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L. Jackson Laslett

December 20, 1965
AN EQUIVALENT DISTRIBUTION OF SURFACE CURRENTS: FOR THE GENERATION
OF A PRESCRIBED STATIC MAGNETIC FIELD WITHIN THE ENCLOSED VOLUME*

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ABSTRACT

It is shown how a specified static magnetic field within a given
volume may be generated through use of a current distribution on the
surface surrounding this volume. The surface distribution includes a
distribution of magnetic moment (oriented tangential to the surface)
that may be interpreted as a double current layer, but no magnetic
poles are introduced. No sources are required external to the surface,
and the exterior field will be zero.
It is known\textsuperscript{1,2} that the external sources of an electrostatic field that is specified within a closed surface \( S \) can be replaced by an equivalent distribution of charge and electric dipole moment on \( S \) (the so-called Green's equivalent stratum). With this replacement, the electric field external to \( S \) vanishes. It appears to be less well known whether an analogous distribution of surface currents may be found to replace the external sources of a static magnetic field, without employing magnetic poles or surface distributions for which no physical interpretation is evident.\textsuperscript{3} It is shown here that there exists a current distribution on \( S \) that will produce the same magnetic field interior to \( S \) as is produced by the external sources (whether the external sources are formed by a current system or are imagined to be magnetic poles), and an explicit prescription is given for determining this current distribution.

The magnetic field, \( \vec{H} \), is considered as given within a closed surface \( S \), together with an associated vector potential, \( \vec{A} \), so that
\[
\nabla \times \vec{A} = \vec{H} \quad \text{and} \quad \nabla \cdot \vec{A} = 0 .
\]
Within \( S \), \( \nabla \times \vec{H} = 0 \) and \( \nabla \cdot \vec{H} = \frac{4\pi}{c} \vec{J} \), with \( \vec{J} = 0 \) if no currents exist in the volume \( V \) interior to \( S \).

One may construct a new vector potential,
\[
\vec{A}' = \vec{A} + \nabla \phi , \tag{1}
\]
where \( \phi \) is specifically selected so that, at all points on \( S \),
\[
(\nabla \phi) \cdot \hat{n} = -A \cdot \hat{n} , \tag{2}
\]
\( \hat{n} \) being the outward-directed unit vector normal to the surface \( S \).
[Such a $\Phi$ may be assumed to exist. In particular, with the stipulation that $\nabla \cdot A = 0$ (so that $\int_S A \cdot \hat{n} \, dS = \int_V \nabla \cdot A \, dv = 0$), $\Phi$ may be taken as a solution of the Neumann problem in which $\nabla^2 \Phi = 0$ and boundary conditions of the second kind apply.] Accordingly, $\nabla \times A' = H$, and $A' \cdot \hat{n} = 0$ on $S$.

One now defines the vector

$$A'' = \iiint_V \frac{J}{r} \, dv + \frac{1}{4\pi} \iiint_S \left\{ \frac{H \times \hat{n}}{r} + \frac{r \times \left[ A' \times \hat{n} \right]}{r^3} \right\} \, dS \quad (3a)$$

$$= \iiint_V \frac{J}{r} \, dv + \frac{1}{4\pi} \iiint_S \left\{ \frac{H \times \hat{n}}{r} + \frac{r \times \left[ A' \times \hat{n} \right]}{r^3} + \frac{r}{r^3} (A' \cdot \hat{n}) \right\} \, dS \quad (3b)$$

that will be shown can serve as a suitable vector potential to give the field $H$ within $S$. The vector $\hat{r}$ is to be understood as extending from the field point $P$ to the surface element $dS$ (Fig. 1), and the two forms given for Eq. (3) are equivalent because the factor $A' \cdot \hat{n}$ that appears in the last term of Eq. (3b) is identically zero on all points of $S$.

The three surface integrals in Eq. (3b) can be transformed to volume integrals as follows:

$$\int_S (H/r) \times \hat{n} \, dS = \iiint_V \left( -[\nabla \times H]/r + [r \times H]/r^3 \right) \, dv$$

$$= -4\pi \iiint_V (\hat{r}/r) \, dv + \iiint_V \left[ (r/r^3) \times [\nabla \times A'] \right] \, dv \; \quad (4a)$$
\[ \iint \left[ \left( \frac{r}{r^3} \right) \times \left( \frac{r^3}{r} \times \hat{n} \right) \right] dS = \iiint_V \left[ A' \cdot \nabla \left( \frac{1}{r^3} \right) + \left( \frac{1}{r^3} \right) \cdot \nabla A' - \nabla \left( \left( \frac{1}{r^3} \right) \cdot A' \right) \right] dv \]

\[ = \frac{4\pi A'}{m} + \iiint_V \left[ \left( \frac{1}{r^3} \right) \cdot \nabla A' - \nabla \left( \left( \frac{1}{r^3} \right) \cdot A' \right) \right] dv \quad (4b) \]

for \( P \) interior to \( S \) [since \( \nabla \left( \frac{1}{r^3} \right) \) may be identified with \( 4\pi \) times the Dirac delta function]; and

\[ \iint_S \left( \frac{1}{r^3} \right) (A' \cdot \hat{n}) dS = \iiint_V \left[ \left( \frac{1}{r^3} \right) (\nabla A') + (A' \cdot \nabla) \left( \frac{1}{r^3} \right) \right] dv \quad (4c) \]

By addition of Eqs. (4), expansion of \( \nabla \left( \left( \frac{1}{r^3} \right) \cdot A' \right) \), and use of \( \nabla \times \left( \frac{1}{r^3} \right) = 0 \), Eq. (3b) reduces to

\[ A'' = A' + \frac{1}{4\pi} \iiint_V \frac{r}{r^3} (\nabla A') dv \quad (5) \]

The curl of the last term in Eq. (5), taken with respect to the coordinates of \( P \), is found to vanish. It thus follows that

\[ \nabla \times A'' = \nabla \times A' = H \quad (6) \]

and \( A'' \) will serve as a vector potential to describe the field \( H \) in the region interior to \( S \).

