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Random shearing by zonal flows and transport reduction

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The physics of random shearing by zonal flows and the consequent reduction of scalar field transport are studied. In contrast to mean shear flows, zonal flows have a finite autocorrelation time and can exhibit complex spatial structure. A random zonal flow with a finite correlation time \( \tau \) decorrelates two nearby fluid elements less efficiently than a mean shear flow does. The decorrelation time is

\[
\tau_D = \left( \frac{\tau_x}{\tau_\Omega} \right)^{1/3} \quad (\tau_x \text{ is the turbulent scattering time, and } \Omega \text{ is the rotation frequency})
\]

and transport in the presence of zonal flows is both relevant and long overdue. Such an analysis is crucial to the long-term goal of relating fluctuations to transport, since direct measurement of turbulent fluxes in the core of relevant plasmas remains too difficult.

The purpose of the paper is to study the effect of random (i.e., broadband) shearing (by zonal flows) on turbulence regulation in a scalar field model. Intuitively, it is clear that shearing becomes ineffective as \( \tau \rightarrow 0 \), since then a shear flow has no time to act on an eddy. In the physically relevant case where \( \tau_\Omega \gg \tau_x \), the correlation time of turbulence, the critical value of the correlation time of the zonal flows, below which the shearing effect is reduced is roughly

\[
\tau_D = \frac{\Omega}{\Omega_{\text{rms}}} \approx \left( \frac{\Omega}{\Omega_{\text{rms}}} \right)^{1/3}
\]

This reduction is an important result that can be used to infer the presence of zonal flows in drift wave turbulence.

Shear flows are ubiquitous in a variety of physical systems, including differential rotation in galaxies and stars, zonal flows in major planets, laboratory plasmas, and earth atmosphere. These coherent structures play a distinctive role in determining transport in plasmas due to the dramatic effect of shear on regulating turbulence (see, e.g., Ref. 3).

The reduction of transport results from the change not only in the turbulence intensity but also in the correlation time and cross phase. For instance, in the case of a passive scalar in the turbulence intensity but also in the correlation time and cross phase. For instance, in the case of a passive scalar

\[
f = \frac{\partial x}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial x}{\partial y}
\]

of the exegesis of the theoretical question of the relation between fluctuation levels and wave turbulence dates back to the early 1960s, and that during the past 10 years, a community consensus as to both the ubiquity and importance of zonal flows in drift wave turbulence has arisen. Thus, analysis of the relation between fluctuations...
Let us consider the transport of scalar field \( \chi \) by random turbulent flow \( v \) and random zonal flow \( U = U(x,t) \hat{y} \) (with \( \langle v \rangle = \langle U \rangle = 0 \)) in the two dimensional \( x \) and \( y \) plane,

\[
[\partial_t + U(x,t) \partial_x] \chi' = -v_x \partial_x \chi_0 + D(\partial_{xx} + \partial_{yy})\chi',
\]

(1)

where \( \chi' \) and \( \chi_0 \) are fluctuating and mean parts of \( \chi \), and \( D \) is the effective diffusivity, including nonlinear interaction. The random turbulent flow is assumed to have characteristic frequency \( \omega_0 \) and correlation time \( \tau_\omega = \chi_0^2 \), while zonal flows have correlation time \( \tau_{ZF} \). Since a zonal flow \( U(x,t) \) is random, the scalar field flux \( \Gamma = \langle \chi'(x,t) \rangle \) should be averaged over the ensemble of zonal flows, in addition to that of the turbulent flow \( v \). We shall use angular brackets \( \langle \cdot \rangle \) to denote the average over either one of the two, and use double angular brackets \( \langle \langle \cdot \rangle \rangle \) to denote the average over both. At this point, the point reader may be somewhat skeptical of the assumption of independent statistics and the probability distribution functions (PDFs) for fluctuations and zonal flows. In this regard, it is useful to think of the fluctuations as complex fields, with amplitude and phase. Zonal flows are driven by fluctuation intensity, via Reynolds stresses, etc. Thus, the fluctuation statistics have a degree of freedom beyond those of the zonal flow, so we speculate that the interdependence of the two may be regarded as weak. Further study of this interesting point is a topic for future investigations. In the following, we focus on the two interesting limits: (a) when a zonal flow is temporally random on time scales \( \tau_{ZF} > \tau_\omega \), with a fixed linear profile \( U(x,t) = x \Omega(t) \), and (b) when the zonal flow is steady, but spatially complex \( U(x,t) = U(x) \) \cite{12,13}.

