Title
An Analysis of the Conformal Formulation of the Einstein Constraint Equations on
Asymptotically Flat Manifolds

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An Analysis of the Conformal Formulation of the Einstein Constraint Equations on Asymptotically Flat Manifolds

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Ali Behzadan

Committee in charge:

Professor Michael Holst, Chair
Professor Bennett Chow
Professor Bruce Driver
Professor Ken Intriligator
Professor Julius Kuti

2015
The dissertation of Ali Behzadan is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2015
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Chapters 2, 3, 4, and 5, in part, and appendices §D, §F, §G, and §H, in full, have been submitted for publication. The material may appear as A. Behzadan and M. Holst, *Rough Solutions of the Einstein Constraint Equations on Asymptotically Flat Manifolds Without Near-CMC Conditions*. The dissertation author was the primary investigator and author of this paper.

Another part of Chapter 3 is currently being prepared for submission for publication. The material may appear as A. Behzadan and M. Holst, *Multiplication in Sobolev-Slobodeckij Spaces, Revisited*. The dissertation author was the primary investigator and author of this paper.
VITA

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ABSTRACT OF THE DISSERTATION

An Analysis of the Conformal Formulation of the Einstein Constraint Equations on Asymptotically Flat Manifolds

by

Ali Behzadan

Doctor of Philosophy in Mathematics

University of California, San Diego, 2015

Professor Michael Holst, Chair

In this thesis we consider the conformal formulation of the Einstein constraint equations on asymptotically flat (AF) manifolds. The conformal method transforms the original underdetermined system of constraint equations into a potentially well-posed nonlinear elliptic system which is referred to as Lichnerowicz-Choquet-Bruhat-York (LCBY) system. We investigate the important properties of weighted Sobolev spaces as the appropriate solution spaces for the LCBY equations on AF manifolds. We combine elliptic estimates, sub- and supersolution constructions, fixed-point theorems, and Fredholm-Riesz-Schauder theory to establish existence of non-CMC
weak solutions of the LCBY equations for AF manifolds of class $W^{s,p}_δ$ where $p \in (1, \infty)$, $s \in (1 + \frac{3}{p}, \infty)$, $-1 < δ < 0$, with metric in the positive Yamabe class.
Introduction

Einstein’s general theory of relativity is an accurate mathematical model for gravitational physics. The Einstein’s field equations relate the curvature of spacetime to the non-gravitational energy present. As it will be discussed in Chapter 2, the theory admits an initial value formulation; Einstein’s field equations can be viewed as a nonlinear system of second-order hyperbolic partial differential equations (PDEs) with components of the metric as the basic variables. The initial data consists of a 3-dimensional Riemannian manifold \((M, \hat{h})\), which, roughly speaking, represents a spacelike surface corresponding to an instant of time in the ambient 4-dimensional spacetime, and a symmetric rank 2 tensor field \(\hat{k}\) which plays the role of the time derivative of \(\hat{h}\). The initial data \(\hat{h}\) and \(\hat{k}\) cannot be freely prescribed; they are necessarily related by a system of nonlinear PDEs known as the Einstein constraint equations. The constraint equations constitute an underdetermined system. To date the most useful approach to the construction of solutions to the Einstein constraint equations has been the conformal method. The standard conformal method transforms the original underdetermined system of constraint equations into a potentially well-posed nonlinear elliptic system which is referred to as Lichnerowicz-Choquet-Bruhat-York
(LCBY) system. In the vacuum case, the LCBY system takes the following form:

\[-8\Delta \phi + R\phi + \frac{2}{3} \tau^2 \phi^5 - [\sigma_{ab} + (\mathcal{L} W)_{ab}][\sigma^{ab} + (\mathcal{L} W)^{ab}]\phi^{-7} = 0,\]

\[-\nabla_a (\mathcal{L} W)^{ab} + \frac{2}{3} \phi^6 \nabla^b \tau = 0,\]

to be solved for the scalar function \(\phi\) and the vector field \(W\) on a given Riemannian manifold \((M, h)\). In the above equations \(\sigma\) is a given rank 2 tensor field and \(\tau\) is a given function on \(M\). \(R\) is the scalar curvature of \((M, h)\) and \(\mathcal{L}\) is the conformal Killing operator. The first equation is referred to as the Lichnerowicz equation or the (conformal formulation of the) Hamiltonian constraint. The second (vector) equation is referred to as the (conformal formulation of the) momentum constraint. All of this will be discussed in detail in Chapter 2.

During the past twenty years major developments have been made in the study of the LCBY equations. The cases where \(\tau\) is constant and \(M\) is either a compact manifold or an asymptotically flat manifold have been extensively studied by several authors [39, 53, 16, 55]. Also the question of existence of solutions to the LCBY equations for nonconstant \(\tau\), and without any smallness assumption on \(\tau\), has been recently studied on closed manifolds and compact manifolds with boundary [37, 38, 36]. A short survey of the well-known results is given in Section 2.4.

In this thesis we describe the mathematics needed to understand and study the constraint equations. This includes areas of differential geometry, functional analysis, and partial differential equations. Of course, our key goal is to provide an answer to the question of existence of solutions to the LCBY equations in the case of asymptotically flat manifolds with very low regularity assumptions on the data and without any smallness assumption on the function \(\tau\).

It is attempted to explain all the concepts as clearly and as rigorously as possible.
In the opinion of the author of this manuscript, details in the proofs of theorems are important in understanding the claim of the theorem, and these details are not trivial at all. Displaying the details also has the advantage that one can quickly identify the possible mistakes. In order to maintain a big picture and avoid getting lost in the details, here we list the fundamental ideas that form the basis of the argument presented in this study and many other works on the well-posedness of the LCBY equations.

- **Abstract interpretation of the differential equation**: We interpret any PDE as an equation of the form $Au = f$ where $A$ is an operator between suitable function spaces. In this view, the existence of a unique solution for all $f$ is equivalent to $A$ being bijective. This abstract interpretation allows one to employ a number of general results from linear and nonlinear analysis.

- **Conformal covariance of the Hamiltonian constraint**: The basic idea is that in the study of existence of solutions to the Hamiltonian constraint, we have some sort of freedom in the choice of the given metric $h$. Note that the coefficient $|\sigma_{ab} + (\mathcal{L}W)_{ab}|^2$ is the only part of the Hamiltonian constraint that depends on the solution of the momentum constraint. Let’s consider the individual Hamiltonian constraint by assuming that $\mathcal{L}W$ is given as data. Clearly, the Hamiltonian constraint depends on the background metric $h$: the differential operator in the Hamiltonian constraint is the Laplacian which is defined using $h$, and also the scalar curvature $R$ is with respect to $h$. An important question that one may ask is “does the existence of solution depend on the background metric $h$”? More specifically, if $h$ and $\tilde{h}$ are two conformally equivalent metrics, does the existence of solution for $\tilde{h}$ imply the existence of solution for $h$? The answer for the general case is, unfortunately, a resounding “NO” [57]. However, the situation is not completely
hopeless. We examine this at length in Appendix D, and we show that one can artificially define the coefficients in the Hamiltonian constraint with respect to the new conformally equivalent metric in such a way that some sort of connection is made between the two equations. Thus in the study of existence of solutions to the Hamiltonian constraint, one may perform a conformal transformation and use a metric in the conformal class whose scalar curvature has “nice” properties. This is exactly why the **Yamabe classes** play an important role in the study of LCBY equations.

- **Fredholm alternative:** If $A$ is a “nice” linear operator (in this context, meaning Fredholm of index zero), then uniqueness implies existence.

- **Maximum Principle:** A linear operator $A$ satisfies the maximum principle if $Au \leq 0$ implies $u \leq 0$ in some suitable pointwise sense. If $A$ satisfies the maximum principle then the solution of $Au = f$ (if it exists) is unique.

- **Sub- and Supersolutions:** Consider the equation $-\Delta \phi + G(\phi) = 0$ where $G$ is a given function. Functions $\phi_+$ and $\phi_-$ satisfying

$$-\Delta \phi_+ + G(\phi_+) \geq 0, \quad -\Delta \phi_- + G(\phi_-) \leq 0$$

are called a **supersolution** and **subsolution**, respectively. One can show that under certain conditions the existence of super- and subsolutions implies the existence of a solution $\phi$ to the PDE.

- **Fixed-Point Theorems:** (in particular the contraction mapping and Schauder theorems) We may reduce the problem of existence of solutions to the problem of existence of fixed points of suitably defined operators.
The Implicit Function Theorem: Although we do not use the implicit function theorem in this manuscript, it is important to know that the implicit function theorem can be used in several different ways to prove existence of solutions. For instance in [16] this theorem has been used to prove the existence of solutions of the coupled constraint equations near a given one. Also the “Continuity Method”, which for instance is used in [11] to study the constraint equations, usually makes use of the implicit function theorem. The basic idea of the continuity method is as follows: let $\Phi(u) = 0$ be the equation to solve. The continuity method consists of the following three steps [4, 29]:

- **Step 1:** Find a continuous family of functions $\Phi_\tau$ with $\tau \in [0, 1]$, such that $\Phi_1(u) = \Phi(u)$, and $\Phi_0(u) = 0$ is a known equation which has a solution $u_0$.

- **Step 2:** Prove that the set $J = \{ \tau \in [0, 1] : \Phi_\tau(u) = 0 \text{ has a solution} \}$ is open. To show this, the implicit function theorem is typically used.

- **Step 3:** Prove that the set $J$ is closed.

Therefore $J$ is a nonempty subset of $[0, 1]$ that is both open and closed. This means $J = [0, 1]$ and in particular $1 \in J$.

The main difficulty is in finding the appropriate function spaces as the domain and codomain of the differential operator $A$, and ensuring that by using those function spaces we are allowed to apply the maximum principle, Fredholm theory, fixed point theorems, and so forth. For elliptic equations on the whole space $\mathbb{R}^n$ (and also for asymptotically flat manifolds), the appropriate spaces are weighted Sobolev spaces.

Having stated the main ideas and our general purpose, we now summarize this work.

In Chapter 1, we summarize the fundamental notions of Riemannian geometry,
with emphasis on those aspects that are most important for the study of Einstein constraint equations. In particular, we carefully discuss the properties of of conformally equivalent metrics. We also introduce some notations and conventions that will be employed throughout the manuscript.

In Chapter 2, we give an overview of the general theory of relativity and the initial value formulation of the Einstein's field equations. We derive the Einstein constraint equations and we describe the conformal decomposition of the constraint equations.

In Chapter 3, we define the weighted Sobolev spaces and discuss their main properties. In particular, multiplication properties of functions in weighted Sobolev spaces, which play a central role in defining a weak formulation of the LCBY equations and also in obtaining \textit{a priori} estimates, are extensively studied in this chapter.

In Chapter 4, we study estimates and related results for elliptic operators on weighted Sobolev spaces.

In Chapter 5, we focus on the problem of existence of solutions in the case of asymptotically flat manifolds with very low regularity assumptions on the data. In Section 5.1, we define weak formulations of the constraint equations that will allow us to develop solution theories for the constraints in the spaces with the weakest possible regularity. In particular, we focus on one of two possible weak formulations of the LCBY equations; a second alternative, which has some advantages but which we do not use in the main body of the manuscript, is described in Appendix H. In Section 5.2, we study the momentum constraint in isolation from the Hamiltonian constraint. We develop some basic technical results for the momentum constraint operator under the weakest possible assumptions on the problem data. In Section 5.3, we study the individual Hamiltonian constraint. We assume the existence of barriers (weak sub- and supersolutions) to the Hamiltonian constraint equation forming a nonempty positive
bounded interval, and then derive several properties of the Hamiltonian constraint that are needed in the analysis of the coupled system. The results are established under the weakest possible assumptions on the problem data. In Section 5.4, we develop a new approach for the construction of global sub- and supersolutions for the Hamiltonian constraint on asymptotically flat manifolds. In particular, we give constructions for both sub- and supersolutions in the positive Yamabe case that have several key features, including: (1) they do not require any smallness assumption on \( \tau \); (2) they require minimal assumptions on the data in order to be used for developing rough solutions; and (3) they have appropriate asymptotic behavior to be compatible with an overall fixed-point argument for the coupled system. Finally, in Section 5.5 we develop our main results for the coupled system. In particular, we clearly state and then prove the main existence result (Theorem 5.22) for rough positive Yamabe solutions to the constraint equations on asymptotically flat manifolds without any smallness assumptions on the function \( \tau \).

For ease of exposition, various supporting technical results are given in several appendices as follows: Appendix §A – a review of some of the basic results from real analysis; Appendix §B – some remarks on ellipticity; Appendix §C – details of the derivation of the LCBY equations; Appendix §D – artificial conformal covariance of the Hamiltonian constraint on asymptotically flat manifolds; Appendix §E – an overview of the Yamabe problem on compact manifolds; Appendix §F – results on Yamabe positive metrics on asymptotically flat manifolds; Appendix §G – some remarks on the alternative use of Bessel Potential spaces; and Appendix §H – an alternative weak formulation of the LCBY system on asymptotically flat manifolds that makes possible additional results that are not developed in this manuscript.
Chapter 1

Riemannian Geometry

1.1 Basic Definitions, Notations, and Conventions

The goal of this section is to review the basic definitions of Riemannian geometry and to establish our notations and conventions for later use. More detail on any topics mentioned here can be found in [10, 49, 48, 30]. Our notations mostly follow those of [48].

1.1.1 Tensors on a Vector Space

Let $V$ be an $n$-dimensional real vector space. Let $V^*$ denote the dual space of $V$. Let $\{E_i\}_{i=1}^n$ be a basis for $V$ and $\{\phi^j\}_{j=1}^n$ be the corresponding dual basis for $V^*$ ($\phi^j(E_i) = \delta^j_i$).

Definition 1.1. Let $k, l \geq 1$ be integers.

- A covariant $k$-tensor on $V$ is a multilinear map

$$F : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \rightarrow \mathbb{R}.$$
The space of all covariant $k$-tensors is denoted by $T^k(V)$.

- A contravariant $l$-tensor is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{\text{l copies}} \to \mathbb{R}.$$  

The space of all contravariant $l$-tensors is denoted by $T^l(V)$.

- A $k$-covariant, $l$-contravariant tensor is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{\text{l copies}} \times \underbrace{V \times \cdots \times V}_{\text{k copies}} \to \mathbb{R}.$$  

The space of all $k$-covariant, $l$-contravariant tensors is denoted by $T^{k\,l}(V)$.

We set $T^0(V) = \mathbb{R}$. Clearly $T^1(V) = V^*$, $T_1(V) = V^{**} = V$, $T^0_0(V) = T^k(V)$, and $T^0_1(V) = T^1_1(V)$. Moreover $T^1_1(V)$ can be identified with the space of linear maps from $V$ to itself (which we denote by $\text{End}(V)$). Indeed, there exists an isomorphism (=a bijective linear map) $\Psi : \text{End}(V) \to T^1_1(V)$: given $\tilde{F} \in \text{End}(V)$, $F = \Psi(\tilde{F})$ is defined by

$$F(\omega, X) = \omega(\tilde{F}(X)).$$

Given $F \in T^1_1(V)$, $\tilde{F} = \Psi^{-1}(F)$ can be obtained as follows

$$\tilde{F} : V \to V$$

$$X \mapsto \sum_{k=1}^n F(\varphi_k, X) E_k.$$

The proof is simple: if $\tilde{F}(X) = \sum_{k=1}^n \alpha^k E_k$, then $\alpha^k = \varphi^k(\tilde{F}(X)) = F(\varphi^k, X)$.

Similarly one can show that there is a natural isomorphism between $T^{k\,l+1}_l(V)$
and the space of multilinear maps

\[ \bigotimes_{\text{l copies}} V^* \times \cdots \times V^* \times \bigotimes_{\text{k copies}} V \times \cdots \times V \to V. \]

**Definition 1.2** (Tensor product). If \( F \in T^k_l(V) \) and \( G \in T^r_s(V) \), the tensor \( F \otimes G \in T^{k+r}_{l+s}(V) \) is defined by

\[
F \otimes G(\omega^1, \ldots, \omega^{l+s}, X_1, \ldots, X_{k+r}) = F(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k) G(\omega^{l+1}, \ldots, \omega^{l+s}, X_{k+1}, \ldots, X_{k+r}).
\]

One can show that the collection of all tensors of the form

\[ E_{j_1} \otimes \cdots \otimes E_{j_l} \otimes \phi_{i_1} \otimes \cdots \otimes \phi_{i_k}, \quad 1 \leq i_p, j_q \leq n \]

is a basis for \( T^k_l(V) \). An arbitrary tensor \( F \in T^k_l(V) \) can be written in terms of this basis as

\[
F = F^{j_1 \cdots j_l}_{i_1 \cdots i_k} E_{j_1} \otimes \cdots \otimes E_{j_l} \otimes \phi_{i_1} \otimes \cdots \otimes \phi_{i_k},
\]

where

\[
F^{j_1 \cdots j_l}_{i_1 \cdots i_k} = F(\phi_{j_1}, \ldots, \phi_{j_l}, E_{i_1}, \ldots, E_{i_k}).
\]

**Remark 1.3.**

- We employ the Einstein summation convention throughout this manuscript: if in any term the same index name appears twice, as both an upper and a lower index, that term is assumed to be summed over all possible values of that index.
- We always choose our index positions so that vectors have lower indices and covectors have upper indices. The components of vectors have upper indices and
the components of covectors have lower indices.

- The components of a mixed tensor may occur in a nonstandard order. For example if we have a multilinear map \( F : V^* \times V \times V^* \to \mathbb{R} \), then

\[
F = F^{i,j,k} E_i \otimes \varphi^j \otimes E_k, \quad F^{i,j,k} = F(\varphi^i, E_j, \varphi^k).
\]

**Definition 1.4 (Trace).**

- Let \( F \in T^1_1(V) \) and let \( \tilde{F} = \Psi^{-1}(F) \) be the corresponding endomorphism. The trace of \( F \), denoted by \( \operatorname{tr} F \), is defined as the trace of \( \tilde{F} \). Note that the trace of an endomorphism is basis-independent.

- More generally, let \( F \in T^k_l(V) \) where \( k, l \geq 1 \). We can define the trace of \( F \) with respect to the pair \((r, s)\) as follows: \( \operatorname{tr} F \) is a tensor in \( T^{k-1}_{l-1}(V) \) defined by

\[
(\operatorname{tr} F)(\omega^1, \ldots, \omega^{r-1}, \omega^{r+1}, \ldots, \omega^l, X_1, \ldots, X_{s-1}, X_{s+1}, \ldots, X_k) := \operatorname{tr} G
\]

where \( G \in T^1_1(V) \) is given by

\[
G(\omega, X) := F(\omega^1, \ldots, \omega^{r-1}, \omega, \omega^{r+1}, \ldots, \omega^l, X_1, \ldots, X_{s-1}, X, X_{s+1}, \ldots, X_k).
\]

Throughout this manuscript, unless otherwise stated, in computing trace we assume \((r, s) = (k, l)\).

Note that in the case where \( F \in T^1_1(V) \), we have

\[
\forall X \in V \quad \tilde{F}(X) = \sum_{i=1}^n F(\varphi^i, X) E_i,
\]
therefore the matrix of $\tilde{F} : V \to V$ is

$$[\tilde{F}(E_1) \ldots \tilde{F}(E_n)] = \begin{bmatrix} F(\phi^1, E_1) & \cdots & F(\phi^1, E_n) \\ \vdots & & \vdots \\ F(\phi^n, E_1) & \cdots & F(\phi^n, E_n) \end{bmatrix}.$$ 

So in terms of a basis,

$$\text{tr} F = \sum_{i=1}^{n} F(\phi^i, E_i) = F_i^i.$$ 

More generally, if $F \in T^k_l(V)$, then the components of $\text{tr} F$ are

$$(\text{tr} F)^{j_1 \ldots j_{l-1}}_{i_1 \ldots i_{k-1} m} = F^{j_1 \ldots j_{l-1} m}_{i_1 \ldots i_{k-1} m}.$$ 

**Definition 1.5.** A covariant $k$-tensor $\phi$ is said to be alternating if

$$\forall \sigma \in S_k \quad \forall X_1, \ldots, X_k \in V \quad \phi(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = (\text{sgn} \sigma) \phi(X_1, \ldots, X_k).$$

Here $S_k$ denotes the group of all bijective maps from $\{1, \ldots, k\}$ to itself. If $\sigma \in S_k$ is an even permutation, then $\text{sgn} \sigma = +1$, otherwise $\text{sgn} \sigma = -1$. Recall that an element of $S_k$ is called an even permutation if it can be obtained as the composition of an even number of exchanges of two elements of $\{1, \ldots, k\}$.

Equivalently, $\phi$ is alternating if it changes sign whenever two arguments are interchanged.

The space of all covariant alternating $k$-tensors on $V$ is denoted by $\Lambda^k(V)$. Each element of $\Lambda^k(V)$ is called a $k$-form. The tensor product of two alternating tensors is not necessarily an alternating tensor. However, there is another product which preserves the alternating tensors, namely the wedge product. In order to define the wedge product, first we define the alternating map.
**Definition 1.6.** The map \( \text{Alt} : T^k(V) \to \Lambda^k(V) \) defined by

\[
\text{Alt} \phi(X_1, \ldots, X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sgn } \sigma) \phi(X_{\sigma(1)}, \ldots, X_{\sigma(k)})
\]

is called the alternating map.

One can show that \( \text{Alt} : T^k(V) \to \Lambda^k(V) \) is surjective; Moreover \( \text{Alt} \) is a projection map, that is \( \text{Alt}^2 = \text{Alt} \) (this is exactly why the coefficient \( \frac{1}{k!} \) is used in the definition).

**Definition 1.7.** If \( \phi \in \Lambda^r(V) \) and \( \psi \in \Lambda^s(V) \), the wedge product (or exterior product) \( \phi \wedge \psi \in \Lambda^{r+s}(V) \) is defined by

\[
\phi \wedge \psi = \frac{(r + s)!}{r! s!} \text{Alt} (\phi \otimes \psi).
\]

### 1.1.2 Vector Bundles, Tensors Fields

Throughout this manuscript, all manifolds are assumed to be smooth \((C^\infty)\), Hausdorff, and second countable. Unless otherwise stated, we assume manifolds are without boundary and all maps are smooth. The tangent space of the manifold \( M \) at point \( p \in M \) is denoted by \( T_p M \), and the cotangent space by \( T^*_p M \). If \( (U, (x^i)) \) is a local coordinate chart and \( p \in U \), we denote the corresponding coordinate basis for \( T_p M \) by \( \left\{ \frac{\partial}{\partial x^i} \right\} \); sometimes we write \( \partial_i \) instead of \( \frac{\partial}{\partial x^i} \). The union of tangent spaces at all points on a manifold can be considered as a manifold in its own right (the **tangent bundle**); the same is true for the union of cotangent spaces (the **cotangent bundle**). Tangent bundle and cotangent bundle are special examples of a common structure in differential geometry which is called **vector bundle**.

**Definition 1.8.** A smooth \( k \)-dimensional vector bundle is a pair of smooth manifolds \( E \)
(the total space) and $M$ (the base), together with a surjective map $\pi : E \to M$, satisfying the following conditions:

- Each set $E_p := \pi^{-1}(p)$ (called the fiber of $E$ over $p$) is a vector space.

- For each $p \in M$, there exists a neighborhood $U$ of $p$ and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $\pi_1 \circ \varphi = \pi$ where $\pi_1 : U \times \mathbb{R}^k \to U$ is the projection onto the first component. The map $\varphi$ is called a local trivialization of $E$.

- The restriction of $\varphi$ to each fiber $\varphi : E_p \to \{p\} \times \mathbb{R}^k$, is an isomorphism of vector spaces.

