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ON ELLIPTIC CURVES WITH AN ISOGENY OF DEGREE 7

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Abstract. We show that if $E$ is an elliptic curve over $\mathbb{Q}$ with a $\mathbb{Q}$-rational isogeny of degree 7, then the image of the 7-adic Galois representation attached to $E$ is as large as allowed by the isogeny, except for the curves with complex multiplication by $\mathbb{Q}(\sqrt{-7})$.

The analogous result with 7 replaced by a prime $p > 7$ was proved by the first author in [8]. The present case $p = 7$ has additional interesting complications. We show that any exceptions correspond to the rational points on a certain curve of genus 12. We then use the method of Chabauty to show that the exceptions are exactly the curves with complex multiplication.

As a by-product of one of the key steps in our proof, we determine exactly when there exist elliptic curves over an arbitrary field $k$ of characteristic not 7 with a $k$-rational isogeny of degree 7 and a specified Galois action on the kernel of the isogeny, and we give a parametric description of such curves.

1. Introduction

Suppose that $E$ is an elliptic curve defined over $\mathbb{Q}$, $p$ is a rational prime, $T_p(E)$ is the $p$-adic Tate module of $E$, and $G_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then there is a natural homomorphism $\rho_{E,p} : G_\mathbb{Q} \to \text{Aut}_{\mathbb{Z}_p}(T_p(E))$ giving the action of $G_\mathbb{Q}$ on $T_p(E)$. Since $T_p(E)$ is a free $\mathbb{Z}_p$-module of rank 2, $\text{Aut}_{\mathbb{Z}_p}(T_p(E))$ can be identified (non-canonically) with $\text{GL}_2(\mathbb{Z}_p)$. Serre [20] showed that if $E$ does not have complex multiplication (CM), then $\rho_{E,p}(G_\mathbb{Q})$ has finite index in $\text{Aut}(T_p(E))$, and $\rho_{E,p}$ is surjective for all but finitely many primes $p$.

Suppose now that $E$ has an isogeny of degree $p$ that is defined over $\mathbb{Q}$ (in other words, $E$ has a $\mathbb{Q}$-rational $p$-isogeny). Mazur [15] showed that $p$ is then in the finite set $\{2, 3, 5, 7, 11, 13, 17, 19, 37, 43, 67, 163\}$, and if further $E$ is non-CM, then $p \in \{2, 3, 5, 7, 11, 13, 17, 37\}$. The kernel of the isogeny is a $\mathbb{Q}$-rational subgroup $\Psi$ of $E(\overline{\mathbb{Q}})$ of order $p$. Since $\text{Aut}(\Psi) \cong \mathbb{F}_p^\times$, the action of $G_\mathbb{Q}$ on $\Psi$ is given by a homomorphism $\psi : G_\mathbb{Q} \to \mathbb{F}_p^\times$. We refer to $\psi$ as the character of the isogeny.

The isogeny and the corresponding character $\psi$ put an obvious constraint on the image of the map $\rho_{E,p}$. In particular, $\rho_{E,p}$ cannot be surjective. If $E$ has complex multiplication, then the additional endomorphisms of $E$ put another constraint on the image of $\rho_{E,p}$. In that case, $\rho_{E,p}(G_\mathbb{Q})$ is a $p$-adic Lie group of dimension 2. We wish to understand whether these are the only constraints, or equivalently, whether there are any non-CM elliptic curves over $\mathbb{Q}$ for which $\rho_{E,p}(G_\mathbb{Q})$ does not contain
a Sylow pro-$p$ subgroup of $\text{Aut}_{\mathbb{Z}_p} (T_p(E))$. This is the motivation for the following definition.

**Definition 1.1.** We will say that a curve $E$ over $\mathbb{Q}$ is $p$-exceptional if $E$ has an isogeny of degree $p$ defined over $\mathbb{Q}$ and the image of $\rho_{E,p}$ does not contain a Sylow pro-$p$ subgroup of $\text{Aut}_{\mathbb{Z}_p} (T_p(E))$.

In other words, a curve $E$ with a $\mathbb{Q}$-rational $p$-isogeny is $p$-exceptional if the index of $\rho_{E,p}(G_\mathbb{Q})$ in $\text{Aut}_{\mathbb{Z}_p} (T_p(E))$ is divisible by $p$. If $E$ is not $p$-exceptional, then the image of $\rho_{E,p}$ is as large as it could be, given the existence of a $\mathbb{Q}$-isogeny for $E$ of degree $p$ with character $\psi$. Note that if $E$ is $p$-exceptional, then so is any curve $\mathbb{Q}$-isogenous to $E$, and if $p > 2$ then so is any quadratic twist of $E$ (see [8]). If $E$ has CM, then as remarked above, $E$ is $p$-exceptional.

If $p < 7$, then non-CM $p$-exceptional curves exist in abundance. Concerning the case $p = 5$, Theorem 2 in [8] describes completely the possible images of $\rho_{E,5}$. Its index cannot be divisible by $5^2$ and is divisible by 5 if and only if $E$ has a cyclic $\mathbb{Q}$-isogeny of degree $5^2$ or two independent $\mathbb{Q}$-isogenies of degree 5.

In this paper we prove that the only 7-exceptional elliptic curves are the elliptic curves with complex multiplication by $\mathbb{Q}(\sqrt{-7})$. The method of proof is as follows. We begin with two results from [8]. Let $\omega : G_{\mathbb{Q}} \to \mathbb{F}_p^\times$ denote the cyclotomic character giving the action of $G_{\mathbb{Q}}$ on $\mu_p$.

**Theorem 1.2** ([8], Theorem 1). Suppose $p \geq 7$ and $E$ is an elliptic curve over $\mathbb{Q}$ with a $\mathbb{Q}$-isogeny of degree $p$. Let $\psi$ denote the character of the isogeny. If $\psi^4 \neq \omega^2$, then $E$ is not $p$-exceptional.

**Proposition 1.3** ([8], Remark 4.2.1). Suppose $p > 7$ and $E$ is an elliptic curve over $\mathbb{Q}$ with a $\mathbb{Q}$-isogeny of degree $p$ and character $\psi$. If $\psi^4 = \omega^2$, then $E$ has CM.

These two results combine to show that there are no non-CM $p$-exceptional curves when $p > 7$. However, Proposition 1.3 fails when $p = 7$ (as can be seen by considering the family of elliptic curves with a $\mathbb{Q}$-isogeny of degree 7 and character $\psi = \omega^5$; see [4]).

Suppose $E$ is a 7-exceptional elliptic curve. By Theorem 1.2 if $\psi$ is the character of the isogeny, then $\psi^4 = \omega^2$. It follows that the $\mathbb{F}_p^\times$-valued character $\psi \omega^{-5}$ has order dividing 2. Thus, replacing $E$ by a quadratic twist if necessary, we may assume that $\psi = \omega^5$. In §2 we show that if $E$ is 7-exceptional, then the ratio of the minimal discriminants of $E$ and its 7-isogenous curve is of the form $7^s w^7$ with $w \in \mathbb{Q}$ and $s \in \mathbb{Z}$.

An explicit description of the family $\{B_v\}$ of elliptic curves over $\mathbb{Q}$ with a $\mathbb{Q}$-rational 7-isogeny with character $\psi = \omega^5$ follows from the results in §3. The curve $E$ is isomorphic over $\mathbb{Q}$ to $B_v$ for some $v \in \mathbb{Q}$. In Corollary 4.6 we show that for $v \in \mathbb{Q}$, the ratio of the minimal discriminants of $B_v$ and its 7-isogenous curve is $7^{\pm 6} g(v)^6$, where $g(v) := (v^3 - 2v^2 - v + 1)/(v^3 - v^2 - 2v + 1)$. Thus the exceptional $E$’s correspond to $\mathbb{Q}$-rational points on the genus 12 curves $C_j : w^7 = 7^j g(v)$. If $7 \nmid j$, then it turns out that $C_j$ has no rational points over $\mathbb{Q}$, and hence none over $\mathbb{Q}$. Thus, the question is reduced to finding the $\mathbb{Q}$-rational points on the curve

$$C_0 : w^7 = (v^3 - 2v^2 - v + 1)/(v^3 - v^2 - 2v + 1).$$

In §7 we use the method of Chabauty to show that

$$C_0(\mathbb{Q}) = \{(0,1), (1,1), (\infty,1), (2,-1), (1/2,-1), (-1,-1)\},$$
and it follows that the only 7-exceptional elliptic curves $E$ are the curves with $j(E) = -15^3$ or $255^3$, i.e., the curves with complex multiplication by $\mathbb{Q}(\sqrt{-7})$.

Now suppose that $k$ is a field of characteristic different from 7, that $E$ is an elliptic curve defined over $k$, and that $E$ has a $k$-rational isogeny of degree 7. Then the kernel of the isogeny is a $k$-rational subgroup $\Psi$ of $E(k^s)$ of order 7, where $k^s$ denotes a fixed separable closure of $k$. Let $G_k = \text{Gal}(k^s/k)$. Since $\text{Aut}(\Psi) \cong \mathbb{F}_7^2$, the action of $G_k$ on $\Psi$ is given by a homomorphism $\psi : G_k \rightarrow \mathbb{F}_7^2$. Again, we refer to $\psi$ as the character of the isogeny. For example, the character $\psi$ is trivial if and only if $\Psi \subset E(k)$.

Now let $\psi : G_k \rightarrow \mathbb{F}_7^2$ be a fixed homomorphism. In [3] we describe all elliptic curves defined over $k$ that have a $k$-rational 7-isogeny with character $\psi$. We will give explicit formulas for a family of elliptic curves $\{A_v\}$, where $v$ varies over an explicit Zariski open subset of the projective line $\mathbb{P}^1$, such that

- for every $v \in k$, the elliptic curve $A_v$ has a $k$-rational 7-isogeny and character $\psi$,
- if $E$ is an elliptic curve over $k$ with a $k$-rational 7-isogeny and character $\psi$, then $E$ is isomorphic over $k$ to $A_v$ for some $v \in k$.

See Theorems 3.6 and 3.10 for more precise statements. One consequence of this explicit description is that if $k \neq \mathbb{F}_2$, and if $\psi$ is any character, then there is an elliptic curve over $k$ that has a $k$-rational 7-isogeny with character $\psi$ (see Corollary 3.12). Note that these explicit families are of a different nature and are constructed differently from the explicit families of elliptic curves with a given mod $N$ representation that were constructed earlier by the second and third authors of this paper.

The route we took to exactly determine $C_0(\mathbb{Q})$ is interesting in itself. (See Remark 6.8 and the appendix for more information.) By Faltings’ proof of the Mordell Conjecture, $|C_0(\mathbb{Q})|$ is finite. An action of the group $S_3$ on $C_0$ shows that $|C_0(\mathbb{Q})|$ is divisible by 6. A descent argument (see [2]) shows that the rank of the Jacobian $J$ of $C_0$ is at most 6. In fact, the 6 known points generate a subgroup of $J(\mathbb{Q})$ of rank 4 and $J$ is $\mathbb{Q}$-isogenous to the square of the Jacobian of a genus 6 curve $D$ defined over $\mathbb{Q}$. Thus the rank of $J(\mathbb{Q})$ must be either 4 or 6.

A Chabauty argument at the prime 2 then gives that $|C_0(\mathbb{Q})|$ is either 6 or 12. At our request, Kiran Kedlaya and Jennifer Balakrishnan, with help from Joseph Wetherell, computed 2-adic Coleman integrals that we hoped would directly rule out the case $|C_0(\mathbb{Q})| = 12$. However, the computed dimensions worked out in such a way that one would need to show that the rank of $J(\mathbb{Q})$ were 4 to use this method to show $|C_0(\mathbb{Q})| \neq 12$.

This gave motivation to determine the rank of $J(\mathbb{Q})$, i.e., to determine the parity of the rank of the Jacobian $\text{Jac}(D)$ of the genus 6 curve $D$. Using data and information provided by Balakrishnan, Sutherland, Kedlaya and the fourth author of this paper, Michael Rubinstein performed computations that gave convincing evidence that the analytic rank of $\text{Jac}(D)$ is 3, so one expects $\text{rank}(J(\mathbb{Q})) = 6$.

This gave motivation to find additional generators of $J(\mathbb{Q})$, which is done at the end of [6] below. We used these additional points to finish the proof that $|C_0(\mathbb{Q})| = 6$, using a Chabauty argument at the prime 5 (see [7]).

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for computations that pointed the way to the correct path to take to achieve Theorem 5.4. We made use of PARI/GP [16] and Magma [1]. The fourth author thanks Wojciech Gajda and the University of Poznań for an invitation to spend a week there, where some of the work on this paper was done.

2. The image of $\rho_{E,p}$

We assume throughout this section that $E$ is an elliptic curve defined over $\mathbb{Q}$ that has a $\mathbb{Q}$-isogeny of prime degree $p \geq 7$. Let $\Psi$ denote the kernel of the isogeny and let $\Phi = E[p]/\Psi$. The actions of $G_\mathbb{Q}$ on $\Psi$ and $\Phi$ are given by characters $\psi, \varphi : G_\mathbb{Q} \to \mathbb{F}_p^\times$, respectively. Let $\omega : G_\mathbb{Q} \to \mathbb{F}_p^\times$ denote the cyclotomic character giving the action of $G_\mathbb{Q}$ on $\mu_p$. That is, if $\sigma \in G_\mathbb{Q}$ and $\zeta_p$ is a primitive $p$-th root of unity in $\mathbb{Q}$, then $\zeta_p^\sigma = \zeta_p^{\omega(\sigma)}$. Since $\psi \varphi = \omega$, which is an odd character, we have $\psi \neq \varphi$. Hence $\Psi \neq \Phi$ as $G_\mathbb{Q}$-modules.

Let $K_\infty = \mathbb{Q}(E[p^{\infty}])$ and let $\rho_{E,p} : G_\mathbb{Q} \to \text{Aut}_{\mathbb{Z}_p}(T_p(E))$ be the homomorphism giving the action of $G_\mathbb{Q}$ on the Tate module $T_p(E)$. Then $\rho_{E,p}$ factors through the Galois group $G := \text{Gal}(K_\infty/\mathbb{Q})$, and defines an injective homomorphism from $G$ into $\text{Aut}_{\mathbb{Z}_p}(T_p(E))$. To simplify the discussion, we identify $G$ with its image in $\text{Aut}_{\mathbb{Z}_p}(T_p(E))$.

