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T-ADIC EXPONENTIAL SUMS OVER FINITE FIELDS

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Abstract. $T$-adic exponential sums associated to a Laurent polynomial $f$ are introduced. They interpolate all classical $p^m$-power order exponential sums associated to $f$. The Hodge bound for the Newton polygon of $L$-functions of $T$-adic exponential sums is established. This bound enables us to determine, for all $m$, the Newton polygons of $L$-functions of $p^m$-power order exponential sums associated to an $f$ which is ordinary for $m = 1$. Deeper properties of $L$-functions of $T$-adic exponential sums are also studied. Along the way, new open problems about the $T$-adic exponential sum itself are discussed.

1. Introduction

1.1. Classical exponential sums. We first recall the definition of classical exponential sums over finite fields of characteristic $p$ with values in a $p$-adic field.

Let $p$ be a fixed prime number, $\mathbb{Z}_p$ the ring of $p$-adic integers, $\mathbb{Q}_p$ the field of $p$-adic numbers, and $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$. Let $q = p^n$ be a power of $p$, $\mathbb{F}_q$ the finite field of $q$ elements, $\mathbb{Q}_q$ the unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, and $\mathbb{Z}_q$ the ring of integers of $\mathbb{Q}_q$.

Fix a positive integer $n$. Let $f(x) \in \mathbb{Z}_q[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial in $n$ variables of the form

$$f(x) = \sum a_u x^u, \quad a_u \in \mu_{q^{2n-1}}, \quad x^u = x_1^{u_1} \cdots x_n^{u_n},$$

where $\mu_k$ denotes the group of $k$-th roots of unity in $\overline{\mathbb{Q}}_p$.

Definition 1.1. Let $\psi$ be a locally constant character of $\mathbb{Z}_p$ of order $p^m$ with values in $\overline{\mathbb{Q}}_p$, and let $\pi_\psi = \psi(1) - 1$. The sum

$$S_{f,\psi}(k) = \sum_{x \in \mu_{q^{2n-1}}} \psi(Tr_{\mathbb{Q}_q/k}(f(x)))$$

is called a $p^m$-power order exponential sum on the $n$-torus $G_m^n$ over $\mathbb{F}_q$. The generating function

$$L_{f,\psi}(s) = L_{f,\psi}(s; \mathbb{F}_q) = \exp\left(\sum_{k=1}^{\infty} S_{f,\psi}(k) s^k\right) \in 1 + s\mathbb{Z}_p[\pi_\psi][[s]]$$

is called the $L$-function of $p^m$-power order exponential sums over $\mathbb{F}_q$ associated to $f(x)$. 

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Note that the above exponential sum for \( m \geq 1 \) is still an exponential sum over a finite field as we just sum over the subset of roots of unity (corresponding to the elements of a finite field via the Teichmüller lifting), not over the whole finite residue ring \( \mathbb{Z}_q/p^m\mathbb{Z}_q \). The exponential sum over the whole finite ring \( \mathbb{Z}_q/p^m\mathbb{Z}_q \) and its generating function as \( m \) varies is the subject of Igusa’s zeta function, see Igusa [17].

In general, the above \( L \)-function \( L_{f,\psi}(s) \) of exponential sums is rational in \( s \). But, if \( f \) is non-degenerate, then \( L_{f,\psi}(s)^{(-1)^{n-1}} \) is a polynomial, as was shown in [1,2] for \( \psi \) of order \( p \), and in [20] for all \( \psi \). By a result of [12], if \( p \) is large enough, then \( f \) is generically non-degenerate. For non-degenerate \( f \), the location of the zeros of \( L_{f,\psi}(s)^{(-1)^{n-1}} \) becomes an important issue. The \( p \)-adic theory of such \( L \)-functions was developed by Dwork, Bombieri [8], Adolphson-Sperber [1,2], the second author [26, 27], and Blache [7] for \( \psi \) of order \( p \). More recently initial part of the theory was extended to all \( \psi \) by Liu-Wei [20] and Liu [19].

The \( p \)-adic theory of the above exponential sum for \( n = 1 \) and \( \psi \) of order \( p \) has a long history and has been studied extensively in the literature. For instance, in the simplest case that \( f(x) = x^d \), the exponential sum was studied by Gauss, see Berndt-Evans [3] for a comprehensive survey. By the Hasse-Davenport relation for Gauss sums, the \( L \)-function is a polynomial whose zeros are given by roots of Gauss sums. Thus, the slopes of the \( L \)-function are completely determined by the Stickelberger theorem for Gauss sums. The roots of the \( L \)-function have explicit \( p \)-adic formulas in terms of \( p \)-adic \( \Gamma \)-function via the Gross-Koblitz formula [13]. These ideas can be extended to treat the so-called diagonal \( f \) case for general \( n \), see Wan [27]. These elementary cases have been used as building bricks to study the deeper non-diagonal \( f(x) \) via various decomposition theorems, which are the main ideas of Wan [26, 27]. In the case \( n = 1 \) and \( \psi \) of order \( p \), further progresses about the slopes of the \( L \)-function were made in Zhu [32, 33], Blache and Ferard [5], and Liu [21].

1.2. \( T \)-adic exponential sums. We now define the \( T \)-adic exponential sum, state our main results, and put forward some new questions.

**Definition 1.2.** For a positive integer \( k \), the \( T \)-adic exponential sum of \( f \) over \( \mathbb{F}_q \) is the sum:

\[
S_f(k,T) = \sum_{x \in \mu_{q^{nk-1}}} (1 + T)^{\text{Tr}_{q^k/q_p}(f(x))} \in \mathbb{Z}_p[[T]].
\]

The \( T \)-adic \( L \)-function of \( f \) over \( \mathbb{F}_q \) is the generating function

\[
L_f(s,T) = L_f(s,T;\mathbb{F}_q) = \exp(\sum_{k=1}^{\infty} S_f(k,T)\frac{s^k}{k}) \in 1 + s\mathbb{Z}_p[[T]][[s]].
\]
The $T$-adic exponential sum interpolates classical exponential sums of $p^m$-order over finite fields for all positive integers $m$. In fact, we have
\[ S_f(k, \pi_\psi) = S_{f, \psi}(k). \]

Similarly, one can recover the classical L-function of the $p^m$-order exponential sum from the $T$-adic L-function by the formula
\[ L_f(s, \pi_\psi) = L_{f, \psi}(s). \]

We view $L_f(s, T)$ as a power series in the single variable $s$ with coefficients in the complete discrete valuation ring $\mathbb{Q}_p[[T]]$ with uniformizer $T$.

**Definition 1.3.** The $T$-adic characteristic function of $f$ over $\mathbb{F}_q$, or $C$-function of $f$ for short, is the generating function
\[ C_f(s, T) = \exp \left( \sum_{k=1}^{\infty} -(q^k - 1)^{-n}S_f(k, T)\frac{s^k}{k} \right) \in 1 + s\mathbb{Z}_p[[T]][[s]]. \]

The $C$-function $C_f(s, T)$ and the $L$-function $L_f(s, T)$ determine each other. They are related by
\[ L_f(s, T) = \prod_{i=0}^{n} C_f(q^i s, T)^{(-1)^{n-i-1}(\binom{n}{i})}, \]
and
\[ C_f(s, T)^{(-1)^{n-1}} = \prod_{j=0}^{\infty} L_f(q^j s, T)^{\binom{n+j-1}{j}}. \]

In §4, we prove

**Theorem 1.4** (analytic continuation). The $C$-function $C_f(s, T)$ is $T$-adic entire in $s$. As a consequence, the $L$-function $L_f(s, T)$ is $T$-adic meromorphic in $s$.

The above theorem tells that the $C$-function behaves $T$-adically better than the $L$-function. In fact, in the $T$-adic setting, the $C$-function is a more natural object than the $L$-function. Thus, we shall focus more on the $C$-function.