We now examine the first case when zonal flows have finite correlation time \( \tau_{ZF} \) with a linear spatial profile, i.e., \( U(x,t) = x \Omega(t) \). The degree to which randomness of zonal flows \( \langle \chi'(x,t) \rangle \) influences the dynamics depends on whether \( \tau_{ZF} \) is smaller or larger than other characteristic time scales, such as the shearing time \( \tau_\Omega \) and decorrelation time \( \tau_D \). As we are interested in the strong shear limit, \( \tau_D \) is taken to be much larger than \( \tau_\Omega \) throughout this paper. Given the uncertainty in \( \tau_{ZF} \), physically relevant cases are likely to be (i) \( \tau_\Omega < \tau_{ZF} \ll \tau_\Omega \approx \tau_D \) and (ii) \( \tau_\Omega \ll \chi_0^2 < \tau_{ZF} \ll \tau_D \). Case (i) corresponds to \( \delta \)-correlated turbulent (and zonal) flow, where the irreversibility arises mainly from the randomness of the flow while in case (ii), the zonal flow-wave resonance is the main source of irreversibility (in the limit \( \tau_{ZF} \rightarrow \infty \)). It is illuminating that even without complicated analysis, the scaling of flux in case (ii) can be easily obtained, since the long time average of the flux does not depend on the dissipation, rendering it legitimate to take \( \tau_{ZF} \rightarrow \infty \). Thus, we can simply use the result for a fixed shear flow \( \langle \chi'(x) \rangle \approx \hat{\delta}(\omega_0 - x \Omega(k)) \), and then take its average over an ensemble of zonal flows. For simplicity, we perform the latter by assuming Gaussian probability for \( \Omega \) as \( dP[\Omega] = (1/\Omega_{rms})d\Omega e^{-\Omega^2/2\Omega_{rms}^2} \). The average is

\[
\langle \langle \chi'(x) \rangle \rangle \equiv \frac{1}{x_k \Omega_{rms}} e^{-\omega_0^2/2x_k^2\Omega_{rms}^2}.
\]

(2)

Thus, a sharp resonance \( \hat{\delta}(\omega_0 - x \Omega(k)) \) becomes a smooth, probabilistic interaction kernel, making the flux maximal for \( \omega_0 = \sqrt{2x_k^2 \Omega_{rms}^2} \), with its value \( \propto \Omega_{rms} \). Thus, the flux has a similar scaling with \( \Omega_{rms} \) as with \( \Omega \) in the case of a fixed shear flow. The same result [Eq. (2)] shall also be obtained below through a more laborious calculation. It is important to note that the same analysis cannot be applied to \( \langle \langle \chi'^2 \rangle \rangle \), since the long time average of \( \langle \chi'^2 \rangle \) for a fixed shear is taken over a time longer than \( \tau_D \), which is much larger than \( \tau_{ZF} \). Thus, a more rigorous analysis is necessary.

To incorporate the shearing effect in Eq. (2), we employ a time-dependent wave number \( k_i(t) \) in the direction of the shear (i.e., shearing coordinate), by assuming

\[
\chi'(x,t) = \left( \frac{1}{2\pi^2} \right) \int d^2k e^{i(k(t)+k_y)\hat{x}} \chi(k_i(t),k_y,t).
\]

(3)

Upon using Eq. (3), and assuming \( \partial_t k_i(t) = -k_i \Omega(t) \), Eq. (1) can be easily solved as

\[
\hat{x}(k_i(t),k_y,t) = -\partial_0 \chi_0 \int d^2k_1 \int d^t \theta(k;1) \times e^{-D\Omega(t)} \tilde{\theta}(k_1,1;k_1,1,t_1).
\]

(4)