Recall the definition of $p$-exceptional from [1] We would like to know whether $\rho_{E,p}(G_\mathbb{Q})$ contains a Sylow pro-$p$ subgroup of $\text{Aut}(T_p(E))$, or equivalently, whether the index $[\text{Aut}_{\mathbb{Z}_p}(T_p(E)) : \rho_{E,p}(G_\mathbb{Q})]$ is prime to $p$.

If we choose a basis for $T_p(E)$ to identify $\text{Aut}(T_p(E))$ with $\text{GL}_2(\mathbb{Z}_p)$, then the Sylow pro-$p$ subgroups of $\text{Aut}(T_p(E))$ are identified with the conjugates of

$$
\begin{pmatrix}
1 + p\mathbb{Z}_p & \mathbb{Z}_p \\
p\mathbb{Z}_p & 1 + p\mathbb{Z}_p
\end{pmatrix}.
$$

There are $p + 1$ such conjugates, all containing $I_2 + pM_2(\mathbb{Z}_p)$.

Let $K = \mathbb{Q}(\Psi, \Phi)$, the fixed field for the intersections of the kernels of $\psi$ and $\varphi$. Then $K$ is an abelian extension of $\mathbb{Q}$ and $[\mathbb{Q}(E[p]) : K]$ is 1 or $p$. Since $[K : \mathbb{Q}]$ divides $(p - 1)^2$, it is not divisible by $p$. Let

$$S := \text{Gal}(K_\infty/K).$$

Then $S$ is a normal subgroup of $G$ and is the (unique) Sylow pro-$p$ subgroup of $G$.

Let

$$E' := E/\Psi.$$

Thus $E'$ has a $\mathbb{Q}$-isogeny of degree $p$ with kernel $\Phi$ and character $\varphi$.

Remark 2.1. The assumption that $p \geq 7$ implies that an elliptic curve over $\mathbb{Q}$ cannot have a $G_\mathbb{Q}$-invariant cyclic subgroup of order $p^2$. This is due to Mazur [15] for most primes, Ligozat [13] or Kenku [10] for $p = 7$, and Kenku [9] for $p = 13$. It follows that an elliptic curve over $\mathbb{Q}$ cannot have two independent $\mathbb{Q}$-isogenies of degree $p \geq 7$. To see this, suppose to the contrary that $E[p] \cong \Psi \times \Phi$. Let $C = \{P \in E : pP \in \Psi\} \subset E[p^2]$, which is obviously $G_\mathbb{Q}$-invariant. Then $C/\Psi$ is a $G_\mathbb{Q}$-invariant cyclic subgroup of $E'$ of order $p^2$, which is not possible. It follows that both of the fields $\mathbb{Q}(E[p])$ and $\mathbb{Q}(E'[p])$ are cyclic extensions of $K$ of degree $p$.

Proposition 2.2 ([3], Proposition 4.3.2). The curve $E$ is $p$-exceptional if and only if $\mathbb{Q}(E[p]) = \mathbb{Q}(E'[p])$. 
The proof of this proposition in [8] is based on the Burnside Basis Theorem. The Frattini quotient of a Sylow pro-$p$ subgroup $S_p$ of $\text{Aut}(T_p(E))$ containing $S$ has $\mathbb{F}_p$-dimension 3. It turns out that the image of $S$ in that Frattini quotient has $\mathbb{F}_p$-dimension 2 if $Q(E[p]) = Q(E'[p])$, and $\mathbb{F}_p$-dimension 3 if those two fields are distinct. In the latter case, one can find a set of topological generators for $S_p$ in $S$, which then implies that $S = S_p$.

The following lemma will provide one way to verify that $Q(E[p]) \neq Q(E'[p])$.

Lemma 2.3. Assume that $\ell$ is a prime and that $\ell \neq p$. Then the ramification degree of $\ell$ in at least one of the two extensions $Q(E[p])/Q$ and $Q(E'[p])/Q$ is prime to $p$.

Proof. Assume that the ramification degree of $\ell$ in $Q(E[p])/Q$ is divisible by $p$. This implies that $E$ has bad reduction at $\ell$. If $E$ had potentially good reduction at $\ell$, then the only primes that could divide the ramification degree for $\ell$ in $Q(E[p])/Q$ are 2 and 3 (see for example the proof of Corollary 2(a) to Theorem 2 of [21]). This contradicts the assumption that $p \geq 7$. Hence, $E$ must have multiplicative or potentially multiplicative reduction at $\ell$. It follows from Proposition 23(b) of [20] that $E$ has multiplicative reduction over $K$ at all primes above $\ell$.

Fix a prime $\lambda$ of $K_\infty$ lying above $\ell$, and let $I$ be the inertia group for $\lambda$ in $S$. The Tate parametrisation shows that for every $n$, the group $E[p^n]^I$ contains a cyclic subgroup of order $p^n$. Since $I$ fixes $K = Q(\Psi, \Phi)$, we have $\Psi \subseteq E[p]^I$. On the other hand, since the ramification degree of $\ell$ in $Q(E[p])/Q$ is divisible by $p$, $I$ acts nontrivially on $E[p]$, and so we have $E[p]^I = \Psi$. Hence $E[p^n]^I$ is cyclic of order $p^n$ for every $n$. In particular, multiplication by $p$ gives an $I$-equivariant isomorphism $E[p^n]^I/\Psi \rightarrow \Psi$. Therefore, we have $I$-equivariant isomorphisms

$$E'[p] = (E/\Psi)[p] \cong E[p]^I/\Psi \times E[p]/\Psi \cong \Psi \times \Phi.$$ 

Since $I$ acts trivially on both $\Phi$ and $\Psi$, it acts trivially on $E'[p]$, so $Q(E'[p])/K$ is unramified above $\ell$. Since $[K : Q]$ is prime to $p$, it follows that the ramification degree of $\ell$ in $Q(E'[p])/Q$ is prime to $p$. $\square$

Remark 2.4. Lemma 2.3 can also be proved by studying how the Tate period for $E$ over $Q_\ell$ changes under the isogeny $E \rightarrow E'$. The advantage of the above proof is that it also could be applied to the $p$-adic representations attached to modular forms, under suitable assumptions.

Lemma 2.5. If $\psi \varphi^{-1}$ has order 2, then $p \equiv 3 \pmod{4}$ and $\psi \varphi^{-1} = \omega^{(p-1)/2}$.

Proof. Since $\psi \varphi = \omega$ and $\varphi$ has order dividing $p - 1$, we have

$$(\psi \varphi^{-1})^{p-1} = (\omega \varphi^{-2})^{p-1} = \omega^{p-1}.$$ 

Since $\omega$ has order $p - 1$, we see that $(\psi \varphi^{-1})^{(p-1)/2}$ is nontrivial. If $\psi \varphi^{-1}$ is quadratic, we conclude that $(p-1)/2$ is odd and hence that $(\psi \varphi^{-1})^{(p-1)/2} = \psi \varphi^{-1}$. The lemma follows. $\square$
Proposition 2.6. Suppose that $\psi_{\varphi^{-1}}$ has order 2. Then $E$ is $p$-exceptional if and only if for every prime $\ell \neq p$, the ramification degrees of $\ell$ in $Q(E[p])/Q$ and $Q(E'[p])/Q$ are both prime to $p$.

Proof. Let $L = Q(E[p])$ and $L' = Q(E'[p])$. By Remark 2.1, $L$ and $L'$ are cyclic extensions of $K$ of degree $p$.

Suppose first that $E$ is $p$-exceptional, and $\ell \neq p$. By Lemma 2.3 the ramification degree of $\ell$ in at least one of $L/Q$ and $L'/Q$ is prime to $p$. But by Proposition 2.2 we have $L = L'$, so the ramification degrees of $\ell$ in $L/Q$ and $L'/Q$ must both be prime to $p$.

Now suppose that for every prime $\ell \neq p$, the ramification degrees of $\ell$ in $L/Q$ and $L'/Q$ are both prime to $p$. Let $\xi = \psi_{\varphi^{-1}}$. Since $\xi = \xi^{-1}$, the action of $\text{Gal}(K/Q)$ on both $\text{Gal}(L/K)$ and $\text{Gal}(L'/K)$ is given by $\xi$. Let $F$ denote the quadratic extension of $Q$ corresponding to $\xi$. Then $F \subset K$, and $F = Q(\sqrt{-p})$ by Lemma 2.5. We can regard $\xi$ as a character of $\text{Gal}(F/Q)$. Since $\text{Gal}(K/F)$ acts trivially on $\text{Gal}(L/K)$ and $\text{Gal}(L'/K)$, it follows that $L$ and $L'$ are abelian extensions of $F$. Since $[K:F]$ is prime to $p$, there exist cyclic extensions $J$ and $J'$ of $F$ of degree $p$ such that $L = KJ$ and $L' = KJ'$. Now $\text{Gal}(F/Q)$ acts on both $\text{Gal}(J/F)$ and $\text{Gal}(J'/F)$ by the character $\xi$, so $J$ and $J'$ are dihedral extensions of $Q$ of degree $2p$.

By our assumption on the ramification of primes $\ell \neq p$, the extensions $J/F$ and $J'/F$ can ramify only at primes above $p$. The class number of $F = Q(\sqrt{-p})$ is not divisible by $p$ (because it is less than $p$; see for example [12, page 365]). Hence, by class field theory, one sees that $F$ has only one cyclic extension of degree $p$ that is both unramified outside of $p$ and dihedral over $Q$. (This extension is the first layer of the so-called “anticyclotomic” $\mathbb{Z}_p$-extension of $F$.) Therefore, we must have $J = J'$, and hence $L = L'$. Now $E$ is $p$-exceptional by Proposition 2.2.

Let $\Delta_{\min}(E)$ and $\Delta_{\min}(E')$ denote the discriminants of minimal integral models for $E$ and $E'$, respectively.

Theorem 2.7. Assume that $\psi_{\varphi^{-1}}$ has order 2 and that $E$ has semistable reduction at all primes $\ell$ dividing the conductor of $E$, except possibly $\ell = p$. Then $E$ is $p$-exceptional if and only if $\Delta_{\min}(E')/\Delta_{\min}(E) = p^a w^p$ for some $a \in \mathbb{Z}$ and $w \in \mathbb{Q}^\times$.

Proof. Suppose first that $\ell \neq p$ is a prime where $E$ has split multiplicative reduction. Then $E$ is a Tate curve over $Q_\ell$. Let $q_{E,\ell}$ denote the corresponding Tate period for $E$. Then we have

$$Q_\ell(E[p]) = Q_\ell(\mu_p, \sqrt[q_{E,\ell}]{q_{E,\ell}})$$

and therefore the ramification degree for $\ell$ in $Q(E[p])$ is divisible by $p$ if and only if $\text{ord}_\ell(q_{E,\ell}) \neq 0 \pmod{p}$. Furthermore, we have (Proposition VII.5.1(b) of [22])

$$\text{ord}_\ell(\Delta_{\min}(E)) = -\text{ord}_\ell(j(E)) = \text{ord}_\ell(q_{E,\ell}).$$

Thus, the ramification degree for $\ell$ in $Q(E[p])/Q$ is divisible by $p$ if and only if $\text{ord}_\ell(\Delta_{\min}(E))$ is not divisible by $p$. This criterion is also valid if $E$ has nonsplit multiplicative reduction at $\ell$, since both the ramification degree for $\ell$ in $Q(E[p])$ and the power of $\ell$ dividing $\Delta_{\min}(E)$ are unchanged by twisting $E$ by a quadratic character that is unramified at $\ell$.

By Lemma 2.3, at least one of the integers $\text{ord}_\ell(\Delta_{\min}(E))$, $\text{ord}_\ell(\Delta_{\min}(E'))$ is divisible by $p$. Therefore, both are divisible by $p$ if and only if their difference is divisible by $p$. Now apply Proposition 2.6.

3. Twisting $X_1(7)$ by characters

Fix a field $k$ of characteristic different from 7. Suppose $\psi : G_k \to \mathbb{F}_7^\times$ is a homomorphism. In this section we will construct the family of all elliptic curves over $k$ with a $k$-rational subgroup of order 7 on which $G_k$ acts via the character $\psi$. The method of our construction is as follows. When $\psi = 1$, we are parametrizing elliptic curves with a point of order 7, so the desired elliptic curves are the fibers of the universal elliptic curve $\mathcal{E}_1$ over the modular curve $X_1(7)$ of genus zero. For general $\psi$, we twist the elliptic surface $\mathcal{E}_1$ to obtain the appropriate elliptic surface $\mathcal{E}_\psi$, and then the $A_\psi$ are the fibers of $\mathcal{E}_\psi$. Theorem 3.6 deals with the case where $\psi$ has order dividing 3. Since any character $\psi$ into $\mathbb{F}_7^\times$ can be written uniquely as the product of a character of order dividing 3 and a character of order dividing 2 (namely, $\psi = \psi^3\psi^5$), we will obtain the family for a general $\psi$ as a quadratic twist of a family with a cubic $\psi$, in Theorem 3.10.

**Definition 3.1.** If $E, E'$ are elliptic curves over $k$, and $P \in E(k^s), P' \in E'(k^s)$ are points of order 7, we say that $\lambda : (E, P) \sim (E', P')$ is an isomorphism if $\lambda$ is an isomorphism from $E$ to $E'$ and $\lambda(P) = P'$. If such a $\lambda$ exists, we say that $(E, P)$ and $(E', P')$ are isomorphic. If further $\lambda : E \sim E'$ is defined over $k$, then we say that $(E, P)$ and $(E', P')$ are isomorphic over $k$.

**Lemma 3.2.** Suppose $E, E'$ are elliptic curves over $k$, $P \in E(k^s), P' \in E'(k^s)$ are points of order 7, and $(E, P)$ is isomorphic to $(E', P')$.

(i) The isomorphism $\lambda : (E, P) \sim (E', P')$ is unique.

(ii) Suppose that the groups $\Psi$ and $\Psi'$ generated by $P$ and $P'$, respectively, are stable under $G_k$. Then $(E, P)$ and $(E', P')$ are isomorphic over $k$ if and only if the two characters

$$G_k \to \text{Aut}(\Psi) \sim \mathbb{F}_7^\times, \quad G_k \to \text{Aut}(\Psi') \sim \mathbb{F}_7^\times$$

are equal.

