Title
Weak capacity in Ahlfors regular metric spaces

Permalink
https://escholarship.org/uc/item/47b387fb

Author
Lindquist, Jeffrey William

Publication Date
2017

Peer reviewed|Thesis/dissertation
Weak capacity in Ahlfors regular metric spaces

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Jeffrey William Lindquist

2017
ABSTRACT OF THE DISSERTATION

Weak capacity in Ahlfors regular metric spaces

by

Jeffrey William Lindquist
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2017
Professor Mario Bonk, Chair

Let $(Z,d,\mu)$ be a compact, connected, Ahlfors $Q$-regular metric space with $Q > 1$. Using a hyperbolic filling of $Z$, we define the notions of the $p$-capacity between certain subsets of $Z$ and of the weak covering $p$-capacity of path families $\Gamma$ in $Z$. We show comparability results and quasisymmetric invariance. We reprove a result due to Tyson on the geometric quasi-conformality of quasisymmetric maps between compact, connected, Ahlfors $Q$-regular metric spaces. Under certain conditions, we identify the Ahlfors regular conformal dimension of $Z$ with critical exponents arising from weak capacity. Following an approach by Mario Bonk and Bruce Kleiner, we prove a necessary and sufficient condition involving weak capacity for an Ahlfors regular metric space that is topologically $S^2$ to be quasisymmetrically equivalent to $S^2$. 
The dissertation of Jeffrey William Lindquist is approved.

John B. Garnett
Ko Honda
Per J. Kraus

Mario Bonk, Committee Chair

University of California, Los Angeles
2017
To my parents

who have always supported me

and continue to do so.
# TABLE OF CONTENTS

1 Introduction and preliminaries ............................................ 1
   1.1 Introduction ............................................................... 1
   1.2 Preliminaries ............................................................. 9

2 Hyperbolic fillings .......................................................... 13
   2.1 Construction .............................................................. 13
   2.2 Bilipschitz equivalence .................................................. 19

3 Weak capacity ............................................................... 31
   3.1 Comparability and quasisymmetric invariance ....................... 31
   3.2 Positivity ................................................................. 44
   3.3 Weak covering capacity .................................................. 48

4 Conformal Dimension and Uniformization ............................... 55
   4.1 Ahlfors regular conformal dimension and critical exponents .... 55
   4.2 Path lemmas .............................................................. 60
   4.3 Control functions that tend to 0 ................................... 63
   4.4 Quasisymmetric uniformization ...................................... 68
   4.5 The Combinatorial Loewner Property ................................ 75
   4.6 Critical exponents when $Z$ attains $\text{ARCDim}$ ............... 77

References ................................................................. 82
ACKNOWLEDGMENTS

There are many people who have had a strong, positive impact on this dissertation.

Primarily, I thank Mario Bonk for countless interesting conversations on both the topics within and mathematics in general. His input and guidance have been immensely valuable and I consider myself extremely lucky to have been able to study under his direction. I also thank John Garnett for his support and discussions of many mathematical ideas.

I have had the pleasure of exploring much of what forms the basis of this thesis with the mathematical community. I thank Jeremy Tyson, Peter Haïssinsky, Kyle Kinneberg, Daniel Meyer, Eero Saksman, Pekka Pankka, Enrico Le Donne, Kai Rajala, Pekka Koskela, Angela Wu, Dimitrios Ntalampekos, Eden Prywes, and many others for this opportunity.

I am grateful to the UCLA mathematics department; my time spent in Los Angeles has been fantastic.

I thank my family for their unending encouragement; they have had a profound effect on my life.

The author was partially supported by NSF grants DMS-1506099 and DMS-1162471.
VITA

2011 B.S., Mathematics, The Ohio State University, *Summa Cum Laude with Honors*

2011-2017 Teaching assistant and graduate student researcher, Mathematics department, UCLA

PUBLICATIONS AND PRESENTATIONS


“Weak capacity and modulus comparability”, Workshop Geometry in May, University of Helsinki, May 2017

“Weak capacity and modulus comparability”, Geometry Seminar, University of Jyväskylä, April 2017

“Lehto’s Condition”, Workshop on Complex Analysis and Probability, Montana State University, August 2016

“Weak capacity and modulus comparability”, Geometry/Analysis Seminar, Rice University, April 2016

“Donsker’s Theorem”, Workshop on Trees in Dynamics, Montana State University, August 2015
“Weak capacity and modulus comparability”, Workshop on Analysis and Geometry in Metric Spaces, ICMAT Spain, June 2015

“Polynomial-like maps”, Workshop on Conformal Mating, Montana State University, August 2014
CHAPTER 1

Introduction and preliminaries

1.1 Introduction

Modulus of path families has become an important tool in studying metric spaces with a rich supply of rectifiable paths. The existence of sufficiently many rectifiable paths, however, is not guaranteed. For instance, starting from a metric space \((X, d)\), one sees the “snowflaked” metric space \((X, d^\alpha)\) with \(\alpha \in (0, 1)\) carries no non-constant rectifiable paths. Accordingly, traditional modulus techniques are insufficient in many cases.

In this paper we will study metric measure spaces \((Z, d, \mu)\) which are compact, connected, and Ahlfors \(Q\)-regular with \(Q > 1\). This means \((Z, d)\) is a separable metric space and \(\mu\) is Borel regular. The last condition is one on the volume growth of balls: a ball \(B\) of radius \(r\) has \(\mu\)-measure comparable to \(r^Q\). Ahlfors regularity is a reasonable assumption to make in studying quasisymmetric invariants; by [He, Corollary 14.15] the class of metric spaces quasisymmetrically equivalent to an Ahlfors regular metric space is precisely those carrying a doubling measure which are uniformly perfect. Both of these properties have important geometric consequences for the hyperbolic filling construction that is used to define weak capacity.

We develop two rough extensions of modulus to a hyperbolic filling associated with a given metric space. A hyperbolic filling \(X = (V_X, E_X)\) of \((Z, d, \mu)\) is a graph with vertices that correspond to metric balls and an edge structure which mirrors the combinatorial structure of our metric space. For a useful picture to have in mind consider the unit disk model of the hyperbolic space \(\mathbb{H}^2\). Here the outer circle \(S^1\) plays the role of our metric measure space \(Z\) and the hyperbolic filling can be interpreted as a graph representing a Whitney
cube decomposition of the interior. In this setting, cubes correspond to vertices and are connected by edges if they intersect. Hyperbolic fillings are Gromov hyperbolic metric spaces when endowed with the graph metric. Moreover, our original space can be identified as the boundary at infinity \( \partial_\infty X = Z \) following a standard construction found in [BuS, Chapter 2].

Hyperbolic fillings are well defined up to a scaling parameter and a choice of a vertex set at each scale. The extensions of modulus presented below are essentially quasi-isometrically invariant; changing the given hyperbolic filling will change the quantities by a controlled multiplicative amount. This multiplicative ambiguity also appears in the modulus comparison results and hence no generality is lost by working with a fixed hyperbolic filling for each metric space. The general construction of hyperbolic fillings follows [BdP] and [BuS] and is detailed in Section 2.1 along with some of the useful properties of such fillings. While it has been known that two hyperbolic fillings of the same metric space are quasi-isometric, we will show as a corollary to Theorem 2.2.1 that the vertex sets of two hyperbolic fillings of the same metric space are actually bilipschitz equivalent.

Generalizations of modulus are not new; in [P1] and [P2] Pansu develops a generalized modulus which is adapted in [Ty]. One key advantage of these generalized notions of modulus, as here, is that proving quasisymmetric invariance is relatively straightforward after setting up the appropriate definitions.

In our definitions we will need the notion of the weak \( \ell^p \)-norm of a function with a countable domain. Let \( X \) be a countable space and \( f : X \to \mathbb{C} \). We define \( \|f\|_{p,\infty} \) as the infimum of all constants \( C > 0 \) such that

\[
\# \{ x : |f(x)| > \lambda \} \leq \frac{C^p}{\lambda^p}
\]

for all \( \lambda > 0 \). We note that in general \( \|f\|_{p,\infty} \) is not a norm but for \( p > 1 \) it is comparable to a norm (see [BnS, Section 2]). We freely interchange the two and refer to \( \|f\|_{p,\infty} \) as the weak \( \ell^p \)-norm of \( f \). The use of the weak norm in the following definitions is motivated by [BnS].

We now define one of the two quantities used in this paper. We work with a compact, connected, Ahlfors \( Q \)-regular metric measure space \((Z,d,\mu)\) with hyperbolic filling \( X = \)
Both quantities are defined in a similar manner as modulus: certain functions defined on the hyperbolic filling are admissible if they give enough length to an appropriate collection of paths. To define the quantity in question we infimize over the $p$-th power of the weak $\ell^p$-norm of all admissible functions.

The first quantity, weak $p$-capacity ($\text{wcap}_p$), is defined both for pairs of open sets with $\text{dist}(A, B) > 0$ and for disjoint continua. A continuum is a compact, connected set that consists of more than one point. We use the notation

$$\text{dist}(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

for the distance between $A$ and $B$. The main idea is that instead of connecting two such sets by paths lying in $Z$, we look at the (necessarily infinite) paths connecting $A$ and $B$ in the hyperbolic filling. More precisely, given $A, B \subseteq Z$ we call a function $\tau : V_E \to [0, \infty]$ admissible for $A$ and $B$ if for all infinite paths $\gamma \subseteq V_E$ with non-tangential limits in $A$ and $B$ we have $\sum_{e \in \gamma} \tau(e) \geq 1$ (see Section 2.1 for boundaries at infinity of hyperbolic fillings and what it means for a path to have non-tangential limits). If $A$ and $B$ are understood we just call $\tau$ admissible.

**Definition 1.1.1.** Given open sets $A, B \subseteq Z$ with $\text{dist}(A, B) > 0$ or disjoint continua $A, B \subseteq Z$, we define the weak $p$-capacity $\text{wcap}_p(A, B)$ between $A$ and $B$ as

$$\text{wcap}_p(A, B) = \inf\{\|\tau\|_{\ell^p}^p : \tau \text{ is admissible for } A \text{ and } B\}.$$ 

Proposition 3.2.5 states that when this is defined for open sets, $\text{wcap}_p(A, B) > 0$. It is also true that for fixed sets $A$ and $B$ we have $\text{wcap}_p(A, B) \to 0$ as $p \to \infty$.

We remark again here that in the definition of $\text{wcap}_p$ there is an implicit choice of a fixed hyperbolic filling and that by changing the hyperbolic filling we may change the value of $\text{wcap}_p$ by a multiplicative constant. That is, if $\text{wcap}_p'$ is defined as $\text{wcap}_p$ with a different hyperbolic filling, then there are constants $c, C > 0$ such that for all open $A$ and $B$ with $\text{dist}(A, B) > 0$ and disjoint continua, one has

$$c \text{wcap}_p(A, B) \leq \text{wcap}_p'(A, B) \leq C \text{wcap}_p(A, B).$$
This follows from the proof of Theorem 1.1.4 as hyperbolic fillings of the same metric space are quasi-isometric. For this reason we ignore the dependence on the hyperbolic filling in the statements of the theorems below.

Our first main result shows that in general \( w_{cap}Q \) is larger than the \( Q \)-modulus of the path family connecting \( A \) and \( B \) (denoted \( \text{mod}_Q(A, B) \); for the definition of \( \text{mod}_Q \) of a path family, see Section 1.2).

**Theorem 1.1.2.** Let \( Q > 1 \) and let \((Z, d, \mu)\) be a compact, connected Ahlfors \( Q \)-regular metric space. Then there exists a constant \( C > 0 \) depending only on \( Q \) and the hyperbolic filling parameters with the following property: whenever \( A, B \subseteq Z \) are either open sets with \( \text{dist}(A, B) > 0 \) or disjoint continua,

\[
\text{mod}_Q(A, B) \leq C w_{cap}Q(A, B).
\]

For a Loewner space (see Section 1.2 or [He, Chapter 8] for the precise definition), \( w_{cap}Q \) is comparable to this modulus.

**Theorem 1.1.3.** Let \( Q > 1 \) and let \((Z, d, \mu)\) be a compact, connected Ahlfors \( Q \)-regular metric space which is also a \( Q \)-Loewner space. Then there exist constants \( c, C > 0 \) depending only on \( Q \) and the hyperbolic filling parameters with the following property: whenever \( A, B \subseteq Z \) are either open sets with \( \text{dist}(A, B) > 0 \) or disjoint continua,

\[
c \text{mod}_Q(A, B) \leq w_{cap}Q(A, B) \leq C \text{mod}_Q(A, B).
\]

Hence \( w_{cap}Q \) is a quantity that agrees with \( \text{mod}_Q \) up to a multiplicative constant, at least for path families connecting appropriate sets, on spaces with a large supply of rectifiable paths. We also prove \( w_{cap}p \) satisfies a quasisymmetric invariance property. Given a homeomorphism \( \eta : [0, \infty) \rightarrow [0, \infty) \), a map \( \varphi : Z \rightarrow W \) is called an \( \eta \)-quasisymmetry if whenever \( z, z', z'' \in Z \) satisfy \( |z - z'| \leq t|z - z''| \), we have \( |\varphi(z) - \varphi(z')| \leq \eta(t)|\varphi(z) - \varphi(z'')| \) where we have used the notation \( |\cdot - \cdot| \) to denote distance in the appropriate metric spaces.

**Theorem 1.1.4.** Let \( Z \) and \( W \) be compact, connected, Ahlfors regular metric spaces and let \( p > 1 \). If \( \varphi : Z \rightarrow W \) is an \( \eta \)-quasisymmetric homeomorphism, then there exist \( c, C > 0 \) depending only on \( \eta \) and the hyperbolic filling parameters with the following property: whenever
\( A, B \subseteq Z \) are either open sets with \( \text{dist}(A, B) > 0 \) or disjoint continua,

\[ c \operatorname{wcap}_p(\varphi(A), \varphi(B)) \leq \operatorname{wcap}_p(A, B) \leq C \operatorname{wcap}_p(\varphi(A), \varphi(B)). \]

We note here that the \( p \) above need not match the Ahlfors regularity dimension of neither \( Z \) nor \( W \) and that \( Z \) and \( W \) might have different Ahlfors regularity dimensions. The quasisymmetric invariance result relies on the fact that a quasisymmetry on compact, connected, metric measure spaces induces a quasi-isometry on corresponding hyperbolic fillings. A map \( F \) between two metric spaces \( X \) and \( Y \) is said to be a quasi-isometry if there are constants \( c, C > 0 \) such that for all \( x, x' \in X \), we have

\[ \frac{1}{C}|x - x'| - c \leq |f(x) - f(x')| \leq C|x - x'| + c \]

and there is a constant \( D > 0 \) such that for all \( y \in Y \), there is an \( x \in X \) such that \( |f(x) - y| \leq D \).

We now define the second quantity: weak covering \( p \)-capacity (\( \operatorname{wcap}_p \)). As before, there is a choice of hyperbolic filling required that introduces a multiplicative ambiguity but which poses no issues for the statements of the theorems. Unlike \( \operatorname{wcap}_p \), the quantity \( \operatorname{wcap}_p \) is defined for all path families in a given metric space.

The vertices \( V_X \) in our hyperbolic filling correspond to balls in \( Z \): we let \( B_v \) denote the ball corresponding to the vertex \( v \in V_X \). A subset \( S \subseteq V_X \) is said to cover \( Z \) if \( Z \subseteq \bigcup_{v \in S} B_v \). Let \( \mathcal{S} = \{S_n\} \) where each \( S_n \subseteq V_X \) is finite and covers \( Z \). We call such an \( \mathcal{S} \) a sequence of covers. We say \( \mathcal{S} \) is expanding if for every finite \( A \subseteq V_X \), we have \( S_n \cap A = \emptyset \) for all large enough \( n \).

For a given subset \( S \subseteq V_X \) that covers \( Z \) and a path \( \gamma : [0, 1] \to Z \) we define a projection \( P : [0, 1] \to V_X \) onto \( S \) as a partition \( 0 = t_0 < t_1 < \cdots < t_m = 1 \) of \( [0, 1] \) and a sequence \( v_1, \ldots, v_m \in S \) such that for all \( k \), \( \gamma([t_{k-1}, t_k]) \subseteq B_{v_k} \).

Given \( \tau : V_X \to [0, \infty] \), we define the \( \tau \)-length of a projection \( P \) of \( \gamma \) on \( S \) as \( \ell_{\tau, P, S}(\gamma) = \sum_k \tau(v_k) \).

Now, let \( \mathcal{S} = \{S_n\} \) be an expanding sequence of covers. Given a rectifiable path
\( \gamma : [0,1] \to Z \), we say \( \tau \) is admissible for \( \gamma \) relative to \( \mathcal{S} \) if

\[
\liminf_{n \to \infty} \left( \inf_P \ell_{\tau,P,S_n}(\gamma) \right) \geq 1
\]

where the infimum \( \inf_P \) is over all projections of \( \gamma \) onto \( S_n \). We say \( \tau \) is admissible for \( \gamma \) if \( \tau \) is admissible relative to \( \mathcal{S} \) for all such \( \mathcal{S} \). We remark that, for a given \( \gamma \), changing the parameterization does not affect the subsequent projected \( \tau \)-length and we will frequently view rectifiable \( \gamma \) as parameterized by arclength.

The main idea is to use subsets (the covers above) of vertices of the hyperbolic filling to approximate \( Z \) in finer and finer resolution. Path families now lie on the boundary \( Z \) and are projected onto these covers in order to test admissibility of functions defined within. By infimizing over all projections onto a given cover and then letting the covers “expand” to become more and more like \( Z \), we arrive at the rough length a given function defined on the filling gives a particular path. Demanding admissibility for all rectifiable paths as with \( \text{mod}_p \) and using the weak norm as with \( \text{wcap}_p \) leads us to our definition.

**Definition 1.1.5.** Given a collection of paths \( \Gamma \) in \( Z \), we define the weak covering \( p \)-capacity \( \text{wc-cap}_p(\Gamma) \) of \( \Gamma \) as

\[
\text{wc-cap}_p(\Gamma) = \inf \{ \| \tau \|_{p,\infty}^p : \tau \text{ is admissible for all } \gamma \in \Gamma \}.
\]

With this quantity we have comparability even without the Loewner condition.

**Theorem 1.1.6.** Let \( Q > 1 \) and let \( (Z,d,\mu) \) be a compact, connected Ahlfors \( Q \)-regular metric space. Then there exist constants \( c, C > 0 \) depending only on \( Q \) and the hyperbolic filling parameters such that for all path families \( \Gamma \),

\[
c \text{mod}_Q(\Gamma) \leq \text{wc-cap}_p(\Gamma) \leq C \text{mod}_Q(\Gamma).
\]

Similarly to \( \text{wcap}_p \), the quantity \( \text{wc-cap}_p \) has a quasisymmetric invariance property.

**Theorem 1.1.7.** Let \( Z \) and \( W \) be compact, connected, Ahlfors regular metric spaces and let \( p > 1 \). If \( \varphi : Z \to W \) is an \( \eta \)-quasisymmetric homeomorphism, then there exist constants
depending only on \(p, \eta,\) and the hyperbolic filling parameters such that for all path families \(\Gamma\) in \(Z\) we have

\[
c \text{wc-cap}_p(\Gamma) \leq \text{wc-cap}_p(\varphi(\Gamma)) \leq C \text{wc-cap}_p(\Gamma).
\]

This quasisymmetric invariance property implies a result due to Tyson \([Ty]\).  

**Corollary 1.1.8.** Let \(Z\) and \(W\) be compact, connected, Ahlfors \(Q\)-regular metric spaces with \(Q > 1\) and let \(\varphi: Z \to W\) be an \(\eta\)-quasisymmetric homeomorphism. Then there exists constants \(c, C > 0\) depending only on \(\eta, Q,\) and the hyperbolic filling parameters such that

\[
c \text{mod}_Q(\Gamma) \leq \text{mod}_Q(\varphi(\Gamma)) \leq C \text{mod}_Q(\Gamma)
\]

for all path families \(\Gamma\) in \(Z\).

Indeed, this follows immediately from combining Theorem 1.1.6 and Theorem 1.1.7 above. Tyson \([Ty]\) shows this result for locally compact, connected, Ahlfors \(Q\)-regular metric spaces, but our framework with hyperbolic fillings is adapted to compact metric spaces. Williams \([Wi]\) also derives this result; his Remark 4.3 relates the conditions in the corollary above to his condition (III) which leads to the conclusion.

The quasisymmetric invariance of weak capacity allows us to study the conformal gauge of \((Z,d)\). This is defined as the set of all metric spaces \((Z', d')\) for which there is a quasisymmetric homeomorphism \(\varphi: Z \to Z'\). An important invariant of the conformal gauge of \(Z\) is the Ahlfors regular conformal dimension \(\text{ARCdim}\), which is the infimum of the Hausdorff dimensions of metric spaces in the conformal gauge of \(Z\). We study two critical exponents which are related to \(\text{ARCdim}\). The first is given by the following definition.

**Definition 1.1.9.**

\[
Q_w = Q_w(Z) = \inf\{p : \text{wcap}_p(A, B) < \infty \text{ for all open } A \text{ and } B \text{ with } \text{dist}(A, B) > 0\}.
\]

Here we require \(\text{wcap}_p(A, B)\) is finite, but we do not require any control on this quantity. This is sufficient when our metric space has the Combinatorial Loewner Property (which is defined in Section 4.5).
Theorem 1.1.10. Let \((Z, d, \mu)\) be an Ahlfors \(Q\)-regular metric space that satisfies the Combinatorial Loewner Property with exponent \(Q\). Then, for all \(p \in (1, Q)\) and open sets \(A, B \subseteq Z\) with \(\text{dist}(A, B) > 0\) we have \(\text{wcap}_p(A, B) = \infty\).

In other situations we require better estimates on \(\text{wcap}_p\). The second critical exponent stipulates that we can control \(\text{wcap}_p(A, B)\) by the relative distance of \(A\) and \(B\).

Definition 1.1.11. The relative distance of two subsets \(A, B \subseteq Z\) is given by
\[\Delta(A, B) = \frac{\text{dist}(A, B)}{\min\{\text{diam}(A), \text{diam}(B)\}}\]

We now state our second critical exponent definition.

Definition 1.1.12. Call a function \(\varphi : (0, \infty) \to [0, \infty)\) a control function if \(\varphi\) is decreasing. Define
\[Q'_w = \inf\{p : \exists \varphi \text{ a control function s.t. } \text{wcap}_p(A, B) \leq \varphi(\Delta(A, B))\}\]
where \(A, B\) above are disjoint open sets with \(\text{dist}(A, B) > 0\).

Remark 1.1.13. While the definition of \(Q'_w\) does not require that the control function satisfies \(\varphi(t) \to 0\) as \(t \to \infty\), it is possible to always choose \(\varphi\) with this property. We will prove this in Section 4.3.

This notion allows us to prove a result analogous to [BnK, Theorem 10.4]. For this result, we need the notion of linear local connectivity.

Definition 1.1.14. A metric space \((Z, d)\) is \(\lambda\)-LLC (linearly locally connected) for \(\lambda \geq 1\) if the following two conditions hold:

(i) Given \(B(a, r) \subseteq Z\) and \(x, y \in B(a, r)\), there exists a continuum \(E \subseteq B(a, \lambda r)\) containing \(x\) and \(y\).

(ii) Given \(B(a, r) \subseteq Z\) and \(x, y \in Z \setminus B(a, r)\), there exists a continuum \(E \subseteq Z \setminus B(a, r/\lambda)\) containing \(x\) and \(y\).
Theorem 1.1.15. Let \((Z,d)\) be a space homeomorphic to \(S^2\) that is linearly locally connected and Ahlfors \(Q\)-regular. Suppose there is a decreasing function \(\varphi: (0,\infty) \to (0,\infty)\) such that for all open sets \(A\) and \(B\) in \(Z\) with \(\text{dist}(A,B) > 0\) one has

\[
\text{wcap}_2(A,B) \leq \varphi(\Delta(A,B)).
\] (1.2)

Then \(Z\) is quasisymmetrically equivalent to \(S^2\). Conversely, if \(Z\) is quasisymmetrically equivalent to \(S^2\), then such a function \(\varphi\) exists.

When the metric space \((Z,d)\) attains its Ahlfors regular conformal dimension, a result due to [KL] (in a form proven in [CP]) allows us to conclude \(Q'_w = \text{ARCDim}\).

Theorem 1.1.16. Let \((Z,d,\mu)\) be an Ahlfors \(Q\)-regular metric measure space with \(\text{ARCdim} = Q\). Let \(p < Q\). Then, there exist open sets \(A_k, B_k\) and a constant \(C > 0\) such that \(\Delta(A_k, B_k) < C\) but \(\text{wcap}_p(A_k, B_k) \to \infty\) as \(k \to \infty\).

We outline the structure of the paper. In the remainder of Chapter 1 we review some preliminaries including the definition of modulus and the statement of a weak \(\ell^p\)-norm comparison result. In Chapter 2 we study hyperbolic fillings. In the first half of this chapter we give our construction and prove some basic properties. In the second half of this chapter we prove the discrete result that shows that quasi-isometric hyperbolic fillings are bilipschitz equivalent. In Chapter 3 we prove the main results related to \(\text{wcap}_p\) and \(\text{wc-cap}_p\) including comparability and quasisymmetric invariance. In Chapter 4 we examine the critical exponents defined above. We prove that control functions can be chosen to tend to 0. Lastly, we prove Theorems 1.1.15, 1.1.10, and 1.1.16.

1.2 Preliminaries

First we define the notion of modulus of path families. Let \((Z,d,\mu)\) be a metric measure space. By a path in \(Z\) we mean a continuous function \(\gamma: I \to Z\) where \(I \subseteq \mathbb{R}\) is an interval. We will use \(\gamma\) to refer to both the path and the image of the path. For a path family \(\Gamma\), we
say a Borel function \( \rho : Z \to [0, \infty] \) is admissible for \( \Gamma \) if for all rectifiable \( \gamma \in \Gamma \) we have
\[
\int_{\gamma} \rho \geq 1.
\]
Here and elsewhere path integrals are assumed to be with respect to the arclength parameterization. We define the \( p \)-modulus of the path family \( \Gamma \) to be
\[
\operatorname{mod}_p(\Gamma) = \inf \left\{ \int_Z \rho^p \right\}
\]
where the infimum is taken over all \( \rho \) admissible for \( \Gamma \) and the integral is against the measure \( \mu \). In the following we will also suppress \( d\mu \) from the notation.

One may think of \( \operatorname{mod}_p \) as an outer measure on path families which is supported on rectifiable paths. The quantity \( \operatorname{mod}_p \) is meaningless in spaces without rectifiable paths such as the snowflaked metric spaces discussed in the introduction. Nonetheless, where rectifiable paths abound \( \operatorname{mod}_p \) is closely related to conformality and quasiconformality. Indeed, a conformal diffeomorphism \( f \) between two Riemannian manifolds \( M \) and \( N \) of dimension \( n \) preserves the \( n \)-modulus of path families [He, Theorem 7.10]. This invariance principle can be used to give an equivalent definition of quasiconformal homeomorphisms between appropriate spaces [He, Definition 7.12] which, while difficult to check, is quite strong. Many basic properties of modulus are detailed in [He, Chapter 7].

Often for clarity and brevity we will make use of the symbols \( \lesssim \) and \( \simeq \). For two quantities \( A \) and \( B \), that may depend on some ambient parameters, we write \( A \lesssim B \) to indicate that there is a constant \( C > 0 \) depending only on these parameters such that \( A \leq CB \). We also write \( A \simeq B \) to indicate that there are constants \( c, C > 0 \) depending only on these parameters such that \( cB \leq A \leq CB \). The exact dependencies of these constants will be clear from the context.

For some results we will make use of [BnS, Lemma 2.2]. For convenience we include the statement here.