A section of $E$ is a map $F : M \to E$ such that $F(p) \in E_p$ for all $p \in M$. $F$ is said to be smooth if it is smooth as a map between manifolds. Suppose $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a local trivialization of $E$. Define $s_i : U \to \pi^{-1}(U)$, $i = 1, \ldots, k$ as follows:

$$\forall \ p \in U \quad s_i(p) = \varphi^{-1}(p, e_i),$$

where $\{e_1, \ldots, e_k\}$ is the standard basis of $\mathbb{R}^k$. From the definition of a vector bundle, it follows that $\{s_1(p), \ldots, s_k(p)\}$ forms a basis for the fiber $E_p$. Note that any section $F$ has a local expression

$$\forall \ p \in U \quad F(p) = F_1(p)s_1(p) + \cdots + F_k(p)s_k(p).$$

A section of the tangent bundle is called a vector field. An important example of a vector bundle is the bundle of $(k \ell)$-tensors on $M$ whose total space is

$$T^k_\ell(M) := \bigsqcup_{p \in M} T^k_\ell(T_p M).$$

A section of this bundle is called a $(k \ell)$-tensor field. $T^k(M)$ and $\Lambda^k(M)$ are defined in a
similar fashion. A section of $\Lambda^k(M)$ is called a differential $k$-form, or just a $k$-form on $M$.

**Remark 1.9.**

- $C^\infty(M)$ denotes the set of all $C^\infty$ real-valued functions on $M$.
- The collection of all vector fields on $M$ is denoted by $\chi(M)$.
- The collection of all $\binom{k}{l}$-tensor fields is denoted by $\tau^k_l(M)$ or $\Gamma^k_l(M)$.
- The collection of all differential $k$-forms on $M$ is denoted by $\Omega^k(M)$ or $\Gamma\Lambda^k(M)$. $\Omega(M)$ denotes the collection of all differential forms on $M$.

Also we set $\tau^k(M) = \tau^k_0(M)$ and $\Omega^0(M) = \tau^0(M) = C^\infty(M)$. For $\phi \in \Omega^r(M)$ and $\psi \in \Omega^s(M)$, the formula $(\phi \wedge \psi)_p = \phi_p \wedge \psi_p$ defines an associative product on $\Omega(M)$ satisfying $\phi \wedge \psi = (-1)^{rs} \psi \wedge \phi$. With this product, $\Omega(M)$ is an algebra over $\mathbb{R}$.

**Remark 1.10.**

- If $f \in C^\infty(M)$ and $X \in \chi(M)$, then $fX \in \chi(M)$ is defined by

$$\forall \ p \in M \quad (fX)_p = f(p)X_p.$$

- If $f \in C^\infty(M)$ and $X \in \chi(M)$, then $Xf \in C^\infty(M)$ is defined by

$$\forall \ p \in M \quad (Xf)(p) = X_p f.$$

- If $X, Y \in \chi(M)$, then the Lie bracket of $X$ and $Y$, $[X, Y] \in \chi(M)$ is defined by

$$\forall \ p \in M \quad [X, Y]_p f = X_p (Y f) - Y_p (X f).$$
• In the remaining of this chapter, by a “local frame” we mean \( n \) vector fields \( \{E_i\}_{i=1}^n \) defined on some open subset \( U \) of \( M \) such that \( \{E_1|_p, \ldots, E_n|_p\} \) form a basis for \( T_pM \) at each point \( p \in U \). By the “dual coframe” we mean \( n \)-1-forms \( \{\eta^j\}_{j=1}^n \) such that \( \eta^j(E_i) = \delta^j_i \).

Note that each smooth section \( T \in \tau^k_l(M) \) induces a \( C^\infty(M) \)-multilinear map:

\[
\tilde{T} : \tau^1(M) \times \cdots \times \tau^1(M) \times \chi(M) \times \cdots \times \chi(M) \to C^\infty(M);
\]

\[
(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k) \mapsto T(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k).
\]

The converse is also true, that is, if

\[
\tilde{T} : \underbrace{\tau^1(M) \times \cdots \times \tau^1(M)}_{l \text{ copies}} \times \underbrace{\chi(M) \times \cdots \times \chi(M)}_{k \text{ copies}} \to C^\infty(M)
\]

is a multilinear map over \( C^\infty(M) \), then it is induced by a \( \binom{k}{l} \)-tensor field \( T \) as above.

Similarly, if a map

\[
\tilde{T} : \underbrace{\tau^1(M) \times \cdots \times \tau^1(M)}_{l \text{ copies}} \times \underbrace{\chi(M) \times \cdots \times \chi(M)}_{k \text{ copies}} \to \chi(M)
\]

is multilinear over \( C^\infty(M) \), then it defines a \( \binom{k}{l+1} \)-tensor field \( T \) as follows

\[
T(\omega^1, \ldots, \omega^l, \omega^{l+1}, X_1, \ldots, X_k) = \omega^{l+1}(\tilde{T}(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k)).
\]

The converse is also true, that is any \( \binom{k}{l+1} \)-tensor field induces a \( C^\infty(M) \)-multilinear map \( \tilde{T} : \underbrace{\tau^1(M) \times \cdots \times \tau^1(M)}_{l \text{ copies}} \times \underbrace{\chi(M) \times \cdots \times \chi(M)}_{k \text{ copies}} \to \chi(M) : \) if \( \{E_i\}_{i=1}^n \) is any local frame
and \( \{ \eta^i \}_{i=1}^n \) is the corresponding dual coframe, then

\[
\tilde{T}(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k) = \sum_{i=1}^n T(\omega^1, \ldots, \omega^l, \eta^1, X_1, \ldots, X_k) E_i.
\]

**Definition 1.11.** Suppose \( F : M \to N \) is a smooth map between manifolds \( M \) and \( N \).

- The differential of \( F \) at \( p \in M \) is the map \( dF_p : T_p M \to T_{F(p)} N \) defined by

\[
\forall X \in T_p M \quad \forall g \in C^\infty(F(p)) \quad (dF_p(X))(g) = X(g \circ F).
\]

Here \( C^\infty(F(p)) \) denotes the germs of \( C^\infty \) functions at \( F(p) \). \( dF_p \) is sometimes denoted by \( F_* p \). If \( F \) is bijective and \( X \) is a vector field on \( M \), then \( F_* X \) is a vector field on \( N \), called the push forward of \( X \) by \( F \).

- Let \( \phi \) be a covariant \( k \)-tensor field on \( N \). The pull back of \( \phi \) by \( F \), \( F^\ast \phi \) is a covariant \( k \)-tensor field on \( M \) defined by

\[
\forall p \in M \quad \forall X_1, \ldots, X_k \in T_p M \quad (F^\ast \phi)_p(X_1, \ldots, X_k) = \phi_{F(p)}(dF_p(X_1), \ldots, dF_p(X_k)).
\]

If \( f : M \to \mathbb{R} \) is smooth and \( p \in M \), then by definition \( df_p : T_p M \to T_{f(p)} \mathbb{R} \). However, \( \mathbb{R} \) is a vector space and \( T_{f(p)} \mathbb{R} \) can be identified with \( \mathbb{R} \). So \( df \) can be considered as an element of \( \Omega^1(M) \). One can easily show that, under this identification, on any local coordinate chart \( (U, (x^i)) \) containing \( p \), we have

\[
df = \frac{\partial f}{\partial x^i} dx^i.
\]

\( \{ dx^i_p : T_p M \to \mathbb{R} \}_{i=1}^n \) form a basis for \( T_p^\ast M \).

**Theorem 1.12.** There exists a unique \( \mathbb{R} \)-linear map \( d_M : \Omega(M) \to \Omega(M) \) such that
1. If \( f \in \Omega^0(M) \), then \( d_M f = df \).

2. If \( \theta \in \Omega^r(M) \) and \( \sigma \in \Omega^s(M) \), then

\[
d_M(\theta \wedge \sigma) = d_M \theta \wedge \sigma + (-1)^r \theta \wedge d_M \sigma.
\]

3. \( d_M \circ d_M = 0 \).

\( d_M \) maps \( \Omega^r(M) \) to \( \Omega^{r+1}(M) \). Also given an \( r \)-form \( \omega \) with representation \( \omega = \alpha_{i_1 \ldots i_r} \, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \) in a local coordinate chart, then \( d_M \omega \) will have the following representation in the same coordinate chart:

\[
d_M \omega = d\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} = \frac{\partial \alpha_{i_1 \ldots i_r}}{\partial x^j} \, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}.
\]

\( d_M \) is called the **exterior derivative**; if the manifold \( M \) is clear from the context, we may drop the subscript \( M \) in \( d_M \).

**Definition 1.13** (Interior Product= Contraction). Let \( X \in \chi(M) \) and \( \omega \in \Omega^k(M) \). The interior product of \( \omega \) by \( X \), \( i_X \omega \), is defined as follows:

1. \( i_X \omega = 0 \) if \( k = 0 \).

2. \( i_X \omega = \omega(X) \) if \( k = 1 \).

3. \( (i_X \omega)_p(v_1, \ldots, v_k) = \omega_p(X_p, v_1, \ldots, v_k) \) if \( k > 1 \) (for all \( p \in M \) and \( v_1, \ldots, v_k \in T_p M \)).

In particular, \( i_X \) maps \( \Omega^k(M) \) to \( \Omega^{k-1}(M) \).

Next we introduce the concept of **connection** as a way of differentiating sections of vector bundles.
**Definition 1.14.** Consider the vector bundle \((E, M, \pi)\) (\(E\) is the total space, \(M\) is the base manifold, and \(\pi : E \rightarrow M\) is the projection map). Let \(\Gamma E\) denote the space of smooth sections of \(E\). A connection in \(E\) is a map

\[
\nabla : \chi(M) \times \Gamma E \rightarrow \Gamma E \\
(X, Y) \mapsto \nabla_X Y
\]

satisfying the following properties:

- \(\nabla_X Y\) is \(C^\infty(M)\)-linear in \(X\) and \(\mathbb{R}\)-linear in \(Y\).
- For all \(f \in C^\infty(M) : \nabla_X (f Y) = f \nabla_X Y + (X f) Y\).

\(\nabla_X Y\) is called the covariant derivative of \(Y\) in the direction of \(X\).

One can show that at each point \(p \in M\), \(\nabla_X Y|_p\) depends only on the value of \(Y\) in a neighborhood of \(p\) and the value of \(X\) at \(p\).

A connection in the tangent bundle is called a linear connection. Let \(\nabla : \chi(M) \times \chi(M) \rightarrow \chi(M)\) be a linear connection. Let \(\{E_i\}\) be a local frame defined on an open subset \(U \subset M\). We can expand \(\nabla_{E_i} E_j\) in terms of this frame:

\[
\nabla_{E_i} E_j = \Gamma^k_{ij} E_k.
\]

The functions \(\Gamma^k_{ij}\) on \(U\), are called the Christoffel symbols of \(\nabla\) with respect to this frame. If a local frame is fixed, we denote \(\nabla_{E_i} X\) by \(\nabla_i X\). As the following theorem shows, any linear connection on \(M\) induces connections on all tensor bundles over \(M\).

**Theorem 1.15.** [48] If \(\nabla\) is a linear connection on \(M\), then there exists a unique connection in each tensor bundle \(T^k_l(M)\), also denoted by \(\nabla\), such that the following conditions are satisfied:
• On the tangent bundle, $\nabla$ agrees with the given connection.

• On $T^0(M)$, $\nabla$ is given by ordinary differentiation of functions, that is, for all real-valued smooth functions $f: M \to \mathbb{R}$: $\nabla_X f = Xf$.

• $\nabla_X (F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$.

• If $\text{tr}$ denotes the trace on any pair of indices, then $\nabla_X (\text{tr} F) = \text{tr} (\nabla_X F)$.

This connection satisfies the following additional property: for any $T \in \tau^k_1(M)$, vector fields $Y_i$, and differential 1-forms $\omega^j$,

$$(\nabla_X T)(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_k) = X(T(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_k))$$

$$\quad - \sum_{j=1}^l T(\omega^1, \ldots, \nabla_X \omega^j, \ldots, \omega^l, Y_1, \ldots, Y_k)$$

$$\quad - \sum_{i=1}^k T(\omega^1, \ldots, \omega^l, Y_1, \ldots, \nabla_X Y_i, \ldots, Y_k).$$

If $\nabla$ is a linear connection on $M$, and $T \in \tau^k_1(M)$, the map

$$\nabla T : \tau^1(M) \times \cdots \times \tau^1(M) \times \chi(M) \times \cdots \times \chi(M) \to C^\infty(M)$$

$$(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_k, X) \mapsto (\nabla_X T)(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_k).$$

is $C^\infty(M)$-multilinear and so it defines a $\left( \frac{k+1}{l} \right)$-tensor field. The tensor field $\nabla T$ is called the total covariant derivative of $T$.

**Remark 1.16** (More on Notation). Let $\{E_i\}$ be a local frame on an open set $U \subset M$ and
\{\eta^j\} be the dual coframe. For \( T \in \tau^k_1(M) \) we have

\[
(\nabla T)^{j_1,\ldots,j_l}_{i_1,\ldots,i_k,m} = (\nabla T)(\eta^{j_1},\ldots,\eta^{j_l}, E_{i_1},\ldots,E_{i_k}, E_m)
\]

\[
= (\nabla_m T)(\eta^{j_1},\ldots,\eta^{j_l}, E_{i_1},\ldots,E_{i_k})
\]

\[
= (\nabla_m T)^{j_1,\ldots,j_l}_{i_1,\ldots,i_k}
\]

\[
=: \nabla_m T_{i_1,\ldots,i_k} =: T^{j_1,\ldots,j_l}_{i_1,\ldots,i_k,m}.
\]

For example

- \( \forall X \in \chi(M) \quad \nabla_i \nabla_j X^m = [\nabla_i (\nabla_j X)]^m. \)

- \( \forall \psi \in \Omega^1(M) \quad \nabla_a \psi_b = (\nabla_a \psi)_b. \)

- \( \forall T \in \tau^2_1(M) \quad \nabla_a T^b_{ce} = (\nabla_a T)_c^b. \)

- \( \forall T \in \tau^1_2(M) \quad \forall S \in \tau^1_3(M) \quad \nabla_a T^{ab} S^{ijkl}_{rm} = [\nabla_a (T \otimes S)]_{r}^{abijkl}. \)

### 1.1.3 Riemannian Manifolds, Covariant Derivative, Curvature

A **Riemannian metric** on a smooth manifold \( M \) is a covariant tensor field \( g \in \tau^2(M) \) that is symmetric and positive definite: \( g(X,Y) = g(Y,X) \) and \( g(X,X) > 0 \) if \( X \neq 0 \). The manifold \( M \) equipped with a Riemannian metric \( g \) is called a Riemannian manifold. If there is no possibility of confusion, we may write \( \langle X,Y \rangle \) or \( \langle X,Y \rangle_g \) instead of \( g(X,Y) \). Given a metric \( g \) on \( M \), one can define the musical isomorphisms as
follows:

\[
\begin{align*}
&\flat : T_pM \to T^*_pM \\
&X \mapsto X^\flat := g(X, \cdot), \\
&\sharp : T^*_pM \to T_pM \\
&\psi \mapsto \psi^\sharp := b^{-1}(\psi).
\end{align*}
\]

Note that the above isomorphisms are well-defined even if \(g\) is only nondegenerate (instead of being positive definite). Using \(\sharp\) we can define \(g^{-1} : \tau^1(M) \times \tau^1(M) \to \mathbb{R}\) as follows

\[
g^{-1}(\psi_1, \psi_2) := g(\psi_1^\sharp, \psi_2^\sharp).
\]

Let \(\{E_i\}\) be a local frame on an open subset \(U \subset M\) and \(\{\eta^i\}\) be the corresponding dual coframe. So we can write \(X = X^i E_i\) and \(\psi = \psi_i \eta^i\). It is standard practice to denote the \(i^{th}\) component of \(X^\flat\) by \(X_i\) and the \(i^{th}\) component of \(\psi^\sharp\) by \(\psi^i\):

\[
X^\flat = X_i \eta^i, \quad \psi^\sharp = \psi^i E_i.
\]

It is easy to show that

\[
X_i = g_{ij} X^j, \quad \psi^i = g^{ij} \psi_j,
\]

where \(g_{ij} = g(E_i, E_j)\) and \(g^{ij} = g^{-1}(\eta^i, \eta^j)\). It is said that \(X^\flat\) is obtained from \(X\) by lowering an index and \(\psi^\sharp\) is obtained from \(\psi\) by raising an index.

**Remark 1.17.**

- Let \(f : M \to \mathbb{R}\) be a smooth function. \(\text{grad } f\) is defined as \((d f)^\sharp\). If \((U, (x^i))\) is any
local coordinate chart, then
\[
d f = \frac{\partial f}{\partial x^i} dx^i, \quad \text{grad} f = [g^{ij} \frac{\partial f}{\partial x^j}] \frac{\partial}{\partial x^j}.
\]

- If \( T \in T^k_1(M) \) and \( \nabla \) is a connection on \( T^k_1(M) \), then \( \nabla^m T := g^{pm} \nabla_p T \).

If \((M, g)\) is a Riemannian manifold, then there exists a unique smoothly varying inner product on each fiber of \( T^k_1(M) \) with the property that for all \( p \in M \), if \( \{e_i\} \) is an orthonormal basis of \( T_p M \) with dual basis \( \{\eta^i\} \), then the corresponding basis of \( T^k_1(T_p M) \) is orthonormal [48]. When there is no ambiguity, we denote this inner product by \( \langle.,.\rangle \). One can show that if \( F \) and \( G \) are \( (k_l) \)-tensor fields, then with respect to any local frame
\[
\langle F, G \rangle = g^{i_1 r_1} \ldots g^{i_k r_k} g_{j_1 s_1} \ldots g_{j_l s_l} F^{i_1 \ldots i_k}_{j_1 \ldots j_l} G^{s_1 \ldots s_l}_{r_1 \ldots r_k} = F_{s_1 \ldots s_l i_1 \ldots i_k} G^{s_1 \ldots s_l i_1 \ldots i_k}.
\]

**Theorem 1.18.** [48][Fundamental Lemma of Riemannian Geometry] Let \((M, g)\) Be a Riemannian manifold. There exists a unique linear connection \( \nabla \) on \( M \) that satisfies the following properties:

- \( \nabla \) is compatible with \( g \):
\[
\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).
\]

- \( \nabla \) is torsion-free:
\[
\nabla_X Y - \nabla_Y X = [X, Y].
\]

This connection is called the *Levi-Civita* connection of \( g \). In the rest of this manuscript, the only connection that we work with is the Levi-Civita connection. We
have the following important formula, which is called the Koszul formula, for the
Levi-Civita connection:

\[ 2\langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle \]
\[ - \langle X, [Y, Z] \rangle - \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle. \]

Our next goal is to define the curvature tensors. Let \((M, g)\) be a Riemannian manifold
and let \(\nabla\) be the corresponding Levi-Civita connection. For all \(X, Y \in \mathfrak{X}(M)\), we define
the curvature operator \(R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)\) by

\[ R(X, Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z. \]

Note that the assignment \((X, Y) \mapsto R(X, Y)\) is antisymmetric, that is, \(R(X, Y) = -R(Y, X)\).
The curvature operator can be seen as a measure of the failure of the second covariant
derivatives to commute. Clearly we can define \(R\) as a map from \(\mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M)\) to
\(\mathfrak{X}(M)\) that sends the triple \((Z, X, Y)\) to \(R(X, Y)Z\). This map is multilinear over \(C^\infty(M)\)
and so it can be viewed as a \(3\)-tensor field as follows:

\[ \forall \psi \in \Omega^1(M) \quad \forall \ X, Y, Z \in \mathfrak{X}(M) \quad R(\psi, Z, X, Y) = \psi(R(X, Y)Z). \]

This tensor field is called the \textit{Riemann curvature tensor}. The covariant Riemann
curvature tensor is defined by

\[ R(W, Z, X, Y) := R(W^\flat, Z, X, Y) = g(W, R(X, Y)Z). \]

\textbf{Remark 1.19.}

- One can find alternative definitions of the curvature operator and the Riemann
curvature tensor in the literature, e.g. the curvature operator may be defined as the negative of the one defined above or Riemann tensor can be defined by  
\[ R(\psi, Z, X, Y) = \psi (R(Z, X) Y) . \]  
As a consequence of the symmetries of the curvature tensor, which will be discussed below, all the various definitions agree up to sign.

• As it is discussed in [30], in dimensions higher than two, the full information of Riemann tensor can be reduced to Gaussian curvatures of certain two-dimensional submanifolds which in turn can be described in terms of the curvature of certain curves.

**Theorem 1.20.** [48, 30][Symmetries of the Riemann Tensor and its Covariant Derivative] Let \( X, Y, Z, V, W \) be arbitrary vector fields on \( M \). Then

- \( R(W, Z, X, Y) = -R(W, Z, Y, X) , \)
- \( R(W, Z, X, Y) = -R(Z, W, X, Y) , \)
- \( R(W, Z, X, Y) = R(X, Y, W, Z) , \)
- **Algebraic Bianchi Identity:**
  \[ R(W, X, Y, Z) + R(W, Y, Z, X) + R(W, Z, X, Y) = 0 . \]
- **Second Bianchi Identity:**
  \[ \nabla_W R(V, Z, X, Y) + \nabla_Z R(W, V, X, Y) + \nabla_V R(Z, W, X, Y) = 0 . \]

Considering the above symmetries, it can be shown that (see e.g. [30]) for \( n \geq 2 \) the number of independent components of the Riemann curvature tensor is \( \frac{n^2(n^2-1)}{12} \).

We define the **Ricci curvature tensor** as the trace of the Riemann curvature tensor with respect to the first and the third components; the **Ricci scalar** is defined as
the trace of the Ricci curvature. In particular, if \( \{E_i\} \) is a local frame with dual coframe \( \{\eta^j\} \), then (the concept of trace of tensor fields will be discussed in detail in Section 1.1.4).

\[
\text{Ric}(X, Y) = \sum_{a=1}^{n} R(\eta^a, X, E_a, Y), \quad (\text{Ric}_{ij} = R^a_{iaj}). \\
R = \sum_{a=1}^{n} \text{Ric}((\eta^a)^\sharp, E_a), \quad (R = g^{ij} \text{Ric}_{ij}).
\]

When there is no possibility of confusion we may denote the component of the Ricci tensor by \( R_{ij} \) instead of \( \text{Ric}_{ij} \).

Finally we briefly review the topic of integration on Riemannian manifolds. On any oriented Riemannian manifold \( (M, g) \) of dimension \( n \), there exists a unique differential \( n \)-form \( dV \) satisfying the property that \( dV(E_1, \ldots, E_n) = 1 \) whenever \( (E_1, \ldots, E_n) \) is an oriented orthonormal basis for some tangent space \( T_p M \). This \( n \)-form \( dV \) is called the \textit{Riemannian volume element}. With respect to any oriented local frame \( \{E_i\} \) with dual coframe \( \{\eta^j\} \) we have

\[
dV = \sqrt{\det(g_{ij})} \eta^1 \wedge \cdots \wedge \eta^n.
\]

Suppose \( (U, (x^i)) \) is a local coordinate chart and \( f : U \to \mathbb{R} \) is a measurable function. We define

\[
\int_U f dV := \int_{x(U)} (f \circ x^{-1}) \sqrt{\det(g_{ij})} \circ x^{-1} dx^1 \ldots dx^n.
\]

The change of variables formula ensures that this integral is well-defined on the intersection of any two charts. Now suppose \( f : M \to \mathbb{R} \) is a measurable function. We define

\[
\int_M f dV := \sum_{a} \int_{U_a} \phi_a f dV,
\]
where \( \{U_a\} \) is a locally finite cover of \( M \) by coordinate neighborhoods and \( \{\phi_a\} \) is a partition of unity subordinate to \( \{U_a\} \) (one can show that this definition is independent of the choice of coordinate neighborhoods and the partition of unity).