Knowing the analytic continuation of $C_f(s, T)$, we are then interested in the location of its zeros. More precisely, we would like to determine the $T$-adic Newton polygon of this entire function $C_f(s, T)$. This is expected to be a complicated problem in general. It is open even in the simplest case $n = 1$ and $f(x) = x^d$ is a monomial if $p \not\equiv 1 \pmod{d}$. What we can do is to give an explicit combinatorial lower bound depending only on $q$ and $\Delta$, called the $q$-Hodge bound $H^p_q(\Delta)$. This polygon will be described in detail in §3.

Let $NP_T(f)$ denote the $T$-adic Newton polygon of the $C$-function $C_f(s, T)$. In §5, we prove
Theorem 1.5 (Hodge bound). We have
\[ \text{NP}_T(f) \geq \text{HP}_q(\Delta). \]

This theorem shall give several new results on classical exponential sums, as we shall see in §2. In particular, this extends, in one stroke, all known ordinariness results for \( \psi \) of order \( p \) to all \( \psi \) of any \( p \)-power order. It demonstrates the significance of the \( T \)-adic \( L \)-function. It also gives rise to the following definition.

Definition 1.6. The Laurent polynomial \( f \) is called \( T \)-adically ordinary if
\[ \text{NP}_T(f) = \text{HP}_q(\Delta). \]

We shall show that the classical notion of ordinariness implies \( T \)-adic ordinariness. But it is possible that a non-ordinary \( f \) is \( T \)-adically ordinary. Thus, it remains of interest to study exactly when \( f \) is \( T \)-adically ordinary. For this purpose, in §6, we extend the facial decomposition theorem in Wan [26] to the \( T \)-adic case. Let \( \Delta \) be the convex closure in \( \mathbb{R}^n \) of the origin and the exponents of the non-zero monomials in the Laurent polynomial \( f(x) \). For any closed face \( \sigma \) of \( \Delta \), we let \( f_\sigma \) denote the sum of monomials of \( f \) whose exponent vectors lie in \( \sigma \).

Theorem 1.7 (\( T \)-adic facial decomposition). The Laurent polynomial \( f \) is \( T \)-adically ordinary if and only if for every closed face \( \sigma \) of \( \Delta \) of codimension 1 not containing the origin, the restriction \( f_\sigma \) is \( T \)-adically ordinary.

In §7, we briefly discuss the variation of the \( C \)-function \( C_f(s, T) \) and its Newton polygon when the reduction of \( f \) moves in an algebraic family over a finite field. The main questions are the generic ordinariness, generic Newton polygon, the analogue of the Adolphson-Sperber conjecture [1], Wan’s limiting conjecture [27], Dwork’s unit root conjecture [10] in the \( T \)-adic and \( \pi \)-adic case. We shall give an overview about what can be proved and what is unknown, including a number of conjectures. Basically, a lot can be proved in the ordinary case, and a lot remain to be proved in the non-ordinary case.

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2. Applications

In this section we give several applications of the \( T \)-adic exponential sum to classical exponential sums.

Theorem 2.1 (integrality theorem). We have
\[ L_f(s, T) \in 1 + s \mathbb{Z}_p[[T]][[s]], \]
and
\[ C_f(s, T) \in 1 + s \mathbb{Z}_p[[T]][[s]]. \]
Proof. Let \(|G^n_m|\) be the set of closed points of \(G^n_m\) over \(F_q\), and \(a \mapsto \hat{a}\) the Teichmüller lifting. It is easy to check that the \(T\)-adic L-function has the Euler product expansion

\[
L_f(s,T) = \prod_{x \in |G^n_m|} \frac{1}{(1 - \left(1 + T\right)^{\text{tr}_{Q_p))(f(\hat{x}))}} \in 1 + s\mathbb{Z}_p[[T]][[s]],
\]

where \(\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)\). The theorem now follows. \(\square\)

The above proof shows that the L-function \(L_f(s,T)\) is the L-function \(L(s,\rho_f)\) of the following continuous \((p,T)\)-adic representation of the arithmetic fundamental group:

\[
\rho_f : \pi_{\text{arith}}^1\left(G^n_m/F_q\right) \rightarrow \text{GL}_1(\mathbb{Z}_p[[T]]),
\]

defined by

\[
\rho_f(\text{Frob}_x) = (1 + T)^{\text{tr}_{Q_p))(f(\hat{x}))}.
\]

The rank one representation \(\rho_f\) is transcendental in nature. Its L-function \(L(s,\rho_f)\) seems to be beyond the reach of \(\ell\)-adic cohomology, where \(\ell\) is a prime different from \(p\). However, the specialization of \(\rho_f\) at the special point \(T = \pi_{\psi}\) is a character of finite order. Thus, the specialization

\[
L(s,\rho_f)|_{T = \pi_{\psi}} = L_{f,\psi}(s)
\]

can indeed be studied using Grothendieck’s \(\ell\)-adic trace formula [14]. This gives another proof that the L-function \(L_{f,\psi}(s)\) is a rational function in \(s\). But the \(T\)-adic L-function \(L_f(s,T)\) itself is certainly out of the reach of \(\ell\)-adic cohomology as it is truly transcendental.

Let \(\text{NP}_T(f)\) denote the \(T\)-adic Newton polygon of the C-function \(C_f(s,T)\), and let \(\text{NP}_{\pi_{\psi}}(f)\) denote the \(\pi_{\psi}\)-adic Newton polygon of the C-function \(C_f(s,\pi_{\psi})\). The integrality of \(C_f(s,T)\) immediately gives the following theorem.

**Theorem 2.2** (rigidity bound). If \(\psi\) is non-trivial, then

\[
\text{NP}_{\pi_{\psi}}(f) \geq \text{NP}_T(f).
\]

**Proof.** Obvious. \(\square\)

A natural question is to ask when \(\text{NP}_{\pi_{\psi}}(f)\) coincides with its rigidity bound.

**Theorem 2.3** (transfer theorem). If \(\text{NP}_{\pi_{\psi}}(f) = \text{NP}_T(f)\) holds for one non-trivial \(\psi\), then it holds for all non-trivial \(\psi\).

**Proof.** By the integrality of \(C_f(s,T)\), the \(T\)-adic Newton polygon of \(C_f(s,T)\) coincides with the \(\pi_{\psi}\)-adic Newton polygon of \(C_f(s,\pi_{\psi})\) if and only if for every vertex \((i,e)\) of the \(T\)-adic Newton polygon of \(C_f(s,T)\), the coefficients of \(s^i\) in \(C_f(s,T)\) differs from \(T^e\) by a unit in \(\mathbb{Z}_p[[T]]^\times\). It follows that if the coincidence happens for one non-trivial \(\psi\), it happens for all non-trivial \(\psi\). The theorem is proved. \(\square\)
Definition 2.4. We call $f$ rigid if $\text{NP}_{\pi,\psi}(f) = \text{NP}_{T}(f)$ for one (and hence for all) non-trivial $\psi$.

In [22], cooperating with his students, the first author showed that $f$ is generically rigid if $n = 1$ and $p$ is sufficiently large. So the rigid bound is the best possible bound. In contrast, the weaker Hodge bound $\text{HP}_q(\Delta)$ is only best possible if $p \equiv 1 \pmod{d}$, where $d$ is the degree of $f$.

We now pause to describe the relationship between the Newton polygons of $C_f(s, \pi, \psi)$ and $L_{f,\psi}(s)^{(-1)^{n-1}}$. We need the following definitions.

Definition 2.5. A convex polygon with initial point $(0, 0)$ is called algebraic if it is the graph of a $\mathbb{Q}$-valued function defined on $\mathbb{N}$ or on an interval of $\mathbb{N}$, and its slopes are of finite multiplicity and of bounded denominator.

Definition 2.6. For an algebraic polygon with slopes $\{\lambda_i\}$, we define its slope series to be $\sum_i t^{\lambda_i}$.