**Proof.** If $\lambda, \lambda' : (E, P) \sim (E', P')$ are isomorphisms over $k^s$, then $\epsilon = \lambda^{-1} \circ \lambda'$ is an automorphism of $E$ fixing $P$, i.e., $(\epsilon - 1)(P) = 0$. But then (viewing $\epsilon$ as a root of unity in an imaginary quadratic field) if $\epsilon \neq 1$ we have

$$7 \leq |\ker(\epsilon - 1)| = \deg(\epsilon - 1) = (\epsilon - 1)(\overline{\epsilon} - 1) = 2 - (\epsilon + \overline{\epsilon}) \leq 4$$

which is impossible. This proves (i).

For (ii), let $\psi$ and $\psi'$ be the characters giving the action of $G_k$ on $\Psi$ and $\Psi'$, respectively. If $\sigma \in G_k$, then $\lambda^\sigma : E \sim E'$ is an isomorphism, and

$$\lambda^\sigma(P) = \lambda^\sigma(\psi^{-1}(\sigma)P\psi) = \psi^{-1}(\sigma)\lambda(P)\psi = \psi^{-1}(\sigma)(P')\psi = \psi^{-1}(\sigma)\psi'(\sigma)P'$$

If $\psi(\sigma) = \psi'(\sigma)$, then $\lambda^\sigma : (E, P) \sim (E', P')$ is an isomorphism, so $\lambda^\sigma = \lambda$ by part (i). On the other hand, if $\psi(\sigma) \neq \psi'(\sigma)$, then $\lambda^\sigma(P) \neq \lambda(P)$, so $\lambda^\sigma \neq \lambda$. This proves (ii).

If $u \in k^s$, define a curve $E_u$ over $k(u)$ by

$$E_u : y^2 - (u^2 - u - 1)xy - (u^3 - u^2)y = x^3 - (u^3 - u^2)x^2.$$  

The discriminant of $E_u$ is

$$\Delta(E_u) = u^7(u - 1)^7(u^3 - 8u^2 + 5u + 1).$$

The next result is #15 in Table 3 on p. 217 of [11].
Theorem 3.3 (11). Let $E_u$ be as above.

(i) If $u \in k$ and $\Delta(E_u) \neq 0$, then $E_u$ is an elliptic curve over $k$ and $(0,0)$ is a point of order 7 in $E(k)$.

(ii) If $E$ is an elliptic curve over $k$ and $P \in E(k)$ is a point of order 7 then there is a unique $u \in k$ such that $(E, P)$ is isomorphic over $k$ to $(E_u, (0,0))$.

Define a linear fractional transformation

$$\eta(v) = 1/(1-v).$$

The following lemma will be used in the proofs of Theorems 3.6 and 5.5 below.

Lemma 3.4. Suppose $u \in k^2$ and $\Delta(E_u) \neq 0$. Then there is a unique isomorphism defined over $k(u)$:

$$(E_{\eta(u)}, 2 \cdot (0,0)) \sim (E_u, (0,0)).$$

Proof. A direct computation shows that the map

$$(x, y) \mapsto ((u-1)^4 x + u^2 - u, (u-1)^6 y + (u-1)^4 (u^2 - 2u)x + u^4 - 2u^3 + u^2)$$

is such an isomorphism. Uniqueness follows from Lemma 3.2(i). \qed

The following lemma is taken from a paper of Washington [27, pp. 64–65].

Lemma 3.5 (Washington [27]). Suppose that $K/k$ is a cyclic cubic extension, and $\sigma$ is a generator of $\text{Gal}(K/k)$. Then there is a $t \in k$ such that

(i) $K$ is the splitting field of the polynomial $f(x) := x^3 - (t+3)x^2 + tx + 1$,

(ii) if $\gamma$ is a root of $f$ then $\gamma^3 = \eta(\gamma)$, where $\eta$ is the linear fractional transformation defined by (3).

Proof. Choose $\alpha \in K$ such that $K = k(\alpha)$. The set $\{1, \alpha, \alpha^2, \alpha^3\}$ is linearly dependent over $k$ (but $\{1, \alpha\}$ is not), so we can find a linear fractional transformation $\phi \in \text{PGL}_2(k)$ such that $\alpha^3 = \phi(\alpha)$. Note that $\phi^3$ fixes $\alpha$, $\alpha^2$, and $\alpha^3$, so $\phi^3 = 1$ in $\text{PGL}_2(k)$.

Let $\phi$ be an element in $\text{GL}_2(k)$ whose image under the map $\text{GL}_2(k) \to \text{PGL}_2(k)$ is $\phi$. We must have $\text{trace}(\phi) \neq 0$. Otherwise, $\phi^2$ would be a scalar matrix and we would then have $\alpha^3 = 1$. Therefore we can choose the lift $\tilde{\phi}$ so that $\text{trace}(\tilde{\phi}) = -1$. Then, writing $I$ for the identity in $\text{GL}_2(k)$,

$$\tilde{\phi}^2 + \tilde{\phi} = -\text{det}(\tilde{\phi})I, \quad \tilde{\phi}^3 = aI$$

for some $a \in k$, so

$$1 - \text{det}(\tilde{\phi}) = a - \text{det}(\tilde{\phi})I.$$

Since $\alpha^3 \neq 1$, $\tilde{\phi}$ cannot be a scalar matrix. It follows that $\text{det}(\tilde{\phi}) = 1$ and the minimal polynomial of $\tilde{\phi}$ over $k$ is $x^2 + x + 1$. Let $\tilde{\eta} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, which is a lift to $\text{GL}_2(k)$ corresponding to $\eta$. Since $\tilde{\eta}$ has the same minimal polynomial as $\tilde{\phi}$, there is a $\tilde{\xi} \in \text{GL}_2(k)$ such that

$$\tilde{\xi}^\top \tilde{\phi} \tilde{\xi}^{-1} = \tilde{\eta}.$$

Set $\gamma = \xi(\alpha)$, where $\xi$ is the linear fractional transformation corresponding to $\tilde{\xi}$. Then $k(\gamma) = k(\alpha) = K$, and by (4) we have

$$\gamma^3 = \eta(\gamma).$$
Theorem 3.6. Suppose that $\sigma$ and $\sigma^2$ to $[\overline{1}]$ shows that $[\overline{4}]$ holds with $\gamma$ replaced by either of its conjugates, and that $\gamma^2 = (\gamma - 1)/\gamma$. We compute

$$(x - \gamma)(x - \gamma^2)(x - \gamma^3) = x^3 - (\frac{3^3 - 3\gamma^2 + 1}{3\gamma - 3})x^2 + (\frac{3^3 - 3\gamma^2 + 1}{3\gamma - 3})x + 1$$

so the lemma holds with $t := \frac{3^3 - 3\gamma^2 + 1}{3\gamma - 3} \in k$. $\square$

Theorem 3.6. Suppose that $\chi : G_k \to \mathbb{F}_7^*$ is a homomorphism, $\chi^3 = 1$, and $E$ is an elliptic curve over $k$. Then $E$ has a $k$-rational subgroup of order 7 on which $G_k$ acts via $\chi$ if and only if there is a $v \in k$ such that $E$ is isomorphic over $k$ to the elliptic curve

$$A_v : y^2 + a_1(v)xy + a_3(v)y = x^3 + a_2(v)x^2 + a_4(v)x + a_6(v)$$

over $k$ defined as follows. If $\chi = 1$, let

$$a_1(v) = -v^2 + v + 1, \quad a_2(v) = a_3(v) = -v^3 + v^2, \quad a_4(v) = a_6(v) = 0.$$ 

If $\chi \neq 1$, then let $K$ be the cubic extension of $k$ cut out by $\chi$, let $\sigma \in \text{Gal}(K/k)$ be the element with $\chi(\sigma) = 4$, fix $t \in k$ satisfying Lemma 3.5 for $K$ and $\sigma$, and let $c = t^2 + 3t + 9$, $f(v) = v^3 - (t + 3)v^2 + tv + 1$,

$$a_1(v) = c(v^2 - v + 1),$$

$$a_2(v) = cf(v)(2v - 1),$$

$$a_3(v) = cf(v)((t^3 - 1)v^3 + (t^3 - 1)v + t^2 - t + 1),$$

$$a_4(v) = c^2f(v)[(-3t^2 - 5t - 2)v^5 + (2t^3 + 8t^2 + 8t - 7)v^4 - (3t^3 + 6t^2 + 5t - 20)v^3 + (2t^3 - t - 23)v^2 + 2(t^2 + 2t + 7)v - t - 1],$$

$$a_6(v) = c^2f(v)^2[(2t^5 + 9t^4 + 23t^3 + 35t^2 + 24t + 11)v^6 + (-t^6 - 6t^5 - 23t^4 - 38t^3 - 33t^2 + 36t)v^5 + (t^6 + 6t^5 + 18t^4 - 6t^3 - 60t^2 - 180t + 13)v^4 + (-t^6 - 2t^5 + 46t^3 + 84t^2 + 142t - 139)v^3 + (-t^5 - 5t^4 - 27t^3 - 15t^2 + 9t + 182)v^2 + (2t^4 + 3t^3 - 10t^2 - 32t - 67)v + 2t^3 + 5t^2 + 11t + 11].$$

Proof. If $\chi = 1$, then $A_v$ is the curve $E_v$ of $[\overline{1}]$, and the theorem follows from Theorem 3.5. Suppose now that $\chi \neq 1$. Let $\gamma \in K$ be a root of $f(x)$. Define

$$U_v := ((\gamma - 1)v + 1)^2(2\gamma^2 - (2t + 5)\gamma + t - 1)^2,$$

$$R_v := cf(v)[(\gamma - t - 3)\gamma v - \gamma^2 + (t + 2)\gamma + 1],$$

$$S_v := ((t + 3)\gamma^2 - (t^2 + 5t + 9)\gamma - 3)v^2 - (2\gamma^2 - 2(t^2 + 3t + 3)\gamma + (t^2 + t + 3))v - 3\gamma^2 + (2t + 6)\gamma - t,$$

$$T_v := cf(v)[(2t^2 + 6t + 5)v^3 - (t^2 + 3t + 9)v^2 - 13v + 2t + 4].$$

If $v \in k$ and $A_v$ is nonsingular, then we compute that $P_v := (R_v, T_v)$ is a point of order 7 in $A_v(K)$, and using Lemma 3.5(ii) we compute that $P_v^7 = 4P_v = (\chi(\sigma)P_v)$. Thus $P_v$ generates a $k$-rational subgroup of order 7 on $A_v$, on which $G_k$ acts via $\chi$. If $E$ is isomorphic over $k$ to $A_v$, then $E$ also has such a subgroup.
Conversely, suppose $E$ is an elliptic curve over $k$ with a $k$-rational subgroup of order 7 on which $G_k$ acts via $\chi$. Let $P \in E(K)$ be a generator of that subgroup (so $P^7 = \chi(\sigma)P = 4P$). By Theorem 3.3(ii) applied with $K$ in place of $k$, $(E, P)$ corresponds to a $K$-rational point of $X_1(7)$, i.e., there is a $u \in K$ and an isomorphism $\varphi : E \simto E_u$ defined over $K$ such that $\varphi(P) = (0,0) \in E_u[7]$.

Let $\delta$ be the linear fractional transformation

$$
\delta(z) = \frac{-z + \gamma}{(\gamma - 1)z + 1}
$$

and let $v = \delta^{-1}(u) \in K$. We compute that the map $\lambda$ defined by

$$
\lambda(x, y) := (U_v^2x + R_v, U_v^3y + U_v^2Sx + T_v)
$$

is an isomorphism over $K$ from $(E_u, (0,0))$ to $(A_v, P_v)$. (Since $\delta(v) = u$, by Lemma 3.2 we have $(\gamma - 1)v + 1 \neq 0$; since also $[k(\gamma) : k] = 3$, we have $U_v \neq 0$.) Therefore $\lambda \circ \varphi$ is an isomorphism from $(E, P)$ to $(A_v, P_v)$. If we show that $v \in k$, then Lemma 3.2(ii) will imply that $(E, P)$ and $(A_v, P_v)$ are isomorphic over $k$.

Suppose $\sigma \in G_k$. Then $\varphi^\sigma$ is an isomorphism from $E$ to $E_{u^\sigma}$, and

$$
\varphi^\sigma(P) = \varphi^\sigma(2P^\sigma) = 2\varphi^\sigma(P^\sigma) = 2\varphi(P)^\sigma = 2(0,0)^\sigma = 2(0,0).
$$

Thus we have isomorphisms

$$(E_{u^\sigma}, (0,0)) \xrightarrow{\varphi^{-1}} (E, P) \xrightarrow{\varphi^\sigma} (E_{u^\sigma}, 2(0,0)) \xrightarrow{\sim} (E_{\eta^{-1}(u^\sigma)}, (0,0))$$

where $\eta$ is defined by (3) and the final isomorphism comes from Lemma 3.4. Thus by the uniqueness of $u$ in Theorem 3.3(ii) (applied with $K$ in place of $k$) we see that $u = \eta^{-1}(u^\sigma)$, so $u^\sigma = \eta(u)$.

Using the definition of $\delta$ and Lemma 3.3(ii), it is easy to check that $\delta^\sigma = \eta \delta$. Hence we have

$$v^\sigma = \delta^{-1}(u^\sigma) = (\delta^\sigma)^{-1}(u^\sigma) = \delta^{-1}\eta^{-1}(\eta(u)) = \delta^{-1}(u) = v.$$  

Therefore $v \in k$, so $A_v$ is defined over $k$, $G_k$ acts on both $P$ and $P_v$ by multiplication by $\chi$, and so the isomorphism $\lambda \circ \varphi : (E, P) \simto (A_v, P_v)$ is defined over $k$ by Lemma 3.2(ii).

**Remark 3.7.** Suppose $\chi \neq 1$ in Theorem 3.6. With notation as in Theorem 3.6 the discriminant of $A_v$ is given by

$$\Delta(A_v) = c^8f(v)^7[(t - 5)v^3 + (5t + 24)v^2 - (8t + 9)v + t - 5].$$

**Remark 3.8.** With notation as in Theorem 3.6 with $\chi \neq 1$, the cubic Galois extension $K$ of $k$ is the splitting field over $k$ of the polynomial $f(x) \in k[x]$, by Lemma 3.5(i). Thus, $f(x)$ is separable and irreducible over $k$. One can compute that $c^2$ is the discriminant of $f$, so $c \neq 0$. It then follows from (1) that $\Delta(A_v) = 0$ for at most six values of $v \in k^\times$.