**Remark 1.21.** Recall that, by definition, \( M \) is second countable; also it follows from the definition of smooth manifolds that \( M \) must be locally compact. Because of these two properties, every open cover of \( M \) has a locally finite refinement [49]. A partition of unity subordinate to the locally finite open cover \( \{U_a\} \) is a family of smooth functions \( \{\phi_a\} \) such that \( \sum_a \phi_a = 1 \) and \( \text{supp} \phi_a \subseteq U_a \).

### 1.1.4 Trace, Divergence, and Laplacian

Let \((M, g)\) be a smooth Riemannian manifold. \( \{E_i\}_{i=1}^n \) is a local frame defined on an open set \( U \subset M \). If we wish to emphasize that the basis is orthonormal, we use \( \{e_i\}_{i=1}^n \) instead of \( \{E_i\}_{i=1}^n \). In both cases \( \{\eta^i\}_{i=1}^n \) denotes the corresponding dual coframe.

**Trace**

The trace operator for mixed tensors (elements of \( T^k_1(V) \) where \( k, l \geq 1 \)) was defined in Section 1.1.1. That definition can be generalized in an obvious way to mixed tensor fields (elements of \( \tau^k_1(M) \) where \( k, l \geq 1 \)). For purely covariant or purely contravariant tensor fields the trace can be defined by first applying the musical isomorphisms; So in this case, as opposed to the case of mixed tensor fields, trace depends on the metric. Here we focus on symmetric \( \binom{2}{0} \)-tensor fields. Our results can be readily extended to \( k \)-covariant tensor fields.

Let \( T \) be a symmetric covariant tensor field of order 2. We can associate to \( T \) a
tensor field \( T^\#: \in \tau^1_1(M) \) as follows:

\[
\forall \omega \in \Omega^1(M) \quad \forall X \in \chi(M) \quad T^\#(\omega, X) := T(\omega^#, X).
\]

Trace of \( T \) is defined by \( \text{tr} T := \text{tr} T^\# \). So \( \text{tr} T \) is a real-valued function on \( M \). With respect to the local frame we have

\[
\text{tr} T = \sum_{i=1}^{n} T^\#(\eta^i, E_i) = \sum_{i=1}^{n} T((\eta^i)^\#, E_i) = \sum_{i=1}^{n} T(g^{ji} E_j, E_i) = g^{ji} T_{ji}.
\]

If \( \{e_i\}_{i=1}^{n} \) is an orthonormal local frame, then \((\eta^i)^\# = e_i \) and

\[
\text{tr} T = \sum_{i=1}^{n} T(e_i, e_i).
\]

**Remark 1.22.** If \( T \) is a general covariant \( k \)-tensor \((k \geq 2)\), then there is more than one way to define \( T^\# \). Throughout this manuscript, unless otherwise stated, we use the following convention:

\[
T^\#(\omega, X_1, \ldots, X_{k-1}) := T(X_1, \ldots, X_{k-1}, \omega^#).
\]

In index notation this means that we raise the last index:

\[
(T^\#)^i_{j_1 \cdots j_{k-1}} = g^{im} T_{j_1 \cdots j_{k-1} m}.
\]

As usual, we may denote the components of \( T^\# \) by \( T^i_{j_1 \cdots j_{k-1}} \) instead of \( (T^\#)^i_{j_1 \cdots j_{k-1}} \).

**Proposition 1.23.** Let \( V \in \chi(M) \) and \( T \in \tau^k_0(M) \). Then

\[
\nabla_V (T^\#) = (\nabla_V T)^\#.
\]
Proof. (Proposition 1.23) Here we will prove the claim for $k = 1$ and $k = 2$. The proof for $k > 2$ is similar.

• $k = 1$

\[
\forall X \in \mathcal{X}(M) \quad \langle (\nabla_V T)^\sharp, X \rangle = (\nabla_V T)(X) = V(T(X)) - T(\nabla_V X) \\
= V(\langle T^\sharp, X \rangle) - \langle T^\sharp, \nabla_V X \rangle \\
= \langle \nabla_V (T^\sharp), X \rangle.
\]

• $k = 2$ For all $X \in \mathcal{X}(M)$ and $\omega \in \Omega^1(M)$ we have

\[
(\nabla_V (T^\sharp))(\omega, X) = V(T^\sharp(\omega, X)) - T^\sharp(\nabla_V \omega, X) - T^\sharp(\omega, \nabla_V X) \\
= V(T(X, \omega^\sharp)) - T((\nabla_V \omega)^\sharp, X) - T(\omega^\sharp, \nabla_V X) \\
= V(T(X, \omega^\sharp)) - T(\nabla_V (\omega^\sharp), X) - T(\omega^\sharp, \nabla_V X) \\
= (\nabla_V T)(X, \omega^\sharp) \\
= (\nabla_V (T^\sharp))(\omega, X).
\]

\[
\Box
\]

Divergence

In general, the divergence of a tensor field is defined as the trace of the covariant derivative:

\[
\text{div} T = \text{tr}(\nabla T).
\]

In what follows we study three cases in more detail:

1. Divergence of a \(\binom{k}{l}\) tensor field where $k \geq 0$ and $l \geq 1$. 
2. Divergence of a \( (k_0) \) tensor field.

3. Divergence of a vector field.

We start with \( T \in \tau^k_l(M) \) where \( k \geq 0 \) and \( l \geq 1 \). \( \nabla T \) is a \( (k+1) \)-tensor field and therefore \( \text{div} T = \text{tr}(\nabla T) \) is a \( (k_l) \)-tensor field. With respect to the local frame

\[
(\text{div} T)(\omega^1, \ldots, \omega^{l-1}, X_1, \ldots, X_k) = \sum_{m=1}^{n} (\nabla T)(\omega^1, \ldots, \omega^{l-1}, \eta^m, X_1, \ldots, X_k, E_m)
\]

\[
= \sum_{m=1}^{n} (\nabla E_m T)(\omega^1, \ldots, \omega^{l-1}, \eta^m, X_1, \ldots, X_k).
\]

Therefore

\[
(\text{div} T)_{j_1 \ldots j_{k+1}} = \sum_{m=1}^{n} (\nabla T)_{j_1 \ldots j_{k+1} m} = \sum_{m=1}^{n} (\nabla m T)_{j_1 \ldots j_{k+1} m} = \sum_{m=1}^{n} T_{j_1 \ldots j_{k+1} m}.
\]

Now let's consider the case of \( (k_0) \)-tensor fields; Suppose \( T \) is such a tensor. \( \nabla T \) is a \( (k+1) \)-tensor field and therefore \( \text{div} T = \text{tr}(\nabla T) \) is a \( (k_l) \)-tensor field. Recall that by definition \( \text{tr}(\nabla T) := \text{tr}((\nabla T)^i) \). In terms of local frame we have

\[
(\text{div} T)(X_1, \ldots, X_{k-1}) = (\text{tr}((\nabla T)^i))(X_1, \ldots, X_{k-1})
\]

\[
= \sum_{i=1}^{n} (\nabla T)^i (\eta^i, X_1, \ldots, X_{k-1}, E_i)
\]

\[
= \sum_{i=1}^{n} (\nabla T)(X_1, \ldots, X_{k-1}, E_i, (\eta^i)^j)
\]

\[
= \sum_{i=1}^{n} (\nabla T)(X_1, \ldots, X_{k-1}, E_i, g^{ij} E_m).
\]
Hence

\[(\text{div} T)_{j_1...j_{k-1}} = (\text{div} T)(E_{j_1}, \ldots, E_{j_{k-1}}) = g^{im}(\nabla T)_{j_1...j_{k-1}i m} \]

\[= g^{im}(\nabla m T)_{j_1...j_{k-1}i m} = (\nabla^i T)_{j_1...j_{k-1}i} \]

\[= g^{im}T_{j_1...j_{k-1};m} = T_{j_1...j_{k-1};m}. \]

Also note that if \(\{e_i\}_{i=1}^n\) is an orthonormal basis, then \((\check{\eta}^i)^j = e_i\) and so

\[(\text{div} T)(X_1, \ldots, X_{k-1}) = \sum_{i=1}^n (\nabla T)(X_1, \ldots, X_{k-1}, e_i, e_i) \]

\[= \sum_{i=1}^n (\nabla e_i T)(X_1, \ldots, X_{k-1}, e_i) \]

In particular,

- if \(\omega \in \Omega^1(M)\), then

\[\text{div} \omega = \sum_{i=1}^n (\nabla \omega)(e_i, e_i) = \sum_{i=1}^n (\nabla e_i \omega)(e_i) = (\nabla_i \omega)^i = \nabla_i \omega = \omega^i_i, \]

- if \(F\) is \(\binom{2}{0}\)-tensor field, then

\[\forall X \in \chi(M) \quad (\text{div} F)(X) = \sum_{i=1}^n (\nabla e_i F)(X, e_i). \]

Finally we consider the divergence of a vector field. Let \(W \in \chi(M)\). As we know, a \(\binom{0}{1}\)-tensor field \(\hat{W}\) can be associated to \(W\) as follows:

\[\hat{W} : \chi^1(M) \to C^\infty_0(M) \]

\[\omega \mapsto \omega(W). \]
The divergence of $W$ is defined by $\text{div} W := \text{div} \hat{W}$. In particular $\text{div} W$ is a function. One can easily show that $\text{div} W = \text{div}(W^b)$. Using the local frame, we can write (in order to avoid any confusion, we do not use the summation convention in the following calculation)

$$\text{div} W = \text{div} \hat{W} = \text{tr}(\nabla \hat{W}) = \sum_{i=1}^{n} (\nabla \hat{W})(\eta^i, E_i)$$

$$= \sum_{i=1}^{n} (\nabla_{E_i} \hat{W})(\eta^i)$$

$$= \sum_{i=1}^{n} \left[ E_i(\hat{W}(\eta^i)) - \hat{W}(\nabla_{E_i} \eta^i) \right]$$

$$= \sum_{i=1}^{n} \left[ E_i(\eta^i(W)) - (\nabla_{E_i} \eta^i)(W) \right]$$

$$= \sum_{i=1}^{n} \eta^i(\nabla_{E_i} W) = \sum_{i=1}^{n} \nabla_i W^i.$$

In particular, if the basis is orthonormal, then

$$\text{div} W = \sum_{i=1}^{n} \eta^i(\nabla_{e_i} W) = \sum_{i=1}^{n} \langle \nabla_{e_i} W, e_i \rangle.$$ 

One can show that $\text{div} W$ is the unique function that satisfies

$$d(i_W dV) = (\text{div} W) dV,$$

where $dV$ is the standard volume element of $(M, g)$.

**Laplacian**

Let $f : M \to \mathbb{R}$ be a smooth function. The Laplacian of $f$, $\Delta f$, is a real-valued function on $M$ that is defined as follows:

$$\Delta f := \text{div}(\text{grad} f).$$
Note that \((\text{grad} f)^\flat = df = \nabla f\), so

\[
\Delta f = \text{div}(\text{grad} f) = \text{div}((\text{grad} f)^\flat) = \text{div}(df) = \text{div}(\nabla f) .
\]

With respect to the local frame \(\{E_i\}_{i=1}^n\) we have

\[
\Delta f = \text{div}(\text{grad} f) = (\nabla_i (\text{grad} f))^i = \nabla_i (\text{grad} f)^i = \nabla_i \nabla_i f .
\]

**Remark 1.24.** Let \((U,(x^i))\) be a local coordinate chart. Here we list some useful formulas \((X \in \chi(M) \text{ and } f, h : M \to \mathbb{R})\)

1. \(\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})\), \(\Gamma^i_{ij} = \frac{\partial_i \sqrt{\text{det} g}}{\sqrt{\text{det} g}}\).

2. \(\text{div} X = X_i^i = \partial_i X^i + \Gamma^i_{ij} X^j = \frac{1}{\sqrt{\text{det} g}} \partial_j (\sqrt{\text{det} g} X^j)\).

3. \(\Delta f = \text{div}(\text{grad} f) = \frac{1}{\sqrt{\text{det} g}} \partial_j (\sqrt{\text{det} g} g_{ij} \partial_i f)\).

4. \(\text{div}(f X) = f \text{div} X + \langle \text{grad} f, X \rangle\).

5. \(\Delta(fh) = f \Delta h + h \Delta f + 2 \langle \text{grad} f, \text{grad} h \rangle\).

6. If \(X\) has compact support, then \(\int_M \text{div} X dV = 0\).

7. If at least one of \(f\) or \(h\) has compact support, then

\[
\int_M h \Delta f dV = \int_M f \Delta h dV = -\int_M \langle \text{grad} f, \text{grad} h \rangle dV .
\]

**1.1.5 Geometry of Hypersurfaces, Gauss and Codazzi Equations**

Let \((\mathcal{M}, g)\) be an \(n + 1\)-dimensional Riemannian manifold. A hypersurface in \(\mathcal{M}\) is an embedded submanifold of \(\mathcal{M}\) of dimension \(n\). Suppose \(M\) is a hypersurface in \(\mathcal{M}\). We equip \(M\) with the induce metric \(h\); this metric is the pull back of \(g\) by the
natural inclusion map and it is sometimes called the first fundamental form of the hypersurface. The first fundamental form encodes the intrinsic geometry of \( M \). Later we will define the second fundamental form which encodes how \( M \) is embedded into \( \mathcal{M} \): it measures the way \( M \) curves within the ambient manifold \( \mathcal{M} \).

At each point \( p \in M \), the ambient tangent space \( T_p \mathcal{M} \) can be decomposed into the direct sum \( T_p \mathcal{M} = T_p M \oplus (T_p M) \perp \). The set \( \mathcal{N} M := \bigsqcup_{p \in M} (T_p M) \perp \) is a vector bundle over \( M \). We assume there exists a smooth section \( N \) of \( \mathcal{N} M \) such that for all \( p \in M \), \( N_p \) has unit length; we say that \( N \) is a unit normal vector field. In particular, for orientable hypersurfaces of the Euclidean space such unit normal vector fields exist. Since at each point \( p \in M \), \((T_p M) \perp \) is a one-dimensional vector space we have \((T_p M) \perp = \text{span} N_p \).

Now let \( \nabla \) and \( D \) denote the Levi-Civita connections of \( g \) and \( h \), respectively. For all \( X, Y \in \chi(M) \) we can write

\[
\nabla_X Y = (\nabla_X Y)^T + (\nabla_X Y)^\perp.
\]

Note that if \( X, Y \in \chi(M) \), then they can be extended to vector fields \( \tilde{X}, \tilde{Y} \in \chi(\mathcal{M}) \); in the above equation by \( \nabla_X Y \) we mean \( \nabla_{\tilde{X}} \tilde{Y} \) (the result is independent of how \( X \) and \( Y \) are extended to vector fields on the ambient manifold \( \mathcal{M} \)).

It can be shown that the assignment \((X, Y) \mapsto (\nabla_X Y)^T \) satisfies all the properties of the Levi-Civita connection on \( M \), therefore by the uniqueness of the Levi-Civita connection, we may conclude that \( D_X Y = (\nabla_X Y)^T \). Also since \((T_p M) \perp = \text{span} N_p \), we have

\[
(\nabla_X Y)^\perp = k(X, Y)N,
\]

where \( k(X, Y) = g(N, \nabla_X Y) \) is called the extrinsic curvature of \( M \) or the (scalar) second fundamental form. Note that if \( X, Y \in \chi(M) \), then \([X, Y] \in \chi(M) \) and so \( g(N, [X, Y]) = 0 \).
Therefore

\[ k(X, Y) = g(N, \nabla_X Y) = g(N, \nabla_Y X + [X, Y]) = g(N, \nabla_Y X) = k(Y, X). \]

Also it is easy to see that \( K \) is bilinear over \( C^\infty(M) \). Therefore \( k \) defines a symmetric \( (0, 2) \)-tensor field on \( M \). Using the fact that \( \nabla \) is compatible with \( g \) and the symmetry of \( k \), we can find the following alternative expressions for \( k \):

\[ k(X, Y) = -g(\nabla_X N, Y) = -g(\nabla_Y N, X). \quad (1.1) \]

Indeed,

\[ g(N, \nabla_X Y) = Xg(Y, N) - g(\nabla_X N, Y) = 0 - g(\nabla_X N, Y) = -g(\nabla_X N, Y). \]

The Weingarten map (also called the shape operator), \( \text{Wein} : \chi M \to \chi(M) \), is defined by

\[ \text{Wein}(X) := \nabla_X N. \]

Note that \( \nabla_X N \) is tangent to \( M \) because \( g(N, N) = 1 \) and

\[ g(\nabla_X N, N) + g(N, \nabla_X N) = X(g(N, N)) = 0 \Rightarrow g(\nabla_X N, N) = 0. \]

Therefore for \( X, Y \in \chi(M) \) we have \( g(\nabla_X N, Y) = h(\nabla_X N, Y) \). We can rewrite (1.1) as follows:

\[ k(X, Y) = -h(\text{Wein}(X), Y) = -h(X, \text{Wein}(Y)). \]

Hence the symmetry of \( k \) is equivalent to the self-adjointness of the Weingarten map with respect to \( h \). At each point \( p \in M \) the \( n \) eigenvalues of \( \text{Wein}_p : T_p M \to T_p M \)
are called the principal curvatures of $M$ at $p$ and the corresponding eigenvectors determine the principal curvature directions.

Now we study the tangential and normal components of $R(X, Y)Z$ where $X, Y, Z \in \chi(M)$ and $R$ is the curvature operator of $(\mathcal{M}, g)$ (with Levi-Civita connection $\nabla$). In what follows we denote the curvature related quantities of the hypersurface $(M, h)$ (with Levi-Civita connection $D$) by $R^D$. We have

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z.$$

Note that

$$\nabla_X(\nabla_Y Z) = \nabla_X[D_Y Z + k(Y, Z)N]$$

$$= \nabla_X(D_Y Z) + \nabla_X(k(Y, Z)N)$$

$$= [D_X(D_Y Z) + k(X, D_Y Z)N] + [X(k(Y, Z))N + k(Y, Z)\nabla_X N].$$

Similarly,

$$\nabla_Y(\nabla_X Z) = [D_Y(D_X Z) + k(Y, D_X Z)N] + [Y(k(X, Z))N + k(X, Z)\nabla_Y N].$$

Also using the fact that the Levi-Civita connection is torsion-free we get

$$\nabla_{[X,Y]}Z = D_{[X,Y]}Z + k([X,Y], Z)N$$

$$= D_{[X,Y]}Z + k(D_X Y, Z)N - k(D_Y X, Z)N.$$
Consequently,

\[ R(X, Y)Z = D_X(D_Y Z) - D_Y(D_X Z) - D_{[X,Y]}Z + k(Y, Z)\nabla_X N - k(X, Z)\nabla_Y N \\
+ [X(k(Y, Z)) - k(D_X Y, Z) - k(Y, D_X Z)]N \\
- [Y(k(X, Z)) - k(X, D_Y Z) - k(D_Y X, Z)]N. \]

Hence

\[ R(X, Y)Z = R^D(X, Y)Z + k(Y, Z)\nabla_X N - k(X, Z)\nabla_Y N \\
+ [(D_X k)(Y, Z) - (D_Y k)(X, Z)]N. \]

It follows that the tangential and normal components of \( R(X, Y)Z \) are

\[ (R(X, Y)Z)^T = R^D(X, Y)Z + k(Y, Z)\nabla_X N - k(X, Z)\nabla_Y N \]
\[ (R(X, Y)Z)^\perp = [(D_X k)(Y, Z) - (D_Y k)(X, Z)]N. \]

Thus for all \( W \in \chi(M) \) we have

\[ R(W, Z, X, Y) = g(R(X, Y)Z, W) = g((R(X, Y)Z)^T, W) \\
= g(R^D(X, Y)Z, W) + k(Y, Z)g(\nabla_X N, W) - k(X, Z)g(\nabla_Y N, W). \]

Therefore

\[ R(W, Z, X, Y) = R^D(W, Z, X, Y) - k(Y, Z)k(X, W) + k(X, Z)k(Y, W). \]

The above equation relates the curvature of the hypersurface to the curvature of the ambient manifold and is called the Gauss equation.
Finally we have

\[ R(N, Z, X, Y) = g(R(X, Y)Z, N) = g((R(X, Y)Z)\perp, N) = (D_X k)(Y, Z) - (D_Y k)(X, Z). \]

The above equation is called the Codazzi (or Codazzi-Mainardi) equation.

1.1.6 The Lie Derivative and The Conformal Killing Operator

So far we have reviewed the definitions of the exterior derivative of differential forms and the covariant derivative of tensor fields. Here we review the notion of Lie derivative.

**Definition 1.25.** Let \( M \) be an \( n \)-dimensional smooth manifold, and let \( X \) be a vector field on \( M \) (\( X \in \chi(M) \)).

- For all smooth functions \( f : M \to \mathbb{R} \), the Lie derivative of \( f \) with respect to \( X \), \( L_X f \), is defined by
  \[ L_X f = X f. \]

- For all \( Y \in \chi(M) \), the Lie derivative of \( Y \) with respect to \( X \), \( L_X Y \in \chi(M) \), is defined by
  \[ L_X Y = [X, Y]. \]

- For all \( T \in \tau^k_0(M) \), the Lie derivative of \( T \) with respect to \( X \), \( L_X T \in \tau^k_0(M) \), is defined by
  \[ (L_X T)(Y_1, \ldots, Y_k) = X(T(Y_1, \ldots, Y_k)) - T([X, Y_1], Y_2, \ldots, Y_k) - \cdots - T(Y_1, \ldots, Y_{k-1}, [X, Y_k]). \]
Remark 1.26. Let $\phi$ denote the flow of the vector field $X$. Then one can equivalently define the Lie derivative as follows:

- For all $p \in M$ and $Y \in \chi(M)$
  \[ (L_X Y)_p = \frac{d}{dt}\big|_{t=0} \phi^*_t(Y_{\phi_t(p)}). \]

- For all $p \in M$ and $T \in \tau^k_0(M)$
  \[ (L_X T)_p = \frac{d}{dt}\big|_{t=0} \phi^*_t(T_{\phi_t(p)}). \]

A covariant tensor field $T$ is said to be invariant under the flow $\phi$ of a vector field $X$ if $\phi^*_t(T_{\phi_t(p)}) = T_p$ for all $(t, p)$ in the domain of $\phi$.

Proposition 1.27. [49] The Lie derivative of a tensor field $T$ with respect to a vector field $X$ is zero if and only if $T$ is invariant under the flow of $X$.

Now suppose $(M, g)$ is a Riemannian manifold and $\nabla$ is the corresponding Levi-Civita connection. For all vector fields $X, Y, Z \in \chi(M)$ we have

\[
L_X g(Y, Z) = X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z])
\]
\[
= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g([X, Y], Z) - g(Y, [X, Z])
\]
\[
= g(\nabla_X Y - [X, Y], Z) + g(Y, \nabla_X Z - [X, Z])
\]
\[
= g(\nabla_Y X, Z) + g(Y, \nabla_Z X). \quad (1.2)
\]

Therefore with respect to any local frame $\{E_i\}$ we have

\[
L_X g_{ij} = \nabla_i X_j + \nabla_j X_i.
\]
By Proposition 1.27 if a vector field \( X \) is such that \( L_X g = 0 \), then \( g \) is invariant under the flow of \( X \). Such a vector field is called a \textit{Killing vector field}.

