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For Reference

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NEW DUALITY CONSTRAINTS ON MESON AMPLITUDES

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ABSTRACT

By requiring s- and t-channel resonance residues to be equal in \( n\pi, nK \), and \( KK \) amplitudes we obtain 18 relations between the resonance couplings (for spins \( \leq 3 \)). All constraints are satisfied within about 10%, with one (possible) exception. Self-consistency requires the resonances to lie on approximately linear Regge trajectories of common slope.

A recent study of the dynamical basis for semi-local duality phenomena led us to suggest certain modifications to the conventional way of imposing duality constraints. In this letter we shall apply these considerations to the \( \pi\pi \rightarrow \pi\pi \), \( nK \rightarrow nK \) and \( KK \rightarrow KK \) scattering amplitudes. As is well-known, the resonance structure in these amplitudes is particularly simple, and Finite Energy Sum Rules (FESR's) have been applied with some success. We also comment on the duality properties of amplitudes involving charmed mesons, and obtain an estimate for the \( D^* \) mass.

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The modifications we propose to the usual (FESR) approach are three-fold. First, duality constraints should only be applied at the (discrete) values of \( t \) corresponding to resonance positions in that channel. This eliminates self-contradictions due to the fact that s-channel and t-channel contributions cannot be equal over a continuous range in \( t \). It also implies that the magnitudes of both the s- and t-channel contributions are given by physical three-particle couplings. The FESR calculations of ref. 2 were actually done at \( t = (\text{resonance mass})^2 \) for precisely this reason. The second modification is that we shall use the exact expression for the t-channel resonance term, proportional to a Legendre polynomial, instead of the usual high-energy Regge form \( \beta(t)\alpha(t) \). This further enhances the symmetry in the treatment of the s- and t-channels. It also provides a solution to an old ambiguity that has plagued semi-local duality considerations, concerning which variable \( (s \) or \( v = \frac{1}{2}(s - u), \) or something else) should be used in the Regge expression. Finally, our third modification is that we shall not integrate, as in an FESR, the t-channel contribution over a range in \( s \). Instead, we simply equate the s- and t-channel resonance residues, each evaluated at the fixed values of \( s \) and \( t \) given by the resonance masses *. This corresponds to replacing the integral by the

* If the resonance spacing is different in the s- and t-channels, this must be taken into account by multiplying each residue by the trajectory slope \( \alpha' \) in that channel.
value of the integrand at the position of the s-channel resonance, multiplied by the resonance spacing \((\sigma_s')^{-1}\). Thus, in our treatment there will be no ambiguities associated with the value of the cut-off in an FESR integral.

This formulation of semi-local duality constraints emerges as a consequence of the more general arguments presented in ref. 1, applied to the pseudo-scalar amplitudes mentioned above. The success of this application is therefore an important indication of the correctness of the basic approach. In their present form, however, the duality constraints used here are not applicable to more complicated scattering amplitudes. We rely in this paper on the fact that the external particles in the \(\pi\pi\), \(\pi K\) and \(K\bar{K}\) amplitudes are spinless, and that there is a single dominating resonance in each \((\text{mass})^2\) interval of \(\leq 1\ \text{GeV}^2\).

We shall restrict ourselves to resonances with spin \(\leq 3\) in the \(\pi\pi\) and \(\pi K\) systems, and to the \(f\) and \(f'\) resonances in \(K\bar{K}\). As stated above, for each choice of an s-channel and a t-channel resonance there is a duality constraint which requires the corresponding residues to be equal. Thus we get \(3 \times 3 = 9\) constraints each from the \(\pi\pi \rightarrow \pi\pi\) and \(\pi K \rightarrow \pi K\) amplitudes, and \(2 \times 3 = 6\) constraints from \(K\bar{K} \rightarrow K\bar{K}\). Because of symmetry, however, only 3 of the \(\pi\pi \rightarrow \pi\pi\) constraints are nontrivial, so that there are all together 18 independent relations. Out of these, 7 are self-consistency relations involving only the masses (and spins) of the particles. Remarkably, they are all satisfied to better than 10%.

Moreover, it turns out (numerically) that the linearity of the Regge trajectories is crucial for this agreement. The remaining 11 relations constrain the couplings of the resonances to the \(\pi\pi\), \(\pi K\) and \(K\bar{K}\) systems. With one exception \(^{*}\) they are all in close agreement with experiment \(^{3}\) and/or \(SU_3\) symmetry.

