of a whole.
considering representations of 1/2, this atypical understanding was not detrimental to her the same way it was for Lisa and Emily. It is likely that Taylor compensated for her “halving” understanding simply by treating 1/2 as just another fraction and explicitly divorcing her drawn representation of 1/2 from her understanding of the notation 1/2. Her compensation for her halving understanding appeared to involve an isolation of her atypical halving understanding from the fractional notation 1/2.

**Uncoordinated fractional values.** Taylor, just like Emily and Lisa had difficulty coordinating the numerator and denominator values. Like Emily, Taylor seemed unable to draw upon a part-whole understanding of fractions to consider the size of the piece for basic comparisons like 1/6 and 1/8. In this section I provide an example that establishes Taylor’s similarity to Lisa and Emily. I then illustrate how Taylor compensated for her inability to coordinate the numerator and denominator. I conclude by considering Taylor’s reflections, which indicate that size was not a relevant dimension for her when considering the denominator values.

**Taylor’s indicator of uncoordinated fractional values.** Taylor did not coordinate the numerator and denominator values of fractions. Like Emily she did not use the denominator to help her reason about the size of the pieces. For example, when asked to compare area models of 2/3 and 2/4, she said she could not determine which was bigger because the denominators were different.

![3) plane](image)

Taylor: This I can’t tell you what’s bigger. Cause, I’m working in like a base of three and a base of four. And those are different.

Taylor was not able to use the denominator values to reason about the relative size of the pieces. This difficulty was similar to Emily’s atypical understanding, in which she did not relate the number of total pieces to the size of the piece.

**Taylor’s compensations.** Although Taylor did not answer the comparison problem presented above, she *was* capable of solving it. Taylor had developed a compensatory strategy for working with fractions that had unlike denominators. She always converted them to a common denominator so they were in the same “base” or “language”. For example, to solve the comparison problem of 1/3 and 1/5, she converted both to the common denominator or 15, and correctly determined that 3/15 was smaller.
Taylor: Um, I’d instantly do this, five over fifteen (writes 5/15), and three over fifteen (writes 3/15) and I know that this one is smaller. (points to 3/15).
Tutor: Uh huh.
Taylor: Because there are three little pieces and there are five little pieces, that’s less than that.

To solve the problem of 1/3 and 1/5, Taylor was not able to directly compare the denominators and infer that 1/3 would be larger. Instead she transformed the denominators of both fractions so that the numerators were directly comparable.

Taylor’s reflections. Taylor relied heavily upon converting fractions to a common denominator and did not conceptualize the denominator value in relationship to size. She referred to the denominator as being a “language.” Fractions using the same language could be compared, whereas fractions using different languages could not be compared. In this sense the denominator took on more of a qualitative rather than quantitative quality. As she reflected upon her solution process for the comparison of 1/3 and 1/5, I asked her if using something like fraction pieces would make the size more apparent. I pulled out a 1/6 and 1/3 piece and she said that she would be inclined to turn them into the same denominator, just as she had with the written notation. In this example, rather than attending to the size, she immediately converted 1/3 to 2/6.

<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript</th>
<th>Screen shots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tutor: You were just comparing one-third and one-fifth, and you went through common denominator. Is seeing these pieces next to each other and sort of thinking of what a third (points to 1/3) would mean with respect to like a static whole (points to 1 whole piece), or what a sixth (points to 1/6) would mean with respect to a static whole (points to 1 whole piece), does that help you reason through the respective sizes of these?</td>
<td>![Screen shot of fraction pieces]</td>
</tr>
</tbody>
</table>
Taylor: No, cause I’d want to do this (*puts two 1/6 pieces on top of the 1/3 piece*) And now they’re all the same.

Taylor seemed to be able to understand and evaluate fractional values only when the fractions had common denominators. She used this compensatory strategy even in contexts where the size of the pieces could be physically manipulated and compared.

**Conclusion.** Just like Lisa and Emily, Taylor was unable to coordinate the numerator and denominator values, particularly when comparing fractions. The size of the pieces was not a salient dimension for Taylor; instead she talked about the denominator in terms of a language or a base. Consequently, fractions with different denominator were impossible to reconcile until they were transformed to the same denominator ("speaking the same language"). This lack of coordination of the numerator and denominator values was not problematic for Taylor primarily because she was able to consistently apply a procedure to transform fractions to a common denominator. Although this was often not the most efficient strategy, it was the one that worked for her consistently.

**Conclusion.** Taylor demonstrated all three indicators of atypicality present in Lisa and Emily at the time of the posttest. That Taylor demonstrates these atypical understandings despite the fact that she was mastering advanced mathematics, suggests that these atypical understandings did not get refined and replaced. Unlike the fifth grade students who resolved their atypical understandings in short order, these atypical understandings appeared to remain core to Taylor’s conceptualization of fractional quantities. She grounded her understanding of fractions in terms of starting with a whole, which highlighted the fractional complement. She understood the fraction 1/2 as a halving or balance between two parts. Finally, she was only able to compare fractions with the same denominators. Unlike Lisa and Emily, these atypical understandings were not problematic for Taylor. She had developed compensatory strategies that allowed her to effectively navigate these atypical understandings. She used the physical or imagined spatial location to maintain focus on the focal fractional quantity. She operated on the fraction 1/2 in the same way as every other fraction, removing the possibility that her halving understanding could disrupt her solution process. She converted all fractions to a common denominator so it was unnecessary to coordinate the numerator and denominator values. Although Taylor displayed the three primary atypical understandings identified in Lisa and Emily, she had compensated so completely that these were never problematic for her.
Discussion and Conclusion

The comparison of the case of Lisa and Emily revealed some surprising similarities in their atypical understandings. Both Lisa and Emily attended to the fractional complement, represented 1/2 by halving a shape, and were unable to coordinate the numerator and denominator values when comparing fractions. These commonalities were all fundamentally concerned with the ways in which Lisa and Emily understood and represented fractional quantity. These similarities provided a preliminary understanding of how MLDs might manifest in similar ways for students with MLDs.

The atypical understandings displayed by Lisa and Emily differed substantially in the case of the fifth grade control students. Although the fifth grade students did demonstrate some similar indicators of atypicality, the identified underlying atypical understanding were different in nature. Unlike Lisa and Emily, these atypical understandings were not persistent or similarly problematic for the fifth grade students. The students that did display an atypical understanding resolved this over the course of the tutoring sessions. It is possible that one of the primary reasons that the atypical understandings were easily addressed is because the fifth grade students had a reasonably stable understanding of fractional quantity. As in Mary’s case, the student’s understanding of fractional quantity may serve a corrective function and assist the student in resolving the atypical understanding. Given the existence of these atypical understandings in the fifth grade control students, the atypical understandings should be understood to have roots in common misinterpretations and misconceptions that students encounter when first learning the topic domain. What makes these understandings atypical for the students with MLDs is the ways in which they become reified, over applied, and absolutely core to the student’s understanding of fractional quantity.

These three atypical understandings were also at the heart of Taylor’s understanding of fractions. Taylor understood the fractional quantity as removed from the whole, she represented 1/2 by halving, and was unable to coordinate the numerator and denominator values when comparing fractions. That these atypical understandings remained central for her was surprising given her advanced mathematical aptitude. Taylor employed specific compensatory strategies, which appeared to help her both navigate and circumvent the ways in which these atypical understandings could potentially become problematic. She grounded her understanding of fractions by always treating fractions in a consistent manner. She maintained her attention on the focal fractional quantity that was “pulled away” by consistently using space and locating the focal quantity on her left side. She treated every fraction the same way and did not attempt to employ more efficient but limited fraction specific strategies to operate on quantities. Finally, she grounded her understanding of fractions by operating from a common base. Each of these compensatory strategies contributed to her ability to maintain a stable understanding of a given fraction. Lisa and Emily did not exhibit any of these compensatory strategies in their interaction with fraction problems. It remains an open question whether similar compensatory approaches could be productive for Lisa and Emily. In the final chapter I consider the implications of these cross case comparisons for the development of screening measures and remediation approaches.
Chapter 6: Discussion

The purpose of this dissertation was to investigate mathematical learning disabilities through a detailed analysis of two students, Lisa and Emily, who were engaged in extensive attempts to learn basic fraction concepts. Both students had a history of unexplained low mathematics achievement and neither student benefited from the tutoring instruction that was effective for fifth grade students. This established that both students have an MLD that renders learning math extremely difficult. At the heart of the difficulties the students experienced were a collection of atypical understandings. These atypical understandings when considered as a whole suggest two overarching findings that characterize the experience of students with MLDs: (1) instability in conceptualization of fractional quantity and (2) inaccessibility of representations. These two findings are woven throughout this chapter and discussed with respect to (1) hypothesized cognitive causes for MLDs posited by prior research, (2) the Vygotskian perspective on disability, and (3) the implications for the identification and remediation of MLDs.

In this chapter I begin by briefly summarizing the results of this study and discuss the findings with respect to the two primary explanations given for the underlying cognitive cause of MLDs: insufficient working memory and an incapacity to process number. The number processing hypothesis, which involves a cognitive difference in the individual’s ability to represent and manipulate numerosities, not only aligns with the findings of this study but provides a plausible explanation for why specific atypical understandings may have emerged for each of the students. To consider the ramifications of these findings, I return to the Vygotskian perspective on disability, which productively frames these issues with respect to qualitative difference and accessibility. I then present the implications of both the case studies and the cross case comparisons for future development of screening tools and remediation approaches. Finally I conclude by discussing the contribution this dissertation makes to the understanding of MLDs.

