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# WAYS OF BRANCHING QUANTIFERS*

## 1. Introduction

Branching quantifiers were first introduced by L. Henkin in his 1959 paper 'Some Remarks on Infinitely Long Formulas'. By 'branching quantifiers' Henkin meant a new, non-linearly structured quantifier-prefix whose discovery was triggered by the problem of interpreting infinitistic formulas of a certain form.[1] The branching (or partially-ordered) quantifier-prefix is, however, not essentially infinitistic, and the issues it raises have largely been discussed in the literature in the context of finitistic logic, as they will be here. Our discussion transcends, however, the resources of standard 1st-order languages and we will consider the new form in the context of 1st-order logic with 1- and 2-place 'Mostowskian' generalized quantifiers.[2]

Eventually we would like to know whether branching quantification is a genuine logical form. But today we find ourselves in an interesting

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[1] Henkin (1959), pp. 179–180.

[2] The Generalized Logic I refer to in this paper is a straightforward extension of Mostowski's system of 'cardinality' quantifiers in 'On a Generalization of Quantifiers'. This extension consists in applying Mostowski's criterion for 1-place quantifiers (over 1-place 1st-order predicates) to 2-place quantifiers (over a pair of 1-place 1st-order predicates). To the best of my knowledge, 2-place 'cardinality' quantifiers were first defined in Lindstrom (1966). However, a formal claim to the effect that the 2nd-order predicates construed as 'cardinality' quantifiers are *logical* already appears in Lindenbaum and Tarski (1935). In my Ph.D. thesis I provide philosophical grounds for a view of logical terms which includes these quantifiers in its extension. The quantifiers in question satisfy 'logicality' (see van-Benthem (1986)) but not other properties attributed to natural language quantifiers in the literature. Since the purpose of the present paper is to develop a notion of branching quantification which is applicable not only to linguistics but also to philosophy and 'pure' logic I feel this wider notion of quantifier is more appropriate. Below I will briefly describe the intended system and explain how to read its formulas for those readers unfamiliar with the subject.

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situation where it is not altogether clear what the branching form is. What happens when you take a collection of quantifiers, order them in an arbitrary partial ordering, and attach the result to a given formula? What truth conditions are to be associated with the resulting expression? Can we define these conditions as a function of the quantifiers involved and their ordering? These questions have not received definite answers in the context of generalized logic. Henkin's work settled the issue of branching standard quantifiers but not of branching generalized quantifiers. Although significant progress in this direction was made by J. Barwise, D. Westerstahl, J. van Benthem and others, the question is largely open. My purpose in this paper is to further clarify the meaning of the branching form. Following the historical development, I will begin with standard quantifiers.

There are two natural ways to approach branching quantification: as a generalization on the ordering of standard quantifier-prefixes, and as a generalization of Skolem Normal Forms.

### 1.1. *Generalizing on the Ordering of Standard Quantifier-Prefixes*

In standard modern logic quantifier-prefixes are linearly ordered, both syntactically and semantically. The syntactic ordering of a quantifier-prefix $\langle(Q_1x_1), \ldots, (Q_nx_n)\rangle$ (where $Q_i$ is either $\forall$ or $\exists$, for $1 \leq i \leq n$) mirrors the sequence of steps used to construct well-formed formulas with that quantifier-prefix. Thus, if

$$(1.1) \qquad (Q_1x_1) \cdots (Q_nx_n)\Phi(x_1, \ldots, x_n)$$

is a well-formed formula, exactly one out of any two quantifiers, $Q_ix_i$ and $Q_jx_j$ $(1 \leq i \neq j \leq n)$ – namely, the innermore – preceeds the other in the syntactic construction of (1.1). The semantic ordering of a quantifier-prefix is the order of determining the truth (satisfaction) conditions of formulas with that prefix, and it is the backward image of the syntactic ordering. The truth of a sentence of the form (1.1) in a model $\mathscr{A}$ with a universe $A$ is determined in the following order of stages:[3]

(i) conditions of truth (in $\mathscr{A}$) for $(Q_1x_1)\Psi_1(x_1)$,
where $\Psi_1 = (Q_2x_2) \cdots (Q_nx_n)\Phi(x_1, x_2, \ldots, x_n)$;
(ii) conditions of truth for $(Q_2x_2)\Psi_2(x_2)$,

[3] For the sake of simplicity I assume that (1.1) has no free variables. I make similar assumptions throughout the paper. I speak of 'truth in a model' rather than of 'satisfaction by an assignment in a model', and I formulate the definitions as if I were dealing only with sentences. It is easy to extend these formulations to the case of formulas with free variables.

where $\Psi_2 = (Q_3x_3) \cdots (Q_nx_n)\Phi(a_1, x_2, x_3, \ldots, x_n)$, and $a_1$ is an arbitrary element of $A$;

⋮

(*n*) conditions of truth for $(Q_nx_n)\Psi_n(x_n)$,
where $\Psi_n = \Phi(a_1, a_2, \ldots, a_{n-1}, x_n)$, and $a_1, \ldots, a_{n-1}$ are arbitrary elements of $A$.

We obtain branched-quantification by relaxing the requirement that quantifier-prefixes be *linearly* ordered, allowing *partial* ordering instead. It is rather clear what renouncing the requirement of linearity means syntactically. But what does it mean semantically? What would a partially-ordered definition of truth for multiply quantified sentences look like? Approaching branching quantifiers as a generalization on the ordering of quantifiers in standard logic leaves the issue of their correct semantic definition an open question.

### 1.2. *Generalizing from the Existence of Skolem Normal Forms*

The Skolem Normal Form Theorem[4] says that every 1st-order formula is logically equivalent to a 2nd-order prenex formula of the form

$$(1.2) \qquad (\exists f_1) \cdots (\exists f_m)(\forall x_1) \cdots (\forall x_n)\Phi,$$

where $x_1, \ldots, x_n$ are individual variables, $f_1, \ldots, f_m$ are functional variables ($m, n \geqslant 0$), and $\Phi$ is a quantifier-free formula. This 2nd-order formula is a *Skolem Normal Form*, and the functions satisfying a Skolem Normal Form are *Skolem Functions*.

The idea is, roughly, that given a formula with an individual existential quantifier in the scope of one or more individual universal quantifiers we obtain its Skolem Normal Form by replacing the former with a functional existential quantifier governing the latter. For example,

$$(1.3) \qquad (\forall x)(\forall y)(\exists z)\Phi(x, y, z)$$

is equivalent to

$$(1.4) \qquad (\exists f^2)(\forall x)(\forall y)\Phi(x, y, f^2(x, y)).$$

The functional variable $f^2$ in (1.4) replaces the individual variable $z$ bound by the existential quantifier $(\exists z)$ in (1.3), and the arguments of $f^2$ are all the individual variables bound by the universal quantifiers governing $(\exists z)$ there. It is characteristic of a Skolem Normal Form of a 1st-order formula

[4] See, for example, Enderton (1972), p. 275.

with more than one existential quantifier that for any two functional variables in it, the set of arguments of one is included in the set of arguments of the other. Consider, for instance, the Skolem Normal Form of

(1.5) $(\forall x)(\exists y)(\forall z)(\exists w)\Phi(x, y, z, w)$

namely,

(1.6) $(\exists f^1)(\exists g^2)(\forall x)(\forall z)\Phi(x, f^1(x), z, g^2(x, z))$.

In general, Skolem Normal Forms of 1st-order formulas are formulas of the form (1.2) satisfying the property:

(1.A) The functional existential quantifiers $(\exists f_1), \ldots, (\exists f_m)$ can be ordered in such a way that for all $1 \leq i, j \leq m$, if $(\exists f_j)$ syntactically precedes $(\exists f_i)$, then the set of arguments of $f_i$ in $\Phi$ is essentially included in the set of arguments of $f_j$ in $\Phi$.[5]

This property reflects what W. J. Walkoe calls the 'essential order' of liner quantifier-prefixes.[6]

The existence of Skolem Normal Forms for all 1st-order formulas is thought to reveal a systematic connection between Skolem Functions and existential individual quantifiers. However, this connection is not 1–1: not all formulas of the form (1.2) – General Skolem Forms – are expressible by standard (i.e., linear) 1st-order formulas. General Skolem Forms not satisfying (1.A) are not.

It is natural to generalize the connection between Skolem Functions and existential quantifiers into a complete correspondence. But such a generalization requires that 1st-order quantifier-prefixes not be, in general, linearly ordered. The simplest Skolem Form not satisfying (1.A) is:

(1.7) $(\exists f^1)(\exists g^1)(\forall x)(\forall z)\Phi[x, f^1(x), z, g^1(z)]$.

Relaxing the requirement of syntactic linearity, we can construct a '1st-order' correlate for it:

(1.8) $$\begin{matrix} (\forall x)(\exists y) & \cdot & \\ & \cdot & \Phi(x, y, z, w). \\ (\forall z)(\exists w) & \cdot & \end{matrix}$$

We see that the semantic structure of a partially-ordered quantifier prefix is introduced, in this approach, *together with* (or even *prior to*) the

[5] Henkin (1959), p. 181.
[6] Walkoe (1970), p. 538.

syntactic structure. The interpretation of a 1st-order branching formula is fixed to begin with by its postulated equivalence to a 2nd-order, linear Skolem Form.