**Lemma 1.2.1.** Let \( H \) and \( K \) be countable sets. Let \( p > 1 \) and \( J \subseteq H \times K \) be such that all sets of the forms \( J_h = \{ k \in K : (h, k) \in J \} \) and \( J^k = \{ h \in H : (h, k) \in J \} \) have at most \( N \)
elements. Then there is a constant \( C(p, N) > 0 \) with the following weak \( \ell^p \) bound property: if \( s = \{s_h\} \) and \( t = \{t_k\} \) are sets of real numbers indexed by \( H \) and \( K \) respectively and

\[
|s_h| \leq \sum_{k \in J_h} |t_k| \tag{1.3}
\]

for all \( h \), then \( \|s\|_{p,\infty} \leq C(p, N) \|t\|_{p,\infty} \).

One useful notion for a metric space to have many rectifiable paths is the Loewner condition introduced in [HK]. This relates the modulus of path families connecting non-intersecting continua, say \( A \) and \( B \), with their relative distance \( \Delta(A, B) \). We let \( \text{mod}_Q(A, B) \) denote the modulus of the path family connecting \( A \) to \( B \).

**Definition 1.2.2.** A metric measure space \((Z, d, \mu)\) is a \( Q \)-Loewner space if there is a decreasing function \( \phi_Q \colon (0, \infty) \to (0, \infty) \) such that for all such \( A \) and \( B \) we have

\[
\phi_Q(\Delta(A, B)) \leq \text{mod}_Q(A, B).
\]

See [He, Chapter 8] for this definition and more information on Loewner spaces. The main intuition here is that the path family connecting two continua with positive relative distance is large enough to carry positive \( Q \)-modulus; that is, there are many rectifiable paths connecting the two sets. This condition cannot be omitted from Theorem 1.1.3 by the quasisymmetric invariance principle for \( \text{wcap}_p \) (Theorem 1.1.4); we can snowflake a Loewner space to construct spaces where the modulus of any path family is 0.

Given a function \( u \) on a metric measure space \((Z, d, \mu)\), we say that a Borel function \( \rho \colon Z \to [0, \infty] \) is an upper gradient for \( u \) if whenever \( z, z' \in Z \) and \( \gamma \) is a rectifiable path connecting \( z \) and \( z' \), we have

\[
|u(z) - u(z')| \leq \int_{\gamma} \rho.
\]

We use the notation that if \( B = B(x, r) \), then \( \lambda B = B(x, \lambda r) \). For a given ball \( B \) and a locally integrable function \( u \) we set \( u_B = \frac{1}{\mu(B)} \int_B u = \int_B u; \) this is the average value of \( u \) over the ball \( B \). We say \((Z, d, \mu)\) admits a \( p \)-Poincaré inequality if there are constants \( C > 0 \) and \( \lambda \geq 1 \) such that

\[
\int_B |u - u_B| \leq C(\text{diam} B) \left( \int_{\lambda B} \rho^p \right)^{1/p} \tag{1.4}
\]
for all open balls $B$ in $Z$, for every locally integrable function $u: Z \rightarrow \mathbb{R}$, and every upper gradient $\rho$ of $u$ in $Z$. This definition and a subsequent discussion can be found in [HKST, Chapter 8]. The $Q$-Poincaré inequality is a regularity condition on our space which will follow from the $Q$-Loewner space hypothesis present in some of the theorems.

We make use of Hausdorff content and Hausdorff measure. Given an exponent $\alpha \geq 0$ and a set $E$, the $\alpha$-Hausdorff content of $E$ is defined as

$$\mathcal{H}_\infty^\alpha(E) = \inf \{ \sum_{A \in \mathcal{A}} \text{diam}(A)^\alpha : \mathcal{A} \text{ is a cover of } E \text{ by balls } A \}.$$ 

The $\alpha$-Hausdorff measure of $E$ is defined as $\mathcal{H}_\delta^\alpha(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^\alpha(E)$ where

$$\mathcal{H}_\delta^\alpha(E) = \inf \{ \sum_{A \in \mathcal{A}} \text{diam}(A)^\alpha : \mathcal{A} \text{ is a cover of } E \text{ by balls } A \text{ with } \text{diam}(A) < \delta \}.$$ 

The Hausdorff dimension of $E$ is defined as $\dim_{\mathcal{H}}(E) = \inf \{ \alpha : \mathcal{H}_\alpha^\alpha(E) = 0 \}.$
CHAPTER 2

Hyperbolic fillings

2.1 Construction

Here we detail the construction of the hyperbolic fillings used to define \(\text{wcap}_Q\) and \(\text{wc-cap}_Q\). The main idea is to construct a graph that encodes the combinatorial data of \(Z\) in finer and finer detail. As remarked in the introduction, it is useful to keep a Whitney cube decomposition of the unit disk model of the hyperbolic plane \(\mathbb{H}^2\) in mind with the vertices corresponding to the Whitney cubes and with edges existing between intersecting cubes.

Recall we work in a compact, connected, Ahlfors \(Q\)-regular metric measure space \((Z, d, \mu)\). Our construction of and proofs of properties involving the hyperbolic fillings of \(Z\) follows [BdP, Section 2.1] almost exactly. There is a minor problem with their construction, however, which we fix by using doubled radii balls inspired a similar construction in [BuS, Section 6.1]. For completeness we include the entire construction here.

By rescaling, we may assume \(\text{diam} Z < 1\). Let \(s > 1\) and, for each \(k \in \mathbb{N}_0\), let \(P_k \subseteq Z\) be an \(s^{-k}\) separated set that is maximal relative to inclusion. We call \(s\) the parameter of the hyperbolic filling. To each \(p \in P_k\), we associate the ball \(v = B(p, 2s^{-k})\) which we will use as our vertices in our graph. We refer to \(k\) as the level of \(v\) and write this as \(\ell(v)\). We also write \(V_{X_k}\) or \(V_k\) for the set of vertices with level \(k\). Note as \(\text{diam}(Z) < 1\), there is a unique vertex in \(V_0\) which we will denote \(O\). We often write \(v\) as \(B_v\) or \(B(v)\) where the level \(k\) and the center \(p\) are understood. We will occasionally make an abuse of notation, however, and write \(B(v, 2s^{-k})\) referring to \(v\) as both the vertex and the center of the ball. We also use the notation \(r(B)\) for the radius of a ball.

We form a graph \(X = (V_X, E_X)\) where the vertex set \(V_X\) is the disjoint union of the
$V_k$ and we connect two distinct $v, w \in V_X$ by an edge if and only if $|\ell(v) - \ell(w)| \leq 1$, and $\overline{B_v} \cap \overline{B_w} \neq \emptyset$. We endow $X$ with the unique path metric in which each edge is isometric to an interval of length 1.

For $v, w \in V_X$, we let $(v, w)$ denote the Gromov product given by

$$(v, w) = \frac{1}{2}(|O - v| + |O - w| - |v - w|),$$

where $O$ is the unique vertex with $\ell(O) = 0$ and $|v - w|$ is the graph distance between $v$ and $w$. Intuitively, $(v, w)$ is roughly the distance between $O$ and any geodesic connecting $v$ to $w$.

**Lemma 2.1.1.** For $v, w \in V_X$, we have $s^{-(v, w)} \simeq \text{diam}(B_v \cup B_w)$.

**Proof.** We find $x \in V_X$ with $\text{diam}(B_x) \simeq \text{diam}(B_v \cup B_w)$ and $B_v, B_w \subseteq B_x$. To do this, we note that there is a number $k \in \mathbb{N}$ with $s^{-k-1} \leq \text{diam}(B_v \cup B_w) \leq s^{-k}$. Then by construction there is a vertex $x$ on level $k$ such that $\frac{1}{2}B_x \cap B_v \neq \emptyset$; this follows from the choice of an $s^{-k}$ separated set for the centers of the balls corresponding to the vertices on level $k$. As $\text{diam}(B_v \cup B_w) \leq s^{-k}$, we see $B_v \cup B_w \subseteq B_x$. Without the extra radius factor this inclusion need not be true; this is the minor oversight in [BdP, Section 2.1]. We choose geodesics $[Ov]$ and $[Ow]$ containing $x$. We see $(v, w) \geq |O - x|$ as

$$(v, w) - |O - x| = \frac{1}{2}(|O - v| - |O - x| + |O - w| - |O - x| - |v - w|)$$

which is non-negative by the triangle inequality. Thus,

$$s^{-(v, w)} \leq s^{-|O - x|} = \frac{1}{2}r(B_x) \simeq \text{diam}(B_v \cup B_w).$$

For the other direction, we follow the notation in [BdP, Lemma 2.2] and set $|v - w| = \ell, |O - v| = m,$ and $|O - w| = n$. Let $[vw]$ be a geodesic segment, which is formed from a sequence of balls $B_k$ for $k \in 0, \ldots, \ell$ with $B_0 = B_v$ and

$$s^{-1}r(B_i) \leq r(B_{i+1}) \leq sr(B_i)$$

14
for all $i$. Hence, for $k \in 0, \ldots, \ell$, we have

$$
\begin{align*}
diam(B_v \cup B_w) & \leq \sum_{i=0}^{\ell} diam(B_i) \\
& = \sum_{i=0}^{k} diam(B_i) + \sum_{j=0}^{\ell-k-1} diam(B_{\ell-j}) \\
& \leq 4 \sum_{i=0}^{k} s^{-m+i} + 4 \sum_{j=0}^{\ell-k-1} s^{-n+j} \\
& = \frac{4}{s-1} \left( s^{-m+k+1} - s^{-m} + s^{-n+\ell-k} - s^{-n} \right) \\
& \leq \frac{4s}{s-1} \left( s^{-m+k} + s^{-n+\ell-k} \right).
\end{align*}
$$

Setting $k = \frac{1}{2}(\ell + m - n)$ (or $k = \frac{1}{2}(\ell + m - n + 1)$ for a comparable bound if this is not an integer), this becomes

$$
\begin{align*}
diam(B_v \cup B_w) & \leq \frac{8s}{s-1} s^{\frac{1}{2}(\ell-m-n)} = \frac{8s}{s-1} s^{-(v,w)}.
\end{align*}
$$

The following lemma involves Gromov hyperbolic metric spaces. For definitions we refer the reader to [BuS, Section 2.1].

**Lemma 2.1.2.** $X$ equipped with the graph metric is a Gromov hyperbolic space.

**Proof.** Let $v, w, x \in V_X$. Then

$$
\begin{align*}
diam(B_v \cup B_w) & \leq diam(B_v \cup B_x) + diam(B_x \cup B_w),
\end{align*}
$$

so by the above lemma there is a constant $D > 0$ independent of $v, w, x$ such that

$$
\begin{align*}
s^{-(v,w)} & \leq D(s^{-(v,x)} + s^{-(x,w)}) \\
& \leq 2D \max(s^{-(v,x)}, s^{-(x,w)}).
\end{align*}
$$

Hence,

$$
-(v, w) \leq \log_s(2D) + \max(-(v, x), -(x, w))
$$

and so

$$
(v, w) \geq \min((v, x), (x, w)) - \log_s(2D)
$$

which is the inequality required in the definition of a Gromov hyperbolic space. \qed
We will also work with the boundary at infinity of our hyperbolic fillings. For completeness we include the standard construction of the boundary at infinity of a Gromov hyperbolic space here and refer the reader to [BuS, Chapter 2] for some of the details as well as more background.

For our given Gromov hyperbolic space $X$, the points of the boundary at infinity $\partial_\infty X$ are equivalence classes of sequences of points “diverging to infinity”. More precisely, we say a sequence of points $\{x_n\}$ diverges to infinity if

$$\lim_{m,n \to \infty} (x_n, x_m) = \infty.$$ 

Two sequences $\{x_n\}$ and $\{y_m\}$ are equivalent if

$$\lim_{m,n \to \infty} (x_n, y_m) = \infty.$$ 

One then may extend the Gromov product to the boundary as in [BuS]. From this one defines a metric $d$ on $\partial_\infty X$ to be a visual metric if there are constants $c, C > 0$ and $\alpha > 1$ such that for all $z, z' \in \partial_\infty X$ we have

$$ca^{-\langle z, z' \rangle} \leq d(z, z') \leq Ca^{-\langle z, z' \rangle}.$$ 

We now relate this construction to $X$ and $Z$.

**Lemma 2.1.3.** With our constructed $X$ above, we can identify $\partial_\infty X$ with $Z$ where the original metric on $Z$ is a visual metric.

**Sketch of proof.** We use

$$s^{-\langle v, w \rangle} \simeq \text{diam}(B_v \cup B_w). \quad (2.1)$$

We see $\{v_n\}$ is a sequence of vertices diverging to infinity if and only if

$$\text{diam}(B_{v_n} \cup B_{v_m}) \to 0$$

and so not only do the diameters satisfy $\text{diam}(B_{v_n}) \to 0$ but the centers $p_n$ of the balls corresponding to $v_n$ also converge to a single point in $z \in Z$. This shows we can view $\partial_\infty X$ as a subset of $Z$ by identifying a sequence of vertices diverging to infinity with the limit.
point of the centers of the corresponding balls. The other inclusion follows by considering that for each \( k \in \mathbb{N}_0 \) the set of balls \( \{ B_v : v \in V_k \} \) covers \( Z \). Hence, for fixed \( z \in Z \) for each \( k \in \mathbb{N}_0 \) we may choose a vertex \( v_k \) with level \( k \) such that \( z \in B_{v_k} \). This creates a sequence in \( X \) diverging to infinity that corresponds to \( z \). Relation (2.1) above also shows our original metric on \( Z \) is in fact a visual metric with respect to \( X \). \[ \square \]

The metric paths in \( X \) that we are interested in travel through many vertices and are often infinite. For this reason we define a path in \( X \) as a (possibly finite) sequence of vertices \( v_k \) such that for all \( k \), the vertices \( v_k \) and \( v_{k+1} \) are connected by an edge. Alternatively we may view a path as a sequence of edges \( e_k \) such that for all \( k \), the edges \( e_k \) and \( e_{k+1} \) share a common vertex; the point of view will be clear from context.

We now specify what it means for a sequence of vertices \( v_n \) to converge to \( z \in Z \): if \( v_n \) is represented by \( B(p_n, r_n) \) then \( v_n \to z \) if and only if \( p_n \to z \) and \( r_n \to 0 \). From this we also see what it means for a path (given by a sequence of vertices \( \{v_k\}_{k \in \mathbb{N}} \) or edges \( \{e_k\}_{k \in \mathbb{N}} \) as discussed above) to converge to a point in \( Z \).

In our definitions we will work with non-tangential limits. Intuitively, a path approaches the boundary non-tangentially if it stays within bounded distance of a geodesic. In our setting this means that the smaller the radii corresponding to vertices on a path are, the closer the centers corresponding to those vertices need to be to the limit point. We state this more precisely as a definition.

**Definition 2.1.4.** A path in \( X \) with vertices \( v_n \) represented by \( B(p_n, r_n) \) converges non-tangentially to \( z \in Z \) if \( p_n \to z \) and \( r_n \to 0 \) and there exists a constant \( C > 0 \) such that for all \( n \in \mathbb{N} \) we have dist(\( z, p_n \)) \( \leq Cr_n \).

For the next lemma, we define the degree of a vertex \( v \) in a graph \( (V, E) \) as the number of edges having \( v \) as a vertex. To say a graph has bounded degree then means that there is a uniform constant \( C \) such that every vertex \( v \in V \) has degree at most \( C \). We write \( \text{deg}(v) \) for the degree of \( v \).
Lemma 2.1.5. The hyperbolic filling $X$ of a compact, connected, Ahlfors $Q$-regular metric measure space has bounded degree.

Proof. Let $v$ be a vertex with level $n > 1$. Let $W$ be the vertices with level $n$ that intersect $v$. We bound $|W|$, the cardinality of the set $W$. Bounds on the number of vertices adjacent to $v$ with levels $n - 1$ and $n + 1$ follow similarly and yield the result.

Here we abuse notation and use $v, w$ as the centers of the balls corresponding to these vertices. We note $\cup_{w \in W} B(w, \frac{1}{3}s^{-n})$ is a disjoint collection of balls by the separation of vertices on level $n$. Moreover, this collection is contained in $B(v, 4s^{-n})$. By Ahlfors regularity there are constants $c, C$ such that for all $z \in Z$ and $r \in (0, \text{diam}(Z)]$ we have

$$cr^Q \leq \mu(B(z, r)) \leq Cr^Q.$$ 

Hence, we see

$$c|W|\frac{1}{3^Q}s^{-nQ} \leq C4^Qs^{-nQ}$$

and so $|W|$ is uniformly bounded above. \hfill \Box

Our final lemma is a construction which extends a quasisymmetric homeomorphism $f$ between two compact, connected, metric measure spaces $Z$ and $W$ to a quasi-isometric map $F$ between corresponding hyperbolic fillings $X$ and $Y$. As we make use of this construction explicitly in Theorem 1.1.7, we record the result here as a lemma as well as a sketch of the proof. The full proof can be found in [BuS, Theorem 7.2.1].

Lemma 2.1.6. Let $Z$ and $W$ be compact, connected, metric measure spaces with corresponding hyperbolic fillings $X$ and $Y$. Let $f : Z \to W$ be a quasisymmetric homeomorphism. Then there is a quasi-isometry $F : X \to Y$ which extends to $f$ on the boundary.

Sketch of proof. It suffices to define $F$ as a map between vertices. Given $x \in X$, we see $f(B_x) \subset W$ and so there is at least one vertex $y \in Y$ of minimal radius (i.e. highest level) that contains $f(B_x)$. We set $F(x) = y$. We show $F$ is a quasi-isometry. For the upper bound in (1.1) we show that for vertices $x, x' \in V_X$ with $|x - x'| \leq 1$, we have uniform control over $|F(x) - F(x')|$. If not, then we have a sequence of such $x, x'$ such that $|F(x) - F(x')|$
gets arbitrarily large. With this, we note $B_{F(x)} \cap B_{F(x')} \neq \emptyset$ and from this deduce that the levels between, and hence the ratio of the radii of the balls $B_{F(x)}$ and $B_{F(x')}$, must become arbitrarily large. Using a common point in $B_{F(x)} \cap B_{F(x')}$ and the quasisymmetry condition, we arrive at a contradiction. For the lower bound one considers $G : Y \to X$ defined as above but using $f^{-1}$ in place of $f$. One then shows that the compositions $G \circ F$ and $F \circ G$ are within a bounded distance from the identity in each case (i.e. there is a $D > 0$ such that for all $x \in X$ one has $|x - (G \circ F)(x)| < D$ and likewise for $F \circ G$). The fact that $F(X)$ is at a bounded distance from any point in $y$ also follows from the compositions being within a bounded distance from the identity.

\section{2.2 Bilipschitz equivalence}

By combining results from [Wh] with results from [MPR] we prove that the vertex sets of hyperbolic fillings of metric spaces are not only quasi-isometric, but actually bilipschitz equivalent. This approach also generalizes a theorem of Papasolglu [Pa] who proves that (the vertices of) $k$-ary homogeneous trees are bilipschitz equivalent whenever $k \geq 3$. Our result applies to a larger class of trees.

A map between metric spaces $f : (X, d_X) \to (Y, d_Y)$ is a \textit{bilipschitz equivalence} if it is a bijection such that there exists a constant $C > 0$ such that for all $x, x' \in X$ we have

$$\frac{1}{C} d_X(x, x') \leq d_Y(f(x), f(x')) \leq C d_X(x, x').$$

Bilipschitz equivalence is a strong property that is not immediate in many situations. We recall the weaker notion of a quasi-isometry which is a map that is bilipschitz at large scales. A natural question to ask is whether a quasi-isometry can be promoted to a bilipschitz equivalence under the right conditions. This was answered affirmatively for some conditions by Whyte [Wh] who showed that a quasi-isometry between $UDBG$ spaces is within bounded distance to a bilipschitz equivalence if a certain homological condition holds (for the definition of a $UDBG$ space, see Definitions $\ref{def:UDBG}$ and $\ref{def:UDBG'}$). Whyte’s results involve boundary estimates which are reminiscent of linear isoperimetric inequalities. Indeed, one can apply
the results in his paper directly to prove Theorem 2.2.1. For this theorem $X$ and $Y$ are imbued with the graph metric which induces the metric on the vertex sets.

**Theorem 2.2.1.** Let $X$ and $Y$ be connected graphs of bounded degree satisfying linear isoperimetric inequalities. If the vertex sets $V_X$ and $V_Y$ are quasi-isometric, then these sets are also bilipschitz equivalent. Moreover, if $f: V_X \to V_Y$ is a quasi-isometry, then there is such a bilipschitz equivalence within bounded distance of $f$.

This theorem allows us to generalize Papasoglu’s result to more exotic trees that satisfy linear isoperimetric inequalities. Such trees were studied in the work of Martínez-Pérez and Rodríguez [MPR]. Their results, together with a quasisymmetric characterization from [DS], yield the following corollary:

**Corollary 2.2.2.** Let $X$ and $Y$ be rooted, pseudo-regular, visual trees of bounded degree. Then, $V_X$ and $V_Y$ are bilipschitz equivalent.

Here a tree is rooted if it has a specified “root” vertex, pseudo-regular if it branches regularly, visual if it does not have arbitrarily long “dead ends”, and of bounded degree if there is a uniform bound on the number of edges connecting to any particular vertex. Precise definitions are found in the following pages.

Theorem 2.2.1 has another corollary when one considers hyperbolic fillings of a given compact, uniformly perfect, Ahlfors regular metric space.

**Corollary 2.2.3.** Let $(Z, d)$ be a compact, uniformly perfect, Ahlfors regular metric space. Let $X$ and $Y$ be hyperbolic fillings of $Z$. Then, $V_X$ and $V_Y$ are bilipschitz equivalent.

It follows from the work in [MPR] that hyperbolic fillings of these spaces satisfy a linear isoperimetric inequality and thus quasi-isometries may be promoted to bilipschitz equivalences in this setting. As two different fillings are quasi-isometric, the corollary follows.

**Definitions and Preliminaries**

We are primarily concerned with graphs, but the results developed in [Wh] hold in the more general setting of uniformly discrete spaces with bounded geometry (denoted UDBG spaces).
Definition 2.2.4. A metric space \((X,d)\) is uniformly discrete if there is a constant \(c > 0\) such that for all \(x,x' \in X\) with \(x \neq x'\) we have \(d(x,x') > c\).

Definition 2.2.5. A metric space \((X,d)\) is said to have bounded geometry if it is uniformly discrete and, for all \(r > 0\) there is a constant \(N_r > 0\) such that for all \(x \in X\) we have \(|B(x,r)| \leq N_r\).

For some statements, it is easier to use the terminology of a dense set.

Definition 2.2.6. A subset of a metric space \(A \subseteq (X,d_X)\) is called \(C\)-dense for some \(C > 0\) if for all \(x \in X\), there exists \(a \in A\) such that \(d_X(x,a) \leq C\). We that \(A\) is dense in \(X\) if there exists a \(C > 0\) such that \(A\) is \(C\)-dense in \(X\).

We also use a uniform notion of closeness of functions.

Definition 2.2.7. Let \(f,g: (X,d_X) \to (Y,d_Y)\) be maps between metric spaces. Given \(r > 0\), we say \(f\) is within \(r\) of \(g\) if for all \(x \in X\) we have \(d_Y(f(x),g(x)) \leq r\). If such an \(r\) exists, we say \(f\) and \(g\) are within bounded distance.

Given a connected graph \(\Gamma = (V,E)\), recall we form a metric space by viewing all edges as intervals of length 1. We refer to this metric as the graph metric.

For one of our applications we consider metric spaces which are rooted trees with some additional properties.

Definition 2.2.8. A rooted graph is a graph with a distinguished vertex \(v_0\). If the graph is also a tree, we call this a rooted tree.

Definition 2.2.9. A rooted graph \(T\) is visual if there is a constant \(C > 0\) such that for every vertex \(v \in V\), there is an infinite geodesic ray \(I\) with endpoint \(v_0\) such that \(d(v,I) \leq C\). In the language of [MPR], this means \(v_0\) is a pole of \(T\).

In [MPR] the combinatorial Cheeger isoperimetric constant \(h(\Gamma)\) of a connected graph \(\Gamma\) is defined. This constant quantifies the existence of an isoperimetric inequality in \(\Gamma\).
**Definition 2.2.10.** Given a connected graph $\Gamma = (V,E)$, we define the combinatorial Cheeger isoperimetric constant of $\Gamma$ as $h(\Gamma) = \inf_A |\partial A|/|A|$ where $A$ ranges over nonempty finite subsets of $V$.

**Remark 2.2.11.** The graph $\Gamma$ satisfies a linear isoperimetric inequality if and only if $h(\Gamma) > 0$.

We also wish to impose a condition on trees that forces regular branching. In [MPR] there is such a condition which they call pseudo-regularity. To fully define this we need some notation, also borrowed from [MPR]. Given a tree $T$ and points $x,y \in T$, we let $[xy]$ denote the (unique) geodesic connecting $x$ to $y$. For a fixed point $v \in T$ and any point $x \in T$, we define

$$T^v_x = \{y \in T : x \in [vy]\}.$$

In the following, $S(v_0,t)$ is the sphere of radius $t$ centered at $v_0$ in the graph metric.

**Definition 2.2.12.** Given a rooted, visual tree $(T,v_0)$ and $K > 0$ we say $(T,v_0)$ is $K$-pseudo-regular if for every $t \in \mathbb{N}$ and every $a \in S(v_0,t)$, there exist at least two points in $S(v_0,t+K) \cap T^v_a$. We say $(T,v_0)$ is pseudo-regular if it is $K$-pseudo-regular for some $K$.

**Remark 2.2.13.** Even though pseudo-regular trees are rooted and visual by definition, we will often refer to rooted, pseudo-regular, visual trees.

We will use the end space definition from [MPR] of the boundary at infinity of a tree. For this, let $(T,v_0)$ be a rooted tree.

**Definition 2.2.14.** The end space of the rooted tree $(T,v_0)$ is

$$\text{end}(T,v_0) = \{F : [0,\infty) \to T : F(0) = v_0 \text{ and } F \text{ is an isometric embedding}\}.$$

We define the Gromov product at infinity of two elements $F,F' \in \text{end}(T,v_0)$ as

$$(F|F')_{v_0} = \sup\{t \geq 0 : F(t) = F'(t)\}.$$

We then define a metric $d = d_{v_0}$ on $\text{end}(T,v_0)$ by $d(F,F') = e^{-\left((F|F')_{v_0}\right)}$. Here $d$ is actually an ultrametric, meaning if $F,G,H \in \text{end}(T,v_0)$, then $d(F,H) \leq \max(d(F,G),d(G,H))$. We often write $\partial_\infty T$ for $\text{end}(T,v_0)$. 

22
In [DS], there is a characterization of metric spaces that are quasisymmetrically equivalent to the standard $1/3$ Cantor set, denoted here as $C_{1/3}$.

**Theorem 2.2.15** ([DS], Theorem 15.11). A compact metric space $(X,d)$ is quasisymmetrically equivalent to $C_{1/3}$ if it is bounded, complete, doubling, uniformly perfect, and uniformly disconnected.

Ultrametric spaces are uniformly disconnected by [DS, Proposition 15.7], so we do not define this term here.

We note complete, bounded, doubling metric spaces are compact as bounded, doubling metric spaces are totally bounded.

The metric spaces we are primarily concerned with are rooted, pseudo-regular, visual trees of bounded degree. We now prove the end spaces of such trees satisfy the criteria in Theorem 2.2.15.

**Lemma 2.2.16.** Let $(T,v_0)$ be a rooted, pseudo-regular, visual tree with bounded degree. Then, $\partial_\infty T$ is quasisymmetrically equivalent to $C_{1/3}$.

**Proof.** Using [DS, Theorem 15.11] it suffices to show $\partial_\infty T$ is bounded, complete, doubling, uniformly perfect, and uniformly disconnected.

From [MPR, Proposition 3.3] $\partial_\infty T$ is a complete, bounded ultrametric space. Thus, $\partial_\infty T$ is also uniformly disconnected by [DS, Proposition 15.7].

The fact that $\partial_\infty T$ is uniformly perfect follows from the fact that $(T,v_0)$ is pseudo-regular. This is proven in [MPR, Proposition 3.20].