We shall now discuss the constraints coming from the various amplitudes in more detail. Our normalization of the three-particle couplings \(T\) is such that the residue of a resonance \(s\) in \(a + b \rightarrow c + d\) is \((a, \cdots, d\) are pseudoscalars)

\[
T(a + b \rightarrow s) T^*(c + d \rightarrow s) J_s \left(\cos \theta_s\right) \propto \delta_n(2J_s + 1)m^2 \left(\frac{\Gamma_1 \Gamma_f}{P_1 P_f}\right)^{1/2} \left(\frac{P_1 P_f}{P_1 P_f}\right)^{1/2} P_s \left(\cos \theta_s\right).
\]

Here \(m, J_s\) are the mass and spin of \(s\), while \(\Gamma_1, \Gamma_f\) and \(P_1, P_f\) are the initial (final) state partial widths and decay momenta, respectively. \(\theta_s\) is the s-channel scattering angle. As in ref. 1, we define isospin reduced couplings \(\tilde{T}\) by

\[
\tilde{T}(a + b \rightarrow s) = \left(\begin{array}{c} I_a \cr I_b \cr I_z \cr I_s \end{array}\right) \left(\begin{array}{cccc} T(a + b \rightarrow s) & & & \\
& T(a + b \rightarrow s) & & \\
& & T(a + b \rightarrow s) & \\
& & & T(a + b \rightarrow s) \end{array}\right) \left(\begin{array}{c} I_a \\
I_b \\
I_z \\
I_s \end{array}\right).
\]

\(^{*}\) We predict the \(g_{\pi\pi}\) coupling to be 30% larger than the currently accepted value \(^{3}\). However, we shall argue below that the experimental uncertainty is larger than the quoted error bars. Hence this discrepancy may, in fact, be spurious.
where \( I, I^2 \) denote total isospin and its third component, and \( \langle \ldots \rangle \) is a 3j symbol.

In this charge configuration of the \( \pi \pi \to \pi \pi \) amplitude the u-channel is exotic and can be ignored. The fact that the s- and t-channel amplitudes are identical means that all constraints where the resonances in the two channels are the same \((s = t = \rho, f \text{ or } g)\) are trivially satisfied. The resonance residues (1) contain, in this case, the same couplings \( T_1 \) and the Legendre functions are also equal, since \( \theta_s = \theta_t \). If we let \( s = f \) and \( t = \rho \) (or vice versa) and require the residues to be equal, we get

\[
\frac{|\mathcal{M}(\pi \pi \to f)|^2}{|\mathcal{M}(\pi \pi \to \rho)|^2} = \frac{P_1(\rho, f)}{P_2(\rho, \rho)} = 0.852.
\]

(3)

Here \( P_1(\rho, f) \) denotes \( P_1(\cos \theta_s) \), with \( \cos \theta_s \) to be evaluated at \( s = m_{\rho}^2, \ t = m_f^2 \). The definition of \( P_2(f, \rho) \) is analogous. For \( s = g \) and \( t = \rho \) we get the condition

\[
\frac{|\mathcal{M}(\pi \pi \to g)|^2}{|\mathcal{M}(\pi \pi \to \rho)|^2} = \frac{P_1(\rho, g)}{P_2(g, \rho)} = 2.32.
\]

(4)

The final \( \pi \pi \to \pi \pi \) constraint (for spin \( \leq 3 \)) results from \( s = g; \ t = f \). However, the \( g\pi \pi/\pi\pi \) coupling ratio being already determined by eqs. (3) and (4), this may be written as a self-consistency

*The numerical values are calculated using the masses given in the PG tables. In the case of \( \pi \) and \( K \) we use the masses of the charged members of the isospin multiplet.