Comparing 1.1 and 1.2, we ask: Do the two generalizations necessarily coincide? Do 2nd-order Skolem Forms provide the only reasonable semantic interpretation for the syntax of partially-ordered quantified formulas? The definition of branching quantifiers by generalized Skolem Functions was propounded by Henkin, who recommended it as 'natural'. Most subsequent writers on the subject took Henkin's definition as given. I was led to reflect on the possibility of alternative definitions by J. Barwise's paper 'On Branching Quantifiers in English' (1979). Barwise shifted the discussion from standard to generalized branching quantifiers, forcing us to rethink the principles underlying the branching structure. Reviewing the earlier controversy around Hintikka's purported discovery of branching-quantifier constructions in natural language and following a course of inquiry begun in my previous work on 1st-order quantifiers, I came to think that both logico-philosophical and linguistic considerations suggest further investigation of the semantics of branching quantifiers.

## 2. Linguistic motivation

In his paper 'Quantifiers vs. Quantification Theory' (1973) J. Hintikka first pointed out that some quantifier constructions in English are branching rather than linear. A well known example is:

(2.1) Some relative of each villager and some relative of each townsman hate each other.[7]

Hintikka says: "This [example] may . . . offer a glimpse of the ways in which branched quantification is expressed in English. Quantifiers occurring in conjoint constituents frequently enjoy independence of each other, it seems, because a sentence is naturally thought of as being symmetrical semantically vis-a-vis such constituents".[8] Another linguistic form of the branching-quantifier structure is illustrated by:

(2.2) Some book by every author is referred to in some essay by every critic.[9]

Hintikka's point is that sentences such as (2.1) or (2.2) contain two

[7] Hintikka (1973), p. 344, (37).

[8] *Ibid.*, ibid.

[9] *Ibid.*, p. 345, (39).

*independent* pairs of iterated quantifiers, the quantifiers in each pair being outside the scope of the quantifiers in the other. A standard 1st-order formalization of such sentences – for instance, that of (2.1) as

$$(2.3) \qquad (\forall x)(\exists y)(\forall z)(\exists w)[Vx \,\&\, Tz \rightarrow \mathrm{REL}yx \,\&\, \mathrm{REL}wz \,\&\, Hyw \,\&\, Hwy]$$

or

$$(2.4) \qquad (\forall x)(\forall z)(\exists y)(\exists w)[Vx \,\&\, Tz \rightarrow \mathrm{REL}yx \,\&\, \mathrm{REL}wz \,\&\, Hyw \,\&\, Hwy]$$

(with the obvious readings of $V$, $T$, REL and $H$) – creates dependencies where none should exist. A branching-quantifier reading, on the other hand –

$$(2.5) \qquad \begin{array}{l} (\forall x)(\exists y) \\ (\forall z)(\exists w) \end{array} \Big\rangle \; Vx \,\&\, Tz \rightarrow \mathrm{REL}yx \,\&\, \mathrm{REL}wz \,\&\, Hyw \,\&\, Hwy$$

simulates accurately the dependencies and independencies involved. Hintikka does not ask what truth conditions should be assigned to (2.5), assuming it is interpreted in the 'usual' way as:

$$(2.6) \qquad (\exists f^1)(\exists g^1)(\forall x)(\forall z)[Vx \,\&\, Tz \rightarrow \mathrm{REL}(f^1(x), x) \,\&\, \mathrm{REL}(g^1(z), z) \,\&\, H(f^1(x), g^1(z)) \,\&\, H(g^1(z), f^1(x))].$$

Hintikka's paper brought forth a lively exchange of opinions, and G. Fauconnier (1975) raised the following objection (which I will formulate in my own words): (2.6) implies that the relation of mutual hatred between relatives of villagers and relatives of townsmen has what we might call *a massive nucleus* – one which contains at least one relative of each villager and one relative of each townsman – and such that *each* villager-relative in the nucleus hates *all* the townsman-relatives in it, and vice versa. However, Fauconnier objects, it is not true that every English sentence with syntactically independent quantifiers implies the existence of a *massive nucleus* of objects standing in the relation quantified. For instance,

(2.7) Some player of every football team is in love with some dancer of every ballet company[10]

does not; it is compatible with the assumption that men are in love with one woman at a time (and that dancers/football-players do not belong to more than one ballet-company/football-team at a time).

[10] Fauconnier (1975), p. 560, (10).

We can illustrate the point graphically as follows:

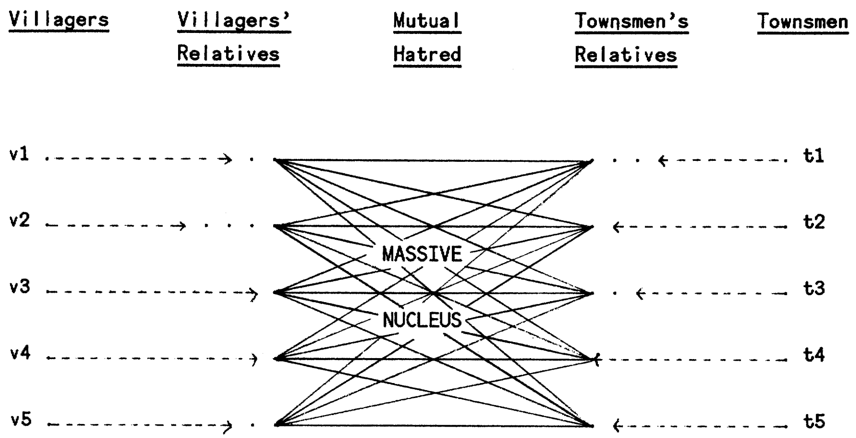


Fig. 1.

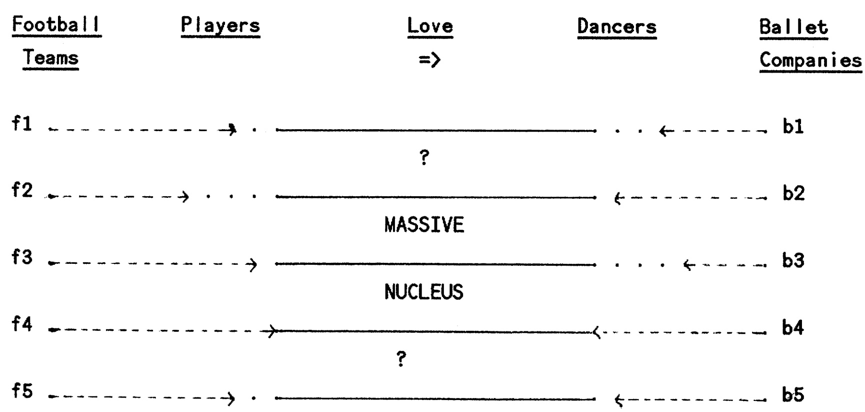


Fig. 2.

Even if Hintikka's interpretation of (2.1) is correct, Fauconnier argues, i.e., (2.1) implies the existence of a *massive nucleus* of villagers and townsmen in mutual hatred, (2.7) *does not* imply the existence of a *massive nucleus* of football players in love with dancers. Hintikka's interpretation, therefore, is not appropriate to all scopally independent quantifiers in natural language.

The point is accentuated in the following examples:

(2.8) Some player of every football team is *the* boyfriend of some dancer of every ballet company,

(2.9) Some relative of each villager and some relative of each townsman are married (to one another).

Is (2.8) logically false? Does (2.9) imply that the community in question is polygamous?

Fauconnier's conclusion is that natural-language constructions with scopally independent quantifiers are sometimes branching and sometimes linear, depending on the context. The correct interpretation of (2.7), for instance, is:

(2.10) $(\forall x)(\forall y)(\exists z)(\exists w)(FTx \& BCy \rightarrow \mathrm{P}zx \& Dwy \& Lzw)$.

Thus, according to Fauconnier, the only alternative to 'massive nucleus' is linear quantification,.

We can, however, approach the matter somewhat differently. Acknowledging the semantic independence of syntactically unnested quantifiers in general, we can ask: Why should independence of quantifiers have anything to do with the existence of a 'massive nucleus' of objects standing in the quantified relation? Interpreting branching quantifiers nonlinearly, yet without commitment to a 'massive nucleus', would do justice both to Hintikka's insight regarding the nature of scopally independent quantifiers and to Fauconnier's (and others') observations regarding the multiplicity of situations which such quantifiers can be used to describe. We are thus led to search for an alternative to Henkin's definition which would avoid the problematical commitment.

## 3. Logico-philosophical motivation

Why are quantifier-prefixes in modern symbolic logic linearly ordered? M. Dummett (1973) ascribes this feature of quantification theory to the genius of Frege. Traditional logic failed because it could not account for the validity of inferences involving multiple quantification. Frege saw that the problem could be solved if we construed multiply quantified sentences as complex *step-by-step* constructions, built by repeated applications of the simple logical operations of universal and/or existential quantification. This step-by-step syntactic analysis of multiply-quantified sentences was to serve as a basis for a corresponding step-by-step semantic analysis, unfolding the truth conditions of one constructional stage – i.e., a *singly quantified formula* – at a time. (See Section I above.) In other words, by Frege's method of logical analysis the problem of defining truth for a quantified many-place relation was reduced to that of defining truth for a series of quantified predicates (one-place relations), a problem whose solution was essentially known.[11] The possibility of such a reduction was

[11] Dummett (1973), pp. 8f.

based, however, on a particular way of representing relations. In Tarskian semantics this form of representation is reflected in the way the linear steps in the definition of truth are 'glued' together, namely, by a relative expression synonymous with 'for each one of which' ('f.e.w.'). Thus, for example, the Fregean–Tarskian definition of truth for

(3.1) $(Q1x)(Q2y)(Q3z)R^3(x, y, z)$

(where $Q1$, $Q2$ and $Q3$ are either $\forall$ or $\exists$) proceeds as follows: (3.1) is true in a model $\mathscr{A}$ with a universe $A$ iff (if and only if) there are $q1\,a$'s in $A$ *f.e.w.* there are $q2\,b$'s in $A$ *f.e.w.* there are $q3\,c$'s in $A$ such that '$R^3(a, b, c)$' is true in $\mathscr{A}$ (where $q1$, $q2$ and $q3$ are the quantifier conditions associated with $Q1$, $Q2$ and $Q3$ respectively).[12]

Intuitively, the view of $R^3$ embedded in the definition of truth for (3.1) is that of a *multiple tree*:

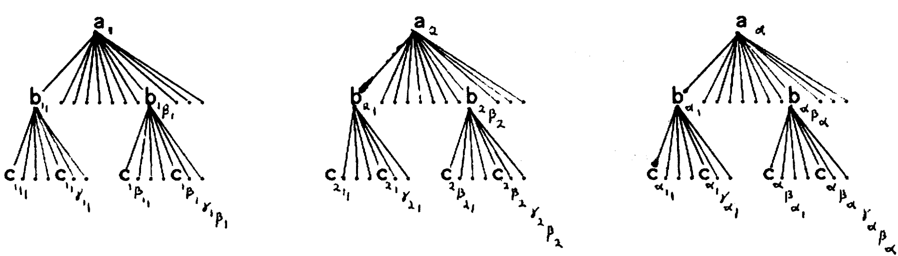

Fig. 3.