It follows from (1.2) that \( \text{tr}(L_X g) = 2 \text{div} X \). Therefore we can decompose \( L_X g \) into the pure trace part and the trace-free part as follows:

\[
L_X g = \left[ \frac{1}{n} (2 \text{div} X) g \right] + \left[ L_X g - \frac{1}{n} (2 \text{div} X) g \right].
\]

The \textit{conformal Killing operator}, \( \mathcal{L} : \chi(M) \rightarrow \tau_2^0(M) \), is defined as follows:

\[
\mathcal{L} X := \text{the contravariant counterpart of the trace-free part of } L_X g.
\]

That is,

\[
(\mathcal{L} X)^{ij} = \nabla^i X^j + \nabla^j X^i - \frac{2}{n} (\text{div} X) g^{ij}.
\]

The elements in the kernel of \( \mathcal{L} \) are called \textit{conformal Killing fields}. \( \text{div} \mathcal{L} \) is sometimes called \textit{vector Laplacian} and is denoted by \( \Delta_L \). In what follows we will discuss some of the important properties of the conformal Killing operator and the vector Laplacian.

**Theorem 1.28.** Let \( S \) be a symmetric traceless \((0,2)\)-tensor field and \( X \) be a compactly supported vector field on the Riemannian manifold \((M, g)\). Then

\[
\int_M \langle \mathcal{L} X, S \rangle dV = -2 \int_M \langle X, \text{div} S \rangle dV.
\]
Proof. (Theorem 1.28)

\[
-2 \int_M \langle X, \text{div} S \rangle \, dV = -2 \int_M g_{ij} X^i (\text{div} S)^j \, dV \\
= -2 \int_M g_{ij} X^i \nabla_k S^j \, dV \\
= -2 \int_M \left[ \nabla_k (g_{ij} X^i S^j) - g_{ij} (\nabla^k X^i) S^{jk} \right] \, dV.
\]

So if we let \( Y^k = g_{ij} X^i S^j \), then

\[
-2 \int_M \langle X, \text{div} S \rangle \, dV = -2 \int_M \nabla_k Y^k \, dV + 2 \int_M g_{ij} (\nabla^k X^i) S^{jk} \, dV \\
= -2 \int_M \text{div} Y \, dV + 2 \int_M g_{ij} (\nabla^k X^i) S^{jk} \, dV \\
= \int_M \langle \mathcal{L} X, S \rangle \, dV.
\]

The last equality holds true because

\[
\langle \mathcal{L} X, S \rangle = g_{ij} g_{kl} (\mathcal{L} X)^i S^j \\
= g_{ij} g_{kl} [\nabla^i X^j + \nabla^j X^i - \frac{2}{n} (\text{div} X) g^{ik} S^j] \\
= g_{ij} g_{kl} [\nabla^i X^j + \nabla^j X^i] S^j - \frac{2}{n} (\text{div} X) g^{ik} g_{ij} g_{kl} S^j \\
= g_{ij} g_{kl} [\nabla^i X^j + \nabla^j X^i] S^j - \frac{2}{n} (\text{div} X) g_{kl} S^j \\
= g_{kl} (\nabla_j X^k) S^j + g_{ij} (\nabla_l X^i) S^l \\
= 2 g_{ij} (\nabla_l X^i) S^l \quad (S \text{ is symmetric}).
\]

\[\square\]

**Theorem 1.29.** \( \Delta_L = \text{div} \mathcal{L} \) is (formally) self adjoint, that is, for all compactly supported
smooth vector fields $X$ and $Y$, we have

$$\langle \Delta_L X, Y \rangle_{L^2} = \langle X, \Delta_L Y \rangle_{L^2}.$$  

$$\langle X, Y \rangle_{L^2} := \int_M g(X, Y) dV$$

Proof. (Theorem 1.29)

$$\langle \Delta_L X, Y \rangle_{L^2} = \langle \text{div} \mathcal{L} X, Y \rangle_{L^2} = -\frac{1}{2} \langle \mathcal{L} Y, \mathcal{L} X \rangle_{L^2}$$

$$= \langle X, \text{div} \mathcal{L} Y \rangle_{L^2} = \langle X, \Delta_L Y \rangle_{L^2}$$

For the second equality we used Theorem 1.28 with $S = \mathcal{L} X$ and for the third equality we applied that theorem with $S = \mathcal{L} Y$.

Theorem 1.30. Let $(M, g)$ be a compact Riemannian manifold. Then

$$\ker \Delta_L = \ker \mathcal{L}.$$  

Proof. (Theorem 1.30) Since $\Delta_L = \text{div} \mathcal{L}$, clearly $\ker \mathcal{L} \subseteq \ker \Delta_L$. It remains to show that $\ker \Delta_L \subseteq \ker \mathcal{L}$. Suppose $X \in \ker \Delta_L$. We have

$$0 = \langle \Delta_L X, X \rangle_{L^2} = -\frac{1}{2} \langle \mathcal{L} X, \mathcal{L} X \rangle_{L^2} = -\frac{1}{2} \| \mathcal{L} X \|_{L^2}^2,$$

which implies that $\mathcal{L} X = 0$. That is, $X \in \ker \mathcal{L}$.

A similar statement will be discussed in the nonsmooth setting on asymptotically flat manifolds in Section 5.2.
1.2 Conformally Equivalent Metrics

Let $M$ be a smooth manifold with a Riemannian metric $g$. We say $\tilde{g}$ and $g$ are conformally equivalent if there exists a positive function $f: M \to \mathbb{R}$ such that $\tilde{g} = f \cdot g$. Suppose $g$ and $\tilde{g}$ are two conformally equivalent metrics on $M$. Let $\nabla$ and $\tilde{\nabla}$ denote Levi-Civita covariant derivatives with respect to the metrics $g$ and $\tilde{g}$. In this section, our goal is to compare $\nabla$ and $\tilde{\nabla}$ and their corresponding curvature tensors; throughout this section all quantities referring to $\tilde{\nabla}$ carry a tilde. In Appendix C we will use these results in the derivation of the conformal formulation of the Einstein constraint equations.

Remark 1.31. In what follows some of the formulas will be simplified if we write $f = e^{2\omega}$ for some function $\omega: M \to \mathbb{R}$ or $f = \varphi^r$ for some positive function $\varphi: M \to \mathbb{R}$ and a suitable exponent $r$. Note that $e^{2\omega} = \varphi^r$ implies that $\omega = \frac{1}{2} r \ln \varphi$.

1.2.1 Relation Between the Linear Connections

By the Koszul formula we have

$$2 g(\nabla_X Y, Z) = X g(Y, Z) + Y g(Z, X) - Z g(X, Y)$$

$$- g(X, [Y, Z]) - g(Y, [Z, X]) + g(Z, [X, Y]),$$

$$2 \tilde{g}(\tilde{\nabla}_X Y, Z) = X \tilde{g}(Y, Z) + Y \tilde{g}(Z, X) - Z \tilde{g}(X, Y)$$

$$- \tilde{g}(X, [Y, Z]) - \tilde{g}(Y, [Z, X]) + \tilde{g}(Z, [X, Y])$$

$$= X(f g(Y, Z)) + Y(f g(Z, X)) - Z(f g(X, Y))$$

$$- f g(X, [Y, Z]) - f g(Y, [Z, X]) + f g(Z, [X, Y])$$

$$= (X f) g(Y, Z) + (Y f) g(Z, X) - (Z f) g(X, Y) + f(2 g(\nabla_X Y, Z)).$$
Therefore

\[2f g(\tilde{\nabla}_X Y - \nabla_X Y, Z) = (X f) g(Y, Z) + (Y f) g(Z, X) - (Z f) g(X, Y) .\]

Noting that \(Z f = g(\text{grad} \ f, Z)\), we get \((Z f) g(X, Y) = g(g(X, Y) \text{grad} \ f, Z)\) and so

\[g(2f(\tilde{\nabla}_X Y - \nabla_X Y) - (X f) Y - (Y f) X + g(X, Y) \text{grad} \ f, Z) = 0.\]

Hence

\[\tilde{\nabla}_X Y - \nabla_X Y = \frac{1}{2f} [(X f) Y + (Y f) X - g(X, Y) \text{grad} \ f]. \tag{1.3}\]

**Remark 1.32.** Note that if \(f = e^{2\omega}\), then \(X e^{2\omega} = 2e^{2\omega} (X \omega)\) and \(\text{grad} \ f = 2e^{2\omega} \text{grad} \omega\). So in terms of \(\omega\) we have

\[\tilde{\nabla}_X Y - \nabla_X Y = (X \omega) Y + (Y \omega) X - g(X, Y) \text{grad} \omega. \tag{1.4}\]

We define \(S : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)\) by \(S(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y\). It follows from (1.3) that \(S\) is symmetric and moreover it is \(C^\infty(M)\)-bilinear; therefore \(S\) defines a \(\mathcal{X}(1)\)-tensor field which is called the *difference tensor*.

### 1.2.2 The Difference Tensor

The goal of this section is to derive some of the important properties of the difference tensor. In order to avoid any confusion, we draw distinction between the \(C^\infty(M)\)-bilinear map \(S : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)\) and the corresponding \(\mathcal{X}(1)\)-tensor field which we denote by \(\hat{S}\). For all \(X, Y \in \mathcal{X}(M)\) and \(\psi \in \Omega^1(M)\) and We have

\[\hat{S}(\psi, X, Y) = \psi(S(X, Y)).\]
Let \((E_i)_{i=1}^n\) denote an arbitrary local frame defined on an open set \(U \subset M\) and let \(\{\eta^i\}_{i=1}^n\) denote the corresponding dual coframe. We have

\[
\hat{S}^k_{ij} = \hat{S}(\eta^k, E_i, E_j) = \eta^k(S(E_i, E_j)).
\]

If \(S(E_i, E_j) = \alpha^k_i E_k\), then \(\alpha^k_i = \eta^k(S(E_i, E_j));\) so \(\alpha^k_i = \hat{S}^k_{ij}\). Consequently,

\[
S(E_i, E_j) = \hat{S}^k_{ij} E_k.
\] (1.5)

More generally, if \(S(X, Y) = \alpha^k E_k\), then \(\alpha^k = \eta^k(S(X, Y)) = \hat{S}(\eta^k, X, Y);\) so

\[
S(X, Y) = \sum_{k=1}^n \hat{S}(\eta^k, X, Y) E_k.
\]

**Remark 1.33.** \(\hat{S}\) is a \(\binom{2}{1}\)-tensor field, therefore

- \(\text{tr}\hat{S}\) is a \(\binom{1}{0}\)-tensor field.

\[
(\text{tr}\hat{S})(X) = \sum_{j=1}^n \hat{S}(\eta^j, X, E_j),
\]

\[
(\text{tr}\hat{S})_i = \hat{S}^i_{ij}.
\]

- \(\text{div}\hat{S}\) is a \(\binom{2}{0}\)-tensor field.

\[
(\text{div}\hat{S})(X, Y) = \sum_{k=1}^n (\nabla_{E_k} \hat{S})(\eta^k, X, Y)
\]

\[
(\text{div}\hat{S})_{ij} = (\nabla_k \hat{S})^k_{ij} = \hat{S}^k_{ij;k}.
\]

- for all vector fields \(V, \nabla_V \hat{S}\) is a \(\binom{2}{1}\)-tensor field and

\[
(\nabla_V \hat{S})(\psi, X, Y) = V(\hat{S}(\psi, X, Y)) - \hat{S}(\nabla_V \psi, X, Y) - \hat{S}(\psi, \nabla_V X, Y) - \hat{S}(\psi, X, \nabla_V Y).
\]
We can associate to the \(H^2\)-tensor field \(\nabla_V \hat{S}\) a \(C^\infty(M)\)-bilinear map from \(\chi(M) \times \chi(M)\) to \(\chi(M)\), denoted by \(\nabla_V S\), as follows:

\[
(\nabla_V S)(X, Y) = \nabla_V (S(X, Y)) - S(\nabla_V X, Y) - S(X, \nabla_V Y).
\]

It is easy to check that

\[
(\nabla_V \hat{S})(\psi, X, Y) = \psi((\nabla_V S)(X, Y)).
\]

Now our goal is to use (1.4) to find a formula for the components of \(\hat{S}\) in terms of the function \(\omega\). According to equation (1.4)

\[
S(X, Y) = (X\omega)Y + (Y\omega)X - g(X, Y)\text{grad}\omega.
\]

We have

\[
\hat{S}^c_{ab} = \hat{S}(\eta^c, E_a, E_b) = \eta^c(S(E_a, E_b))
= \eta^c((E_a\omega)E_b + (E_b\omega)E_a - g(E_a, E_b)\text{grad}\omega)
= (\nabla_a\omega)\delta^c_b + (\nabla_b\omega)\delta^c_a - g_{ab}\eta^c(\text{grad}\omega).
\]

Therefore

\[
\hat{S}^c_{ab} = (\nabla_a\omega)\delta^c_b + (\nabla_b\omega)\delta^c_a - g_{ab}g^{cd}\nabla_d\omega.
\]

Using \(\omega = \frac{1}{2} r \ln \varphi\) (recall that \(\hat{g} = e^{2\omega} g = \varphi^r g\)), we can rewrite the above formula in terms of \(\varphi\):

\[
\hat{S}^c_{ab} = \frac{1}{2} (\nabla_a(\ln \varphi))\delta^c_b + \frac{1}{2} (\nabla_b(\ln \varphi))\delta^c_a - g_{ab}g^{cd}\frac{r}{2} \nabla_d(\ln \varphi).
\] (1.6)
1.2.3 Relation Between the Covariant Derivatives

Let \( \{E_i\}_{i=1}^n \) denote an arbitrary local frame defined on an open set \( U \subset M \) and let \( \{\eta^i\}_{i=1}^n \) denote the corresponding dual coframe. The goal of this section is to find a relation between \( \tilde{\nabla}_a T \) and \( \nabla_a T \) where \( T \) is an arbitrary covariant or contravariant tensor field.

**Proposition 1.34.**

1. \( \forall \ Y \in \chi(M) \quad \tilde{\nabla}_a Y^c = \nabla_a Y^c + \hat{S}^c_{ab} Y^b. \)

2. \( \forall \ \psi \in \Omega^1(M) \quad \tilde{\nabla}_a \psi_c = \nabla_a \psi_c - \hat{S}^b_{ac} \psi_b. \)

3. \( \forall \ T \in \tau^k_0(M) \quad \tilde{\nabla}_a T_{j_1...j_k} = \nabla_a T_{j_1...j_k} - \hat{S}^m_{a j_1} T_{m j_2...j_k} - \cdots - \hat{S}^m_{a j_k} T_{j_1...j_{k-1} m}. \)

4. \( \forall \ T \in \tau^0_1(M) \quad \tilde{\nabla}_a T^{i_1...i_l} = \nabla_a T^{i_1...i_l} + \hat{S}^i_{a m} T^{m i_2...i_l} + \cdots + \hat{S}^i_{a m} T^{i_1...i_{l-1} m}. \)

**Proof.** (Proposition 1.34)

1. \( \tilde{\nabla}_a Y - \nabla_a Y = S(E_a, Y); \) so the \( c^{th} \) component of \( \tilde{\nabla}_a Y - \nabla_a Y \) is \( \eta^c (S(E_a, Y)) = \hat{S}(\eta^c, E_a, Y). \) Therefore

\[
(\tilde{\nabla}_a Y)^c - (\nabla_a Y)^c = \hat{S}(\eta^c, E_a, Y) = \sum_b Y^b \hat{S}(\eta^c, E_a, E_b) = \hat{S}^c_{ab} Y^b.
\]
2.

\[(\tilde{\nabla}_a \psi)_c = (\tilde{\nabla}_a \psi)(E_c) = E_a(\psi(E_c)) - \psi(\tilde{\nabla}_a E_c)\]
\[= E_a(\psi(E_c)) - \psi(\nabla_a E_c + S(E_a, E_c))\]
\[= E_a(\psi(E_c)) - \psi(\nabla_a E_c) - \psi(S(E_a, E_c))\]
\[= (\nabla_a \psi)(E_c) - \check{S}(\psi, E_a, E_c)\]
\[= (\nabla_a \psi)(E_c) - \check{S}(\psi b^b, E_a, E_c)\]
\[= \nabla_a \psi_c - \check{S}_{ac}\psi_b.\]

3.

\[(\tilde{\nabla}_a T)_{j_1...j_k} = (\tilde{\nabla}_a T)(E_{j_1}, ..., E_{j_k})\]
\[= E_a(T(E_{j_1}, ..., E_{j_k})) - T(\tilde{\nabla}_a E_{j_1}, ..., E_{j_k}) - \cdots - T(E_{j_1}, ..., \tilde{\nabla}_a E_{j_k})\]
\[= E_a(T(E_{j_1}, ..., E_{j_k}))\]
\[= T(E_{j_1}, ..., E_{j_k}) - T(\nabla_a E_{j_1}, ..., E_{j_k}) - \cdots - T(E_{j_1}, ..., \nabla_a E_{j_k})\]
\[= \nabla_a T_{j_1...j_k} - T(\check{S}_{a j_1}^m E_m, E_{j_2}, ..., E_{j_k}) - \cdots - T(E_{j_1}, E_{j_2}, ..., \check{S}_{a j_k}^m E_m)\]
\[= \nabla_a T_{j_1...j_k} - \check{S}_{a j_1}^m T_{m j_2...j_k} - \cdots - \check{S}_{a j_k}^m T_{j_1...j_k-1 m}.\]

4.

\[(\tilde{\nabla}_a T)^{i_1...i_l} = (\tilde{\nabla}_a T)(\eta^{i_1}, ..., \eta^{i_l})\]
\[= E_a(T(\eta^{i_1}, ..., \eta^{i_l})) - T(\tilde{\nabla}_a \eta^{i_1}, ..., \eta^{i_l}) - \cdots - T(\eta^{i_l}, ..., \tilde{\nabla}_a \eta^{i_l}).\]
Using item 2., one can easily show that for any element $\eta^d$ of the dual coframe we have

$$\tilde{\nabla}_a \eta^d = \nabla_a \eta^d - \tilde{\hat{S}}^d_{ab} \eta^b.$$ 

The rest of the proof is completely analogous to the proof of item 3.

\[ \square \]

1.2.4 Relation Between the Gradients

**Proposition 1.35.** Suppose $\tilde{g} = \varphi^r g$. If $h : M \to \mathbb{R}$ is a smooth function, then

$$\tilde{\text{grad}} h = \varphi^{-r} \text{grad} h.$$

**Proof.** (Proposition 1.35) For all $X \in \chi(M)$ we have

$$X h = \tilde{g}(\tilde{\text{grad}} h, X) = \varphi^r g(\text{grad} h, X) = g(\varphi^r \tilde{\text{grad}} h, X).$$

Therefore, for all $X \in \chi(M)$

$$X h = g(\text{grad} h, X) = g(\varphi^r \tilde{\text{grad}} h, X).$$

It follows that, $\text{grad} h = \varphi^r \tilde{\text{grad}} h$.  \[ \square \]

1.2.5 Relation Between the Divergences

We are interested in the following 3 cases

1. Conformal transformation of the divergence of a vector field.
2. Conformal transformation of the divergence of a symmetric traceless \( (2,0) \)-tensor field.

3. Conformal transformation of the divergence of a symmetric traceless \( (0,2) \)-tensor field.

Let \( \{E_i\}_{i=1}^n \) denote an arbitrary local frame defined on an open set \( U \subset M \) and let \( \{\eta^i\}_{i=1}^n \) denote the corresponding dual coframe. First we note that equation (1.6) can be rewritten as follows:

\[
\hat{S}_{ab} = \frac{r}{2\varphi}(\nabla_{a}\varphi)\delta_{b} - \frac{r}{2\varphi}(\nabla_{b}\varphi)\delta_{a} - g_{ab}g^{cd} \frac{r}{2\varphi}(\nabla_{d}\varphi). \tag{1.7}
\]

Also as a direct consequence we have

\[
\hat{S}_{ac} = \frac{r}{2\varphi}(\nabla_{a}\varphi)\delta_{c} + \frac{r}{2\varphi}(\nabla_{c}\varphi)\delta_{a} - \frac{r}{2\varphi} g_{ac}g^{bd} \frac{2}{2\varphi}(\nabla_{d}\varphi) = \frac{nr}{2\varphi} \nabla_{c}\varphi. \tag{1.8}
\]

**Case 1:** Conformal transformation of the divergence of a vector field.

Let \( Y \in \chi(M) \). By Proposition 1.34 we have

\[
\text{div} \tilde{g} Y = \tilde{\nabla}_{a} Y^{a} = \nabla_{a} Y^{a} + \hat{S}_{ab} Y^{b} = \text{div} g Y + \hat{S}_{ab} Y^{b}.
\]

Since \( \hat{S}_{ab} = \frac{nr}{2\varphi} \nabla_{b}\varphi \), we get

\[
\text{div} \tilde{g} Y = \text{div} g Y + \frac{nr}{2\varphi} \nabla_{b}\varphi Y^{b} = \text{div} g Y + \frac{nr}{2\varphi} \langle \text{grad} \varphi, Y \rangle g. \tag{1.9}
\]

**Case 2:** Conformal transformation of the divergence of a symmetric traceless \( (2,0) \)-tensor field.
Let \( T \in \tau^2_0(M) \) be a symmetric traceless tensor field. Let \( \tilde{T}_{ab} := \varphi^s T_{ab} \). By Proposition 1.34 we have

\[
(\text{div}_g \tilde{T})_b = \tilde{g}^{ac} \tilde{\nabla}_c \tilde{T}_{ba}
\]

\[
= \tilde{g}^{ac} [\nabla_c \tilde{T}_{ba} - \tilde{S}_{cb}^m \tilde{T}_{ma} - \tilde{S}_{ca}^m \tilde{T}_{bm}]
\]

\[
= \varphi^{-r} \tilde{g}^{ac} [\nabla_c \tilde{T}_{ab} - \tilde{S}_{cb}^m \tilde{T}_{ma} - \tilde{S}_{ca}^m \tilde{T}_{bm}].
\]

Note that

\[
\varphi^{-r} \tilde{g}^{ac} \nabla_c \tilde{T}_{ab} = \varphi^{-r} \tilde{g}^{ac} \nabla_c (\varphi^s T_{ab}) = \varphi^{-r} \tilde{g}^{ac} [s \varphi^{s-1} \nabla_c \varphi T_{ab} + \varphi^s \nabla_c T_{ab}]
\]

\[
= s \varphi^{s-r-1} \nabla_a \varphi T_{ab} + \varphi^{s-r} (\tilde{g}^{ac} \nabla_c T_{ab})
\]

\[
= s \varphi^{s-r-1} \nabla_a \varphi T_{ab} + \varphi^{s-r} (\text{div}_g T)_b.
\]

\[
\varphi^{-r} \tilde{g}^{ac} \tilde{S}_{cb}^m \tilde{T}_{ma} = \varphi^{-r} \tilde{g}^{ac} \frac{r}{2 \varphi} [\nabla_c \varphi \delta^m_b + \nabla_b \varphi \delta^m_c - \tilde{g}_{cb} \tilde{g}^{md} \nabla_d \varphi] \varphi^s T_{ma}
\]

\[
= \frac{r}{2} \varphi^{s-r-1} \tilde{g}^{ac} [\nabla_c \varphi T_{ba} + \nabla_b \varphi T_{ca} - \tilde{g}_{cb} \nabla^m \varphi T_{ma}]
\]

\[
= \frac{r}{2} \varphi^{s-r-1} [\nabla^a \varphi T_{ba} + \nabla^b \varphi \tilde{g}^{ac} T_{ac} - \delta^a_b \nabla^m \varphi T_{ma} = 0.
\]

\[
\varphi^{-r} \tilde{g}^{ac} \tilde{S}_{ca}^m \tilde{T}_{bm} = \varphi^{-r} \tilde{g}^{ac} \frac{r}{2 \varphi} [\nabla_c \varphi \delta^m_a + \nabla_a \varphi \delta^m_c - \tilde{g}_{ca} \tilde{g}^{md} \nabla_d \varphi] \varphi^s T_{bm}
\]

\[
= \frac{r}{2} \varphi^{s-r-1} \tilde{g}^{ac} [\nabla_c \varphi T_{ba} + \nabla_a \varphi T_{bc} - \tilde{g}_{ca} \nabla^m \varphi T_{bm}]
\]

\[
= \frac{r}{2} \varphi^{s-r-1} [\nabla^a \varphi T_{ba} + \nabla^c \varphi T_{bc} - \tilde{g}^{ac} \tilde{g}_{ac} \nabla^m \varphi T_{bm}].
\]
Therefore

$$(\text{div}_g \tilde{T})_b = \varphi^{s-r} (\text{div}_g T)_b + \varphi^{s-r-1} (-r + s + \frac{n r}{2}) \nabla^a \varphi T_{ba}.$$ 

In particular, if we let $s = r - \frac{n r}{2}$, then

$$\text{div}_g \tilde{T} = \varphi^{s-r} \text{div}_g T.$$ 

If $r = \frac{4}{n-2}$ and $s = r - \frac{n r}{2} = -2$, then

$$\text{div} \frac{4}{\varphi^{n-2} g} (\varphi^{-2} T) = \varphi^{-2} \text{div}_g T.$$ 

**Case 3:** Conformal transformation of the divergence of a symmetric traceless $\mathcal{T}^{(0)}_{\mathcal{2}}$-tensor field.