Discussion of findings

This dissertation has provided a detailed view of how MLDs may manifest in a more complex mathematical domain. Unlike the number fact retrieval difficulties classically documented in students with MLDs, the difficulties experienced by students in this study were conceptual and representational in nature. A small collection of atypical understandings was identified for each student (see Figure 151), providing an explanation for almost all of the difficulties that the students experienced. These understandings were atypical in that they were unlike the documented difficulties that students often experience when learning fractions (see Lamon, 2007 for a review), and that they were often in conflict with the conventions used in the topic domain. For both Lisa and Emily, these atypical understandings were pervasive across the tutoring sessions and resistant to standard instruction. Although each student demonstrated some unique atypical understandings, there were surprising similarities between the students with MLDs (fractional complement, halving, uncoordinated fractional values; see Figure 151). These similarities suggest that there may be a common underlying cognitive cause, which was responsible for the atypical understandings the students demonstrated.
**Relationship to prior research on MLDs.** Two conflicting hypotheses have been posited for the neurological cause of MLDs with respect to student difficulties with basic number facts: a difference in domain-general working memory capacity (e.g., Geary, 2007), versus a difference in a domain-specific number processing ability (e.g., Butterworth & Reigosa, 2007). When discussing the relationship of the findings of this dissertation to prior research on MLDs, I use the term “MLDs” to refer to studies that purport to be studying mathematical learning disabilities. These studies often have used only a low achievement criterion to identify students with MLDs. As mentioned in the introductory chapter, this is problematic for a whole host of reasons. Nevertheless, it is important to discuss this research with respect to prior studies of MLDs. Therefore, I use the term “MLDs” in this section more loosely, to refer to research on MLDs, even when the researchers’ definitions explicitly do not meet my more stringent operational definition of MLDs. I do this to draw connections between the findings of this study and what others studies have found using a low achievement proxy for MLD. It is worth noting that the research has varying definitions of MLDs and this inconsistency is one of the primary issues hindering the field (Mazzocco, 2007).

In this section I explore the alignment of each of these hypotheses with respect to the findings of this study. Although a working memory hypothesis may provide a partial explanation for particular atypical understandings, it does not provide a comprehensive explanation, and does not explain why the variety of atypical understandings may have emerged. In contrast a number processing hypothesis aligns with the findings of this dissertation and provides a reasonable explanation for the kinds of atypical understandings that were evident in the cases of Lisa and Emily.

**The insufficient working memory hypothesis.** An insufficient working memory capacity has been proposed as the primary and underlying cognitive cause of MLDs (Geary, 2007; Swanson 2011). From, this perspective the number fact difficulties characteristic of MLDs are understood to result from the individual’s inability to attend to all the information involved in a basic number fact problem. The premise is that while
calculating the answer to the problem, the student cannot attend to the operands and the answer at the same time and therefore the “fact” is never stored in long term memory. From this perspective, the fundamental cause of MLDs is the capacity of the student’s working memory. In this section I consider the explanatory power of this working memory hypothesis for the atypical understandings common for both Lisa and Emily (i.e., indicators of atypicality). For the three common atypical understandings, I present how a working memory explanation could be applied and discuss the limitations of such an explanation. I conclude by considering how this hypothesis is further challenged by the data collected with Taylor, who appeared to demonstrate an exceptional working memory capacity while at the same time displaying all three indicators of atypicality.

Fractional complement. Lisa and Emily sometimes attended to the fractional complement rather than the focal fractional quantity. A working memory explanation might argue that the student did not have a sufficient working memory to maintain focus on the focal quantity and consequently the student sometimes attended to the fractional complement. Although on the surface this may appear like a plausible explanation, the data do not support this memory based explanation. For example, when Lisa was having difficulty comparing 7/12 and 1/2, she did not forget the quantities she should be comparing. She repeated the fraction values several times during the solution process. Instead, her tendency to focus on the fractional complement was due to her understanding of the shaded region as “taken”. Her comparison of the fractional amount involved a comparison of what was “left” (5 pieces for 7/12 and 6 pieces for 1/2). Similarly, Emily sometimes attended to the fractional complement because she understood the fraction to be represented by the part comprised of the fewer pieces. It is likely that her understanding of the representational entailments influenced her interpretations of fractional representations in terms of the complement. Lisa’s and Emily’s tendency to attend to the fractional complement did not appear to result from an inability to maintain the given fractional value in memory, but instead resulted from an alternative understanding of the representational conventions.

Halving. Lisa’s and Emily’s halving understanding involved representing the fraction 1/2 by partitioning a shape and omitting the shading. There are two possible working memory explanations for this atypical understanding. First, one could argue that the student’s working memory while representing 1/2 was over burdened and she was not able to self-monitor to realize she had not completed the representation. Alternatively, one could argue that during prior exposure to area models, the student’s working memory capacity was not sufficient to enable the student to attend to and store the representational conventions in long-term memory. Neither of these explanations is particularly satisfying. First, the cognitive demands involved in drawing a representation of 1/2 are relatively low, as compared to the other partitioning and representational activities the students performed during the sessions. Second, the students did shade the numerator quantity on subsequent problems, which suggests that they had stored and had access to this particular representational convention. The omission of the shading in the drawing of 1/2 was not due to a memory issue, but instead, the students understood 1/2 as the process of halving a shape. Rather than lacking particular knowledge of the representational conventions, the students had an alternative understanding of the
fraction $1/2$. Therefore, an insufficient working memory capacity does not provide a plausible explanation for this atypical understanding.

Uncoordinated fraction values. When comparing fractions, Lisa and Emily judged the magnitude of a fraction based on one value alone, either the numerator or denominator value. A working memory explanation might argue that the student’s working memory was insufficient to attend to and compare the four values (i.e., two numerators and two denominators) involved in the comparison. Lisa may have attended solely to the size of the pieces because she did not have sufficient resources to attend to the number of pieces, and Emily may have attended solely to the number of pieces because she did not have sufficient resources to attend to the size of the pieces. Admittedly when comparing fractions one must coordinate several different kinds of information at once. However, both students experienced difficulties even when the students were comparing fractions using area models and would have been able to directly compare the shaded regions without needing to hold the fractional values in memory. Lisa misinterpreted the meaning of the shading and Emily misinterpreted the meaning of the total number of pieces. Although a taxed working memory may provide a partial explanation, it does not appear to be the only factor at play for this atypical understanding.

Atypical understandings and sufficient working memory. The data collected with Taylor suggests that these atypical understandings can occur despite a well developed working memory. Taylor’s working memory capacity was demonstrated in her solution to the problem $8 \times 3 =$, “two eights is sixteen, and then I’m adding [another] eight. Sixteen plus what equals 20? Sixteen plus four. Now the eight, minus the four, and there is four left over. Twenty plus four, twenty-four.” In addition to the two operands, 8 and 3, she also maintained in working memory the number of eights she had added, along with four other intermediate sums (16, 20, 4, and 4), a total of seven values. Taylor’s solution to this problem is taken as evidence of a sufficient working memory capacity\textsuperscript{81}. However, despite her working memory capacity, she also displayed all three indicators of atypicality. This suggests that, for her, working memory was not the underlying cause of her MLD or the cause of the atypical understandings she demonstrated.

Conclusion. The insufficient working memory hypothesis does not align with the findings from this study. Neither fractional complement nor halving are sufficiently explained by working memory issues. Although the inability to coordinate the numerator and denominator values might have been the result of an overtaxed working memory, it does not provide a comprehensive explanation for the ways in which this

\textsuperscript{81} Not only did Taylor appear to have sufficient working memory capacity, her solution procedure for the problem $8 \times 3 =$ calls into question the assertion that working memory is the cause of number fact difficulties for students with MLD. Although Taylor did not retrieve the solution to $8 \times 3 =$ from memory, her lack of memory storage of this fact was not due to an insufficient working memory. Instead, her working memory was taxed because she was solving the problem in a much more cognitively demanding way. It is possible that working memory is implicated for basic arithmetic precisely because the students are solving problems in different ways, and in ways that place huge demands on their working memory.
atypical understanding occurred across representational types. In short, the working memory hypothesis explains only a small part of the documented atypical understandings and does not provide a compelling reason for why these atypical understandings might have arisen. For the students in this study, an insufficient working memory does not appear to be the underlying cause of their MLDs.

**The insufficient number processing hypothesis.** The second hypothesis for the underlying cause of MLDs involves cognitive differences in the number module for students with MLDs, which results in difficulty mentally representing or processing any form of numerical information (Butterworth, Varma & Laurillard, 2011). The number fact difficulties, characteristic of MLDs, are thought to be caused by an inability to mentally represent or relate various numerical quantities. Students with MLDs are understood to have difficulty creating a mental representation of magnitude for a number like “3” (Butterworth et al., 2011), and also difficulty directly perceiving three dots without counting each of the dots (i.e., subitizing; Moeller, Neuberger, Kaufmann, Landerl, & Neurk, 2009). The number processing issues are thought to contribute to the individual’s inability to learn and memorize number facts, because they are unable to mentally represent and relate the various quantities. In this section I consider the explanatory power of the number processing hypothesis for the findings from this study. The hypothesis that MLDs are due to a cognitive difference in numerical processing is aligned with the findings from this study and is a productive overarching lens for considering why the atypical understandings might have arisen for these students.