We show $\partial_\infty T$ is doubling, which will follow from our bounded degree assumption. Let $B(F,r)$ be a ball in $\partial_\infty T$. Let

$$M = \sup\{m : F(k) = G(k) \text{ for all } G \in B(F,r), k \leq m\}.$$ 

Then, $r \geq e^{-\left(M+1\right)}$ and so $r/2 \geq e^{-\left(M+2\right)}$. As $T$ has bounded degree, say $\deg(v) \leq \mu$ for $v \in V$, it follows that $B(F,r)$ is contained in $\mu^2$ balls of radius $r/2$. 

23
The quasisymmetries induced on the end spaces of these trees give rises to quasi-isometries
between the trees themselves as our lemma will show. The proof strategy is to show maximal
godesically complete subtrees are quasi-isometric first, and that these subtrees are quasi-
isometric to the original trees. We follow the terminology in [MPR].

**Definition 2.2.17.** A rooted tree \((T, v_0)\) is geodesically complete if whenever \(f: [0, t] \rightarrow T\)
is an isometric embedding with \(f(0) = v_0\), there is an isometric embedding \(F: [0, \infty) \rightarrow T\)such that \(f(s) = F(s)\) for all \(s \in [0, t]\).

**Definition 2.2.18.** Given a rooted tree \((T, v_0)\), we define \((T_\infty, v_0)\) as the unique geodesically
complete subtree with the same root \(v_0\) that is maximal under inclusion.

The fact that such a tree exists and is unique follows from an application of Zorn’s
Lemma, see [MPM, Theorem 10.1].

**Remark 2.2.19.** We note that if \((T, v_0)\) is visual then from [MPR, Proposition 3.8] it follows
that there is a quasi-isometry \((T, v_0) \rightarrow (T_\infty, v_0)\).

**Lemma 2.2.20.** Let \((T, v_0)\) and \((U, w_0)\) be rooted, pseudo-regular, visual trees with bounded
degree. Then \(T\) and \(U\) are quasi-isometric.

It is important that the trees we work with are visual so as to apply Remark 2.2.19. The
construction of a quasi-isometry between \(T_\infty\) and \(U_\infty\) is given by [BuS, Theorem 7.2.1], we
provide the main idea of the construction here.

*Sketch of proof of Lemma 2.2.20.* Let \(\varphi: \partial_\infty T \rightarrow \partial_\infty U\) be a quasisymmetry. This exists by
Lemma 2.2.16. From Remark 2.2.19 it suffices to construct a quasi-isometry \(f\) between
\((T_\infty, v_0)\) and \((U_\infty, w_0)\). For \(v \in T_\infty\) set

\[B_v = \{ F \in \partial_\infty T_\infty : v \in F([0, \infty))\}\]

and likewise define \(B_w\) for \(w \in U_\infty\). Define \(f(v) = w\) where \(w \in U_\infty\) is a vertex of maximal
distance from \(w_0\) such that \(\varphi(B_v) \subseteq B_w\) (such a vertex exists as \(B_{w_0} = \partial_\infty U_\infty\)). We then show
there is a constant \(C > 0\) such that if \(v, v' \in T_\infty\) with \(|v - v'| \leq 1\), then \(|f(v) - f(v')| \leq C\).
From our tree structure we may assume without loss of generality that \( B_v' \subseteq B_v \). Let \( w = f(v) \) and \( w' = f(v') \). We conclude that \( \varphi(B_v) \cap B_w \neq \emptyset \) and, by using a common point and the quasisymmetry condition, that there is a uniform bound on \( |w - w'| \). Constructing \( g: (U_\infty, w_0) \to (T_\infty, v_0) \) similarly and checking that \( f \) and \( g \) are coarse inverses of one another completes the proof.

We will also work with some homological terminology as in [Wh]. In what follows we define what is needed, sometimes adapting definitions to our more specific setting.

Given a graph \( \Gamma = (V, E) \) with bounded degree, we define a 0-chain \( c \) as a formal sum \( c = \sum_{v \in V} c_v v \) with \( c_v \in \mathbb{Z} \). Likewise, a 1-chain \( b \) is a formal sum \( b = \sum_{e \in E} b_e e \) with \( b_e \in \mathbb{Z} \). We call a chain bounded if its coefficients are bounded. Let \( C^b_0 = C^b_0(\Gamma) \) denote the set of bounded 0-chains and \( C^b_1 = C^b_1(\Gamma) \) the set of bounded 1-chains. Given an orientation on \( E \), meaning we view each edge as an ordered pair \( e = (e_+, e_-) \), we define the boundary map \( \partial: C^b_1 \to C^b_0 \) by defining \( \partial e = e_+ - e_- \) and extending linearly.

We define an equivalence relation on \( C^b_0 \) by setting \( c \sim c' \) if and only if there exists \( b \in C^b_1 \) such that \( \partial b = c - c' \). We let \( [c] \) denote the equivalence class of \( c \) under this relation. The fundamental class \( [V] \) is defined as the equivalence class of \( \sum_{v \in V} v \).

Given a graph \( \Gamma = (V, E) \) equipped with the graph metric and a constant \( r > 0 \), we can define another graph \( \Gamma_r = (V_r, E_r) \) where \( V_r = V \) and

\[
E_r = \{(x, y) : x, y \in V, d(x, y) \leq r\}.
\]

This is the 1-dimensional subcomplex of the \( r \)-Rips complex as defined in [Wh]. In [Wh] homology is defined as above on \( \Gamma_r \) for sufficiently large \( r \) and the sets of chains are denoted \( C^a_0 \) and \( C^a_1 \). To apply Whyte’s result, we need the following lemma.

**Lemma 2.2.21.** Let \( \Gamma = (V, E) \) be a connected graph with bounded degree. Let \( c \in C^a_0 \) be a boundary, so \( c = \partial b \) for some \( b \in C^a_1 \). Then, \( c \in C^b_0 \) is a boundary, so \( c = \partial b' \) for some \( b' \in C^b_1 \).

**Proof.** We fix \( R > 0 \) so that the Rips complex is given by \( \Gamma_R = (V_R, E_R) \). Then, any edge \( e \in E_R \) connects two vertices \( x, y \) with \( d(x, y) \leq R \). As the metric on \( \Gamma_R \) is geodesic, this
means there are edges $e_1, \ldots, e_m \in E$ connecting $x$ and $y$ with $m \leq R$. This observation lets us define a map $\Phi : E_R \to E^{\leq R}$ (where $E^{\leq R}$ denotes the set of finite strings of edges from $E$ consisting of at most $R$ edges) such that for each $e \in E_R$, the sequence $\Phi(e)$ connects the boundary vertices of $e$ in $\Gamma$. The function $\Phi$ induces a map $C^u_{1} \to C^b_{1}$ by sending the chain consisting solely of an edge $e \in E_R$ to the chain consisting of the edges in $\Phi(e)$ and extending linearly. We see $\partial e = \partial \Phi(e)$ with this definition. As $\Gamma$ has bounded degree and $R < \infty$, it follows that any edge in $E$ occurs as an element of $\Phi(e)$ for a uniformly bounded number of $e \in E_R$. Hence, if $c = \partial b$ with $b \in C^u_{1}$, then $\Phi(b) \in C^b_{1}$ and $c = \partial \Phi(b)$. 

It is clear that if $c \in C^b_{0}$ is a boundary then $c \in C^u_{0}$ is a boundary as the $1$-Rips complex contains the graph $(V, E)$. Hence, in the results that follow on connected graphs we may freely think of the $C^b$ homology in place of the $C^u$ homology, even though [Wh] works with the $C^u$ homology.

**Results from Whyte and Martínez-Pérez and Rodríguez**

Here we state the results from [Wh] and [MPR] relevant for our setting. We indicate some ideas as to the proofs where applicable, but we leave most of the details to the papers. We first state a key result of [Wh].

**Theorem 2.2.22** ([Wh], Theorem 4.1). Let $f : X \to Y$ be a quasi-isometry between UDBG spaces with $f_*([X]) = [Y]$. Then, there is a bilipschitz map at bounded distance from $f$.

To apply this, we need a condition that implies $[f_*([X]) - Y] = 0$. This is achieved using an isoperimetric inequality and Theorem 2.2.24. To prove the direction we need from this theorem we will use Hall’s selection theorem. For convenience we state the theorem here as it appears in [Wh].

**Theorem 2.2.23** (Hall’s Selection Theorem). Let $\alpha : A \to \text{Fin}(B)$. Then, there is an injective map $\phi : A \to B$ such that $\phi(a) \in \alpha(a)$ for all $a \in A$ if and only if for all finite $S \subseteq A$ we have

$$|S| \leq |\bigcup_{s \in S} \alpha(s)|$$
Theorem 2.2.24 ([Wh], Theorem 7.6). Let \( Z \) be a UDBG space, \( c \in C^0_f(Z) \). Then, \([c] = 0\) if and only if there are \( r, C > 0 \) such that for all finite \( S \subseteq Z \) we have

\[ |\sum_S c| \leq C|\partial_r(S)|.\]

Proof of the direction we need (if). This proof is a more explicit version of what appears in [Wh]. We show that if such \( r, C \) exist, then \([c] = 0\). Let \( n \) such that \( n \geq C \) and such that for all \( z \in Z \) we have \( c_z + n \geq 1 \). Let \( S \subseteq Z \) be finite. We consider the chains \( c + n[Z] \) and \( n[Z] \). Since \( \sum_S c \leq C|\partial_r(S)| \), we have

\[ \sum_S (c + n[Z]) \leq \sum_{N_r(S)} n[Z] \tag{2.2} \]

where \( N_r(S) = \{ z \in Z : d(z, S) < r \} \) and \( \partial_r(S) = N_r(S) \setminus S \). We apply Hall’s selection theorem to the spaces

\[ A = \bigcup_{z \in Z} (z \times \{1, \ldots, c_z + n\}) \]

and

\[ B = \bigcup_{z \in Z} (z \times \{1, \ldots, n\}) \]

where \( \alpha((z, k)) = \bigcup_{w \in N_r(z)} (w \times \{1, \ldots, n\}) \). Then inequality (2.2) yields the relevant condition in Hall’s selection theorem so there is an injective map \( \phi: A \to B \). For each \( a \in A \) with \( a \neq \phi(a) \), we connect the \( z\)-component of \( a \) with that of its image.

We create an injection the other way in a similar manner. For this, we let \( m = \min(c_z) \) and require \( n \geq C + |m| \). Start with \( \sum_S (-c) \leq C|\partial_r(S)| \) which gives

\[ \sum_S (-c) \leq \sum_{\partial_r(S)} c + n[Z] \]

and so

\[ \sum_S n[Z] \leq \sum_{N_r(S)} (c + n[Z]) \]

which lets us run the Hall’s Selection Theorem argument in this direction. Thus, if \( n \) is large enough, we have injections in both directions between \( A \) and \( B \) with \( Z \) projections within bounded distance of the identity. This gives us a bijection between the sets with this
property. Connecting the $z$-coordinates of the images of points with the $z$-coordinates of the points themselves forms an element $b \in C^1_\text{uf}(Z)$ as $Z$ has bounded geometry and the $z$-coordinate of any point $a$ and its image differs by at most $r$. We see $c = \partial b$, so $[c] = 0$. □

The main result from [MPR] that concerns us is the following.

**Theorem 2.2.25** ([MPR], Theorem 4.15). Let $\Gamma$ be a (Gromov) hyperbolic, rooted, visual graph of bounded degree. Then, $h(\Gamma) > 0$ if and only if $\partial_\infty \Gamma$ is uniformly perfect for some visual metric.

Here $\partial_\infty \Gamma$ refers to the Gromov boundary of $\Gamma$ with a visual metric. The proof, while not overly complicated, is notationally involved and will be omitted. We instead summarize the main ideas. This summary will use results from [MPR]; the exact theorems and lemmas used in their proof can be found in their paper.

**Summary of proof of Theorem 2.2.25.** It suffices to prove the result for a hyperbolic approximation $\Gamma'$ in place of $\Gamma$. The boundary at infinity, $\partial_\infty \Gamma = \partial_\infty \Gamma'$ has strongly bounded geometry. From this and the uniformly perfect condition, one studies the combinatorics of $\Gamma'$. By using a refinement of $\Gamma'$, one passes to a hyperbolic approximation $\Gamma''$ for which the map $f: V_{\Gamma''} \to \mathbb{R}$ defined by $f(v) = k$ for all $v \in V_k$ satisfies $|\nabla_{xy} f| \leq c_1$ and $\Delta f(x) \geq c_2 > 0$ for some $c_1, c_2 > 0$. This lets us conclude that $h(\Gamma'') > 0$ and so $h(\Gamma) > 0$. □

**Proofs of results**

We now combine the results above to prove the main theorem. In what follows $X$ and $Y$ are assumed to be quasi-isometric so one of these spaces satisfies a linear isoperimetric inequality if and only if the other does.

**Proof of Theorem 2.2.1.** Let $f: V_X \to V_Y$ be a quasi-isometry. By assumption $Y$ supports a linear isoperimetric inequality with constant $h = h(Y)^{-1} > 0$. That is, for any finite set $S \subseteq V_Y$, we have $|S| \leq h|\partial S|$. As $f$ is a quasi-isometry and $X$ has bounded degree, there
is an $A > 0$ such that for each vertex $v \in V_Y$, we have $|(f_*([X]) - [Y])(v)| \leq A$. Hence, for $S \subseteq V_Y$ finite,

$$|\sum_S (f_*([X]) - [Y])| \leq A|S| \leq Ah|\partial S|$$

and so $[f_*([X]) - [Y]] = 0$ by Theorem 2.2.24. Thus, by Theorem 2.2.22, $f$ is within bounded distance of a bilipschitz equivalence.

We now prove the corollaries of Theorem 2.2.1 and discuss the conditions in Corollary 2.2.2.

**Proof of Corollary 2.2.2.** By Lemma 2.2.20 there exists a quasi-isometry $f : V_X \to V_Y$. By \[\text{MPR, Proposition 3.20}\] $\partial_\infty Y$ is uniformly perfect. As trees are hyperbolic, we may apply Theorem 2.2.25 to conclude $Y$ supports a linear isoperimetric inequality. Thus, the result follows from Theorem 2.2.1.

We examine the conditions in Corollary 2.2.2, namely that $X$ and $Y$ are pseudo-regular, visual trees of bounded degree. Recall the fact that our trees were visual was important for Lemma 2.2.20. There is a similar condition in \[\text{MPR, Theorem 3.16}\] which must be satisfied for our tree to even support a linear isoperimetric inequality. Pseudo-regularity guarantees that our trees branch regularly; without this condition, one tree, say $X$, will have arbitrarily long segments with no branching. If this is not the case for $Y$, then the size of the number of vertices in balls of radius $R$ in $Y$ grow exponentially in $R$. If a bilipschitz map $g$ existed between vertices with bilipschitz constant $C > 0$, then one could consider a non-branching segment of length $M$ in $X$. Letting $x$ denote its midpoint vertex, the ball $B(g(x), R)$ has at least $c^R$ vertices for some $c > 1$ that depends only on $Y$. Assuming $M > 3CR$, there are at most $2CR$ vertices on our non-branching segment that could be the image of these $c^R$ vertices. As $2CR/c^R < 1$ for large $R$, it follows that $g$ cannot exist. Indeed, similar reasoning shows that such trees cannot even be quasi-isometric.

The bounded degree condition is similar; if the degree of $Y$ is at most $\mu < \infty$, then the size of any ball $B(y, R)$ is bounded by $c\mu^{R+1}$ for some $c > 0$. Hence, if $X$ had unbounded degree
and a bilipschitz equivalence $g$ existed, we could arrive at a contradiction by considering $g(B(x_n, 1))$ for a sequence of vertices $x_n$ with increasing degree in $X$.

\textit{Proof of Corollary 2.2.3} From Lemma 2.1.6 we know that $V_X$ and $V_Y$ are quasi-isometric (the identity in this case provides the relevant quasisymmetry). Theorem 2.2.25 shows $Y$ satisfies a linear isoperimetric inequality, so Theorem 2.2.1 applies. \qed
CHAPTER 3

Weak capacity

3.1 Comparability and quasisymmetric invariance

Now we prove the main results involving weak capacity: Theorems 1.1.2, 1.1.3, and 1.1.4. Recall we consider compact, connected, Ahlfors $Q$-regular metric measure spaces $(Z,d,\mu)$. After the proofs of the main theorems we prove that for open $A,B \subseteq Z$ with $\text{dist}(A,B) > 0$ we always have $\text{wcap}_p(A,B) > 0$. We prove Theorem 1.1.2 for disjoint continua first. For this we need a technical lemma.

Lemma 3.1.1. Let $p \geq 1$, let $f \in \ell^p(V_X)$, and let $A \subseteq Z$ be a continuum. Let $\{B_{v_n}\}$ be a sequence of balls corresponding to vertices $v_n$ with $\ell(v_n) \to \infty$. Suppose that there is a constant $c > 0$ such that for all large enough $n$ we have

$$\mathcal{H}^1_{\infty}(A \cap B_{v_n}) \geq cs^{-\ell(v_n)}.$$

Then for each $\delta > 0$ there is an $N$ such that for all $n \geq N$, there is a vertex path $\sigma_n \subseteq X$ that starts from $v_n$ and has limit in $A$ which satisfies $\sum_{\sigma_n} f(v) \leq \delta$.

Proof. We note if $f$ has finite support then the result is immediate. Hence, we assume without loss of generality that $f$ does not have finite support. We also may assume all balls in consideration have radius bounded above by 4 as $\text{diam}(Z) \leq 1$. For $\epsilon > 0$ and fixed $n$ we define

$$K_n = K_n(\epsilon) = \{z \in A: \exists v \in V_X \text{ with } z \in B_v, f(v)^p \geq \epsilon r(B_v) \text{ and } \ell(v) \geq \ell(v_n)\}.$$

We then may cover $K_n$ by $\{B_{v_z}\}_{z \in K_n}$ where $f(v_z)^p \geq \epsilon r(v_z)$. As the balls corresponding to these vertices have uniformly bounded diameter there is a countable subcollection of vertices
$G_n$ such that
\[ \bigcup_{z \in K_n} B_{v_z} \subseteq \bigcup_{v \in G_n} 5B_v \]
and such that if $v, w \in G_n$ are distinct, then $B_v \cap B_w = \emptyset$ (see [He, Theorem 1.2]). Hence, $K_n \subseteq \bigcup_{v \in G_n} 5B_v$. Thus, we have
\[ H^1_\infty (K_n) \leq \sum_{G_n} 10 r(B_v) \leq 10 \sum_{G_n} \frac{f(v)^p}{\epsilon} \leq \frac{10}{\epsilon} \| f(\ell(v_n)) \|_p \]
where $f_j$ is $f$ restricted to vertices of level $j$ and higher.

In the above we use $\epsilon = \epsilon_n$ defined by
\[ \frac{10}{\epsilon_n} \| f(\ell(v_n)) \|_p = \frac{1}{2} c s^{-\ell(v_n)} \]
which is possible as $\| f(\ell(v_n)) \|_p > 0$ for all $n$ as $f$ does not have finite support. We conclude that for large enough $n$ we have
\[ H^1_\infty (K_n) \leq \frac{1}{2} H^1_\infty (A \cap B_{v_n}) \]
Hence, there is a point $z \in (A \cap B_{v_n}) \setminus K_n$.

For this $z$ we form a vertex path $\sigma$ from $v_n$ with limit $z$ by choosing a vertex $w_k$ for each $k > \ell(v_n)$ such that $z \in B_{w_k}$. As $z \notin K_n$, we estimate the sum
\[ \sum_{\sigma} f(v) \leq (\epsilon_n^{1/p}) \sum_{k=\ell(v_n)}^\infty (2s^{-k})^{1/p} \]
\[ = (2\epsilon_n)^{1/p} \sum_{k=\ell(v_n)}^\infty (s^{1/p})^{-k} \]
\[ \lesssim (2\epsilon_n)^{1/p} s^{-\ell(v_n)/p} \]
\[ \lesssim (\epsilon_n s^{-\ell(v_n)})^{1/p} \]
From our choice of $\epsilon_n$ we have
\[ \| f(\ell(v_n)) \|_p \simeq \epsilon_n s^{-\ell(v_n)} \]
and as $\| f(\ell(v_n)) \|_p \to 0$ as $n \to \infty$, the result holds. \qed
Proof of Theorem 1.1.2. We begin by showing the result for disjoint continua \( A \) and \( B \). Let \( \tau : E_X \to [0, \infty] \) be an admissible function for \( \text{wcap}_Q(A, B) \). Define \( f : V_X \to \mathbb{R} \) by

\[
 f(v) = \sum_{e \sim v} \tau(e) \quad \text{where the sum is over all edges with } v \text{ as an endpoint.}
\]

We note \( \| f \|_{Q, \infty} \simeq \| \tau \|_{Q, \infty} \) follows from Lemma 1.2.1. Indeed, in this case our set \( J \subseteq V_X \times E_X \) consists of vertex-edge pairs \((v, e)\) with \( e \) having \( v \) as a boundary vertex where \( s_v = f(v) \) and \( t_e = \tau(e) \). As \( X \) has bounded degree, Lemma 1.2.1 applies to give us one direction of comparability. By using edge-vertex pairs \((e, v)\) and adjusting the definitions of the \( s_v \) and \( t_e \) we also get the other bound. We also note that if \( \| f \|_{Q, \infty} < \infty \) this means \( f \in \ell^p \) whenever \( p > Q \).

Consider the functions \( u_n : Z \to \mathbb{R} \) given by

\[
 u_n = \sum_{v \in V_n} \frac{f(v)}{r(B_v)} \chi_{2B_v}, \tag{3.1}
\]

where we recall \( V_n \) are the vertices on level \( n \). We claim \( 2u_n \) is admissible for \( Q \)-modulus between \( A \) and \( B \) for large enough \( n \). Suppose this fails for some sequence \( n_i \). Then there is a rectifiable path \( \gamma_{n_i} \) connecting \( A \) and \( B \) with \( \int_{\gamma_{n_i}} u_{n_i} \leq \frac{1}{2} \). The endpoints of \( \gamma_{n_i} \) lie in balls \( \frac{1}{2} B_{v_{n_i}}^A \) and \( \frac{1}{2} B_{v_{n_i}}^B \) corresponding to vertices of level \( n_i \). We now apply Lemma 3.1.1 with \( \delta = \frac{1}{16} \) to obtain, for all large enough \( n_i \), paths \( \sigma_{n_i}^A \) and \( \sigma_{n_i}^B \) with

\[
 \sum_{\sigma_{n_i}^A} f(v) < \frac{1}{16} \quad \text{and} \quad \sum_{\sigma_{n_i}^B} f(v) < \frac{1}{16}.
\]

From the definition of \( f \) it is clear that summing along the edges of these paths gives a \( \tau \)-length of less than \( \frac{1}{16} \) as well. We note the hypothesis in Lemma 3.1.1 follows once \( s^{-n_i} < \min(\text{diam}(A), \text{diam}(B)) \).

We show this violates the admissibility of \( \tau \). Indeed, if \( v_0, \ldots, v_m \) is a path of vertices in \( V_{n_i} \) where \( \gamma_{n_i} \) passes through each \( \frac{1}{2} B_{v_k} \) and \( \ell(\gamma_{n_i}) \geq 8s^{-n_i} \), then

\[
 \int_{\gamma_{n_i}} u_{n_i} \geq \sum_{j=0}^{m} \frac{r(B_{v_j}) f(v_j)}{2r(B_{v_j})} = \sum_{j=0}^{m} \frac{1}{2} f(v_j) \geq \sum_{j=1}^{m} \tau(e_j),
\]

33
where $e_j$ denotes the edge connecting $v_{j-1}$ to $v_j$. Choosing such a path with $v_0$ and $v_m$ corresponding to $B_{v_A}$ and $B_{v_B}$ and combining this path with $\sigma^A_{n_i}$ and $\sigma^B_{n_i}$ yields a path in the graph with non-tangential boundary limits in $A$ and $B$ with $\sum \tau(e) < 1$, contradicting the admissibility of $\tau$.

We have just shown that for large enough $n$ the function $2u_n$ is admissible for $\text{mod}_Q(A, B)$. It remains to compute $\|u_n\|^Q_Q$. We have

$$\|u_n\|^Q_Q = \int_Z \left( \sum_{v \in V_n} \frac{f(v)}{r(B_v) \chi_{2B_v}} \right)^Q$$

$$\lesssim \int_Z \left( \sum_{v \in V_n} \frac{f(v)^Q}{r(B_v)^Q \chi_{2B_v}} \right)^Q$$

$$\lesssim \sum_{v \in V_n} f(v)^Q$$

(3.2)

where we have used the bounded degree of our hyperbolic filling (Lemma 2.1.5) for the first inequality and Ahlfors $Q$-regularity for the second. From [BnS, Proof of Theorem 1.4], for any $N > 1$ there is an $n \in [N, 2N]$ such that

$$\sum_{v \in V_n} f(v)^Q \lesssim \|f\|^Q_{Q, \infty}.$$ 

Thus, for specific large enough $n$, we see that $u_n$ is admissible for the modulus between $A$ and $B$ and satisfies $\|u_n\|^Q_Q \lesssim \|\tau\|^Q_{Q, \infty}$. Infimizing over admissible $\tau$ yields the result for continua.

Now, for open sets $A$ and $B$ we use the same technique, but the setup is more involved: we need to work safely inside the open sets to be able to satisfy the hypothesis of Lemma 3.1.1. For $\lambda > 0$ define $A_\lambda = \{ a \in A : d(a, Z \setminus A) > \lambda \}$ and define $B_\lambda$ similarly. Fix $\lambda > 0$ such that $A_\lambda$ and $B_\lambda$ are nonempty (all small enough $\lambda$ will satisfy this). We claim for such $\lambda$ we have $\text{mod}_Q(A_\lambda, B_\lambda) \lesssim \text{wcap}_Q(A, B)$ with an implicit constant independent of $\lambda$. Indeed, as above we claim $2u_n$ is admissible for $\text{mod}_Q(A_\lambda, B_\lambda)$ for large enough $n$, where we recall $u_n$ is defined in equation (3.1).

If $2u_n$ is not admissible, then we may find a path $\gamma_n$ on level $n$ with $\int_{\gamma_n} u_n \leq \frac{1}{2}$. We note for large enough $n$, the endpoints $v_A^n$ and $v_B^n$ of $\gamma_n$ satisfy $B_{v_A^n} \subseteq A$ and $B_{v_B^n} \subseteq B$. We then apply the above procedure to create a short $\tau$-path connecting $A$ and $B$ which contradicts
the admissibility of \( \tau \). From the norm computation above, by infimizing over admissible \( \tau \) we conclude \( \text{mod}_Q(A_\lambda, B_\lambda) \lesssim w\text{cap}_Q(A, B) \) with an implicit constant independent of \( \lambda \).

We show now that this implies \( \text{mod}_Q(A, B) \lesssim w\text{cap}_Q(A, B) \). Let \( \lambda_n = \frac{1}{n} \) and, for each \( n \), let \( \sigma_n \geq 0 \) be an admissible function for \( \text{mod}_Q(A_{\lambda_n}, B_{\lambda_n}) \) such that \( \|\sigma_n\|_Q^Q \lesssim w\text{cap}_Q(A, B) \).

By Mazur’s Lemma [HKST, p. 19], there exist convex combinations \( \rho_n \) of \( \sigma_k \) with \( k \geq n \) and a limit function \( \rho \) such that \( \rho_n \to \rho \) in \( L_Q \). By Fuglede’s Lemma [HKST, p. 131], after passing to a subsequence of the \( \rho_n \), we may assume that for all paths \( \gamma \) except in a family \( \Gamma_0 \) of \( Q \)-modulus 0, \( \int_\gamma \rho_n \to \int_\gamma \rho \). As the \( Q \)-modulus of \( \Gamma_0 \) is 0, there exists a function \( \sigma \geq 0 \) with \( \int_\gamma \sigma = \infty \) for all \( \gamma \in \Gamma_0 \) and \( \int_Z \sigma^Q < \infty \).