Let $T \in \mathcal{T}^{(0)}_{\mathcal{2}}(M)$ be a symmetric traceless tensor field. Let $\tilde{T}^{ab} := \varphi^s T^{ab}$. By Proposition 1.34 we have

$$(\text{div}_g \tilde{T})^b = \tilde{\nabla}_a \tilde{T}^{ab}$$

$$= \nabla_a \tilde{T}^{ab} + \tilde{S}^{ab}_a + \tilde{S}^{ab}_m \tilde{T}^{am}.$$ 

Note that

$$\nabla_a \tilde{T}^{ab} = \nabla_a (\varphi^s T_{ab}) = s \varphi^{s-1} \nabla_a \varphi T^{ab} + \varphi^s \nabla_a T^{ab}$$

$$= s \varphi^{s-1} \nabla_a \varphi T^{ab} + \varphi^s (\text{div}_g T)^b.$$ 

$$\tilde{S}^{ab}_a \tilde{T}^{mb} = \frac{n r}{2 \varphi} (\nabla_m \varphi) \varphi^s T^{mb} = \frac{n r}{2} \varphi^{s-1} \nabla_m \varphi T^{mb}.$$
\[ \hat{S}_{am}^b \hat{T}^{am} = \frac{r}{2\varphi} \left[ \nabla_a \varphi \delta^b_m + \nabla_m \varphi \delta^b_a - g_{am} g^{bd} \nabla_d \varphi \right] \varphi^s \hat{T}^{am} \]
\[ = \frac{r}{2} \varphi^{s-1} \nabla_a \varphi T^{ab} + \frac{r}{2} \varphi^{s-1} \nabla_m \varphi T^{bm} - \nabla^b \varphi \frac{g_{am} T^{am}}{0}. \]

Therefore

\[ (\text{div}_\tilde{g} \hat{T})^b = \varphi^s (\text{div}_g T)^b + \varphi^{s-1} (s + \frac{nr}{2} + r) \nabla_a \varphi T^{ab}. \]

In particular, if we let \( s = -r \left( \frac{n}{2} + 1 \right) \), then

\[ \text{div}_\tilde{g} \hat{T} = \varphi^s \text{div}_g T. \]

If \( r = \frac{4}{n-2} \) and \( s = -r \left( \frac{n}{2} + 1 \right) = -\frac{2(n+2)}{n-2} \), then

\[ \text{div} \left( \varphi^{\frac{2(n+2)}{n-2}} g \right) (\varphi^{-\frac{2(n+2)}{n-2}} T) = \varphi^{-\frac{2(n+2)}{n-2}} \text{div}_g T. \]

Note that if \( n = 3 \), then the above choices for \( r \) and \( s \) correspond to \( r = 4 \) and \( s = -10 \).
1.2.6 Relation Between the Laplacians

Let $h : M \to \mathbb{R}$ be a smooth function. By Proposition 1.35 and equation (1.9) we have

\[
\Delta \tilde{g} h = \text{div}_{\tilde{g}} (\tilde{\nabla} h) = \text{div}_{\tilde{g}} (\varphi^{-r} \nabla h)
\]

\[
= \text{div}_{\tilde{g}} (\varphi^{-r} \nabla h) + \frac{n r}{2 \varphi} \langle \nabla \varphi, \varphi^{-r} \nabla h \rangle_{g}
\]

\[
= \varphi^{-r} \text{div}_{\tilde{g}} (\nabla h) + \langle \nabla (\varphi^{-r}), \nabla h \rangle_{g} + \frac{n r}{2} \varphi^{-r-1} \langle \nabla \varphi, \nabla h \rangle_{g}
\]

\[
= \varphi^{-r} \Delta_{\tilde{g}} h - r \varphi^{-r-1} \langle \nabla \varphi, \nabla h \rangle_{g} + \frac{n r}{2} \varphi^{-r-1} \langle \nabla \varphi, \nabla h \rangle_{g}
\]

\[
= \varphi^{-r} \Delta_{\tilde{g}} h + \left( \frac{n r}{2} - r \right) \varphi^{-r-1} \langle \nabla \varphi, \nabla h \rangle_{g}.
\]

1.2.7 Relation Between the Conformal Killing Operators

Let $X \in \chi(M)$. In any local frame we have

\[
(\mathcal{L}_{\tilde{g}} X)^{ij} = \tilde{\nabla}^{i} X^{j} + \tilde{\nabla}^{j} X^{i} - \frac{2}{n} (\text{div}_{\tilde{g}} X) \tilde{g}^{ij}
\]

\[
= \tilde{g}^{ia} \tilde{\nabla}_{a} X^{j} + \tilde{g}^{ja} \tilde{\nabla}_{a} X^{i} - \frac{2}{n} \left[ \text{div}_{\tilde{g}} X + \frac{n r}{2 \varphi} \langle \nabla \varphi, X \rangle_{g} \right] \varphi^{-r} g^{ij}
\]

\[
= \varphi^{-r} g^{ia} \left[ \nabla_{a} X^{j} + \tilde{S}_{ab}^{j} X^{b} \right] + \varphi^{-r} g^{ja} \left[ \nabla_{a} X^{i} + \tilde{S}_{ab}^{i} X^{b} \right]
\]

\[
- \frac{2}{n} \varphi^{-r} g^{ij} \left[ \text{div}_{\tilde{g}} X + \frac{n r}{2 \varphi} \langle \nabla \varphi, X \rangle_{g} \right]
\]

\[
= \varphi^{-r} \left[ \nabla^{i} X^{j} + g^{i a} \tilde{S}_{a b}^{j} X^{b} \right] + \varphi^{-r} \left[ \nabla^{j} X^{i} + g^{j a} \tilde{S}_{a b}^{i} X^{b} \right]
\]

\[
- \frac{2}{n} \varphi^{-r} \left[ \text{div}_{\tilde{g}} X g^{ij} + \frac{n r}{2 \varphi} g^{ij} \langle \nabla \varphi, X \rangle_{g} \right]
\]

\[
= \varphi^{-r} (\mathcal{L}_{g} X)^{ij} + \varphi^{-r} \left[ g^{i a} \tilde{S}_{a b}^{j} X^{b} + g^{j a} \tilde{S}_{a b}^{i} X^{b} - \frac{r}{\varphi} g^{ij} \langle \nabla \varphi, X \rangle_{g} \right].
\]
Now note that

\[ \varphi^{-r} g^{ia} \hat{S}_{ab} X^b = \varphi^{-r-1} g^{ia} \frac{r}{2} \left[ \nabla_a \varphi \delta^j_b + \nabla_b \varphi \delta^j_a - g_{ab} g^{jd} \nabla_d \varphi \right] X^b \]

\[ = \varphi^{-r-1} \frac{r}{2} \left[ \nabla^i \varphi X^j + (\nabla_b \varphi X^b) g^{ij} - \delta^i_b \nabla^j \varphi X^b \right] \]

\[ = \frac{r}{2} \varphi^{-r-1} \left[ \nabla^i \varphi X^j - \nabla^j \varphi X^i + g^{ij} \langle \text{grad} \varphi, X \rangle_g \right]. \]

By switching the roles of \( i \) and \( j \) we get

\[ \varphi^{-r} g^{i} \hat{S}^j_{ab} X^b = \frac{r}{2} \varphi^{-r-1} \left[ \nabla^j \varphi X^i - \nabla^i \varphi X^j + g^{ij} \langle \text{grad} \varphi, X \rangle_g \right]. \]

Hence

\[ \varphi^{-r} g^{ia} \hat{S}_{ab} X^b + g^{i} \hat{S}^j_{ab} X^b - \frac{r}{\varphi} g^{ij} \langle \text{grad} \varphi, X \rangle_g \]

\[ = \frac{r}{2} \varphi^{-r-1} \left[ 2 g^{ij} \langle \text{grad} \varphi, X \rangle_g \right] - r \varphi^{-r-1} g^{ij} \langle \text{grad} \varphi, X \rangle_g \]

\[ = 0. \]

Therefore,

\[ (\mathcal{L}_g X)^{ij} = \varphi^{-r} (\mathcal{L}_g X)^{ij}. \]
1.2.8 Relation Between the Curvature Tensors

\[ \tilde{R}(X, Y)Z = \tilde{\nabla}_X(\tilde{\nabla}_Y Z) - \tilde{\nabla}_Y(\tilde{\nabla}_X Z) - \tilde{\nabla}_{[X,Y]} Z \]
\[ = \tilde{\nabla}_X(\nabla_Y Z + S(Y, Z)) - \tilde{\nabla}_Y(\nabla_X Z + S(X, Z)) - (\nabla_{[X,Y]} Z + S([X, Y], Z)) \]
\[ = \nabla_X(\nabla_Y Z + S(Y, Z)) + S(X, \nabla_Y Z + S(Y, Z)) - \nabla_Y(\nabla_X Z + S(X, Z)) - S(Y, \nabla_X Z + S(X, Z)) \]
\[ - \nabla_{[X,Y]} Z - S([X, Y], Z) \]
\[ = R(X, Y)Z + \nabla_X(S(Y, Z)) - \nabla_Y(S(X, Z)) + S(X, S(Y, Z)) - S(Y, S(X, Z)) + S(X, \nabla_Y Z) - S(Y, \nabla_X Z) - S(\nabla_X Y - \nabla_Y X, Z) \]
\[ = R(X, Y)Z + \nabla_X(S(Y, Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z) - [\nabla_Y(S(X, Z)) - S(\nabla_Y X, Z) - S(X, \nabla_Y Z)] \]
\[ + S(X, S(Y, Z)) - S(Y, S(X, Z)). \]

Therefore

\[ \tilde{R}(X, Y)Z = R(X, Y)Z + (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) \]
\[ + S(X, S(Y, Z)) - S(Y, S(X, Z)). \tag{1.10} \]

Recall that the curvature tensor can be viewed as a \( \left( \begin{array}{c} \partial^3 \\ \partial \end{array} \right) \)-tensor field:

\[ R(\psi, Z, X, Y) := \psi(R(X, Y)Z), \quad \tilde{R}(\psi, Z, X, Y) := \psi(\tilde{R}(X, Y)Z). \]
In order to find a relation between the above \( (3) \)-tensor fields, we apply \( \psi \) to both sides of (1.10):

\[
\hat{R}(\psi, Z, X, Y) = R(\psi, Z, X, Y) + \psi[\nabla_X S(Y, Z)] - \psi[\nabla_Y S(X, Z)]
\]

\[
+ \psi[S(X, S(Y, Z))] - \psi[S(Y, S(X, Z))]
\]

\[
= R(\psi, Z, X, Y) + (\nabla_X \hat{S})(\psi, Y, Z) - (\nabla_Y \hat{S})(\psi, X, Z)
\]

\[
+ \hat{S}(\psi, X, S(Y, Z)) - \hat{S}(\psi, Y, S(X, Z))
\]

If \( \{E_i\} \) is any local frame with dual coframe \( \{\eta^i\} \), we get

\[
\hat{R}(\eta^d, E_a, E_b, E_c) = R(\eta^d, E_a, E_b, E_c) + (\nabla_b \hat{S})(\eta^d, E_c, E_a) - (\nabla_c \hat{S})(\eta^d, E_b, E_a)
\]

\[
+ \hat{S}(\eta^d, E_b, S(E_c, E_a)) - \hat{S}(\eta^d, E_c, S(E_b, E_a)).
\]

According to (1.5), \( S(E_c, E_a) = \hat{S}^e_{ca} E_a \) and \( S(E_b, E_a) = \hat{S}^e_{ba} E_a \), so

\[
\hat{R}_d^{abc} = R_d^{abc} + (\nabla_b \hat{S})_{ca}^d - (\nabla_c \hat{S})_{ba}^d + \hat{S}^e_{ca} \hat{S}^b_{be} - \hat{S}^e_{ba} \hat{S}^b_{ce}.
\]

Now noting that \( \text{Ric}^{ac} = R_{abc}^b \), we obtain the following formula for Ricci tensors

\[
\hat{\text{Ric}}^{ac} = \text{Ric}^{ac} + (\nabla_b \hat{S})_{ca}^b - (\nabla_c \hat{S})_{ba}^b + \hat{S}^e_{ca} \hat{S}^b_{be} - \hat{S}^e_{ba} \hat{S}^b_{ce}.
\]

Finally, recall that \( R = g^{ac} \text{Ric}^{ac} \); so we can multiply both sides of the above equation by \( g^{ac} \) to find a relation between the two Ricci scalars (note that \( g^{ac} = \phi^r g^{rc} \)).

\[
\phi^r g^{ac} \hat{\text{Ric}}^{ac} = g^{ac} \text{Ric}^{ac} + g^{ac} (\nabla_b \hat{S})_{ca}^b - g^{ac} (\nabla_c \hat{S})_{ba}^b + g^{ac} \hat{S}^e_{ca} \hat{S}^b_{be} - g^{ac} \hat{S}^e_{ba} \hat{S}^b_{ce}.
\]
Therefore

\[ \tilde{R} = \varphi^{-r} \left[ R + g^{ac} (\nabla_b \hat{S})^b_{ca} - (\nabla^a \hat{S})^b_{ba} + g^{ae} \hat{S}^b_{ea} \hat{S}^b_{be} - g^{ae} \hat{S}^b_{ba} \hat{S}^b_{ce} \right]. \]

Using equation (1.6) we can rewrite the above equation as follows:

\[ \tilde{R} = \varphi^{-r-1} \left[ \varphi R - r(n-1)\Delta \varphi + \frac{r(n-1)}{4\varphi} (4 - (n-2)r) \left| \nabla \varphi \right|^2 \right]. \quad (1.11) \]

In particular, notice that if we set \( r = \frac{4}{n-2} \), the above equation will be considerably simplified:

\[ \tilde{R} = \varphi^{1-p} \left( -a \Delta \varphi + R \varphi \right), \]

where \( p = 2 + \frac{4}{n-2} \) and \( a = 4 \frac{n-1}{n-2} \).
Chapter 2

The Initial Value Problem in General Relativity

Most physical models admit an initial value formulation: Newtonian mechanics, the wave and heat equations, Maxwell’s equations all admit initial value problems. Einstein’s general theory of relativity relates the curvature of spacetime to the non-gravitational energy present through the Einstein’s field equations. An effective way of studying solutions of Einstein’s field equations is via the initial value formulation; hidden in the geometric formulation of the theory is an initial value problem analogous to the initial value problem of Newtonian mechanics. As we shall see, rather than specifying the positions and momenta of gravitational sources as in the Newtonian problem, the initial data consists of a 3-dimensional Riemannian manifold \((M, \hat{h})\) determining the geometry of a spacelike slice of spacetime, together with a symmetric rank 2 tensor field \(\hat{k}\) which represents the second fundamental form of the embedding of \(M\) in the ambient spacetime. Unlike the Newtonian problem, where the initial configuration is essentially unconstrained, one cannot arbitrarily prescribe the metric \(\hat{h}\) and the second fundamental form \(\hat{k}\); they are necessarily related by a system.
of partial differential equations known as the Einstein constraint equations. These equations reflect the Gauss and Codazzi equations inherited from the embedding of $(M, \hat{h})$ into an ambient spacetime which satisfies the field equations.

There are several analytical techniques which have been developed to obtain solutions of the constraint equations. The one which has been most widely and successfully applied is the conformal method, initially formulated by Lichnerowicz [50] and substantially extended by Choquet-Bruhat and York [17]. Beyond this there are others which have met with some success, including gluing techniques [42, 21, 22, 18, 19, 58] and the “conformal thin-sandwich” method [73] (which is in fact completely equivalent to the conformal method [57]).

In section 2.1 we review some of the fundamental notions of Lorentzian geometry. In section 2.2 we study the Einstein’s field equations and we derive the Einstein constraint equations. In section 2.3 we discuss the standard conformal method in detail. In section 2.4 we briefly review some of the well-known results.

2.1 Preliminaries

2.1.1 Lorentzian Manifolds and Spacetime

Let $\mathcal{M}^{n+1}$ be a connected smooth manifold. A symmetric $(2,0)$-tensor field $g$ on $\mathcal{M}$ is said to be a nondegenerate metric on $\mathcal{M}$ if for all $p \in \mathcal{M}$ and $0 \neq X_p \in T_p \mathcal{M}$, there exists $Y_p \in T_p \mathcal{M}$ such that $g(X_p, Y_p) \neq 0$. Suppose $\mathcal{M}$ is equipped with a nondegenerate metric $g$. Suppose $p \in \mathcal{M}$ and let $\{e_0, \cdots, e_n\}$ be an orthogonal basis for $T_p \mathcal{M}$. Let $A = \{g(e_0, e_0), \cdots, g(e_n, e_n)\}$. Since $g$ is nondegenerate, elements of $A$ are nonzero. Let $r$ denote the number of positive elements of $A$ and $q$ denote the number of negative elements of $A$ ($r + q = n + 1$). One can show that the numbers $r$ and $q$ are independent of the chosen point $p$ and the basis $\{e_i\}$. We say that the metric
$g$ is of index $q$, or $g$ is of signature $(q,r)$. An orthogonal basis $\{e_0, \cdots, e_n\}$ of $T_pM$ is called orthonormal (or $g$-orthonormal) if $g(e_i, e_i) = -1$ whenever $g(e_i, e_i) < 0$, and $g(e_i, e_i) = 1$ whenever $g(e_i, e_i) > 0$.

**Definition 2.1.** A Lorentzian manifold $(\mathcal{M}, g)$ is a smooth connected manifold $\mathcal{M}$ equipped with a nondegenerate metric $g$ of index 1.

As a matrix, with respect to a Lorentzian orthonormal basis we have $[g_{ij}] = \text{diag}(-1, +1, \cdots, +1)$.

Throughout the remaining of this chapter we take $n = 3$. For any $p \in \mathcal{M}$, we have a classification of vectors $V \in T_p\mathcal{M}$ into timelike, null or spacelike as follows:

- $V$ is timelike if $g(V, V) < 0$.
- $V$ is null if $g(V, V) = 0$.
- $V$ is spacelike if $g(V, V) > 0$.

We say $V$ is causal if $V$ is either timelike or null. We can extend this notion to smooth paths $\gamma : (a, b) \to \mathcal{M}$ as follows:

- $\gamma$ is timelike if $\gamma'(t)$ is timelike $\forall \ t \in (a, b)$.
- $\gamma$ is null if $\gamma'(t)$ is null $\forall \ t \in (a, b)$.
- $\gamma$ is spacelike if $\gamma'(t)$ is spacelike $\forall \ t \in (a, b)$.

We say that $\gamma$ is causal if $\gamma'(t)$ is either timelike or null for all $t \in (a, b)$. Also we can extend this notion to hypersurfaces $\Sigma$ of $\mathcal{M}$. Let $i : \Sigma \to \mathcal{M}$ be the inclusion map and let $\hat{h} = i^* g$. We say

- $\Sigma$ is spacelike if the metric $\hat{h}$ is Riemannian, i.e. has signature $(0,3)$.
- $\Sigma$ is timelike if the metric $\hat{h}$ is Lorentzian, i.e. has signature $(1,2)$.
• $\Sigma$ is null if the metric $\hat{h}$ is degenerate.

We say that the Lorentzian manifold $(\mathcal{M}, g)$ is time-orientable if it admits a continuous timelike vector field.

**Definition 2.2.** A spacetime $(\mathcal{M}, g)$ is a time-orientable Lorentzian manifold.

Let $T$ denote a timelike vector field defining the time orientation on $\mathcal{M}$. For any nonzero causal vector $V \in T_p \mathcal{M}$, $g(V, T_p)$ is either positive or negative. If $g(V, T_p) < 0$, we say that $V$ is future pointing, and if $g(V, T_p) > 0$ we say that $V$ is past pointing. A causal path $\gamma$ is said to be future pointing if $\gamma'$ is future pointing at each point along $\gamma$.

A causal path $\gamma$ is said to be inextendible if there does not exist a smooth path $\tilde{\gamma}$ which contains $\gamma$ as a proper subset. A Cauchy surface in $\mathcal{M}$ is a subset of $\mathcal{M}$ that is met exactly once by every inextendible timelike curve in $\mathcal{M}$. It can be shown that a Cauchy surface is necessarily a topological hypersurface of $\mathcal{M}$. A spacetime $(\mathcal{M}, g)$ that admits a Cauchy surface is said to be globally hyperbolic.

The Levi-Civita connection on $(\mathcal{M}, g)$ and the curvature tensors are defined exactly as in the case of Riemannian manifolds (see Chapter 1).

**2.1.2 The Geometry of Spacelike Hypersurfaces**

Let $(\mathcal{M}, g)$ be a spacetime and let $M$ be an embedded smooth spacelike hypersurface. We assume $M$ to be orientable. We denote the induced Riemannian metric on $M$ by $\hat{h}$. Let $n$ be a unit (timelike) vector field normal to $M$. We denote the Levi-Civita connection on $(\mathcal{M}, g)$ by $\tilde{\nabla}$ and the Levi-Civita connection on $(M, \hat{h})$ by $\hat{\nabla}$. The curvature tensors of $(\mathcal{M}, g)$ carry a tilde; the curvature tensors of $(M, \hat{h})$ carry a hat. We denote the second fundamental form of $M$ as an embedded hypersurface of $\mathcal{M}$ by $\hat{k}$. The proof of the following results is completely analogous to their Riemannian counterparts that were discussed in Chapter 1. For all $X, Y, Z, W \in \chi(M)$
• $\tilde{\nabla}_X Y = \tilde{\nabla}_X Y + \hat{k}(X, Y)n$.

• $\hat{k}(X, Y) = -g(n, \tilde{\nabla}_X Y)$.

• $\hat{k}(X, Y) = \hat{k}(Y, X)$.

• $\hat{k}(X, Y) = \hat{h}(\tilde{\nabla}_X n, Y) = \hat{h}(X, \tilde{\nabla}_Y n)$.

