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NAMING AN INDISCERNIBLE SEQUENCE IN NIP THEORIES

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Abstract. In this short note we show that if we add predicate for a dense complete indiscernible sequence in a dependent theory then the result is still dependent. This answers a question of Baldwin and Benedikt and implies that every unstable dependent theory has a dependent expansion interpreting linear order.

Introduction

Let \( T \) be an NIP theory in a language \( L \). Consider a model \( M \) and a small indiscernible sequence \( I \) indexed by a dense complete linear order (small means that \( M \) is \(|I|^+\)-saturated). We consider the language \( L_P \) with a unary predicate \( P \) added for the sequence \( I \), and let \( T_P = Th(M,I) \).

Definition 1. We say that an \( L_P \)-formula is bounded if it is of the form
\[(Q_1x_1 \in P)(Q_2x_2 \in P)...(Q_nx_n \in P)\phi(x_1,...,x_n,\bar{y}),\]
where \( \phi \) is an \( L \)-formula and each \( Q_i \) is either \( \exists \) or \( \forall \).

In [BB00] Baldwin and Benedikt prove the following.

Theorem 2. Assume \( T \) is NIP.
1) For each dense complete indiscernible sequence \( I \) and formula \( \phi(\bar{a},y) \), there is some \( c \in I \) such that for every \( b \in I \), the truth value of \( \phi(\bar{a},b) \) is totally determined by the quantifier-free order type of \( b \) over \( \bar{c} \).
2) Every formula in \( T_P \) is equivalent to a bounded one.
3) \( Th(M,I) \equiv Th(M,J) \) if and only if EM(\( I \)) = EM(\( J \)).
4) \( P \) is stably embedded and the \( L_P \)-induced structure (traces on \( P \) of all \( L_P \)-definable sets with parameters from \( M \)) is that of a pure linear order.

Remark 3. Point 1) is the Theorem 5.2 there, but for a simplified proof see [Adl08], Section 3. 2) is Theorem 3.3, 3) is Theorem 8.1, 4) is Corollary 3.6.

They prove that if \( T \) is stable then \( T_P \) is stable as well, and ask whether \( T_P \) is always dependent when \( T \) was. In the next section we answer this question positively. Throughout the paper we assume Martin’s Axiom (MA).

Dependence of \( T_P \)

First a trivial combinatorial observation.

Lemma 4. Let \( h_k : \omega \to I, k \leq m \) be monotone functions, \( h_0(n) < ... < h_m(n) \) for all \( n \) and let \( a_1,...,a_n \in I \). Then for some \( p \leq 2mn + 1 \) both \( (h_k(p))_{k \leq m} \) and \( (h_k(p+1))_{k \leq m} \) have the same order type over \( a_1,...,a_n \).
Corollary 6. Let \((a_i)_{i \in \omega} \in \mathbb{M}, (b_i)_{i \in \omega} \in \mathbb{P}\) be given and \(d \in \mathbb{M}\). Then there is some sequence \((a'_i b'_i)_{i \in \omega}\) \(L_P\)-indiscernible over \(d\) and such that for every \(\psi \in L_P\)

Proof. Suppose not. Then for each \(p \leq 2mn + 1\) there is some \(i \leq n, j \leq m\) with \(h_i(p) < a_j, h_i(p+1) \geq a_j\) or \(h_i(p) > a_j, h_i(p+1) \leq a_j\) or \(h_i(p) = a_j, h_i(p+1) \neq a_j\), and by monotonicity for every pair of \(i, j\) there can be only up to two such \(p\) - a contradiction. □

Next a crucial technical lemma.

Proposition 5. 1) Assume \(P\) is ordered by some \(L_P\)-definable \("<\"\). Let \(I = (b_i)_{i < \omega}\) be an \(L_P\)-indiscernible sequence and \(E \subset \omega\) the set of even numbers. Assume that \(f : E \rightarrow P(\mathbb{M})\), \(n < \omega\) even and \(\phi(x_1, ..., x_n; y_1, ..., y_n) \in L_P\) such that for any sequence \(k_1 < k_2 < ... < k_n \in E\) we have

\[\models \phi(b_{k_1}, ..., b_{k_n}; f(k_1), ..., f(k_n)).\]

Then there is \(g : \omega \rightarrow P(\mathbb{M})\) extending \(f\) and \(l_1, ..., l_n \in \omega\) with \(l_i \equiv i \pmod{2}\) and

\[\models \phi(b_{l_1}, ..., b_{l_n}; g(l_1), ..., g(l_n)).\]

2) Same claim but assuming that the \(L_P\)-induced structure on \(P\) is just the equality.

Proof. 1) Since by Theorem 2 the \(L_P\)-induced structure on \(P\) is just that of linear order by compactness there is some \(m < \omega\) such that given any \((a_1, ..., a_n) \in \mathbb{M}\) there are some \((c_1 < ... < c_m) \in P(\mathbb{M})\) such that for any \((d_1, ..., d_n) \in P(\mathbb{M})\) the truth value of \(\phi(a_1, ..., a_n; d_1, ..., d_n)\) is totally determined by the order type of \(d\) over \(\bar{c}\).