From Eq. (3a), which served to define \( A'' \), it is seen that this potential would arise from such currents \( J \) as may exist in the region interior to \( S \), supplemented by the following surface distributions:

(1) a surface current
\[ J = \frac{1}{4\pi} [H \times \hat{n}] \text{ abamp/cm} \]  

(7a)

and

(ii) a double layer of current, visualized as formed of currents parallel and antiparallel to \( A' \) on the inner and outer surfaces of an infinitesimally thin shell, that is describable by a surface distribution of magnetic moment

\[ p_n = \frac{1}{4\pi} [A' \times \hat{n}] \text{ abamp}. \]  

(7b)

These surface distributions therefore may be employed in place of sources external to \( S \). The surface-current distribution of Eq. (7a), when supplemented by volume currents \( J \) that terminate on leaving the interior region, are such that the steady-state equation of continuity is satisfied.

If the foregoing analysis is applied to evaluate \( A'' \) at a point \( P \) external to \( S \), subject to \( A' \) being characteristic of a field whose sources are confined to a finite region of space, Eqs. (4) will still apply except that the term \( 4\pi A' \) will be absent from Eq. (4b). In this case the term \( A' \) will be absent from Eq. (5) and the curl of \( A'' \) will vanish. The current distribution stipulated in the preceding paragraph therefore produces no external field.

In examples for which the given vector potential is such that \( A \cdot \hat{n} = 0 \) on the boundary \( S \), the scalar function \( \phi \) of course need not be introduced. Thus one may characterize a uniform interior field,

\[ H = H_0 \hat{e}_z = H_0 [\cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta] = H_0 [P_\perp (\cos \theta) \hat{e}_r - P_\perp (\cos \theta) \hat{e}_\theta], \]

(8)
with \( J = 0 \), by the vector potential

\[
A_1 = \frac{1}{2} H_0 (-y \hat{e}_x + x \hat{e}_y) = \frac{1}{2} H_0 r \sin \theta \hat{e}_\phi = \frac{1}{2} H_0 r P_1 (\cos \theta) \hat{e}_\phi,
\]

in which the last forms shown in Eqs. (8) and (9) are expressed in spherical coordinates. If \( S \) is selected to be a sphere of radius \( a \), concentric with the origin of the coordinate system, the vector potential \( A_1 \) is such that \( A_1 \cdot \hat{n} = 0 \) on this surface and \( A_1' = A_1 \). Equations (7) then immediately give the surface distributions

\[
j \cdot \hat{m} = \frac{H_0}{4\pi} \sin \theta \hat{\phi} \quad \text{and} \quad \frac{P}{\hat{m}} = \frac{H_0 a}{\sin \theta} \hat{\phi}
\]

that will produce the specified uniform internal field. The form of the surface current \( j \) is such that this current alone will produce a uniform interior field; the respective contributions, to the internal field, due to the current and magnetic-moment distributions of Eqs. (10) are in the ratio 2:1 and the individual contributions to the external field cancel.

Alternatively, if the field of Eq. (8) where characterized by the vector potential

\[
A_2 = -H_0 y \hat{e}_x = -H_0 r (\sin^2 \theta \sin \phi \cos \phi \hat{e}_r + \sin \theta \cos \phi \sin \phi \cos \phi \hat{e}_\theta - \sin \theta \sin^2 \phi \hat{e}_\phi),
\]

\( A_2 \cdot \hat{n} = -H_0 a \sin^2 \theta \sin \phi \cos \phi \) and one may take \( \phi \) as the (harmonic) function
(12)

\[ \phi = \frac{1}{2} H_0 r^2 \sin^2 \theta \sin \varphi \cos \varphi \]

to satisfy Eq. (2). With this form for \( \phi \),

\[ A_2' = A_2 + \nabla \phi = \frac{1}{2} H_0 r \sin \theta \hat{e}_\theta = A_1 \]

(13)

and Eqs. (7) lead to the expressions for \( j \) and \( p \) given before by Eqs. (10).

This work was begun as a result of stimulating conversations with my colleague, Dr. A. M. Sessler, concerning the possibility of realizing certain field configurations intended to reduce aberrations in beta-ray spectrometers.
FOOTNOTES AND REFERENCES

* Work assisted by the U. S. Atomic Energy Commission.


2. W. R. Smythe, Static and Dynamic Electricity, Ed. 1 (McGraw-Hill, New York, 1939), Sec. 3.12.

3. See J. A. Stratton, Electromagnetic Theory, Ed. 1 (McGraw-Hill, New York, 1941), Sec. 4.15---esp. Eq. (14), in which the last term is difficult to interpret in physical terms, and Eq. (23), in which the last term represents (in rationalized MKS units) the field of a distribution of magnetic poles.

4. Unrationalized electromagnetic units are employed.

5. W. R. Smythe, Ref. 2, Sec. 7.051; Ed. 2 (1950), Sec. 7.12.
Fig. 1. The surface $S$ to which the vector $\mathbf{r}$ is drawn from the field point $P$. 
Fig. 1
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