• Gauss equation:

$$\tilde{R}(W, Z, X, Y) = \hat{R}(W, Z, X, Y) + \hat{k}(Y, Z)\hat{k}(X, W) - \hat{k}(X, Z)\hat{k}(Y, W).$$

• Codazzi equation:

$$\tilde{R}(n, Z, X, Y) = g(\tilde{R}(X, Y) Z, n) = (\tilde{\nabla}_X \hat{k})(Y, Z) - (\tilde{\nabla}_Y \hat{k})(X, Z).$$

The sign difference between some of the above items and their Riemannian counterparts (Chapter 1) is due to the fact that $n$ is a unit timelike vector field and $\langle n, n \rangle = -1$.

The trace of the second fundamental form, $\text{tr}_h \hat{k}$, is called the **mean (extrinsic) curvature** of $M$.

### 2.1.3 Foliation of Spacetime

It seems that in order to cast the general relativity into an initial value problem we need to be able to have a notion of evolution with respect to a given external time; however, time is part of the fabric of spacetime. Therefore, to develop an initial value formulation we need to somehow decompose the spacetime into space and an arbitrary choice of (i.e. artificial notion of) time. In more detail we wish to be able to write $\mathcal{M}$ as a disjoint union of spacelike hypersurfaces such that each hypersurface
represents a constant instant of time. There is a celebrated theorem by Geroch [28] which states that this can be done in globally hyperbolic spacetimes.

**Theorem 2.3** (Geroch). Let \((\mathcal{M}, g)\) be a globally hyperbolic spacetime. If \(S\) is a Cauchy surface for \(\mathcal{M}\), then \(\mathcal{M}\) is homeomorphic to \(\mathbb{R} \times S\).

**Theorem 2.4** (Bernal and Sanchez). If we assume the spacetime \((\mathcal{M}, g)\) admits a smooth spacelike Cauchy hypersurface \(S\), then in the previous theorem “homeomorphic” can be replaced by “diffeomorphic”. There exists a real-valued function \(t : \mathcal{M} \rightarrow \mathbb{R}\) with nowhere vanishing gradient such that each level surface of \(t\) is a smooth Cauchy surface and all these Cauchy surfaces are diffeomorphic.[7, 66]

It can be shown that if \((\mathcal{M}, g)\) is an orientable globally hyperbolic spacetime, then it admits a smooth spacelike Cauchy hypersurface [66]. Thus the conclusion is that if \((\mathcal{M}, g)\) is an orientable globally hyperbolic spacetime, then it can be foliated by a family of nonintersecting smooth orientable spacelike hypersurfaces, that is, there exist a 3-dimensional orientable Riemannian manifold \(\Sigma\) and a one-parameter family of embeddings

\[F_s : \Sigma \rightarrow \mathcal{M}, \quad \Sigma_s := F_s(\Sigma) \subset \mathcal{M}\]

such that \(\bigcup \Sigma_s = \mathcal{M}\). \(\Sigma_s\) is called the leaf corresponding to the value \(s \in \mathbb{R}\). Each point in \(\mathcal{M}\) is contained in exactly one leaf. The real-valued function \(t : \mathcal{M} \rightarrow \mathbb{R}\) assigns to each point in \(\mathcal{M}\) the parameter value of the leaf it lies on:

\[t(p) = s \iff p \in \Sigma_s.\]

The fact that each hypersurface in the foliation is spacelike implies that \(\text{grad} t = (dt)^\sharp\)
is timelike. Let \( n \) denote the unit vector field perpendicular to the leaves defined by

\[
n = \frac{\text{grad} t}{\sqrt{-g(\text{grad} t, \text{grad} t)}}. \tag{2.1}
\]

Notice that \( n \) is past pointing if \( t \) is increasing towards the future. Clearly if \( \{x^1, x^2, x^3\} \) are local coordinates on \( \Sigma \), then \( \{t, x^1, x^2, x^3\} \) form local coordinates for \( \mathcal{M} \) (we call them \emph{adapted} local coordinates). A fixed point \( q \in \Sigma \) defines the following path in \( \mathcal{M} \)

\[
s \mapsto \mathcal{F}_s(q)
\]

\( \frac{\partial}{\partial t}\big|_{\mathcal{F}_s(q)} \) is the vector tangent to this path at \( p = \mathcal{F}_s(q) \). So for all \( f \in C^\infty(\mathcal{M}) \) we have

\[
\frac{\partial}{\partial t}\big|_{\mathcal{F}_s(q)} f = \frac{df(\mathcal{F}_{s'}(q))}{ds'} \bigg|_{s'=s}.
\]

\( \frac{\partial}{\partial t} \) is a vector field that defines the same point in space at different instants of time. Clearly at each point \( \frac{\partial}{\partial t} \) can be decomposed into its horizontal component that is tangential to the leaves of the given foliation and its normal component:

\[
\frac{\partial}{\partial t} = \frac{N}{\text{normal}} + \underbrace{X}_{\text{horizontal}}.
\]

The real-valued function \( N \) is called the \emph{lapse} function and the vector field \( X = X^i \frac{\partial}{\partial x^i} \) is called the \emph{shift} vector field. If we denote the induced Riemannian metric on each leaf by \( \hat{h} \) and the corresponding second fundamental form by \( \hat{k} \), then in terms of the adapted local coordinates we have ([30])

\[
g = -N^2 dt^2 + \hat{h}_{ij}(dx^i + X^i dt)(dx^j + X^j dt) \quad (1 \leq i, j \leq 3), \tag{2.2}
\]

\[
\frac{\partial \hat{h}_{ij}}{\partial t} = L_X \hat{h}_{ij} + 2N \hat{k}_{ij} \quad (1 \leq i, j \leq 3). \tag{2.3}
\]
2.2 Einstein's Field Equations: A PDE Perspective

What did Einstein want to achieve by the theory of general relativity? Roughly speaking his goal was to describe the gravitational field by 10 components of the spacetime metric tensor in a form invariant with respect to general transformations of spacetime coordinates. After several years of unsuccessful attempts, Einstein finally discovered in 1915 the equation according to which gravitation is produced by matter

\[ \text{Ric}_g - \frac{1}{2} R_g g + \Lambda g = 8 \pi \frac{G}{c^4} T \]  

(2.4)

where \( g \) is a nondegenerate metric of index 1, \( \text{Ric}_g \) is the Ricci curvature of \( g \), and \( R_g \) is the scalar curvature of \( g \). \( \Lambda \) is the cosmological constant which may be zero but is currently thought to be positive in our universe. The \((3,0)\)-tensor field \( T \) is the stress-energy-momentum tensor. This encodes the non-gravitational physics (e.g. electromagnetic field, fluids, etc.). \( G \) is the Gravitation constant and \( c \) is the speed of light. The equation (2.4) is called the Einstein's field equation. It is a single tensor equation but sometimes in order to emphasize that it comprises several component equations we refer to (2.4) as Einstein's field equations.

We may summarize the assumptions that lead to the derivation of Einstein's law of gravitation as follows [35]:

- The gravitational field is fully described by the spacetime metric tensor \( g \).
- The source of the gravitational field is the symmetric stress-energy-momentum tensor \( T \) of all matter and non-gravitational fields present in space. \( T \) satisfies the conservation law \( \text{div} T = 0 \).
- The gravitational field is produced by the stress-energy-momentum tensor \( T \).
according to the second order field equations

\[ \mathcal{G} = T \]

where \( \mathcal{G} \) is a \( (2, 0) \)-tensor field constructed entirely from the metric tensor \( g \) and its first and second derivatives \((\mathcal{G}_{\mu\nu}(g_{\theta\sigma}, g_{\theta\sigma,\alpha}, g_{\theta\sigma,\alpha\beta}))\) in a way that does not depend on the system of coordinates.

It follows from the above assumptions that the second rank tensor field \( \mathcal{G} \) must have the following form

\[ \mathcal{G} = \kappa^{-1}(\text{Ric}_g - \frac{1}{2}R_g g) + \kappa^{-1}\Lambda g, \]

where \( \kappa \) and \( \Lambda \) are constant numbers. The value \( \kappa = 8\pi G c^4 \) is obtained by considering the Newtonian limit, so that the Einstein's field equations reduce to Newtonian gravity at speeds much slower than the speed of light.

The above-mentioned assumptions are by no means necessary. By modifying them, one gets alternative theories of gravitation. Throughout this manuscript we take \( \Lambda = 0 \).

**Remark 2.5.** *In the case where \( T \equiv 0 \) (the vacuum Einstein's field equations), by taking traces, one can show that \( R_g = 0 \), so that the Einstein's equations reduce to

\[ \text{Ric}_g = 0. \]

Now comes an important question: how can we view general relativity as describing the time evolution of some quantity? In other theories of classical physics the spacetime background is given and our task is to determine the time evolution of quantities in the background from initial data. However, **in general relativity we are solving for the spacetime itself.** That is, the unknowns constitute of a smooth
4-dimensional manifold $\mathcal{M}$ and a nondegenerate metric $g$ such that $(\mathcal{M}, g)$ is a spacetime and equation (2.4) holds. Considering this, what should be the quantity or quantities to prescribe initially in general relativity in order that spacetime structure be determined? In order to answer this question, we recall that if $(\mathcal{M}, g)$ is an orientable globally hyperbolic spacetime, then $\mathcal{M}$ can be foliated by smooth spacelike Cauchy surfaces, $M_t$, parametrized by a function $t : \mathcal{M} \to \mathbb{R}$. Moreover we know that all these Cauchy surfaces are diffeomorphic to $M_0$ and $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times M_0$.

The spacetime metric, $g$, induces a Riemannian metric $\hat{h}$ on each $M_t$; if we identify the hypersurfaces $M_0$, $M_t$, we may view the effect of moving forward in time in an orientable globally hyperbolic spacetime as that of changing the Riemannian metric on a fixed 3-dimensional manifold $M$ from $\hat{h}(0)$ to $\hat{h}(t)$. Thus, the metric is the dynamical variable and we may view an orientable globally hyperbolic spacetime $(\mathcal{M}, g)$ as representing the time evolution of a Riemannian metric on a fixed 3-dimensional manifold. Note that, considering equation (2.3), the second fundamental form of the initial spacelike hypersurface can be viewed as a representative of the initial time derivative of the metric. Therefore it makes sense to say that an initial data set for the Einstein’s theory of gravity consists of a Riemannian manifold $(M, \hat{h} = g|_M)$ together with a symmetric rank 2 tensor field, $\hat{k}$, which represents the second fundamental form of the embedding of $M$ in the ambient spacetime $(\hat{k}$ plays the role of $\dot{g}|_M$). As we shall see, one cannot freely prescribe the initial data; they must satisfy the Einstein constraint equations. The best way to understand the origin of the constraints is to assume that we have a globally hyperbolic spacetime solution and to consider the induced data on a smooth spacelike hypersurface that represents an instant of time.

**Theorem 2.6.** If $(\mathcal{M}, g)$ is an oriented globally hyperbolic spacetime satisfying the Einstein's field equations (2.4), and $M$ is a smooth spacelike hypersurface with induced
Riemannian metric $\hat{h}$ and second fundamental form $\hat{k}$ then

\[ R_{\hat{h}} - |\hat{k}|^{2}_{\hat{h}} + (\text{tr}_{\hat{h}} \hat{k})^{2} = 2\kappa \hat{\rho}, \quad (2.5) \]
\[ \text{div}_{\hat{h}} \hat{k} - d(\text{tr}_{\hat{h}} \hat{k}) = \kappa \hat{J}, \quad (2.6) \]

where $R_{\hat{h}}$ is the scalar curvature of $\hat{h}$, and where $\hat{\rho}$ is a nonnegative scalar field and $\hat{J}$ is a 1-form on $M$, representing the energy and momentum densities of the matter and non-gravitational fields, respectively ($\hat{\rho} := T(n, n)$ and $\hat{J}(W) := -T(W, n)$ for all $W \in \chi(M)$). $\kappa = 8\pi \frac{G}{c^4}$ is a constant.

**Remark 2.7.** The equations (2.5) and (2.6) are called the **Einstein constraint equations**. (2.5) is called the **Hamiltonian constraint** and (2.6) is called the **momentum constraint**.

**Proof.** (Theorem 2.6) In this proof all the curvature tensors of $\mathcal{M}$ will have a subscript $g$. Also to avoid any confusion, we denote the covariant Riemann curvature tensor by “Riem”. The Levi-Civita connection on $(M, \hat{h})$ is denoted by $\nabla$.

It is enough to show that the equations hold at an arbitrary point $p \in M$. Let $\{e_0, e_1, e_2, e_3\}$ be an adapted orthonormal frame defined on a neighborhood $U$ of $p$, that is, $e_0 = n$ (see equation (2.1)) and $e_1, e_2, e_3$ are unit orthogonal vectors tangent to $M$ at all points of $U \cap M$. We begin with the right hand side of equation (2.5):

\[ 2\kappa \hat{\rho} = 2\kappa T(n, n) = 2\kappa \left[ \frac{1}{k} \left( \text{Ric}_{g}(n, n) - \frac{1}{2} R_{g} g(n, n) \right) \right]. \]

But $g(n, n) = -1$, so

\[ 2\kappa \hat{\rho} = R_{g} + 2\text{Ric}_{g}(n, n). \]
Now note that
\[ R_g = \text{tr}_g \text{Ric}_g = -\text{Ric}_g(n, n) + \sum_{i=1}^{3} \text{Ric}_g(e_i, e_i). \]

Therefore
\[
2\kappa \hat{\rho} = R_g + 2\text{Ric}_g(n, n) = \text{Ric}_g(n, n) + \sum_{i=1}^{3} \text{Ric}_g(e_i, e_i)
\]
\[
\quad = \text{Ric}_g(n, n) + \sum_{i=1}^{3} \left[ -\text{Riem}_g(n, e_i, n, e_i) + \sum_{j=1}^{3} \text{Riem}_g(e_j, e_i, e_j, e_i) \right]
\]
\[
\quad = \left[ \text{Ric}_g(n, n) - \sum_{i=1}^{3} \text{Riem}_g(n, e_i, n, e_i) \right] + \sum_{i,j=1}^{3} \text{Riem}_g(e_j, e_i, e_j, e_i)
\]
\[
\quad = \sum_{i,j=1}^{3} \text{Riem}_g(e_j, e_i, e_j, e_i)
\]
\[
\quad = \sum_{i,j=1}^{3} \text{Riem}_h(e_j, e_i, e_j, e_i) + \hat{k}(e_j, e_j)\hat{k}(e_j, e_j) - \hat{k}(e_j, e_j)k(e_j, e_j)
\]
\[
\quad = R_h + \left[ \sum_{i=1}^{3} \hat{k}(e_i, e_i) \right] \left[ \sum_{j=1}^{3} \hat{k}(e_j, e_j) \right] - |\hat{k}|^2_h
\]
\[
\quad = R_h + (\text{tr}_h \hat{k})^2 - |\hat{k}|^2_h.
\]

For the fifth equality we used the fact that \(\text{Riem}_g(n, e_0, n, e_0) = \text{Riem}_g(n, n, n, n) = 0\) and so
\[
\text{Ric}_g(n, n) - \sum_{i=1}^{3} \text{Riem}_g(n, e_i, n, e_i) = \text{Ric}_g(n, n) - \sum_{i=0}^{3} \text{Riem}_g(n, e_i, n, e_i)
\]
\[
\quad = \text{Ric}_g(n, n) - \text{Ric}_g(n, n) = 0.
\]

The sixth equality is a consequence of the Gauss equation. This finishes the proof of (2.5).
Now we prove (2.6). Let $W$ be an arbitrary vector field on $M$. We have

$$\kappa \hat{J}(W) = -\kappa T(W, n) = -\kappa T(W, n) - \frac{1}{2} R_g g(W, n) = -\text{Ric}_g(W, n).$$

Therefore

$$\kappa \hat{J}(W) = -\text{Ric}_g(W, n) = - \sum_{i=0}^{3} \text{Riem}_g(e_i, W, e_i, n)$$

$$= - \sum_{i=1}^{3} \text{Riem}_g(e_i, W, e_i, n) = \sum_{i=1}^{3} \text{Riem}_g(n, e_i, e_i, W)$$

$$= \sum_{i=1}^{3} [\hat{\nabla}_{e_i} \hat{k}(W, e_i) - (\hat{\nabla}_W \hat{k})(e_i, e_i)]$$

$$= (\text{div}_h \hat{k})(W) - \text{tr}_h (\hat{\nabla}_W \hat{k}) = (\text{div}_h \hat{k})(W) - \hat{\nabla}_W (\text{tr}_h \hat{k})$$

$$= (\text{div}_h \hat{k})(W) - d(\text{tr}_h \hat{k})(W).$$

The third equality is true because

$$\text{Riem}_g(e_0, W, e_0, n) = \text{Riem}_g(n, W, n, n) = 0.$$

The fifth equality is a consequence of the Codazzi equation. So we have proved that for all $W \in \chi(M)$

$$\kappa \hat{J}(W) = (\text{div}_h \hat{k})(W) - d(\text{tr}_h \hat{k})(W),$$

which clearly implies (2.6).

**Remark 2.8.** The proof of the above theorem shows that the constraint equations are obtained by considering the the normal-normal and normal-tangential components of the Einstein’s equations, i.e. $\mathcal{G}_0 = T_0 (0 \leq \mu \leq 4)$. If $(\mathcal{M}, g)$ is globally hyperbolic and the smooth spacelike hypersurface $M$ is a leaf of the foliation, then it can be shown that the other 6 components of the field equations (the tangential-tangential components)
will result in the following evolution equation [30]:

$$\frac{\partial \hat{k}_{ij}}{\partial t} = L_X \hat{k}_{ij} + \hat{\nabla}_i \hat{\nabla}_j N + N\{2\hat{k}_{ij} \hat{k}_{lj} - (\text{tr}_h \hat{k}) \hat{k}_{ij} - \hat{R}_{ij}\} + N_2 \{\hat{h}_{ij}(\rho - S) + 2T_{ij}\}, \quad (2.7)$$

where $S$ is the trace of the horizontal projection of $T$, that is, $S = \hat{h}^{ij} T_{\mu\nu} \hat{h}_i^\mu \hat{h}_j^\nu$. The components are in terms of an adapted local coordinates; Greek letters vary among spacetime indices $(0 \leq \mu, \nu \leq 3)$, Roman letters are used for spatial indices $(1 \leq i, j, l \leq 3)$.

It was first shown by Yvonne Choquet-Bruhat in [13] that if smooth tensor fields $\hat{h}$ and $\hat{k}$ satisfy the vacuum Einstein constraint equations on a 3-dimensional manifold $M$, then $M$ can be embedded as a hypersurface in a 4-dimensional manifold corresponding to a solution of the Einstein’s field equations, and the push forward of $\hat{h}$ and $\hat{k}$ represent the first and second fundamental forms of the embedded hypersurface. As for uniqueness, it can be shown that there exists a unique (up to diffeomorphism) maximal globally hyperbolic spacetime $(\mathcal{M}, g)$ that contains $(M, \hat{h})$ as a spacelike Cauchy hypersurface with the second fundamental form $\hat{k}$: any other spacetime that is a solution of the Einstein’s field equations and contains $(M, \hat{h})$ as a spacelike hypersurface with the second fundamental form $\hat{k}$ is diffeomorphic to a subset of $(\mathcal{M}, g)$ [15, 66]. These existence and uniqueness results can be extended to allow nonsmooth tensor fields $\hat{h}$ and $\hat{k}$ [46]. More details about the well-posedness of the initial value formulation of the Einstein’s field equations can be found in the review by Rendall [65].

Considering the above discussion, it makes sense to define an initial data set for the initial value formulation of the Einstein’s field equations as follows:

**Definition 2.9.** A triple $(M, \hat{h}, \hat{k})$ is said to be an initial data set for the initial value formulation (Cauchy formulation) of the Einstein’s field equations iff $(M, \hat{h})$
is a 3-dimensional Riemannian manifold and $\hat{k}$ is a symmetric covariant tensor of order 2 on $M$ that satisfy the Einstein constraint equations.

To sum up, in the globally hyperbolic setting, the Einstein's general theory of relativity can be viewed as an initial value problem via the following steps:

1. We choose a 3-dimensional manifold $M$.

2. We find a Riemannian metric $\hat{h} \in \tau_0^2(M)$ and a symmetric covariant $(\mathcal{O}_0)$-tensor field $\hat{k}$ that satisfy the Einstein constraint equations.

3. We freely choose a time dependent lapse function $N(x, t)$ and a time dependent shift vector field $X(x, t)$ on $M$.

4. We use the evolution equations (2.3) and (2.7) to evolve the initial data $(\hat{h}, \hat{k})$ into a one parameter family of Riemannian metrics $\hat{h}(t)$ and symmetric $(\mathcal{O}_0)$-tensor fields $\hat{k}(t)$.

5. We construct the nondegenerate metric $g$ on the spacetime manifold $\mathcal{M} = \mathbb{R} \times M$ via (2.2).

It is important to note that the evolution according to (2.3) and (2.7) preserves the constraints, that is, if $(\hat{h}, \hat{k})$ satisfies the constraints, then $(\hat{h}(t), \hat{k}(t))$ will satisfy the constraints for all $t$. The proof is based on the Bianchi identities and can be found in [31].

**Remark 2.10.** In this section we obtained the Einstein constraint equations by applying the Gauss and Codazzi equations to some components of the Einstein's field equations. Alternatively, the constraint equations appear quite naturally in the Hamiltonian formulation of general relativity as secondary first class constraints. Moreover, if we consider $Q = \text{Riem}(M)$ (the space of Riemannian metrics on $M$) as the configuration
space, the group \( G = \text{Diff}(M) \) (the diffeomorphism group of \( M \)) acts on \( Q \). This action can be lifted to an action on \( T^*Q \) and one can consider the corresponding momentum map. Momentum constraints coincide with the zero momentum map for this group action. More details on this point of view can be found in [30] and [9].

2.3 The Conformal Method

The first step in constructing a spacetime solution of the Einstein’s field equations via the initial value problem is to obtain initial data sets which satisfy the Einstein constraint equations. Given a 3-dimensional smooth manifold \( M \), a scalar function \( \hat{\rho} \) and a vector valued function (or 1 form) \( \hat{J} \), we need to find a Riemannian metric \( \hat{h} \) and a symmetric rank 2 tensor field \( \hat{k} \), such that the triple \( (\hat{h}, \hat{k}) \) forms an initial data set for the Einstein constraint equations, i.e., such that \( (\hat{h}, \hat{k}) \) satisfies the constraint equations. Using any local frame we may write the constraint equations as follows:

\[
R_{\hat{h}} + (\hat{h}^{ab}\hat{k}_{ab})^2 - \hat{k}_{ab}\hat{k}^{ab} = 2\kappa \hat{\rho}, \\
\hat{\nabla}^b (\hat{h}^{ac}\hat{k}_{ac}) - \hat{\nabla}^a\hat{k}^{ab} = -\kappa \hat{J}^b, \quad 1 \leq b \leq 3.
\]

The constraint equations constitute an underdetermined system of equations (the number of unknowns is 12, whereas the number of equations is 4). These equations admit a wide variety of solutions and therefore it is desirable to find intrinsic parameters describing the set of solutions of the constraint equations. To date the most widely used approach to the parametrization and construction of solutions to the Einstein constraint equations has been the conformal method. The main idea of the conformal method is to divide the initial data on \( M \) into two sets: the Free (Conformal) Data, and the Determined Data, such that given a choice of free data, the constraint equations
become a determined system to be solved for the determined data [6]. There are several ways to do this; here we focus on the “semi-decoupling split”, and examine briefly how the method works. More details can be found in Appendix C.