Here, \( Q(t+2t) = k^2(t-t_1) + \int_{-t}^t d^t k^2(t') \), and \( \theta(k,t;k_1,t_1) = \delta(k_i - k_1) \tilde{\theta}(k_1-k_1+i_1,1,t_2) \) is the Green’s function for the evolution of \( \chi' \). From Eq. (4), the flux and mean square amplitude of \( \chi' \), when averaged over the statistics of turbulent flow \( v \), are,

\[
\langle \chi'(x,t) \rangle = -\partial_0 \chi_0 \int d^2k_1 \int d^t \theta(k;1) e^{-i(k+\Omega(t)\hat{x})d^t(\hat{x})} \phi(k_1,t-1),
\]

(5)

\[
\langle \chi^2 \rangle = \left( \frac{\partial_0 \chi_0}{\Omega_{rms}} \right)^2 \int d^2k_1 d^t \theta(k_1,1) e^{-i(k+\Omega(t)\hat{x})d^t(\hat{x})} \phi(k_1,t-1).
\]

(6)

For case (i), we can take \( \phi(k_1,2t-1) = \tau_\Omega \delta(t_2-1) \phi(k) \) since \( \tau_\Omega \) is the shortest time scale in the system. Then, it is trivial to see that the effect of shearing for the flux vanishes as \( \langle \chi'(x,t) \rangle = \langle \chi'(x) \rangle = \langle \chi'(x) \rangle = -(\tau_\Omega \delta(x/(2\pi^2)) \int d^k \delta(k) \rangle \), which is consistent with the result for a steady shear flow. On the other hand, the effect of dissipation \( D \), enhanced by zonal flow shearing, is critical to determining the amplitude \( \langle \chi'^2 \rangle \). This is because of the generation of fine scales in \( x \) (or large \( k_x \)), and requires the following quantity to be averaged over an ensemble of zonal flows:

\[
I_x = \langle e^{-Dd^t\hat{x}^2(t')} \rangle.
\]

(7)

Since the argument of the exponential is quadratic in \( \Omega \) with nonvanishing mean value \( \langle \chi'(x) \rangle = -k_i \Omega(t) \), the average can be evaluated for each term by assuming Gaussian statistics for \( \Omega \), after expanding the exponential function. Of course, other forms of the probability distribution function should be considered as well. For this average, shearing can be treated as a random walk over \( \tau_D \), since the former changes many times, so long as \( \tau_{ZF} \ll \tau_D \). For instance, it is reasonable to
TABLE I. Summary of results for zonal flows with finite correlation time $\tau_D$.

<table>
<thead>
<tr>
<th>$\tau_c &lt; \tau_{ZF} &lt; \tau_{ms}$</th>
<th>$\tau_{D} &lt; \tau_c &lt; \tau_{ZF} &lt; \tau_{D}$</th>
<th>$\tau_{D} &lt; \tau_{ZF}$</th>
<th>$\tau_{ZF} &lt; \tau_{D}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \chi' \rangle$</td>
<td>$\Omega_{ms}^{-1}$</td>
<td>$\Omega_{ms}^{-1}$</td>
<td>$\Omega_{ms}^{-1}$</td>
</tr>
<tr>
<td>$\langle \chi^2 \rangle$</td>
<td>$\Omega_{ms}^{-1}$</td>
<td>$\Omega_{ms}^{-2} D^{1/2}$</td>
<td>$\Omega_{ms}^{-5/3} D^{1/3}$</td>
</tr>
</tbody>
</table>

Note that zonal flow shearing was taken to be coherent over

for $\omega_k = \sqrt{2k_x \gamma^2 \Omega^2}$, (which replaces the resonance condition), and is $\sim \Omega_{ms}^{-1}$.