**Definition 3.9.** If $E$ is an elliptic curve over $k$ and $\epsilon : G_k \to \{\pm 1\} \subseteq \text{Aut}(E)$ is a homomorphism, then the (quadratic) twist of $E$ by $\epsilon$ is an elliptic curve $E^{(\epsilon)}$ over $k$ such that there is an isomorphism $\lambda : E^{(\epsilon)} \to E$ over $k^\times$ with $\lambda^\sigma \circ \lambda^{-1} = \epsilon(\sigma)$ for all $\sigma \in G_k$.

If $\text{char}(k) \neq 2$, $E$ is an elliptic curve over $k$ defined by an equation of the form $y^2 = F(x)$, and $k(\sqrt{d})$ is the field cut out by such a character $\epsilon$, then $E^{(\epsilon)}$ is isomorphic over $k$ to the curve defined by $dy^2 = F(x)$, i.e., the quadratic twist of $E$ by $d$. 
Theorem 3.10. Suppose that \( \psi : G_k \to \mathbf{F}_7^\times \) is a homomorphism. If \( E \) is an elliptic curve over \( k \), then \( E \) has a \( k \)-rational subgroup of order 7 on which \( G_k \) acts via \( \psi \) if and only if there is a \( v \in k \) such that \( E \) is isomorphic over \( k \) to the twist of \( A_v \) by \( \psi^3 \), where \( A_v \) is as in Theorem 3.6 for the character \( \chi = \psi^4 \).

In particular, if \( \text{char} (k) \neq 2 \) and \( k(\sqrt{d}) \) is the field cut out by \( \psi^3 \), then the twist of \( A_v \) by \( \psi^3 \) is

\[
A_v^{(d)} : y^2 = x^3 + db_2(v)x^2 + 8d^2b_4(v)x + 16d^3b_6(v),
\]

where \( b_2 = a_1^2 + 4a_2, \ b_4 = 2a_4 + a_1a_3, \ b_6 = a_3^2 + 4a_6 \) are the usual invariants of the curve \( A_v \).

Proof. Since \( \psi^6 = 1 \), we have \((\psi^4)^3 = 1 \), so we can apply Theorem 3.6 and we can also twist by \( \psi^3 \) as in Definition 3.9. Let \( \lambda : E(\psi^3) \to E \) be as in Definition 3.9.

For \( P \in E(k^s) \) and \( \sigma \in G_k \), \( P^\sigma = \psi(\sigma)P \) if and only if \( \lambda^{-1}(P)^\sigma = \psi^4(\sigma)\lambda^{-1}(P) \). Thus by Theorem 3.6

\[
E \text{ has a } k \text{-rational subgroup of order 7 on which } G_k \text{ acts via } \psi \\
\iff E(\psi^3) \text{ has a } k \text{-rational subgroup of order 7 on which } G_k \text{ acts via } \psi^4 \\
\iff E(\psi^3) \text{ is isomorphic over } k \text{ to } A_v \text{ for some } v \in k \\
\iff E \text{ is isomorphic over } k \text{ to } A_v^{(\psi^3)} \text{ for some } v \in k.
\]

If \( \text{char}(k) \neq 2 \), then \( A_v^{(d)} \) is a Weierstrass model for \( A_v^{(\psi^3)} \).

Lemma 3.11. Let \( A_v^{(d)} \) be as in Theorem 3.10, and let \( \eta \) be as defined by (3). Then for every \( v \), we have \( A_v^{(d)} \cong A_v^{(\psi^3)} \) over \( k(v) \).

Proof. This can be shown by exhibiting an explicit isomorphism. We will give a slightly less computational way to deduce the lemma from Lemma 3.4. We can easily reduce to the case \( d = 1 \).

Let \( \delta \) be the linear fractional transformation defined by (6) in the proof of Theorem 3.6. The proof of Theorem 3.6 showed that \( (A_v, P_v) \cong (E_{\delta(v)}, (0,0)) \) for every \( v \), so we have

\[
(A_v, P_v) \cong (E_{\delta(v)}, (0,0)) \cong (E_{\eta\delta(v)}, 2 \cdot (0,0)) \cong (A_{\delta^{-1}\eta\delta(v)}, 2 \cdot P_{\delta^{-1}\eta\delta(v)}),
\]

where the middle isomorphism is from Lemma 3.4. A simple calculation shows that \( \delta^{-1}\eta\delta(v) = \eta^2 = \eta^{-1} \), and so \( A_v \cong A_{\eta^{-1}(v)} \) over \( k(v) \) by Lemma 3.2(ii).

Corollary 3.12. Suppose that \( \psi : G_k \to \mathbf{F}_2^\times \) is a homomorphism. If \( k \neq \mathbf{F}_2 \), then there exists an elliptic curve \( E \) over \( k \) with a \( k \)-rational subgroup of order 7 on which \( G_k \) acts via \( \psi \). If \( k = \mathbf{F}_2 \), then there exists an elliptic curve \( E \) over \( \mathbf{F}_2 \) with an \( \mathbf{F}_2 \)-rational subgroup of order 7 on which \( G_{\mathbf{F}_2} \) acts via \( \psi \) and only if \( \psi \) is \( \omega^{-1} \) or \( \omega^{-1} \epsilon \), where \( \epsilon \) is the unique character of \( G_{\mathbf{F}_2} \) of order 2 and \( \omega \) is the cyclotomic character.

Proof. Let \( A_v \) be as defined in Theorem 3.6 for the (at most cubic) character \( \psi^4 \). If \( v \in k \) is not a zero of the discriminant \( \Delta(A_v) \), then Theorem 3.10 shows that the twist of \( A_v \) by \( \psi^3 \) has the desired property. Suppose \( k \neq \mathbf{F}_2 \). We need only show that there exists a \( v \in k \) such that \( \Delta(A_v) \neq 0 \).
Suppose $\psi^4 = 1$. Then $A_v = E_v$, so $\Delta(A_v) = \Delta(E_v)$ is given by (2). That polynomial has at most 5 roots, and it is easy to check that it has only the roots 0 and 1 if $|k| \leq 5$. Hence, there are $v \in k$ with $\Delta(A_v) \neq 0$ if and only if $k \neq F_2$.

Now suppose $\psi^4 \neq 1$. Then $\Delta(A_v)$ is the polynomial given in (7). By Remark 3.8, $c \neq 0$, and $f(v) \neq 0$ for every $v \in k$. Thus for $v \in k$, $\Delta(A_v) = 0$ if and only if $(t - 5)v^3 + (5t + 24)uv^2 - (8t + 9)v + t - 5 = 0$. If $t \neq 5$, then $\Delta(A_v) \neq 0$ when $v = 0$. If $t = 5$, then $\Delta(A_v) = 0$ only when $v = 0$ or 1 (since char($k$) $\neq 7$), so there are $v \in k$ with $\Delta(A_v) \neq 0$ if and only if $k \neq F_2$.

When $k = F_2$, the stated result follows from the above proof, the fact that $\omega$ has order 3, and the fact that $t = 1$ when $\psi^4 = \omega$ while $t = 0$ when $\psi^4 = \omega^{-1}$. (Alternatively, it can also be deduced from the fact that no elliptic curve defined over $F_2$ has a rational point of order 7, which follows from the Weil bounds.) Note that the remaining characters are 1, $\omega$, $\epsilon$, and $\omega\epsilon$; for each of these characters there is no elliptic curve over $F_2$ with an $F_2$-rational subgroup of order 7 on which $G_{F_2}$ acts via that character.

Remark 3.13. Here is another interpretation of Theorem 3.10. Suppose $\Psi$ is a cyclic group of order 7 with an action of $G_k$. Consider isomorphism classes (in the obvious sense) of pairs $(E, f)$ where $E$ is an elliptic curve and $f : \Psi \to E[7]$ is an injection. We say that $(E, f)$ is $k$-rational if $(E^\sigma, f^\sigma)$ is isomorphic to $(E, f)$ for every $\sigma \in G_k$, and we let $X(\Psi)$ denote the moduli space of such isomorphism classes. If $K$ is an extension of $k$ such that $G_K$ acts trivially on $\Psi$, and $P$ is a generator of $\Psi$, then the map

$$(E, f) \mapsto (E, f(P))$$

induces an isomorphism from $X(\Psi)$ to $X_1(7)$ defined over $K$. Thus $X(\Psi)$ is a twist of $X_1(7)$.

Let $\psi : G_k \to Aut(\Psi) \cong F_7^\times$ be the character giving the action of $G_k$ on $\Psi$. Let $A_v$ be as defined in Theorem 3.6 with $\chi = \psi^4$, and let $f_v : \Psi \to A_v[7]$ be the unique homomorphism with $f_v(P) = P_v$, where $P_v$ is as in the proof of Theorem 3.6. Then Theorem 3.10 says that the elliptic surface $A_v$ is the universal elliptic curve over $X(\Psi)$, in the following sense. For every $v \in k^\circ$ such that $\Delta(A_v) \neq 0$, the pair $(A_v, f_v)$ is $k(v)$-rational, and conversely for every pair $(E, f)$ with $E$ defined over $k^\circ$, there is a unique $v \in k^\circ$ such that $(E, f)$ is isomorphic to $(A_v, f_v)$.

If $a \in F_7^\times/(\pm 1)$ there is a natural automorphism of $X(\Psi)$ induced by $(E, f) \mapsto (E, af)$. The proof of Lemma 3.11 shows that the automorphism corresponding to $a = 2$ is $\eta$, in the sense that $(A_{\eta(v)}, f_{\eta(v)}) \cong (A_v, 2f_v)$.

4. The special case where $k = \mathbb{Q}$ and $\psi = \omega^5$

As before, let $\omega : G_{\mathbb{Q}} \to F_7^\times$ denote the cyclotomic character, and if $E$ is an elliptic curve defined over $\mathbb{Q}$, let $\Delta_{\text{min}}(E)$ denote the discriminant of a minimal model of $E$.

Define an elliptic curve

$$(8) \quad B_v : y^2 + xy = x^3 - x^2 + \alpha(v)x + \beta(v),$$
where

\[
\begin{align*}
\alpha(v) := & -\frac{35}{16} v^8 + \frac{63}{4} v^7 - \frac{833}{2} v^6 + 42 v^5 + 245 v^4 - \frac{49}{2} v^3 + \frac{343}{24} v^2 + \frac{7}{5} v - 2, \\
\beta(v) := & -\frac{49}{32} v^{12} + \frac{637}{48} v^{11} - \frac{1617}{32} v^{10} + \frac{44142}{432} v^9 - \frac{165552}{192} v^8 - \frac{1477}{56} v^7 + \frac{1911}{16} v^6 \\
& - \frac{8183}{144} v^5 - \frac{2009}{192} v^4 + \frac{7007}{432} v^3 - \frac{147}{16} v^2 + \frac{14}{3} v - 1.
\end{align*}
\]

The discriminant of \( B_v \) is

\[
(9) \quad \Delta(B_v) = -7^3(v^3 - 2v^2 - v + 1)(v^3 - v^2 - 2v + 1)^7.
\]

**Theorem 4.1.** Let \( B_v \) be as above.

(i) Suppose \( E \) is an elliptic curve over \( \mathbb{Q} \). Then \( E \) has a rational subgroup of order 7 on which \( G_{\mathbb{Q}} \) acts via \( \omega^5 \) if and only if there is a \( v \in \mathbb{Q} \) such that \( E \cong B_v \) over \( \mathbb{Q} \).

(ii) If \( p \) is a prime and \( v \in \mathbb{Q} \) is integral at \( p \), then the above model for \( B_v \) is integral at \( p \) and minimal at \( p \).

(iii) If \( p \) is a prime and \( v \in \mathbb{Q} \) is not integral at \( p \), then

\[
\text{ord}_p(\Delta(B_v)) = \text{ord}_p(\Delta(B_v)) - 24 \text{ord}_p(v).
\]

**Proof.** We will deduce part (i) from Theorem 3.10. The cyclotomic character \( \omega \) has order 6. Let \( \psi = \omega^5 \). The cubic character \( \psi^3 = \omega^2 \) cuts out the unique cubic subfield \( K := \mathbb{Q}(\mu_2)^+ \) of \( \mathbb{Q}(\mu_5) \). The automorphism \( \sigma \in \text{Gal}(K/\mathbb{Q}) \) that sends \( \zeta_7 + \zeta_7^{-1} \) to \( \zeta_7^2 + \zeta_7^{-2} \) satisfies \( \omega(\sigma) = 2 \), so \( \psi(\sigma) = 2^5 = 4 \). Using the construction of \( \gamma \) and \( t \) in the proof of Lemma 3.5, we obtain \( t = -2 \) and \( \gamma = -(\zeta_7 + \zeta_7^{-1}) \).

Further, the quadratic character \( \psi^3 = \omega^3 \) cuts out the unique quadratic subfield \( \mathbb{Q}(\sqrt{-7}) \) of \( \mathbb{Q}(\mu_7) \). The map \( (x,y) \mapsto (x',y') \) where

\[
x' = \frac{1}{4} x + \frac{3}{4} v^4 - \frac{5}{6} v^3 - \frac{15}{4} v^2 + \frac{23}{6} v - 1, \quad y' = \frac{1}{2} y - \frac{1}{2} x'
\]

is an isomorphism from the curve \( A_v^{(d)} \) of Theorem 3.10 with \( t = -2 \) and \( d = -1/7 \), to the curve \( B_v \) of Theorem 3.10 (recall that \( t \) is in the definition of \( A_v \) in Theorem 3.10, which is used to define \( A_v^{(d)} \) in Theorem 3.10). Now (i) follows from Theorem 3.10.

The polynomials \( \alpha(v) \) and \( \beta(v) \) take \( \mathbb{Z} \) to \( \mathbb{Z} \), as can be seen, for example, by expressing them as integral linear combinations of binomial coefficient polynomials \( \binom{n}{k} \). It follows that if \( v \in \mathbb{Q} \) is integral at \( p \), then the model \( (8) \) is integral at \( p \).

Suppose first that \( p \neq 7 \). With \( c_4 \) the usual invariant (see [22] III.1.1), we check that the polynomial \( c_4(B_v) \) is in \( \mathbb{Z}[v] \). The resultant of the polynomials \( \Delta(B_v) \) and \( c_4(B_v) \) is \( 7^{98} \), which is a unit at \( p \). Hence if \( v \) is integral at \( p \), then \( \Delta(B_v) \) and \( c_4(B_v) \) cannot both vanish mod \( p \), so by Tate’s algorithm (or [22] Proposition III.1.4(ii)), the equation defining \( B_v \) is minimal at \( p \).