It is clear that an algebraic polygon is uniquely determined by its slope series. So the slope series embeds the set of algebraic polygons into the ring $\lim \mathbb{Z}[[t^\frac{1}{d}]]$. The image is $\lim \mathbb{N}[[t^\frac{1}{d}]]$. It is closed under addition and multiplication. Therefore one can define an addition and a multiplication on the set of algebraic polygons.

Lemma 2.7. Suppose that $f$ is non-degenerate. Then the $q$-adic Newton polygon of $C_f(s, \pi, \psi; \mathbb{F}_q)$ is the product of the $q$-adic Newton polygon of $L_{f,\psi}(s; \mathbb{F}_q)^{(-1)^{n-1}}$ and the algebraic polygon $\frac{1}{(1-t)^n}$.

Proof. Note that the $C$-value $C_f(s, \pi, \psi)$ and the $L$-function $L_{f,\psi}(s)$ determine each other. They are related by

$$L_{f,\psi}(s) = \prod_{i=0}^{n} C_f(q^i s, \pi, \psi)^{(-1)^{n-i-1}\binom{n}{i}},$$

and

$$C_f(s, \pi, \psi)^{(-1)^{n-1}} = \prod_{j=0}^{\infty} L_{f,\psi}(q^j s)^{\binom{n+j-1}{j}}.$$ 

Suppose that

$$L_{f,\psi}(s)^{(-1)^{n-1}} = \prod_{i=1}^{d} (1 - \alpha_i s).$$

Then

$$C_f(s, \pi, \psi) = \prod_{j=0}^{\infty} \prod_{i=1}^{d} (1 - \alpha_i q^j s)^{\binom{n+j-1}{j}}.$$
Let \( \lambda_i \) be the \( q \)-adic order of \( \alpha_i \). Then the \( q \)-adic order of \( \alpha_i q^j \) is \( \lambda_i + j \). So the slope series of the \( q \)-adic Newton polygon of \( L_{f,\psi}(s)(-1)^{n-1} \) is
\[
S(t) = \sum_{i=1}^{d} t^{\lambda_i},
\]
and the slope series of the \( q \)-adic Newton polygon of \( C_f(s, \pi_\psi) \) is
\[
\sum_{j=0}^{+\infty} \sum_{i=0}^{d} \binom{n+j-1}{j} t^{\lambda_i+j} = \frac{1}{(1-t)^n} S(t).
\]
The lemma now follows. \( \square \)

We combine the rigidity bound and the Hodge bound to give the following theorem.

**Theorem 2.8.** If \( \psi \) is non-trivial, then
\[
NP_{\pi_\psi}(f) \geq NP_T(f) \geq HP_q(\Delta).
\]

**Proof.** Obvious. \( \square \)

If we drop the middle term, we arrive at the Hodge bound
\[
NP_{\pi_\psi}(f) \geq HP_q(\Delta)
\]
of Adolphson-Sperber [2] and Liu-Wei [20].

**Theorem 2.9.** If \( NP_{\pi_\psi}(f) = HP_q(\Delta) \) holds for one non-trivial \( \psi \), then \( f \) is rigid, \( T \)-adically ordinary, and the equality holds for all non-trivial \( \psi \).

**Proof.** Suppose that \( NP_{\pi_{\psi_0}}(f) = HP_q(\Delta) \) for a non-trivial \( \psi_0 \). Then, by the last theorem, we have
\[
NP_{\pi_{\psi_0}}(f) = NP_T(f) = HP_q(\Delta).
\]
So \( f \) is rigid and \( T \)-adically ordinary, and
\[
NP_{\pi_\psi}(f) = NP_T(f) = HP_q(\Delta)
\]
holds for all nontrivial \( \psi \). The theorem is proved. \( \square \)

**Definition 2.10.** We call \( f \) ordinary if \( NP_{\pi_\psi}(f) = HP_q(\Delta) \) holds for one (and hence for all) non-trivial \( \psi \).

The notion of ordinarity now carries much more information than what we had known. From this, we see that the \( T \)-adic exponential sum provides a new framework to study all \( p^m \)-power order exponential sums simultaneously. Instead of the usual way of extending the methods for \( \psi \) of order \( p \) to the case of higher order, the \( T \)-adic exponential sum has the novel feature that it can sometimes transfer a known result for one non-trivial \( \psi \) to all non-trivial \( \psi \). This philosophy is carried out further in the paper [22].
Example 2.1. Let
\[ f(x) = x_1 + x_2 + \cdots + x_n + \frac{\alpha}{x_1 x_2 \cdots x_n}, \quad \alpha \in \mu_{q-1}. \]
Then, by the result of Sperber [25] and our new information on ordinariness, we have
\[ \text{NP}_{\pi_\psi}(f) = \text{HP}_q(\Delta) \]
for all non-trivial \( \psi \).

3. The \( q \)-Hodge Polygon

In this section, we describe explicitly the \( q \)-Hodge polygon mentioned in the introduction. Recall that \( f(x) \in \mathbb{Z}_q[x_1, x_2, \ldots, x_n] \) is a Laurent polynomial in \( n \) variables of the form
\[ f(x) = \sum_{u \in \mathbb{Z}^n} a_u x^u, \quad a_u \in \mathbb{Z}_q, \quad a_u^q = a_u. \]
We stress that the non-zero coefficients of \( f(x) \) are roots of unity in \( \mathbb{Z}_q \), thus correspond in a unique way to Teichmüller liftings of elements of the finite field \( \mathbb{F}_q \). If the coefficients of \( f(x) \) are arbitrary elements in \( \mathbb{Z}_q \), much of the theory still holds, but it is more complicated to describe the results. We have made the simplifying assumption that the non-zero coefficients are always roots of unity in this paper.

Let \( \Delta \) be the convex polyhedron in \( \mathbb{R}^n \) associated to \( f \), which is generated by the origin and the exponent vectors of the non-zero monomials of \( f \). Let \( C(\Delta) \) be the cone in \( \mathbb{R}^n \) generated by \( \Delta \). Define the degree function \( u \mapsto \deg(u) \) on \( C(\Delta) \) such that \( \deg(u) = 1 \) when \( u \) lies on a codimensional 1 face of \( \Delta \) that does not contain the origin, and such that \( \deg(ru) = r \deg(u), \quad r \in \mathbb{R}_{\geq 0}, \quad u \in C(\Delta) \).

We call it the degree function associated to \( \Delta \). We have \( \deg(u + v) \leq \deg(u) + \deg(v) \) if \( u,v \in C(\Delta) \), and the equality holds if and only if \( u \) and \( v \) are co-facial. In other words, the number
\[ c(u,v) := \deg(u) + \deg(v) - \deg(u + v) \]
is 0 if \( u,v \in C(\Delta) \) are co-facial, and is positive otherwise. We call that number \( c(u,v) \) the co-facial defect of \( u \) and \( v \). Let
\[ M(\Delta) := C(\Delta) \cap \mathbb{Z}^n \]
be the set of lattice points in the cone \( C(\Delta) \). Let \( D \) be the denominator of the degree function, which is the smallest positive integer such that
\[ \deg M(\Delta) \subset \frac{1}{D} \mathbb{Z}. \]
For every natural number \( k \), we define
\[ W(k) := W_\Delta(k) = \# \{ u \in M(\Delta) \mid \deg(u) = k/D \} \]
to be the number of lattice points of degree $\frac{k}{2}$ in $M(\Delta)$. For prime power $q = p^a$, the $q$-Hodge polygon of $f$ is the polygon with vertices $(0,0)$ and

$$\left(\sum_{j=0}^{i} W(j), a(p-1) \sum_{j=0}^{i} \frac{j}{D} W(j)\right), \ i = 0, 1, \cdots.$$ 

It is also called the $q$-Hodge polygon of $\Delta$ and denoted by $\text{HP}_{q}(\Delta)$. It depends only on $q$ and $\Delta$. It has a side of slope $a(p-1)\frac{4}{p}$ with horizontal length $W(j)$ for each non-negative integer $j$.