**Fractional quantity.** One overall finding of the case studies was that the atypical understandings displayed by the students with MLDs did not support a conceptual understanding of fractions as quantities. This is aligned with a number processing hypothesis, which would argue that students with MLDs would have difficulty processing both symbolic and non-symbolic representations of number. If a student with MLDs experiences difficulty in processing the numeral “4”, this has obvious ramifications for a student’s ability to process a quantity like “3/4”. Not only does a fraction involve two numbers, but the magnitude of the fraction is determined by the relationship between the two numbers rather than the numbers themselves. Similarly, if a student has difficulty processing an array of dots (e.g., □□□□□), consider the ramifications for a student’s attempts to work with an area model. Not only must the student attend to the number of shaded pieces, but the number of total pieces and the relative size of the pieces. The

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82 I use the terminology “cognitive difference”, but in the research these are exclusively referred to as “cognitive deficits”. Because a deficit model is incompatible with my framing I refer to these as differences.

83 The “number module” is understood to be a dedicated brain area and circuitry for processing numbers in both animals and humans, which is believed to exist in the interparietal sulcus (Butterworth et al., 2011; Dahaene, 1997).

84 In this body of research symbols are understood to be part of a symbol system (e.g., letters, numbers). “Non-symbolic” is used to refer to any representation that is not composed of numbers or letters. I use the terms “symbolic” and “non-symbolic” in a way consistent with this established usage.
atypical understandings can be thought of as manifestations of the student’s difficulty conceptualizing fractional quantity in both symbolic and non-symbolic forms.

Indeed, the three common atypical understandings identified for both Lisa and Emily allowed for instability in the understanding of fractional quantity. Consider the vacillating meanings of fractional quantity for each of the primary atypical understandings. “Fractional complement” allowed students to understand the focal fractional quantity as the amount “left” (e.g., understanding the area model of 3/4 as showing 1/4 left). “Halving” involved a representation of the fractional quantity (1/2) by a splitting or partitioning action. Finally, “uncoordinated fractional values” did not involve the coordination of the two values comprising the fraction, but judging the magnitude of the fraction based on one parameter alone, either the number of pieces of the size of the pieces. Whereas area models would typically represent fractional quantity in a consistent manner across problem contexts, the atypical understanding reflects the inconsistency in how the students with MLDs understood fractional quantity (see Figure 152). Rather than understanding fractional quantity as stable, fractions were understood through the atypical understandings as being variable amounts, in which 3/4 could become 1/4, 1/2 was created by dividing something in two, and a fraction’s value could be judged based on the number of piece or the size of the piece alone. These atypical understandings supported a shifting understanding of fractional quantity, suggesting that difficulties conceptualizing fractional quantity might be at the heart of MLDs.
Figure 152. Illustration of the typical understanding of fractional quantity (FQ) as one fixed amount, versus the varying and shifting understanding of fractional quantity associated with each of the primary atypical understandings.

*Fractional quantity in the cross case comparisons.* The number processing hypothesis also explains the findings of the cross case comparison. The primary factor differentiating the students with MLDs from the fifth grade controls was the ability to understand and manipulate fractional quantity. Fractional quantity appeared to serve a corrective function for the fifth grade controls students. In contrast, Taylor’s sophisticated compensatory strategies appeared to circumvent the need to conceptualize fractional quantity. These findings further suggest that ability to process and manipulate quantities is at the heart of the issues for students with MLDs.

Lisa’s and Emily’s understanding of fractional quantity can be contrasted with the findings from the fifth grade control students. Not only did the fifth grade control students appear to have a more sophisticated and stable understanding of fractional quantity, but when an atypical understanding surfaced it was precisely their understanding of fractional quantity that appeared to help them resolve their atypical understanding. For example, Mary was able to resolve her “taking” atypical understanding by drawing upon her understanding of the enactment of adding two
fractional quantities together to reconcile her inconsistent use of shading. Her ability to conceptualize and represent fractional quantities enabled her to identify potential ambiguity in her use of the shading for area models and resolve it completely. When the atypical understandings arose for the fifth graders, rather than being detrimental to their understanding of fractional quantity, their understanding of fractional quantity appeared to allow them to address their atypical understandings. Fractional quantity appeared to serve a corrective function for these students.

Fractional quantity did not appear to serve the same corrective function for Taylor. Although Taylor did compensate for the atypical understandings, unlike the fifth grade control students, she did not rely upon an understanding of fractional quantity to successfully navigate her atypical understandings. Recall that she similarly (1) understood the numerator as being taken away, (2) represented 1/2 by halving, and (3) was unable to coordinate both the size and number of pieces to compare fractions. Nevertheless she was able to successfully operate on fractional values. Her primary compensatory strategy was to understand and relate fractional quantities only after converting them to a common denominator. By converting the fractions in the same base, she could relate the fractions using the numerator values alone. Taylor’s ability to work with fractional quantities was not based on an ability to mentally conceptualize quantity. Instead, she had established a stable foundation based upon something other than fractional quantity. That her compensatory strategies circumvented fractional quantity provides additional evidence that number processing is the fundamental cognitive difference for students with MLDs.

Conclusion. The number processing hypothesis is aligned with the findings of this study. A cognitive difference in the way in which a student can mentally represent and manipulate quantities suggests that the student with MLDs might have difficulty developing a stable understanding of fractional quantity. Indeed ability to conceptualize fractional quantities appeared to be the primary differentiating factor between the control students and the students with MLDs. Whereas the fifth grade control students used fractional quantity to reconcile their atypical understandings when they occurred, this was not the case for the students with MLDs. Taylor’s compensatory strategies suggest that students with MLDs will likely continue to have difficulty conceptualizing basic fractional quantities and may need to ground themselves in an alternative way.

Conclusion. Of the two proposed hypotheses for the underlying cause of MLDs: insufficient working memory and number processing issues, only the latter aligned with the findings from this study. Working memory issues did not appear to be the cause of the difficulties that the students with MLDs experienced during the tutoring sessions and the atypical understandings were not explained by an underlying difference in working memory capacity. The findings of this study did align with the numerical processing hypothesis, which suggests that an inability to mentally represent and manipulate quantity is the underlying cause of MLDs. Difficulty processing quantity provides a plausible explanation for the collection of atypical understandings documented for the students with MLDs. While this hypothesis provides a productive overarching lens to consider how MLDs may manifest, this hypothesis is firmly situated from a deficit model, and consequently provides limited traction in understanding the implications of this dissertation research. In the following section, I discuss how the implications of the
number processing hypothesis can be further elaborated when contextualized by the Vygotskian approach to disability.

**MLDs in terms of difference and inaccessibility.** The Vygotskian approach to disability provides for a contextualization of these number processing differences that goes beyond understanding MLDs in terms of deficits and deficiencies. The Vygotskian perspective that framed this study conceptualized disability with respect to the themes of difference and accessibility. Vygotsky (1993) argued that students with disabilities are not simply less developed, but develop differently. An understanding of that individual necessitates identifying not what they lack, but what they have, appreciating that it might be qualitatively different than typically developing individuals. In addition, because human activity is understood to be primarily mediated, Vygotsky (1993) noted that for students with disabilities, culturally developed meditational tools might be inaccessible or incompatible with their natural biological development. Just as spoken language is incompatible with a Deaf child’s biological development, standard mathematical representations may be incompatible with the cognitive differences of the students with MLDs. This perspective situates the findings of this study with respect to understanding MLDs through the lens of difference and inaccessibility.

**Different conception of quantity.** The Vygotskian perspective on disability provides an elaboration of the number processing difficulties characteristic of students with MLDs. Vygotsky argues that individuals with a disability will develop different, because of that disability. The number processing hypothesis suggests that the core of MLDs is a cognitive difference in the student’s ability to represent and manipulate numerical information. Consider the implications of these two perspectives of MLDs together. If the underlying cause of a student MLDs involves an inability to process or manipulate quantities and students with disabilities to develop differently, it is likely that a student with an MLD will develop atypical ways of interacting with contexts that require the conceptualization of quantity. If the student experiences difficulties even when representing and manipulating whole number quantities, it is likely that when asked to represent and manipulate fractional quantities atypicalities may emerge. If a student is unable to mentally represent quantity it is possible that she will understand representations of fractional quantity as something other than quantity. This framing of number processing difficulties in terms of difference allows the implications of atypical understandings to be more fully explored. Each of the primary atypical understandings can be reconceptualized as the student representing something other than quantity.

Quantity is most clearly absent in the case of halving. Lisa, Emily and Taylor all represented the fraction 1/2 by drawing a shape and splitting it in two pieces. Rather than representing the quantity 1/2 in the drawing, it was represented by the partitioning action. All three students identified the partition line, itself, as the representation of 1/2. For most students the fraction 1/2 is the best understood and the most familiar fractional quantity (Hunting & Davis, 1991). It is consequential that when asked to represent this common fraction, that the students used an area model representation, but did not signify one of the two pieces as the quantity 1/2. This strongly suggests that quantity was not central to their understanding of fractions.

Similarly, fractional complement can be thought of as using shading to represent an action rather than a quantity. Lisa understood the shading of an area model as a
removal of pieces. She represented a removal action rather than a focal quantity. Similarly, Emily understood the construction of a representation as resulting in an answer. Her drawing of 3/4 might result in a fractional representation that she interpreted as 1/4. Although not representing a taking action in the same way as Lisa, her representative act produced something that was not the fractional quantity she was drawing. Therefore, she too, appeared to represent something other than fractional quantity.