In order to compute the amplitude for case (ii), we note that the effect of dissipation enters on a long time scale ($\tau_D$), compared to other time scales. Thus, we envision taking average of zonal flows over two different time scales $T_1$ and $T_2$, where $\tau_{ZF}, \tau_c < T_1 < \tau_D$ and $T_2 > \tau_D$. Thus, we can first ignore the dissipation (i.e., $D=0$) and take average over $T_1$, and then take average over $T_2$ with $D \neq 0$. Then, analyses similar to those given above yield

$$\langle \chi^2 \rangle = \frac{\langle \partial_\chi \rangle^2}{(2\pi)^2} dt \frac{1}{(\Omega^2)(D k^2 \tau_{ZF})^{1/2}},$$

(11)

where again $C = \int dsl(s)$ is a convergent integral. Equation (11) reveals that the amplitude in the case of random shear is enhanced due to the inefficiency of turbulence regulation. This is due to the shear’s random character, as compared to a steady shear, where $\langle \chi^2 \rangle \propto \tau_D / \Omega$. For instance, the amplitude increases as $\tau_{ZF}$ becomes small, as expected. The upper limit on the amplitude is, however, again given by $\tau_D = (D k^2 \Omega^2)^{1/2}$. On the other hand, as $\tau_{ZF}$ becomes larger than $\tau_D$, the parameter $\tau_D$ in Eq. (11) should be replaced by $\tau_D$, since zonal flows can then be treated as steady flows.

We will now show that in the case of a steady zonal flow with complex spatial dependence, the results are similar to those in the case of linear shear flow, provided that $\Omega$ is replaced by the rms shearing rate $\Omega_{ms} = \langle (\partial_\chi U)^2 \rangle^{1/2}$, in agreement with Ref. 12. Since it is plausible that $l_{ZF}$, the correlation length of zonal flows, can be comparable to $l_s$, that of the turbulence, we can no longer Fourier decompose $\chi'$ in $x$. Therefore, we Fourier transform only in $y$ and introduce a phase function $g(x,t)$ as

$$\chi'(x,t) = \frac{1}{2\pi} \int dk e^{-iky} e^{iky \eta(x,t)} \tilde{\chi}(k, y, t).$$

(12)

By assuming that $|\partial_\eta g| \gg |\partial_\eta \tilde{\chi}|$, and by using the usual Fourier transform for $u_y$ as $u_y(x,t) = 1/2 \pi \int dk e^{iky} \tilde{\chi}(k, y, t)$, Eq. (1) can easily be solved for $\tilde{\chi}$ as

$$\tilde{\chi}(k, y, t) = -\partial_\chi \int dt e^{-\partial_\chi \tilde{\chi}(k, t)} \tilde{\chi}(k, y, t) \eta'(k, y, t).$$

(13)

Here, $\tilde{\chi}(k, t) = k^2 U + (k^2 U')^2 + (3/2) k' U'' U'' + (2/2)$ with $U' = \partial_\chi U$ and $U'' = \partial_{\chi^2} U$. By exploiting the conditions of “steady and homogeneous turbulence” with the correlation function

$\langle \tilde{\chi}(k, y, t_1) \tilde{\chi}(k', y, t_2) \rangle = (2\pi) \partial_\chi k \partial_\chi k' \delta(k_{y, t_2} - k_{y, t_1})$, we obtain

$$\langle \chi' \rangle = -\partial_\chi \int dt dky e^{-iky \eta(x,y) \tilde{\chi}(k, y, t) \partial_\chi \tilde{\chi}(k, y, t) \eta(x,y,t)}.$$

(14)
First, it is easy to see that for a \( \delta \) correlated flow \( v_x \), i.e., \( \tau_\delta \ll \tau_{d1} \), the flux is independent of \( U \). However, the computation of \( \langle \chi'^2 \rangle \) requires an average (over zonal flows) like \( \langle e^{-2D(k,U)^2(\tau_{d1}/3)} \rangle \). Since the scaling of the amplitude with the rms shear is of greatest interest, this computation can be done by following an analysis similar to that done previously to obtain Eq. (8), and then scaling the time with \((Dk^2_s(U''^2))^{1/3}\). The result is

\[
\langle \chi'^2 \rangle \approx \frac{(\hbar \chi)(2 \pi)^3}{(2 \pi)^3} \int dk_y \phi(k_y) \frac{1}{[Dk^2_s(U''^2)]^{1/3}}.
\]

Therefore, the scaling of the amplitude with rms shear and \( D \) are the same as those in the case of a linear shear flow, provided \( \langle U''^2 \rangle^{1/2} \) replaces constant \( \Omega \).