4. Analytic continuation

In this section, we prove the $T$-adic analytic continuation of the $C$-function $C_f(s, T)$. The idea is to employ Dwork’s trace formula in the $T$-adic case.

Note that the Galois group $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ is cyclic of order $a = \log_p q$. There is an element in the Galois group whose restriction to $\mu_{q-1}$ is the $p$-power morphism. It is of order $a$, and is called the Frobenius element. We denote that element by $\sigma$.

We define a new variable $\pi$ by the relation $E(\pi) = 1 + T$, where

$$E(\pi) = \exp\left(\sum_{i=0}^{\infty} \frac{\pi^p^i}{p^i}\right) \in 1 + \pi \mathbb{Z}_p[[\pi]]$$

is the Artin-Hasse exponential series. Thus, $\pi$ and $T$ are two different uniformizers of the $T$-adic local ring $\mathbb{Q}_p[[T]]$. It is clear that for $\alpha \in \mathbb{Z}_q$, we have

$$E(\pi \alpha) \in 1 + \pi \mathbb{Z}_q[[\pi]],$$

and for $\beta \in \mathbb{Z}_p$, we have

$$E(\pi^\beta) \in 1 + \pi \mathbb{Z}_p[[\pi]].$$

The Galois group $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ can act on $\mathbb{Z}_q[[\pi]]$ but keeping $\pi$ fixed. The Artin-Hasse exponential series has a kind of commutativity expressed as the following lemma.

**Lemma 4.1 (Commutativity).** We have the following commutative diagram

$$\begin{array}{ccc}
\mu_{q-1} & \xrightarrow{E(\pi \cdot)} & \mathbb{Z}_q[[\pi]] \\
\text{Tr} \downarrow & & \downarrow \text{Norm} \\
\mu_{p-1} & \xrightarrow{E(\pi \cdot)} & \mathbb{Z}_p[[\pi]].
\end{array}$$

That is, if $x \in \mu_{q-1}$, then

$$E(\pi^{x+p+\cdots+p^{a-1}}) = E(\pi x) E(\pi x^{p}) \cdots E(\pi x^{p^{a-1}}).$$

**Proof.** Since for $x \in \mu_{q-1}$,

$$\sum_{j=0}^{a-1} x^{p^j} = \sum_{j=0}^{a-1} x^{p^{j+i}}.$$
we have
\[ E(\pi)^{x + x^p + \cdots + x^p^{a-1}} = \exp(\sum_{i=0}^{\infty} \frac{\pi^p^i}{p^i} \sum_{j=0}^{a-1} x^{p^i+j}) = E(\pi x)E(\pi x^p) \cdots E(\pi x^p^{a-1}). \]

The lemma is proved. □

**Definition 4.2.** Let \( \pi^{1/D} \) be a fixed \( D \)-th root of \( \pi \). Define
\[ L(\Delta) = \left\{ \sum_{u \in M(\Delta)} b_u \pi^{\deg(u)} x^u : b_u \in \mathbb{Z}_q[\lbrack \lbrack \pi^{1/D} \rbrack \rbrack] \right\}, \]
and
\[ B = \left\{ \sum_{u \in M(\Delta)} b_u \pi^{\deg(u)} x^u \in L(\Delta), \ \text{ord}_T(b_u) \to +\infty \text{ if } \deg(u) \to +\infty \right\}. \]

The spaces \( L(\Delta) \) and \( B \) are \( T \)-adic Banach algebras over the ring \( \mathbb{Z}_q[\lbrack \lbrack \pi^{1/D} \rbrack \rbrack] \). The monomials \( \pi^{\deg(u)} x^u \) (\( u \in M(\Delta) \)) form an orthonormal basis (resp., a formal basis) of \( B \) (resp., \( L(\Delta) \)). The algebra \( B \) is contained in the larger Banach algebra \( L(\Delta) \). If \( u \in \Delta \), it is clear that \( E(\pi x^u) \in L(\Delta) \). Write
\[ E_f(x) := \prod_{a_u \neq 0} E(\pi a_u x^u), \text{ if } f(x) = \sum_{u \in \mathbb{Z}_q^n} a_u x^u. \]

This is an element of \( L(\Delta) \) since \( L(\Delta) \) is a ring.

The Galois group \( \text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p) \) can act on \( L(\Delta) \) but keeping \( \pi^{1/D} \) as well as the variables \( x_i \)'s fixed. From the commutativity of the Artin-Hasse exponential series, one can infer the following lemma.

**Lemma 4.3** (Dwork’s splitting lemma). If \( x \in \mu_{q^k-1} \), then
\[ E(\pi)^{\text{Tr}_{\mathbb{Q}_q^{k}/\mathbb{Q}_p}(f(x))} = \prod_{i=0}^{ak-1} E^{\sigma^i}(x^p^i), \]
where \( a \) is the order of \( \text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p) \).

**Proof.** We have
\[ E(\pi)^{\text{Tr}_{\mathbb{Q}_q^{k}/\mathbb{Q}_p}(f(x))} = \prod_{a_u \neq 0} E(\pi)^{\text{Tr}_{\mathbb{Q}_q^{k}/\mathbb{Q}_p}(a_u x^u)} \]
\[ = \prod_{a_u \neq 0} \prod_{i=0}^{ak-1} E(\pi (a_u x^u)^{p^i}) = \prod_{i=0}^{ak-1} E^{\sigma^i}(x^p^i). \]
The lemma is proved. □

**Definition 4.4.** We define a map
\[ \psi_p : L(\Delta) \to L(\Delta), \sum_{u \in M(\Delta)} b_u x^u \mapsto \sum_{u \in M(\Delta)} b_{pu} x^u. \]

It is clear that the composition map \( \psi_p \circ E_f \) sends \( B \) to \( B \).
Lemma 4.5. Write
\[ E_f(x) = \sum_{u \in M(\Delta)} \alpha_u(f) \pi^{\deg(u)} x^u. \]

Then, \( \psi_p \circ E_f(\pi^{\deg(u)} x^u) \)
\[ = \sum_{w \in M(\Delta)} \alpha_{pw-u}(f) \pi^{c(pw-u, u)} \pi^{(p-1)\deg(w)} \pi^{\deg(w)} x^w, \ u \in M(\Delta), \]
where \( c(pw-u, u) \) is the co-facial defect of \( pw-u \) and \( u \).

Proof. This follows directly from the definition of \( \psi_p \) and \( E_f(x) \). \( \square \)

Definition 4.6. Define
\[ \psi := \sigma^{-1} \circ \psi_p \circ E_f : B \rightarrow B, \]
and its \( a \)-th iterate
\[ \psi^a = \psi_p^a \circ \prod_{i=0}^{a-1} E_f^{\sigma^i}(x^{p^i}). \]

Note that \( \psi \) is linear over \( \mathbb{Z}_p[[\pi^{1/D}]] \), but semi-linear over \( \mathbb{Z}_q[[\pi^{1/D}]]. \) On the other hand, \( \psi^a \) is linear over \( \mathbb{Z}_q[[\pi^{1/D}]]. \) By the last lemma, \( \psi^a \) is completely continuous in the sense of Serre [24].