Each row in the multiple tree represents one domain of $R^3$ (the extension of one argument place of $R^3$); each tree represents the restriction of $R^3$ to some one element of the domain listed in the upper row. In this way, the extension of the second domain is represented *relative* to that of the first, and the extension of the third, relative to the (already relative) representation of the second. Different quantifier-prefixes allow different multiple-tree views of relations, but Frege's linear quantification limits the expressive power of quantifier-prefixes to properties of relations which are discernible in a multiple-tree representation.

We can describe the sense in which (all but the outermost) quantifiers in a linear prefix are semantically dependent as follows: a *linearly dependent* quantifier assigns a property not to a complete domain of the relation quantified, but to a domain relativized to individual elements of another

[12] An alternative reading: $q1\,a$'s in $A$ are such that for each one of them [$q2\,b$'s in $A$ are such that for each one of them [$q3\,c$'s in $A$ are such that for each one of them '$R^3(a, b, c)$' is true in $\mathscr{A}$]].

domain, higher up in the multiple tree. It is characteristic of a linear quantifier-prefix that each quantifier (but the outermost) is directly dependent on exactly one other quantifier. We shall therefore call linear quantifiers *uni- or simply-dependent.*

There are two natural alternatives to simple dependence: (i) no dependence, i.e., *independence*, and (ii) *complex dependence*. These correspond to two ways in which we can view relations in a non-linear manner: we can view each domain separately, complete and unrelativised; or we can view a whole cluster of domains at once, in their mutual relationships.

Syntactically, we will represent an *independent* quantification by:

$$(3.2)\qquad \left.\begin{array}{c}(Q_1x_1)\\ \vdots\\ (Q_nx_n)\end{array}\right| R^n(x_1, \ldots, x_n),$$

and a *complex* quantification by:

$$(3.3)\qquad \left.\begin{array}{c}(Q_1x_1)\\ \vdots\\ (Q_nx_n)\end{array}\right\rangle \!\!\!\Rightarrow R^n(x_1, \ldots, x_n).$$

Of course there are many complex patterns of dependence among quantifiers. These can be represented by partially-ordered prefixes.

Our analysis indicates that the concept of independent quantification is different from that of complex quantification. Therefore, the first question regarding the correct interpretation of natural-language sentences with branching quantifiers is: Are the quantifiers in these sentences independent or complex?

## 4. Independent branching quantifiers

It is easy to give a precise definition of *independent quantification*:

$$(4.1)\qquad \left.\begin{array}{c}(Q1x)\\ (Q2y)\end{array}\right| \Phi(x, y) =_{\mathrm{Df}} (Q1x)(\exists y)\Phi(x, y) \;\&\; (Q2y)(\exists x)\Phi(x, y),$$

or, more generally:

$$(4.2)\quad \left.\begin{matrix}(Q_{1_1}x_{1_1})\cdots(Q_{1_{m_1}}x_{1_{m_1}})\\ \vdots \\ (Q_{n_1}x_{n_1})\cdots(Q_{n_{m_n}}x_{n_{m_n}})\end{matrix}\right| \Phi(x_{1_1},\ldots,x_{1_{m_1}},\ldots,x_{n_1},\ldots,x_{n_{m_n}}) =_{\mathrm{Df}}$$

$$(Q_{1_1}x_{1_1})\cdots(Q_{1_{m_1}}x_{1_{m_1}})(\exists x_{2_1})\cdots(\exists x_{n_{m_n}})$$
$$\Phi(x_{1_1},\ldots,x_{1_{m_1}},\ldots,x_{n_1},\ldots,x_{n_{m_n}})\ \&\cdots\&$$
$$(Q_{n_1}x_{n_1})\cdots(Q_{n_{m_n}}x_{n_{m_n}})(\exists x_{1_1})\cdots(\exists x_{n-1_{m_{n-1}}})$$
$$\Phi(x_{1_1},\ldots,x_{1_{m_1}},\ldots,x_{n_1},\ldots,x_{n_{m_n}}).$$

This analysis gives the notion of branching quantification a sense which is very different from that of Henkin's. Independent quantification is essentially 1st-order. It does not involve commitment to a 'massive nucleus' or to any other complex structure of objects standing in the quantified relation. Therefore it enables us to analyze natural-language branching quantifications in a straightforward manner, and without forcing any independent quantifier into a nested position. We thus propose (4.1) as a definition of branching quantifiers qua *independent* quantifiers. Linguistically, this construal is supported by the fact that 'and' often appears as a 'quantifier connective' in natural-language branching structures in a way which might indicate a shift from the 'original' position as a sentential connective. Moreover, natural-language branching quantifiers are symmetrical in much the same way that the conjuncts in our definition are. An English sentence with standard quantifiers which appears to exemplify independent quantification is:

(4.3) Nobody loves nobody,

understood as 'Nobody loves anybody'.[13] We will symbolize it as

$$(4.4)\quad \left.\begin{matrix}(\sim\exists x)\\(\sim\exists y)\end{matrix}\right| Lxy$$

and interpret it by:

$$(4.5)\quad \sim(\exists x)(\exists y)Lxy\ \&\ \sim(\exists y)(\exists x)Lxy.$$

Extending our logical apparatus by adding 1-place Mostowskian general-

[13] I wish to thank Robert May for this example.

ized quantifiers[14] we will be able to interpret the following English sentences as independent branching quantifications:

(4.6) Three frightened elephants were chased by a dozen hunters.
(4.7) Four Martians and five Humans exchanged insults.
(4.8) [Can] an odd number of beds serve an even number of patients [?]

The 'independent' interpretation of (4.6)–(4.7) reflects a 'cumulative' reading,[15] under which no massive nucleus, or any other complex relationship between the domain and range of the relation in question, is intended. We thus understand (4.6) as saying that the relation 'a frightened elephant $x$ was chased by a hunter $y$' includes three individuals in its domain and a dozen individuals in its range. And this reading is captured by (4.1). Similarly, (4.1) yields the cumulative interpretations of (4.7) and (4.8).

To extend the applicability of our definition further, we will allow 2-place Mostowskian quantifiers,[16] yielding branching quantifications of the

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[14] A 1-place Mostowskian quantifier is, syntactically, an operator $Q$ such that if $x$ is an individual variable and $\Phi$ is a formula, $(Qx)\Phi$ is also a formula. Semantically, $Q$ assigns to every model $\mathscr{A}$ for (the non-logical vocabulary of) the language a function $q_{\mathscr{A}}$ such that: (i) if $A$ is the universe of $\mathscr{A}$ and $B$ is a subset of $A$, then $q_{\mathscr{A}}(B) \in \{T, F\}$, and (ii) for any models $\mathscr{A}$, $\mathscr{A}'$ with universes $A$, $A'$, respectively, if $B \subseteq A$, $B' \subseteq A'$ and the structures $\langle A, B\rangle$, $\langle A', B'\rangle$ are isomorphic, then $q_{\mathscr{A}}(B) = q_{\mathscr{A}'}(B')$. Intuitively a Mostowskian quantifier $Q$ is logical because it distinguishes only the *structure* of the predicates over which it ranges. As was proved by Mostowski, such a quantifier is a 'cardinality' quantifier in that the value it assigns to any 1-place predicate $P$ in a given model $\mathscr{A}$ depends only on the *cardinality* of the extensions of $P$ and its complement in $\mathscr{A}$. Among 1-place Mostowskian quantifiers we find 'Exactly/at-least/at-most $n$' for every natural number $n$, 'between $m$ and $n$', 'most', 'few', 'one-half', 'infinitely many', 'countably/uncountably many', 'an even number of', etc. Given a 1-place 1st-order predicate $P$ and a model $\mathscr{A}$, '(Exactly-$n\,x)Px$' is true in $\mathscr{A}$ if and only if the extension of $Px$ in $\mathscr{A}$ is of cardinality $n$; '(Most $x)Px$' is true in $\mathscr{A}$ if and only if the extension of $Px$ in $\mathscr{A}$ is larger than the extension of $\sim Px$ in $\mathscr{A}$; etc.

[15] See van Benthem (1988).