Finally, uncoordinated fractional values can be conceptualized as relating two representations not in terms of fractional quantity, but in terms of a comparison of one-dimension of variance. Lisa focused exclusively on the size of the pieces, and judged fraction magnitude based on the fraction with the larger pieces. Emily focused exclusively on number, and judged the fraction based solely on the fraction comprised of the largest number of pieces. Although both size and number involve a comparison of magnitude, both students attended solely to one dimension. The comparison involved comparing properties of each fraction but not comparing fractional quantity. Therefore, uncoordinated fractional values involved comparisons of fractions based upon one dimension of variance rather than a comparison of fractional quantity.

The atypical understandings did not support a stable understanding of fractions, and it is possible that fractional quantities themselves were so intractable for these students that their attempts to use representations of fractions involved representing something other than quantity. Presuming that students with MLDs experience significant difficulty with number processing, in conjunction with a Vygotskian perspective on disability, provides a reasonable explanation for why the atypical understandings identified might have arisen. That all three students demonstrated some similarities in atypical understandings suggests that these atypical understandings are a natural consequence of this kind of underlying cognitive difference. Rather than conceptualized as deviant from the “norm”, these students are better understood as developing in accord with their cognitive difference.

**Incompatibility of representations.** The Vygotskian understanding of disability also provides a way of contextualizing the difficulties that students experienced working with mathematical representations. The number processing hypothesis argues that students with MLDs will experience difficulty processing non-symbolic forms of number. Rather than casting these difficulties as deficits, the Vygotskian approach would argue that for students with MLDs these meditational tools might be inaccessible. In this section I summarize the incompatibilities, which I argue renders the representation at least partially inaccessible for the students. For the sake of clarity and succinctness I will note the associated atypical understandings in parentheses as I discuss how the student understood the representation.

As demonstrated in the individual case studies the atypical understandings were incompatible and in conflict with the primary representations used during the tutoring sessions. Area models were incompatible with both Lisa and Emily’s atypical understandings. They both used shading inconsistently (fractional complement) and omitted shading when representation one-half (halving). In addition, Lisa had difficulty with partitioning (partitioning) and often ignored the importance of the size of the pieces (discrete set) whereas Emily had difficulties when pieces were perceptually similar to
quarters (quarters) and sometimes understood the area models as representing two distinct parts, rather than a part out of a whole (part-part). The collection of atypical understandings rendered area models inaccessible for both Lisa and Emily.

The fraction pieces were intended to support both the denominator and numerator concepts by making the size of the pieces and the number of pieces tactically accessible. However, both students attempted to fill the empty space (fractional complement) when interpreting fraction pieces. Lisa understood the symbolic label on the piece as a representation of the number in the set rather than the size of the piece (discrete set) whereas Emily ignored the size of the pieces (more pieces). Given the atypical understandings, the fraction pieces can be considered at least partially inaccessible for Lisa and Emily.

Finally, the area model squares with transparency overlays did not appear to be particularly useful for Lisa or Emily. In addition to all the previously discussed difficulties associated with drawn area models, the transparency overlay added another level of complexity. Findings suggest that this novel representational tool might have inadvertently supported both Lisa and Emily’s reliance upon halving and have contributed to Lisa’s partitioning issues. Therefore, this representational tool also interacted with the atypical understandings in ways that were not ultimately productive.

**Conclusion.** The canonical representational conventions for area models, fraction pieces, and transparency overlays were all in conflict with the atypical understandings displayed by each student. The mismatch of the atypical understandings and representational conventions suggests that these meditational tools were inaccessible for the students. This recontextualizes the number processing hypothesis by casting this inability to use non-symbolic representations of quantity, as an issue of accessibility for students with MLDs.

**Conclusion.** The detailed analysis indicated that the students with MLDs had atypical understandings of fractional quantities and that the primary representations used were inaccessible for the students. These findings were consistent with both the number processing hypothesis and a Vygotskian notion of disability. All the atypical understandings can be seen as examples of or a consequence of difficulties with numerical processing. However, these difficulties can be most productively conceptualized in terms of a difference rather than a deficient development. Students with MLDs had qualitatively different ways of approaching and understanding fractional quantity. In addition representations were inaccessible for them because they were incompatible with the student’s atypical understandings. These issues of fractional quantity and representational inaccessibility may have operated in tandem for the students with MLDs. The students were unable to address the ambiguity inherent in the representations because of their lack of understanding of fractional quantity, and their understanding of fractional quantity was not sufficiently developed because the representations intended to support fractional quantity were inaccessible. Therefore, at the core of the MLD is instability of fractional quantity and inaccessibility of representations, in the next section I consider the implications of these two findings for considering the design of screening measures and remediation approaches.
Implications

The findings from this study reveal novel avenues to explore with respect to subject identification and remediation. Rather than documenting deficits in the student’s performance (e.g., student cannot interpret fractions, student cannot compare fractions, student cannot add fractions), this study has identified differences in the form of atypical understandings. This has significant implications for the design and development of screening measures and remediation approaches. For screening measures, rather than documenting deficient performance, which cannot distinguish low achievement due to an MLD from low achievement due to other factors, the screening measure can document the presence of an atypicality, which may suggest the existence of an underlying MLD. Similarly, for remediation approaches, rather than providing more standard instruction in areas where the student is judged to be “deficient”, remediation can be tailored to address and build upon the atypical understandings. In this section I discuss the implications of these findings for future work. I begin by briefly describing how these findings are a first step towards the development of a screening measure and discuss future work involving a pilot screening measure and reflect on the limitations of this kind of MLD identification approach. I then consider the implications of these atypical understandings for remediation and present an overview of future work in which a remediation program was designed for Lisa to specifically build upon her atypical understandings and provide her with a more grounded understanding of fractional quantity.

Development of screening measures. This dissertation represents a first step toward the development of a screening measure that may be able to differentiate low achievement from MLDs. Whereas prior subject identification techniques have relied solely on low math achievement scores, which cannot differentiate low achievement due to an MLD from low achievement due to other factors, the findings from this dissertation present an alternative. The students with MLDs consistently demonstrated the three indicators of atypicality, whereas the fifth grade students did not. Therefore, it may be possible to design an assessment to document the presence of an atypicality, which may suggest the existence of an underlying MLD. A pilot version of a written assessment has been designed, which attempts to capture all three indicators of atypicality in written responses to questions (see Appendix 7).

The paper and pencil screening measure could be group administered and would take approximately 15 minutes to complete. This would allow teachers and/or researchers to quickly assess which students demonstrate atypicalities, which may be indicative of an MLD. Future work will involve an evaluation of this screening measure to consider whether it is possible to use the three indicators of atypicality to screen students for potential MLDs.

Although this represents a step toward accurate subject identification, it is only a modest step and several limitations should be acknowledged. First, this screening measure is not a subject classification tool; instead it is intended to help flag in a large sample the students who may have an MLD. Those students who display indicators of atypicality might have an underlying atypical understanding, and that underlying atypical understanding might be due to an MLD. For those low achieving students who demonstrate indicators of atypicality additional data should be collected to determine if the student does in fact have an underlying atypical understanding that is resistant to
standard instructional approaches. Second, this screening tool would be most effective when used with students who had received prior instruction in fractions. It is expected that some students when first learning the topic domain will experience misunderstandings or misinterpretations that are consistent with some of the atypical understandings demonstrated. Ensuring that students have had an adequate opportunity to resolve these potential misunderstandings will increase the effectiveness of the screening measure. Third, this collection of indicators may simply be characteristic of one particular subtype of MLD. Other cognitive differences may result in different kinds of atypicalities, which were not captured by the detailed analysis of Lisa’s and Emily’s cases. Although this screening measure marks a step forward in considering the identification of students with MLDs, because it involves looking for specific markers of MLDs, rather than simply identifying students by what the student lacks (e.g., performance deficits), much work remains to be done to empirically validate the screening measures for identifying students with MLDs.

**Development of remediation approaches.** The findings from this dissertation suggest novel approaches to explore for remediation of MLDs. Standard instructional approaches were ineffective for Lisa and Emily. Not only had classroom instruction been ineffective, but the one-on-one tutoring session focused on fractions did not address their problematic atypical understandings which were evident at the time of the posttest. In this section I first reframe the term “remediation” with respect to the Vygotskian perspective informing this study. Second, I describe a remediation approach that was designed and implemented with Lisa, which attempted to build upon her atypical understandings to provide a grounded understanding of fractional quantity. Third, I discuss the preliminary analysis of these sessions, which suggests the merit of approaching remediation in this way.

**Vygotskian approach to remediation.** The term remediation is often understood to have negative connotations and associated with “fixing” a perceived “defect” within the individual. Rather than shy away from this value laden term, I recast remediation from a perspective of mediation and disability. As previously discussed, Vygotsky (1986) argued that almost all human activity was mediated by culturally established tools. Any individual solving a math problem is relying upon culturally developed tools (e.g., pencil and paper, Hindu-arabic digits and symbols, place value, reading from left to right, etc.) all of which mediate the individual’s activity. Therefore, mathematical problem solving, by nature, can be understood as a mediated activity. Given that mediation is central to all individual’s mathematical activity and for students with MLDs the standard forms of cultural tools may be inaccessible, the natural conclusion is that those students may need alternative meditational tools. I understand remediation to be exactly that, re-remediating, or providing alternative meditational tools, which render the mathematical content accessible for those students. The goal of remediation is therefore to create tools, which maintain a fidelity to the canonical mathematical topic domain, and yet allow for the student to build upon and compensate for their atypical understandings.