The dependence of the ratio (5) on the resonance masses is clearly very nonlinear. It is thus quite surprising that the condition for this ratio to be near 1 appears (numerically) to be that the resonances lie on a linear Regge trajectory. Thus, if instead of the physical \( g \) mass \((m_g = 1.690 \text{ GeV})\) we use the mass value \((1.623 \text{ GeV})\) corresponding to a linear trajectory through \( \rho \) and \( f \), the ratio (5) equals 1.02. It is also interesting to note that if we extend the analysis to the \( j = 4 \) recurrence on the linear trajectory, we get two more self-consistency conditions which are valid at the 5% level. Finally, we may study the dependence of the ratio (5) on the slope \( \alpha' \) of the linear trajectory (assumed to pass through the \( \rho \)). For \( \alpha' = 0.1 \text{ GeV}^{-2} \) the ratio is 0.954, while for \( \alpha' = 10 \text{ GeV}^{-2} \) it is 1.002 (as \( \alpha' \to \infty \) the ratio tends to 1). Thus the ratio is almost independent of \( \alpha' \) in this range, which is again quite surprising, considering that some of the factors in (5) change by more than a factor 30. This should be contrasted with the sensitivity of the ratio on deviations from a linear trajectory: we saw above that a fairly small change in the \( g \) mass \((\Delta m_g^2 = 0.22 \text{ GeV}^2)\) altered the ratio by 10%. Hence the self-consistency conditions imply that the Regge trajectories must be nearly linear, but do not determine the slope.
The experimental value for the coupling ratio (3) is
0.84 ± 0.09, in very good agreement with our duality condition. On
the other hand, the ratio (4) is measured to be 1.37 ± 0.23,
considerably smaller than our prediction. Equivalently, we could
say that eq. (4) predicts the elasticity of g to be
x = 0.41 ± 0.07, whereas the PDG value is 0.24 ± 0.01.
It should be noted, however, that this determination is based on
analyses of data on \( \pi^+p \rightarrow \pi^+\pi^-n \), extrapolated to the pion exchange pole. The
experiments which have determined the g elasticity directly by
observing both the 2π and 4π decay modes actually obtain values
close to ours: x = 0.4 ± 0.1. Thus we believe that a definite
comparison of eq. (4) with data must be postponed until the two ways
of experimentally measuring the elasticity give compatible results.

\[ \pi^+K^- \rightarrow \pi^+K^- \]

By taking ratios of the constraints for \( s = K^*(892) \),
\( K^{2*}(1420) \) \(^*\) and \( K^{3*}(1780) \) \(^**\) we get

\[ \frac{\Gamma(nK \rightarrow K^{*})}{\Gamma(nK \rightarrow K)} = \frac{P_1(K^*, t) P_2(t, K^{*})}{P_1(t, K^*) P_2(K, t)} \quad (6a) \]

\[ \frac{\Gamma(nK \rightarrow K^{*})}{\Gamma(nK \rightarrow K^{*})} = \frac{P_1(K^*, t) P_2(t, K^{*})}{P_1(t, K^*) P_2(K, t)} \quad (6b) \]

Thus there are three different expressions for each ratio, cor-
responding to \( t = \rho, \pi \) and g. For the ratio (6a), these three
choices of \( t \) give the numerical values 2.64, 2.86 and 2.69,
respectively. This is to be compared with the experimental value
2.43 ± 0.22 for the same ratio. The \( SU_3 \) prediction \(^*\), using eq.
(3), is 3.04. The corresponding values for the ratio (6b) are 1.23,
1.25 and 1.20, again quite consistent with each other and with the
\( SU_3 \) prediction 1.07 (using eqs. (3) and (4)). A value 1.23 for the
ratio (6b) would imply a \( K^{*}(1780) \) elasticity x = 0.37 ± 0.12. The
present data do not, however, allow a quantitative comparison of the
elasticity with experiment.

The \( s = K^*, t = \rho \) constraint implies

\[ \frac{\Gamma(nK \rightarrow \rho)}{\Gamma(nK \rightarrow \rho)} = \frac{P_1(K^*, \rho)}{P_2(K^*, \rho)} \quad (7) \]

\(^*\) The error represents the biggest deviation introduced by changing
any one of the experimental quantities within its error bars, as
given in the PDG tables \(^3\).

\(^**\) We use the mass and width of the \( K^{3*}(1780) \) determined \(^6\) from
data on \( K^-p \rightarrow K^{0}n^-p \) at 14.3 GeV/c: \( m = (1.780 \pm 0.030) \) GeV,
\( \Gamma = (0.120 \pm 0.040) \) GeV.