Theorem 4.7 (Dwork’s trace formula). For every positive integer \( k \),
\[ (q^k - 1)^{-n} S_f(k, T) = \text{Tr}_{B/\mathbb{Z}_q[[\pi^{1/D}]]}(\psi^a). \]

Proof. Let \( g(x) \in B \). We have
\[ \psi^a(g) = \psi_p^a(g \prod_{i=0}^{a-1} E_f^{\sigma^i}(x^{p^i})). \]

Write
\[ \prod_{i=0}^{a-1} E_f^{\sigma^i}(x^{p^i}) = \sum_{u \in M(\Delta)} \beta_u x^u. \]

One computes that
\[ \psi^a(\pi^{\deg(v)} x^v) = \sum_{u \in M(\Delta)} \beta_{u-v} \pi^{\deg(v)} x^u. \]

Thus,
\[ \text{Tr}(\psi^a | B/\mathbb{Z}_q[[\pi^{1/D}]] = \sum_{u \in M(\Delta)} \beta_{(q^k-1)u}. \]

But, by Dwork’s splitting lemma, we have
\[ (q^k - 1)^{-n} S_f(k, T) = (q^k - 1)^{-n} \sum_{x \in \mu_{q^k-1}} \prod_{i=0}^{a-1} E_f^{\sigma^i}(x^{p^i}) = \sum_{u \in M(\Delta)} \beta_{(q^k-1)u}. \]

The theorem now follows. \( \square \)
**Theorem 4.8** (Analytic trace formula). We have
\[ C_f(s, T) = \det(1 - \psi^s | B/\mathbb{Z}_q[[\pi^{1/2}]]). \]
In particular, the \( T \)-adic \( C \)-function \( C_f(s, T) \) is \( T \)-adic analytic in \( s \).

**Proof.** It follows from the last theorem and the well known identity
\[ \det(1 - \psi^s | B/\mathbb{Z}_q[[\pi^{1/2}]]) = \exp(- \sum_{k=1}^{\infty} \text{Tr}(\psi^k s^k)). \]

This theorem gives another proof that the coefficients of \( C_f(s, T) \) and \( L_f(s, T) \) as power series in \( s \) are \( T \)-adically integral.

**Corollary 4.9.** For each non-trivial \( \psi \), the \( C \)-value \( C_f(s, \pi \psi) \) is \( p \)-adic entire in \( s \) and the \( L \)-function \( L_{f, \psi}(s) \) is rational in \( s \).

**Proof.** Obvious. \( \Box \)

5. The Hodge bound

The analytic trace formula in the previous section reduces the study of \( C_f(s, T) \) to the study of the operator \( \psi^s \). We consider \( \psi \) first. Note that \( \psi \) operates on \( B \) and is linear over \( \mathbb{Z}_p[[\pi^{1/2}]] \).

**Theorem 5.1.** The \( T \)-adic Newton polygon of \( \det(1 - \psi^s | B/\mathbb{Z}_p[[\pi^{1/2}]] \) lies above the polygon with vertices \((0, 0)\) and \((a \sum_{k=0}^{i} W(k), a(p-1) \sum_{k=0}^{i} k D W(k)), i = 0, 1, \cdots \).

**Proof.** Let \( \xi_1, \xi_2, \cdots, \xi_a \) be a normal basis of \( \mathbb{Q}_q \) over \( \mathbb{Q}_p \). Write
\[ (\xi_j \alpha_{p^w-u}(f))^{n-1} = \sum_{i=0}^{a-1} \alpha_{(i,w),(j,u)}(f) \xi_i, \ \alpha_{(i,w),(j,u)}(f) \in \mathbb{Z}_p[[\pi^{1/2}]]. \]
Then \( \psi(\xi_j \pi^{-\deg(u)} x^u) \)
\[ = \sum_{i=0}^{a-1} \sum_{w \in M(\Delta)} \alpha_{(i,w),(j,u)}(f) \pi^{c(p^w-u)} \pi^{(p-1) \deg(u)} \xi_i \pi^{-\deg(u)} x^u. \]
That is, the matrix of \( \psi \) over \( \mathbb{Z}_p[[\pi^{1/2}]] \) with respect to the orthonormal basis \( \{\xi_j \pi^{-\deg(u)} x^u\}_{0 \leq j < a, u \in M(\Delta)} \) is
\[ A = (\alpha_{(i,w),(j,u)}(f) \pi^{c(p^w-u)} \pi^{(p-1) \deg(u)}(i,w),(j,u)). \]
So, the \( T \)-adic Newton polygon of \( \det(1 - \psi^s | B/\mathbb{Z}_p[[\pi^{1/2}]] \) lies above the polygon with vertices \((0, 0)\) and \((a \sum_{k=0}^{i} W(k), a(p-1) \sum_{k=0}^{i} k D W(k)), i = 0, 1, \cdots \).
Theorem 5.1 is proved.
We are now ready to prove the Hodge bound for the Newton polygon.

**Theorem 5.2.** We have

\[\text{NP}_T(f) \geq \text{HP}_q(\Delta).\]

*Proof.* By the above theorem, it suffices to prove that the \(T\)-adic Newton polygon of \(\det(1 - \psi^a s^a \mid B/\mathbb{Z}_q[[\pi^{1/2}]])\) coincides with that of \(\det(1 - \psi s \mid B/\mathbb{Z}_p[[\pi^{1/2}]])\). Note that

\[\det(1 - \psi^a s \mid B/\mathbb{Z}_p[[\pi^{1/2}]]) = \text{Norm}(\det(1 - \psi^a s \mid B/\mathbb{Z}_q[[\pi^{1/2}]])).\]

where the norm map is the norm from \(\mathbb{Z}_q[[\pi^{1/2}]]\) to \(\mathbb{Z}_q[[\pi^{1/2}]]\). The theorem now follows from the equality

\[
\prod_{\zeta^a=1} \det(1 - \zeta \psi s \mid B/\mathbb{Z}_p[[\pi^{1/2}]]) = \det(1 - \psi^a s^a \mid B/\mathbb{Z}_p[[\pi^{1/2}]]).
\]

\[\blacksquare\]

6. **Facial decomposition**

In this section, we extend the facial decomposition theorem in [26]. Recall that the operator \(\psi = \sigma^{-1} \circ (\psi \circ E_f)\) is only semi-linear over \(\mathbb{Z}_q[[\pi^{1/2}]]\). But its second factor \(\psi \circ E_f\) is clearly linear and so \(\det(1 - (\psi \circ E_f) s \mid B/\mathbb{Z}_q[[\pi^{1/2}]])\) is well defined. We begin with the following theorem.

**Theorem 6.1.** The \(T\)-adic Newton polygon of \(f\) coincides with \(\text{HP}_q(\Delta)\) if and only if the \(T\)-adic Newton polygon of \(\det(1 - (\psi \circ E_f) s \mid B/\mathbb{Z}_q[[\pi^{1/2}]])\) coincides with the polygon with vertices \((0,0)\) and

\[
\left( \sum_{k=0}^{i} W(k), (p-1) \sum_{k=0}^{i} \frac{k}{D} W(k) \right), \ i = 0, 1, \ldots.
\]

*Proof.* In the proof of Theorem 5.2, we showed that the \(T\)-adic Newton polygon of \(C_f(s^a, T)\) coincides with that of \(\det(1 - \psi s \mid B/\mathbb{Z}_p[[\pi^{1/2}]])\). Note that

\[\det(1 - (\psi \circ E_f) s \mid B/\mathbb{Z}_p[[\pi^{1/2}]]) = \text{Norm}(\det(1 - (\psi \circ E_f) s \mid B/\mathbb{Z}_q[[\pi^{1/2}]])).\]

where the norm map is the norm from \(\mathbb{Z}_q[[\pi^{1/2}]]\) to \(\mathbb{Z}_q[[\pi^{1/2}]]\). The theorem is equivalent to the statement that the \(T\)-adic Newton polygon of \(\det(1 - \psi s \mid B/\mathbb{Z}_p[[\pi^{1/2}]])\) coincides with the polygon with vertices \((0,0)\) and

\[
\left( \sum_{k=0}^{i} a W(k), a(p-1) \sum_{k=0}^{i} \frac{k}{D} W(k) \right), \ i = 0, 1, \ldots
\]

if and only if the \(T\)-adic Newton polygon of \(\det(1 - (\psi \circ E_f) s \mid B/\mathbb{Z}_p[[\pi^{1/2}]])\) does. Therefore it suffices to show that the determinant of the matrix

\[
(a_{(i,w),(j,u)}(f)\pi^{|pw-u|})_{0 \leq i,j < a, \deg(w), \deg(u) \leq \frac{k}{D}}
\]
is not divisible by $T$ in $\mathbb{Z}_q[[\frac{1}{\sigma}]]$ if and only if the determinant of the matrix
\[
(\alpha_{pw-u}(f)\pi^{c(pw-u,u)})_{\deg(w),\deg(u)\leq \frac{k}{D}}
\]
is not divisible by $T$ in $\mathbb{Z}_q[[\frac{1}{\sigma}]]$. The theorem now follows from the fact that the latter determinant is the norm of the former from $\mathbb{Q}_q[[\frac{1}{\sigma}]]$ to $\mathbb{Q}_p[[\frac{1}{\sigma}]]$ up to a sign.