If \( p = 7 \) and \( v \in \mathbb{Q} \) is integral at \( 7 \), then Lemma 4.4(c) below shows that \( \text{ord}_7(\Delta(B_v)) < 12 \), so the equation defining \( B_v \) is minimal at \( 7 \). This proves (ii).

Now suppose that \( v \) is not integral at \( p \). Then \( w := 1/v \) is integral at \( p \), and via the change of variables

\[
(x,y) \mapsto (w^4 x - \frac{w^4-1}{4}, w^6 y + \frac{w^4(w^2-1)}{2} x + \frac{w^4-1}{8}),
\]

\( B_v \) has a model

\[
\hat{B}_v : y^2 + xy = x^3 - x^2 + (\hat{\alpha}(w) + \frac{3}{16} (1 - w^8)) x
\]

\[
+ \hat{\beta}(w) + \frac{w^4-1}{4} \hat{\alpha}(w) - \frac{2w^{12} - 3w^8 + 1}{64}
\]
where \( \tilde{\alpha}(z) := z^8\alpha(1/z), \quad \tilde{\beta}(z) := z^{12}\beta(1/z) \in \mathbb{Q}[z] \) with \( \alpha \) and \( \beta \) as in \( \text{(8)} \). Again, one can check that the polynomials

\[
\tilde{\alpha}(z) + 3(1-z^8)/16 \quad \text{and} \quad \tilde{\beta}(z) + (z^4 - 1)\tilde{\alpha}(z)/4 + (-2z^{12} + 3z^8 - 1)/16
\]
take \( \mathbb{Z} \) to \( \mathbb{Z} \). Hence \( \tilde{B}_v \) is integral at \( p \). Exactly as for (ii) one can show that \( \tilde{B}_v \) is minimal at \( p \), and hence \( \Delta_{\min}(B_v) = \Delta(\tilde{B}_v) = v^{-24}\Delta(B_v) \). This proves (iii). \( \square \)

**Remark 4.2.** Theorem \( \text{(4.1)(i)} \) shows that for \( v \in \mathbb{Q} \), the representation of \( G_{\mathbb{Q}} \) acting on \( B_v[7] \) is of the form \( (\omega^5 \omega^3) \).

**Corollary 4.3.** If \( v \in \mathbb{Q} \) has denominator \( d \), then

\[
\Delta_{\min}(B_v) = \Delta(B_v)d^{24} = -7^4d^{24}(v^3 - 2v^2 - v + 1)(v^3 - v^2 - 2v + 1)^7.
\]

**Proof.** This follows directly from Theorem \( \text{(4.1)(ii,iii)} \). \( \square \)

**Lemma 4.4.** Let \( f_1(v) = v^3 - 2v^2 - v + 1 \) and \( f_2(v) = v^3 - v^2 - 2v + 1 \). For \( v \in \mathbb{Q} \) and \( B_v \) as above, we have the following table:

<table>
<thead>
<tr>
<th>( v ) integral at 7</th>
<th>( v ) not integral at 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v \equiv 3 \pmod{7} )</td>
<td>( v \equiv 5 \pmod{7} )</td>
</tr>
<tr>
<td>( \text{ord}_7(f_1(v)) )</td>
<td>( \text{ord}_7(f_2(v)) )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \text{ord}_7(f_2(v)) )</td>
<td>0</td>
</tr>
<tr>
<td>( \text{ord}_7(\Delta(B_v)) )</td>
<td>4</td>
</tr>
<tr>
<td>( \text{ord}<em>7(\Delta</em>{\min}(B_v)) )</td>
<td>4</td>
</tr>
<tr>
<td>( \geq 2 )</td>
<td>( \geq 5 )</td>
</tr>
</tbody>
</table>

**Proof.** If \( v \) is not integral at 7, then \( \text{ord}_7(f_1(v)) = \text{ord}_7(f_2(v)) = 3\text{ord}_7(v) \). If \( v \) is integral at 7, then a direct computation shows that \( f_1(v), f_2(v) \not\equiv 0 \pmod{7^2} \).

Since \( f_1(v) \equiv (v - 3)^3 \pmod{7} \) and \( f_2(v) \equiv (v - 5)^3 \pmod{7} \), (a) and (b) follow.

By \( \text{(9)} \) we have \( \Delta(B_v) = -7^3f_1(v)f_2(v)^7 \), so (c) follows from (a) and (b). Assertion (d) follows directly from (c) and Theorem \( \text{(4.1)(ii,iii)} \).

We compute that

\[
j(B_v) = -\frac{[(v^2 - 2v - 3)(v^2 - v + 1)(3v^2 - 9v + 5)(5v^2 - v - 1)^3]}{f_1(v)f_2(v)^7}.
\]

If \( v \equiv 5 \pmod{7} \), then each quadratic factor in the numerator vanishes modulo 7. If \( v \equiv 3 \pmod{7} \), then \( v^2 - v + 1 \equiv 0 \pmod{7} \). If \( v \equiv 3 \) or \( 5 \pmod{7} \), then none of the factors in the numerator vanish modulo 7. These remarks together with (a) and (b) imply the assertions in (e). \( \square \)

**Proposition 4.5.** For \( v \in \mathbb{Q} \) and \( B_v \) as above, let \( \Psi_v \) be the \( \mathbb{Q} \)-rational subgroup of \( B_v \) of order 7 on which \( G_{\mathbb{Q}} \) acts via \( \omega^3 \). Let \( B_v' \) be the quotient of \( B_v \) by \( \Psi_v \), so there is an isogeny from \( B_v \) to \( B_v' \) defined over \( \mathbb{Q} \). Then the isogenous curve \( B_v' \) is isomorphic over \( \mathbb{Q} \) to the twist of \( B_{1-v} \) by \( \omega^3 \).

**Proof.** One can verify this by a direct calculation, using the formulas of Vélu \( \text{[26]} \) (see \( \text{[2]} \) \( \text{[4.1]} \)) to exhibit the isogeny. (See especially Proposition 4.1 of \( \text{[2]} \) and the formulas for \( A \) and \( \tilde{B} \) in the paragraph after its proof.) \( \square \)

Note that \( B_v' \) has a subgroup of order 7 on which \( G_{\mathbb{Q}} \) acts via \( \omega^2 \), namely, \( B_v[7]/\Psi_v \). Also, the twist of \( B_{1-v} \) by \( \omega^3 \) is the quadratic twist of \( B_{1-v} \) by \(-7 \).
Corollary 4.6. Suppose $v \in \mathbb{Q}$. Then:

(i) $\frac{\Delta_{\text{min}}(B'_v)}{\Delta_{\text{min}}(B_v)} = 7^{s_v} \left( \frac{v^3 - 2v^2 - v + 1}{v^3 - v^2 - 2v + 1} \right)^6$ for some $s_v \in \{\pm 6\}$, and

(ii) $\text{ord}_7 \left( \frac{\Delta_{\text{min}}(B'_v)}{\Delta_{\text{min}}(B_v)} \right) = \begin{cases} 0 & \text{if } v \text{ is integral at } 7 \text{ and } v \equiv 3 \text{ or } 5 \pmod{7}, \\ 6 & \text{otherwise.} \end{cases}$

Proof. Let $f_1(v) = v^3 - 2v^2 - v + 1$ and $f_2(v) = v^3 - v^2 - 2v + 1$ as in Lemma 4.4 let $B'_{1-v}$ denote the quadratic twist of $B_{1-v}$ by $-7$, and let

$$s_v := \text{ord}_7 \left( \frac{\Delta_{\text{min}}(B'_{1-v})}{\Delta_{\text{min}}(B_v)} \right) \in \{\pm 6\}. \tag{11}$$

Note that $f_1(1 - v) = -f_2(v)$ and $f_2(1 - v) = -f_1(v)$. Since $v$ and $1 - v$ have the same denominator, by Corollary 4.3 we have

$$\Delta_{\text{min}}(B_{1-v})/\Delta_{\text{min}}(B_v) = f_1(v)/f_2(v)^6.$$

By Proposition 4.5 we have $B'_v \cong B'_{1-v}$. Thus

$$\frac{\Delta_{\text{min}}(B'_v)}{\Delta_{\text{min}}(B_v)} = \frac{\Delta_{\text{min}}(B'_{1-v})}{\Delta_{\text{min}}(B_v)} = 7^{s_v} \frac{\Delta_{\text{min}}(B_{1-v})}{\Delta_{\text{min}}(B_v)} = 7^{s_v} \left( \frac{f_1(v)}{f_2(v)} \right)^6, \tag{12}$$

proving (i).

To prove (ii) we need to compute $s_v$. Using Lemma 4.4(e), it follows that

$$\text{ord}_7(j(B_{1-v})) = \text{ord}_7(j(B_{1-v})) \geq 0.$$

Hence, by Tate’s algorithm (see for example [22, Table 15.1]), we have

$$0 \leq \text{ord}_7(\Delta_{\text{min}}(B'_{1-v})) \leq 10.$$

It follows that the integer $s_v$ of (11) satisfies

$$s_v = \begin{cases} 6 & \text{if } \text{ord}_7(\Delta_{\text{min}}(B_{1-v})) < 6, \\ -6 & \text{if } \text{ord}_7(\Delta_{\text{min}}(B_{1-v})) \geq 6. \end{cases}$$

Now Lemma 4.4(d) shows that $s_v = -6$ if $v$ is both integral at 7 and congruent to 3 (mod 7), and $s_v = 6$ otherwise. Assertion (ii) now follows from (12) and Lemma 4.4(a,b). \qed

Proposition 4.7. Suppose that $v \in \mathbb{P}^1(\mathbb{Q})$. Then:

(i) The conductor of the elliptic curve $B_v$ is of the form $49 \prod_{i=1}^m \ell_i$, where the $\ell_i$’s are distinct primes such that $\ell_i \equiv \pm 1 \pmod{7}$.

(ii) The curve $B_v$ has potentially good reduction at 7. Further, if $v$ is integral at 7 and $v \equiv 3$ or 5 (mod 7), then $B_v$ has potentially ordinary reduction at 7, and for all other $v \in \mathbb{P}^1(\mathbb{Q})$, $B_v$ has potentially supersingular reduction at 7.

(iii) The conductor of $B_v$ is 49 if and only if $v \in \{0, 1, \infty, 2, 1/2, -1\}$.

Proof. Lemma 4.4(e) shows that $j(B_v)$ is integral at 7 for all $v \in \mathbb{P}^1(\mathbb{Q})$. Hence $B_v$ always has potentially good reduction at 7, giving (ii). However, $B_v$ cannot have good reduction at 7. One sees this by considering the action of $G_{\mathbb{Q}}$ on $B_v[7]$. If $B_v$ had good ordinary reduction at 7, then $B_v[7]$ would have a nontrivial unramified quotient over $\mathbb{Q}_7$ (by [20, Proposition 11]), which is not the case since $\omega^2$ and $\omega^5$
are ramified characters of $G_{Q}$. If $B_{v}$ had good supersingular reduction, then $B_{v}[7]$ would be irreducible over $Q_{7}$ (by [20] Proposition 12(c)), which is also not the case. It follows that the conductor of $B_{v}$ is $49M$, where $M$ is not divisible by 7.

By examining the elliptic curves over $F_{7}$, we see that an elliptic curve $E$ over $Q$ has supersingular or potentially supersingular reduction at 7 if and only if $j(E) \equiv -1 \pmod{7}$. If $v$ is integral at 7 and satisfies $v \equiv 3 \text{ or } 5 \pmod{7}$, then Lemma 4.4(e) shows that $j(B_{v}) \equiv 0 \pmod{7}$, and so $B_{v}$ has potentially ordinary reduction at 7. For the other $v$’s, the formula for $j(B_{v})$ in [10] shows that we indeed have $j(B_{v}) \equiv -1 \pmod{7}$. This proves (ii).

Suppose $\ell$ is a prime dividing $M$. If $B_{v}$ has additive reduction at $\ell$, then $B_{v}$ becomes semistable over $Q(B_{v}[7])$ and the ramification degree of $\ell$ in $[Q(B_{v}[7]) : Q]$ is divisible by 2 or 3. (A good summary of the ramification properties in the non-semistable case can be found in [20], §5.6, especially Proposition 23(b).) This contradicts the facts that $\ell$ is unramified in $Q(\mu_{7})/Q$ and $[Q(B_{v}[7]) : Q(\mu_{7})]$ is (by Remark 4.2) 1 or 7. Thus, $B_{v}$ has multiplicative reduction at $\ell$. It follows that $M$ is not divisible by $\ell^{3}$.

The action of $G_{Q}$ on $B_{v}[7^{\infty}]$ can be described by the Tate parametrization. One sees that $B_{v}[7]$, or an unramified quadratic twist of $B_{v}[7]$, has composition factors isomorphic over $Q_{v}$ to $\mu_{7}$ and $Z/7Z$. Let $\omega_{v}$ denote the restriction of $\omega$ to $G_{Q}$. Thus, $G_{Q}$ acts on the composition factors by two $F_{7}$-valued characters whose ratio is $\omega_{v}$, or its inverse. On the other hand, $G_{Q}$ acts on the composition factors via $\omega_{v}^{2}$ and $\omega_{v}^{5}$, whose ratio is $\omega_{v}^{2+3}$, a character of order 1 or 2. Therefore $\omega_{v}$ has order 1 or 2, so $\ell \equiv \pm 1 \pmod{7}$, giving (i).

There are exactly two $j$-invariants of curves of conductor 49. Using [10] we see that these correspond precisely to the six values of $v$ listed in (iii). □

Remark 4.8. There is an $S_{3}$-action on $P^{1}(Q)$ defined by the linear fractional transformations $\eta$ of $\{3\}$ and $\tau$ defined by $\tau(v) = 1 - v$. Since the fixed points of $\eta$ (the primitive sixth roots of unity) are not in $Q$, the orbits under the action of $\eta$ always have length 3. There are just two orbits of length 3 under the action of $S_{3}$. For $v$ is in such an orbit if and only if $1 - v = \eta^{i}(v)$ for some $i \in \{0, 1, 2\}$. One easily determines the possible orbits of that type: $\{0, 1, \infty\}$ is one, $\{2, 1/2, -1\}$ is the other. By Proposition 4.7(iii) the corresponding curves $B_{v}$ have conductor 49, and hence have complex multiplication. One can also explain this as follows.