- **Step 1**: The original unknowns, \( \hat{h} \) and \( \hat{k} \), each has six distinct components, therefore we have twelve unknowns. We can decompose \( \hat{k} \) into the trace-free and the pure trace parts:

\[
\hat{k}^{ab} = s^{ab} + \frac{1}{3} (\text{tr}_{\hat{h}} \hat{k}) \hat{h}^{ab}.
\]

Clearly \( \text{tr}_{\hat{h}} \hat{s} = 0 \).

- **Step 2: Conformal rescaling.** Let

\[
\hat{h}_{ab} = \phi^r h_{ab}, \quad \hat{s}^{ab} = \phi^s s^{ab}, \quad \text{tr}_{\hat{h}} \hat{k} = \phi^t \tau,
\]

where \( r, s, \) and \( t \) are fixed but arbitrary integers. In Appendix C we will show that the above equations imply that \( \hat{s}_{ab} = \phi^{2r+s} s_{ab} \). Note that if \( t = 0 \) then \( \tau \) is the mean (extrinsic) curvature. We denote the Levi-Civita connection for \( h \) by \( \nabla \). We will assume \( h \) and \( \tau \) are given (i.e we consider them as free data) so we are left with 7 unknowns (components of \( s_{ab} \) and \( \phi \)).

- **Step 3: York decomposition.** In what follows \( \nabla, \mathcal{L} \) (the conformal Killing operator) and \( \text{div} \) are all taken with respect to the metric \( h \). For closed manifolds and asymptotically flat manifolds, one can show that (see e.g. [39, 55]) under suitable smoothness assumptions, if \( \psi \) is a symmetric traceless contravariant tensor of order 2, then there exists \( W \in \chi(M) \), uniquely determined up to conformal Killing fields, such that \( \Delta_L W = \text{div}\psi \) where \( \Delta_L = \text{div}\mathcal{L} \) is the vector Laplacian. Therefore,
there exists \( W \in \chi(M) \) such that

\[
\Delta L W = \text{divs} \left( \nabla_c (\mathcal{L} W)^{ac} = \nabla_c s^{ac} \right).
\]

Now define \( \sigma^{ab} := s^{ab} - (\mathcal{L} W)^{ab} \). Clearly, \( \text{div} \sigma = 0 \). It is easy to check that \( \sigma \) is trace-free as well. So in fact \( \sigma \) is a transverse-traceless (TT) tensor. The decomposition \( s^{ab} = \sigma^{ab} + (\mathcal{L} W)^{ab} \) is called York splitting and it is unique up to addition of a conformal Killing field to \( W \).

**Step 4:** We assume \( \sigma^{ab} \) is given, i.e, we will consider it as part of the free data; now we are left with four unknowns (components of the vector field \( W^a \) and the scalar function \( \phi \)).

Therefore, the set of free (conformal) data consists of a background Riemannian metric \( h \), a transverse-traceless symmetric tensor \( \sigma \), and a function \( \tau \). The set of determined data consists of a positive function \( \phi \) and a vector field \( W \). The transformed system consists of the *Lichnerowicz-Choquet-Bruhat-York (LCB-Y) equations*. For the semi-decoupling split we set \( r = 4 \), \( s = -10 \), \( t = 0 \). When energy and momentum densities of matter and non-gravitational fields are present, one also takes \( \rho = \phi^8 \hat{\rho} \) and \( J^b = \phi^{10} \hat{J}^b \).

**The conformal formulation of the Einstein constraint equations.** Applying the conformal method by following **Steps 1–4** above, one produces a coupled nonlinear elliptic system for the unknown conformal factor \( \phi \in C^\infty(M) \) and \( W \in \chi(M) \):

\[
-8\Delta \phi + R \phi + \frac{2}{3} \tau^2 \phi^5 - [\sigma_{ab} + (\mathcal{L} W)_{ab}] [\sigma^{ab} + (\mathcal{L} W)^{ab}] \phi^{-7} = 2\kappa \rho \phi^{-3}, \tag{2.8}
\]

\[
-\nabla_a (\mathcal{L} W)^{ab} + \frac{2}{3} \phi^6 \nabla^b \tau = -\kappa J^b, \tag{2.9}
\]
where the first equation (2.8) is referred to as the Lichnerowicz equation or the (conformal formulation of the) Hamiltonian constraint, and the second equation (2.9) is referred to as the (conformal formulation of the) momentum constraint. In the vacuum case, the right-hand sides of both equations vanish. If \((\phi, W)\) solves the LCBY equations, then the initial data \((\hat{h}_{ab}, \hat{k}_{ab})\) constructed by the formulas

\[
\hat{h}_{ab} = \phi^4 h_{ab},
\]

\[
\hat{k}_{ab} = \hat{s}_{ab} + \frac{1}{3} (\text{tr}_h \hat{k}) \hat{h}_{ab} = \phi^{-2} (\sigma_{ab} + \mathcal{L} W_{ab}) + \frac{1}{3} \phi^4 h_{ab} \tau,
\]

satisfies the Einstein constraint equations.

**Remark 2.11.** In the above discussion we explained the machinery of the conformal method, but we did not discuss the original motivation for such procedure. The best way to understand the origins of the semi-decoupling version of the conformal method is by considering the vacuum case with constant mean curvature \(\tau\). In the very special case where \(\tau = 0\), the constraint equations reduce to

\[
R_{\hat{h}} = |\hat{k}|_{\hat{h}}^2,
\]

\[
\text{div}_{\hat{h}} \hat{k} = 0.
\]

The goal is to solve the above equations for \((\hat{h}, \hat{k})\). Note that if \((\hat{h}, \hat{k})\) is a solution, then by assumption \(\text{tr}_{\hat{h}} \hat{k} = \tau = 0\) and by the second equation \(\text{div}_{\hat{h}} \hat{k} = 0\). Therefore \(\hat{k}\) is a transverse-traceless (TT) tensor. Now the key observation which was made by Lichnerowicz is the following (see Section 1.2.5)

\[
\sigma_{ab} \text{ is TT with respect to } h \iff \phi^{-2} \sigma_{ab} \text{ is TT with respect to } \phi^4 h
\]

The above observation leads naturally to the following recipe for finding a vacuum
solution \((\hat{h}, \hat{k})\) in the case \(\tau = 0\):

- Fix \(h_{ab}\) and \(\sigma_{ab}\) such that \(\sigma_{ab}\) is TT with respect to \(h_{ab}\).

- Seek solutions of the form \(\hat{h}_{ab} = \phi^4 h_{ab}\) and \(\hat{k}_{ab} = \phi^{-2} \sigma_{ab}\).

The big advantage is that now the momentum constraint \(\text{div}_h \hat{k} = 0\) is automatically satisfied and so we just need to use the Hamiltonian constraint to determine \(\phi\). This shows how TT tensors naturally appear in the study of constraint equations.

A similar argument can be made for the more general case where \(\tau = \tau_0\) (\(\tau_0\) is a given nonzero constant). In this case the vacuum constraint equations will take the following form

\[
R_h = |\hat{k}|^2_h - \tau_0^2,
\]
\[
\text{div}_h \hat{k} = 0.
\]

This time \(\hat{k}\) is divergence-free but it is not trace-free. However, if we let \(\hat{\sigma}\) to be the trace-free part of \(\hat{k}\), that is,

\[
\hat{k}_{ab} = \hat{\sigma}_{ab} + \frac{\tau_0}{3} \hat{h}_{ab},
\]

then \(\hat{\sigma}_{ab}\) is TT with respect to \(\hat{h}_{ab}\). Therefore by the key observation that was mentioned above, we have the following recipe for finding a solution:

- Fix \(h_{ab}\) and \(\sigma_{ab}\) such that \(\sigma_{ab}\) is TT with respect to \(h_{ab}\).

- Seek solutions of the form \(\hat{h}_{ab} = \phi^4 h_{ab}\) and \(\hat{k}_{ab} = \phi^{-2} \sigma_{ab} + \frac{\tau_0}{3} \phi^4 h_{ab}\).

Again the momentum constraint will be automatically satisfied and the Hamiltonian constraint will be used to determine \(\phi\).

The key role of TT tensors in the above special cases has been a principal motivation for the development of the conformal method in the form it was described in this
2.4 An Overview of the Known Results

Ideally, one wishes to show that for any given set of free data \((h, \sigma, \tau)\) on a smooth 3-dimensional manifold \(M\), there exists a unique solution of the LCBY equations, or if this is not the case, to determine conditions on sets of free data for which these equations have unique solutions and then, for those data sets which do not satisfy these conditions, to give a complete description of the nonexistence or the multiplicity of solutions.

To describe what is known about solving the LCBY equations, it is useful to categorize the given data using the following criteria:

- **Manifold**: Compact, Compact with boundary, Asymptotically flat, Asymptotically hyperbolic, Asymptotically cylindrical

- **Metric conformal classes**: Yamabe positive, Yamabe zero, Yamabe negative

- **Mean curvature \((\tau)\)**: Constant (CMC), Near constant (size of the partial derivatives of \(\tau\) are sufficiently small), Non-constant

2.4.1 CMC Data on Compact Manifolds (Without Boundary)

The CMC (constant mean curvature \(\tau\)) assumption produces a decoupling of the constraint equations. In the vacuum case, the momentum constraint can be solved separately and the resulting vector fields \(W\) necessarily satisfy \(\mathcal{L} W = 0\). Therefore one is left only to study the single Lichnerowicz equation

\[
-8\Delta \phi + R\phi - |\sigma|^2 \phi^{-7} + \frac{2}{3} \tau^2 \phi^5 = 0.
\]
The solvability of (2.10) on compact manifolds is completely understood [39]; the analysis of the Lichnerowicz equation in the CMC vacuum case is based on four fundamental facts:

- The Lichnerowicz equation in the CMC vacuum case is conformally covariant in the sense that if \( \phi \) is a solution for a set of free data \((h_{ab}, \sigma_{ab}, \tau)\), then for any smooth positive function \( \eta \), the function \( \eta^{-1}\phi \) is a solution to the Lichnerowicz equation for the free data \((\eta^4 h_{ab}, \eta^{-2} \sigma_{ab}, \tau)\).

- According to the Yamabe theorem, every smooth metric on a compact manifold is conformally equivalent to a metric of constant scalar curvature; moreover, the sign of that constant scalar curvature is unique. We will discuss the Yamabe theorem in more details in Appendix E. As a consequence of the Yamabe theorem, the set of Riemannian metrics on a given compact manifold can be partitioned into three Yamabe classes, depending on the sign of the corresponding constant scalar curvatures.

- As a direct consequence of the maximum principle on compact manifolds (without boundary), the equation \( \Delta \phi = G(x, \phi) \), with \( G(x, \phi) \) either nonvanishing and nonpositive or nonvanishing and nonnegative, has no positive solution.

- The sub- and supersolution theorem: roughly speaking, if there exist a pair of positive functions \( \phi_- \leq \phi_+ \) such that

\[
-8\Delta \phi_+ + R\phi_+ - |\sigma|^2 \phi_+^7 + \frac{2}{3} \tau^2 \phi_+^5 \geq 0,
\]

\[
-8\Delta \phi_- + R\phi_- - |\sigma|^2 \phi_-^7 + \frac{2}{3} \tau^2 \phi_-^5 \leq 0,
\]

then there exists a solution \( \phi \) of the Lichnerowicz equation with \( \phi_- \leq \phi \leq \phi_+ \).
Conditions for solvability involve the Yamabe class of the metric \( h \), whether or not the constant \( \tau \) is zero, and whether or not \( |\sigma| \) is zero. One can show that a solution exists except for the following cases:

- \( h \) is in the negative Yamabe class and \( \tau = 0 \).
- \( h \) is in the zero Yamabe class and exactly one of \( \tau \) or \( |\sigma| \) is zero.
- \( h \) is in the positive Yamabe class and \( |\sigma| \equiv 0 \).

The nonexistence in all the above cases follows from the maximum principle. In other cases where at least one solution exists, solution is unique except for the case where \( h \) is in the Yamabe zero class and \( \tau = 0, |\sigma| \equiv 0 \); for this data there is a one-parameter family of solutions. The main technique employed in the existence proofs is the sub- and supersolution theorem.

The above ideas for studying the CMC solutions of the vacuum constraints can be extended to the non-vacuum constraints. In fact the analysis of the conformal method for many forms of non-gravitational fields coupled to Einstein's equations is essentially indistinguishable from its analysis for the vacuum Einstein theory [41].

### 2.4.2 CMC Data on Asymptotically Flat Manifolds

Roughly speaking, an asymptotically flat manifold is a Riemannian manifold which, outside a compact set, is diffeomorphic to the complement of a ball in Euclidean space. Furthermore, in the asymptotic coordinates induced by the diffeomorphism, the metric and its derivatives decay fast enough to a flat metric. The precise definition, which is slightly more general, will be given in Chapter 3. The decay conditions which are normally used to define a set of data to be asymptotically flat require that the mean curvature (\( \tau \)), if constant, be zero [55]. In this case, one can
show that the Lichnerowicz equation admits a solution for asymptotically flat free data if and only if the metric is conformally equivalent to a metric whose scalar curvature vanishes everywhere [14, 55]. Such metrics on asymptotically flat manifolds are called \textit{Yamabe positive}. See Appendix F for more details.

Results on the solvability of the Lichnerowicz equation for asymptotically hyperbolic data and asymptotically cylindrical data can be found in [3, 20].

\subsection{2.4.3 Non-CMC Data}

\textbf{Near-CMC Data}

The first partial result for the case that CMC assumption is not used, referred to as the non-CMC (nonconstant mean curvature) case, leaving the two constraints coupled, was established by Isenberg and Moncrief in 1996 [43]. Their result required the near-CMC (near-constant mean curvature) assumption, in the sense that $\frac{dr}{r}$ is sufficiently small in some appropriate sense. The key idea is to use an iterative scheme that replaces the LCBY equations with a sequence of PDE systems which are easier to handle. In the vacuum case, we consider the following PDE systems:

\begin{align}
-8\Delta \phi(n) + R\phi(n) + \frac{2}{3} r^2 \phi^5(n) - [\sigma_{ab} + (\mathcal{L} W(n))_{ab}] [\sigma^{ab} + (\mathcal{L} W(n))^{ab}] \phi^{-7}(n) &= 0, \quad (2.11) \\
-\nabla_a (\mathcal{L} W(n))^{ab} + \frac{2}{3} \phi^{6(n-1)} \nabla^b r &= 0, \quad (2.12)
\end{align}

The scheme starts by choosing a suitable value for $\phi(0)$. Then using Fredholm theory and sub- and supersolution theorem one shows that a sequence $(\phi(n), W(n))$ of solutions exist. Next, using the near-CMC assumption and elliptic estimates, one proves that there are uniform upper and lower bounds for the sequence $(\phi(n), W(n))$. Finally, using the uniform bounds and the near-CMC condition, one applies a contraction mapping argument to show that the sequence $(\phi(n), W(n))$ converges to a limit $(\phi, W)$.
which solves the LCBY equations.

The known results for the solvability of LCBY equations for near-CMC data on compact manifolds can be summarized as follows [43, 2, 44, 23]:

- There exists a unique solution to the LCBY equations if
  - $h$ is in the positive Yamabe class and $\sigma$ is not identically zero.
  - $h$ is in the zero Yamabe class and $\sigma$ is not identically zero.
  - $h$ is in the negative Yamabe class and $\tau$ is nowhere zero.

- A solution does not exist if $h$ is in the positive Yamabe class or the zero Yamabe class, $\sigma$ is identically zero, and $\tau$ is nowhere 0.

The proof of nonexistence is based on elliptic estimates and maximum principle.

Similar results for the asymptotically flat case and the asymptotically hyperbolic case are discussed in [16, 45].

**Far-From-CMC Data**

The analysis of the LCBY equations is much more difficult without the CMC or near-CMC conditions and much less is known in this case. The first non-CMC result without near-CMC condition (sometimes we use the label *far-from-CMC* for this case) appeared in [37, 38] for compact manifolds (without boundary), which demonstrated that for a Yamabe positive metric admitting no conformal Killing fields, and for any desired mean curvature function, if the transverse traceless tensor $\sigma \neq 0$ is sufficiently small, and if there is a small but nonvanishing amount of matter present, then there exists at least one solution of the LCBY equations. This result was extended shortly afterward to handle the vacuum case [56]. These results are based on combining sub- and supersolution technique, *a priori* estimates, and a topological fixed-point
argument based on compactness arguments rather than contraction mapping arguments. The analysis techniques first developed in [37, 38] have been intensively studied and extended to a number of other cases including the important case of compact manifolds with boundary [25, 36].

Topological fixed-point arguments also play a role in the analysis developed in [23] which in particular shows that in the compact case if the background metric has no conformal Killing fields, $\tau$ has constant sign and no zeros, and if $\sigma \neq 0$, then LCBY equations have a solution if the equation

$$\Delta W^a = |\mathcal{L} W| \frac{\nabla^a \tau}{\tau}$$

(which is called the limit equation) does not admit a solution.

A more comprehensive report on the known results for the Einstein constraint equations (due to the conformal method or other techniques) can be found in the survey of Bartnik and Isenberg [6] and the recent review by Isenberg [40]. In Chapter 5 we will study the question of existence of rough far-from-CMC solutions to the LCBY equations on asymptotically flat manifolds.

Chapter 2, in part, has been submitted for publication. The material may appear as A. Behzadan and M. Holst, *Rough Solutions of the Einstein Constraint Equations on Asymptotically Flat Manifolds Without Near-CMC Conditions.*
Chapter 3

Weighted Sobolev Spaces

Unweighted Sobolev spaces work well in studying partial differential operators on bounded domains of \( \mathbb{R}^n \) or compact manifolds, but they do not work quite as well when working with the entire \( \mathbb{R}^n \) or open manifolds. For example, if \( \Omega \subset \mathbb{R}^n \) is a bounded open set with smooth boundary, then \( \Delta : \tilde{H}^2(\Omega) \to L^2(\Omega) \) is an isomorphism; however, a simple scaling argument shows that Laplacian as an operator from \( H^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \) does not have a closed range and is not an isomorphism onto its image \([52]\). As we will discuss, the good mapping properties of (elliptic) partial differential operators on bounded domains can be regained for unbounded regions if one uses weighted Sobolev spaces instead of unweighted Sobolev spaces. Historically, the idea of using weighted Sobolev spaces (with weights similar to the ones that will be introduced in this chapter) to study (elliptic) partial differential operators on \( \mathbb{R}^n \) was introduced by Nirenberg and Walker \([63]\), and later developed by McOwen \([60]\).

**Notation.** Throughout the rest of this manuscript we use the notation \( A \preceq B \) to mean \( A \leq cB \) where \( c \) is a positive constant that does not depend on the non-fixed parameters appearing in \( A \) and \( B \). We use the notation \( X \hookrightarrow Y \) to mean \( X \subseteq Y \) and the inclusion map is continuous.
3.1 Basic Definitions

In this short section we briefly review some basic definitions related to the Sobolev space theory with emphasis on fractional order Sobolev spaces.

3.1.1 Unweighted Sobolev Spaces

Definition 3.1. Let $k \in \mathbb{N}_0$, $1 < p < \infty$. The Sobolev space $W^{k,p}(\mathbb{R}^n)$ is defined as follows:

$$\quad W^{k,p}(\mathbb{R}^n) = \{ u \in L^p(\mathbb{R}^n) : \|u\|_{W^{k,p}(\mathbb{R}^n)} := \sum_{|\nu| \leq k} \|\partial^\nu u\|_p < \infty \}$$

For $k \in \mathbb{N}$, the Sobolev space $W^{-k,p}(\mathbb{R}^n)$ is defined as $(W^{k,p}(\mathbb{R}^n))^* \quad \left( \frac{1}{p} + \frac{1}{p'} = 1 \right)$.

Remark 3.2.

- For the real-valued function $u(x_1, \ldots, x_n)$ and $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n$,

$$\quad |\nu| := \nu_1 + \cdots + \nu_n, \quad \partial^\nu u := \frac{\partial^{|
u|} u}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}}, \quad \|u\|_p := \left( \int_{\mathbb{R}^n} |u|^p \, dx \right)^{\frac{1}{p}}.$$

- The Sobolev norm is defined so that $\partial^\alpha : W^{k,p}(\mathbb{R}^n) \to W^{k-|\alpha|,p}(\mathbb{R}^n)$ becomes a continuous operator. It can be shown that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$. In fact, $W^{k,p}(\mathbb{R}^n)$ is the completion of the space of smooth function with respect to $\|\cdot\|_{W^{k,p}(\mathbb{R}^n)}$.

- Clearly, if $k_1 \geq k_0$, then $W^{k_1,p}(\mathbb{R}^n) \subseteq W^{k_0,p}(\mathbb{R}^n)$

There are nonequivalent ways to generalize the above definition to allow non-integer exponents. We can define Sobolev spaces with noninteger exponents as

1. Slobodeckij spaces, or,
2. Bessel potential spaces.
There are three equivalent methods to define each of the above spaces:

1. Original definition

2. Definition based on interpolation theory

3. Definition based on Littlewood-Paley theory

1-Original Definitions

**Definition 3.3.** Let \( s \in \mathbb{R} \) and \( p \in [1, \infty] \). The Sobolev-Slobodeckij space \( W^{s,p}(\mathbb{R}^n) \) is defined as follows:

- If \( s = k \in \mathbb{N}_0, p \in [1, \infty] \),
  \[
  W^{k,p}(\mathbb{R}^n) = \{ u \in L^p(\mathbb{R}^n) : ||u||_{W^{k,p}(\mathbb{R}^n)} := \sum_{|\nu| \leq k} ||\partial^\nu u||_p < \infty \}
  \]

- If \( s = \theta \in (0, 1), p \in [1, \infty] \),
  \[
  W^{\theta,p}(\mathbb{R}^n) = \{ u \in L^p(\mathbb{R}^n) : |u|_{W^{\theta,p}(\mathbb{R}^n)} := \left( \int \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\theta p}} \, dx \, dy \right)^{\frac{1}{p}} < \infty \}
  \]

- If \( s = \theta \in (0, 1), p = \infty \),
  \[
  W^{\theta,\infty}(\mathbb{R}^n) = \{ u \in L^\infty(\mathbb{R}^n) : |u|_{W^{\theta,\infty}(\mathbb{R}^n)} := \text{ess sup}_{x,y \in \mathbb{R}^n, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\theta} < \infty \}
  \]

- If \( s = k + \theta, k \in \mathbb{N}_0, \theta \in (0, 1), p \in [1, \infty] \),
  \[
  W^{s,p}(\mathbb{R}^n) = \{ u \in W^{k,p}(\mathbb{R}^n) : ||u||_{W^{s,p}(\mathbb{R}^n)} := ||u||_{W^{k,p}(\mathbb{R}^n)} + \sum_{|\nu| \leq k} ||\partial^\nu u||_{W^{\theta,p}(\mathbb{R}^n)} < \infty \}
  \]
• If $s < 0$ and $p \in (1, \infty)$,
\[
W^{s,p}(\mathbb{R}^n) = (W^{-s,p'}(\mathbb{R}^n))^* = \left( \frac{1}{p} + \frac{1}{p'} = 1 \right).
\]

Alternatively, for $s \in \mathbb{R}$ and $1 < p < \infty$ one can define Sobolev spaces as Bessel potential spaces $H^{s,p}(\mathbb{R}^n)$:
\[
H^{s,p}(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) : ||u||_{W^{s,p}(\mathbb{R}^n)} := ||\mathcal{F}^{-1}(\langle \xi \rangle^s \mathcal{F} u) ||_{L^p} < \infty \},
\]
where $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$. It is a well known fact that $H^{s,p}(\mathbb{R}^n) = (H^{-s,p'}(\mathbb{R}^n))^*$ and for $k \in \mathbb{Z}$ the two definitions agree [32, 72, 68]. Also for $s \in \mathbb{R}$ and $p = 2$ the two definitions agree[32, 68]. $H^{s,2}(\mathbb{R}^n)$ is often denoted by $H^s(\mathbb{R}^n)$.