**Design of remediation.** A pilot remediation was designed and implemented for Lisa which attempted to address the two primary issues identified in this dissertation: (1) instability of fractional quantity and (2) inaccessibility of standard representations of fractional quantity. It is worth noting it is not just a matter of finding the “right” standard
representation for fractions (e.g., number lines, Cuisenaire rods). Lisa not only applied her “taking” understanding to number lines but also had difficulty identifying the whole and unintentionally obscured the meaning of the denominator by plotting non-essential points (like 1/2). This suggests that not only would Lisa bring her atypical understandings to bear for any new representation of fractional quantity, but new representational issues might also arise. Therefore true re-mediation involves creating an alternative representation, one that presumes the existence of and is compatible with the student’s atypical understandings.

The goal of the remediation was to help Lisa develop a more stable understanding of fractional quantity in an alternative representational context, while specifically allowing her to build upon her “taking” understanding. The fundamental design question was how to allow her to “take” the numerator quantity, but then have her refocus on the taken amount? The remediation was designed around the idea of a scale to be used to weigh fractional amounts that were “taken” and placed on the scale. Not only did the scale allow for a place for the “taken” quantity to reside, the “scale” analogy had the benefits of allowing us to discuss the “weight” of a fraction, which served as a proxy for quantity.

The scale remediation involved representing fractional quantities on a bathroom scale, and then later on a two sided balance. The fractional quantities were represented by using pre-cut colored index cards, which were adhered to transparency “wholes” with re-positional glue. When the scale remediation was introduced during the tutoring sessions Lisa was asked to use the scale to represent a fractional amount weighed (see Figure 153). For example, to construct the fraction 1/3, Lisa would select the colored card pre-partitioned into 3 pieces, and then literally “take” one of the pieces and stick it to a transparency holder on the scale. Once the area model was constructed on the scale, Lisa was asked to evaluate the amount weighed and record that amount on the scale. She was then asked to draw a picture of the scale with the associated fractional “weight.” The goal of the remediation was to build upon her “taking” understanding but repurpose it to enable “taking” to (1) lead to a stable understanding of fractional quantity and (2) provide a context in which area models worked with, rather than against, her taking understanding.

Although this remediation was designed before the interview with Taylor, there were several parallels in the scale remediation and Taylor’s compensatory processes. First, in each case the student started with a whole and created a fractional amount from that whole. Second, the focal fractional quantity was given a constant place to reside. For Taylor, she mentally “held” the quantity on her left side, for Lisa, she placed the quantity on the scale. Finally, as in Taylor’s case, this scale remediation did not address the fact that the fractional complement exists as part of the construction process (e.g., after 1/3 is placed on the scale, the fractional complement 2/3 remains on the original transparency). Instead, this process attempted to circumvent that potential issue by having Lisa focus on the amount weighed and providing a consistent way of attending to the focal fractional value.

Lisa was required to select both the appropriately partitioned card and the appropriately partitioned transparency and set both of those up. This was never problematic for her.
After Lisa became comfortable with using the scale to represent and interpret fractional quantities, the scale model was extended to a two-sided scale (represented by envelopes) and Lisa was asked to compare the fractional quantities. As depicted in Figure 154, Lisa was still asked to record the fractional values on the paper when she drew and compared the amounts on the scales. The scale remediation was used to represent fractions, compare fractions, create equivalent fractions, and perform fraction operations.

This remediation involved designing a non-standard representational system that would enable Lisa to focus on the focal fractional quantity. In addition, there was an explicit attempt to increase the accessibility of area model representations. By requiring that Lisa draw the fractional amounts on the scale, the meaning of the shading in area models could potentially be recontextualized with reference to the amount weighed. This remediation involved designing an alternative representational tool, which built upon Lisa’s atypical understanding and attempted to provide a more stable understanding of fractional quantity and increase the accessibility of area model representations.
Preliminary analysis of remediation approach. Preliminary analysis of these remediation sessions indicates that this approach holds promise. The 5 weekly hour-long remediation sessions were conducted 13 months after the completion of the original tutoring sessions. Because this remediation was based upon Lisa’s “taking” understanding it was essential to determine if her “taking” understanding still was evident. Despite the lapse in time, during the first remediation session, Lisa continued to rely upon her “taking” understanding. When working with area models, she identified the shaded pieces as “taken” and the non-shaded pieces as “left.” She also judged 1/2 to be larger than 7/12 because 7/12 would have “5 pieces left.” Therefore, at the commencement of the remediation sessions, Lisa still relied upon her “taking” understanding, and this understanding continued to be problematic for her.

Preliminary analysis of the remediation sessions indicated that the scale model was effective in orientating Lisa to fractional quantity. Over the course of the remediation she shifted her focus from the non-shaded pieces (“pieces left”) to the shaded pieces (“pieces weighed”; see Figure 155).

![Figure 155. Recreation of Lisa’s labeling of area models before an after the using the scale remediation.](image)

By the time of the final remediation session she had shifted her understanding of the shading of the area model and correctly determined that 7/12 was larger than 1/2, justifying her answer with area models and the scale analogy.

Lisa: If we were looking at the scale, this (pointing to the 7/12 drawing) would be heavier, like the seven pieces, although they are thinner, I feel like there would be more of them, thus making it heavier.

Although the scale was not available during her solution to this question, she not only used the scale to justify her answer, but was able to reason through her interpretation of her drawn area models focusing on the relative “weight” of the two different fractions.
Preliminary analysis indicated that the scale model was effective in orientating Lisa to fractional quantity. An alternative representational form building upon her atypical understanding enabled her to make sense of fractional quantity in a way the standard representations had failed to do. Moreover, she was able to use the scale model to generate equivalent fractions and fraction operations, something that she had been unable to accomplish with area models alone. Future work will involve a systematic and detailed analysis of these remediation sessions to investigate how this model interacted with Lisa’s atypical understanding. The preliminary analysis suggests that a detailed analysis of the student’s atypical understandings is a necessary and productive first step for the design of effective remediation.

**Conclusion.** The findings from this dissertation allow for innovative approaches for the identification and remediation of students with MLDs. The atypicalities identified suggest that students with MLDs may be able to be identified by qualitative differences in their understanding of number. This potentially allows for a differentiation of low achievement due to an MLD from low achievement due to other factors. Additionally, these findings suggest that effective remediation may involve the design of alternative instructional tools that build on the productive parts of student’s atypical understandings. If the atypical understandings will persist despite explicit instruction, remediation must consider how to work with and around those atypical understandings. Preliminary analysis of the remediation sessions has suggested that alternative instructional tools were effective for Lisa. This would suggest that students with MLDs do not simply need more instruction, but they may in fact need different kinds of instruction, one that accounts for their atypical understandings and acknowledges the inaccessibility of particular representations.

**Conclusion**

This dissertation presents a novel approach to the study of MLDs, which has the potential to identify and address the challenges students with MLDs face. While prior research on MLDs has focused on low achieving students’ difficulties with basic number facts, this dissertation extends and reconceptualizes the study of MLDs. In contrast with previous research, I carefully selected students who demonstrated an evident MLD, explored the mathematically rich topic domain of fractions, and conducted a comprehensive analysis of the videotaped data to uncover the reasons that the students with MLDs did not learn.

Atypical understandings were identified for two students with MLDs. These atypical understandings were unlike those documented in prior research on rational number and were not problematic for the fifth grade control students. The atypical understandings reoccurred across the sessions, were resistant to standard instructional approaches, and proved to be highly consequential for the student’s ability to understand more complex fraction concepts. When considered together the atypical understandings caused the standard representations of fractions to be inaccessible and undermined their ability to conceptualize and manipulate fractional quantities. The findings of this dissertation are consistent with prior research that suggests that the underlying cognitive cause of MLDs may be an inability to represent and manipulate quantities. However, the atypical understandings identified suggest that the difficulties that students with MLDs
encounter are far more complex and nuanced than memory problems with basic number facts.

The consistency documented between the students with MLDs suggests that this alternative methodological approach is a productive path toward identifying characteristic features of MLDs. It is no longer sufficient to use low achievement as a proxy for MLD. Future research on MLDs must focus on and improve the subject identification procedures. This dissertation provides a model for employing a response-to-intervention model to differentiate MLDs from low achievement due to other factors. Future research should continue to explore more complex mathematical domains where procedural, conceptual, and representational issues are at play. The ultimate goal of research on MLDs must be to identify students for the purpose of addressing their underlying challenges through re-mediations that are specifically intended to make mathematics more accessible.

In this dissertation I made a conscious move away from the deficit model, which is the predominant frame used to understand learning disabilities. By shifting the analytic focus away from documenting performance deficits and towards identifying qualitative difference, the students with MLDs were not defined by what they lacked. The identification of qualitative differences is a necessary first step towards the development of valid subject identification methods and effective remediation approaches. Qualitative differences will be the key to developing screening measures that identify specific characteristics of MLDs. Understanding the differences in students with MLDs is a critical first step toward understanding why standard instruction has consistently failed them, and developing alternative instructional approaches, which acknowledge their difference and makes the mathematics accessible. That the students have failed to learn the most basic mathematics points to a failure to acknowledge and accommodate that student’s cognitive difference, rather than a deficit within the individual.
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Appendix 1. Middle School and High School Student Initial Interview

Academic Background:
- What is your favorite subject in school?
- What do you like about it?
- What is your least favorite subject?
- What don’t you like about it?