For \( \tau_{d1} < \tau_\delta < \tau_D \), we again use a Lorentzian frequency spectrum for \( v_x \). After straightforward algebra using Gaussian statistics for \( U, U' \), and \( U'' \), and \( \langle UU' \rangle = \langle U'U'' \rangle = 0 \), we can obtain the following scalings:

\[
\langle \chi' v_x \rangle = \frac{-\hbar \chi(2 \pi)^3}{(2 \pi)^3} \int dk_y \phi(k_y) \frac{e^{-\omega_1^2/[2k^2_s(U''^2)]}}{2k^2_s(U''^2)^{1/2}},
\]

\[
\langle \chi'^2 \rangle \approx \frac{(\hbar \chi)(2 \pi)^3}{(2 \pi)^3} \int dk_y \phi(k_y) \frac{e^{-\omega_1^2/[2k^2_s(U''^2)]}}{2k^2_s(U''^2)^{1/2}[Dk^2_s(U'')^{1/3}]^{1/3}}.
\]

Due to the spatial randomness of the zonal flow pattern, resonance between zonal flow and turbulence is smoothed out, with the maximum flux occurring when \( \omega_0 = \sqrt{2k^2_s(U''^2)} \), as in temporally random case [see Eq. (10)]. Therefore, the scalings of the amplitude with shear and \( D \) are basically the same as those for the case of a linear shear flow, provided \( \langle U'' \rangle^{1/2} \) is replaced by \( \Omega \). We note that the curvature effect \( U'' \) does not appear in the final amplitude, since a strong shear limit \( D^2k^2_s(U'')^2[Dk^2_s(U''^2)]^{4/3} \sim [(Dk^2_s)^2/(U'')^2]^{4/3} \) and \( Dk^2_s(U''^2)^{3/2} \) are obtained as in the case of a steady linear shear flow (with \( (U'')^{1/2} \) replacing \( \Omega \)).

More interesting results were found for zonal flows with finite correlation time \( \tau_{ZF} \) [i.e., \( U(x,t) = x\Omega(t) \)] (see Table I). For \( \tau_\delta \ll \tau_{ZF} \ll \tau_D \), the flux becomes independent of shear to leading order, while \( \langle \chi'^2 \rangle \approx \Omega_{rms}^{-1} \).

The physically more interesting case where \( \tau_\delta \ll \tau < \tau_{ZF} \ll \tau_D \), \( \langle \chi'^2 \rangle \approx \Omega_{rms}^{-1} \), while \( \langle \chi'^2 \rangle \approx \Omega_{rms}^{-1} \). The scaling of the latter, which is different from \( \langle \chi'^2 \rangle \approx \Omega^{-5/3}D^{-1/3} \) in the case of coherent shearing, \( \Omega \), is a result of the longer effective decorrelation time of fluid elements \( \tau_{ZF} > \tau_\delta \), induced by finite zonal flow autocorrelation time \( \tau_{ZF} \). As \( \tau_{ZF} \) exceeds \( \tau_D \), zonal flows can be considered to be steady in time, thus recovering previous results.

The results of this paper highlight the great importance of the determination of both the frequency spectrum (in particular, the correlation time \( \tau_{ZF} \)) and PDF of zonal flows, in both simulations and physical experiments. In particular, we have assumed a Gaussian PDF of zonal flows throughout this paper, but there are likely cases for which the PDF of zonal flows is exponential or even power law. Experimentally, a useful estimate on \( \tau_{ZF} \) can be obtained from constructing the average two time correlation function of a zonal flow \( V_E \), i.e., \( \tau_{ZF} = \int_0^\infty dt (\langle V_E(t) V_E(t+\tau) \rangle) / \langle V_E(t)^2 \rangle \) or from the width of the \( m=0 \) frequency spectrum. We finally note that the methodology and approach of this work is relevant to geodesic acoustic modes, but the detailed analysis would require the formulation of a toroidal model and the inclusion of a broader range of zonal flow frequencies.

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