[16] A 2-place Mostowskian quantifier is, syntactically, an operator $Q^2$ such that if $x$ is an individual variable and $\Phi$, $\Psi$ are formulas, $(Q^2x)(\Phi, \Psi)$ is a formula. Semantically, $Q^2$ assigns to every model $\mathscr{A}$ a function $q^2_{\mathscr{A}}$ such that: (i) if $A$ is the universe of $\mathscr{A}$ and $B$, $C$ are subsets of $A$, then $q^2_{\mathscr{A}}(B, C) \in \{T, F\}$, and (ii) for any models $\mathscr{A}$, $\mathscr{A}'$ with universes $A$, $A'$, respectively, if $B, C \subseteq A$, $B'$, $C' \subseteq A'$ and the structures $\langle A, (B, C)\rangle$, $\langle A', (B', C')\rangle$ are isomorphic, then $q^2_{\mathscr{A}}(B, C) = q^2_{\mathscr{A}'}(B', C')$. 2-place Mostowskian quantifiers are *logical* in the same sense as 1-place Mostowskian quantifiers are. They are also 'cardinality' quantifiers since $q^2_{\mathscr{A}}(B, C)$ depends (aside from the order of the pair $(B, C)$) only on the cardinalities of the atoms of the Boolean algebra generated by $B$ and $C$ in $A$. (The atoms are $B - C$, $B \cap C$, $C - B$, and $A - (B \cup C)$.) Among 2-place Mostowskian quantifiers we find 'All$^2$', 'Exactly-$n^2$', 'Most$^2$', 'Few$^2$', 'Only$^2$', 'Half$^2$', 'There are as many --- as ***', 'There are fewer --- than ***', etc. Given a model $\mathscr{A}$ with a universe $A$ and two 1-place 1st-order predicates $P1$ and $P2$, '(All$^2$ $x)(P1x, P2x)$' is true in $\mathscr{A}$ iff the extension of $P1x$ & $\sim P2x$ in $\mathscr{A}$ is empty; '(Exactly-$n^2$ $x)$ $(P1x, P2x)$' is true in $\mathscr{A}$ iff the extension of $P1x$ & $P2x$ in $\mathscr{A}$ has exactly $n$ elements;

form:

(4.9) $\begin{array}{c|l} (Q_1^2 x) & \Psi_1 x, \\ (Q_2^2 y) & \Psi_2 y, \end{array} \Phi xy.$

Here, however, we can apply the notion of independent quantification in several ways. Given a binary relation $R$, two sets, $A$ and $B$, and two quantifier-conditions $q1$ and $q2$, we can say:

(a) The relation $R$ has $q1$ $A$'s in its domain and $q2$ $B$'s in its range;
(b) The relation $A \upharpoonleft R \upharpoonright B$ has $q1$ elements in its domain and $q2$ elements in its range (where $A \upharpoonleft R \upharpoonright B$ is obtained from $R$ by restricting its domain to $A$ and its range to $B$);
(c) The relation $A \upharpoonleft R \upharpoonright B$ has $q1$ $A$'s in its domain and $q2$ $B$'s in its range;
(d) The relation $R \upharpoonright B$ has $q1$ $A$'s in its domain and $q2$ $B$'s in its range;
etc.

As the reader can verify, (a)–(d) above are not equivalent. However, for the examples discussed in this paper it suffices to define (4.9) for case (c). We thus propose the following definition of a pair of 2-place independent quantifiers:

(4.10) $$\begin{array}{c|l} (Q_1^2 x) & \Psi_1 x, \\ (Q_2^2 y) & \Psi_2 y, \end{array} \Phi xy =_{\mathrm{Df}} (Q_1^2 x)[\Psi_1 x, (\exists y)(\Psi_1 x \,\&\, \Psi_2 y \,\&\, \Phi xy)] \,\&\, (Q_2^2 y)[\Psi_2 y, (\exists x)(\Psi_1 x \,\&\, \Psi_2 y \,\&\, \Phi xy)].$$

When $Q1_1^2$ and $Q_2^2$ are *conservative* – i.e., $(Q_1^2 x)(\Phi x, \Psi x)$ is logically equivalent to $(Q_1^2 x)(\Phi x, \Phi x \,\&\, \Psi x)$ – we can replace (4.10) by the simpler

(4.11) $$\begin{array}{c|l} (Q_1^2 x) & \Psi_1 x, \\ (Q_2^2 y) & \Psi_2 y, \end{array} \Phi xy =_{\mathrm{Df}} (Q_1^2 x)[\Psi_1 x, (\exists y)(\Psi_2 y \,\&\, \Phi xy)] \,\&\, (Q_2^2 y)[\Psi_2 y, (\exists x)(\Psi_1 x \,\&\, \Phi xy)].$$

Using this definition, we can interpret (4.12)–(4.13) below as independent

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'$(\mathrm{Most}^2 x)(P1x, P2x)$' is true in $\mathscr{A}$ iff the extension of $P1x$ & $P2x$ in $\mathscr{A}$ is larger than the extension of $P1x$ & $\sim P2x$ in $\mathscr{A}$; '$(\mathrm{Few}^2 x)(P1x, P2x)$' is true in $\mathscr{A}$ iff the extension of $P1x$ & $P2x$ in $\mathscr{A}$ is 'small with respect to the extension of $P1x$ in $\mathscr{A}$' (where 'small . . .' receives some quantitative interpretation, say 'less than a third of the extension of $P1x$ in $\mathscr{A}$'); '$(\mathrm{Only}^2 x)(P1x, P2x)$' is true in $\mathscr{A}$ iff the extension of $\neg P1x$ & $P2x$ in $\mathscr{A}$ is empty; '$(\text{As-many-as}^2 x)$ $(P1x, P2x)$' is true in $\mathscr{A}$ iff the extension of $P1x$ in $\mathscr{A}$ is of the same cardinality as the extension of $P2x$ in $\mathscr{A}$; etc. In the body of the paper I will sometimes omit the superscripts of 2-place quantifiers.

(4.12) All the boys ate all the apples.[17]
(4.13) Two boys ate half the apples.

We could also analyze (4.6)–(4.8) as independent quantifications of the form (4.11).[18]

What about Hintikka's (2.1) and Fauconnier's (2.7)? Should we interpret these as independent branching quantifications of the form (4.11)? Under such an interpretation (2.1) would say that the relation of mutual hatred between relatives of villagers and relatives of townsmen includes at least one relative of each villager in its domain and at least one relative of each townsman in its range; (2.7) would be understood as saying that the relation of love between football players and ballet dancers includes at least one player of each football team in its domain and at least one dancer from each ballet company in its range. Such interpretations would be compatible both with Figure 1 and with Figure 2. Later on we will suggest a test to determine whether the intended interpretation of a given natural-language sentence with branching quantifiers is that of an independent or complex quantification, and this might give us a clue regarding Hintikka's and Fauconnier's sentences. As for the linear option, here the question is whether one pair of quantifiers is within the scope of the other. Generally, I would say that when 'and' appears as a quantifier connective – '$Q1$ $A$'s *and* $Q2$ $B$'s stand in the relation $R$' – the quantification is not linear. However, when the quantification is of the form '$Q1$ $A$'s $R$ $Q2$ $B$'s', the situation is less clear. For relevant discussions see May (1987) and van Benthem (1988).[19] Here I would just comment that sometimes the method

[17] I would like to thank an anonymous referee for suggesting (4.12) as an example for an independent quantification which, unlike (4.6)–(4.8), cannot be analyzed by (4.1).

[18] However, to analyze 'Mostly women were elected to the vacant seats in Congress' we will have to define independent quantification of type (d):

$$\begin{array}{c|l} (Q_1^2 x) & \Psi_1 x, \\ (Q_2^2 y) & \Psi_2 y, \end{array} \Phi xy =_{\mathrm{Df}} (Q_1^2 x)(\Psi_1 x, (\exists y)(\Psi_2 y \mathbin{\&} \Phi xy)) \mathbin{\&} (Q_2^2 y)(\Psi_2 y, (\exists x)\Phi xy).$$

Construing 'mostly' and the plural 'the' as 2-place Mostowskian quantifiers defined by: '$(\mathrm{The}_P^2 x)(P1x, P2x)$' is true in a model $\mathscr{A}$ iff the extension of $P1x$ in $\mathscr{A}$ is not empty and the extension of $P1x$ & $\sim P2x$ in $\mathscr{A}$ is empty; '$(\mathrm{Mostly}^2 x)(P1x, P2x)$' is true in $\mathscr{A}$ iff the extension of $P1x$ & $P2x$ in $\mathscr{A}$ is larger than the extension of $\sim P1x$ & P2x in $\mathscr{A}$ – we get the intended reading of 'Mostly women . . .'.

[19] Johan van Benthem suggests that we characterize independent quantifiers as 'scope free', where '$Q1$ $A$'s stand in the relation $R$ to $Q2$ $B$'s' is scope-free iff it satisfies (i) invariance under passive transformations: '$Q1$ $A$'s stand in the relation $R$ to $Q2$ $B$'s' is logically equivalent to '$Q2$ $B$'s stand in the relation $\breve{R}$ to $Q1$ $A$'s', where $\breve{R}$ is the converse of $R$, and (ii) domain/range invariance: if $S$ is a relation such that DOM(R) = DOM(S) and RAN(R) = RAN(S), then '$Q1$ $A$'s stand in the relation $R$ to $Q2$ $B$'s' is logically equivalent to '$Q1$ $A$'s stand in the

of semantic representation itself forces upon us one interpretation over the other. For example, in standard semantics relations are represented in such a way that it is impossible for the range of a given binary relation to be empty while its domain is not empty. Thus a quantification of the form 'Three $A$'s stand in the relation $R$ to zero $B$'s' would be logically false if interpreted as independent branching quantification. To render it logically contingent we may reduce it to a series of quantifications over 1-*place predicates*, and this gives us the linear reading.

We now turn to *complex quantification*. Evidently, Henkin's quantifiers belong in this category. We ask: What kind of information on a quantified relation does a complex quantifier-prefix give us? To create a more general context for our investigation we will, following Barwise, discuss the issue within the framework of generalized logic.