(If hasn’t already talked about math: “What do you think about math? What do you like about it? What don’t you like about it?”)

Nature of the student’s difficulty:
- What are you working on in math class right now?
- Can you give me an example?
- What about learning and/or doing math was hard for you? Can you give an example?
- What about learning and/or doing math was easy for you? Can you give an example?

Effort:
- Do you get a lot of homework in math?
- When do you do your homework?
- Do you tend to do all your homework and turn it in?

Resources Questions:
- If you get stuck on a problem, what do you do?
- Have you ever gone to math tutoring before? Where?

Language Fluency:
- What language do you tend to speak at home?
Appendix 2. Adult interview questions.

General Background Info:
- Did you grow up / go to high school in the area?
- Did you take any time off between high school and college?
- How long have you been at [name of community college]?
- Did you go to any other colleges?

Math Background Information:
- What was your experience learning math?
- What age were you when you started having difficulty with math?
- What is the highest level of math you have taken? (algebra, trig, pre-calc, calculus, college calculus, etc)
- Have you ever had learning disability testing done by a professional?
  If so:
    o What test(s) did they use?
    o What was did the professional tell you about the results of the testing?

Experience with Math:
- What about learning and/or doing math was hard for you? Can you give an example?
- What about learning and/or doing math was easy for you? Can you give an example?
- What was your experience in math classes like?
- What kind of teaching, assignments, experiences was helpful for you?
- What kind of teaching, assignments, experiences were not helpful and/or frustrating?
- Is there a time when you remember something just clicking and making sense?

Specific Math Topic Questions:
- Did you have problems memorizing number facts (5+4=9 or 6x7=42)?
- What strategies did you use to memorize number facts?
- Did you have difficulty working with fractions? If so, what made fractions difficult for you?
- Did you have difficulty learning algebra (solving for variables, graphing, etc)? If so, what made algebra difficult for you?

Effort:
Do you get a lot of homework in math?
When do you do your homework?
Do you tend to do all your homework and turn it in?

Resources Questions:
If you get stuck on a problem, what do you do?
Have you ever gone to math tutoring before? Where?

**Language Fluency:**
- What language do you tend to speak at home?
Appendix 3. Pretest/Posttest Protocol

Problem 1:
Prompt: Draw/write the fraction ___. Can you think of another way to draw or write it?
Fraction values: 1/2, 3/4, 2/5, 1 3/8, 5/4
Stop rule: Stop administering task when student has more than 2 incorrect answers or two “I don’t know” answers.

Problem 2:
Prompt: Circle all the pictures that you think are the same as ___. Explain why you choose (answer). Explain why you didn’t choose (answer).
Fraction values: 1/2, 1/3, 4/5, 5/3
Problem 3:
Description: This activity consists of the student and successively halving two pieces of paper. I intentionally cut my paper in a different way so that we end up with triangular and rectangular halves and quarters to compare.

Illustration:

Prompts:
Can you find 2 pieces of paper the same size? I will take one and you can take one. Can you fold your paper in half? I am going to fold my paper like this (diagonally). Now, let’s both cut along the fold. (both cut paper in half) If we imagine that this is cake and I had this whole cake and you had that whole cake would we have the same amount or would one of us have more? Now let’s each give away one of our pieces of cake.
a) Does one of us have more or do we have the same amount?

b) What fraction name would you give that piece?

Let’s fold our piece of paper in half again. Let’s cut along the fold. Now if we give away another piece of cake.

c) Does one of us have more or do we have the same amount?

d) What fraction name would you give that piece?

e) If we put our pieces together how much of a whole cake do we have?

Problem 4:
Prompt: For each problem, circle the picture that shows the bigger amount, if they are equal, write equal. How did you decide that [those were equal / this one was bigger]?

Problem 5:
Prompt: Which is more ___ or ____? How do you know?

a) 1/6 or 1/8
b) 2/7 or 2/5
c) 2/8 or 5/8
d) 3/6 or 5/10
e) 3/2 or 7/9
f) 4/5 or 2/3

g) 2/6 or 1/2

h) 2/5 or 3/10

i) 3/7 or 2/3

j) 5/8 or 2/3

Stop rule: Stop administering task when student has more than 2 incorrect answers or two “I don’t know” answers.

Problem 6:
Prompts: Can you come up with a fraction equal to ____? Can you come up with another fraction equal to ____? How many fractions are there equal to ____?

a) 1/2

b) 1/3

c) 2/5

d) 8/12

Stop rule: Stop administering task when student has more than 2 incorrect answers or two “I don’t know” answers.

Problem 7:
Prompt: How would you solve the problem (write down following problems one at a time)

a) 1/3 + 1/3 =

b) 3/4 - 1/4 =

c) 3/5 + 4/5 =

d) 1/2 + 1/4 =

e) 3/4 - 1/8 =

f) 1/3 + 1/2 =

g) 2/5 + 2/3 =

Stop rule: Stop administering task when student has more than 2 incorrect answers or two “I don’t know” answers.
Appendix 4. Overview of challenges administered during the tutoring sessions.

Session #1

**Overview:** In this session the student and I will explore the meaning of the numerator and denominator. We will use foam fraction pieces (both circular and rectangular) to discuss the meaning of the denominator, meaning of the numerator and the importance of identifying the whole.

<table>
<thead>
<tr>
<th>Challenges</th>
<th>Representations / Tools</th>
<th>Rationale</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>How would you explain to someone else how they labeled each of these fraction pieces? I took the 1/8 pieces out of this set, could you make a replacement 1/8 piece? How many thirds does it take to make a whole?</td>
<td>Circular fraction pieces with the 1/8 pieces removed.</td>
</tr>
<tr>
<td>2</td>
<td>I copied some fraction pieces, but I had them upside down. Can you help me figure out what name each should be?</td>
<td>Circular fraction pieces</td>
</tr>
<tr>
<td>3</td>
<td>How can you show what 3/4 would look like with fraction pieces? How would you explain how to make 3/4 to someone else? If someone said this (two 1/10 and one 1/8 pieces) was 3/10, would you agree or disagree?</td>
<td>Circular fraction pieces</td>
</tr>
<tr>
<td>4</td>
<td>This rectangular set of fraction pieces is missing the one whole piece. Figure out how to make a replacement one whole piece.</td>
<td>Rectangular fraction pieces with 1 whole and 1/2 pieces removed</td>
</tr>
<tr>
<td>5</td>
<td>For each set of cards put the fractions in order from least to greatest.</td>
<td>Card set 1: 1/10, 1/3, 1/100 Card set 2: 2/5, 2/7, 2/16 Card set 3: 3/7, 6/7, 18/7</td>
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<td></td>
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</tr>
<tr>
<td><strong>6</strong></td>
<td>How many ways can you find to make 1/2 using other fractions pieces? Come up with your own way to record all the ways you find that work.</td>
<td>Rectangular and circular fraction pieces.</td>
</tr>
<tr>
<td><strong>7</strong></td>
<td>Without using the fraction pieces, can you make each of the fractions close to, but NOT equal to 1/2?</td>
<td>Written problems: __/10, __/18, <strong>/100, <strong>/7, 3/</strong><em>, 8/</em></strong></td>
</tr>
<tr>
<td><strong>8</strong></td>
<td>Put the fractions on these cards in order from least to greatest.</td>
<td>Fraction values 0, 1/100, 1, 8/8, 3/2, 9/10, 7/12, 1 2/5, 1/2, 2/4</td>
</tr>
<tr>
<td><strong>Journal</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>- What is something that made sense to you today?</td>
<td>Journal</td>
</tr>
<tr>
<td></td>
<td>- How would you explain it to yourself if you forgot? / explain to someone else?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Write down an example</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Is there something that is still confusing – what is confusing about it?</td>
<td></td>
</tr>
<tr>
<td><strong>Game</strong></td>
<td>Bingo game in which students are asked to create equivalent fraction relationships (e.g., make 1/2 using only eighth pieces)</td>
<td>[Diagram of Bingo game board]</td>
</tr>
<tr>
<td><strong>9</strong></td>
<td>Are these fraction pieces the same or different amounts?</td>
<td>[Image of fraction pieces]</td>
</tr>
</tbody>
</table>
Session #2

Overview: The goal of this session is to explore how the student chooses to pictorially represent fractions. Given the student’s representation I want to highlight issues of ambiguity and convention through a discussion in which the student chooses how to represent a fraction and I interpret the drawn representation (3.2). The remainder of the session involves the student interpreting area models for fractions. The central concepts are (1) meaning of the denominator, (2) meaning of the numerator, and (3) importance of considering the size of the whole.