\(^*\) All \( SU_3 \) comparisons are made assuming ideal mixing. The symmetry
is applied to regularized couplings \( T(a+b-s)/\lambda^n(a^2_b, s) \),
where the power n of the \( \lambda \) function is the appropriate one for
removing threshold singularities.
If we multiply both sides of eq. (7) by the regularizing factor
\[ \frac{1}{\lambda'(m_{\pi}^2, m_{\pi}^2, m_{\rho}^2)\lambda'(m_{K}^2, m_{K}^2, m_{\rho}^2)} \] and use the experimental \( \pi K^* \) and \( \pi \pi \rho \) couplings the numerical value of the ratio is 0.52 \( \pm \) 0.02, in agreement with the \( SU_3 \) prediction 0.5. The \( s = K^*, t = f \) constraint gives
\[
\frac{\hat{T}(\pi K^* \to f)}{\hat{T}(\pi \pi \to f)} = \frac{\hat{T}(\pi K^* \to f)}{\hat{T}(\pi \pi \to f)} = 1.76 \pm 0.10 \text{ GeV. (8)}
\]
This corresponds to a \((3.4 \pm 0.4)\%\) branching fraction for \( f \to KK \), to be compared with the experimental value \((2.7 \pm 0.6)\%\). As our final prediction from \( nK \to nK \) we may take the ratio of the \( s = K^*, t = 0 \) and \( s = K^*, t = g \) constraints, giving
\[
\frac{\hat{T}(\pi K^* \to 0)}{\hat{T}(\pi K^* \to g)} = \frac{\hat{T}(\pi \pi \to 0)}{\hat{T}(\pi \pi \to g)} = \frac{\hat{T}(\pi K^* \to 0)}{\hat{T}(\pi \pi \to 0)} = 27.0 \text{ GeV}^4 (9)
\]
where \( \lambda^3/2(m_{K}^2, m_{K}^2, m_{\rho}^2) \) and we used eq. (4).

The \( SU_3 \) prediction for the ratio (9), again using eq. (4), is 26.3.

We also note that the \( s = K^*, t = g \) constraint by itself implies a branching fraction \((5.3 \pm 0.9)\%\) for \( g \to KK \), consistent with the small number indicated by experiment.

\( \chi K^- \chi K^0 \)

In this charge exchange mode of the \( KK \to KK \) amplitude only the \( \rho \to A_2 \to g \) states contribute in the \( t \)-channel, while the \( \rho, \omega \) and \( f, A_2 \) contributions cancel in the \( s \)-channel. This leads to the standard duality relation between the \( \rho \) states in the \( t \)-channel and \( \chi K^* \) states \((\rho, \rho')\) in the \( s \)-channel. Taking ratios of the constraints with \( s = \rho \) and \( s = f' \), we get three different expressions for the \( f'KK/\chi KK \) coupling ratio:
\[
\frac{\hat{T}(\chi KK \to f')}{\hat{T}(\chi KK \to 0)} = \frac{\hat{T}(\chi KK \to f')}{\hat{T}(\chi KK \to 0)} = \frac{\hat{T}(\chi KK \to f')}{\hat{T}(\chi KK \to 0)} = 7.9 \pm 1.1 \text{ GeV}^2. (10)
\]

The numerical values for \( t = 0, A_2 \) and \( g \) are 9.77, 10.4 and 10.2, respectively. The experimental value is 9.7 \( \pm \) 2.4, while eq. (3) and \( SU_3 \) gives 11.8.

The \( s = \rho, t = A_2 \) constraint gives
\[
\frac{\hat{T}(\chi KK \to A_2)}{\hat{T}(\chi KK \to \rho)} = \frac{\hat{T}(\chi KK \to A_2)}{\hat{T}(\chi KK \to \rho)} = \frac{\hat{T}(\chi KK \to A_2)}{\hat{T}(\chi KK \to \rho)} = 7.9 \pm 1.1 \text{ GeV}^2. (11)
\]

This corresponds to a branching fraction of \((5.1 \pm 0.7)\%\) for \( A_2 \to KK \), to be compared with the experimental value \((4.7 \pm 0.5)\%\). Dividing the \( s = \rho, t = 0 \) constraint with that for \( s = \rho, t = g \) we get another expression for the ratio (9):
\[
\frac{\hat{T}(\chi KK \to 0)}{\hat{T}(\chi KK \to g)} = \frac{\hat{T}(\chi KK \to 0)}{\hat{T}(\chi KK \to g)} = \frac{\hat{T}(\chi KK \to 0)}{\hat{T}(\chi KK \to g)} = 25.9 \text{ GeV}^4. (12)
\]