## 5. Barwise's Generalization of Henkin's Quantifiers

Barwise (1979) generalized Henkin's definition of standard branching quantifiers to 1-place monotone-increasing Mostowskian quantifiers[20] in the following way:

(5.1) $$\begin{matrix}(Q1x)\\(Q2y)\end{matrix}\!\!>\!\Phi xy =_{\mathrm{Df}} (\exists X)(\exists Y)[(Q1x)Xx \,\&\, (Q2y)Yy \,\&\, (\forall x)(\forall y)(Xx \,\&\, Yy \rightarrow \Phi xy)].$$ [21]

Technically, the generalization is based on a relational reading of the Skolem functions in Henkin's definition. Thus, the Barwise equivalent of Henkin's (1.7) is:

(5.2) $$(\exists R)(\exists S)[(\forall x)(\exists y)Rxy \,\&\, (\forall z)(\exists w)Szw \,\&\, (\forall x)(\forall y)(\forall z)(\forall w)(Rxy \,\&\, Szw \rightarrow \Phi(x, y, z, w)].$$

Clearly, Barwise's quantifiers are, like Henkin's, *complex*, not independent branching quantifiers.

Barwise suggested that this generalization enables us to give English sentences with unnested monotone-increasing generalized quantifiers a

relation $S$ to $Q2$ $B$'s'. (This definition applies to independent quantifiers of types (a)–(c).) van Benthem suggests that if we have evidence that a natural language sentence of the above form satisfies the two invariance conditions (i) and (ii), its logical form is that of independent quantification.

[20] A quantifier $Q$ is monotone-increasing iff $(Qx)\Phi x$ and $(\forall x)(\Phi x \rightarrow \Psi x)$ imply $(Qx)\Psi x$.

[21] Barwise (1979), p. 63.

'Henkinian' interpretation similar to Hintikka's interpretation of (2.1) and (2.2). Here are two of his examples:[22]

(5.3) Most philosophers and most linguists agree with each other about branching quantification.

(5.4) Quite a few boys in my class and most girls in your class have all dated each other.

In order to interpret (5.3) and (5.4) based on Barwise's (5.1), we have to extend (5.1) to 2-place quantifiers: Let $Q_1^2$ and $Q_2^2$ be 2-place monotone-increasing quantifiers, then:

(5.5)
$$\begin{array}{ll} (Q_1^2 x) \cdot \ \Psi_1 x & \\ \qquad\quad \cdot & \Phi xy =_{\mathrm{Df}} \\ (Q_2^2 y) \cdot \ \Psi_2 y & \end{array}$$

$$(\exists X)(\exists Y)\{(Q_1^2 x)[\Psi_1 x, Xx] \,\&\, (Q_2^2 y)[\Psi_2 y, Yy] \\ \& (\forall x)(\forall y)[Xx \,\&\, Yy \rightarrow \Phi xy]\}.$$

We can now interpret (5.3) by:

(5.6)
$$\begin{array}{ll} (M^2 x) \cdot \ Px, & \\ \qquad\quad \cdot & Axy \,\&\, Ayx =_{\mathrm{Df}} \\ (M^2 y) \cdot \ Ly, & \end{array}$$

$$(\exists X)(\exists Y)\{(M^2 x)[Px, Xx] \,\&\, (M^2 y)[Ly, Yy] \\ \& (\forall x)(\forall y)[Xx \,\&\, Yy \rightarrow Axy \,\&\, Ayx]\}$$

with the obvious readings of $P$, $L$, $A$, and where $M^2$ stands for the 2-place 'most'. We interpret (5.4) in a similar manner.

Barwise emphasized that his definition of branching monotone-increasing generalized quantifiers is not applicable to monotone-decreasing, non-monotone[23] and mixed branching quantifiers. This is easily explained by the absurd results of applying (5.1) to such quantifiers: (5.1) would render any monotone-decreasing branching formula vacuously true (by taking $X$ and $Y$ to be the empty set); it would render false non-monotone branching formulas true, as in the case of 'Exactly one $x$ and exactly one $y$ stand in the relation $R$', where $R$ is universal and the cardinality of the universe is larger than 1.

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[22] Barwise (1979), p. 60, (21) and (22).

[23] $Q$ is monotone-decreasing iff $(Qx)\Phi x$ and $(\forall x)(\Psi x \rightarrow \Phi x)$ imply $(Qx)\Psi x$; $Q$ is non-monotone iff it is neither monotone-increasing nor monotone-decreasing.

Barwise proposed the following definition for a pair of 1-place monotone-decreasing branching quantifiers:

(5.7) $$\begin{matrix}(Q1x)\\(Q2y)\end{matrix}\!\!>\!\Phi xy =_{\mathrm{Df}} (\exists X)(\exists Y)\{(Q1x)Xx \,\&\, (Q2y)Yy \,\&\, (\forall x)(\forall y)[\Phi xy \rightarrow Xx \,\&\, Yy]\}.$$[24]

(5.7) provides an intuitively correct semantics for English sentences with a pair of unnested monotone-decreasing quantifiers. Consider, for instance:

(5.8) Few philosophers and few linguists agree with each other about branching quantification.

As to non-monotone and mixed branching quantifiers, Barwise left the former unattended, remarking skeptically about the latter: "there is no sensible way to interpret

(s) $$\begin{matrix}Q1x\\Q2y\end{matrix}\!\!>\!A(x, y)$$

when one [quantifier] is increasing and the other is decreasing. Thus, for example,

(29)? Few of the boys in my class and most of the girls in your class have all dated each other.

appears grammatical, but it makes no sense."[25]

Barwise's work suggests that the semantics of branching quantifiers depends on the monotonicity properties of the quantifiers involved. The truth conditions for a sentence with branching monotone-increasing quantifiers are altogether different from the truth conditions for a sentence with branching monotone-decreasing quantifiers, and truth for sentences with mixed branching quantifiers is simply undefinable. Is the meaning of branching quantification as intimately connected with monotonicity as Barwise's analysis may lead one to conclude?

First I would like to observe that Barwise interprets branching monotone-decreasing quantifiers simply as *independent* quantifiers: when $Q1$ and $Q2$ are monotone-decreasing (5.7) is logically equivalent to our (4.1). The latter definition, as we have seen, has meaning – the same meaning – for all quantifiers irrespective of monotonicity. On this 1st-order reading

[24] Barwise (1979), p. 64.

[25] Barwise (1979), pp. 65–66.

(5.8) says that the relation of mutual agreement about branching quantification between philosophers and linguists includes (at most) few philosophers in its domain and (at most) few linguists in its range.

Barwise explained the limited applicability of (5.1) in the following way: Every fomula of the form

(5.9) $(Qx)\Phi x$,

where $Q$ is monotone-increasing, is logically equivalent to a 2nd-order formula of the form

(5.10) $(\exists X)[(Qx)Xx \,\&\, (\forall x)(Xx \rightarrow \Phi x)]$,

which is structurally similar to (5.1). This fact establishes (by analogy?) (5.1) as the correct definition of branching monotone-increasing quantifiers. However, (5.10) is not a 2nd-order representation of quantified formulas with non monotone-increasing quantifiers. Hence, (5.1) does not apply to branching quantifiers of the latter kind. The definition of branching monotone-decreasing quantifiers by (5.7) is explained in a similar manner: When $Q$ is monotone-decreasing, (5.9) is logically equivalent to

(5.11) $(\exists X)[(Qx)Xx \,\&\, (\forall x)(\Phi x \rightarrow Xx)]$,

which is structurally similar to (5.7).[26]

I do not find this explanation convincing. Linear quantifiers vary with respect to monotonicity as much as branching quantifiers do, yet the semantic definition of linear quantifiers is the same for *all* quantifiers, irrespective of monotonicity. Linear quantification is also meaningful for all combinations of quantifiers (including mixed-monotone). Why shouldn't branching quantification be the same? Moreover, if the 2nd-order representation of 'simple' 1st-order quantifications is significant for the analysis of branching quantifications, Barwise has not shown that there is no 2nd-order representation of (5.9) which applies *universally*, without regard to monotonicity.

## 6. A GENERAL DEFINITION OF COMPLEX, HENKIN–BARWISE BRANCHING QUANTIFIERS

The Henkin–Barwise conception of complex branching quantification embedded in (5.1) sets the following truth conditions for branching formulas

[26] Barwise (1979), pp. 62–64.

of the form

(6.1) $\begin{matrix}(Q1x)\\(Q2y)\end{matrix}\!\!>\Phi xy,$

(where Q1 and $Q2$ are monotone-increasing): (6.1) is true in a model $\mathcal{A}$ with a universe $A$ iff:

(6.A) THERE IS AT LEAST ONE PAIR, $\langle X, Y\rangle$, OF SUBSETS OF $A$ FOR WHICH (1)–(3) BELOW HOLD:
(1) $X$ SATISFIES THE QUANTIFIER-CONDITION $Q1$;
(2) $Y$ SATISFIES THE QUANTIFIER-CONDITION $Q2$;
(3) *EACH* ELEMENT OF $X$ STANDS IN THE RELATION $\Phi^{\mathcal{A}}$ TO *ALL* THE ELEMENTS OF $Y$.

The condition expressed by (3) we shall call the *each-all* (Or *all-all*) *condition on* $\langle X, Y\rangle$ *with respect to* $\Phi^{\mathcal{A}}$. We will then be able to express (6.A) more succinctly as:

(6.B) THERE IS AT LEAST ONE PAIR OF SUBSETS OF THE UNIVERSE SATISFYING THE EACH-ALL CONDITION WITH RESPECT TO $\Phi^{\mathcal{A}}$ WITH ITS FIRST ELEMENT SATISFYING $Q1$ AND ITS SECOND ELEMENT SATISFYING $Q2$.