<table>
<thead>
<tr>
<th>Challenges</th>
<th>Representations / Tools</th>
<th>Rationale</th>
</tr>
</thead>
<tbody>
<tr>
<td>Journal review: Does this still make sense? Can you explain / give me an example?</td>
<td>Student’s previous journal entries</td>
<td>Student is asked to reflect on whether their journal entry makes sense to them, and reminds the student of what was covered during the last session.</td>
</tr>
<tr>
<td>Review use of fraction pieces: Can you come up with a fraction and show me how you would make it with the fraction pieces? (if the student does not volunteer a non-unit fraction, ask the student to construct one, e.g., 2/5, 7/8).</td>
<td>Circular fractions pieces</td>
<td>Revisit how to use the fraction pieces to make or interpret a given fractional amount.</td>
</tr>
<tr>
<td>1 Without writing down any numbers can, you draw a picture of 3/4 so that other people would know that it’s a picture of 3/4?</td>
<td>Fraction cards values: 1/2, 2/3, 3/5, 4/4, 1/6, 7/10, 3/8, 2/5, ¼, 5/3, 1½, 4/7, 7/4, 5/8, 4/9</td>
<td>Determine how the student chooses to draw 3/4. Is this a canonical area model?</td>
</tr>
<tr>
<td>2 Help me guess 5 fractions on the back of the cards. (Student draws card with a fraction value, writes it down, draws a picture, and then hides fraction value so I can “guess”. I follow student-generated rules to “guess” the fraction the student drew.) (note: I intentionally misinterpret rules to highlight ambiguity and help the student further clarify their understanding for fraction interpretation).</td>
<td>3 pieces of 4”x8” paper.</td>
<td>The goal of this problem is to raise issues of ambiguity in the representation and come to some agreement with the student about the conventions we should follow for drawing/interpreting pictures of fractions.</td>
</tr>
<tr>
<td>3 Imagine each of these pieces of paper is a cake. Four people want to share the cake evenly. How many different ways can you split the cake into 4 pieces? (student can cut/fold paper)</td>
<td>Paper from previous challenge</td>
<td>This problem sets up problem 4, where the student is asked to compare two different shaped 1/4 pieces.</td>
</tr>
<tr>
<td>4 If you are given 1 piece of each of the</td>
<td>3 pieces of 4”x8” paper.</td>
<td>Can the student reason</td>
</tr>
<tr>
<td>cakes, are they the same amount of cake or different amounts?</td>
<td>partitioned in several ways about different shaped pieces using part-whole understanding, or does the student rely on a figurative understanding (which piece looks bigger)</td>
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<tr>
<td>Show how you would convince someone of your answer.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Game: Board game in which students must determine which piece of cake is bigger the orange or the green (taken from Armstrong and Larson)</td>
<td>This game is similar to Armstrong &amp; Larson’s study, but the goal is to use these problems to generate a list of important features to consider when comparing pictures like these (i.e., number of partitions, number of shaded pieces, and size of the whole).</td>
<td></td>
</tr>
<tr>
<td>If these were both cakes how can you share each cake between 3 people? Cut out 1 piece of cake, are they the same amounts or different amounts?</td>
<td>Paper (paper 3x6 and 6x9) This is a similar paper folding activity, but focused on the importance of the whole. How does the student reason about fractional pieces when the cakes themselves are different sizes?</td>
<td></td>
</tr>
<tr>
<td>Journal: Write down something that made sense, how would you explain it to yourself if you forgot? What things were challenging – what do you want to work on more?</td>
<td>Journal Determine what is most salient for the student and how he/she is able to communicate what he/she understands or doesn’t understand.</td>
<td></td>
</tr>
</tbody>
</table>

**Session #3**

**Overview:** During this session transformation of area models (to generate equivalent fractions) is explored first through the context of fair sharing (partitioning entire cakes). This fair sharing idea is then extended to area models of fractions with unequal partitions. Ultimately the goal is for the student to become comfortable partitioning area models and begin to explore the idea of equivalent fractions.

<table>
<thead>
<tr>
<th>Challenges</th>
<th>Representations / Tools</th>
<th>Rationale</th>
</tr>
</thead>
</table>
| Journal review: Does this still make sense? Can you explain / give me an example? | Student’s previous journal entries | Student is asked to reflect on whether their journal entry makes sense to them, and reminds the student of what was covered during the
<table>
<thead>
<tr>
<th>Warm up:</th>
<th>Student is asked to both construct and interpret an area model representation to remind students of how we use pictures of fractions.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Last time we were talking about using pictures to show fractions. - How would you draw a picture of 3/5 - (draw 5/6) what fraction name would you give this drawing?</td>
<td>This problem uses fair sharing ideas to motivate the necessity of even sized pieces.</td>
</tr>
<tr>
<td>1. Someone has started cutting up these cakes to share, but they haven’t finished the job. Finish cutting up the cakes so that they can be shared evenly. How many people could evenly share? How much would each person get?</td>
<td>The goal is to transition from the previous problem (importance of fair sharing) to thinking about how to identify a fractional piece of an area model.</td>
</tr>
<tr>
<td>2. Here are some cakes. Some have chocolate frosting and some have vanilla frosting. Your challenge is to figure out how much has chocolate frosting.</td>
<td>This problem begins to introduce the concept of transforming an area model representation. Can the student parse the new representation as both 2/3 &amp; 4/6? Introduce equivalent fraction term.</td>
</tr>
<tr>
<td>3. Using a whole sheet of paper, draw 2/3 of a cake. Figure out how you can cut the cake again so you still have even pieces. What has change and what has stayed the same?</td>
<td>Fraction area model squares with transparency overlays (illustration below)</td>
</tr>
<tr>
<td>4. To complete this challenge you must come up with 5 equivalent fraction pairs. Draw a card and figure out the fraction of cake shaded. Draw a transparency and use it to help you cut the cake into smaller pieces. Decide how you want to record your progress.</td>
<td>This problem uses the transparency overlay to enable an area model to be partitioned and then un-partitioned, to help the student “see” both fractions (e.g., 2/3 and 4/6).</td>
</tr>
</tbody>
</table>
### Session #4

**Overview:** The goal of this session is to explore fraction operations using the representational tools in previous sessions (fraction pieces and area models). This session is structured in a way that a challenging problem is posed (\(1/2 + 1/3 = \)) which then motivates the exploration of easier problems (e.g., \(1/3 + 1/3 = \)) to help the student build their understanding.

<table>
<thead>
<tr>
<th>Challenges</th>
<th>Representations / Tools</th>
<th>Rationale</th>
</tr>
</thead>
<tbody>
<tr>
<td>Journal review:</td>
<td>Student’s previous journal entries</td>
<td>Student is asked to reflect on whether their journal entry makes sense to them, and</td>
</tr>
<tr>
<td>Does this still make sense? Can you explain / give me an example?</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
reminds the student of what was covered during the last session.

<table>
<thead>
<tr>
<th>Warm up: Last time we were talking about equivalent fractions, what do you remember about equivalent fractions? What fraction is this(1/3)? Can we divide it into 6ths? How? What fraction is this (1/2)? Can we divide it into 10ths? How?</th>
<th>Student is asked to construct two equivalent fractions using area models to remind the student how area models can be used to generate equivalent fraction values.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fraction pieces</td>
<td>Fraction pieces</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>How much do we have there? (1/2 and 1/3 piece) Can you write it as an addition problem? If student solves algorithmically ask the student to represent the problem with fraction pieces and area models. If the student doesn’t know how to start the problem, go onto the sub-problems below. Sub-problems 1/3 + 1/3 = (problem with the denominators constant) 1/2 + 1/4 = (problem where 1 denominator multiplies into other using ½ ) 1/6 + 1/3 = (problem where 1 denominator multiplies into other) Questions: - How would you solve this problem using fraction pieces? - How would you solve this problem using pictures? If you look back at these problems does that help you solve the original problem 1/2+1/3? - What other ways can we write ½? - What other ways can we write 1/3?</td>
</tr>
<tr>
<td>Fraction pieces</td>
<td>Fraction pieces</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Solve: 2/5 + 3/10= using fraction pieces and area models. Sub-problems: 2/5+3/5= and 2/10+3/10=</td>
</tr>
<tr>
<td>Fraction pieces</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Solve: 7/8 – 3/4 = using fraction pieces and area models.</td>
</tr>
<tr>
<td>Fraction pieces</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>( \frac{2}{3} + \frac{1}{4} = )</td>
<td>This is the first problem for which the common denominator (12) is not available for the fraction pieces. Students will need to transition away from using the pieces to relying on area models to solve this problem.</td>
</tr>
<tr>
<td>5</td>
<td>Game: Fraction land – students solve fraction addition/subtraction problems and reduce the fraction to figure out the next move.</td>
<td>This is intended to provide students practice with adding/subtracting fractions and thinking about equivalence as part of reducing fractions.</td>
</tr>
<tr>
<td></td>
<td>Journal: Write down something that made sense, how would you explain it to yourself if you forgot? What things were challenging – what do you want to work on more?</td>
<td>Journal Determine what is most salient for the student and how he/she is able to communicate what he/she understands or doesn’t understand.</td>
</tr>
</tbody>
</table>
## Appendix 5. Pretest/Posttest Scoring Rubric With an Example of Problem Scoring

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>Problem</th>
<th>Stop Rule</th>
<th>Scoring</th>
<th>Example Problem Scoring</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 – Draw the fraction</td>
<td>Draw/write the fraction ___. Can you think of another way to draw or write it?</td>
<td>If student gets more than 2 wrong.</td>
<td><strong>Total possible: 15 points</strong>&lt;br&gt;For each iteration a-e: 1 point for correctly producing:&lt;br&gt;- drawing of the fraction (area model, number line, etc of 1/2)&lt;br&gt;- drawing of an equivalent fraction (area model, number line, etc of 2/4).&lt;br&gt;- a numerical rep of an equivalent fraction (2/4, 7/14)&lt;br&gt;(note: decimal and percentage representations are not considered here because they were not the focus of the tutoring sessions)</td>
<td>Subtotal: 3/3&lt;br&gt;(draws both correct fraction and equivalent fraction and produces numerical equivalent fraction)</td>
</tr>
<tr>
<td>2 – Fraction Interpretation</td>
<td>Can you circle all the pictures that you think are the same as ____?</td>
<td>If the number of distracters identified is greater than or equal to the number of correct answers.</td>
<td><strong>Total possible: 15 points</strong>&lt;br&gt;Total negative points: 12&lt;br&gt;1 point for every correct area model or discrete set answer circled. (note: number lines answers were not calculated into student’s total score because they were not the focus of the tutoring sessions)&lt;br&gt;-1 for every distracter answer circled.</td>
<td>Total Points:&lt;br&gt;Correct: 4 points&lt;br&gt;Distracter: -1 points&lt;br&gt;Subtotal: 3/4&lt;br&gt;(continue administration of this task for b-d)</td>
</tr>
<tr>
<td>3 – Paper halving Folding/cutting activity</td>
<td>This activity consists of the student and successively halving two pieces of paper. I intentionally cut</td>
<td>(no stopping rule)</td>
<td><strong>Total possible: 5 points</strong>&lt;br&gt;1 point if the student says the triangular and rectangular halves are the same size.</td>
<td>(cut piece of paper in half)&lt;br&gt;T: So… if we look at both of our cakes, we think of these as cakes, I guess. If we look at both of ours, do we have the same amount or different amounts?</td>
</tr>
</tbody>
</table>
my paper in a different way so that we end up with triangular and rectangular halves and quarters to compare.