We show \( \rho + c\sigma \) is admissible for \( \text{mod}_Q(A, B) \) for any \( c > 0 \). Let \( \gamma \) be a path connecting \( A \) and \( B \). If \( \gamma \in \Gamma_0 \), then \( \int_\gamma c\sigma = \infty \), so suppose \( \gamma \notin \Gamma_0 \). Then, as \( A \) and \( B \) are open, \( \gamma \) connects \( A_{\lambda_n} \) and \( B_{\lambda_n} \) for some \( n \). As \( A_{\lambda_n} \subseteq A_{\lambda_k} \) for \( n \leq k \), and likewise for \( B \), we see \( \int_\gamma \sigma_k \geq 1 \) whenever \( n \leq k \). Hence, \( \int_\gamma \rho_m \geq 1 \) for all \( m \geq n \), and so \( \int_\gamma \rho \geq 1 \). Thus, \( \rho + c\sigma \) is admissible for \( \text{mod}_Q(A, B) \). Now,

\[
\|\rho + c\sigma\|_Q^Q \lesssim \|\rho\|_Q^Q + \|c\sigma\|_Q^Q \lesssim w\text{cap}_Q(A, B) + \|c\sigma\|_Q^Q
\]

and as \( c \) may be taken arbitrarily small we have \( \text{mod}_Q(A, B) \lesssim w\text{cap}_Q(A, B) \).

**Proof of Theorem 1.1.3.** Recall that for this direction, in addition to working in a compact, connected Ahlfors \( Q \)-regular metric measure space \( (Z, d, \mu) \) we also assume that \( Z \) is a \( Q \)-Loewner space. We first assume \( A \) and \( B \) are open sets with \( \text{dist}(A, B) > 0 \). For the continua case the proof will be the same except for one detail. This is noted in the following proof and the required modifications will follow from Lemma 3.1.3.

If \( \text{mod}_Q(A, B) = \infty \) there is nothing to prove so suppose \( \text{mod}_Q(A, B) < \infty \). Let \( \rho : Z \to [0, \infty) \) be \( Q \)-integrable and admissible for modulus. Define \( v : Z \to \mathbb{R} \) by

\[
v(z) = \inf \left\{ \int_\gamma \rho : \gamma \in \Gamma_z \right\}
\]

where \( \Gamma_z \) is the set of all rectifiable paths with one endpoint in \( A \) and the other endpoint equal to \( z \). It is clear that \( \rho \) is an upper gradient for \( v \); that is, given \( y, z \in Z \), we have

\[\|\rho + c\sigma\|_Q^Q \lesssim w\text{cap}_Q(A, B) + \|c\sigma\|_Q^Q \]

and as \( c \) may be taken arbitrarily small we have \( \text{mod}_Q(A, B) \lesssim w\text{cap}_Q(A, B) \).
\(|v(y) - v(z)| \leq \int_\gamma \rho\) whenever \(\gamma\) is a rectifiable path connecting \(y\) and \(z\). Hence, as \(\rho\) is \(Q\)-integrable, it follows from [HKST, Theorem 9.3.4] that \(v\) is measurable. Set \(u = \min \{v, 1\}\). We see \(\rho\) is an upper gradient for \(u\) as well. Clearly \(u(a) = 0\) whenever \(a \in A\) and, as \(\rho\) is admissible for \(Q\)-modulus, we see \(u(b) = 1\) whenever \(b \in B\).

By [HK, Theorem 5.12], we note that \(Z\) supports a \(Q\)-Poincaré inequality for continuous functions. This is equivalent to a \(Q\)-Poincaré inequality for locally integrable functions by [HKST, Theorem 8.4.1]. Following [HKST, Theorem 12.3.9], first seen in [KZ], this promotes to a \(p\)-Poincaré inequality for functions which are integrable on balls for \(p\) slightly smaller than \(Q\).

Now, consider the function \(\tau: E_X \rightarrow \mathbb{R}\) by

\[
\tau(e) = r(B_{e+}) \left( \int_{KB_{e+}} \rho^p \right)^{1/p} + r(B_{e-}) \left( \int_{KB_{e-}} \rho^p \right)^{1/p}
\]

with \(K\) to be chosen and \(B_{e+}, B_{e-}\) the balls representing the vertices of \(e\). As before, \(KB_{e+}\) denotes the ball with the same center as \(B_{e+}\) and with radius \(Kr(B_{e+})\). We claim with appropriate \(K\) that \(\tau\) is admissible. We note that for intersecting balls \(B'\) and \(B''\) with a constant \(k \geq 1\) such that \(B' \subseteq k B''\) and \(B'' \subseteq k B'\), we have

\[
|u_{B''} - u_{k B'}| = \left| \int_{B''} u - \frac{1}{|B''|} \int_{B''} u_{k B'} \right| \\
\leq \frac{1}{|B''|} \int_{B''} |u - u_{k B'}| \\
\leq \frac{1}{|B''|} \int_{k B'} |u - u_{k B'}| \\
= \frac{|k B'|}{|B''|} \int_{k B'} |u - u_{k B'}| \\
\lesssim \int_{k B'} |u - u_{k B'}|
\]

and similarly \(|u_{B'} - u_{k B''}| \lesssim \int_{k B'} |u_{k B''} - u|\). Hence, by the triangle inequality,

\[
|u_{B'} - u_{B''}| \lesssim \int_{k B'} |u - u_{k B'}|
\]

and so

\[
|u_{B'} - u_{B''}| \lesssim \int_{k B'} |u - u_{k B'}| + \int_{k B''} |u - u_{k B''}|.
\]
In our graph, there is a uniform $k$ such that the above holds whenever $B'$ and $B''$ are vertices for a given edge. Thus, using the above notation where $e$ is an edge with $e_+$ and $e_-$ as the vertices of $e$, we have

$$
|u_{B_{e_+}} - u_{B_{e_-}}| \lesssim \int_{K_{B_{e_+}}} |u - u_{kB_{e_+}}| + \int_{K_{B_{e_-}}} |u - u_{kB_{e_-}}| \\
\lesssim r(B_{e_+}) \left( \int_{K_{B_{e_+}}} \rho^p \right)^{1/p} + r(B_{e_-}) \left( \int_{K_{B_{e_-}}} \rho^p \right)^{1/p} \\
= \tau(e),
$$

where $K = \lambda k$ arises from the Poincaré inequality (inequality (1.4)). Thus, if $\gamma$ is a path in the hyperbolic filling with limits in $A$ and $B$ we have, summing over the edges $e$ in $\gamma$,

$$\sum_{\gamma} |u_{B_{e_+}} - u_{B_{e_-}}| \lesssim \sum_{\gamma} \tau(e). \quad (3.4)
$$

Now, $\gamma$ has boundary limits in $A$ and $B$. We recall that if a sequence of vertices $B_n$ along a path approaches a limit $z \in Z = \partial_\infty X$, then the centers $p_n$ of the $B_n$ satisfy $p_n \to z$. Thus, as $A$ and $B$ are open, for edges sufficiently close to $A$ we have $u_{B_{e_+}} = 0$ and for edges sufficiently close to $B$ we have $u_{B_{e_+}} = 1$. We remark here that this fact is not true in the case that $A$ and $B$ are disjoint continua; this is where we will use Lemma 3.1.3. Hence, $1 \lesssim \sum_{\gamma} \tau(e)$ with constant independent of $\gamma$. Thus, for suitable $c$ depending only on the constants $\lambda$ and $C$ from the Poincaré inequality and $k$ above, $c\tau$ is admissible.

It remains to compute $\|\tau\|_{Q,\infty}$. We follow a proof similar to [BnS, Proposition 5.3]. We estimate the number of edges $e$ with an endpoint labeled as $v$ that belong to

$$V(\alpha) = \{v \in V_X : r(B_v) \left( \int_{K_{B_v}} \rho^p \right)^{1/p} > \frac{\alpha}{2} \}$$

where $\alpha > 0$. We will bound $\|\tau\|_{Q,\infty}$ by bounding $\#V(\alpha)$. Now,

$$\#V(\alpha) = \int_Z \left( \sum_{v \in V(\alpha)} \frac{1}{|B_v|} \chi_{B_v} \right)$$

and, by the geometric structure of our graph, we have, for $z \in Z$,

$$\sum_{v \in V(\alpha)} \frac{1}{|B_v|} \chi_{B_v}(z) \lesssim \frac{1}{|B_z|}$$
where $B_z$ is the ball in $V(\alpha)$ of smallest radius that contains $z$ (such a ball exists for almost every $z$ as $\rho \in L^p$).

We note that for $v \in V(\alpha)$ and $z \in KB_v$, we have

$$\left( \frac{1}{|KB_v|} \int_{KB_v} \rho^p \right)^{1/p} \leq M(\rho^p)(z)^{1/p}$$

where $M$ denotes the (uncentered) Hardy-Littlewood maximal function (see for instance [He, Chapter 2]). Thus,

$$r(B_v) M(\rho^p)^{1/p}(z) > \frac{\alpha}{2}$$

for all $z \in KB_v$. For such $v$, we see

$$\frac{1}{r(B_v)^Q} \leq \frac{2^Q M(\rho^p)(z)^{Q/p}}{\alpha^Q}.$$  \hfill (3.5)

Hence,

$$\#V(\alpha) \lesssim \int_{Z} \left( \sum_{v \in V(\alpha)} \frac{1}{|B_v|} \chi_{B_v} \right)$$

$$\lesssim \int_{Z} \frac{1}{|B_z|}$$

$$\lesssim \int_{Z} \frac{M(\rho^p)(z)^{Q/p}}{\alpha^Q}$$

$$\lesssim \frac{1}{\alpha^Q} \int_{Z} (\rho^p)^{Q/p}$$

$$= \frac{1}{\alpha^Q} \|\rho\|_Q^Q$$

where we have used Ahlfors regularity with inequality (3.5) to bound $\frac{1}{|B_z|}$ and the fact that $Q/p > 1$ to bound the maximal function as an operator on $L^{Q/p}(Z)$. Now, if an edge $e$ satisfies $\tau(e) > \alpha$ then at least one of its vertices must belong to $V(\alpha)$. As $X$ has bounded degree, there is an $L > 0$ such that each vertex can only occur as the boundary of at most $L$ edges. Thus,

$$\#\{ e \in E_X : \tau(e) > \alpha \} \leq L \#V(\alpha) \lesssim \frac{L}{\alpha^Q} \|\rho\|_Q^Q$$

From this we see $\|\tau\|_{Q,\infty}^Q \lesssim \|\rho\|_Q^Q$ and hence $\text{wcap}_Q(A, B) \lesssim \text{mod}_Q(A, B)$ with a constant depending only on $Z$ and the hyperbolic filling.  \hfill $\square$
We now establish a lemma required to complete the proof of Theorem 1.1.3 in the case of disjoint continua. We continue to work with an admissible \( \rho \in L^Q(Z) \). For this lemma we need the following fact.

**Lemma 3.1.2.** Let \( \eta > 0 \) and set

\[
E = E(\eta, \rho) = \left\{ z \in Z : \text{there exists } r > 0 \text{ with } \int_{B(z, r)} \rho^Q \geq \eta r^Q \right\}.
\]

Then \( \mathcal{H}^Q_\infty(E) \leq \frac{10^Q}{\eta} \int_Z \rho^Q \).

**Proof.** For each \( z \in E \) there is a ball \( B_z = B(z, r_z) \) for which \( \int_{B_z} \rho^Q \geq \eta r_z^Q \). As \( Z \) is bounded it is clear we may assume the balls \( B_z \) have uniformly bounded radius. Hence, we may find a disjoint collection of these balls \( B_z \) such that \( E \subseteq \bigcup_i 5B_{z_i} \). Thus,

\[
\mathcal{H}^Q_\infty(E) \leq \sum_i (10r_{z_i})^Q = 10^Q \sum_i r_{z_i}^Q \leq \frac{10^Q}{\eta} \sum_i \int_{B_{z_i}} \rho^Q \leq \frac{10^Q}{\eta} \int_Z \rho^Q.
\]

We note that \( Z \) here may be replaced by an appropriate smaller ambient space \( Z' \) as long as we stipulate that we only consider radii \( r \) for which \( B(z, r) \subseteq Z' \). Indeed, in our case we will apply this to \( Z' = c_0 B_v = B(z_0, c_0 R) \) with \( c_0 \) a constant that depends only on our path and the Loewner function. Thus, in this case the conclusion reads \( \mathcal{H}^Q_\infty(E) \leq \frac{10^Q}{\eta} \int_{c_0 B_v} \rho^Q \).

We also use \( u \) from the proof of Theorem 1.1.3. Recall this means \( u = \min(v, 1) \) where \( v \) is defined in equation (3.3). We also will carefully keep track of constants. We will denote the Ahlfors regularity constants as \( c_Q \) and \( C_Q \). That is, for balls \( B' \) with radius \( r \) bounded above by \( \text{diam}(Z) \) we have \( c_Q r^Q \leq \mu(B') \leq C_Q r^Q \).

**Lemma 3.1.3.** Consider a sequence of balls \( B_v \rightarrow a \in A \) non-tangentially. Then \( u_{B_v} = \int_{B_v} u \rightarrow 0 \).

A rough outline of the proof is as follows: for a ball \( B_v \) in the sequence we consider the set \( M_v = \{ u \geq \epsilon \} \cap B_v \). For balls close enough to \( A \), if \( M_v \) is too large in \( B_v \) we will use the Loewner condition to construct a path connecting \( A \) to \( M_v \) with short \( \rho \)-length. When the \( \rho \)-length is less than \( \epsilon \) this contradicts the definition of \( u \).
Proof. Let $B_v = B(z_0, R)$ be a ball in the sequence. As our sequence approaches non-tangentially, there is a constant $c_1 > 0$ depending only on our sequence such that $\text{dist}(z_0, A) \leq c_1 R$. We assume $B_v$ is close enough to $A$ that $4(1 + c_1) R < \text{diam}(A)$. Fix $\epsilon \in (0, 1)$. Set $M_v = \{ u \geq \epsilon \} \cap B_v$. We then note that by setting $\delta = \frac{\mu(M_v)}{\mu(B_v)}$ we have

$$u_{B_v} \leq \frac{1}{\mu(B_v)} (\mu(M_v) + \epsilon \mu(B_v \setminus M_v)) = \delta + \epsilon (1 - \delta).$$

(3.6)
as $u \leq 1$. We assume $\mu(M_v) > 0$ as our conclusion holds if $\mu(M_v) = 0$.

We now relate the measure of $M_v$ with its Hausdorff $Q$-content. Indeed, if $B_i$ is a collection of balls covering $M_v$ with radii $s_i$, we have $\mu(M_v) \leq \sum_i \mu(B_i) \leq C_Q \sum_i s_i^Q$ and infimizing over all such collections yields $\frac{\mu(M_v)}{C_Q} \leq \mathcal{H}_Q^Q(M_v)$.

Now by applying Lemma 3.1.2 we see that the Hausdorff $Q$-content of the set

$$E = E(\eta, v)$$

$$= \left\{ z \in M_v : \text{there exists } r > 0 \text{ such that } B(z, r) \subseteq c_0 B_v \text{ and } \int_{B(z, r)} \rho^Q \geq \eta r^Q \right\}$$
satisfies $\mathcal{H}_Q^Q(E) \leq \frac{10Q}{\eta} \int_{c_0 B_v} \rho^Q$.

If $\int_{c_0 B_v} \rho^Q = 0$ then we set $\eta = 1$ and otherwise we set

$$\eta = \frac{2C_Q 10^Q \int_{c_0 B_v} \rho^Q}{\mu(M_v)}. \quad (3.7)$$

Hence,

$$\mathcal{H}_Q^Q(E) \leq \frac{10^Q}{\eta} \int_{c_0 B_v} \rho^Q = \frac{\mu(M_v)}{2C_Q} \leq \frac{\mu(M_v)}{C_Q} \leq \mathcal{H}_Q^Q(M_v).$$

It follows from $\mathcal{H}_Q^Q(E) < \mathcal{H}_Q^Q(M_v)$ that there is an $x \in M_v \setminus E$ such that for all $r > 0$ with $B(x, r) \subseteq c_0 B_v$ we have $\int_{B(x, r)} \rho^Q < \eta r^Q$.

Recall $\text{dist}(z_0, A) \leq c_1 R$ and so $\text{dist}(x, A) \leq (1 + c_1) R$. We set $B_0 = B(x, c_2 R)$ and $B_j = 2^{-j} B_0$ where $c_2 = 2(1 + c_1)$. As $4(1 + c_1) R < \text{diam}(A)$ there is a subcontinua $E_0 \subseteq A$ that satisfies $E_0 \subseteq B_0$ and $\text{diam}(E_0) \geq \frac{c_1}{2} R$. Indeed, by possibly removing some of $E_0$ we may also assume $E_0 \subseteq B_0 \setminus B_1$. As $Z$ is a complete and doubling metric measure space that supports a Poincarè inequality, $Z$ is quasiconvex (see [HKST, Theorem 8.3.2]) and hence rectifiably path connected. Thus, there is a rectifiable path $\beta$ connecting $x$ to $E_0$, say
$\beta : [0, 1] \rightarrow Z$ with $\beta(0) = x$ and $\beta(1) \in E_0$. We define continua $E_j$ as follows: given $j > 0$, let $t_j^-$ denote the first time after which $\beta$ does not return to $B_{2j+1}$ and let $t_j^+$ denote the first time $\beta$ leaves $B_{2j}$. Then set $E_j = \beta([t_j^-, t_j^+])$. We note that for each $j$ we have $E_j \subseteq B_{2j}$ and $\text{diam}(E_j) \geq \frac{1}{10} \text{diam}(B_{2j})$. Hence, it follows that

$$\Delta(E_j, E_{j+1}) = \frac{\text{dist}(E_j, E_{j+1})}{\text{min}\{\text{diam}(E_j), \text{diam}(E_{j+1})\}} \leq \frac{1}{10} \frac{\text{diam}(B_{2j})}{\text{diam}(B_{2j+2})} \leq \frac{10}{2} \frac{2^{-2j}c_2R}{2(2^{-2j+2}c_2R)} = 40.$$ 

As we are in a $Q$-Loewner space, this means $\text{mod}_Q(E_j, E_{j+1}) \geq \varphi(40)$, where $\varphi$ is the $Q$-Loewner function associated to $Z$. As $Z$ is Ahlfors $Q$-regular, it follows from [HKST, Proposition 5.3.9] that there is a constant $c_3 > 0$ such that

$$\text{mod}_Q(\overline{B}(x_0, s), Z \setminus B(x_0, S)) \leq c_3 \left(\log \frac{S}{s}\right)^{1-Q}$$

when $0 < 2s < S$. In particular, we can find a constant $c_4 > 2$ such that if $\Gamma^*(E_j, E_{j+1})$ is the path family of rectifiable paths connecting $E_j$ to $E_{j+1}$ that leaves $c_4B_{2j}$, we have $\text{mod}_Q(\Gamma^*(E_j, E_{j+1})) \leq \frac{\varphi(40)}{2}$. Thus, the $Q$-modulus of the family of rectifiable paths connecting $E_j$ to $E_{j+1}$ which stay inside $c_4B_{2j}$ is at least $\frac{\varphi(40)}{2}$. In particular, this means for each $j$ that one can find a path $\alpha_j$ that connects $E_j$ and $E_{j+1}$, stays inside $c_4B_{2j}$, and satisfies

$$\int_{\alpha_j} \rho \leq \left(\frac{4 \int_{c_4B_{2j}} \rho^Q}{\varphi(40)}\right)^{1/Q}.$$ 

Here if $\int_{c_0B_v} \rho^Q = 0$ then instead for each $\nu > 0$ we can find $\alpha_j(\nu)$ such that $\int_{\alpha_j} \rho \leq \nu$ and the following argument works by choosing values of $\nu$ that are sufficiently small. Recall we also use $\alpha_j$ to denote the image of $\alpha_j$. Hence, each $\alpha_j$ is a continuum,

$$\text{dist}(\alpha_j, \alpha_{j+1}) \leq \text{diam}(E_{j+1}) \leq 2(2^{-2j+2})c_2R,$$

and

$$\text{diam}(\alpha_j) \geq \text{dist}(E_j, E_{j+1}) \geq \frac{1}{4} \text{diam}(B_{2j+1}) \geq \frac{1}{2} 2^{-2j+1}c_2R.$$
Thus, we see
\[ \Delta(\alpha_j, \alpha_{j+1}) \leq \frac{2(2^{-(2j+2)}c_2 R)}{2^{-(2j+3)}c_2 R} = 8 < 40. \]

Hence we may perform the same procedure as above to find paths \( \beta_j \) connecting \( \alpha_j \) to \( \alpha_{j+1} \) with
\[ \int_{\beta_j} \rho \leq \left( \frac{4 \int_{c_4 B_{2j}} \rho^Q}{\varphi(40)} \right)^{1/Q} \]
where we note the \( c_4^2 \) arises as \( \alpha_j \subseteq c_4 B_{2j} \). From the way these paths were constructed it is clear we can extract a rectifiable path \( \gamma \) from \( \bigcup_j (\alpha_j \cup \beta_j) \) connecting \( M_v \) to \( A \) such that
\[ \int_{\gamma} \rho \leq \sum_j \left( \frac{4 \int_{c_4 B_{2j}} \rho^Q}{\varphi(40)} \right)^{1/Q} \left( \int_{c_4 B_{2j}} \rho^Q \right)^{1/Q} . \]

As \( x \notin E(\eta) \), we know that \( \int_{cB_{2j}} \rho^Q \leq \eta(c2^{-(2j)}c_2 R)^Q \) for each \( c > 0 \) with \( cB_0 \subseteq c_0 B_v \). We note here that this is the requirement on \( c_0 \), namely that \( B(x, c_4^2 c_2 R) \subseteq c_0 B_v \). Hence,
\[ \int_{\gamma} \rho \leq \sum_j \left( \frac{4 \int_{c_4 B_{2j}} \rho^Q}{\varphi(40)} \right)^{1/Q} \left( \frac{4 \int_{c_4 B_{2j}} \rho^Q}{\varphi(40)} \right)^{1/Q} \lesssim \eta^{1/Q} R \]
with constant only depending on \( c_4 \) and \( \varphi(40) \) (in the case that \( \int_{c_0 B_v} \rho^Q = 0 \) we can instead make \( \int_{\gamma} \rho \) as small as we like). Recall the definition of \( \eta \) given by (3.7) which gives
\[ \int_{\gamma} \rho \lesssim \left( \int_{c_0 B_v} \rho^Q \right)^{1/Q} \left( \frac{1}{\mu(M_v)^{1/Q}} \right) R. \]

Now, \( \delta = \frac{\mu(M_v)}{\mu(B_v)} \) and \( \mu(B_v) \geq c_Q R^Q \), from which we conclude
\[ \int_{\gamma} \rho \lesssim \left( \int_{c_0 B_v} \rho^Q \right)^{1/Q} \delta^{-1/Q}. \]

Now, as \( x \in M_v \) and \( \gamma \) is a path connecting \( A \) to \( x \), we must have \( \int_{\gamma} \rho \geq \epsilon \). Thus,
\[ \epsilon \delta^{1/Q} \leq \left( \int_{c_0 B_v} \rho^Q \right)^{1/Q} . \]

For fixed \( \epsilon > 0 \) the right hand side tends to 0 as \( v \to a \in A \). Thus, we must have \( \delta \to 0 \) as \( v \to a \). Hence, from inequality (3.6) we conclude \( u_{B_v} \to 0 \) as \( v \to a \).

The above argument can be adapted to show that as \( B_v \to B \) we have \( u_{B_v} \to 1 \). To do this, we would instead use \( M_v = \{ u \leq 1 - \epsilon \} \cap B_v \) and argue as above that if \( \delta = \frac{\mu(M_v)}{\mu(B_v)} \) was
large then there would exist a path from $M$ to $B$ with short $\rho$-length. From the definition of $u$ this would produce a path $\gamma$ connecting $A$ to $B$ with total length less than 1, contradicting the admissibility of $\rho$.

This completes the proof of the continua case by bounding below the quantity in inequality (3.4).

Lastly we prove Theorem 1.1.4, the quasisymmetric invariance property for $w\text{cap}_p$.

**Proof of Theorem 1.1.4** Recall $Z$ and $W$ are compact, connected, Ahlfors regular metric spaces and $\varphi : Z \to W$ is an $\eta$-quasisymmetry. Let $X = (V_X, E_X)$ and $Y = (V_Y, E_Y)$ be corresponding hyperbolic fillings. The proof for open sets and continua is the same, so let $A$ and $B$ either be open sets with $\text{dist}(A, B) > 0$ or disjoint continua. As $\varphi^{-1}$ is an $\eta'$-quasisymmetry with $\eta'$ depending on $\eta$, it suffices to show $w\text{cap}_p(\varphi(A), \varphi(B)) \lesssim w\text{cap}_p(A, B)$.

Let $G : Y \to X$ be the quasi-isometry induced by $\varphi^{-1}$ as in Lemma 2.1.6. We note that $G$ maps vertices to vertices. Let $D > 0$ be such that for all adjacent vertices $y, w \in V_Y$, $|G(y) - G(w)| \leq D$.

Let $\tau \geq 0$ be admissible for $w\text{cap}_p(A, B)$. We construct $\sigma$ on $E_Y$ as follows: given an edge $e' \in E_Y$ with vertices $e'_+$ and $e'_-$, we set

$$
\sigma(e') = \sum_{|x - G(e'_+)| \leq D} \left( \sum_{e \sim x} \tau(e) \right) + \sum_{|x - G(e'_-)| \leq D} \left( \sum_{e \sim x} \tau(e) \right)
$$

where $e \sim x$ means $e$ is an edge that has the vertex $x$ as an endpoint.

We show that $\sigma$ is admissible for $w\text{cap}_p(\varphi(A), \varphi(B))$. Indeed, if $\gamma$ is a path in $Y$ with limits in $\varphi(A)$ and $\varphi(B)$, then we construct a path in $X$ with limits in $A$ and $B$ that serves as a suitable image of $\gamma$. Each vertex $y \in \gamma$ corresponds to a point $G(y) \in V_X$. By our choice of $D$, if two vertices $y$ and $y'$ are connected by an edge in $\gamma$, then $|G(y) - G(y')| \leq D$.

We choose a path connecting $G(y)$ and $G(y')$ that stays in the ball of radius $D$ centered at $G(y)$. By doing this for all connected vertices in $\gamma$, we produce a path $\gamma_X$ in $X$ with limits in $A$ and $B$. Now, by construction,

$$
\sum_{e' \in \gamma} \sigma(e') \geq \sum_{y \in \gamma} \left( \sum_{|x - G(y)| \leq D} \left( \sum_{e \sim x} \tau(e) \right) \right) \geq \sum_{e \in \gamma_X} \tau(e) \geq 1,
$$
where we have viewed $\gamma$ as both a sequence of vertices and a sequence of edges. The last inequality follows as $\tau$ was assumed admissible for $\text{wcap}_p(A, B)$.

It remains to show $\|\sigma\|_{p, \infty} \lesssim \|\tau\|_{p, \infty}$. For this we use Lemma 1.2.1. Our set $J \subseteq E_Y \times E_X$ consists of pairs $(e', e)$ for which $e$ appears as a summand in the definition of $\sigma(e')$. Following the notation from Lemma 1.2.1, for $e' \in E_Y$ the set $J_{e'}$ is the set of edges $e$ that appear as a summand in the definition of $\sigma(e')$. The cardinality $|J_{e'}|$ is bounded independent of $e'$ as $X$ has bounded degree. Similarly, for $e \in E_X$ the set $J_e$ is the set of $e'$ for which $e$ contributes to the sum in the definition of $\sigma(e')$. We show that a given $e$ can only contribute to a bounded number of such $\sigma(e')$. Indeed, if $(e', e) \in J$, then one of the vertices of $e$ must lie in the $D$ radius ball around the image under $G$ of one of the vertices of $e'$. As $G$ is a quasi-isometry, we see there is a constant $C > 0$ such that if $y, y' \in V_Y$ and $|y - y'| > C$, then $|G(y) - G(y')| > 2D + 1$. Hence, for edges $e''$ far enough away from $e'$ in $Y$, we must have $e'' \notin J_e$. As $Y$ has bounded degree by Lemma 2.1.5, we see $|J_e|$ is bounded independent of $e$.