Set theoretically, (6.B) says that $\Phi^{\mathcal{A}}$ includes at least one *Cartesian Product* of two subsets of the universe satisfying $Q1$ and $Q2$ respectively. (The 'massive nucleus' of Section 2 above was an informal term for a Cartesian Product.)

Is the complex quantifier-condition expressed by (6.B) meaningful only with respect to monotone-increasing quantifiers? I think that what (6.B) says makes sense no matter what quantifiers $Q1$ and $Q2$ are. However, (6.B), as it stands, is not general enough. It fails to capture the intended condition when $Q1$ and/or $Q2$ are not monotone-increasing. In that case $Q1$ and/or $Q2$ set a limit on the size of the sets $X$ and/or $Y$ such that $\langle X, Y\rangle$ satisfies the each-all condition with respect to $\Phi^{\mathcal{A}}$: (6.1) is true only if there is a *small* enough Cartesian Product included in $\Phi^{\mathcal{A}}$. But whenever there are two subsets of the universe, $X$ and $Y$, whose cardinalities *exceed* the limits set by $Q1$ and $Q2$ but whose Cartesian Product is included in $\Phi^{\mathcal{A}}$, (6.B) is automatically satisfied. (This is because for two non-empty sets, $A$ and $B$, if $A \times B$ is a Cartesian Product included in $\Phi^{\mathcal{A}}$ so is $A' \times B'$, where $A'$ and $B'$ are any proper subsets of $A$ and $B$ respectively.) The difficulty, however, appears to be technical. What we need is an additional condition on the Cartesian Product in question: Only *maximal* Cartesian Products should count.

We therefore add a maximality condition to (6.A), arriving at the following semantic definition for complex branching quantifications of the form (6.1), where no restriction is set on $Q1$ and $Q2$:

(6.C) THERE IS AT LEAST ONE PAIR, $\langle X, Y\rangle$, OF SUBSETS OF $A$ FOR WHICH (1)–(4) BELOW HOLD:
(1) $X$ SATISFIES THE QUANTIFIER-CONDITION $Q1$;
(2) $Y$ SATISFIES THE QUANTIFIER-CONDITION $Q2$;
(3) *EACH* ELEMENT OF $X$ STANDS IN THE RELATION $\Phi^{\mathscr{A}}$ TO *ALL* THE ELEMENTS OF $Y$.
(4) THE PAIR $\langle X, Y\rangle$ IS A MAXIMAL PAIR SATISFYING (3).

(6.C) is formally correct. (I.e., given a model $\mathscr{A}$ with a universe $A$, a binary relation $\Phi^{\mathscr{A}}$ and two subsets, $B$ and $C$, of $A$ s.t. $B \times C \subseteq \Phi^{\mathscr{A}}$, there are subsets $B'$ and $C'$ of $A$ s.t. $B \subseteq B'$, $C \subseteq C'$ and $B' \times C'$ is a maximal Cartesian Product included in $\Phi^{\mathscr{A}}$.)

Referring to (3)–(4) as 'the *maximal each-all* condition on $\langle X, Y\rangle$ with respect to $\Phi^{\mathscr{A}}$', we reformulate (6.C) more concisely as:

(6.D) THERE IS AT LEAST ONE PAIR OF SUBSETS IN THE UNIVERSE SATISFYING THE *MAXIMAL* EACH-ALL CONDITION WITH RESPECT TO $\Phi^{\mathscr{A}}$ SUCH THAT ITS FIRST ELEMENT SATISFIES $Q1$ AND ITS SECOND ELEMENT SATISFIES $Q2$.

We thus propose to replace (5.1) by

(6.2)
$$\begin{matrix}(Q1x)\\(Q2y)\end{matrix}\!\!>\Phi xy =_{\mathrm{Df}}$$
$$(\exists X)(\exists Y)\{(Q1x)Xx \,\&\, (Q2y)Yy \,\&\, (\forall x)(\forall y)[Xx \,\&\, Yy \rightarrow \Phi xy] \,\&$$
$$(\forall X')(\forall Y')[(\forall x)(\forall y)(Xx \,\&\, Yy \rightarrow X'x \,\&\, Y'y)]$$
$$\&\, (\forall x)(\forall y)(X'x \,\&\, Y'y \rightarrow \Phi xy) \rightarrow$$
$$(\forall x)(\forall y)(Xx \,\&\, Yy \leftrightarrow X'x \,\&\, Y'y)]\}$$

as the generalized definition of Henkin–Barwise complex branching quantifiers.

We can rewrite (6.2) more shortly as:

(6.3)
$$\begin{matrix}(Q1x)\\(Q2y)\end{matrix}\!\!>\Phi xy =_{\mathrm{Df}}$$
$$(\exists X)(\exists Y)\{(Q1x)Xx \,\&\, (Q2y)Yy \,\&\, X \times Y \subseteq \Phi \,\&$$
$$(\forall X')(\forall Y')[X \times Y \subseteq X' \times Y' \subseteq \Phi \rightarrow X \times Y = X' \times Y']\}.$$

More concisely yet, we have:

$$(6.4)\quad \begin{matrix}(Q1x)\\(Q2y)\end{matrix}\!\!>\Phi xy =_{\text{Df}}$$

$$(\exists X)(\exists Y)\{(Q1x)Xx\&\ (Q2y)Yy\&\ (\forall X')(\forall Y')[X\times Y\subseteq X'\times Y'\subseteq\Phi\leftrightarrow X\times Y=X'\times Y']\}.$$

It is easy to see that whenever $Q1$ and $Q2$ are monotone-increasing, (6.3) is logically equivalent to (5.1). At the same time, (6.3) avoids the problems which arise when (5.1) is applied to non-monotone-increasing quantifiers.

Is the maximality condition ad-hoc? I think it is not. The Henkin–Barwise branching quantifier-prefix expresses a certain condition on a subset of the relation quantified. It seems to me that when we talk about sets it is generally maximal sets that we are interested in. Indeed, any condition on a set is, unless otherwise specified, a condition on a maximal set: Consider, for instance, the statement 'Three students passed the test'. Would this statement be true had 10 students passed the test? It would be if the quantifier '∃!3' set a condition on a non-maximal set: a partial extension of '$x$ is a student who passed the test' would satisfy that condition. Consider also 'No student passed the test' and 'Two people live in America'.

The fact that quantification in general sets a condition on maximal sets (relations) is reflected by the equivalence of any 1st-order formula of the form

$$(5.9)\quad (Qx)\Phi x\ -$$

no matter what quantifier $Q$ is (monotone-increasing, monotone-decreasing or non-monotone) – to

$$(6.5)\quad (\exists X)\{(Qx)Xx\ \&\ (\forall x)(Xx\to\Phi x)\ \&\ (\forall X')[X\subseteq X'\ \&\ (\forall x)(X'x\to\Phi x)\to X'=X]\},$$

which expresses a maximality condition. (Thus, if we accept arguments by analogy, the logical equivalence of (5.9) to (6.5) provides a further justification for the reformulation of (5.1) as (6.3).)[27]

We have seen that the two conceptions of non-linear quantification

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[27] Maximality conditions are very common in mathematics. Generally, when a structure is maximal it is 'complete' in some relevant sense. Thus, the structure of a maximal consistent set of formulas gives us enough information to construct a syntactic model as in Henkin's proof of the completeness of standard 1st-order logic. (I wish to thank Charles Parsons for this example.) In set theory, Zorn's lemma expresses a sufficient condition for the existence of a maximal set, and the numerous uses of this powerful lemma present abundant evidence for the importance of maximality. Etc.

discussed so far – *independence* (1st-order), and *complex dependence* (2nd-order) – have little to do with monotonicity or its direction. The two conceptions lead to entirely different definitions of the branching quantifier-prefix, both, however, universally applicable.

Linguistically, our suggestion is that to determine the truth conditions of natural language sentences with a non-linear quantifier-prefix one has to ask not whether the quantifiers involved are monotone-increasing, monotone-decreasing, etc., but whether the prefix is independent or complex. Our analysis points to the following clue: Complex, Henkin–Barwise quantifications always include an inner *each-all* condition, explicit or implicit. Independent quantifications, on the other hand, do not include any such condition.

Barwise actually gave several examples of branching sentences with an explicit *each-all* condition:

(5.4) Quite a few boys in my class and most girls in your class have *all* dated each other.[28]

(6.6) Most of the dots and most of the stars are *all* connected by lines.[29]

Such an explicit 'all' also appears in his

(29) Few of the boys in my class and most of the girls in your class have *all* dated each other.[30]

I therefore suggest that we interpret (29) as an instance of (6.3).

Some natural examples of Henkin–Barwise complex branching quantifiers in English involve non-monotone quantifiers. E.g.,

(6.7) A couple of boys from my class and a couple of girls from your class were *all* dating each other.

(6.8) An even number of dots and an odd number of stars are *all* connected by lines.

Another expression which seems to point to a complex branching quantification (by indicating a 2nd-order structure) is 'the same'. Consider:

(6.9) Most of my friends have *all* applied to the *same* few graduate programs.

[28] The italicization is mine.
[29] Barwise (1979), p. 62 (23). See also (25). The italicization is mine.
[30] My italicization.