We now define the open facial decomposition $F(\Delta)$. It is the decomposition of $C(\Delta)$ into a disjoint union of relatively open cones generated by the relatively open faces of $\Delta$ whose closure does not contain the origin. Note that every relatively open cone generated by co-facial vectors in $C(\Delta)$ is contained in a unique element of $F(\Delta)$.

**Lemma 6.2.** Let $\sigma \in F(\Delta)$, and $u \in \sigma$. Then $\alpha_u(f_\sigma) \equiv \alpha_u(f) \pmod{\pi^{1/D}}$, where $f_\sigma$ is the sum of monomials of $f$ whose exponent vectors lie in the closure $\bar{\sigma}$ of $\sigma$.

**Proof.** Let $v_1, \ldots, v_j$ be exponent vectors of monomials of $f$ such that $a_1v_1 + \cdots + a_jv_j = u$ with $a_1 > 0, \ldots, a_j > 0$. It suffices to show that either $v_1, \ldots, v_j$ lie in the closure of $\sigma$, or their contribution to $\alpha_u(f)$ is $\equiv 0 \pmod{\pi^{1/D}}$. Suppose that their contribution to $\alpha_u(f)$ is $\not\equiv 0 \pmod{\pi^{1/D}}$. Then $v_1, \ldots, v_j$ must be co-facial. So the interior of the cone generated by those vectors is contained in a unique element of $F(\Delta)$. As that interior has a common point $u$ with $\sigma$, it must be $\sigma$. It follows that $v_1, \ldots, v_j$ lie in the closure of $\sigma$. The lemma is proved.

**Lemma 6.3.** Let $\sigma, \tau \in F(\Delta)$ be distinct. Let $w \in \sigma$, and $u \in \tau$. Suppose that the dimension of $\sigma$ is no greater than that of $\tau$. Then $pw - u$ and $u$ are not co-facial, i.e., $c(pw - u, u) > 0$.

**Proof.** Suppose that $pw - u$ and $u$ are co-facial. Then the interior of the cone generated by $pw - u$ and $u$ is contained in a unique element of $F(\Delta)$. As that interior has a common point $w$ with $\sigma$, it must be $\sigma$. It follows that $u$ lies in the closure of $\sigma$. As $\sigma$ and $\tau$ are distinct, $u$ lies in the boundary of $\sigma$. This implies that the dimension of $\tau$ is less than that of $\sigma$, which is a contradiction. Therefore $pw - u$ and $u$ are not co-facial. The lemma is proved.

For $\sigma \in F(\Delta)$, we define
\[
M(\sigma) = M(\Delta) \cap \sigma = \mathbb{Z}^n \cap \sigma
\]
be the set of lattice points in the cone $\sigma$.

**Theorem 6.4** (Open facial decomposition). The $T$-adic Newton polygon of $f$ coincides with $\text{HP}_q(\Delta)$ if and only if for every $\sigma \in F(\Delta)$, the determinants of the matrices
\[
\{\alpha_{pw-u}(f_\sigma)\pi^{c(pw-u,u)}\}_{w, u \in M(\sigma), \deg(w), \deg(u) \leq \frac{k}{D}}, k = 0, 1, \ldots
\]
are not divisible by $T$ in $\mathbb{Z}_q[[\frac{1}{\sigma}]]$, where $\bar{\sigma}$ is the closure of $\sigma$. 
Proof. By Theorem 6.1, the \( T \)-adic Newton polygon of \( C_f(s, T) \) coincides with the \( q \)-Hodge polygon of \( f \) if and only if the determinants of the matrices

\[
A^{(k)} = \{ \alpha_{p^k-u}(f) \pi^{c(p^k-u,u)} \}_{u \in M(\Delta), \deg(u) \leq \frac{k}{T}}, \quad k = 0, 1, \ldots
\]

are not divisible by \( T \) in \( \mathbb{Z}_q[[\pi^T]] \). Write

\[
A^{(k)}_{\sigma,\tau} = \{ \alpha_{p^k-u}(f) \pi^{c(p^k-u,u)} \}_{u \in M(\sigma), u \in M(\sigma), \deg(u) \leq \frac{k}{T}}.
\]

The facial decomposition shows that \( A^{(k)} \) has the block form \((A^{(k)}_{\sigma,\tau})_{\sigma,\tau} \). The last lemma shows that the block form modulo \( \pi^T \) is triangular if we order the cones in \( F(\Delta) \) in dimension-increasing order. It follows that \( \det A^{(k)} \) is not divisible by \( T \) in \( \mathbb{Z}_q[[\pi^T]] \) if and only if for all \( \sigma \in F(\Delta) \), \( \det A^{(k)}_{\sigma,\sigma} \) is not divisible by \( T \) in \( \mathbb{Z}_q[[\pi^T]] \). By Lemma 6.2, \( \det A^{(k)}_{\sigma,\tau} \) is congruent to the matrix

\[
\{ \alpha_{p^k-u}(f_\sigma) \pi^{c(p^k-u,u)} \}_{u \in M(\sigma), \deg(u) \leq \frac{k}{T}}.
\]

So \( \det A^{(k)}_{\sigma,\tau} \) is not divisible by \( T \) in \( \mathbb{Z}_q[[\pi^T]] \) if and only if the determinant of the matrix

\[
\{ \alpha_{p^k-u}(f_\sigma) \pi^{c(p^k-u,u)} \}_{u \in M(\sigma), \deg(u) \leq \frac{k}{T}}
\]

is not divisible by \( T \) in \( \mathbb{Z}_q[[\pi^T]] \). The theorem is proved.

The closed facial decomposition Theorem 1.7 follows from the open decomposition theorem and the fact that

\[
F(\Delta) = \bigcup_{\sigma \in F(\Delta): \dim \sigma = \dim \Delta} F(\sigma).
\]

A similar \( \pi_{\psi} \)-adic facial decomposition theorem for \( C_f(s, \pi_{\psi}) \) can be proved in a similar way. Alternatively, it follows from the transfer theorem together with the \( \pi_{\psi} \)-adic facial decomposition in [26] for \( \psi \) of order \( p \).

7. Variation of C-functions in a family

Fix an \( n \)-dimensional integral convex polytope \( \Delta \) in \( \mathbb{R}^n \) containing the origin. For each prime \( p \), let \( P(\Delta, \mathbb{F}_p) \) denote the parameter space of all Laurent polynomials \( f(x) \) over \( \mathbb{F}_p \) such that \( \Delta(f) = \Delta \). This is a connected rational variety defined over \( \mathbb{F}_p \). For each \( f \in P(\Delta, \mathbb{F}_p)(\mathbb{F}_q) \), the Teichmüller lifting gives a Laurent polynomial \( \tilde{f} \) whose non-zero coefficients are roots of unity in \( \mathbb{Z}_q \). The C-function \( C_f(s, T) \) is then defined and \( T \)-adically entire. For simplicity of notation, we shall just write \( C_f(s, T) \) for \( C_f(s, T) \), similarly, \( L_f(s, T) \) for \( L_f(s, T) \). Thus, our C-function and L-function are now defined for Laurent polynomials over finite fields, via the Teichmüller lifting. We would like to study how \( C_f(s, T) \) varies when \( f \) varies in the algebraic variety \( P(\Delta, \mathbb{F}_p) \).
Recall that for a closed face $\sigma \in \Delta$, $f_{\sigma}$ denotes the restriction of $f$ to $\sigma$. That is, $f_{\sigma}$ is the sum of those non-zero monomials in $f$ whose exponents are in $\sigma$.