Now for each \(k \leq m\) let \(h_k : \mathbb{M}^n \rightarrow \mathbb{P}\) be the \(L_P\)-definable function sending \((a_1, ..., a_n)\) to the corresponding \(c_k\) (W.l.o.g. we assume there is a constant \(\rho \in P\). If for some \(\bar{a}\) there are \(k' < k\) alternations we let \(h_j(\bar{a}) = \rho\) for \(j > k'\).

We have :

\((*)\) for any \((a_1, ..., a_n) \in \mathbb{M}, (d_1, ..., d_n), (d_1', ..., d_n') \in P,\)

\(d\) and \(d'\) have the same order type over \((h_k(d))_{< m} \implies \phi(\bar{d}; d) = \phi(\bar{d}; d')\)

From this by \(L_P\)-indiscernibility of \(I\) :

\((**):\) for any \(b_1, ..., b_n\) and \(b'_1, ..., b'_n\) increasing sequences from \(I\) and \(\bar{d}, \bar{d}'\) in \(P,\)

\(d\) has the same order type over \((h_k(b))_{< m}\) as \(d'\) over \((h_k(b'))_{< m} \implies \phi(\bar{d}; d) = \phi(\bar{d}; d').\)

Choose some \((0 < l_2 < ... < l_{n-2} < l_n) \in E^{n/2}\) with \(l_{2(i+1)} - l_{2i} > 2mn + 1\).

Define \(h' : \omega \rightarrow P(\mathbb{M})\) by \(h'_k(p) = h_k(b_{p_0}, b_{l_2}, b_{l_2+p_0}, ..., b_{l_n-2+p_0}, b_{l_n}).\)

By \(L_P\)-indiscernibility of \(I\) the \(h'_k\)'s are monotonic (at least in the interval \([1, 2mn+1]\) which is all that matters). Thus by Lemma 4 we find some \(g(p_0), g(l_2 + p_0), ..., g(l_{n-2} + p_0) \in P(\mathbb{M})\) such that \(g(p_0), f(l_2), ..., g(l_{n-2} + p_0), f(l_n)\) has the same order-type over \((h'_k(p_0))_{< m}\) as \(f(p_0+1), f(l_2), ..., f(l_{n-2} + p_0+1), f(l_n)\) over \((h'_k(p_0 + 1))_{< m}\), and so by \((**):\)

\[\models \phi(b_{p_0}, b_{l_2}, ..., b_{l_n-2+p_0}, b_{l_n}; g(p_0), f(l_2), ..., g(l_{n-2}+p_0), f(l_n))\]

and we are done. □

2) Analogously.

This gives us a Ramsey-like result on completing indiscernible sequences of triangles.

Corollary 6. Let \((a_i)_{i \in \omega} \in \mathbb{M}, (b_i)_{i \in \omega} \in \mathbb{P}\) be given and \(d \in \mathbb{M}\). Then there is some sequence \((a'_i b'_i)_{i \in \omega}\) \(L_P\)-indiscernible over \(d\) and such that for every \(\psi \in L_P\)
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(+) $\psi((a'_{2i} a'_{2i+1} b'_{2i})_{i<n}, d) \Rightarrow \psi((a_{2k_i} a_{2k_i+1} b_{2k_i})_{i<n}, d)$ for some $k_0 < k_1 < \ldots < k_{n-1} \in \omega$.

Proof. First by Ramsey find an $L_P$-indiscernible sequence $(a''_i, b''_i)_{i<\omega}$ with property (+). Now let $I = (a''_i)_{i<\omega}$ and $f(a''_{2i}) = b''_{2i}$ and use Proposition with compactness to conclude. \[\square\]

Finally we are ready to prove our main result.

Theorem 7. $T_P$ is dependent.

Proof. First note that by Theorem the $L_P$-induced structure on $P$ is equality or it is ordered by some $L$-formula (with parameters).

We prove by induction on the number of bounded quantifiers that all $L_P$-formulas are dependent, and since the set of formulas with NIP is closed under boolean combinations it is enough to consider adding single existential bounded quantifier to a dependent formula.

So assume $\phi(x; y) = (\exists z \in P)\psi(x, y, z)$ has IP where $\psi(x, y, z)$ is an $L_P$-formula. Then there is some $L_P$-indiscernible sequence $(a_i)_{i<\omega}$ and $d$ such that $\phi(d, a_i)$ holds if and only if $i$ is even, and so for $i = 2k$ let $b_i \in P$ be such that $\psi(d, a_i, b_i)$ holds. By Lemma we find some sequence $(a'_i, b'_i)_{i<\omega}$ which is $L_P$-indiscernible and (using (+)) still $\psi(d, a'_{2i}, b'_{2i})$ and $\neg\psi(d, a'_{2i+1}, b'_{2i+1})$ hold. But this means that $\psi(d; y, z)$ has infinite alternation - contradicting the inductive assumption. \[\square\]

Question 8. Assuming $T$ is strongly-dependent, is $T_P$ strongly-dependent?

Remark 9. Note however that unsurprisingly $dp$-minimality is not preserved in general after naming an indiscernible sequence. By [Goo09], Lemma 3.3, in an ordered $dp$-minimal group, there is no infinite definable nowhere-dense subset, but of course every small indiscernible sequence is like this.

Corollary 10. Every unstable dependent theory has a dependent expansion interpreting an infinite linear order.

Proof. Just take a small indiscernible sequence that is not an indiscernible set, mark it by a predicate and use Theorem \[\square\]

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References