We set $s_{e'} = \sigma(e')$ and $t_e = \tau(e)$. Inequality (1.3) in Lemma 1.2.1 then follows from the definition of $\sigma$. Hence, we conclude $\|\sigma\|_{p, \infty} \lesssim \|\tau\|_{p, \infty}$.

### 3.2 Positivity

We now show the positivity of $\text{wcap}_p$: whenever $A, B \subseteq Z$ are open sets with $\text{dist}(A, B) > 0$, we have $\text{wcap}_p(A, B) > 0$. Here only $p \geq 1$ is assumed. Recall we work with a fixed hyperbolic filling $X = (V_X, E_X)$ with parameter $s > 1$. To prove this result, Proposition 3.2.5 we first detail a construction. For this we need the following lemma.

**Lemma 3.2.1.** There exists a constant $M > 0$ depending on the hyperbolic filling parameter $s$ and the Ahlfors regularity constants such that whenever $v \in V_X$ is a vertex in $X$, there exist two vertices $v_1, v_2$ with levels $\ell(v_j) = \ell(v) + M$, $2B_{v_1} \cap 2B_{v_2} = \emptyset$, and $B_{v_1}, B_{v_2} \subseteq B_v$.

**Proof.** We write $B_v = B(z_v, r_v)$. We recall $Z$ is Ahlfors $Q$-regular, so there exist constants $c, C > 0$ such that for all $z \in Z$ and $r \in (0, \text{diam}(Z))$, we have $cr^Q \leq \mu(B(z, r)) \leq Cr^Q$. Fix
small \( k > 0 \) such that \((\frac{3}{4})^Q > \frac{c}{c'}\). Then

\[
\mu(B(z_v, (3/4)r_v) \setminus B(z_v, kr_v)) \geq c(3/4)^Q r_v Q - CkQ r_v Q
\]

and by our choice of \( k \) we see

\[
c(3/4)^Q r_v Q - CkQ r_v Q > 0.
\]

Hence, there is a point \( z_1 \in B(z_v, (3/4)r_v) \setminus B(z_v, kr_v) \). There is also a point \( z_2 \in B(z_v, \frac{k}{2}r_v) \).

For example, let \( z_2 = z_v \).

Let \( M \) be such that \( s^{-M} < \frac{k}{16} \), where we recall \( s > 1 \) was a parameter in the construction of the hyperbolic filling. Then as the balls of level \( \ell(v) + M \) cover \( Z \), there must be balls \( B_j \) of radius \( 2s^{-(\ell(v)+M)} \) with \( z_j \in B_j \). Now, \( \text{dist}(z_1, z_2) \geq \frac{k}{2}r_v \) by construction. As \( r_v = 2s^{-\ell(v)} \), this is \( \text{dist}(z_1, z_2) \geq ks^{-\ell(v)} \). Hence, the sum of the diameters of the balls \( 2B_j \) is bounded by

\[
16s^{-(\ell(v)+M)} \frac{k}{16} = ks^{-\ell(v)} \leq \text{dist}(z_1, z_2)
\]

and so \( 2B_1 \cap 2B_2 = \emptyset \). \( \square \)

We now construct our path structure. Let \( v \in V_X \). Let \( G = \{0, 1\}^* \) be the set of finite sequences of elements of \( \{0, 1\} \). For an element \( g \in G \), we let \( g0 \) and \( g1 \) denote the concatenations of the symbols 0 and 1 to the right hand side of \( g \). We associate elements of \( G \) to vertices in \( V_X \) inductively as follows: let \( \emptyset \) correspond to \( v \) and, given an element \( g \in G \) with corresponding vertex \( v_g \), apply Lemma 3.2.1 to \( B_{v_g} \) to obtain \( B_{v_{g0}} \) and \( B_{v_{g1}} \) which correspond to \( g0 \) and \( g1 \). We also choose \( \gamma_{g0} \) (and likewise \( \gamma_{g1} \)) to be an edge path connecting \( v_g \) to \( v_{g0} \) with length \( M \). To construct such a path, one can choose a point in \( B_{v_{g0}} \) and select vertices corresponding to balls containing that point with levels between those of \( v_g \) and \( v_{g0} \). To form \( \gamma_{g0} \) one then uses the edges between these vertices.

**Remark 3.2.2.** We note from the fact that \( 2B_1 \cap 2B_2 = \emptyset \) in Lemma 3.2.1 that, if \( g, h \in G^N \), then \( \gamma_g \cap \gamma_h \neq \emptyset \) can only happen if the first \( N - 1 \) entries of \( g \) match those of \( h \).

**Definition 3.2.3.** Given a vertex \( v \in V_X \), we call a path structure as constructed above a binary path structure and denote it \( T_v \). We call \( M \) the splitting constant of \( T_v \).
We call an edge path \( \gamma = (e_k) \) ascending if the levels of the endpoints of successive edges is strictly increasing. That is, if \((v_k)\) is the sequence of vertices that \( \gamma \) travels through, then \( \ell(v_{k+1}) = \ell(v_k) + 1 \) for all \( k \). We consider functions \( \tau : E_X \to [0, \infty] \) which are admissible on ascending edge paths originating from \( v \). That is, for such paths \( \gamma \) we require \( \sum_{e_k} \tau(e_k) \geq 1 \).

We claim such functions cannot have too small weak \( \ell^p \)-norm.

**Lemma 3.2.4.** Let \( \tau : E_X \to [0, \infty] \) and \( v \in V_X \). Let \( T_v \) be a binary path structure with splitting constant \( M \). Then, there is a constant \( S(p) = S(p, M) < \infty \) and an ascending edge path \( \gamma = (e_k)_{k=0}^\infty \) from \( v \) to \( B_v \subseteq \mathbb{Z} \) with \( \tau \)-length bounded above by \( \| \tau \|_{p, \infty} S(p) \).

Functions that are admissible on ascending edge paths must give at least \( \tau \)-length 1 to such paths, so for these functions we have \( \| \tau \|_{p, \infty} \geq 1/S(p) \).

**Proof.** We may assume \( \| \tau \|_{p, \infty} < \infty \). Let \( T_v \) be a binary path structure originating from \( v \). For \( N \in \mathbb{N} \), let \( G^N = \{0, 1\}^N \) be the set of finite strings of elements of \( \{0, 1\} \) of length \( N \). For \( g \in G^N \) and \( k < N \), let \( g_k \) denote the element of \( G^k \) which matches the first \( k \) entries of \( g \). We also set \( \gamma'_g = \bigcup_k \gamma_{g_k} \) to be the ascending edge path formed by concatenating \( \gamma_{g_1}, \ldots, \gamma_g \).

The average \( \tau \)-length of the paths \( \gamma'_g \), where \( g \in G^N \), is given by

\[
\frac{1}{2^N} \sum_{g \in G^N} \sum_{e \in \gamma'_g} \tau(e) = \frac{1}{2^N} \sum_{g \in G^N} \sum_{k=1}^N \left( \sum_{e \in \gamma_{g_k}} \tau(e) \right) = \frac{1}{2^N} \sum_{k=1}^N 2^{N-k} \sum_{h \in G^k} \left( \sum_{e \in \gamma_h} \tau(e) \right). \tag{3.8}
\]

We bound this average above using \( \| \tau \|_{p, \infty} \). For ease of notation, let \( a = \| \tau \|_{p, \infty} \). By definition, we have

\[
\# \{ e : \tau(e) > \lambda \} \leq \frac{a^p}{\lambda^p}.
\]

Hence, for \( j \in \mathbb{N} \) the function \( \tau \) can take values \( \tau(e) \geq \frac{a}{j^{1/p}} \) for at most \( j \) edges \( e \). From Remark 3.2.2 for fixed \( k > 1 \) there are at least \((2^{(k-1)} - 1)M \) distinct edges contributing to the right hand side of equation (3.8) belonging to paths \( \gamma_f \) with \( f \in G^\ell \) where \( \ell < k \). With the weighting factors \( 2^{N-k} \) it follows that the average increases the more \( \tau \)-mass is located
on paths with smaller associated $k$ values. Using these observations we conclude that to bound equation (3.8) above, for $k > 1$ and $h \in G^k$ we bound the sum $\sum_{e \in \gamma_h} \tau(e)$ above by

$$M \frac{a}{((2^{(k-1)} - 1)M)^{(1/p)}}.$$

From this, we obtain the upper bound

$$\frac{1}{2} \sum_{h \in G^1} \left( \sum_{e \in \gamma_h} \tau(e) \right) + \sum_{k=2}^{N} 2^{-k} \sum_{h \in G^k} M \frac{a}{((2^{(k-1)} - 1)M)^{(1/p)}}.$$

We bound the first term by noting that $\tau(e) \leq a$ for all $e \in E_X$. As $G^k$ has $2^k$ elements, our bound becomes

$$Ma + \sum_{k=2}^{N} M \frac{a}{((2^{(k-1)} - 1)M)^{(1/p)}}.$$

Setting

$$S(p) = M + \sum_{k=2}^{\infty} M \frac{1}{((2^{(k-1)} - 1)M)^{(1/p)}} < \infty$$

we thus bound (3.8) above by $aS(p) = \|\tau\|_{p,\infty} S(p)$. As this is a bound on the average $\tau$-length of the paths $\gamma_g'$, where $g \in G^N$, we conclude that for each $N$ there is a $g_N \in G^N$ such that $\gamma'_{g_N}$ has $\tau$-length bounded above by $\|\tau\|_{p,\infty} S(p)$.

To complete the proof we construct a path $\gamma$ from the paths $\gamma_{g_N}'$. For each $N$ we have $g_N \in \{0, 1\}^N$. Hence, there is a subsequence of the $g_N$ that has the property that all strings in this subsequence have the same first element. From this subsequence, we may extract another subsequence that consists of strings that all have the same first two elements. Continuing in this manner and then diagonalizing produces an infinite sequence $h \in \{0, 1\}^\mathbb{N}$ for which, for infinitely many $k \in \mathbb{N}$, the first $k$ elements of $h$ match $g_k$. It is clear how to associate $h$ to an ascending path $\gamma$ which, from the bounds on the $\tau$-lengths of the paths $\gamma_{g_k}'$, satisfies the conclusion of the lemma.

We perform a similar analysis on a path of edges of length $L$. As above, if $\|\tau\|_{p,\infty} = b$, then $\tau$ can take values $\tau(e) \geq \frac{b}{j^{(1/p)}}$ at most $j$ times. Thus, the maximum $\tau$-length our line can have is $\sum_{k=1}^{L} \frac{b}{k^{(1/p)}}$. This is bounded above (for $p > 1$) by $b(\int_0^L \frac{1}{x^{(1/p)}} dx)$, which is $\frac{bL^{1-(1/p)}}{1-(1/p)}$. For $p = 1$ this is bounded above by $b(1 + \log(L))$.

We are now ready to prove positivity.
Proposition 3.2.5. Let $p \geq 1$ and let $A, B \subseteq \mathbb{Z}$ be open sets with $\text{dist}(A, B) > 0$. Then, $\text{wcap}_p(A, B) > 0$.

Proof. We first assume $p > 1$. Let $v, w \in V_X$ be vertices such that $\overline{B_v} \subseteq A$ and $\overline{B_w} \subseteq B$ and such that $\ell(v) = \ell(w)$. We connect $v$ and $w$ by an edge path $\gamma$ contained in $\{x \in X : \abs{x - O} \leq \ell(v)\}$, where $O$ is the unique vertex with $\ell(O) = 0$. We let $L$ denote the length of $\gamma$. Recall $T_v$ and $T_w$ denote binary path structures originating from $v$ and $w$. Now, if $\tau$ is admissible for $\text{wcap}_p(A, B)$, then $\tau$ is admissible on paths contained in $T_v \cup T_w \cup \gamma$. Let $\tau_A, \tau_B,$ and $\tau_\gamma$ denote the restrictions of $\tau$ to $T_v, T_w,$ and $\gamma$. We then must have

$$\|\tau_A\|_{p, \infty} S(p) + \|\tau_B\|_{p, \infty} S(p) + \|\tau_\gamma\|_{p, \infty} \frac{L^{1-(1/p)}}{1 - (1/p)} \geq 1.$$ 

As $\|\tau\|_{p, \infty}$ is larger than the norm of each of these restrictions, we see

$$\|\tau\|_{p, \infty} S(p) + \|\tau\|_{p, \infty} S(p) + \|\tau\|_{p, \infty} \frac{L^{1-(1/p)}}{1 - (1/p)} \geq 1.$$ 

Hence,

$$\|\tau\|_{p, \infty} \geq \frac{1}{2S(p) + \frac{L^{1-(1/p)}}{1 - (1/p)}}.$$ 

The above analysis also applies to the $p = 1$ case with the appropriate bound modification. In this case, this bound becomes

$$\|\tau\|_{1, \infty} \geq \frac{1}{2S(1) + (1 + \log(L))}.$$ 

Both cases thus yield a lower bound on $\|\tau\|_{p, \infty}$ for admissible $\tau$, as desired. \qed

3.3 Weak covering capacity

Here we state and prove some basic properties and the main theorems involving $\text{wc-cap}_p$.

Recall we work on a compact, connected, Ahlfors $Q$-regular metric measure space $(Z, d, \mu)$ with $Q > 1$ and with a fixed hyperbolic filling $X = (V_X, E_X)$ of $(Z, d, \mu)$ with scaling parameter $s > 1$. We first prove that if $p \geq Q$, then $\text{wc-cap}_p$ is supported on rectifiable paths.
Lemma 3.3.1. Let $p \geq Q$ and let $\Gamma_\infty$ be the set of all infinite-length paths $\gamma : [0, 1] \to \mathbb{Z}$. Then $\text{wc-cap}_p(\Gamma_\infty) = 0$.

Proof. We show that for any $\epsilon > 0$, the functions $\tau_\epsilon(v) = r(B_v)\epsilon$ are admissible for $\Gamma_\infty$ and that $\|\tau_\epsilon\|_{p, \infty} \to 0$ as $\epsilon \to 0$. From this it follows that $\text{wc-cap}_p(\Gamma_\infty) = 0$. Fix $\epsilon > 0$.

Let $\gamma \in \Gamma_\infty$. Let $t_0 < t_1 < \cdots < t_m$ be a partition of $[0, 1]$ such that the points $\gamma(t_k)$ are distinct and $\sum_k d(\gamma(t_{k-1}), \gamma(t_k)) > 4/\epsilon$. Set $\gamma_k = \gamma|_{[t_{k-1}, t_k]}$. Let $N$ be such that $400s^{-N} \leq \min_k d(\gamma(t_{k-1}), \gamma(t_k))$ and $m2\epsilon s^{-N} < 1$. Let $\{S_j\}$ be an expanding sequence of covers. As $\{S_j\}$ is expanding, for large enough $j$ we see that the balls in $S_j$ all have radius bounded above by $2s^{-N}$. Thus, for such a $j$, if $P_k$ is any projection of $\gamma_k$ on $S_j$ with balls $\{B_i\}$ we have

$$\ell_{\tau_\epsilon, P_k, S_j}(\gamma_k) = \sum_i \tau_\epsilon(B_i) = \sum_i r(B_i)\epsilon \geq \frac{\epsilon}{2} d(\gamma(t_{k-1}), \gamma(t_k)).$$

(3.9)

Now, let $P$ be any projection of $\gamma$ on $S_j$. By adding the values $t_0, \ldots, t_m$ to $P$ we obtain a partition $P'$ from $P$ and subpartitions $P_k$ of $P'$ consisting of the values between $t_{k-1}$ and $t_k$. It is clear that $P'$ has at most $m$ more intervals than $P$. As $\tau(v) \leq 2\epsilon s^{-N}$ for all $v \in S_j$, it follows that

$$\ell_{\tau_\epsilon, P', S_j}(\gamma) - \ell_{\tau_\epsilon, P, S_j}(\gamma) \leq m2\epsilon s^{-N}.$$ 

(3.10)

We note $\ell_{\tau_\epsilon, P', S_j}(\gamma) = \sum_k \ell_{\tau_\epsilon, P_k, S_j}(\gamma_k)$. Combining this with (3.9) and (3.10) yields

$$\ell_{\tau_\epsilon, P, S_j}(\gamma) \geq \frac{\epsilon}{2} \left( \sum_k d(\gamma(t_{k-1}), \gamma(t_k)) \right) - m2\epsilon s^{-N}$$

$$> 2 - m2\epsilon s^{-N}$$

$$\geq 1.$$

As this holds for all large enough $j$, we conclude that $\tau_\epsilon$ is admissible for each $\gamma \in \Gamma$ and hence for $\Gamma_\infty$.

It remains to show $\|\tau_\epsilon\|_{p, \infty} \to 0$ as $\epsilon \to 0$. As our hyperbolic filling has bounded degree (Lemma 2.1.5), we see the number of vertices with level $n$ is comparable to $s^{nQ}$ up to a fixed
multiplicative constant. Thus, for $\lambda = 2\varepsilon s^n \leq 1$ we have
\[
\# \{ v \in V_X : \tau(v) > \lambda \} \lesssim s^{nQ} \lesssim \frac{\epsilon^Q}{\lambda^Q} \lesssim \frac{\epsilon^p}{\lambda^p}
\]
with implicit constants independent of $n$. From this the limiting behavior of $\|\tau_\epsilon\|_{p,\infty}$ follows.

We remark here that the above result does not hold for $p < Q$. Indeed, by quasisymmetric invariance (Theorem 1.1.7) there are spaces with path families $\Gamma$ of non-rectifiable curves for which $\text{wc-cap}_p(\Gamma) > 0$ holds for some $p$.

We also note if $\Gamma_1$ and $\Gamma_2$ are arbitrary path families, then\[
\text{wc-cap}_Q(\Gamma_1) \leq \text{wc-cap}_Q(\Gamma_1 \cup \Gamma_2) \leq \text{wc-cap}_Q(\Gamma_1) + \text{wc-cap}_Q(\Gamma_2).
\]
This follows as if $\tau$ is admissible for $\Gamma_1 \cup \Gamma_2$, then $\tau$ is admissible for $\Gamma_1$ and if $\tau_1, \tau_2$ are admissible for $\Gamma_1, \Gamma_2$, then $\max\{\tau_1, \tau_2\}$ is admissible for $\Gamma_1 \cup \Gamma_2$. With this observation and Lemma 3.3.1 it follows that for any path family $\Gamma$ one has $\text{wc-cap}_Q(\Gamma) = \text{wc-cap}_Q(\Gamma \setminus \Gamma_\infty)$. Thus, we may assume in the following that all path families $\Gamma$ consist solely of paths with finite length.

Now we prove Theorems 1.1.6 and 1.1.7. We start with Theorem 1.1.6. Recall we work in a compact, connected, Ahlfors $Q$-regular metric measure space $Z$ with hyperbolic filling $X = (V_X, E_X)$. We also work with a fixed path family $\Gamma$ such that every $\gamma \in \Gamma$ has finite length.

**Proof of Theorem 1.1.6** We first prove $\text{wc-cap}_Q(\Gamma) \lesssim \text{mod}_Q(\Gamma)$. Let $\rho : Z \to [0, \infty]$ be an admissible function for the $Q$-modulus of $\Gamma$. As $Z$ is compact, we may assume $\rho$ is lower semicontinuous (this follows from the Vitali-Carathéodory theorem, see [HKST, Section 4.2]). Define $\tau : V_X \to \mathbb{R}$ by\[
\tau(v) = r(B_v) \left( \int_{B_v} 10\rho \right)
\]
for $v \in V_X$. We show that $\tau$ is admissible for covering capacity.

Fix $\gamma \in \Gamma$. We recall we assume $\gamma$ has finite length $\ell(\gamma) > 0$. Set $I = [0, \ell(\gamma)]$; we work with partitions of $I$ and the arclength parameterization of $\gamma$ as in the remark in the
Let \( \epsilon = \frac{1}{\ell(\gamma) + 1} \). As \( Z \) is compact, \( f \) is uniformly continuous. Hence there is a \( \delta_1 > 0 \) such that if \( d(x, y) < \delta_1 \), then \( |f(x) - f(y)| < \epsilon \).

We find a partition of \( [0, \ell(\gamma)] \) given by \( 0 = x_0 < \cdots < x_p = \ell(\gamma) \) with \( x_{k+1} - x_k < \delta_1 \) and find a \( \delta_2 > 0 \) such that for each \( i \), every \( x, y \) in the \( \delta_2 \) neighborhood of \( \gamma_i = \gamma([x_{i-1}, x_i]) \) satisfies \( |f(x) - f(y)| < \epsilon \). The existence of this partition and of \( \delta_2 \) follow from the uniform continuity of \( f \) on \( Z \). We also set \( m_i \) to be the infimum of the values of \( f \) on the \( \delta_2 \) neighborhood of \( \gamma_i \). We further partition each \( [x_{i-1}, x_i] \) as \( x_{i-1} = y'_0 < \cdots < y'_q = x_i \) such that \( \ell(\gamma_i) - \sum_j d(\gamma(y'_j), \gamma(y'_j)) < \frac{\epsilon}{M_0} \). Set \( \delta_3 = \frac{1}{10} \min_{i, j} d(\gamma(y'_j), \gamma(y'_j)) \) which we may assume is positive by appropriately choosing \( y'_j \).

Set \( \delta = \min(\frac{1}{3} \delta_1, \delta_2, \delta_3) \).

Let \( \mathcal{S} = \{S_n\} \) be an expanding sequence of covers. Then, as \( \mathcal{S} \) is expanding, for large enough \( n \) it follows that \( r(B_v) < \delta \) for all \( v \in S_n \). We work with one of these covers \( S_n \) with large \( n \) which we denote \( S \). Let \( P: [0, \ell(\gamma)] \to V \) be a projection of \( \gamma \) onto \( S \) with \( t_0, \ldots, t_m \) partitioning \( [0, \ell(\gamma)] \) and \( v_1, \ldots, v_m \) vertices such that \( \gamma([t_{k-1}, t_k]) \subseteq B_{v_k} \).

We have
\[
\int_{\gamma} f = \sum_i \int_{\gamma_i} f
\]
and we see
\[
\left| \sum_i \left( \int_{\gamma_i} f - m_i \ell(\gamma_i) \right) \right| \leq \epsilon \sum_i \ell(\gamma_i) < 1
\]
and so
\[
\sum_i m_i \ell(\gamma_i) \geq 6.
\]

Now we group the \( t_k \) into \( T_1, \ldots, T_p \) where \( T_i = \{t_k : x_{i-1} \leq t_k \leq x_i\} \) and similarly write \( K_i = \{k : t_k \in T_i\} \). By our choice of \( \delta \) above, we see for each \( i \) that
\[
|m_i - f(\gamma(t_k))| < \epsilon
\]
whenever $k \in K_i$. Using $\ell(\gamma_i) - \sum_j d(\gamma(y^i_{j-1}), \gamma(y^i_j)) < \frac{\epsilon}{M^p}$, we deduce
\[
\sum_i m_i \left( \sum_j d(\gamma(y^i_{j-1}), \gamma(y^i_j)) \right) \geq 6 - \epsilon \geq 4.
\]

Now, as $\delta \leq \delta_3$ we may replace $\sum_j d(\gamma(y^i_{j-1}), \gamma(y^i_j))$ in the above sum with twice the sum of the radii of balls $B_v$ from our partition $P$ with corresponding intervals intersecting $\bigcup_j [y^i_{j-1}, y^i_j]$. That is, we have
\[
\sum_i \sum_{k \in K_i} m_i 2 r(B_{v_k}) \geq 4.
\]
Thus,
\[
\sum_i \sum_{k \in K_i} m_i r(B_{v_k}) \geq 2.
\]

We note that for $k \in K_i$ we have
\[
\int_{B_{v_k}} f - m_i \geq 0.
\]
From this we conclude
\[
\sum_i \sum_{k \in K_i} \left( \int_{B_{v_k}} f \right) r(B_{v_k}) \geq \sum_i \sum_{k \in K_i} m_i r(B_{v_k}) \geq 2.
\]
Lastly we deal with the overestimation possible from having $k \in K_i$ for more than one $i$. This only happens if $t_k = x_i$ for some $i$, which happens at most $p + 1$ times (recall our partition is $x_0, \ldots, x_p$). We note that $f$ is bounded and that in an expanding sequence of covers we have $r(B_{v_k}) \to 0$ in the above sum. Thus, if $f \leq M$ and $r(B_{v_k}) \leq \nu(n)$, double counting such $k$ adds at most $(p + 1)M\nu(n)$ to our estimate. We conclude
\[
\sum_{k=1}^m \tau(v_k) \geq \sum_{k=1}^m \left( \int_{B_{v_k}} f \right) r(B_{v_k}) \geq 2 - (p + 1)M\nu(n).
\]
From this we see that for large enough $n$ the $\tau$-length of any partition $P$ of $\gamma$ onto $S_n$ is at least 1. That is, $\tau$ is admissible for $\gamma$ relative to $\mathcal{S}$. As $\mathcal{S}$ was arbitrary, it follows that $\tau$ is admissible for $\gamma$.

As this holds for all $\gamma \in \Gamma$ we see $\tau$ is admissible for covering capacity. It remains to show $\|\tau\|_{Q,\infty}^Q \lesssim \|\rho\|_{Q}^Q$ but this follows as in the proof of Theorem 1.1.3 with $p = 1$. 

52
We now prove the other direction, namely \( \text{mod}_Q \lesssim \text{wc-cap}_Q \).

Let \( \tau : V_X \to \mathbb{R} \) be admissible for covering capacity. Let

\[
\sigma_n = 2 \sum_{v \in V_n} \frac{\tau(v)}{r(B_v)} \chi_{2B_v}
\]

As in the proof that \( \text{mod}_Q(A, B) \lesssim \text{wc-cap}_Q(A, B) \) for open sets, we note that there is a subsequence \( \sigma_{n_i} \) with \( \|\sigma_{n_i}\|^Q_Q \lesssim \|\tau\|^Q_Q \). Applying Mazur’s Lemma to this subsequence, as in the proof of Theorem 1.1.2, we get convex combinations \( \rho_k \) of \( \sigma_{n_i} \) with \( i \geq k \) and a limit function \( \rho \) with \( \rho_k \to \rho \) in \( L^Q \). Similarly to that proof, by applying Fuglede’s Lemma we may pass to a subsequence and assume that for all paths \( \gamma \) except in a family \( \Gamma_0 \) of \( Q \)-modulus 0 we have \( \int_{\gamma} \rho_n \to \int_{\gamma} \rho \). We note that \( \|\rho\|^Q_Q \lesssim \|\tau\|^Q_Q \).

As the \( Q \)-modulus of \( \Gamma_0 \) is 0, there exists a function \( \sigma \geq 0 \) with \( \int \sigma < \infty \) such that for \( \gamma \in \Gamma_0 \) we have \( \int_{\gamma} \sigma = \infty \). We claim that \( \rho + c\sigma \) is admissible for modulus for any \( c > 0 \). For \( \gamma \in \Gamma_0 \), admissibility is clear, so suppose \( \gamma \notin \Gamma_0 \). We see if \( B_{v_1}, \ldots, B_{v_M} \) is a sequence of balls which \( \gamma \) passes through and that \( \ell(\gamma \cap 2B_{v_k}) \geq r(B_{v_k}) \) for each \( v_k \), then

\[
\int_{\gamma} \sigma_n = 2 \sum_{v \in V_n} \frac{\ell(\gamma \cap 2B_v)}{r(B_v)} \frac{\tau(v)}{r(B_v)} \geq 2 \sum_{k=1}^{M} \frac{r(B_{v_k})}{r(B_{v_k})} \frac{\tau(v_k)}{r(B_{v_k})}.
\]

Let \( S_j = \{ v \in V_X : j \leq \ell(v) \leq 2j \} \) be the set of all vertices with levels between \( j \) and \( 2j \). As \( \tau \) is admissible, it is admissible for the expanding sequence of covers \( \{S_j\} \) and hence our integral is bounded below by 1 for large enough \( n \). Thus, \( \int_{\gamma} \rho = \lim_{n \to \infty} \int_{\gamma} \rho_n \geq 1 \). We conclude that \( \rho + c\sigma \) is admissible for modulus and, as \( \|\rho + c\sigma\|^Q_Q \lesssim \|\rho\|^Q_Q + c\|\sigma\|^Q_Q \), this shows \( \text{mod}_Q(\Gamma) \lesssim \text{wc-cap}_Q(\Gamma) \).