**Definition 7.1.** A Laurent polynomial $f \in P(\Delta, \mathbb{F}_p)$ is called non-degenerate if for every closed face $\sigma$ of $\Delta$ of arbitrary dimension which does not contain the origin, the system

$$\frac{\partial f_\sigma}{\partial x_1} = \cdots = \frac{\partial f_\sigma}{\partial x_n} = 0$$

has no common zeros with $x_1 \cdots x_n \neq 0$ over the algebraic closure of $\mathbb{F}_p$.

The non-degenerate condition is a geometric condition which insures that the associated Dwork cohomology can be calculated. In particular, it implies that, if $\psi$ is of order $p^m$, then the L-function $L_{f, \psi}(s)^{(-1)^{n-1}}$ is a polynomial in $s$ whose degree is precisely $n! \text{Vol}(\Delta)p^{n(m-1)}$, see [20]. As a consequence, we deduce

**Theorem 7.2.** Let $f \in P(\Delta, \mathbb{F}_p)(\mathbb{F}_q)$. Write

$$L_f(s, T)^{(-1)^{n-1}} = \sum_{k=0}^{\infty} L_{f,k}(T)s^k, \quad L_{f,k}(T) \in \mathbb{Z}_p[[T]].$$

Assume that $f$ is non-degenerate. Then for every positive integer $m$ and all positive integer $k > n! \text{Vol}(\Delta)p^{n(m-1)}$, we have the following congruence in $\mathbb{Z}_p[[T]]$:

$$L_{f,k}(T) \equiv 0 \pmod{(1 + T)^{p^m} - 1}. $$

**Proof.** Write

$$\frac{(1 + T)^{p^m} - 1}{T} = \prod (T - \xi).$$

The non-degenerate assumption implies that

$$L_f(s, \xi)^{(-1)^{n-1}} = \sum_{j=0}^{\infty} L_{f,j}(\xi)s^j,$$

is a polynomials in $s$ of degree $\leq n! \text{Vol}(\Delta)p^{n(m-1)} < k$. It follows that $L_{f,k}(\xi) = 0$ for all $\xi$. That is, $L_{f,k}(T)$ is divisible by $(T - \xi)$ for $\xi$. The theorem now follows. \qed

**Definition 7.3.** Let $N(\Delta, \mathbb{F}_p)$ denote the subset of all non-degenerate Laurent polynomials $f \in P(\Delta, \mathbb{F}_p)$.

The subset $N(\Delta, \mathbb{F}_p)$ is Zariski open in $P(\Delta, \mathbb{F}_p)$. It can be empty for some pair $(\Delta, \mathbb{F}_p)$. But, for a given $\Delta$, $N(\Delta, \mathbb{F}_p)$ is Zariski open dense in $P(\Delta, \mathbb{F}_p)$ for all primes $p$ except for possibly finitely many primes depending on $\Delta$. It is an interesting and independent question to classify the primes $p$ for which $N(\Delta, \mathbb{F}_p)$ is non-empty. This is related to the GKZ discriminant [12]. For simplicity, we shall only consider non-degenerate $f$ in the following.
7.1. **Generic ordinariness.** The first question is how often \( f \) is \( T \)-adically ordinary when \( f \) varies in the non-degenerate locus \( N(\Delta, \mathbb{F}_p) \). Let \( U_p(\Delta, T) \) be the subset of \( f \in N(\Delta, \mathbb{F}_p) \) such that \( f \) is \( T \)-adically ordinary, and \( U_p(\Delta) \) the subset of \( f \in N(\Delta, \mathbb{F}_p) \) such that \( f \) is ordinary. One can prove

**Lemma 7.4.** The set \( U_p(\Delta) \) is Zariski open in \( N(\Delta, \mathbb{F}_p) \).

One can ask if \( U_p(\Delta, T) \) is also Zariski open in \( N(\Delta, \mathbb{F}_p) \). We do not know the answer.

Our question is for which \( p \), \( U_p(\Delta) \) and \( U_p(\Delta, T) \) are Zariski dense in \( N(\Delta, \mathbb{F}_p) \). The rigidity bound as well as the Hodge bound imply that

\[
U_p(\Delta) \subseteq U_p(\Delta, T).
\]

It follows that if \( U_p(\Delta) \) is Zariski dense in \( N(\Delta, \mathbb{F}_p) \), then \( U_p(\Delta, T) \) is also Zariski dense in \( N(\Delta, \mathbb{F}_p) \).

The Adolphson-Sperber conjecture [1] says that if \( p \equiv 1 \pmod{I(\Delta)} \), then \( U_p(\Delta) \) is Zariski dense in \( N(\Delta, \mathbb{F}_p) \). This conjecture was proved to be true in [26] [27] if \( n \leq 3 \). In particular, this implies

**Theorem 7.5.** If \( p \equiv 1 \pmod{D} \) and \( n \leq 3 \), then \( U_p(\Delta, T) \) is Zariski dense in \( N(\Delta, \mathbb{F}_p) \).

For \( n \geq 4 \), it was shown in [26] [27] that there is an effectively computable positive integer \( D^*(\Delta) \) depending only on \( \Delta \) such that if \( p \equiv 1 \pmod{D^*(\Delta)} \), then \( U_p(\Delta) \) is Zariski dense in \( N(\Delta, \mathbb{F}_p) \). Thus, we obtain

**Theorem 7.6.** For each \( \Delta \), there is an effectively computable positive integer \( D^*(\Delta) \) such that if \( p \equiv 1 \pmod{D^*(\Delta)} \), then \( U_p(\Delta, T) \) is Zariski dense in \( N(\Delta, \mathbb{F}_p) \).

The smallest possible \( D^*(\Delta) \) is rather subtle to compute in general, and it can be much larger than \( D \). We now state a conjecture giving reasonably precise information on \( D^*(\Delta) \).

**Definition 7.7.** Let \( S(\Delta) \) be the monoid generated by the degree 1 lattice points in \( M(\Delta) \), i.e., those lattice points on the codimension 1 faces of \( \Delta \) not containing the origin. Define the exponent of \( \Delta \) by

\[
I(\Delta) = \inf \{ d \in \mathbb{Z}_{>0} | dM(\Delta) \subseteq S(\Delta) \}.
\]

If \( u \in M(\Delta) \), then the degree of \( Du \) will be integral but \( Du \) may not be a non-negative integral combination of degree 1 elements in \( M(\Delta) \) and thus \( DM(\Delta) \) may not be a subset of \( S(\Delta) \). It is not hard to show that \( I(\Delta) \geq D \). In general they are different but they are equal if \( n \leq 3 \). This explains why the Adolphson-Sperber conjecture is true if \( n \leq 3 \) and it can be false if \( n \geq 4 \). The following conjecture is a modified form, and it is a consequence of Conjecture 9.1 in [26].

**Conjecture 7.8.** If \( p \equiv 1 \pmod{I(\Delta)} \), then \( U_p(\Delta) \) is Zariski dense in \( N(\Delta, \mathbb{F}_p) \). In particular, \( U_p(\Delta, T) \) is Zariski dense in \( N(\Delta, \mathbb{F}_p) \) for such \( p \).
By the facial decomposition theorem, in proving the above conjecture, it is sufficient to assume that \( \Delta \) has only one codimension 1 face not containing the origin.