To give the above sentences a formal interpretation we have to extend (6.3) to 2-place quantifiers. As in the case of 2-place independent quantifiers we can apply the notion of complex, 'each all' quantification in more than one way. (See Section 4 above.) We will define '$Q1$ $A$'s and $Q2$ $B$'s *all* stand in the relation $R$' by: 'There is at least one maximal Cartesian Product included in $A \upharpoonleft R \upharpoonright B$ such that its domain includes $Q1$ $A$'s and its range includes $Q2$ $B$'s'. In symbols:

(6.10) $$\begin{matrix}(Q_1^2 x) \cdot \Psi_1 x, \\ \cdot \qquad \Psi xy =_{\mathrm{Df}} \\ (Q_2^2 y) \cdot \Psi_2 y,\end{matrix}$$

$$(\exists X)(\exists Y)\{(Q_1^2 x)[\Psi_1 x, Xx] \,\&\, (Q_2^2 y)[\Psi_2 y, Yy] \,\&\, (\forall X')(\forall Y')[X \times Y \subseteq X' \times Y' \subseteq \Psi_1 \upharpoonleft \Phi \upharpoonright \Psi_2 \leftrightarrow X \times Y = X' \times Y']\}.$$

Going back to the controversy regarding Hintikka's reading of natural language sentences with non-nesting quantifiers, we can reformulate Fauconnier's criticism as follows: Some natural language sentences with unnested quantifiers do not appear to contain, explicitly or implicitly, an inner *each-all* quantifier condition. On our analysis, these are not Henkin-Barwise branching quantifications. Whether Hintikka's (2.1) includes an implicit each-all condition, I leave an open question.

Remark: The analysis of branching quantifiers in this paper is intended mainly to clarify the nature of a new *logical* form. As such, this is not a linguistic analysis. It appears, however, that natural language sentences with a pair of unnested quantifiers and an *explicit* inner 'each-all' condition are naturally interpreted as Henkin–Barwise complex quantifications. (Note that the inner 'all' does not bind any new individual variables in addition to those bound by $Q1$ and $Q2$, and therefore a 'standard' reading of such sentences would be very problematic.) The question, however, arises how to treat English statements with a pair of unnested quantifiers when an explicit 'each-all' condition does not occur. Our discussion so far indicates three possible readings: as independent quantifications, as linear quantifications (the syntactic unnesting is misleading), or as Henkin–Barwise complex quantifications (the 'each-all' condition is taken to be implicit). In the next section we will propose further alternatives.

## 7. Branching Quantifiers: A Family of Interpretations

The Henkin–Barwise definition of branching quantifiers – in its original as well as generalized form – includes two quantifier conditions in addition to those explicit in the definiendum: the outer quantifier condition 'there is at least one pair $\langle X, Y\rangle$', and the inner (maximal) 'each-all' quantifier condition. By generalizing these conditions we arrive at a new definition-schema whose instances comprise a family of semantic interpretations for multiple quantifiers. Among the members of this family are both the independent branching quantifiers of Section 4 above and the Henkin–Barwise complex quantifiers of Section 6 above. This generalized definition-schema delineates a (certain) totality of *forms of quantifier-dependence*. Degenerate dependence is independence; linear dependence is a particular case of the (non-degenerate) Henkin–Barwise dependence.[31]

We arrive at our definition-schema in two steps. First we generalize the inner 'each-all' quantifier condition:

(7.A) FOR AT LEAST ONE PAIR, $\langle X, Y\rangle$, OF SUBSETS OF THE UNIVERSE SATISFYING THE MAXIMAL QUANTIFIER CONDITION $\mathcal{Q}_1$ WITH RESPECT TO $\Phi^{\mathcal{A}}$ THE FOLLOWING HOLDS: $X$ SATISFIES $Q1$ AND $Y$ SATISFIES $Q2$,

where $\mathcal{Q}_1$ represents any (1st-order) maximal quantifier-condition on a pair of subsets of the universe with respect to $\Phi^{\mathcal{A}}$. The following are a few instances of $\mathcal{Q}_1$:

(7.a) Maximal *one-one* quantifier condition: $\langle X, Y\rangle$ is a maximal pair such that each element of $X$ stands in the relation $\Phi^{\mathcal{A}}$ to exactly one element of $Y$, and for each element of $Y$ there is exactly one element of $X$ which stands to it in the relation $\Phi^{\mathcal{A}}$.

(7.b) Maximal *each – two-or-more* quantifier condition: $\langle X, Y\rangle$ is a maximal pair such that each element of $X$ stands in the relation $\Phi^{\mathcal{A}}$ to two or more elements of $Y$, and for each element of $Y$ there is an element of $X$ which stands to it in the relation $\Phi^{\mathcal{A}}$.

(7.c) Maximal *each – more-than ...* quantifier condition: $\langle X, Y\rangle$ is a maximal pair such that each element of $X$ stands in the relation $\Phi^{\mathcal{A}}$ to more than ... elements of $Y$, and for each element of $Y$ there is an element of $X$ which stands to it in the relation $\Phi^{\mathcal{A}}$.

[31] To see that linear quantification is a particular instance of the Henkin–Barwise complex quantification we have to express the conception of branching embedded in (6.3) more generally, so that it applies to any partially-ordered quantifier prefix. I will not discuss the nature of such a definition here, but in the case of '$(Q1x)(Q2y)\Phi xy$' the definition I have in mind will yield the following 2nd-order counterpart: $(\exists X)(\exists R)\{(Q1x)Xx$ & $X$ is a max. set s.t. $(\forall x)[Xx \rightarrow (Q2y)Rxy]$ & $R$ is a max. relation s.t. $(\forall x)(\forall y)[Rxy \rightarrow \Phi xy]\}$.

(7.d) Maximal *each – at-least-half/at-least-half – each* quantifier condition: $\langle X, Y\rangle$ is a maximal pair such that each element of $X$ stands in the relation $\Phi^{\mathscr{A}}$ to at least half the elements of $Y$ and to each element of $Y$ at least half the elements of $X$ stand in the relation $\Phi^{\mathscr{A}}$.

We can find natural language sentences which exemplify the instances of (7.A) obtained by substituting (7.a)–(7.d) respectively for $\mathscr{Q}_1$:

(7.1) Most of my right hand gloves and most of my left hand gloves match one to one.

(7.2) Most of my friends saw at least two of the same few Truffaut movies.

(7.3) The same few characters repeatedly appear in many of her early novels.

(7.4) Most of the boys and most of the girls in this party are such that *each* boy has chased *at least half* the girls, and *each* girl has been chased by *at least half* the boys.

The adaptation of (7.A) to 2-place quantifiers, needed in order to give (7.1)–(7.4) a formal interpretation, is analogous to (6.10).

We can verify the correctness of our interpretations by checking whether they can be put in the following canonical forms:

(7.5) Most of my right hand gloves and most of my left hand gloves are such that each of the former matches exactly one of the latter and vice versa.

(7.6) Most of my friends and few of Truffaut's movies are such that each of the former saw at least two of the latter, and each of the latter was seen by at least one of the former.

(7.7) Few of her characters and many of her early novels are such that each of the former appears in more than . . . of the latter and each of the latter includes at least one of the former.

(7.4) is already in canonical form.

By replacing $\mathscr{Q}_1$ in (7.A) with (7.e) below we get the *independent quantification* of Section 4 above.

(7.e) Maximal *each-some/some-each* quantifier condition: $\langle X, Y\rangle$ is a maximal pair such that each element of $X$ stands in the relation $\Phi^{\mathscr{A}}$ to some element of $Y$, and for each element of $Y$ there is some element of $X$ which stands to it in the relation $\Phi^{\mathscr{A}}$

Thus, both independent branching quantifiers and complex, Henkin–Barwise branching quantifiers fall under the general schema (7.A).

The second generalization abstracts from the outermost existential condition:

(7.B) FOR $\mathcal{Q}_2$ PAIRS, ⟨X, Y⟩, OF SUBSETS OF THE UNIVERSE SATISFYING THE MAXIMAL QUANTIFIER CONDITION $\mathcal{Q}_1$ WITH RESPECT TO $\Phi^{\mathcal{A}}$ THE FOLLOWING HOLDS: $X$ SATISFIES $Q1$ AND $Y$ SATISFIES $Q2$.[32]

The following sentences exemplify two instances of (7.B) obtained by substituting 'by and large' (interpreted as 'most') and 'at most few' for $\mathcal{Q}_2$ respectively ($\mathcal{Q}_1$ is the 'each-all' condition):

(7.8) *By and large*, no more than a few boys and a few girls all date one another.

(7.9) There are *at most few* cases of more than a couple Eastern delegates and more than a couple Western delegates who are all on speaking terms with one another.

The family of branching structures delineated above enlarges considerably the array of interpretations available for natural language sentences with multiple quantifiers. The task of selecting the right alternative for a given natural language quantification is easier if an explicit inner/outer quantifier-condition occurs in the sentence, but is more complicated otherwise. One could of course be assisted by 'context',[33] but linguists will be interested in formulating general guidelines based on constant regularities as well. Indeed, we may look at Barwise's claims regarding monotone-increasing and monotone-decreasing English branching quantifiers in this light: According to Barwise, in English monotone-increasing branching quantifiers regularly appear in complex quantifications of the 'each-all' type (with 'some' as the outer quantifier-condition); monotone-decreasing quantifiers regularly appear in 'each-some, some-each' quantifications, which are, as we have seen, equivalent to independent branching quantifications. Thus Barwise's claims can be expressed as conjectures in terms of the general definition schema of multiple quantification (7.B).[34]

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[32] The quantifiers over which $\mathcal{Q}_2$ ranges are higher-order Mostowskian quantifiers which 'treat' pairs as single elements.

[33] Thus, contextual considerations may lead us on some occasions to read (4.8) as a complex quantification with an inner 'one-one' quantifier condition (and an outer 'some') rather than as an independent quantification.