Lastly we prove Theorem 1.1.7

Proof of Theorem 1.1.7 Recall \( Z \) and \( W \) are compact, connected, Ahlfors regular metric spaces and \( \varphi : Z \to W \) is an \( \eta \)-quasisymmetry. Let \( X = (V_X, E_X) \) and \( Y = (V_Y, E_Y) \) by corresponding hyperbolic fillings. Fix a path family \( \Gamma \) in \( Z \). Let \( \tau \) be admissible for
wc-cap_p(Γ). Let \( G: Y \to X \) denote the quasi-isometry induced by \( \varphi^{-1} \) from Lemma 2.1.6. We define \( \sigma: V_Y \to [0, \infty] \) by \( \sigma(y) = \tau(G(y)) \).

We claim \( \sigma \) is admissible for wc-cap_p(\( \varphi(\Gamma) \)). To prove this, let \( \mathcal{S}' = \{ S'_n \} \) be an expanding sequence of covers in \( Y \). We note that if \( \{ v_k \} \) is the set of vertices in \( S'_n \) then \( W \subseteq \bigcup_k B_{v_k} \).

Recall that for a vertex \( v \in V_Y \) we defined \( G(v) \) such that \( \varphi^{-1}(B_v) \subseteq B_{G(v)} \). Hence,

\[
Z = \varphi^{-1}(W) \subseteq \bigcup_k \varphi^{-1}(B_{v_k}) \subseteq \bigcup_k B_{G(v_k)}
\]

and we see that \( \{ S_n \} = \{ G(S'_n) \} \) is an expanding sequence of covers.

We fix a rectifiable \( \gamma' = \varphi(\gamma) \in \varphi(\Gamma) \). Now, let \( P \) be a projection of \( \gamma' \) onto \( S'_n \), say with balls \( B_{y_1}, \ldots, B_{y_m} \). From the above, \( \varphi^{-1}(B_{y_k}) \subseteq B_{G(y_k)} \) and so the sequence \( B_{G(y_k)} \) forms a partition of \( \gamma \) using balls in \( S_n \). Hence, for large enough \( n \), we see \( \sum_k \tau(G(y_k)) \geq 1 \). As \( \tau(G(y_k)) = \sigma(y_k) \), it follows that \( \sigma \) is admissible for wc-cap_p(\( \varphi(\Gamma) \)).

It remains to show \( \| \sigma \|_{p, \infty} \lesssim \| \tau \|_{p, \infty} \) for which we use Lemma 1.2.1. To apply Lemma 1.2.1 we set \( J \subseteq V_Y \times V_X \) where \( (y, x) \in J \) if \( x = G(y) \). We see for \( J^x = \{ y : (y, x) \in J \} \) we have

\[
|J^x| = |\{ y : x = G(y) \}|.
\]

If \( x = G(y) = G(y') \) then, as \( G \) is a quasi-isometry, it follows that there is a there is a fixed \( D' > 0 \) such that \( |y - y'| \leq D' \). From Lemma 2.1.5 it follows there is a uniformly bounded number of such \( y \); that is, \( |J^x| \) is uniformly bounded.

Now, \( J_y = \{ x : (y, x) \in J \} = \{ G(y) \} \) so \( |J_y| = 1 \). Lastly, we use \( s_y = \sigma(y) \) and \( t_x = \tau(x) \) for our sequences. We have

\[
\sigma(y) = \tau(G(y)) = \sum_{x \in J_y} \tau(x)
\]

as \( G(y) \in J_y \). Thus, \( \| \sigma \|_{p, \infty} \lesssim \| \tau \|_{p, \infty} \) and so wc-cap_p(\( \varphi(\Gamma) \)) \( \lesssim \) wc-cap_p(\( \Gamma \)). The other inequality follows from considering \( \varphi^{-1} \) in place of \( \varphi \). \( \square \)
CHAPTER 4

Conformal Dimension and Uniformization

4.1 Ahlfors regular conformal dimension and critical exponents

In this section we relate the Ahlfors regular conformal dimension of our space \((Z,d)\) with weak capacity.

**Definition 4.1.1.** The *Ahlfors regular conformal dimension* of \((Z,d)\) will be denoted by \(\text{ARCdim}\). Let
\[
\mathcal{G} = \{ \theta : \theta \text{ is a metric on } Z \text{ with } (Z, \theta) \sim_{qs} (Z, d) \}.
\]
Then \(\text{ARCdim} = \inf \text{dim}_H (Z, \theta)\) where the infimum is taken over all \(\theta \in \mathcal{G}\) such that \((Z, \theta)\) is Ahlfors regular. Here \((Z, \theta) \sim_{qs} (Z, d)\) means that the identity map is a quasisymmetry.

We define two critical exponents relating to \(\text{wcap}\) which are motivated by a critical exponent defined in [BdK]. This is not the first attempt to define meaningful critical exponents using hyperbolic fillings. For example, see [CP].

The First Critical Exponent

**Definition 4.1.2.** Let
\[
Q_w = \inf \{ p : \text{wcap}_p(A, B) < \infty \text{ for all open } A \text{ and } B \text{ with } \text{dist}(A, B) > 0 \}.
\]

**Lemma 4.1.3.** We have \(Q_w \leq \text{ARCdim}\).

**Proof.** From quasisymmetric invariance, it suffices to show for any \(p > \text{ARCdim}\) and any open sets \(A\) and \(B\) with \(\text{dist}(A, B) > 0\) we have \(\text{wcap}_p(A, B) < \infty\). Fix such parameters and
let $\theta$ be an Ahlfors regular metric on $Z$ such that $\dim_H(Z, \theta) \leq p$. Let $X = (V_X, E_X)$ be a hyperbolic filling for $(Z, \theta)$ with parameter $s > 1$ and consider $f : V_X \to \mathbb{R}$ defined by

$$f(v) = \frac{4r(B_v)}{\text{dist}(A, B)}.$$ 

It follows from $\dim_H(Z, \theta) \leq p$ that the number of vertices on level $n$ is bounded above by $Cs^n p^{n}$ for some $C > 0$ and so $f \in \ell^p(V_X)$. Define $\tau : E_X \to \mathbb{R}$ by $\tau(e) = f(e^+) + f(e^-)$. Hence, $\|\tau\|_{p, \infty} \lesssim \|f\|_{p, \infty} < \infty$ follows from Lemma 1.2.1. Thus, we need only show $\tau$ is admissible.

Now, if $\gamma$ is any finite chain of vertices, say $\{v_0, \ldots, v_N\}$ (where $v_k$ is connected to $v_{k+1}$ for all $k$), then

$$\sum_{e \in \gamma} \tau(e) \geq \sum_{k} f(v_k) \geq \frac{2 \text{dist}(B_{v_0}, B_{v_N})}{\text{dist}(A, B)}.$$ 

Thus, as an infinite $\gamma$ with non-tangential limits in $A$ and $B$ has a finite subpath $\gamma_0 = \{v_0, \ldots, v_N\}$ with $B_{v_0} \cap A \neq \emptyset$ and $B_{v_N} \cap B \neq \emptyset$ and small enough radii so that $2 \text{dist}(B_{v_0}, B_{v_N}) \geq \text{dist}(A, B)$, we have

$$\sum_{e \in \gamma} \tau(e) \geq \sum_{e \in \gamma_0} \tau(e) \geq \frac{2 \text{dist}(B_{v_0}, B_{v_N})}{\text{dist}(A, B)} \geq 1$$

so $\tau$ is admissible.

In view of this inequality, one question is how does $Q_w$ relate to $\text{ARCdim}$? If $Q_w \neq \text{ARCdim}$, then can one define a similar critical exponent that is $\text{ARCdim}$? The following result shows that for some metric spaces we do have equality.

**Lemma 4.1.4.** Let $(Z, d, \mu)$ be a compact, connected metric measure space that is Ahlfors $Q$-regular, $Q > 1$, and such that there exists $1 \leq p \leq Q$ and a family of paths $\Gamma$ with $\text{mod}_p(\Gamma) > 0$. Then there exist open balls $A$ and $B$ with $\text{dist}(A, B) > 0$ such that for all $q < Q$ we have $\text{wcap}_q(A, B) = \infty$.

In such metric spaces [MT] Proposition 4.1.8] shows $\text{ARCdim}(Z, d) = Q$ and so we have $Q_w = \text{ARCdim}$.

**Proof.** From [MT] Proposition 4.1.6 (vii)] it follows that $\text{mod}_Q(\Gamma) > 0$. By potentially taking subpaths, we may assume every path in $\Gamma$ has distinct end points (i.e. no paths in $\Gamma$ are
loops). By writing $\Gamma = \bigcup (\Gamma_m)$ where $\Gamma_m = \{ \gamma \in \Gamma : \ell(\gamma) > \frac{1}{m} \}$, we may assume the paths in $\Gamma$ have lengths uniformly bounded from below (we need $Q > 1$ for this, see \textbf{MT}, Proposition 4.1.6 (iv)). By covering $Z$ with a finite number balls of small enough radius and writing $\Gamma$ as the union of paths connecting two disjoint balls with positive separation, we may assume $\Gamma$ connects two open balls $A$ and $B$ with $\text{dist}(A, B) > 0$. Refining this slightly allows us to assume $\Gamma$ connects $A_\lambda$ and $B_\lambda$ for some $\lambda > 0$ as in the proof of Theorem 1.1.2.

Now suppose $\tau : E_X \to \mathbb{R}$ is admissible for $\text{wcap}_q(A, B)$ and satisfies $\|\tau\|_{q, \infty} < \infty$, where $q < Q$. Define $f : V_X \to \mathbb{R}$ by $f(v) = \sum_{e \sim v} \tau(e)$ and, as in the proof of Theorem 1.1.2 set $u_n = \sum_{v \in V_n} f(v) \chi_{2B_v}$. Then, as before, for specific large enough $n$ the function $2u_n$ is admissible for $\Gamma$. Our estimate for $\|u_n\|_{Q}^Q$ is also the same as in inequality (3.2):

$$\|u_n\|_{Q}^Q \lesssim \sum_{v \in V_n} f(v)^Q.$$ 

As $f \in \ell^{q, \infty}(V)$, it follows that $f \in \ell^Q(V)$ and so $\sum_{v \in V_n} f(v)^Q \to 0$ as $n \to \infty$. This shows $\text{mod}_Q(\Gamma) = 0$, a contradiction. Hence, no such admissible function exists and $\text{wcap}_q(A, B) = \infty$.

\[ \square \]

The second critical exponent

Recall in Definition 1.1.12 we defined the second critical exponent $Q'_w$ by demanding control for $\text{wcap}_p(A, B)$ in terms of $\Delta(A, B)$. We see $Q_w \leq Q'_w$. We will show $Q'_w \leq \text{ARCDim}$. First, we estimate the effect a quasisymmetric map has on $\Delta(A, B)$ when $A, B$ are open with $\text{dist}(A, B) > 0$. This is similar to \textbf{BnK}, Lemma 3.2. We state the result for compact metric spaces as this is our current setting.

\textbf{Lemma 4.1.5.} Let $(Z, d)$ and $(Z', d')$ be compact metric spaces. Let $f : Z \to Z'$ be an $\eta$-quasisymmetric homeomorphism. Then there is an increasing homeomorphism $\Phi : (0, \infty) \to (0, \infty)$ with $\lim_{t \to \infty} \Phi(t) = \infty$ such that for all open $A, B \subseteq Z$ with $\text{dist}(A, B) > 0$ we have $\Delta(f(A), f(B)) \geq \Phi(\Delta(A, B))$.  

57
Proof. Throughout the proof we use ′ to denote the images of quantities under f. Let $a'_n$ and $b'_n$ be sequences of points in $A'$ and $B'$ such that $\text{dist}(A', B') = \lim_{n \to \infty} d'(a'_n, b'_n)$. Let $c'_n$ be a point in $A'$ such that $\text{diam}(A')/3 \leq d'(a'_n, c'_n)$ and $d'_n$ a point in $B'$ such that $\text{diam}(B')/3 \leq d'(b'_n, d'_n)$. Then, as $\eta$ is increasing,

$$\eta(\Delta(A, B)^{-1}) \geq \eta \left( \frac{d(a_n, c_n) \wedge d(b_n, d_n)}{d(a_n, b_n)} \right) \geq \frac{d'(a'_n, c'_n) \wedge d'(b'_n, d'_n)}{d'(a'_n, b'_n)} \geq \frac{\text{diam}(A') \wedge \text{diam}(B')}{3d'(a'_n, b'_n)}.$$ 

Letting $n \to \infty$ we see $3\eta(\Delta(A, B)^{-1}) \geq \Delta(A', B')^{-1}$ and so $\Delta(A', B') \geq 1/\eta((3\Delta(A, B))^{-1})$. Hence we may set $\Phi(t) = 1/\eta((3t)^{-1})$. \hfill $\square$

Remark 4.1.6. We may apply the above result to $f^{-1}$ as well to conclude that there exists an increasing function $\Phi': (0, \infty) \to (0, \infty)$ with $\lim_{t \to \infty} \Phi'(t) = \infty$ such that for all open $A, B \subseteq Z$ with $\text{dist}(A, B) > 0$ we have $\Delta(A, B) \geq \Phi'(\Delta(f(A), f(B)))$. Thus,

$$\Phi(\Delta(A, B)) \leq \Delta(f(A), f(B)) \leq \Phi^{-1}(\Delta(A, B))$$

Lemma 4.1.7. $Q'_w \leq \text{ARCDim}.$

Proof. Let $p > \text{ARCDim}$. Let $\theta$ be an Ahlfors regular metric on $Z$ such that $\text{dim}_H(Z, \theta) \leq p$. We work in $(Z, \theta)$, which is justified by Lemma 4.1.5. Let $A$ and $B$ be open with $\text{dist}(A, B) > 0$. Without loss of generality assume $\text{diam}(A) \leq \text{diam}(B)$ and let $\Delta = \Delta(A, B) > 0$. Set $m = \text{diam}(A)$ and $D = \text{dist}(A, B)$, so $\Delta = D/m$. Let $X = (V_X, E_X)$ be a hyperbolic filling for $(Z, \theta)$ with parameter $s > 1$. Consider $g: V_X \to \mathbb{R}$ defined by

$$g(v) = \frac{r(B_v)}{\theta(v, A) \vee D/4} \chi_{\{\theta(v, a) \leq 3D/4\}}$$

where $x \vee y = \max\{x, y\}$ and we have used $v$ as both the vertex in $V$ and the center of the corresponding ball $B_v$.

We estimate $\|g\|_{p, \infty}^p$. Observe that as $\text{dim}_H(Z, \theta) \leq p$ there is a $C > 0$ such that the number of vertices on level $n$ with centers lying in a ball of radius $R$ is bounded above by $CR^n s^{np}$. For $\lambda > 0$ we have $g(v) > \lambda$ if and only if

$$r(B_v)/\lambda > \theta(v, A) \vee D/4 \text{ and } \theta(v, A) \leq 3D/4.$$
Writing \( r(B_v) = 2s^{-k} \), this is

\[
2s^{-k}/\lambda > \theta(v, A) \lor D/4 \text{ and } \theta(v, A) \leq 3D/4.
\]

As \( 3D/4 > D/4 \), if \( 2s^{-k}/\lambda > 3D/4 \) we see all vertices with \( \theta(v, A) \leq 3D/4 \) on level \( k \) satisfy these inequalities. From our observation above, there are at most

\[
C((3D/4) + m)p s^{kp}
\]

such vertices on level \( k \). If \( 2s^{-k}/\lambda \leq D/4 \) we see no vertices on level \( k \) satisfy these inequalities. For \( k \) in between we have \( \theta(v, A) < 2s^{-k}/\lambda \leq 3D/4 \) and so there are at most

\[
C((2s^{-k}/\lambda) + m)p s^{kp} = C((2/\lambda) + ms^k)p
\]

many vertices satisfying these inequalities. In this case, \( D/4 < 2s^{-k}/\lambda \) so \( s^k < 8/D\lambda \). Thus, we obtain the upper bound

\[
C((2/\lambda) + 8m/D\lambda)p \lesssim (2 + 8\Delta^{-1})p/\lambda^p.
\]

Combining these estimates, we have

\[
\#\{v : g(v) > \lambda\} \lesssim \sum_{2s^{-k}/\lambda > 3D/4} C((3D/4) + m)p s^{kp} + \sum_{D/4 < 2s^{-k}/\lambda \leq 3D/4} (2 + 4\Delta^{-1})p/\lambda^p.
\]

The first sum is geometric and hence estimated by the last term. This is bounded by a constant times \( C((3D/4) + m)p(8/3\lambda D)p \) which is \( C(2 + \Delta^{-1}8/3)p/\lambda^p \). The second sum has a number of terms independent of \( \lambda \). Hence, \( \|g\|_{p,\infty} \lesssim (1 + \Delta^{-1})^p \).

We now investigate admissibility for \( g \). Consider a path of vertices \( \gamma \) with limits in \( A \) and \( B \). Consider the subpath \( \gamma' \) of \( \gamma \) denoted \( v_0, v_1, \ldots, v_N \) such that \( v_0 \) is the last vertex with \( \theta(v_0, A) \leq D/4 \) and \( v_N \) is the first vertex after \( v_0 \) with \( \theta(v_N, A) > 3D/4 \). With \( B_{v_j} = B(v_j, r_j) \), we have \( \theta(v_j, v_{j+1}) < r_j + r_{j+1} < (1 + (1 + s))r_j \). It follows that \( \theta(v_{j+1}, A) \leq \theta(v_j, A) + (2 + s)r_j \). Thus,

\[
\sum_{j=0}^{N-1} \frac{(2 + s)r_j}{\theta(v_j, A) \lor D/4} \geq \int_{D/4}^{3D/4} \frac{dx}{x}.
\]
We now estimate the sum of $g$ over $\gamma$:

$$
\sum_{\gamma} (2 + s)g(v) \geq \sum_{j=0}^{N-1} \frac{(2 + s) r_j}{\theta(v_j, A) \vee D/4} \geq \int_{D/4}^{3D/4} \frac{dx}{x} = \log(3).
$$

Now, set $f = (2 + s)g/\log(3)$. Set $\tau(e) = f(e_+) + f(e_-)$. The above shows $\tau$ is admissible for $\text{wcap}_p(A, B)$. We have $\|\tau\|^p_{p, \infty} \lesssim \|f\|^p_{p, \infty}$ and so

$$
\|\tau\|^p_{p, \infty} \lesssim \|g\|^p_{p, \infty} \lesssim (1 + \Delta^{-1})^p.
$$

It follows that $Q'_w \leq p$ and, as $p > \text{ARCdim}$ was arbitrary, we conclude $Q'_w \leq \text{ARCdim}$. \qed

### 4.2 Path lemmas

In what follows it is useful to refine the path lemma in Section 3.2. Our main lemma is Lemma 4.2.3. We first start with a modification of Lemma 3.2.4.

**Lemma 4.2.1.** Let $v \in V_X$ and let $T_v$ be a binary path structure with splitting constant $M$. Let $p > 1$. Then, there is a function $S(a, \beta) = S(a, \beta, p, M) < \infty$ with the following property: whenever $\tau : E_X \to [0, \beta]$ is a function with $\|\tau\|^p_{p, \infty} \leq a$, then there is an ascending edge path with $\tau$-length bounded above by $S(a, \beta)$. Moreover, $S(a, \beta) \to 0$ as $a \to 0$ for fixed $\beta$ and as $\beta \to 0$ for fixed $a$.

The proof is much the same as Lemma 3.2.4.

**Proof.** Suppose $T_v$ is as above and $\tau : E_X \to [0, \beta]$ is a function with $\|\tau\|^p_{p, \infty} \leq a$. Note that there are at least $M$ edges in the first $M$ levels of $T_v$, at least $2M$ edges in levels $M + 1$ to $2M$ of $T_v$, and in general at least $2^{k-1}M$ edges in the levels $(k - 1)M + 1$ to $kM$ in $T_v$. Hence, the total number of edges up to level $kM$ is at least

$$
\sum_{j=1}^{k} 2^{j-1}M = (2^k - 1)M.
$$

As before, we bound the total path length $\tau$ gives to paths. That is, if $\Gamma_k$ is the collection of ascending paths starting at $v$ and ending at level $kM$, then we bound

$$
\sum_{\gamma \in \Gamma_k} \sum_{e \in \gamma} \tau(e).
$$
This is increased the more mass is placed on lower level edges, which use to bound this quantity. From the weak norm definition, we have

$$\# \{ e : \tau(e) > \lambda \} \leq a^p / \lambda^p. \quad (4.1)$$

Choose $N$ such that $(2^N - 1)M < a^p / \beta^p \leq (2^{N+1} - 1)M$. We overestimate by assuming all edges up to level $(N+1)M$ carry weight $\beta$. Inequality (4.1) then tells us all other weights satisfy $\tau(e) \leq a((2^{N+1} - 1)M)^{-1/p}$. Using this bound for the next (at least) $2^{N+1}M$ edges up to level $(N+2)M$, we see the weights placed on edges with levels more than $(N+2)M$ satisfy $\tau(e) \leq a((2^{N+2} - 1)M)^{-1/p}$. Continuing in this manner, we obtain the following bound:

$$\sum_{\gamma \in \Gamma_k} \sum_{e \in \gamma} \tau(e) \leq 2^k \left( M \beta(N + 1) + \sum_{j=N+1}^{k-1} \frac{Ma^p}{[(2^j - 1)M]^{1/p}} \right)$$

To produce $f$, we average and diagonalize as before to conclude that there is an ascending edge path with length bounded above by

$$S(a, \beta) = \left( M \beta(N + 1) + \sum_{j=N+1}^{\infty} \frac{Ma^p}{[(2^j - 1)M]^{1/p}} \right).$$

We also compute from $(2^N - 1)M < a^p / \beta^p$ that $N < \log(1 + a^p / \beta^p M) / \log(2)$. Hence, $\beta N \to 0$ as $\beta \to 0$ and as $a \to 0$. Likewise, from our other estimate, $N \geq \log(1 + a^p / \beta^p M) / \log(2) - 1$ and so $N \to \infty$ as $\beta \to 0$ for fixed $a$. These observations are what is needed for the claimed behavior of $S$.

In what follows we will need a stronger conclusion. Above we bound the average path length from $\tau$ by $S(a, \beta)$. Instead, we wish to guarantee at least $m$ paths have a “small” bounded length. For this, we first prove the result abstractly and then use

$$S_k(a, \beta) = \left( M \beta(N + 1) + \sum_{j=N+1}^{k-1} \frac{Ma^p}{[(2^j - 1)M]^{1/p}} \right),$$

the average bound we found above up to to level $k$ with $k > N + 2$.

**Lemma 4.2.2.** Suppose $\Gamma$ is a finite collection of paths and $\tau : E_X \to [0, \infty)$ is a function such that the average $\tau$-length of paths in $\Gamma$ is $\alpha$. Let $h > \alpha$. Then, there are at least $|\Gamma|(1 - \alpha/h)$ paths with $\tau$-length bounded above by $h$. 

61
Proof. The total \( \tau \)-length given to paths is \(|\Gamma|\alpha\). Suppose \( m \) paths have \( \tau \)-length greater than \( h \). Then, we must have \( mh \leq |\Gamma|\alpha \) and so \( m \leq |\Gamma|\alpha/h \). Thus, there are at least \(|\Gamma| - |\Gamma|\alpha/h = |\Gamma|(1 - \alpha/h) \) paths with \( \tau \)-length bounded above by \( h \).

In our situation, \( \Gamma = \Gamma_k \) and so \(|\Gamma| = 2^k\). We also have \( \alpha \leq S_k(a, \beta) \leq S(a, \beta) \). Hence, there are at least
\[
2^k(1 - S(a, \beta)/h)
\]
paths with \( \tau \)-length bounded above by \( h \).

We now state and prove our main path lemma.

Lemma 4.2.3 (Main Path Lemma). Let \( T_v \) be a binary path structure with splitting constant \( M \) and let \( p > 1 \). Suppose \( \tau : E_X \to [0, 1] \) satisfies \( \|\tau\|_{p, \infty} \leq a \). Let \( \delta > 0 \). Then, there is a real number \( \beta > 0 \) and level \( \ell_0 = \ell_0(a, \beta, M) \) with the following property: if \( \tau(e) \leq \beta \) for all \( e \) with \( \ell(e) \leq \ell_0 \) (where \( \ell(e) \) is the maximal level of the endpoints of \( e \)), then there is an infinite ascending path \( \gamma \) with \( \sum_{e \in \gamma} \tau(e) < \delta \).

Intuitively the \( \beta \) bound forces the initial portions of ascending paths to have small \( \tau \)-length while the bound on \( \|\tau\|_{p, \infty} \) together with the exponential growth of \( T_v \) forces there to be many paths with ending portions with small \( \tau \)-length.

Proof of Lemma 4.2.3. Let \( T_v \), \( M \), \( p \), and \( \tau \) be as above. Let \( \beta \) be such that \( 3S(a, \beta) < \delta \). If \( \tau \leq \beta \) up to level \( Mk \) then there are \( 2^k \) paths with average path length bounded above by \( S(a, \beta) \). These paths have the property that their corresponding subtrees (i.e. the subtrees starting at the end vertices of these paths) are disjoint. Let \( h = 2S(a, \beta) \) in Lemma 4.2.2 and choose \( k \) such that \( 2^{k-1} > a^p/\beta^p + 1 \). Then, there are at least \( 2^k(1 - S(a, \beta)/2S(a, \beta)) = 2^{k-1} \) paths with length bounded above by \( 2S(a, \beta) \). As \( |\{e : \tau(e) > \beta\}| \leq a^p/\beta^p \) it follows that the restriction of \( \tau \) to at least one of these subtrees is bounded by \( \beta \). On this subtree we apply Lemma 4.2.1 to conclude there is an ascending path with \( \tau \)-length bounded above by \( S(a, \beta) \). Concatenate these paths and set \( \ell_0 = kM \).

Remark 4.2.4. For fixed \( a \), as \( \delta \to 0 \) we have \( \beta \to 0 \) and \( \ell_0 \to \infty \)
4.3 Control functions that tend to 0

In this section we will show that control functions \( \varphi \) for \( Q_w' \) can always be assumed to satisfy \( \varphi(t) \to 0 \) as \( t \to \infty \).

**Theorem 4.3.1.** Suppose \((Z, d, \mu)\) is an Ahlfors \( Q \)-regular metric space. Let \( p > 1 \). Suppose there is a decreasing function \( \varphi : (0, \infty) \to (0, \infty) \) such that for all open sets \( A \) and \( B \) in \( Z \) with \( \text{dist}(A, B) > 0 \) one has \( \text{wcap}_p(A, B) \leq \varphi(\Delta(A, B)) \). Then, there is a decreasing homeomorphism \( \varphi' : (0, \infty) \to (0, \infty) \) with the same property.

To prove the result we will use the notion of the hull of a set in a hyperbolic filling \( X \). Recall here \((Z, d)\) is our metric space and \( X = (V_X, E_X) \) is a given hyperbolic filling with parameter \( s > 1 \).

**Definition 4.3.2.** Let \( A \subseteq Z \). Define \( j_A \in \mathbb{N} \) by

\[
-\log_s(\text{diam}(A)) < j_A \leq -\log_s(\text{diam}(A)) + 1
\]

so \( \text{diam}(A)/s \leq s^{-j_A} < \text{diam}(A) \). The hull of \( A \) in \( X \) is defined as

\[
H(A) = H_A = \{ v \in V_X : \ell(v) \geq j_A \text{ and } B_v \cap A \neq \emptyset \}.
\]

**Lemma 4.3.3.** Let \( K > 0 \). There are constants \( c_0(K) > 0 \) and \( R_{\text{max}} \) such that if \( 0 < r_1 < r_2 < R_{\text{max}} \) satisfy \( r_2/r_1 > c_0 \), then for any \( z \in Z \) we have

\[
\rho(H(B_{r_1}(z)), X \setminus H(B_{r_2}(z))) \geq K.
\]

where \( \rho \) designates distance in \( X \).