### 7.2. Generic Newton polygon.

In the case that \( U_p(\Delta, T) \) is empty, we expect the existence of a generic \( T \)-adic Newton polygon. For this purpose, we need to re-scale the uniformizer. For \( f \in N(\Delta, \mathbb{F}_p)(\mathbb{F}_{p^a}) \), the \( T^{a(p-1)} \)-adic Newton polygon of \( C_f(s, T; \mathbb{F}_{p^a}) \) is independent of the choice of \( a \) for which \( f \) is defined over \( \mathbb{F}_{p^a} \). We call them the absolute \( T \)-adic Newton polygon of \( f \).

**Conjecture 7.9.** There is a Zariski open dense subset \( G_p(\Delta, T) \) of \( N(\Delta, \mathbb{F}_p) \) such that the absolute \( T \)-adic Newton polygon of \( f \) is constant for all \( f \in G_p(\Delta, T) \). Denote this common polygon by \( \text{GNP}_p(\Delta, T) \), and call it the generic Newton polygon of \((\Delta, T)\).

More generally, one expects that much of classical theory for finite rank \( F \)-crystals extends to a certain nuclear infinite rank setting. This includes the classical Dieudonne-Manin isogeny theorem, the Grothendieck specialization theorem, the Katz isogeny theorem [18]. All these are essentially understood in the ordinary infinite rank case, but open in the non-ordinary infinite rank case.

Similarly, for each non-trivial \( \psi \), there is a Zariski open dense subset \( G_p(\Delta, \psi) \) of \( N(\Delta, \mathbb{F}_p) \) such that the \( \pi^a s(p-1) \)-adic Newton polygon of the \( C \)-value \( C_f(s, \pi^a \psi; \mathbb{F}_{p^a}) \) is constant for all \( f \in G_p(\Delta, \psi) \). Denote this common polygon by \( \text{GNP}_p(\Delta, \psi) \), and call it the generic Newton polygon of \((\Delta, \psi)\).

The existence of \( G_p(\Delta, \psi) \) can be proved. Since the non-degenerate assumption implies that the \( C \)-function \( C_f(s, \pi^a \psi) \) is determined by a single finite rank \( F \)-crystal via a Dwork type cohomological formula for \( L_{f, \psi}(s) \). In the \( T \)-adic case, we are not aware of any such finite rank reduction.

Clearly, we have the relation

\[ \text{GNP}_p(\Delta, \psi) \geq \text{GNP}_p(\Delta, T) \]

**Conjecture 7.10.** If \( p \) is sufficiently large, then

\[ \text{GNP}_p(\Delta, \psi) = \text{GNP}_p(\Delta, T). \]

This conjecture is proved in the case \( n = 1 \) in [22].

Let \( \text{HP}(\Delta) \) denote the absolute Hodge polygon with vertices \((0, 0)\) and

\[(\sum_{k=0}^{i} W(k), \sum_{k=0}^{i} \frac{k}{D} W(k)), \quad i = 0, 1, \cdots. \]

Note that \( \text{HP}(\Delta) \) depends only on \( \Delta \), not on \( q \) any more. It is re-scaled from the \( q \)-Hodge polygon \( \text{HP}_q(\Delta) \). Clearly, we have the relation

\[ \text{GNP}_p(\Delta, \psi) \geq \text{GNP}_p(\Delta, T) \geq \text{HP}(\Delta). \]

**Conjecture 7.8** says that if \( p \equiv 1 \pmod{I(\Delta)} \), then both \( \text{GNP}_p(\Delta, \psi) \) and \( \text{GNP}_p(\Delta, T) \) are equal to \( \text{HP}(\Delta) \). In general, the generic Newton polygon
lies above $\text{HP}(\Delta)$ but for many $\Delta$ it should be getting closer and closer to $\text{HP}(\Delta)$ as $p$ goes to infinity. We now make this more precise. Let $E(\Delta)$ be the monoid generated by the lattice points in $\Delta$. This is a subset of $M(\Delta)$. Generalizing the limiting Conjecture 1.11 in [27] for $\psi$ of order $p$, we have

**Conjecture 7.11.** If the difference $M(\Delta) - E(\Delta)$ is a finite set, then for each non-trivial $\psi$, we have

$$\lim_{p \to \infty} \text{GNP}_p(\Delta, \psi) = \text{HP}(\Delta).$$

In particular,

$$\lim_{p \to \infty} \text{GNP}_p(\Delta, T) = \text{HP}(\Delta).$$

This conjecture is equivalent to the existence of the limit. This is because for all primes $p \equiv 1 \pmod{D^*(\Delta)}$, we already have the equality $\text{GNP}_p(\Delta, \psi) = \text{HP}(\Delta)$ by Theorem 7.6. A stronger version of this conjecture (namely, Conjecture 1.12 in [27]) has been proved by Zhu [32] [33] [34] in the case $m = 1$ and $n = 1$, see also Blache and Férard [5] [6] and Liu [21] for related further work in the case $m = 1$ and $n = 1$, Hong [15] [16] and Yang [31] for more specialized one variable results. For $n \geq 2$, the conjecture is clearly true for any $\Delta$ for which both $D \leq 2$ and the Adolphson-Sperber conjecture holds, because then $\text{GNP}_p(\Delta, \psi) = \text{HP}(\Delta)$ for every $p > 2$. There are many such higher dimensional examples [27]. Using free products of polytopes and the above known examples, one can construct further examples [7].

### 7.3. $T$-adic Dwork Conjecture

In this final subsection, we describe the $T$-adic version of Dwork’s conjecture [10] on pure slope zeta functions.

Let $\Lambda$ be a quasi-projective subvariety of $N(\Delta, \mathbb{F}_p)$ defined over $\mathbb{F}_p$. Let $f_\lambda$ be a family of Laurent polynomials parameterized by $\lambda \in \Lambda$. For each closed point $\lambda \in \Lambda$, the Laurent polynomial $f_\lambda$ is defined over the finite field $\mathbb{F}_{p^\deg(\lambda)}$. The $T$-adic entire function $C_{f_\lambda}(s, T)$ has the pure slope factorization

$$C_{f_\lambda}(s, T) = \prod_{\alpha \in \mathbb{Q}_{\geq 0}} P_\alpha(f_\lambda, s),$$

where each $P_\alpha(f_\lambda, s) \in 1 + s\mathbb{Z}_p[[T]][s]$ is a polynomial in $s$ whose reciprocal roots all have $T^{\deg(\lambda)(p-1)}$-slope equal to $\alpha$.

**Definition 7.12.** For $\alpha \in \mathbb{Q}_{\geq 0}$, the $T$-adic pure slope $L$-function of the family $f_\lambda$ is defined to be the infinite Euler product

$$L_\alpha(f_\lambda, s) = \prod_{\lambda \in |\Lambda|} \frac{1}{P_\alpha(f_\lambda, s^{\deg(\lambda)})} \in 1 + s\mathbb{Z}_p[[T]][[s]],$$

where $|\Lambda|$ denotes the set of closed points of $\Lambda$ over $\mathbb{F}_p$.

The $T$-adic version of Dwork’s conjecture is then the following

**Conjecture 7.13.** For $\alpha \in \mathbb{Q}_{\geq 0}$, the $T$-adic pure slope $L$-function $L_\alpha(f_\lambda, s)$ is $T$-adic meromorphic in $s$. 
In the ordinary case, this conjecture can be proved using the methods in [28] [29] [30]. It would be interesting to prove this conjecture in the general case. The $\pi^\psi$-adic version of this conjecture is essentially Dwork’s original conjecture, which can be proved as it reduces to finite rank $F$-crystals. The difficulty of the $T$-adic version is that we have to work with infinite rank objects, where much less is known in the non-ordinary case.

References


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