By Proposition 4.7, the elliptic curves $B_{1-v}$ and $B_{v}/\Psi_{v}$ are $Q$-isomorphic for every $v \in P^{1}(Q)$. However, if $1 - v \in \{v, \eta(v), \eta^{2}(v)\}$, then Lemma 3.11 shows that $B_{1-v}$ is $Q$-isomorphic to $B_{v}$, so there is an isomorphism $B_{v}/\Psi_{v} \cong B_{v}$ defined over $F = Q(\sqrt{-7})$. Therefore, $B_{v}$ has an endomorphism of degree 7 defined over $F$. This means that $B_{v}$ has CM by $F$. Furthermore, since a CM curve has no primes of multiplicative reduction, Proposition 4.7 shows that the the conductor of $B_{v}$ is 49. If $v \in \{0, 1, \infty\}$, then $j(B_{v}) = -15^{7}$ and $\text{End}(B_{v})$ is the maximal order in $F$. If $v \in \{2, 1/2, -1\}$, then $j(B_{v}) = 255^{3}$ and $\text{End}(B_{v})$ is the nonmaximal order of conductor 2 in $F$.

5. THE IMAGE OF $\rho_{E,7}$

Suppose $E$ is an elliptic curve over $Q$ with a $Q$-isogeny of prime degree $p \geq 7$. We retain the notation of [2]. Note that since $\psi \varphi = \omega$, we have that $\psi \varphi^{-1}$ has order 2 if and only if $\psi^{4} = \omega^{2}$. 


By Theorem 1.2 ([8, Theorem 1]), if $E$ is $p$-exceptional then $\psi \varphi^{-1}$ has order 2. If $p > 7$ then Proposition 1.3 ([8, Remark 4.2.1]) says that if $\psi \varphi^{-1}$ has order 2 then $E$ has CM by $\mathbb{Q}(\sqrt{-p})$. If $E$ has CM, then $\text{image}(\rho_E,p)$ is a $p$-adic Lie group of dimension 2, and so it cannot contain a Sylow pro-$p$ subgroup of $\text{Aut}(T_p(E))$. Thus for $p > 7$, an elliptic curve over $\mathbb{Q}$ is $p$-exceptional if and only if it has CM by $\mathbb{Q}(\sqrt{-p})$.

However, for $p = 7$, it is possible for $\psi \varphi^{-1}$ to have order 2 even if $E$ does not have complex multiplication. For example, for every $v \in \mathbb{Q}$, the curve $B_v$ of §4 has a $\mathbb{Q}$-isogeny of degree 7 with $\psi = \omega^3$ and $\psi \varphi^{-1} = \omega^3$ of order 2. In this section we will use Theorem 2.7 to study 7-exceptional curves. We assume from now on that $p = 7$.

For $j \in \mathbb{Z}$, let $C_j$ denote the curve

$$w^7 = 7^j \left( \frac{v^3 - 2v^2 - v + 1}{v^3 - v^2 - 2v + 1} \right).$$

**Lemma 5.1.** Let $C_j$ be as above.

(i) For every $j \in \mathbb{Z}$, the curve $C_j$ has genus 12.

(ii) If $7 \nmid j$, then $C_j(\mathbb{Q}) = \emptyset$.

**Proof.** The curves $C_j$ are degree 7 covers of $\mathbb{P}^1(\mathbb{C})$ with six branch points, each with ramification degree 7, so by the Riemann-Hurwitz formula they have genus 12.

To prove (ii), we will show that $C_j(\mathbb{Q}_7) = \emptyset$ if $7 \nmid j$. The map $(v, w) \mapsto (1/v, 1/w)$ defines an isomorphism from $C_j$ to $C_{-j}$. Since $C_j \cong C_{j'}$ if $j \equiv j' \pmod{7}$, it suffices to consider just $j = 1, 2, 3$. Then if $(v, w)$ is a point on $C_j$ or $C_{-j}$ defined over $\mathbb{Q}_7$, and $v \in \mathbb{Z}_7$, then $w \in \mathbb{Z}_7^\times$; this follows immediately from Lemma 4.4(a, b), which is clearly valid even for $v \in \mathbb{Q}_7$. Thus, to show that $C_j(\mathbb{Q}_7)$ is empty, it suffices to show that neither of the equations

$$v^3 - 2v^2 - v + 1 = 7^j w^7(v^3 - v^2 - 2v + 1),$$
$$v^3 - v^2 - 2v + 1 = 7^j w^7(v^3 - 2v^2 - v + 1)$$

has solutions $v, w \in \mathbb{Z}_7$. If $j = 2$ or 3, this follows from Lemma 4.4(a, b) since the powers of 7 on the two sides differ. If $j = 1$, then one finds easily that neither equation has a solution modulo $7^3$. □

Recall the elliptic curve $B_v$ defined by [8], and the definition of $p$-exceptional in Definition 1.1.

**Proposition 5.2.** Suppose $v \in \mathbb{Q}$. Then the following are equivalent:

(i) $B_v$ is $7$-exceptional;

(ii) there is a $w \in \mathbb{Q}$ such that $(v, w) \in C_0(\mathbb{Q})$.

**Proof.** By Proposition 4.7, $B_v$ has multiplicative reduction at all primes of bad reduction different from 7. Thus we can apply Theorem 2.7 to $B_v$ with $p = 7$ to
conclude that

\[ \frac{\Delta_{\min}(B'_e)}{\Delta_{\min}(B_e)} \in 7\mathbb{Z} \cdot (\mathbb{Q}^\times)^7 \]

\[ \Longleftrightarrow \left( \frac{v^3 - 2v^2 - v + 1}{v^3 - v^2 - 2v + 1} \right)^6 \in 7\mathbb{Z} \cdot (\mathbb{Q}^\times)^7 \]

\[ \Longleftrightarrow \frac{v^3 - 2v^2 - v + 1}{v^3 - v^2 - 2v + 1} \in 7\mathbb{Z} \cdot (\mathbb{Q}^\times)^7, \]

the middle equivalence by Corollary 4.6(i). This in turn is equivalent to saying there is a point \((v, w) \in C_j(\mathbb{Q})\) for some \(j\) with \(0 \leq j \leq 6\). But by Lemma 5.1(ii), \(C_j(\mathbb{Q})\) is empty if \(1 \leq j \leq 6\). This proves the lemma.

**Theorem 5.3.** Suppose that \(E\) is an elliptic curve over \(\mathbb{Q}\) with a \(\mathbb{Q}\)-isogeny of degree 7. Then the following are equivalent:

(i) \(E\) is 7-exceptional;

(ii) \(E\) is a quadratic twist of \(B_v\) for some \((v, w) \in C_0(\mathbb{Q})\).

**Proof.** Suppose (i), i.e., the image of \(\rho_{E, 7}\) does not contain a Sylow pro-7 subgroup of \(\text{Aut}_Z(T_7(E))\). By Theorem 1.2 (\(\mathbb{R}\) Theorem 1), \(\psi \varphi^{-1}\) has order 2, so by Lemma 2.5 we have \(\psi \varphi^{-1} = \omega^3\). Since \(\psi \varphi = \omega\), we have \(\psi^2 = \varphi^3 = \omega^4\). Let \(\epsilon = \psi \omega\). Then \(\epsilon\) is a quadratic character, \(\psi = \omega^3 \epsilon\), and \(\varphi = \omega^2 \epsilon\). Replacing \(E\) by its quadratic twist by \(\epsilon\), we may assume that \(\psi = \omega^5\) and \(\varphi = \omega^2\). By Theorem 4.1(i), we have that \(E \cong B_v\) for some \(v \in \mathbb{Q}\). Now the theorem follows from Proposition 5.2. □

The curve

\[ C_0 : w^7 = \frac{v^3 - 2v^2 - v + 1}{v^3 - v^2 - 2v + 1} \]

of \(\mathbb{P}^3\) has a nonsingular model \(C \subset \mathbb{P}^3 \times \mathbb{P}^3\) with coordinates \(((v : u), (w : z))\) (that we will abbreviate as \((v, w))\) given by

\[ w^7(v^3 - v^2 u - 2vu^2 + u^3) = z^7(v^3 - 2v^2 u - vu^2 + u^3), \]

which has good reduction outside of 7. By Theorem 5.3 we wish to determine \(C(\mathbb{Q})\). One finds easily the following rational points on \(C\). Let

\[ P_0 = (0, 1), \quad P_1 = (1, 1), \quad P_\infty = (\infty, 1), \]

\[ P_2 = (2, -1), \quad P_3 = (\frac{1}{2}, -1), \quad P_4 = (-1, -1) \]

and

\[ Z = \{ P_0, P_1, P_\infty, P_2, P_3, P_4 \} \subseteq C(\mathbb{Q}). \]

**Theorem 5.4.** \(C(\mathbb{Q}) = Z\).

We will prove Theorem 5.4 using the method of Chabauty, as made explicit in [24]. Before that we deduce the following consequence.

**Theorem 5.5.** If \(E\) is an elliptic curve over \(\mathbb{Q}\) with a \(\mathbb{Q}\)-rational subgroup of order 7, and \(E\) is exceptional for 7, then \(E\) has CM by \(\mathbb{Q}(\sqrt{-7})\), i.e., \(j(E) \in \{-153, 2553\}\), i.e., \(E\) is a quadratic twist of an elliptic curve of conductor 49.

**Proof.** By Theorem 5.3 \(E\) is a quadratic twist of \(B_v\) for some \((v, w) \in C(\mathbb{Q})\). By Theorem 5.4 \((v, w) \in Z\). If \(v \in \{0, 1, \infty\}\), then \(B_v\) is isomorphic to the curve 49A1 in Cremona's tables [6]. If \(v \in \{2, 1/2, -1\}\), then \(B_v\) is isomorphic to the curve 49A2. □
6. The rank of $J(\mathbb{Q})$

Let $J$ be the Jacobian of $C$. The first step in bounding $C(\mathbb{Q})$ is to compute the rank of $J(\mathbb{Q})$. In this section we prove that the rank is 6. We first prove that 6 is an upper bound, following the method described by Poonen and Schaefer in [17]. To keep our notation as close as possible to theirs, we replace $C$ by the (birationally) isomorphic curve

$$X : y^7 = (x^3 - 2x^2 - x + 1)(x^3 - x^2 - 2x + 1)^6.$$ 

Let $\zeta$ be a primitive 7-th root of unity, $k = \mathbb{Q}(\zeta)$, $\mathcal{O} = \mathbb{Z}[\zeta]$, and $\pi = \zeta - 1 \in \mathcal{O}$, a generator of the prime ideal of $\mathcal{O}$ above 7. We identify $\mathcal{O}$ with a subring of $\text{End}_k(J)$ by sending $\zeta$ to the automorphism of $J$ induced by the automorphism $(x, y) \mapsto (x, \zeta y)$ of $X$. We will use [17] to compute an upper bound for the size of $J(k)/\pi J(k)$.

Define

$$f(x) = (x^3 - 2x^2 - x + 1)(x^3 - x^2 - 2x + 1)^6,$$
$$f_0(x) = (x^3 - 2x^2 - x + 1)(x^3 - x^2 - 2x + 1).$$

A calculation in PARI/GP shows that the roots of $x^3 - 2x^2 - x + 1$ are $\alpha_i := 1 + \zeta^i + \zeta^{-i} \in k$ for $1 \leq i \leq 3$, and the roots of $x^3 - x^2 - 2x + 1$ are $\alpha_i := -\zeta^i - \zeta^{-i} \in k$ for $4 \leq i \leq 6$. In particular, $f$ and $f_0$ factor into linear factors in $k[x]$.

Suppose $K$ is a field containing $k$. Let $\text{Div}(X/K)$ denote the group of $K$-rational divisors on $X$, i.e., the group of $\mathbb{Z}$-linear combinations of points in $X(\bar{k})$ that are fixed by $G_K$, let $\text{Div}^0(X/K)$ denote the subgroup of divisors of degree zero, and let $\text{Pic}^0(X/K) = \text{Div}^0(X/K)/P(X/K)$ where $P(X/K)$ is the group of divisors of $K$-rational functions on $X$ (i.e., the principal divisors). Since $X(k)$ is nonempty, there is a natural isomorphism $\text{Pic}^0(X/K) \cong J(K)$, and we will identify these two groups.

If $R$ is a (multiplicative) abelian group, let

$$V(R) := (R/R^7)^6/(R/R^7)$$

where $R^7$ denotes seventh powers in $R$, and $R/R^7$ is embedded diagonally in the direct product $(R/R^7)^6$.

In [17 §5], Poonen and Schaefer define what they call the “$(x - T)$ map” for every field $K$ containing $k$:

$$(x - T)_K : J(K)/\pi J(K) \longrightarrow V(K^\times).$$

This map is characterized as follows. If $D = \sum_P n_P P \in \text{Div}^0(X)$ is supported on points $P \in X(\bar{k})$ with $x$-coordinate $x(P) \notin \{\alpha_i : 1 \leq i \leq 6\} \cup \{\infty\}$, then

$$(x - T)_K(D) := \prod_P ((x(P) - \alpha_1)^{n_1}, \ldots, (x(P) - \alpha_6)^{n_6}).$$

**Lemma 6.1** (Poonen-Schaefer [17]). Suppose $K$ is a field containing $k$, and $P = (x(P), y(P)) \in X(K)$. Let $\infty$ denote the rational point with $x = \infty$, i.e., the point corresponding to $(\infty, 1)$ on the nonsingular model $C$ of $X$. If $x(P) \notin \{\alpha_i : 1 \leq i \leq 6\} \cup \{\infty\}$ then

$$(x - T)_K(P - \infty) = (x(P) - \alpha_1, \ldots, x(P) - \alpha_6).$$

**Proof.** This follows from [17 Proposition 5.1].
There is a natural localization map from $V(k^\times)$ to $V(k^\times)$, where $k_\pi$ is the completion of $k$ at $\pi$. Let $N$ be the “weighted norm” map from §6 of [17]:

$$N : V(k^\times) \to k^\times/(k^\times)^7; \quad (z_1, z_2, z_3, z_4, z_5, z_6) \mapsto z_1 z_2 z_3 (z_4 z_5 z_6)^6.$$

**Theorem 6.2 (Poonen-Schaefer [17]).** In the commutative diagram

$$\begin{array}{ccc}
J(k)/\pi J(k) & \xrightarrow{(x-T)_{k_\pi}} & V(k^\times) \\
\downarrow & & \downarrow \text{loc}_\pi \\
J(k_\pi)/\pi J(k_\pi) & \xrightarrow{(x-T)_{k_\pi}} & V(k^\times)
\end{array}$$

the maps $(x-T)_k$ and $(x-T)_{k_\pi}$ are injective, and the image of $(x-T)_k$ is contained in

$$V(O[1/\pi]^\times) \cap \ker(N) \cap \text{loc}_\pi^{-1}(\text{image}((x-T)_{k_\pi})).$$

**Proof.** That the maps are injective follows from [17] Theorem 11.3, since $X$ has $k$-rational points and $f(x)$ factors into linear factors in $k[x]$.