Finally, from the \( s = \rho, t = g \) duality constraint we may calculate \( \hat{T}(\chi KK \to g) \) in terms of \( \hat{T}(\chi KK \to \rho) \). Using the experimental value for the \( \phi KK \) coupling one obtains a branching fraction of \((3.4 \pm 0.6)\%\) for \( g \to KK \), compatible with the value found above from the \( nK \to nK \) amplitude.
We have seen that all the 18 relations considered here are consistent with each other and with experiment (with one exception, which need not, however, be considered definite). Furthermore, the accuracy by which the conditions are satisfied is typically better than 10%, quite remarkable in view of the fact that there are no free parameters or ambiguities in the relations. This provides support for the conjecture\(^1\) that the more approximate duality relations of the usual FESR formulation are indeed consequences of dynamical constraints connecting the residues of s- and t-channel resonances. We do not yet know the general form of these constraints. Consider, for example, amplitudes where a semilocal average has to be taken over two neighboring resonances in the s-channel (as opposed to the cases considered above, where only one resonance was important in each mass bin). Our present approach suggests that a t-channel residue should be evaluated at the two (discrete) values of \(s\) corresponding to the masses of the neighboring resonances. Presumably, the two terms so obtained should be weighted according to the relative importance of the two s-channel resonances. A natural way of doing this is to form products of s- and t-channel residues. This agrees with the conclusions of ref. 1, where the need for such products (called "tetrahadrons") was recognized from a rather different line of reasoning based on symmetry in the treatment of spin.

The self-consistency of our scheme is made possible by the fact that ratios of the type (5) are numerically very close to unity when the particles lie on linear Regge trajectories. It would clearly be important to understand the mathematical reason behind this fact. Similarly, we saw that the ratios (6) and (10) assumed almost the same value for different choices of the t-channel particle. One may verify that this is so only when the slopes of the \(K^* - K^{**}\) and \(\phi - f'\) trajectories are similar to that of the \(\rho - f'\) trajectory.\(^*)\n
Considering the highly nonlinear dependence of the ratios on the masses, and that the kinematics (i.e., the dependence of \(\cos \theta\) on the resonance masses) is quite different in the three reactions, the simplicity of these numerical results is quite striking.

It is interesting to ask whether the duality constraints used in this work can be applied also to amplitudes involving charmed particles.\(^7\) Thus for the \(\pi D \rightarrow \pi D\) amplitude we would have a relation of the form (5a) (with \(K \rightarrow D\) everywhere). The \(D^{**}\) mass being unknown, we can check whether there is any choice of the mass for which the three expressions (corresponding to \(t = \rho, f'\) and \(g\)) assume the same numerical value. Using \(m_D = 1.865\) GeV and \(m_{D^*} = 2.006\) GeV we find that the three values are within 3% of each other when \(m_{D^{**}} = 2.580\) GeV. Allowing the expressions to differ by at most 20% leads to the mass range \(2.515 < m_{D^{**}} < 2.655\). This mass assignment corresponds to a rather small value for the effective slope of the \(D^{**}\) trajectory: \(\alpha' = (0.38 \pm 0.05)\) GeV\(^{-2}\). A similar slope can be deduced in the charmonium sector from the known masses of the \(\psi(3095)\) and \(X(3850)\): \(\alpha' \approx 0.33\) GeV\(^{-2}\).

In applying our constraints to the \(D^{*} \rightarrow D\) amplitude we find that most of the Legendre functions are small, and some are nearly zero.

\(^*)\ We constrained the trajectories to pass through the first physical state (\(\rho(773), K^*(892)\) and \(\phi(1020)\), respectively) as their slope was varied.
This means that the contributions from states on the leading trajectory are suppressed, and secondary terms from scalar mesons etc. must be taken into account. In contrast, in our studies of the noncharmed amplitudes the Legendre functions were always large, suggesting a dominance of the peripheral resonances. Hence we shall not attempt here to impose the above duality constraints on the $\bar{D}D \rightarrow \bar{D}D$ amplitude.

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* The $\pi D \rightarrow \pi D$ amplitude is an intermediate case, with one Legendre function below unity ($|P_1(n,p^*)| = 0.63$).

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