[34] Another conjecture expressible in terms of our general definition schema was suggested by an anonymous referee. Compare:

## 8. Conclusion

Our investigation has yielded a general definition-schema for a pair of branching, or partially-ordered, generalized quantifiers. The existing definitions, due to Barwise, constitute particular instances of this schema. The next task would be to extend the schema, or particular instances thereof (especially (6.3)), to arbitrary partially-ordered quantifier-prefixes. This work is, however, beyond the scope of the present paper.

In a recent article, 'Branching Quantifiers and Natural Language', D. Westerstahl proposed a general definition for a pair of branching quantifiers different from the ones discussed in this paper. Although Westerstahl's motivation is similar to mine: dissatisfaction with the multiplicity of partial definitions – he approaches the problem in a different way. Accepting Barwise's definitions for monotone-increasing and monotone-decreasing branching quantifiers, plus van Benthem's definition for non-monotone quantifiers of the form 'exactly *n*', Westerstahl constructs a general formula which yields the above definitions when the quantifiers plugged in have the 'right' kind of monotonicity. That is, Westerstahl is looking for an 'umbrella' under which the various partial existent definitions fall. When Westerstahl's article was published the present paper had already been written. It seemed to me better not to include a discussion of Westerstahl's work since from the point of view of the issues discussed here his approach is very similar to Barwise's. As for van Benthem's proposal for the analysis of non-monotone branching quantifiers, his definition is:

(8.1)
$$\begin{array}{ll}(\text{Exactly-}n\,x) & \cdot\ Ax, \\ & \cdot\qquad Rxy =_{\text{Df}} \\ (\text{Exactly-}m\,y) & \cdot\ By,\end{array}$$
$$(\exists X)(\exists Y)(X \subseteq A \;\&\; Y \subseteq B \;\&\; |X| = n \;\&\; |Y| = m \;\&\; R = X \times Y).$$ [35]

For 1-place quantifiers, the definition would be:

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(i) In the class, most of the boys and most of the girls *all* like each other;
(ii) In the class, most of the boys and most of the girls like each other.

Conjecture: the difference between (i) and (ii) is in the intended inner quantifier-condition. The presence of the explicit 'all' in (i) indicates that the inner quantifier-condition is 'each-all'. However, the absence of 'all' in (ii) signifies that there the inner quantifier-condition is weaker. The reviewer suggests that this condition is, however, stronger than 'each-some, some-each' (independence). 'Each-most' appears appropriate.

[35] Westerstahl (1987), p. 274.

$$\text{(8.2)}\quad \begin{matrix}(\text{Exactly-}n\ x)\\(\text{Exactly-}m\ y)\end{matrix} > Rxy =_{\mathrm{Df}} (\exists X)(\exists Y)(|X| = n \,\&\, |Y| = m \,\&\, R = X \times Y).$$

Given the equivalence of (8.2) to (8.3) below, we can express van Benthem's proposal in terms of our general definition-schema (7.B) as follows: Quantifiers of the form 'Exactly *n*' tend to occur in complex quantifications obtained from (7.B) by substituting 'each-all' for $\mathcal{Q}_1$ and the singular 'the' for $\mathcal{Q}_2$.[36]

(8.3) *The* (*only*) pair $\langle X, Y\rangle$ of subsets of the universe satisfying the *maximal each-all* condition with respect to $R$ is such that $X$ has *exactly n* elements and $Y$ has *exactly m* elements.

I would like to end with a philosophical note. Russell divided the enterprise of logic into two tasks: the discovery of universal 'templates' of truth and the discovery of new, philosophically significant, logical forms. In this context branching quantifiers offer a striking example of an altogether new way of building formulas. Speaking about the new form H. B. Enderton said: "We speak in real time, and real time progresses linearly . . . But formal languages are not spoken (at least not easily). So there is no reason to be influenced by the linearity of time into being narrow-minded about formulas. And linearity is the ultimate in narrowness".[37] Thus, according to Enderton, the passage from linear to partially-ordered quantifier-prefixes signifies real progress in our understanding of the possibilities of language. One cannot, however, avoid asking: When does a generalization of a particular linguistic structure lead to a new, more general form of *language* and when does it lead to a formal system which can no longer be considered language? Henkin, for instance, mentioned the possibility of constructing a densely-ordered quantifier-prefix. Would this be language? What about a finite collection of quantifiers positioned in a circle around a formula? Etc.

With respect to *partially-ordered* quantifiers, I see the situation as follows: Henkin demonstrated the feasibility of introducing standard branching quantification into abstract language. As for natural language, al-

[36] the singular 'the' (or 'the only') is a 2-place Mostowskian quantifier defined by: '(The $x$)($P1x$, $P2x$)' is true in $\mathcal{A}$ iff the extension of $P1x$ in $\mathcal{A}$ is a unit set and is included in the extension of $P2x$ in $\mathcal{A}$. In (8.3) we have the 2nd-order version of 'the', where $\langle X, Y\rangle$ is taken as a single element.

[37] Enderton (1970), p. 393.

though various natural language constructions do appear to exemplify branching structures, we have to admit that these are particularly simple instances of the general form and as such could probably be construed as exemplifying other logical principles. But quite apart from this, the question arises whether, in principle, we can introduce new forms of quantification into natural language. It is common to say that present day languages do not fully use their resources, at least as far as lexicon and grammatical complexity go. The case of branching quantifiers makes one wonder whether logical form is another unexhausted resource.

Other general questions regarding branching quantifiers have been raised by Barwise. Do branching quantifiers commit us to a 2nd-order ontology? Do they show that language is not compositional in Frege's sense? It appears that before we can determine whether branching quantifiers are genuinely logical operators, we have to engage in a critical examination of our concept of language. Indeed, branching quantifiers offer an interesting starting point for such an investigation.

## References

→ Barwise, J.: 1979, 'On Branching Quantifiers in English', *Journal of Philosophical Logic* **8**, 47–80.

→ Barwise, J. and R. Cooper: 1981, 'Generalized Quantifiers and Natural Language', *Linguistics and Philosophy* **4**, 159–219.

van Benthem, J.: 1986, *Essays in Logical Semantics*, D. Reidel, Dordrecht.

van Benthem, J.: 1989, 'Polyadic Quantifiers', *Linguistics and Philosophy* **12**, 437–464.

→ Boolos, G.: 1984, 'To Be Is to Be a Value of a Variable (or to Be Some Values of Some Variables)', *Journal of Philosophy* **81**, 430–449.

Dummett, M.: 1973, *Frege: Philosophy of Language*, Harper & Row, New York.

Enderton, H. B.: 1970, 'Finite Partially-Ordered Quantifiers', *Z. f. math. Logik und Grundlagen d. Math.* **16**, 393–397.

Enderton, H. B.: 1972, *A Mathematical Introduction to Logic*, Academic Press, New York.

Fauconnier, G.: 1975, 'Do Quantifiers Branch?', *Linguistic Inquiry* **6**, 555–567.

Frege, G.: 1884, *The Foundations of Arithmetic*, tr. by J. L. Austin, 2nd revised ed., Northwestern University Press, Evanston, 1968.

Henkin, L.: 1959, 'Some Remarks on Infinitely Long Formulas', *Infinitistic Methods, Proceedings of the Symposium on Foundations of Mathematics*, Panstwowe Wydawnictwo Naukowe and Pergamon Press, Warsaw, 1961, pp. 167–183.

Higginbotham, J. and R. May: 1981, 'Questions, Quantifiers and Crossing', *The Linguistic Review* **1**, 41–79.

Hintikka, J.: 1973, 'Quantifiers vs. Quantification Theory', *Dialectica* **27**, 329–358; *Linguistic Inquiry* **5**, 1974, 153–177.

Hintikka, J.: 1976, 'Partially Ordered Quantifiers vs. Partially Ordered Ideas', *Dialectica* **30**, 89–99.

Lindenbaum, A. and A. Tarski: 1935, 'On the Limitations of the Means of Expression of Deductive Theories', in A. Tarski, *Logic, Semantics, Metamathematics*, 2nd ed., Hackett, 1983.

Lindstrom, P.: 1966, 'First Order Predicate Logic with Generalized Quantifiers', *Theoria* **32**, 186–195.

May, R.: 1985, *Logical Form: Its Structure and Derivation*, MIT Press, Cambridge, Ma.

May, R.: 1989, 'Interpreting Logical Form', *Linguistics and Philosophy* **12**, 387–435.

Mostowski, A.: 1957, 'On a Generalization of Quantifiers', *Fundamenta Mathematicae* **44**, 12–36.

Quine, W. V. O.: 1970, *Philosophy of Logic*, Prentice-Hall, Englewood Cliffs, N.J.

Sher, G.: 1989, 'A Conception of Tarskian Logic', *Pacific Philosophical Quarterly* **70**, 341–368.

Sher, G.: 1989, 'Generalized Logic: A Philosophical Perspective with Linguistic Applications', Ph.D. thesis, Columbia University.

Stenius, E.: 1976, 'Comments on Jaakko Hintikka's paper "Quantifiers vs. Quantification Theory"', *Dialectica* **30**, 67–88.

Tarski, A.: 1956, *Logic*, *Semantics*, *Metamathematics*, 2nd. ed., 1983, edited and introduced by J. Corcoran, Hackett.

Tarski, A.: 1966, 'What are Logical Notions?', edited and introduced by J. Corcoran, *History and Philosophy of Logic* **7**, 1986, 143–154.

Walkoe, W. J., Jr.: 1970, 'Finite Partially-Ordered Quantification', *Journal of Symbolic Logic* **35**, 535–555.

Westerstahl, D.: 1987, 'Branching Generalized Quantifiers and Natural Language', in *Generalized Quantifiers: Linguistic and Logical Approaches*, D. Reidel, Dordrecht.

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