Let $U$ denote the image of $(x-T)_k$. Then $U \subseteq V(O[1/\pi]^\times)$ by [17] Proposition 12.4 since $J$ has good reduction outside of 7, and $U \subseteq \ker(N)$ by [17] Proposition 12.1. The commutativity of the diagram shows that $U \subseteq \text{loc}_\pi^{-1}(\text{image}((x-T)_{k_\pi}))$.

**Lemma 6.3 (Poonen-Schaefer [17]).** We have

(i) $\dim_{F_\pi} J(k)[\pi] = 4$,

(ii) $\dim_{F_\pi} J(k_\pi)/\pi J(k_\pi) = 16$.

**Proof.** Assertion (i) is [17] Lemma 12.9, since $f(x)$ factors into linear factors in $k[x]$, of which 6 are distinct.

Similarly, [17] Lemma 12.9 shows that $\dim_{F_\pi} J(k_\pi)[\pi] = 4$, and then [17] Lemma 12.10 shows that

$$\dim_{F_\pi} J(k_\pi)/\pi J(k_\pi) = g + \dim_{F_\pi} J(k_\pi)[\pi] = 12 + 4 = 16,$$

where $g = 12$ is the genus of $X$. □

**Remark 6.4.** We observe that $C$ has an action of the group $\Sigma \cong S_3$ generated by the two involutions

$$(v, w) \mapsto (v^{-1}, w^{-1}) \quad \text{and} \quad (v, w) \mapsto (1 - v, w^{-1}).$$

The group $\Sigma$ has the two orbits $\{P_0, P_1, P_\infty\}$ and $\{P_2, P_3, P_1\}$ on the set $Z$ of known rational points.

**Proposition 6.5.** $\text{rank}_O J(k) \leq 6$.

**Proof.** We will use Theorem 6.2 to bound the $O$-rank of $J(k)$. All of the terms in Theorem 6.2 are $F_\pi$-vector spaces, and we need to compute them explicitly.

It follows from Theorem 5.1 of Chapter 3 of [12], and the fact that $\mathbb{Q}(\zeta + \zeta^{-1})$ has class number one, that $O[1/\pi]^\times$ is generated by the roots of unity, the cyclotomic units, and $\pi$. Thus an $F_\pi$-basis of $O[1/\pi]^\times/(O[1/\pi]^\times)^7$ is given by $\{\zeta, 1 + \zeta, 1 + \zeta + \zeta^2, \pi\}$. 

Using PARI/GP and Lemma 6.1, we compute \( \dim F_i(\text{image}((x - T)_{k_x})) = 16 \). Using PARI/GP, we find points \( Q_i = (x_i, y_i) \in X(k_x) \) for \( 1 \leq i \leq 6 \) with \( x \)-coordinates:

\[
x_1 = 0, \quad x_2 = -1, \\
x_3 = 3 + 4\pi^2 + 5\pi^3 + \pi^4 + 4\pi^5 + 2\pi^6 + 6\pi^7 + 5\pi^8 + 5\pi^9 + 5\pi^{10}, \\
x_4 = 3 + \pi^2 + 5\pi^3 + 5\pi^4 + 4\pi^5 + 2\pi^6 + 5\pi^7 + 5\pi^8 + \pi^{10}, \\
x_5 = 3 + \pi^2 + 2\pi^4 + 4\pi^5 + 2\pi^6 + \pi^7 + 2\pi^8, \\
x_6 = 3 + 2\pi^2 + 5\pi^3 + \pi^4 + 6\pi^7 + 2\pi^8 + 6\pi^9 + 2\pi^{10} + 5\pi^{11} + 4\pi^{12} + 2\pi^{14} + 2\pi^{15} + 6\pi^{16} + \pi^{17}.
\]

Using PARI/GP and Lemma 6.1, we compute \( (x - T)_{k_x}(\sigma(Q_i) - \infty) \) for \( 1 \leq i \leq 6 \) and for all \( \sigma \in \Sigma \), and we find that those values generate an \( F_7 \)-subspace of \( V(k_7^\times) \) of dimension 16. (We work inside the \( F_7 \)-vector space \( k_7^\times/(k_7^\times)^7 \), using the basis \( \{ \pi, 1 + \pi, 1 + \pi^2, 1 + \pi^3, 1 + \pi^4, 1 + \pi^5, 1 + \pi^6, 1 + \pi^7 \} \).)

It follows that we have found the full image of \( (x - T)_{k_x} \).

Using the above information, a linear algebra computation in PARI/GP now shows that

\[
\dim F_7(V(\mathcal{O}[1/\pi]^\times) \cap \ker(N) \cap \text{loc}_\pi^{-1}(\text{image}((x - T)_{k_x}))) = 10.
\]

Therefore by Theorem 6.2 we have \( \dim F_7 J(k)/\pi J(k) \leq 10 \). Since

\[
\dim F_7 J(k)/\pi J(k) = \text{rank}_\mathcal{O} J(k) + \dim F_7 J(k)[\pi],
\]

and \( \dim F_7 J(k)[\pi] = 4 \) by Lemma 6.3(i), we conclude that \( \text{rank}_\mathcal{O} J(k) \leq 6 \).

\( \square \)

**Lemma 6.6.** \( \text{rank}_\mathcal{O} J(Q) \leq 6 \).

**Proof.** This follows from Proposition 6.5 and [17, Lemma 13.4]. \( \square \)

**Remark 6.7.** The involution \( (v, w) \mapsto (v^{-1}, w^{-1}) \) has exactly the two fixed points \( P_1 \) and \( P_4 \), therefore the quotient of \( C \) by this involution is a curve \( D \) of genus 6. The Jacobian \( J \) of \( C \) is isogenous to a product of two copies of the Jacobian of \( D \). The genus 6 curve \( D \) is

\[
Y^7 - 7Y^5 + 14Y^3 - 7Y = (2X^3 - 6X^2 - 7X + 24)/(X^3 - 3X^2 - 4X + 13)
\]

which by a change of variables is

\[
(-2 + Y)(-1 - 2Y + Y^2 + Y^3)^2 = X/(1 - 4X + 3X^2 + X^3).
\]

**Remark 6.8.** We now elaborate on the path that led to the proof that \( |C(Q)| = 6 \) (although it isn’t actually used in our proof). The subgroup of \( J(Q) \) generated by differences of known rational points on \( C \) has rank 4. Since \( J \) is \( Q \)-isogenous to \( \text{Jac}(D)^2 \), the rank of \( J(Q) \) is even, so it must be either 4 or 6. The \( S_3 \)-action shows that \( |C(Q)| \) is divisible by 6. A Chabauty argument at the prime 2, using [24], then gives that \( |C(Q)| \) is 6 or 12. The argument is as follows. Consider the pairing \( J(Q_2) \times \Omega(C/Z_2) \to Q_2 \) defined by \( G, \omega \mapsto (G, \omega) := \int_G \omega \) where \( \Omega(C/L) \) is the set of holomorphic differentials on \( C \) over \( L \) and \( G \) is a degree 0 divisor on \( C \). View \( Z \subset C(Q) \subset J(Q) \) (fixing a basepoint in \( Z \)) and let

\[
V := \{ \omega \in \Omega(C/Z_2) : (J(Q), \omega) = 0 \} \subseteq V_0 := \{ \omega \in \Omega(C/Z_2) : (Z, \omega) = 0 \}.
\]
Let $\tilde{V}$ (resp., $\tilde{V}_0$) be the image of $V$ (resp., $V_0$) under the reduction map $\Omega(C/\mathbb{Z}_2) \to \Omega(C/\mathbb{F}_2)$. Then $\dim_{\mathbb{F}_2} \tilde{V} = \text{rank}_{\mathbb{Z}_2} V \geq 12 - \text{rank } J(Q) \geq 6$. Let

$$W := \{ \omega \in \Omega(C/\mathbb{F}_2) : \text{ord}_P(\omega) \geq 2 \text{ for all } P \in \{R_0, P_1, P_\infty\} \}.$$

If $|C(Q)| = 12$, each fiber of $C(Q) \to C(\mathbb{F}_2)$ has size 4, and it follows (using [24]) that $\tilde{V} \subseteq W \cap \tilde{V}_0$. So if $\dim(W \cap \tilde{V}_0) < 6$, then $|C(Q)| = 6$. Balakrishnan, using ideas of Kedlaya and Wetherell, computed $J^0_\omega$ for $\omega \in \Omega(C/\mathbb{Z}_2)$ and $P,Q \in \mathbb{Z}$. This gives $V_0$ and $\tilde{V}_0$. Unfortunately, $\dim(W \cap \tilde{V}_0) = 6$. However, if rank $J(Q)$ were 4, then $\dim_{\mathbb{F}_2} \tilde{V} = \text{rank}_{\mathbb{Z}_2} V \geq 12 - \text{rank } J(Q) = 8$. This would imply that $|C(Q)| = 6$, since $\tilde{V} \subseteq W \cap \tilde{V}_0$ when $|C(Q)| = 12$. As explained in Appendix A, we had reason to believe that rank$_{\mathbb{Z}} J(Q) = 6$. This gave motivation to find additional generators of $J(Q)$, which we do in the proof of the following theorem.

**Theorem 6.9.** $\text{rank}_{\mathbb{Z}} J(Q) = 6$.

**Proof.** By Lemma 6.6 we have $\text{rank}_{\mathbb{Z}} J(Q) \leq 6$.

We look for closed points of higher degree on the genus 6 quotient $D$ of $C$ defined in Remark 6.7. For $D$, we use the model in $\mathbb{P}^3$ obtained as the image of the map sending a point $(v, w)$ on $C$ to $(v + \frac{1}{7} : w + \frac{1}{w} : \frac{v}{w} + \frac{w}{v} : 1)$. This model is a smooth curve of genus 6 (and degree 10) still having bad reduction only at 7. We intersect $D$ with hyperplanes and quadrics of increasing height and split the resulting divisor as a sum of prime divisors. If it splits, we check if the new prime divisor leads to a larger subgroup in the Jacobian of $D$. For this check, we use the homomorphism

$$\text{Pic}(D) \to \prod_{p \in S} \text{Pic}(D/\mathbb{F}_p)$$

where $S$ is a suitable set of primes (we used $S = \{2, 3, 5, 11, 13, 17\}$); compare [24] for details.

In this way, we find a point of degree 4 that leads to a larger group that is still of rank 2, and a point of degree 8 that increases the rank to 3. Pulling back these points to $C$, we obtain a point of degree 8 of the form

$$(-\alpha^7 + 2\alpha^6 + \alpha^4 + \alpha^2 + 2, \alpha)$$

with $\alpha$ a root of

$$t^8 - 2t^7 - t^5 - t^3 - 2t + 1,$$

and a point of degree 16 of the form

$$t^{16} + 5t^{15} + 2t^{14} + 4t^{13} + 5t^{12} + 10t^{11} + 11t^{10} + 13t^9 + 6t^8 + 13t^7 + 11t^6 + 10t^5 + 5t^4 + 4t^3 + 2t^2 + 5t + 1.$$ 

Denote the corresponding prime divisors on $C$ by $Q_1$ and $Q_2$, respectively. Let $Q_3 = \sigma(Q_1)$ and $Q_4 = \sigma(Q_2)$, where $\sigma \in \Sigma$ is the map $(v, w) \mapsto (\frac{1}{v}, w)$. Let $R \subset J(Q)$ be the subgroup generated by degree-zero linear combinations of the ten prime divisors:

$$\{P_0, P_1, P_\infty, P_2, P_3, P_4, Q_1, Q_2, Q_3, Q_4\}.$$
torsion. (By considering additional reductions one can show that $|J(\mathbb{Q})|_{\text{tors}}$ divides $7^2$.) In addition, we check that the image of

$$H \rightarrow J(\mathbb{F}_2) \times J(\mathbb{F}_3) \times J(\mathbb{F}_{11})$$

has a quotient isomorphic to $(\mathbb{Z}/5\mathbb{Z})^6$. It follows that $\text{rank}_{\mathbb{Z}}H \geq 6$, so $\text{rank}_{\mathbb{Z}}J(\mathbb{Q}) = 6$ and $H$ has finite index in $J(\mathbb{Q})$. \hfill $\Box$

Remark 6.10. In the course of the calculations in the proof of Theorem 6.9, we showed also that the subgroup $H \subseteq J(\mathbb{Q})$ is free of rank 6, and that the relations among the divisors (14) are generated by:

$$P_0 + P_1 + P_\infty \sim P_2 + P_3 + P_4,$$

$$P_1 + P_4 + Q_1 \sim P_2 + P_\infty + Q_3,$$

$$5P_0 + 3P_1 + 4P_\infty + 4P_2 + 5P_3 + 3P_4 \sim 2Q_1 + Q_3.$$  

7. Proof of Theorem 5.4

Since the Mordell-Weil rank of $J(\mathbb{Q})$ is smaller than the genus of $C$, we can apply Chabauty’s method (see [4, 15, 24]) to calculate $C(\mathbb{Q})$.

Define divisors on $C$:

$$D_x = \sum_{\zeta \in \mu_7} (x, \zeta) \text{ for } x \in \{0, 1, \infty\}, \quad G_1 = \sum_{i=1}^{3} (\alpha_i, 0), \quad G_2 = \sum_{i=4}^{6} (\alpha_i, \infty),$$

where (as before), $\alpha_1, \alpha_2, \alpha_3$ (resp., $\alpha_4, \alpha_5, \alpha_6$) are the roots of $v^3 - 2v^2 - v + 1$ (resp., $v^3 - v^2 - 2v + 1$). One checks easily that the divisors of the rational functions $v, w,$ and $v^3 - v^2 - 2v + 1$ are given by

$$v = D_0 - D_\infty, \quad w = G_1 - G_2, \quad (v^3 - v^2 - 2v + 1) = 7G_2 - 3D_\infty.$$  

If $L$ is a field of characteristic different from 7, let $\Omega(C/L)$ denote the $L$-vector space of holomorphic differentials on $C/L$. If $\omega \in \Omega(C/L)$, let $(\omega)$ denote the divisor of $\omega$.