**Proof.** We choose \( R_{\text{max}} \) such that for any \( z \in Z \) the set \( X \setminus H(B_{r_2}(z)) \) is nonempty. Fix \( z \in Z \) and let \( H_i = H(B_{r_i}(z)) \) for \( i \in \{1, 2\} \), so we wish to show \( \rho(H_1, X \setminus H_2) \geq K \). Consider a chain of neighboring vertices \( v_0, v_1, v_2, \ldots, v_m \) such that \( v_0 \in H_1 \) and \( v_m \notin H_2 \).

From the definition of the hulls, we have integers \( j_i = j_{B_{r_i}(z)} \). There are two cases, either (i) \( \ell(v_m) < j_2 \) or (ii) \( \ell(v_m) \geq j_2 \) and \( B_{v_m} \cap B_{r_2}(z) = \emptyset \).
Assume (i). We have $\ell(v_0) \geq j_1$ and so $m \geq j_1 - j_2$. We have
\[ j_1 - j_2 \gtrsim \log_s(\text{diam}(B_{r_2})/\text{diam}(B_{r_1})) \gtrsim \log_s(cr_2/r_1) \]
for a constant $c > 0$ as Ahlfors regular spaces are uniformly perfect. Thus, if $r_2/r_1$ is large then so is $m$.

Now, assume (ii). View the vertices $v_k$ as the centers of their respective balls in $Z$. Then, by the triangle inequality we have
\[ d(v_0, v_m) \geq d(z, v_m) - d(z, v_0) \geq r_2 - r_1 - r(B_{v_0}). \]

We see
\[ r(B_{v_0}) \leq 2s^{-j_1} < 2 \text{diam}(B_{r_1}) \leq 4r_1 \]
and so
\[ d(v_0, v_m) \geq r_2 - 5r_1. \]

We also have
\[ d(v_k, v_{k+1}) \leq 2s^{-\ell(v_k)} + 2s^{-\ell(v_k)+1}. \]

Hence,
\[ d(v_0, v_m) \leq \sum_{k=1}^m 4s^{-j_1+k} = 4s^{-j_1} \frac{s^m - 1}{s - 1} \lesssim r_1 s^{m+1}. \]

Combining these two inequalities, we have
\[ \frac{r_2}{r_1} - 5 \lesssim s^m \]
and so if $r_2/r_1$ is large then so is $m$. \hfill \Box

We record the following observation that paths connecting the separated regions $H_1$ and $X \setminus H_2$ as in Lemma 4.3.3 must at some point be far from both sets.

**Lemma 4.3.4.** Suppose $0 < r_1 < r_2 < R_{\max}$ and let $z \in Z$. Set $H_i = H_{B(z, r_i)}$. Let $L \in \mathbb{N}$ and suppose $\rho(H_1, X \setminus H_2) \geq 2L + 1$. Define $R = H_2 \setminus H_1$. Let $\gamma$ be an edge path in $X$ such that there are edges $e_1, e_2 \in \gamma$ with $e_1 \in H_1$ and $e_2 \notin H_2$. Then, there is an edge $e \in E_X$ with $e \in H_2 \setminus H_1$ such that $\rho(e, H_1) \geq L$ and $\rho(e, X \setminus H_2) \geq L$.  

64
Proof. This follows immediately as the vertices of a given edge are at distance 1 from one another; tracing $\gamma$ from $e_1$ to $e_2$ we must encounter such an edge.

Definition 4.3.5. If $\gamma, H_1, H_2$ and $R$ are defined as above, we say $\gamma$ travels through $R$.

Lemma 4.3.6. Let $N, K \in \mathbb{N}$ and $c_1 > 0$. Then, there exists $c_2 = c_2(K, N)$ such that if $A, B \subseteq Z$ satisfy $\text{diam}(B) \leq \text{diam}(A)$ and $\Delta(A, B) \geq c_2$ then, given $b \in B$, we can construct a set of $N + 1$ concentric balls $B_k = B_{r_k}(b)$ with $r_0 = \text{diam}(B)$ such that $A \cap B_N = \emptyset$, the radii satisfy $r_{k+1}/r_k \geq c_0(K)$, and for each $0 \leq k < N$ we have $\Delta(10B_k, Z \setminus B_{k+1}) \geq c_1$.

Proof. In what follows we note we may assume $r_2$ is small enough such that $Z \setminus B_{r_2}(b) \neq \emptyset$. We see that if $r_2 > 10r_1$ then

$$\Delta(10B_{r_1}(b), Z \setminus B_{r_2}(b)) \geq \frac{r_2 - 10r_1}{20r_1} = \frac{r_2}{20r_1} - \frac{1}{2}.$$ 

Thus, to satisfy the relative distance condition and the $c_0(K)$ condition it suffices to choose the successive ratios $r_{k+1}/r_k$ to be large enough, say $r_{k+1}/r_k \geq c_3(K)$. We choose $r_0 = \text{diam}(B)$ and $r_{k+1} = c_3r_k$. As $\text{diam}(Z) \leq 1$ and $\text{diam}(B) \leq \text{diam}(A)$, we see $\text{diam}(B) \leq 1/\Delta(A, B)$. Thus, for fixed $N$, we can guarantee for $c_2$ large enough that $r_N < R_{\text{max}}$. If $B_N \cap A \neq \emptyset$ then $\text{dist}(A, B) \leq r_N = c_3^N \text{diam}(B)$ and so

$$\Delta(A, B) \leq \frac{\text{dist}(A, B) \leq \text{diam}(B) \leq c_3^N}. $$

Thus $c_2 = c_3^N + 1$ satisfies the claim.

Lemma 4.3.7. Let $(Z, d, \mu)$ be a compact, connected, Ahlfors $Q$-regular metric measure space and let $X = (V_X, E_X)$ be a hyperbolic filling of $Z$ with parameter $s$.

(i) Let $z \in Z$ and set $B = B_r(z)$. Suppose $v \in V_X$ satisfies $v \in H(B)$. Then, there is a binary path structure $T_v$ with splitting constant $M$ depending only on $(Z, d, \mu)$ and $s$, originating at $v$, such that all ascending paths have boundary limits in $10B$.

(ii) There exists $R'_{\text{max}} > 0$ with the following property. Let $z \in Z$ and let $B = B_r(z)$ and $B' = B_t(z)$ with $0 < r < t < R'_{\text{max}}$. Suppose $v \in V_X$ satisfies $v \in H(B') \setminus H(B)$. Then there is a binary path structure $T_v$ with splitting constant $M'$ depending only on $(Z, d, \mu)$ and $s$,
originating at \( v \), such that all ascending paths (in this case geodesic paths in \( T_v \) originating at \( v \)) have boundary limits in \( Z \setminus \overline{B} \).

**Proof.** We first show (i). As \( v \in H(B) \) we see \( B_v \cap B \neq \emptyset \) and \( \text{diam}(B_v) \leq 8r \). Thus, \( B_v \subseteq 10B \). The result then follows from repeatedly applying Lemma 3.2.1 to \( B_v \).

We now show (ii). As \( v \notin H(B) \), either \( B_v \cap B = \emptyset \) or \( B_v \cap B \neq \emptyset \) and \( \ell(v) < j_B \).

If \( B_v \cap B = \emptyset \) we form our structure as in (i). To guarantee the limits lie in \( Z \setminus \overline{B} \) we first pass to a vertex \( w \in V_X \) with \( \ell(w) = \ell(v) + 1 \) and \( v \in B_w \). This guarantees \( B_w \cap B = \emptyset \), so we may apply this construction.

Suppose \( B_v \cap B \neq \emptyset \) and \( \ell(v) < j_B \). Consider a parent \( x \) of \( v \) on level \( \ell(v) - j \), where we define \( R'_{\text{max}} \) such that \( \ell(v) - j > 0 \). The maximal \( j \) required will only depend on \((Z,d,\mu)\) and \( s \) below, so \( R'_{\text{max}} \) can be defined this way. We have \( r(B_x) = r(B_v)s^j \). Let \( c_Q \) and \( C_Q \) denote the Ahlfors regularity constants of \( Z \), so for every ball \( B_r \) of radius \( r \leq \text{diam}(Z) \) we have \( c_Q r^Q \leq \mu(B_r) \leq C_Q r^Q \). Then,

\[
\mu(B_x \setminus (4 + s)B_v) \geq c_Q r(B_v)^Q s^j Q - C_Q((4 + s)r(B_v))^Q
\]

which is positive for large enough \( j \) depending only on \((Z,d,\mu)\) and \( s \). Now, choose a point \( z' \in B \setminus (4 + s)B_v \). There is a vertex \( w \) with level \( \ell(w) = \ell(v) \) such that \( z' \in B_w \). We claim \( \overline{B_w} \cap \overline{B} = \emptyset \). As \( \ell(v) < j_B \), we have

\[
2 \text{diam}(B)/s \leq 2s^{-j_B} \leq 2s^{-\ell(v)} = r(B_v).
\]

Thus, as every point \( b \in B \) satisfies \( d(b,v) \leq r(B_v) + \text{diam}(B) \), we have \( \overline{B} \subseteq (1 + s)B_v \). We see \( d(v,z') \geq (4 + s)r(B_v) \) and so if \( w' \in B_w \) we have

\[
d(w',v) \geq d(v,z') - d(z',w') \geq (4 + s)r(B_v) - 2r(B_v) = (2 + s)r(B_v).
\]

Hence, \( \text{dist}(w',(1 + s)B_v) \geq r(B_v) \). Thus, \( \overline{B_w} \cap \overline{B} = \emptyset \). The result now follows from applying Lemma 3.2.1 to \( B_w \). The key here is that there is a controlled number of steps to get from \( B_v \) to \( B_w \) going through \( B_x \) which impacts the resulting \( M' \).
Remark 4.3.8. By choosing the maximum of the values $M$ and $M'$ above, we may assume there is a uniform $M$ that works in both situations.

Proof of Theorem 4.3.1 Let $\epsilon > 0$ and $p > 1$. We will show if $\Delta(A, B)$ is large enough, then $\text{wcap}_p(A, B) < \epsilon$. We assume $\text{diam}(B) \leq \text{diam}(A)$. Choose $\beta$ from Lemma 4.2.3 with $\delta = 1/4$, $a = \varphi(1)$ and $M$ from Remark 4.3.8. Let $K = 2\ell_0(\beta) + 3$ where $\ell_0(\beta)$ is from Lemma 4.2.3. Let $n_1, n_2 \in \mathbb{N}$ be such that $2^p\varphi(1)/n_1^{p-1} < \epsilon$ and $n_2 > 2((\varphi(1)^2/\beta^2) + 2)$. Let $N = n_1(2n_2 + 2)$.

Let $\Delta(A, B) > c_2$ from Lemma 4.3.6 with $N, K$ defined above and $c_1 = 1$. Fix $b \in B$ and construct the balls $B_k$ as in that lemma. From this we obtain $N$ rings $H(B_{k+1}) \setminus H(B_k)$ in $X$; we partition these in increasing order into $n_1$ different packages $P_i$ of $2n_2 + 2$ rings, say package $i$ contains the rings $R_j = R_j(i) = H(B_{q_i+j}) \setminus H(B_{q_i+j-1})$ for $q_i = i(2n_2 + 2)$ and $j \in \{1, \ldots, 2n_2 + 2\}$. Consider the ring $R_{q_i+n_2+1}$. By construction,

$$\Delta(10B_{q_i+n_2}, Z \setminus \overline{B_{q_i+n_2+1}}) \geq 1$$

and so there is an admissible $\tau$ for $\text{wcap}_p(10B_{q_i+n_2}, Z \setminus \overline{B_{q_i+n_2+1}})$ such that $\|\tau\|_{p, \infty} \leq \varphi(1)$. We claim $2\tau_i = 2\tau|_{P_i}$ is admissible for $\text{wcap}_p(10B_{q_i+n_2}, Z \setminus \overline{B_{q_i+n_2+1}})$.

Let $\gamma$ be an edge path in $X$ connecting $10B_{q_i+n_2}$ and $Z \setminus \overline{B_{q_i+n_2+1}}$. As the rings $R_j$ are disjoint, it follows from the definition of $n_2$ that there are $k \in \{1, \ldots, n_2\}$ and $k' \in \{n_2 + 2, \ldots, 2n_2 + 2\}$ such that for all $e \in R_{q_i+k}, R_{q_i+k'}$ we have $\tau(e) \leq \beta$. Assume the path $\gamma$ travels through at least one of $R_{q_i+k}$ and $R_{q_i+k'}$; otherwise it only lies in $P_i$ and $2\tau_i$ defined below is admissible for $\gamma$. From any vertex in $R_{q_i+k}$ or $R_{q_i+k'}$ at distance at least $(K - 1)/2$ from the boundaries of these sets we may form a binary path structure as in Lemma 4.3.7 with limits in $10B_{q_i+n_2}$ for $R_{q_i+k}$ and in $Z \setminus \overline{B_{q_i+n_2+1}}$ for $R_{q_i+k'}$. Such a path structure avoids $\beta$ for long enough to satisfy the conditions in Lemma 4.2.3 and hence contains an ascending edge path of $\tau$-length $< 1/4$. Thus, the portion of $\gamma$ in $P_i$ must have $\tau$-length $\geq 1/2$. Hence, $2\tau_i$ is admissible. As $A \subseteq Z \setminus \overline{B_{q_i+n_2+1}}$ and $B \subseteq 10B_{q_i+n_2}$, it follows that $2\tau_i$ is admissible for $\text{wcap}_p(A, B)$ as well. Note that the functions $2\tau_i$ have disjoint support.
Now, set
\[ T = \frac{1}{n_1} \sum_{i=1}^{n_1} 2\tau_i. \]
As each \(2\tau_i\) is admissible for \(w\text{cap}_p(A, B)\) it follows that \(T\) is as well. We estimate \(\|T\|_{p,\infty}\)
recalling that the functions \(\tau_i\) have disjoint support:
\[
\#\{e : T(e) > \lambda\} = \#\{e : \exists i \text{ s.t. } 2\tau_i(e)/n_1 > \lambda\} \\
\leq n_1\#\{e : \tau_i(e) > \lambda n_1/2\} \\
\leq 2^p \varphi(1)n_1/(\lambda n_1)^p \\
= (2^p \varphi(1)/n_1^{p-1})(1/\lambda^p)
\]
which, by our choice of \(n_1\), is bounded by \(\epsilon/\lambda^p\).

### 4.4 Quasisymmetric uniformization

In this section we prove Theorem 1.1.15, a result analogous to \[BnK\] Theorem 10.4.

The converse follows from Theorem 1.1.4 and Remark 4.1.6 as \(S^2\) is a Loewner space; alternatively this follows from Lemma 4.1.7. Inequality (1.2) states that \(Q_w'\leq 2\).

Comparing this to \[BnK\] Theorem 10.4 we have the following observations. The control function \(\Psi(t)\) in [BnK] Theorem 10.4 was required to tend to 0 as \(t \to \infty\). Here this is not required a priori as Theorem 4.3.1 provides a function with this limiting behavior. The statement of [BnK] Theorem 10.4 is also fairly technical. While the statement of Theorem 1.1.15 has technicalities as well, particularly in the definition of the hyperbolic filling, it has the advantage of being simpler to parse.

To prove Theorem 1.1.15, we will use a number of results from [BnK]. We list the full statements with some discussion in the next few pages, but we omit proofs.

**Results from [BnK] and preliminaries**

It will be easier to work with quasi-Möbius maps than quasisymmetric maps in what follows. In a metric space \((Z, d)\), the cross-ratio of four distinct points \(z_1, z_2, z_3, z_4 \in (Z, d)\) is given...
by
\[ [z_1, z_2, z_3, z_4] = \frac{d(z_1, z_3)d(z_2, z_4)}{d(z_1, z_4)d(z_2, z_3)}. \]

**Definition 4.4.1.** Given a homeomorphism \( \eta: [0, \infty) \to [0, \infty) \), a homeomorphism \( f: (Z, d) \to (Z', d') \) is \( \eta \)-quasi-Möbius if for every set of four distinct points \( z_1, z_2, z_3, z_4 \in (Z, d) \) with images \( f(z_i) = z'_i \in (Z', d') \) we have
\[ [z'_1, z'_2, z'_3, z'_4] \leq \eta([z_1, z_2, z_3, z_4]). \]

A quantity related to the cross-ratio, called the modified cross-ratio, is given by
\[ \langle z_1, z_2, z_3, z_4 \rangle = \frac{d(z_1, z_3) \wedge d(z_2, z_4)}{d(z_1, z_4) \wedge d(z_2, z_3)}. \]

By [BnK, Lemma 2.3], the function \( \eta_0(t) = 3(t \vee \sqrt{t}) \) has the property that for every set of four distinct points \( z_1, z_2, z_3, z_4 \in (Z, d) \) we have
\[ \langle z_1, z_2, z_3, z_4 \rangle \leq \eta_0([z_1, z_2, z_3, z_4]). \]

**Remark 4.4.2.** By permuting \( z_1, z_2, z_3, z_4 \) there is similar control of the modified cross-ratio by the cross-ratio from below.

On bounded spaces quasi-Möbius maps are quasisymmetric maps (cf. [Vä]). Thus, the limiting map attained from [BnK, Lemma 3.1] at the end of the proof will be our desired quasisymmetric map.

The basic strategy of the proof of Theorem 1.1.15 follows that of [BnK]. We briefly state how each of the lemmas here will be used in the proof that follows.

First, we will set up denser and denser graphs in \( Z \) and \( S^2 \) which are triangulations by using the following proposition.

**Proposition 4.4.3** ([BnK], Proposition 6.7). Suppose \((Z, d)\) is a metric space homeomorphic to \( S^2 \). If \((Z, d)\) is \( C_0 \)-doubling and \( \lambda \)-LLC, then for given \( 0 < r < \text{diam}(Z) \) and any maximal \( r \)-separated set \( A \subseteq Z \) there exists an embedded graph \( G = (V, E) \) which is the 1-skeleton of a triangulation \( T \) of \( Z \) such that:
The valence of $G$ is bounded by $K$.

(ii) The vertex set $V$ of $G$ contains $A$.

(iii) If $e \in E$ then $\text{diam}(e) < Kr$. If $u,v \in V$ and $d(u,v) < 2r$ then $k_G(u,v) < K$.

(iv) For all balls $B(a,r) \subseteq Z$ we have $\#(B(a,r) \cap V) \leq K$.

The constant $K \geq 1$ depends only on $C_0$ and $\lambda$.

Property (iii) will be used to show that we can combine these triangulations into a structure $Y$ which is quasi-isometric to our hyperbolic filling. To use other results in [BnK], we need to know that these structures form $K$-approximations. This is done in $Z$ by associating the objects $p(y) = y$, $r(y) = s^{-n}$, and $U_y = B(p(y), Kr(y))$ to form $p, r$, and $\mathcal{U}$.

**Corollary 4.4.4 ([BnK], Corollary 6.8).** $(G, p, r, \mathcal{U})$ is a $K'$-approximation of $Z$, where $K'$ depends only on $\lambda$ and $C_0$.

In $S^2$ we use properties of normalized circle packings.

**Lemma 4.4.5 ([BnK], Lemma 5.1).** Suppose $G$ is combinatorially equivalent to a 1-skeleton of a triangulation of $S^2$, and $\mathcal{C}$ is a normalized circle packing realizing $G$. Then $(G, p, r, \mathcal{U})$ is a $K$-approximation of $S^2$ with $K$ depending only on the valence of $G$.

We use these structures to define coarse quasi-Möbius maps between vertex subsets. This is done with the aid of the following Lemma.

**Lemma 4.4.6 ([BnK], Lemma 4.7).** Suppose $(Z, d)$ is a connected metric space and $((V, \sim), p, r, \mathcal{U})$ is a $K$-approximation of $Z$. Suppose $L \geq K$ and $W \subseteq V$ is a maximal set of combinatorially $L$-separated vertices. Then $M = p(W) \subseteq Z$ is weakly $\lambda$-uniformly perfect with $\lambda$ depending only on $L$ and $K$.

We show a subsequence of these maps is actually uniformly quasi-Möbius. If not, then [BnK] Lemma 3.3] would fail.

**Lemma 4.4.7 ([BnK], Lemma 3.3).** Suppose $(X, d_X)$ and $(Y, d_Y)$ are metric spaces, and $f: X \to Y$ is a bijection. Suppose that $X$ is weakly $\lambda$-uniformly perfect, $Y$ is $C_0$-doubling,
and there exists a function $\delta_0: (0, \infty) \to (0, \infty)$ such that

$$[f(x_1), f(x_2), f(x_3), f(x_4)] < \delta_0(\epsilon) \implies [x_1, x_2, x_3, x_4] < \epsilon$$

whenever $\epsilon > 0$ and $(x_1, x_2, x_3, x_4)$ is a four-tuple of distinct points in $X$. Then $f$ is $\eta$-quasi-Möbius with $\eta$ depending only on $\lambda, C_0$, and $\delta_0$.

To show that this lemma holds, we argue by contradiction. To use the wcap\textsubscript{2} condition, we must extract open sets from our data. This is done by taking small open neighborhoods of the continua given by the following lemma.

**Lemma 4.4.8** ([BnK], Lemma 2.10). Suppose $(Z, d)$ is $\lambda$-LLC. Then there exist functions $\delta_1, \delta_2: (0, \infty) \to (0, \infty)$ depending only on $\lambda$ with the following properties. Suppose $\epsilon > 0$ and $(z_1, z_2, z_3, z_4)$ is a four-tuple of distinct points in $Z$.

(i) If $[z_1, z_2, z_3, z_4] < \delta_1(\epsilon)$, then there exist continua $E, F \subseteq Z$ with $z_1, z_3 \in E$, $z_2, z_4 \in F$, and $\Delta(E, F) \geq 1/\epsilon$.

(ii) If there exist continua $E, F \subseteq Z$ with $z_1, z_3 \in E$, $z_2, z_4 \in F$ and $\Delta(E, F) \geq 1/\delta_2(\epsilon)$, then $[z_1, z_2, z_3, z_4] < \epsilon$.

One finds a contradiction by using the Loewner property in $S^2$ and the following lemma to appropriate sets. Hence, our maps must have been uniformly quasi-Möbius.

**Proposition 4.4.9** ([BnK], Proposition 8.1). Let $(Z, d, \mu)$ be a $Q$-regular metric measure space, $Q \geq 1$, and let $\mathscr{A}$ be a $K$-approximation of $Z$. Then there exists a constant $C \geq 1$ depending only on $K$ and the data of $Z$ with the following property:

If $E, F \subseteq Z$ are continua and if $\text{dist}(V_E, V_F) \geq 4K$, then

$$\text{mod}_Q(E, F) \leq C \text{mod}_Q^G(V_E, V_F).$$

Lastly, we may find a subsequence which converges to our desired map.

**Lemma 4.4.10** ([BnK], Lemma 3.1). Suppose $(X, d_X)$ and $(Y, d_Y)$ are compact metric spaces, and $f_k: D_k \to Y$ for $k \in \mathbb{N}$ is an $\eta$-quasi-Möbius map defined on a subset $D_k$ of $X$. Suppose

$$\lim_{k \to \infty} \sup_{x \in X} \text{dist}(x, D_k) = 0$$
and that for \( k \in \mathbb{N} \) there exist triples \((x_k^1, x_k^2, x_k^3)\) and \((y_k^1, y_k^2, y_k^3)\) of points in \( D_k \subseteq X \) and \( Y \), respectively, such that \( f(x_k^i) = y_k^i \), \( k \in \mathbb{N}, i \in \{1, 2, 3\} \),

\[
\inf \{ d_x(x_k^i, x_j^i) : k \in \mathbb{N}, i, j \in \{1, 2, 3\}, i \neq j \} > 0,
\]

and

\[
\inf \{ d_x(x_k^i, x_j^i) : k \in \mathbb{N}, i, j \in \{1, 2, 3\}, i \neq j \} > 0.
\]

Then the sequence \((f_k)\) subconverges uniformly to an \( \eta \)-quasi-Möbius map \( f : X \to Y \), i.e. there exists an increasing sequence \((k_n)\) in \( \mathbb{N} \) such that

\[
\lim_{n \to \infty} \sup_{x \in D_{k_n}} d_y(f(x), f_{k_n}(x)) = 0.
\]

**Proof of uniformization**

*Proof of Theorem 1.1.15.* Let \( X \) be a hyperbolic filling of \( Z \) with parameter \( s \). We apply [BnK] Proposition 6.7 to the vertices \( V_{X^n} \) to form a graph \( Y^n \), so \( V_{X^n} \subseteq V_{Y^n} \) and \( Y^n \) embeds as a triangulation of \( Z \). As in [BnK], this induces triangulations \( T^n \) on \( S^2 \) with the same incidence pattern as \( Y^n \) and for which the incidence pattern is realized as that of a normalized circle packing. This allows us to define functions \( \varphi_n : V_{T^n} \to Z \) where we map a vertex to its corresponding image point in the triangulation. We note each \( Y^n \) forms a \( K \)-approximation of \( Z \) in the language of [BnK] when associating the objects \( p(y) = y \), \( r(y) = s^{-n} \), and \( U_y = B(p(y), Kr(y)) \) by [BnK] Corollary 6.8] and \( T^n \) forms a \( K' \)-approximation of \( S^2 \) by [BnK] Lemma 5.1].

We form an infinite graph \( Y \) from the graphs \( Y^n \) by connecting vertices \( y \in Y^n \) and \( y' \in Y^{n+1} \) if \( y \in X^n \), \( y' \in X^{n+1} \), and \( y, y' \) are connected in \( X \). We imbue \( Y \) with the graph metric.

We claim \( X \) and \( Y \) are quasi-isometric. Define \( F : Y \to X \) on vertices by sending a given vertex \( y \in Y^n \) to the closest vertex (in \( Y^n \)) to \( y \) that belongs to \( X^n \). If \( x, x' \in X^n \) are the closest vertices to \( y, y' \), then by [BnK] Proposition 6.7] (iii) it follows that \( d(x, x') < (2K + 1)Ks^{-n} \). Hence there is a constant \( C(K) \) such that \( C(K)B_x \cap B_{x'} \neq \emptyset \). Thus, when \( |y - y'|_Y = 1 \) it follows that \( |x - x'|_X \) is bounded by a constant depending only on \( K \). Now
suppose the images \( x, x' \in X^n \) satisfy \( |x - x'|_X = 1 \). Then, \( B_x \cap B_{x'} \neq \emptyset \), so \( d(y, y') < 6s^{-n} \). As noted in [BnK], from (iii) we conclude \( |y - y'|_Y \leq C(6, K, C_0, \lambda) \). Thus, there is a constant \( C > 0 \) such that for all \( y, y' \in Y^n \), we have
\[
\frac{1}{C} |y - y'|_Y - C \leq |x - x'|_X \leq C |y - y'|_Y + C.
\]
This holds for all \( y \in Y \) as the only connections between levels are those already there from \( X \) and \( F(x) = x \) whenever \( x \in X \).

For the coarse inverse \( G \), we set \( G(x) = x \in Y \). It follows from [BnK, Proposition 6.7] (iii) that \( G \circ F \) and \( F \circ G \) are within bounded distance to the identity and that every point in \( Y \) is within \( K \) of a point in \( X \).

We claim the family \( \{ \varphi_n \} \) restricted to suitable subsets of \( V_{Y^n} \) is uniformly quasi-Möbius. For this we apply [BnK, Lemma 4.7] to \( Y^n \) with \( L = K \) to generate \( W = W^n \), a set of vertices such that \( p(W) \) is weakly uniformly perfect. We restrict \( \varphi_n \) to these subsets, which we denote as \( \varphi_n^W \). Suppose that \( \varphi_n^W \) is not uniformly quasi-Möbius. Then, the condition given in [BnK, Lemma 3.3] must fail. Hence, there exists \( \epsilon > 0 \) such that for each \( \delta_k = 1/k \) there exists \( \varphi_{n_k}^W \) and four points \( x_{1,k}, x_{2,k}, x_{3,k}, x_{4,k} \in W^{n_k} \) such that
\[
[x_{1,k}', x_{2,k}', x_{3,k}', x_{4,k}'] < \delta_k
\]
but
\[
[x_{1,k}, x_{2,k}, x_{3,k}, x_{4,k}] \geq \epsilon
\]
where \( x' \) denotes the image of \( x \) under \( \varphi_{n_k}^W \). Note we may assume \( n_k \to \infty \). Applying [BnK, Lemma 2.10] (i) we see that if \( \delta_k \) is small enough then there are continua \( E'_k \) and \( F'_k \) in \( Z \) such that \( x_{1,k}', x_{3,k}' \in E'_k, x_{2,k}', x_{4,k}' \in F'_k \), and \( \Delta(E'_k, F'_k) \) is large quantitatively. We consider open neighborhoods \( N_\rho(E'_N) \) and \( N_\rho(F'_N) \) where \( N \) is large enough and \( \rho \) is small enough such that \( \Delta(N_\rho(E'_N), N_\rho(F'_N)) \) is large and hence \( \varphi(\Delta(N_\rho(E'_N), N_\rho(F'_N))) \) is small. Fix \( N \) and \( \rho \).