(half paper)
a) Does one of us have more or do we have the same amount?

(b) What fraction name would you give that piece?

(half paper)
c) Does one of us have more or do we have the same amount?

d) What fraction name would you give that piece?

e) If we put our pieces together how much of a whole cake do we have?

1 point if the student can produce the fraction name for the rectangular half piece. (1/2)

1 point if the student knows the quarter pieces are the same size.

1 point if the student can produce the fraction name for the rectangular quarter piece. (1/4)

1 point if the student can determine how much of the whole cake is produced by the addition of the triangular and rectangular quarter pieces. (1/2)
<table>
<thead>
<tr>
<th>4 – Area model comparisons</th>
<th>Which is bigger or are they equal? Circle the bigger amount or write equal.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(no stop rule)</td>
<td>Total Possible: 8 points</td>
</tr>
<tr>
<td>1 point for each correct answer.</td>
<td>1 point for each correct justification.</td>
</tr>
<tr>
<td>Total: 3/5</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>5 – Fraction Comparison</th>
<th>Which is more ___ or ____? a) 1/6 or 1/8 b) 2/7 or 2/5 c) 2/8 or 5/8 d) 3/6 or 5/10 e) 3/2 or 7/9 f) 4/5 or 2/3 g) 2/6 or 1/2 h) 2/5 or 3/10 i) 3/7 or 2/3 j) 5/8 or 2/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>After 2 incorrect answers.</td>
<td>Total Possible: 10 points</td>
</tr>
<tr>
<td>1 point for each correct answer and explanation.</td>
<td></td>
</tr>
<tr>
<td>Total: 2/10</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>6 – Equivalent fractions</th>
<th>Can you come up with a fraction equal to ___? a) 1/2 b) 1/3 c) 2/5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Can you come up with another fraction equal to ___?</td>
<td>Total Possible: 8 points</td>
</tr>
<tr>
<td>How many fractions are there equal to ___? a) 1/2 b) 1/3 c) 2/5</td>
<td>1 point for each equivalent fraction (up to 2) they can generate for the given fraction.</td>
</tr>
<tr>
<td>If student is not able to produce an equivalent fraction.</td>
<td>Total: 2/8</td>
</tr>
</tbody>
</table>

| 1 point – correct comparison of halves. | 1 point – correct naming of half pieces. |
| 1 point – correct comparison of quarters. | 0 points – incorrect naming of quarter pieces. |
| 0 points – incorrect addition of quarter pieces. |                                                      |
## 7 – Fraction Operations

<table>
<thead>
<tr>
<th></th>
<th>How would you solve the problem:</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>$\frac{1}{3} + \frac{1}{3}$ =</td>
</tr>
<tr>
<td>b)</td>
<td>$\frac{3}{4} - \frac{1}{4}$ =</td>
</tr>
<tr>
<td>c)</td>
<td>$\frac{3}{5} + \frac{4}{5}$ =</td>
</tr>
<tr>
<td>d)</td>
<td>$\frac{1}{2} + \frac{1}{4}$ =</td>
</tr>
<tr>
<td>e)</td>
<td>$\frac{3}{4} - \frac{1}{8}$ =</td>
</tr>
<tr>
<td>f)</td>
<td>$\frac{1}{3} + \frac{1}{2}$ =</td>
</tr>
<tr>
<td>g)</td>
<td>$\frac{2}{5} + \frac{2}{3}$ =</td>
</tr>
</tbody>
</table>

If the student gets 2 or more answers incorrect.

**Total Possible: 7 points**

1 point for each correct answer and explanation.

<table>
<thead>
<tr>
<th></th>
<th>Total: 3/7</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>$\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$</td>
</tr>
<tr>
<td>b)</td>
<td>$\frac{3}{4} - \frac{1}{4} = \frac{2}{4}$ (incorrect)</td>
</tr>
<tr>
<td>c)</td>
<td>$\frac{3}{5} + \frac{4}{5} = \frac{7}{5}$</td>
</tr>
<tr>
<td>d)</td>
<td>$\frac{1}{2} + \frac{1}{4} = \frac{2}{4}$ (incorrect)</td>
</tr>
<tr>
<td>e)</td>
<td>$\frac{3}{4} - \frac{1}{8} = \frac{5}{8}$ (incorrect)</td>
</tr>
<tr>
<td>f)</td>
<td>$\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$</td>
</tr>
<tr>
<td>g)</td>
<td>$\frac{2}{5} + \frac{2}{3} = \frac{12}{15} = \frac{4}{5}$</td>
</tr>
</tbody>
</table>

Stopped administration because of 2 incorrect answers.

Total: 3/7
Appendix 6. Transcript conventions.

<table>
<thead>
<tr>
<th>Punctuation</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>…</td>
<td>Voice trailing off</td>
</tr>
<tr>
<td>-</td>
<td>Interruption</td>
</tr>
<tr>
<td>(parenthesis)</td>
<td>Non-verbal action</td>
</tr>
<tr>
<td><em>italics</em></td>
<td>Word(s) stressed by speaker</td>
</tr>
<tr>
<td>!</td>
<td>Exclamation</td>
</tr>
<tr>
<td>?</td>
<td>Raised intonation</td>
</tr>
<tr>
<td>.</td>
<td>Falling intonation</td>
</tr>
<tr>
<td>,</td>
<td>Pause</td>
</tr>
<tr>
<td></td>
<td>Note: all pauses of significant duration are noted in non-verbal notation, e.g., “(pause 7 seconds)”</td>
</tr>
</tbody>
</table>
## Appendix 7. Written assessment screening measure intended to identify indicators of atypicality.

<table>
<thead>
<tr>
<th>Question</th>
<th>Indicator of atypicality</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1</strong> Draw a picture of 1/2.</td>
<td>Drawing 1/2 with no shading (halving).</td>
</tr>
<tr>
<td>Draw another way to show 1/2.</td>
<td>Drawing 3/5 by shading 2 pieces (fractional complement)</td>
</tr>
<tr>
<td>Draw a picture of 3/5.</td>
<td></td>
</tr>
<tr>
<td>Draw a picture of .</td>
<td></td>
</tr>
<tr>
<td><strong>2</strong> Circle all the pictures that you think show 1/2?</td>
<td>Circling 1/2 with no shading (halving).</td>
</tr>
<tr>
<td>![Image of various fraction representations]</td>
<td></td>
</tr>
<tr>
<td><strong>3</strong> Which is more 1/6 or 1/8? Explain your answer.</td>
<td>Comparing fractions based on 1 value (uncoordinated fractional value)</td>
</tr>
<tr>
<td>Which is more 2/8 or 5/8? Explain your answer.</td>
<td>Answering that 2/8 is more than 5/8 because there is more “left” (fractional complement)</td>
</tr>
<tr>
<td>![Image of various fraction representations]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Lisa drew this picture.</td>
</tr>
<tr>
<td>---</td>
<td>------------------------</td>
</tr>
<tr>
<td></td>
<td>What fraction does this drawing show?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Which is bigger? Explain your answer</th>
<th>Judging magnitude based on non-shaded pieces (fractional complement)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Which is bigger? Explain your answer</td>
<td>Or</td>
</tr>
<tr>
<td></td>
<td>Judging magnitude based on non-shaded pieces (fractional complement)</td>
<td>Comparing fractions based on 1 value (uncoordinated fractional value)</td>
</tr>
<tr>
<td></td>
<td>(A)</td>
<td>(B)</td>
</tr>
<tr>
<td></td>
<td>(C) They are equal</td>
<td>(C) They are equal</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Solve the problem 1/2 + 1/4 = using pictures.</th>
<th>Drawing 1/2 without shading (halving)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Solving the problem 1/2 + 1/4 = using pictures.</td>
<td>Drawing 1/2 without shading (halving)</td>
</tr>
<tr>
<td></td>
<td>(A)</td>
<td>(B)</td>
</tr>
<tr>
<td></td>
<td>(C) They are equal</td>
<td>(C) They are equal</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>What fraction does the picture show? ___</th>
<th>Identifying the fraction using the empty space (fractional complement)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What fraction does the picture show? ___</td>
<td>Identifying the fraction using the empty space (fractional complement)</td>
</tr>
</tbody>
</table>