Lemma 7.1. Suppose $L$ is a field of characteristic different from 7.

(i) The divisor of the differential $dv$ is $(dv) = 6G_1 + 6G_2 - 2D_\infty$.

(ii) A basis for the $L$-vector space $\Omega(C/L)$ is given by

$$\omega_{i,j} := \frac{v^i w^j}{(v^3 - v^2 - 2v + 1) w^6} dv, \quad 0 \leq i \leq 1, \quad 0 \leq j \leq 5.$$  

Proof. The function $v$ has (simple) poles at each of the 7 points $\{(\infty, \zeta) : \zeta \in \mu_7\}$, and no other poles. Hence $\text{ord}_{(\infty, \zeta)}(dv) = -2$, and $\text{ord}_P(dv) \geq 0$ for all other points $P$. If $\alpha$ is a root of $v^3 - 2v^2 - v + 1$, then the equation for $C$ shows that $\text{ord}_{(\alpha, 0)}(v - \alpha)$ is a (positive) multiple of 7. Since the polar divisor of $v$ is $D_\infty$, we conclude that $\text{ord}_{(\alpha, 0)}((v - \alpha)) = 7$, and

$$\text{ord}_{(\alpha, 0)}(dv) = \text{ord}_{(\alpha, 0)}(d(v - \alpha)) = 6.$$  

Similarly, if $\alpha$ is a root of $v^3 - v^2 - 2v + 1$ then $\text{ord}_{(\alpha, \infty)}(dv) = 6$. Since the divisor $(dv)$ has degree $2g - 2 = 22$, we conclude that $(dv) = 6G_1 + 6G_2 - 2D_\infty$, giving (i).

It now follows from (15) that the differential $\omega_{i,j}$ has divisor

$$(\omega_{i,j}) = iD_0 + (1 - i)D_\infty + jG_1 + (5 - j)G_2.$$
In particular $\omega_{ij}$ is holomorphic if (and only if) $0 \leq i \leq 1$ and $0 \leq j \leq 5$.

Since $v$ is transcendental over $L$ and $w$ has degree 7 over $L(v)$, the set \{ $v^iw^j : 0 \leq i \leq 1, 0 \leq j \leq 5$ \} is linearly independent over $L$, so \{ $\omega_{ij} : 0 \leq i \leq 1, 0 \leq j \leq 5$ \} is linearly independent over $L$. Since $\dim_L(\Omega(C/L)) = \text{genus}(C) = 12$, we have (ii).

Let $\Omega(C/Z_5)$ be the $Z_5$-span of the differentials $\omega_{i,j}$ with $0 \leq i \leq 1$ and $0 \leq j \leq 5$. Consider the bilinear pairing

$$J(Q_5)/J(Q_5)_{\text{tors}} \times \Omega(C/Z_5) \to Q_5$$

of free $Z_5$-modules of rank 12 with trivial left and right kernel that is used on p. 1210 of [24]. Let $V \subset \Omega(C/Z_5)$ be the orthogonal complement under this pairing of (the closure of) $J(Q) \subset J(Q_5)$. By Theorem 6.9 we have $\text{rank}_{Z}J(Q) = 6$.

Let $\tilde{V} \subset \Omega(C/F_5)$ be the image of $V$ under the (surjective) reduction map $\text{red}_5 : \Omega(C/Z_5) \to \Omega(C/F_5)$. Since $\text{rank}_{Z_5}(\Omega(C/Z_5)) = 12 = \dim_{F_5}(\Omega(C/F_5))$, we have $\ker(\text{red}_5) = 5\Omega(C/Z_5)$. Since $\Omega(C/Z_5)/V$ is torsion-free, we have $5V = V \cap 5\Omega(C/Z_5) = \ker(\text{red}_5|_{V})$. Thus $\tilde{V} \cong V/5V$, so $\dim_{F_5}(\tilde{V}) = \text{rank}_{Z_5}(V) = 6$.

We will show that for each point $P \in C(F_5)$, there exists a differential $\omega_P \in \tilde{V}$ that does not vanish at $P$. Proposition 6.3 of [24] then shows that there is at most one point in $C(Q)$ that reduces to $P$. Since the set $Z \subseteq C(Q)$ bijects onto $C(F_5)$ via the reduction map, that will show $Z = C(Q)$, as desired.

Next we determine the space $\tilde{V}$ explicitly.

Using lattice basis reduction (with higher weights on the higher-degree points), we find the following generators of the intersection of the known finite-index subgroup of $J(Q)$ and the kernel of reduction mod 5:

- $B_1 = P_1 - P_\infty + P_2 - P_4 - 6Q_2 + 6Q_4$,
- $B_2 = -4P_0 + 9P_\infty + 5P_2 - 3P_3 + P_4 + 4Q_1 - 7Q_3 + Q_4$,
- $B_3 = -4P_0 + 9P_1 + P_2 - 3P_3 + 5P_4 - 7Q_1 + Q_2 + 4Q_3$,
- $B_4 = -2P_0 - 2P_1 + 4P_\infty - 11P_2 + 13P_3 - 2P_4 - 4Q_1 + Q_2 - 2Q_3 + 2Q_4$,
- $B_5 = -6P_1 + 6P_\infty - 9P_2 + 9P_4 - 2Q_1 - Q_2 + 2Q_3 + Q_4$,
- $B_6 = 10P_0 - 8P_1 - 14P_\infty + P_2 - 7P_3 - 6P_4 - 12Q_1 + 11Q_2 - 15Q_3 + 4Q_4$.

For each of these divisors $B_m$, we compute the Riemann-Roch space of $\pm B_m + 12P_0$ (with sign chosen in an attempt to make the computation more efficient). We check that it is of dimension one and that the same is true for the corresponding Riemann-Roch space over $F_5$. Let $f_m$ be a function spanning the space; then it follows that the divisor of $f_m$ must be $\mp B_m - 12P_0 + D_m$ with an effective divisor $D_m$ of degree 12 supported on points having the same reduction mod 5 as $P_0$. (Note that the divisor of the reduction of $f_m$ mod 5 must be the reduction of $\mp B_m$, since this reduction is a principal divisor and the Riemann-Roch space of the reduction of $\pm B_m + 12P_0$ has dimension one.)

We write each basis element $\omega_{ij} \in \Omega(C/Z_5)$ as a power series in $t$ times $dt$, where $t = v$ is a uniformizer at $P_0$ (and also a uniformizer at the reduction of $P_0$ mod 5). We then integrate formally to obtain the corresponding logarithms $\lambda_{ij}$ as power series in $t$. These power series converge 5-adically on points in the same residue class as $P_0$. The expansion of $f_m$ as a Laurent series in $t$ is of the form $t^\alpha F_m(t)$ where $\alpha = -12 \mp$ the multiplicity of $P_0$ in $B_m$. After scaling by a power of 5 if necessary, the series $F_m(t)$ is in $Z_5[[t]]$, the coefficients of $t^k$ with $k < 5$ have
positive valuation, and the coefficient of $t^{12}$ is a 5-adic unit. This reflects the facts that $D_m$ has the same reduction mod 5 as $12P_0$, and $P_0$ is not in the support of the reductions mod 5 of any of the $Q_i$ or $P_j$ with $j \neq 0$ (as is easily checked). We multiply this series by an invertible series in $\mathbb{Z}_5[t]$ such that we obtain a polynomial of degree 12 (to sufficient 5-adic precision). The roots of this polynomial are then the values of degree 12 that are in the support of $D_m$. We can then easily compute the power sums of these values (by looking at the logarithmic derivative of the reverse of the polynomial, see again [25]) and therefore evaluate the integrals

$$\int_{[D_m-12P_0]} \omega_{ij} = \sum_{k=1}^{12} \lambda_{ij}(t_k).$$

The values are in $5\mathbb{Z}_5$. Dividing by 5 and reducing mod 5, we obtain a linear relation that every differential in $\tilde{V}$ has to satisfy.

We used Magma [11] for these computations. A recent version (2.17-1) spends less than 16 hours on current hardware to produce the relations we are looking for. We obtain a 6-dimensional space of relations generated by the following elements (in terms of the basis dual to $\omega_{00}, \ldots, \omega_{05}, \omega_{10}, \ldots, \omega_{15}$):

$$\begin{align*}
(1, 0, 0, 0, 0, 0, 0, 4, 1, 1, 3, 3, 4), \\
(0, 1, 0, 0, 0, 0, 1, 1, 2, 3, 0), \\
(0, 0, 1, 0, 0, 0, 3, 4, 0, 0, 4, 2), \\
(0, 0, 0, 1, 0, 0, 3, 1, 0, 1, 1, 2), \\
(0, 0, 0, 0, 1, 0, 0, 2, 3, 0, 0, 4), \\
(0, 0, 0, 0, 0, 1, 1, 2, 4, 4, 2).
\end{align*}$$

As stated above, Theorem 5.4 would follow if we can find, for $P \in C(\mathbb{F}_5)$, a differential $\omega_P \in \tilde{V}$ that does not vanish at $P$. Next we give an explicit differential $\omega$ that works for all $P \in C(\mathbb{F}_5)$. Let $h(v, w) = 3 + w + 3w^3 + 2w^4 + v(w^2 + 2w^3)$, $f_2(v) = v^3 - v^2 - 2v + 1$, and

$$\omega := 3\omega_{00} + \omega_{01} + 3\omega_{03} + 2\omega_{04} + \omega_{12} + 2\omega_{13} = \frac{h(v, w)}{w^6} \cdot \frac{dv}{f_2(v)} \in \Omega(C/\mathbb{F}_5).$$

The coefficients of the $\omega_{ij}$ that define $\omega$ satisfy the relations above, so $\omega \in \tilde{V}$.

If $0 \leq i \leq 4$ then (15) and Lemma 7.1(i) show that $dv/f_2(v)$ does not vanish at $P_i$. By inspection $h(v, w)/w^6$ does not vanish at $P_i$, so $\omega$ does not vanish at $P_i$.

Similarly, $dv/f_2(v)$ vanishes at $P_\infty$ but $v dv/f_2(v)$ does not, by (15) and Lemma 7.1(i). Since $w^2 + 2w^3$ also does not vanish at $P_\infty$, it follows that $\omega$ does not vanish at $P_\infty$. This concludes the proof of Theorem 5.4.

As an independent check, we have performed a similar computation on the curve $D$. There are four points in $D(\mathbb{F}_5)$ that are images of points in $C(\mathbb{F}_5)$. For each of these points, we find a differential that kills the Mordell-Weil group of $D$ and whose reduction mod 5 does not vanish at the given point. This shows that $D$ can have at most four rational points that are in the image of $C(\mathbb{Q})$; this number is accounted for by the known points.

**Appendix A. Conjectural determination of the rank**

Recall that $J$ is isogenous to the square of the Jacobian $J_D$ of a curve $D$ over $\mathbb{Q}$ of genus 6 defined in Remark 6.7. In this appendix, we explain the computations
that led us to believe that the Mordell-Weil rank of $J_D$ is 3 and hence that the rank of $J(JD)$ is 6, assuming standard conjectures on $L$-series and the conjecture of Birch and Swinnerton-Dyer for $J_D$.

Since we already knew that the rank of $J_D(\mathbb{Q})$ must be either 2 or 3, it would be sufficient to verify that a positive sign in the functional equation for the $L$-series of $J_D$ is not consistent with standard conjectures. The result would be even more convincing if it could also be shown that a negative sign, and analytic rank 3 in particular, is consistent with the conjectures.

Two of the main ingredients for the computation are the conductor of $J_D$ and the Fourier coefficients of its $L$-series. Since $D$ (and therefore $J_D$) has bad reduction only at 7, the conductor is of the form $7^n$ for some $n$. Since the reduction of $J_D$ at 7 is totally unipotent (this is shown by computing a proper regular model of $D$ over $\mathbb{Z}_7$, whose special fiber turns out to be tree-like and without components of positive genus), it follows that $n \geq 2 \cdot \text{genus}(D) = 12$. An upper bound $n \leq 26$ follows from [14] or [3].

The Fourier coefficients of the $L$-series can be obtained by counting points on $D$ or $C$ over all finite fields $\mathbb{F}_p$ for $p$ below some bound $N$ (and $e \leq \text{genus}(D) = 6$, making use of the Weil conjectures). This provides the first $N$ coefficients. In our case, Balakrishnan, Sutherland, and Kedlaya provided these coefficients for $N = 10^7$ and $p \neq 7$. The Euler factor at 7 is trivial, since the reduction is totally unipotent. This many coefficients turned out to be enough to produce satisfactory results (“a handful” of digits of accuracy, according to Rubinstein).

Rubinstein performed the $L$-series computations for us, using his lcalc package [19]. He first checked that the rank of $J_D(\mathbb{Q})$ being 2 is not consistent with the standard conjectures, by computing the central $L$-value assuming the root number to be +1 and the conductor to be $7^{12}, 7^{13}, \ldots, 7^{26}$. In each case the result is clearly non-zero. Since we know that the rank is 2 or 3, the Birch and Swinnerton-Dyer conjecture leads to the conclusion that the rank must be 3. This computation is based on the approximate functional equation; see [18, Thm. 1].

As a further check, Rubinstein used the approximate functional equation to compute the first 17 zeros on the critical line (normalized to be $s = 1/2$) with positive imaginary part, assuming the root number to be $-1$ and the conductor to be $7^{26}$. He then compared the two sides of the “explicit formula”, assuming a triple zero at the critical point and using the function $\phi(x) = \left(\frac{\sin 4x}{4x}\right)^4$ (whose Fourier transform has compact support of a size that allows essentially exact computation of the right hand side of the explicit formula with the known Fourier coefficients). The values obtained for $\sum_\gamma \phi(\gamma - t)$, where $\gamma$ runs through the zeros and $0 \leq t \leq 3$, agree almost perfectly. See Figure 1 for the corresponding calculation using only 16 zeros away from the real axis, where the two graphs can be seen to begin to diverge for $t > 2.8$.

Heuristic considerations indicate that the discriminant of $C$ should be $7^{52}$. The given model of $C$ is already regular at 7, so the exponent of the conductor should equal that of the discriminant. Since the conductor of $C$ is the square of that of $D$, this provides further evidence that $D$ has conductor $7^{26}$.

Note that the global root number of the $L$-function of $J_D$ should be the product of local root numbers, which in our case will all be +1 except at $p = 7$. So it would be sufficient to determine the local root number at 7 in order to get the parity
of the rank. However, computing such a local root number when the reduction is
unipotent seems to be rather hard.

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