For ease of notation, we write \( x_i = x_{i,N} \) and \( x'_i = x'_{i,N} \). Consider \( V_{E'_j} \subseteq V_{Y^j} \) defined by \( V_{E'_j} = V_{E'_j}(N) = \{ y \in Y^j : U_y \cap E'_N \neq \emptyset \} \) and likewise for \( V_{F'_j} \). From the corresponding vertices in \( T^j \) we define \( V_{E_j} \) and \( V_{F_j} \). From the incidence pattern in \( V_{E_j} \) and \( V_{F_j} \) and the fact
that $Y^j$ embeds in $Z$ we extract continua $E_j$ and $F_j$ such that $x_1, x_3 \in E_j$ and $x_2, x_4 \in F_j$. From this it follows that $\Delta(E_j, F_j) \leq (x_1, x_2, x_3, x_4)^{-1}$. As $S^2$ is a Loewner space, it follows that there is a lower bound $0 < a \leq \text{mod}_2(E_j, F_j)$. For large $j$, the combinatorial distance between $E_j$ and $F_j$ is large in $T^j$ as these sets were constructed from $Y^j$ and the sets there have large combinatorial distance. Thus, we may apply [BnK, Proposition 8.1] to conclude there is a constant $c > 0$ depending only on $\tau$ such that for large $j$ we have $c < \text{mod}_2(V_{E_j}, V_{F_j}, T^j)$. As the graphs are the same, this means for large $j$ we have $c < \text{mod}_2(V_{E_j'}, V_{F_j'}, Y^j)$.

Let $\tau$ be admissible for $\text{wcap}_2(N_\rho(E'_N), N_\rho(F'_N))$ and satisfy $\|\tau\|_{2,\infty} < \infty$. From Lemma 4.2.3 for large enough levels $j$ from any vertex in $V_{E'_j}$ we can guarantee the existence of a path connecting down to $N_\rho(E'_N)$ with $\tau$-length bounded above by $1/4$ (just avoid the required large values $\beta$ for enough levels; these occur on finitely many levels). Then, $f_j : V_{Y^j} \to [0, \infty)$ defined by $f_j(v) = \sum_{e \sim v} 2\tau(e)k(e)$ must be admissible for $\text{mod}_2(N_\rho(E'_N), N_\rho(F'_N), Y^j)$. The norms $\|f_j\|_{2,\infty}$ and $\|2\tau|_j\|_{2,\infty}$ are comparable, so we conclude that there is a constant $c' > 0$ depending only on $c$ such that $\|\tau|_j\|_2^2 \geq c'$ for these levels.

From [BnS, Proof of Theorem 1.4], for any $N' > 1$ there is an $n \in [N', 2N']$ such that

$$\|\tau|_n\|_2^2 = \sum_{e \in E_N} \tau(e)^2 \lesssim \|\tau\|_{2,\infty}^2,$$

but the left hand side of this is bounded below by $c' > 0$ for large $n$ and the right hand side tends to 0 with different choices of $N$ and $\rho$. Thus, our maps $\varphi^W_n$ are uniformly quasi-Möbius.

Lastly, we apply [BnK, Lemma 3.1] to the sequence $\varphi^W_n$. Because we normalized our circle packings, we need only know that the subsets $W^n$ have the property that

$$\lim_{n \to \infty} \sup_{x \in X} \text{dist}(x, W^n) = 0.$$

The vertices in $W^n$ are a maximal combinatorially $K$-separated subset of $Y^n$. Thus, by [BnK, Proposition 6.7] every vertex in $y \in V_{Y^n}$ satisfies $d(y, W^n) < K^2s^{-n}$. In particular, every $x \in V_{X^n}$ has this property and every point $z \in Z$ is $s^{-n}$ close to some point $x \in V_{X^n}$, so every $z \in Z$ is within $(K^2 + 1)s^{-n}$ of some point in $W^n$. As quasi-Möbius maps on bounded spaces are quasisymmetric, the result follows.

\[74\]
4.5 The Combinatorial Loewner Property

In [BdK] the Combinatorial Loewner Property (CLP) is investigated. This property is a discretized version of Definition 1.2.2; it requires that appropriately defined discrete modulus of connecting path families is controlled both below and above by relative distance. We show that if an Ahlfors $Q$-regular has the CLP with exponent $Q$, then $Q_w = Q' = ARCDim = Q$.

First we define some terms from [BdK] to make this precise.

Preliminary definitions

We work in $(Z, d, \mu)$ a compact, connected Ahlfors $Q$-regular metric measure space. For $k \in \mathbb{N}$ and $\kappa \geq 1$, a finite graph $G_k = (V_k, E_k)$ is a $\kappa$-approximation on scale $k$ if it has the following properties: each $v \in V_k$ corresponds to an open set $U_v \subseteq Z$, we have $Z = \bigcup_{v \in V} U_v$, vertices $v$ and $w$ are connected by an edge if and only if $U_v \cap U_w \neq \emptyset$, there is an $s > 1$ such that for every $v \in V_k$ there is a $z_v \in Z$ with

$$B(z_v, \kappa^{-1}s^{-k}) \subseteq U_v \subseteq B(z_v, \kappa s^{-k})$$

and, if $v, w \in V_k$ are distinct, then

$$B(z_v, \kappa^{-1}s^{-k}) \cap B(z_w, \kappa^{-1}s^{-k}) = \emptyset.$$

In [BdK] the value $s = 2$ is used. Let $\rho : V_k \rightarrow [0, \infty)$. Given a path $\gamma \subseteq Z$, the function $\rho$ is admissible for $\gamma$ if

$$\sum_{U_v \cap \gamma \neq \emptyset} \rho(v) \geq 1.$$

If $\Gamma$ is a path family in $Z$, we say $\rho$ is admissible for $\Gamma$ if it is admissible for all $\gamma \in \Gamma$. The $G_k$-combinatorial $p$-modulus of the path family $\Gamma$ is defined by

$$\text{Mod}_p(\Gamma, G_k) = \inf \{ \sum_{v \in V_k} \rho(v)^p : \rho \text{ admissible for } \Gamma \}.$$

Given two sets $A, B \subseteq Z$, we write $\text{Mod}_p(A, B, G_k)$ for the $G_k$-combinatorial $p$-modulus of the path family connecting $A$ and $B$. 

75
Definition 4.5.1. The metric space $Z$ satisfies the Combinatorial Loewner Property with exponent $p > 1$ if there exists decreasing functions $\phi, \psi: (0, \infty) \to (0, \infty)$ with $\lim_{t \to \infty} \phi(t) = \lim_{t \to \infty} \psi(t) = 0$ such that for all disjoint continua $A, B \subseteq Z$ and all $k$ with $s^{-k} \leq \min\{\text{diam}(A), \text{diam}(B)\}$ we have

$$\phi(\Delta(A, B)) \leq \text{Mod}_p(A, B, G_k) \leq \psi(\Delta(A, B)).$$

CLP exponent equivalence

Proof of Theorem 1.1.10. Let $X$ be a hyperbolic filling for $Z$ with parameter $s$. Fix $1 < p < Q$. From the monotonicity of $\text{wcap}_p$ we may assume $A = B(a, r_a)$ and $B = B(b, r_b)$ are balls. Let $\phi$ be as in Definition 4.5.1. We note that $Z$ is linearly connected by [BdK, Proposition 2.5]. Hence we may find continua $E \subseteq \frac{1}{2}A$ and $F \subseteq \frac{1}{2}B$.

Suppose there exists $\tau: E_X \to [0, 1]$ which is admissible for $\text{wcap}_p(A, B)$ with $\|\tau\|_{p, \infty} < \infty$. Define $\sigma: V_X \to [0, \infty)$ by $\sigma(v) = \sum_{e \sim v} \tau(e)$. As $\tau \in \ell^{p, \infty}(E_X)$, it follows from the bounded degree of $X$ that $\sigma \in \ell^{p, \infty}(V_X)$ and hence $\sigma \in \ell^Q(V_X)$ as $p < Q$. Thus, there is a level $N_1$ such that for all $n \geq N_1$ we have $\|2\sigma|_n\|_Q^Q < \phi(\Delta(E, F))$, where $2\sigma|_n$ denotes the restriction of $\sigma$ to vertices on level $n$. For large such levels, it follows that $2\sigma|_n$ is not admissible for $\text{Mod}_p(E, F, X^n)$, where $X^n$ is the subgraph of $X$ consisting of vertices on level $n$ and the edges in $E_X$ which connect such vertices. We see $X^n$ is a $\kappa$-approximation on scale $n$ with $\kappa = 2$. Hence, there is an $N_2 \geq N_1$ such that if $n \geq N_2$ then there exists a path $\gamma_n$ in $Z$ connecting $E$ and $F$ with

$$\sum_{v \in V_{\gamma_n}} \sigma(v) < 1/2$$

where $V_{\gamma_n} = \{v \in X^n : B_v \cap \gamma_n \neq \emptyset\}$. The path $\gamma_n$ gives rise to an edge path connecting $E$ and $F$ on level $n$: there is a chain of distinct vertices $v_0, v_1, \ldots, v_m$ with $B_{v_0} \cap E \neq \emptyset$ and $B_{v_m} \cap F \neq \emptyset$ such that $B_{v_k} \cap \gamma_n \neq \emptyset$ and $B_{v_k} \cap B_{v_{k-1}} \neq \emptyset$ for $k \in \{1, \ldots, m\}$ as otherwise we could create a disconnection of $\gamma_n$. For this edge path $\gamma'_n$ we have

$$\sum_{e \in \gamma'_n} \tau(e) \leq \sum_{v \in V_{\gamma_n}} \sigma(v) < 1/2$$

76
by construction. Choose $\beta$ such that $S(\|\tau\|_{p,\infty}, \beta) < 1/5$ from Lemma 4.2.1. As $\tau \in L^{p,\infty}(E_X)$, it follows that there is a level $N_3(\beta) \geq N_2$ such that for all edges $e$ connecting vertices on levels $\geq N_3$ we have $\tau(e) < \beta$. As $E \subseteq \frac{1}{2}A$ and $F \subseteq \frac{1}{2}B$ there is an $N_4 \geq N_3$ such that if $n \geq N_4$ and $v \in V_{X^n}$ satisfies $B_v \cap E \neq \emptyset$ or $B_v \cap F \neq \emptyset$ then $B_v \subseteq A$ or $B_v \subseteq B$.

Thus, concatenating the edge path $\gamma'_n$ provided $n \geq N_4$ with those guaranteed from Lemma 4.2.1 produces an edge path $\gamma$ connecting $A$ and $B$ with

$$\sum_{\gamma} \tau(e) < 1/2 + 1/5 + 1/5 < 1,$$

contradicting the admissibility of $\tau$. $\square$

**Corollary 4.5.2.** In the above setting, $Q_w = Q'_w = \text{ARCdim} = Q$.

**Proof.** Theorem 1.1.10 shows that if $p < Q_w = Q'_w$. Thus, $Q \leq Q_w$. As $Z$ is $Q$-regular, it follows that $\text{ARCdim} \leq Q$. Hence, $Q \leq Q_w \leq Q'_w \leq \text{ARCdim} \leq Q$. $\square$

### 4.6 Critical exponents when $Z$ attains $\text{ARCdim}$

In this section we assume the metric space $(Z, d)$ attains its Ahlfors regular conformal dimension. In this case we will show $Q'_w = \text{ARCdim}$. To do so, we will work with a curve family of positive modulus in a weak tangent of $Z$. The construction of what we will use appears in [CP], while the original positive modulus result appears in [KL].

**Preliminaries**

We first define weak tangents (cf. [CP, Definition 3.4]).

**Definition 4.6.1.** Compact pointed metric spaces $(Z_k, d_k, \mu_k, z_k)$ converge to the pointed metric space $(Z_\infty, d_\infty, \mu_\infty, z_\infty)$ if there is a pointed metric space $(W, d_W, q)$ and isometric embeddings $\iota_k : Z_k \to W$ and $\iota : Z_\infty \to W$ with $\iota_n(z_k) = \iota(z_\infty) = q$ such that $\iota_n(Z_k) \to \iota(Z_\infty)$ in the sense of Gromov-Hausdorff convergence and $(\iota_k)_* \mu_k \to \iota_* \mu_\infty$ weakly.

When the spaces considered are of the form $Z_k = (Z, r_k^{-1}d, z_k)$ and the above limit
(Z_\infty, d_\infty, z_\infty) exists, we call this limit a weak tangent of Z. We assume Z is Ahlfors Q-regular. In this case the weak limit \mu_\infty of \{r_k^{-Q}\mu\} is a measure on Z_\infty such that (Z_\infty, d_\infty, \mu_\infty) is Ahlfors Q-regular. We consider Z_\infty equipped with this measure.

We also need some theorems relating modulus to discrete modulus. This is similar to what appeared in our discussion of the Combinatorial Loewner Property.

**Definition 4.6.2** \(\text{[Ha], Definition B.1} \). Given a metric space \((Z, d)\) and \(K \geq 1\), a cover \(\mathcal{S}\) is a \(K\)-quasi-packing if for each \(s \in \mathcal{S}\) there exists a point \(z_s \in s\) and a number \(r_s > 0\) such that

(i) \(B(z_s, r_s) \subseteq s \subseteq B(z_s, K r_s)\)

(ii) Every internal ball \(B(z_s, r_s)\) intersects at most \(K\) other internal balls \(B(z'_s, r'_s)\).

**Lemma 4.6.3** \(\text{[Ha], Lemma B.3} \). Let \((Z, d, \mu)\) be an Ahlfors Q regular metric measure space, \(\mathcal{S}\) a \(K\)-quasi-packing, and \(\Gamma\) a family of paths. Suppose that there exists a constant \(\kappa > 0\) for which, for every \(s \in \mathcal{S}\) and \(\gamma \in \Gamma\), if \(\gamma \cap s \neq \emptyset\) then \(\text{diam}(\gamma \cap (2K)B(s)) \geq \kappa \text{diam}(B(s))\), where \(B(s)\) is the internal ball of \(s\). Then,

\[
\text{mod}_Q \Gamma \lesssim \text{mod}_Q(\Gamma, \mathcal{S}).
\]

Here \(\text{mod}_Q(\Gamma, \mathcal{S})\) is the discrete \(Q\)-modulus of the curve family \(\Gamma\) in the graph induced by the incidence pattern of \(\mathcal{S}\); see Section 4.5 for this definition.

**The main result**

**Remark 4.6.4.** From this theorem we conclude that in this setting \(Q'_w \geq Q = \text{ARCdim}\). Hence, \(Q'_w = \text{ARCdim}\) in this case.

**Proof of Theorem 1.1.16** From \[\text{CP, Corollary 3.10} \] (originally in \[\text{KL} \], but we use the precise statement in \[\text{CP} \]) it follows that there exists a weak tangent \((Z_\infty, d_\infty)\) of \(Z\) and two bounded open sets \(A = B(z_\infty, 1)\) and \(B = B(z_\infty, 3) \setminus \overline{\text{B}(z_\infty, 2)}\) in \(Z_\infty\) with \(\text{dist}(A, B) > 0\) such that \(\text{mod}_Q(A, B) > 0\). By choosing a subsequence if necessary, we may assume the Gromov-Hausdorff distance of \(t_k(Z_k)\) and \(\nu(Z_\infty)\) is \(< 1/k\). We also assume \(r_k \to 0\), as oth-
erwise we are dealing with a rescaled metric of our space and the equality of ARCDim and $Q'_w$ follows from Lemma 4.1.4.

Let $X$ be a hyperbolic filling for $Z$ with parameter $s$ and write $X^n$ for the subgraph of $X$ consisting of vertices with level $n$ and edges connecting these vertices.

For $k \in \mathbb{N}$, let $\epsilon = 2/k$. Let $c = 99/100$. Define a map from $V_{X^n}$ to subsets of $Z_\infty$ by associating to $v$ the set

$$S_v = \iota^{-1}(N_c(\iota_k(cB_v))).$$

Let $\mathscr{S}_n = \{S_v : S_v \cap B(z_\infty, 4) \neq \emptyset\}$. Let $r = r(n,k) = r_k^{-1}2s^{-n} > 0$ be the radius of $B_v$ in the metric of $Z_k$. Thus, $1/kr = r_k s^n/2k \to 0$ as $k \to \infty$.

We show $\mathscr{S}_n$ forms a $K$-quasi-packing of $Z_\infty \cap B(z_\infty, 4)$ for appropriate $n$. Note $\mathscr{S}_n$ is a cover of $Z_\infty \cap B(z_\infty, 4)$ as any point $z' \in Z_\infty$ has a corresponding point $z \in Z_k$ such that $d_W(\iota_k(z), \iota(z')) < 1/k < \epsilon$ and $\cup B_v$ is a cover of $Z$.

Given $v \in V_{X^n}$, we abuse notation and write $B_v = B(v, 2s^{-n})$. Define $v' \in S_v$ as follows: by definition, there exists $v'_W \in \iota(Z_\infty)$ such that $d_W(\iota_k(v), v'_W) < 1/k$. Set $v' = \iota^{-1}(v'_W)$.

Now, let $z' \in B(v', r/8)$. Let $z \in Z_k$ be such that $d_W(\iota_k(z), \iota(z')) < 1/k$. Then,

$$d_{Z_k}(v, z) = d_W(\iota_k(v), \iota_k(z))$$

$$\leq d_W(\iota_k(v), \iota(v')) + d_W(\iota(v'), \iota(z')) + d_W(\iota(z'), \iota_k(z))$$

$$< 2/k + r/8.$$

Hence, $z \in (2/kr + 1/8)B_v$. For large enough $k$ depending on $n$ this is contained in $cB_v$, and hence for such $k$ we have $B(v', r/8) \subseteq S_v$.

Now, suppose $z' \in S_v$, so $\iota(z') \in N_c(\iota_k(cB_v))$. Thus, there exists $z_W \in \iota_kc(B_v)$ such that $d_W(\iota(z'), z_W) < \epsilon = 2/k$, so $d_W(\iota(z'), \iota_k(v)) \leq d_W(\iota(z'), z_W) + d_W(z_W, \iota_k(v)) < 2/k + cr$. As $d_W(\iota_k(v), \iota(v')) < 1/k$ it follows that $d_\infty(z', v') = d_W(\iota(z'), \iota(v')) < 3/k + cr$. Thus, $S_v \subseteq B(v', Kr)$ where $K = (3/kr) + 1$. For large enough $k$ we then have $S_v \subseteq B(v', 2r)$.

For condition (ii), suppose $u, v \in V_{X^n}$ and $z' \in B(u', r/8) \cap B(v', r/8)$, where $u'$ is defined as $v'$ is above. We note $d_{Z_k}(u, v) \geq r/2$ by construction. We see

$$d_W(\iota_k(v), \iota(z')) \leq d_W(\iota_k(v), \iota(v')) + d_W(\iota(v'), \iota(z')) < 1/k + r/8.$$
and likewise for $d_W(\iota_k(u), \iota(z'))$. Hence, $r/2 < 2/k + r/4$ and so $1/4 < 2/kr$ which is impossible for $k$ large enough. Thus, $B(v', r/8) \cap B(u', r/8) = \emptyset$ for $k$ large.

We conclude $S_v$ forms a $K$-quasi-packing of $Z_\infty \cap B(z_\infty, 4)$ with internal balls $B(v', r/8)$ and $K = 16$. We now look at the incidence pattern of $\mathcal{I}_n$. Suppose $z' \in S_v \cap S_u$. Then, $\iota(z') \in N_\iota(\iota_k(cB_v)) \cap N_\iota(\iota_k(cB_u))$. Let $z \in Z_k$ be such that $d_W(\iota_k(z), \iota(z')) < 1/k$. Then, $d_{Z_k}(z, cB_v) < 3/k$ and likewise for $cB_u$. Thus, if $3/k < r/100$ we have the implication $S_v \cap S_u \neq \emptyset \implies B_v \cap B_u \neq \emptyset$.

Note all of the above conclusions were true for $k$ large enough that $1/kr$ is small. We apply [Ha] Lemma B.3], which will give us an upper bound on $k$ depending on $n$. We note any path $\gamma$ joining $A$ and $B$ in $Z_\infty$ has diam$(\gamma) > 1/2$. In [Ha] Lemma B.3] we have diam$(B(s)) \leq r/4$ and if $\gamma \cap S_v \neq \emptyset$ and diam$(\gamma) > r$ then diam$(\gamma \cap (2K)B(s)) \geq r$. Hence, if $1/2 > r$ we may apply this lemma to the path family $\Gamma$ connecting $A$ and $B$ to conclude that there is a constant $C_{mod} > 0$ such that $C_{mod} < \text{mod}_Q(\Gamma, \mathcal{I}_n)$.

Let $N \in \mathbb{N}$ and $\theta > 0$. Recall $r = r_k^{-1}2s^{-n}$. We show that we can choose $k$ large enough so that both $1/kr < \theta$ and $r < 1/2$ for at least $N$ values of $n$. Substituting yields the conditions $r_k s^n/2k < \theta$ and $4 < r_k s^n$. Hence, $s^n \in [4/r_k, 2\theta k/r_k]$. Taking logarithms, there are at least $\log_s(\theta kr_k/(8r_k)) - 1 = \log_s(\theta k/8) - 1$ such values of $n$, so such a $k$ exists. In what follows we will use a lower interval bound of $32/r_k$ instead of $4/r_k$, but the same conclusion holds.

We set $V^k_A = \{v : \ell(v) \geq \log_s(32/r_k) \text{ and } d_W(\iota_k(B_v), \iota(A)) < 1/4\}$, set $A_k = \bigcup_{v \in V^k_A} B_v$, and define $V^k_B$ and $B_k$ similarly. Let $v \in V^k_A$ and $u \in V^k_B$ and let $z_v \in B_v$ and $z_u \in B_u$. We estimate $d(z_v, z_u)$. Note for $n_k = \log_s(32/r_k)$ we have diam$(\iota_k(B_v)) \leq 2r_k^{-1}2s^{-n_k}$. Hence, $d_W(\iota_k(z_v), \iota(A)) \leq 2r_k^{-1}2s^{-n_k} + 1/4$. A similar bound holds for $\iota_k(z_u)$ and $\iota(B)$ and we know $d_W(\iota(A), \iota(B)) \geq 1$. Hence, $d_W(\iota_k(z_v), \iota_k(z_u)) \geq 1 - 2(3r_k^{-1}4s^{-n_k} + 1/4)$. It follows from the fact that $d_{Z_k} = r_k^{-1}d$ that $d(z_v, z_u) \geq r_k(1/2 - 8r_k^{-1}4s^{-n_k})$. This is positive as $s^{n_k} \geq 32/r_k$.

Now, the minimal diameter of $A_k$ and $B_k$ is bounded below by the diameter of a ball in the smallest level of $V^k_A$ and $V^k_B$, which is comparable to $r_k$ as $s^{n_k} \simeq 32/r_k$ for such balls. Hence,

$$\Delta(A_k, B_k) \leq \frac{r_k(1/2 - 8r_k^{-1}4s^{-n_k})}{r_k}$$
and so the sets $A_k$ and $B_k$ have controlled relative distance as $k \to \infty$.

Let $a > 0$. We show that if $k$ is large enough, then $\text{wcap}_p(A_k, B_k) > a$. Apply Lemma 4.2.3 with $\delta = 1/5$ and $M$ given by Lemma 3.2.1 applied to $X$ to obtain $\beta$ and $\ell_0$. For $k \in \mathbb{N}$, let $N_k$ be the number of consecutive values $n$ such that $s^n \geq 32/r_k$ and the combinatorial properties derived for $\mathcal{S}_n$ above hold, so $N_k \to \infty$ as $k \to \infty$ by the above. Call these levels the admissible range of $k$. We choose $k$ so that $N_k$ is large enough for the following argument; this will only depend on $a$, the hyperbolic filling parameters, $\beta$, and $\ell_0$.

Suppose that $\tau : E_X \to [0, 1]$ is an edge function with $\|\tau\|_{p,\infty}^p \leq a$. For $v \in V_X$ set $f(v) = \sum_{e \ni v} \tau(e)$. It follows from Lemma 1.2.1 that $\|f\|_{p,\infty}^p \leq a$. As $p < Q$, it follows that there exists a constant $C(a)$ such that $\|f\|_Q^Q < C(a)$. Hence, there exists $L_1 = L_1(a, C_{\text{mod}}) \in \mathbb{N}$ such that at most $L_1$ levels $n$ satisfy $\|2f|_{V_X^n}\|_Q^Q \geq C_{\text{mod}}$. Similarly, as there exists $L_2 = L_2(a, \beta)$ such that at most $L_2$ levels have a connecting edge $e$ such that $\tau(e) \geq \beta$. Thus, if $N_k$ is large enough, there exists a level $n_0$ with $\|2f|_{V_X^{n_0}}\|_Q^Q < C_{\text{mod}}$, the property that any edge $e$ connecting to levels $n_0, \ldots, n_0 + \ell_0 + 1$ has $\tau(e) < \beta$, and all such levels lie in the admissible range of $k$.

Define $g : \mathcal{S}_{n_0} \to [0, \infty)$ by $g(S_v) = 2f(v)$. Now, as $\|2f|_{V_X^{n_0}}\|_Q^Q < C_{\text{mod}}$, the function $g$ is not admissible for discrete modulus connecting $A$ and $B$. Hence, there is a chain of vertices $\hat{\gamma} = \{v_0, \ldots, v_J\}$ with $S_{v_0} \cap A \neq \emptyset$, $S_{v_J} \cap B \neq \emptyset$, and $\sum_j g(S_{v_j}) < 1/2$. We claim $B_{v_0} \subseteq A_k$ and $B_{v_J} \subseteq B_k$ as long as $k > 8$. We prove the first of these claims as the second is similar. Let $z' \in S_{v_0} \cap A$. Then, $d_W(\iota_k(B_{v_0}), \iota(A)) \leq d_W(\iota_k(B_{v_0}), \iota(z'))$. As $z' \in S_{v_0}$, it follows that $d_W(\iota(z'), \iota_k((99/100)B_{v_0})) < 2/k$. Thus, $d_W(\iota_k(B_{v_0}), \iota(A)) < 2/k < 1/4$ if $k > 8$, so $B_{v_0} \subseteq A_k$.

Recall that if $S_v \cap S_u \neq \emptyset$ then $B_v \cap B_u \neq \emptyset$. Thus, the same path $\hat{\gamma}$ can be traced in $V_X^{n_0}$ and, from the definition of $f$, we extract from this an edge path $\gamma$ connecting $B_{v_0}$ and $B_{v_J}$ such that $\sum_{e \in \gamma} \tau(e) < 1/2$. We apply Lemma 4.2.3 to find two paths connecting $v_0$ and $v_J$ to $A_k$ and $B_k$ with $\tau$-length $< 1/5$. Concatenating these paths yields an edge path connecting $A_k$ to $B_k$ with $\tau$-length $< 1$, so $\gamma$ cannot be admissible for $\text{wcap}_p(A_k, B_k)$. As $\tau$ with $\|\tau\|_{p,\infty}^p \leq a$ was arbitrary, it follows that $\text{wcap}_p(A_k, B_k) > a$. Hence, $\lim_{k \to \infty} \text{wcap}_p(A_k, B_k) = \infty$.  

81
REFERENCES