2-Definitions Based on Interpolation Theory

A short introduction to interpolation theory in Banach spaces is given in Section 3.4.1. Suppose $s \in \mathbb{R} \setminus \mathbb{Z}$, $1 < p < \infty$, and let $\theta := s - [s]$.

• $W^{s,p}(\mathbb{R}^n) = (W^{[s],p}(\mathbb{R}^n), W^{[s]+1,p}(\mathbb{R}^n))_{\theta,p}$.

• $H^{s,p}(\mathbb{R}^n) = [H^{[s],p}(\mathbb{R}^n), H^{[s]+1,p}(\mathbb{R}^n)]_{\theta}$.

3-Definitions Based on Littlewood-Paley Theory

Consider an open cover of $\mathbb{R}^n$ that consists of the following sets:
\[
B_2, \quad B_4 \setminus B_1, \quad B_8 \setminus B_2, \ldots, B_{2^{j+1}} \setminus B_{2^j-1},
\]
where $B_r$ is the open ball of radius $r$ centered at the origin. Consider the following
partition of unity subordinate to the above cover of \( \mathbb{R}^n \):

\[
\varphi_0 = 1 \quad \text{on} \quad B_1, \quad \text{supp} \varphi_0 \subseteq B_2,
\]

\[
\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi) \quad (\text{supp} \varphi \subseteq B_2, \quad \varphi = 0 \text{ on } B_2),
\]

\[
\forall \ j \geq 1 \quad \varphi_j(\xi) = \varphi(2^{-j} \xi).
\]

One can easily check that \( \sum_{j=0}^{\infty} \varphi_j(\xi) = 1 \).

**Definition 3.4.**

- For \( s \in \mathbb{R}, 1 \leq p < \infty, \) and \( 1 \leq q < \infty \) (or \( p = q = \infty \)) we define the Triebel-Lizorkin space \( F_{p,q}^s(\mathbb{R}^n) \) as follows

\[
F_{p,q}^s(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) : \| u \|_{F_{p,q}^s(\mathbb{R}^n)} = \bigg( \| 2^{sj} \mathcal{F}^{-1}(\varphi_j \mathcal{F} u) \|_{L^p(\mathbb{R}^n)} \bigg)^{1/p} < \infty \}
\]

- For \( s \in \mathbb{R}, 1 \leq p < \infty, \) and \( 1 \leq q < \infty \) we define the Besov space \( B_{p,q}^s(\mathbb{R}^n) \) as follows

\[
B_{p,q}^s(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) : \| u \|_{B_{p,q}^s(\mathbb{R}^n)} = \bigg( \| 2^{sj} \mathcal{F}^{-1}(\varphi_j \mathcal{F} u) \|_{L^p(\mathbb{R}^n)} \bigg)^{1/q} < \infty \}
\]

We have the following relations [72, 68, 12]

- \( L^p = F_{p,2}^0, \ 1 < p < \infty. \)
- \( B_{p,p}^s = F_{p,p}^s, \ s \in \mathbb{R}, 1 < p < \infty. \)
- \( H^{s,p} = F_{p,2}^s, \ s \in \mathbb{R}, 1 < p < \infty. \)
- \( W^{k,p} = H^{k,p} = F_{p,2}^k, \ k \in \mathbb{Z}, 1 < p < \infty. \)
- \( W^{s,p} = B_{p,p}^s = F_{p,p}^s, \ s \in \mathbb{R} \setminus \mathbb{Z}, 1 < p < \infty. \)
- If \( k \in \mathbb{N}, \) then \( B_{p,p}^k \hookrightarrow W^{k,p} \) for \( 1 \leq p \leq 2 \) and \( W^{k,p} \hookrightarrow B_{p,p}^k \) for \( p \geq 2. \)
Definition 3.5. Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ with Lipschitz continuous boundary. Suppose $s \geq 0$ and $1 \leq p < \infty$. $W^{s,p}(\Omega)$ is defined as the restriction of $W^{s,p}(\mathbb{R}^n)$ to $\Omega$ and is equipped with the following norm:

$$
||u||_{W^{s,p}(\Omega)} = \inf_{v \in W^{s,p}(\mathbb{R}^n), v|_{\Omega} = u} ||v||_{W^{s,p}(\mathbb{R}^n)}.
$$

$W^{s,p}_0(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{s,p}(\Omega)$. $W^{s,2}_0(\Omega)$ is often denoted by $\dot{H}^s(\Omega)$.

Remark 3.6.

- One may define $H^{s,p}(\Omega)$, $B^{s,q}_p(\Omega)$, and $F^{s}_p(\Omega)$ in a similar fashion.

- It can be shown that for $k \in \mathbb{N}_0$ the above definition of $W^{k,p}(\Omega)$ agrees with the following intrinsic definition [72]

$$
W^{k,p}(\Omega) = \{u \in L^p(\Omega) : ||u||_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} ||\partial^\alpha u||_{L^p(\Omega)} < \infty\}.
$$

For $s < 0$ and $1 < p < \infty$ we define $W^{s,p}(\Omega) := (W^{-s,p}_0(\Omega))^*$. When there is no ambiguity about the domain we may write

- $W^{s,p}$ instead of $W^{s,p}(\Omega)$,

- $\| \cdot \|_{W^{s,p}}$ or $\| \cdot \|_{s,p}$ instead of $\| \cdot \|_{W^{s,p}(\Omega)}$.

3.1.2 Weighted Sobolev Spaces, Asymptotically Flat Manifolds

Consider the partition of unity $\{\phi_j\}$ introduced in the previous section. For $s \in \mathbb{R}$, $p \in (1, \infty)$, the weighted Sobolev space $W^{s,p}_\delta(\mathbb{R}^n)$ is defined as follows:

$$
W^{s,p}_\delta(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : \|u\|_{W^{s,p}_\delta(\mathbb{R}^n)}^p = \sum_{j=0}^\infty 2^{-p\delta j} \|\partial^\alpha_j (\phi_j u)\|_{W^{s,p}(\mathbb{R}^n)}^p < \infty\}.
$$
Alternatively, we could define weighted Sobolev spaces as weighted Bessel potential spaces. We denote the corresponding weighted spaces by $H^s_{\delta}$. For $k \in \mathbb{N}_0$, the norm on $W^k_{\delta}(\mathbb{R}^n)$ is equivalent to the following norm \([52, 70]\): (since the norms are equivalent we use the same notation for both norms)

$$\|u\|_{W^k_{\delta}(\mathbb{R}^n)} = \sum_{|\beta| \leq k} \|\langle x \rangle^{-\frac{n}{p} + |\beta|}\partial^\beta u\|_{L^p(\mathbb{R}^n)}.$$  

When $s = 0$, we denote $W^s_{\delta}(\mathbb{R}^n)$ by $L^p_{\delta}(\mathbb{R}^n)$. In particular we have

$$\|u\|_{L^p_{\delta}(\mathbb{R}^n)} = \|\langle x \rangle^{-\frac{n}{p}} u\|_{L^p(\mathbb{R}^n)}.$$  

**Remark 3.7.** We take a moment to make the following three observations.

- Considering the above formula for the norm, it is obvious that if $\delta \leq -\frac{n}{p}$ then $\langle x \rangle^{-\delta - \frac{n}{p} + |\beta|} \geq 1$ and therefore $\|u\|_{W^k_{\delta}(\mathbb{R}^n)} \leq \|u\|_{W^k_{\delta}(\mathbb{R}^n)}$ and $W^k_{\delta}(\mathbb{R}^n) \hookrightarrow W^k_{\delta}(\mathbb{R}^n)$.

- Note that if $u \in L^p_{\delta}(\mathbb{R}^n)$ and $v \in L^\infty(\mathbb{R}^n)$, then

$$\|uv\|_{L^p_{\delta}(\mathbb{R}^n)} = \|\langle x \rangle^{-\frac{n}{p}} v u\|_{L^p(\mathbb{R}^n)} \\
\leq \|v\|_{L^\infty} \|\langle x \rangle^{-\frac{n}{p}} u\|_{L^p(\mathbb{R}^n)} \\
= \|v\|_{L^\infty} \|u\|_{L^p_{\delta}(\mathbb{R}^n)}.$$  

- It is easy to show that for $p \in (1, \infty)$, $\langle x \rangle^{\delta'} \in L^p_{\delta}(\mathbb{R}^n)$ for every $\delta' < \delta$, but $\langle x \rangle^{\delta} \notin L^p_{\delta}(\mathbb{R}^n)$ \([52]\).

**Remark 3.8.** Note that our definition of $W^s_{\delta}(\mathbb{R}^n)$ for $s < 0$ is not based on duality. Nevertheless, as it is stated in Theorem 3.16, $(W^s_{\delta}(\mathbb{R}^n))^* \text{ can be identified with } W^{s'}_{-n-\delta}(\mathbb{R}^n)$. This identification can be done by defining a suitable bilinear form $W^{s'}_{-n-\delta}(\mathbb{R}^n) \times W^s_{\delta}(\mathbb{R}^n) \rightarrow \mathbb{R}$ \([71]\).
Remark 3.9. In the literature, the growth parameter $\delta$ has been incorporated in the definition of weighted spaces in more than one way. Our convention for the growth parameter agrees with Bartnik’s convention [5] and Maxwell’s convention [52, 55, 54]. The following items describe how our definition corresponds with the other related definitions of weighted spaces in the literature:

- For $s \in \mathbb{Z}$ our spaces $W_{\delta}^{s,p}(\mathbb{R}^n)$ correspond with the spaces $h_{p,s,p-\delta-n}^s(\mathbb{R}^n)$ in [70, 71] and $H_{\delta}^{s,p}(\mathbb{R}^n)$ in [52].

- For $s \not\in \mathbb{Z}$ our spaces $W_{\delta}^{s,p}(\mathbb{R}^n)$ correspond with the spaces $b_{p,p,s-p-\delta-n}^s(\mathbb{R}^n)$ in [70, 71] and $W_{p,s-\delta-n}^p(\mathbb{R}^n)$ in [11].

- For $s \in \mathbb{R}$ and $p = 2$ our spaces $W_{\delta}^{s,p}(\mathbb{R}^n)$ correspond with the spaces $H_{\delta}^s(\mathbb{R}^n)$ in [52, 55].

The space $W_{\delta}^{s,p}_{loc}(\mathbb{R}^n)$ is defined as the set of distributions $u \in D'(\mathbb{R}^n)$ for which $\chi u \in W^{s,p}(\mathbb{R}^n)$ for all $\chi \in C_\infty(\mathbb{R}^n)$. $W_{\delta}^{s,p}_{loc}(\mathbb{R}^n)$ is a Frechet space with the topology defined by the seminorms $p_\chi(u) = \|\chi u\|_{W^{s,p}(\mathbb{R}^n)}$ for $\chi \in C_\infty(\mathbb{R}^n)$ [34]. Also $C_\infty(\mathbb{R}^n)$ is dense in $W_{\delta}^{s,p}_{loc}(\mathbb{R}^n)$.

Definition 3.10. Let $\Omega$ be an open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary and $s \geq 0$. $W_{\delta}^{s,p}(\Omega)$ is defined as the restriction of $W_{\delta}^{s,p}(\mathbb{R}^n)$ to $\Omega$ and is equipped with the following norm:

$$\|u\|_{W_{\delta}^{s,p}(\Omega)} = \inf_{v \in W_{\delta}^{s,p}(\mathbb{R}^n), v|_{\Omega} = u} \|v\|_{W_{\delta}^{s,p}(\mathbb{R}^n)}.$$

When there is no ambiguity about the domain we may write

- $W^{s,p}$ instead of $W^{s,p}(\Omega)$,

- $W_{\delta}^{s,p}$ instead of $W_{\delta}^{s,p}(\Omega)$,
Using weighted spaces, we can give a precise definition of asymptotically flat manifolds.

**Definition 3.11.** Let $M$ be an $n$-dimensional smooth connected oriented manifold and let $h$ be a metric on $M$ for which $(M, h)$ is complete. Let $E_r = \{ x \in \mathbb{R}^n : |x| > r \}$. We say $(M, h)$ is asymptotically flat (AF) of class $W^{s,p}_\delta$ (where $s \geq 0$, $p \in (1, \infty)$, and $\delta < 0$) if

1. $h \in W^{s,p}_{\text{loc}}$.

2. There is a finite collection $\{U_i\}_{i=1}^m$ of open sets of $M$ and diffeomorphisms $\phi_i : U_i \to E_1$ such that $M \setminus (\bigcup_{i=1}^m U_i)$ is compact.

3. There exists a constant $\theta \geq 1$ such that for each $i$

   $$\forall x \in E_1 \forall y \in \mathbb{R}^n \quad \theta^{-1}|y|^2 \leq ((\phi_i^{-1})^* h)_{r,s}(x) y^s y^s \leq \theta |y|^2. \quad (\text{see Remark 3.14})$$

4. There exists a positive constant $\omega$ such that for each $i$, $(\phi_i^{-1})^* h - \omega \bar{h} \in W^{s,p}_{\delta}(E_1)$, where $\bar{h}$ is the Euclidean metric.

The charts $(U_i, \phi_i)$ are called end charts, and the corresponding coordinates are called end coordinates.

**Definition 3.12.** Let $(M, h)$ be an $n$-dimensional AF manifold of class $W^{\alpha,\gamma}_\rho$. In addition, let $\{(U_i, \phi_i)\}_{i=1}^m$ be the collection of end charts. We can extend this set to an atlas $\{(U_i, \phi_i)\}_{i=1}^k$ such that for $i > m$ the set $\bar{U}_i$ is compact and $\phi_i(U_i) = B_1 := \{ x \in \mathbb{R}^n : |x| < 1 \}$.

Let $\{\chi_i\}_{i=1}^k$ be a partition of unity subordinate to the cover $\{U_i\}_{i=1}^k$. The weighted Sobolev
space $W^{s,p}_\delta(M)$ is the subset of $W^{s,p}_{loc}(M)$ consisting of functions $u$ that satisfy

$$
\|u\|_{W^{s,p}_\delta(M)} := \sum_{i=1}^{m} \|(\phi_i^{-1})^*(\chi_i u)\|_{W^{s,p}(\mathbb{R}^n)} + \sum_{i=m+1}^{k} \|(\phi_i^{-1})^*(\chi_i u)\|_{W^{s,p}(B_1)} < \infty
$$

The collection $\{(U_i, \phi_i)\}_{i=1}^{k}$ is called an AF atlas for $M$.

**Remark 3.13.** The above definition of $W^{s,p}_\delta(M)$ does not depend on the metric $h$ and its class and it is also independent of the chosen partition of unity, but it is based on the specific charts that were introduced in the definition of AF manifolds. This definition is not necessarily coordinate independent (of course see Remark 3.14). Indeed, as for the case of compact manifolds, one can easily show that different choices for $\{(U_i, \phi_i)\}_{i=m+1}^{k}$ result in equivalent norms; but the dependence of the norm on the end charts is more critical. In what follows we always assume that one fixed AF atlas is given and we just work with that fixed atlas.

**Remark 3.14.** Item (3) in the definition of AF manifolds (Definition 3.11) guarantees that $L^{p}_\delta(M)$ is independent of the chosen AF atlas and in fact $\|u\|_{L^{p}_\delta(M)}$ agrees with the following norm that uses the natural volume form of $M$ [5]:

$$\|u\|_{L^{p}_\delta(M)} = \|x^{-\delta-n/p} u\|_{L^{p}(M)}. \quad (\|u\|_{L^{p}(M)} = (\int_{M} |u|^p dV_h)^{1/p}).$$

Of course it is not necessary to single out weighted Lebesgue spaces and require their definition to be coordinate independent. One may choose to treat the spaces $L^{p}_\delta(M)$ as general $W^{s,p}_\delta(M)$ spaces are treated. This is the reason why in some of the literature item (3) in Definition 3.11 is not considered as part of the definition.

**Remark 3.15.** Let $\pi : E \rightarrow M$ be a smooth vector bundle over $M$. Completely analogous to Definition 3.12, one can define the Sobolev space $W^{s,p}_\delta(E)$ of sections of $E$ by using a finite trivializing cover of coordinate charts and a partition of unity subordinate to the
3.2 Properties of Weighted Sobolev Spaces

Theorem 3.16. [52, 55, 54, 5, 11, 70, 71] Let $p_1, p_2, p, q \in (1, \infty)$, $\delta, \delta_1, \delta_2, \delta' \in \mathbb{R}$.

1. If $p \geq q$ and $\delta' < \delta$ then $L^p_{\delta'}(\mathbb{R}^n) \subseteq L^q_{\delta}(\mathbb{R}^n)$ is continuous.

2. For $s \geq s'$ and $\delta \leq \delta'$ the inclusion $W^{s, p}_{\delta}(\mathbb{R}^n) \subseteq W^{s', p}_{\delta}(\mathbb{R}^n)$ is continuous.

3. For $s > s'$ and $\delta < \delta'$ the inclusion $W^{s, p}_{\delta}(\mathbb{R}^n) \subseteq W^{s', p}_{\delta'}(\mathbb{R}^n)$ is compact.

4. If $0 \leq sp < n$ then $W^{s, p}_{\delta}(\mathbb{R}^n) \subseteq L^r_{\delta}(\mathbb{R}^n)$ is continuous for every $r$ with $\frac{1}{p} - \frac{s}{n} \leq \frac{1}{r} \leq \frac{1}{p}$.

5. If $sp = n$ then $W^{s, p}_{\delta}(\mathbb{R}^n) \subseteq L^r_{\delta}(\mathbb{R}^n)$ is continuous for every $r \geq p$.

6. If $sp > n$ then $W^{s, p}_{\delta}(\mathbb{R}^n) \subseteq C^0_{\delta}(\mathbb{R}^n)$ is continuous for every $r \geq p$. Moreover $W^{s, p}_{\delta}(\mathbb{R}^n) \subseteq C^0_{\delta}(\mathbb{R}^n)$ is continuous where $C^0_{\delta}(\mathbb{R}^n)$ is the set of continuous functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ for which $\|f\|_{C^0_{\delta}} := \sup_{x \in \mathbb{R}^n} (\langle x \rangle^{-\delta} |f|) < \infty$.

7. If $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} < 1$, then pointwise multiplication is a continuous bilinear map $L^{p_1}_{\delta_1}(\mathbb{R}^n) \times L^{p_2}_{\delta_2}(\mathbb{R}^n) \rightarrow L^r_{\delta_1 + \delta_2}(\mathbb{R}^n)$.

8. Pointwise multiplication is a continuous bilinear map $C^0_{\delta_1}(\mathbb{R}^n) \times L^p_{\delta_2}(\mathbb{R}^n) \rightarrow L^p_{\delta_1 + \delta_2}(\mathbb{R}^n)$.

9. For $s \in \mathbb{R}$ (and $p \in (1, \infty)$), $W^{s, p}_{\delta}(\mathbb{R}^n)$ is a reflexive space and $(W^{s, p}_{\delta}(\mathbb{R}^n))^* = W^{-s, p'}_{-n-\delta}(\mathbb{R}^n)$.

10. **Real Interpolation**: Suppose $\theta \in (0, 1)$. If

$$s = (1 - \theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \delta = (1 - \theta)\delta_0 + \theta \delta_1$$

cover.
then \( W^{s_0,p}_δ(\mathbb{R}^n) = (W^{s_0,p_0}_δ(\mathbb{R}^n), W^{s_1,p_1}_δ(\mathbb{R}^n))_0 \), unless \( s_0, s_1 \in \mathbb{R} \) with \( s_0 \neq s_1 \) and \( s \in \mathbb{Z} \). In the case where \( s_0 \) and \( s_1 \) are not both positive and exactly one of \( s_0 \) and \( s_1 \) is an integer, we additionally assume that \( p_0 = p_1 = p \).

11. **Complex Interpolation:** Suppose \( \theta \in (0,1) \). If

\[
    s = (1 - \theta) s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \delta = (1 - \theta) \delta_0 + \theta \delta_1
\]

then \( W^{s,p}_δ(\mathbb{R}^n) = [W^{s_0,p_0}_δ(\mathbb{R}^n), W^{s_1,p_1}_δ(\mathbb{R}^n)]_0 \) provided \( s_0, s_1, s \in \mathbb{Z} \) or \( s_0, s_1, s \not\in \mathbb{Z} \).

**Note:** The above interpolation facts do not say anything about the case where \( s_0 \) or \( s_1 \in \mathbb{R} \setminus \mathbb{Z} \) and \( s \in \mathbb{Z} \).

12. **\( C^∞_c(\mathbb{R}^n) \) is dense in \( W^{s,p}_δ(\mathbb{R}^n) \) for all \( s \in \mathbb{R} \).**

**Remark 3.17.** We define \( L^∞_δ(\mathbb{R}^n) \) as follows: \( f \in L^∞_δ(\mathbb{R}^n) \) if and only if \( \langle x \rangle^{-\delta} f \in L^∞(\mathbb{R}^n) \). We equip this space with the norm \( \| f \|_{L^∞_δ(\mathbb{R}^n)} := \| \langle x \rangle^{-\delta} f \|_{L^∞(\mathbb{R}^n)} \). More generally, for all \( k \in \mathbb{N}_0 \)

\[
    W^{k,∞}_δ(\mathbb{R}^n) := \{ u \in L^∞_δ(\mathbb{R}^n) : \partial^α u \in L^∞_{δ-|α|}(\mathbb{R}^n) \quad \forall \ |α| \leq k \},
\]

\[
    \| u \|_{W^{k,∞}_δ(\mathbb{R}^n)} = \sum_{|α| \leq k} \| \partial^α u \|_{L^∞_{δ-|α|}(\mathbb{R}^n)}.
\]

It is easy to show that \( C^0_δ(\mathbb{R}^n) \) is a subspace of \( L^∞_δ(\mathbb{R}^n) \), pointwise multiplication is a continuous bilinear map \( L^∞_δ(\mathbb{R}^n) \times L^p_δ(\mathbb{R}^n) \to L^p_δ(\mathbb{R}^n) \) and the inclusion \( L^∞_δ(\mathbb{R}^n) \subseteq L^p_δ(\mathbb{R}^n) \) is continuous for \( \delta < \delta \) and \( p \in (1,\infty) \) [5]. Also if \( sp > n \), then the inclusions \( W^{s,p}_δ(\mathbb{R}^n) \subseteq C^0_δ(\mathbb{R}^n) \subseteq L^∞_δ(\mathbb{R}^n) \) are continuous.

**Note that if we let \( r := \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}} \), then for \( u \in L^∞_δ(\mathbb{R}^n) \) we have**

\[
    \| u \|_{L^∞_δ(\mathbb{R}^n)} = \text{ess sup}_{x \in \mathbb{R}^n} (r^{-\delta}|u|) \implies |u| \leq r^δ \| u \|_{L^∞_δ(\mathbb{R}^n)} \text{ a.e.}
\]
Remark 3.18. By using partition of unity arguments one can prove all the items in Theorem 3.16 for AF manifolds (see below; also for item 9. there are several ways to construct an isomorphism between \((W_s^{p,M})^*\) and \(W_{-p}^{s,M}(M)\), see Section 3.3 for a discussion about duality pairing). Of course note that for instance we have \(\|f\|_{C^0_\delta(M)} := \sup_{x \in \Omega(M)} (1 + |x|^2)^{1/2} - \delta |f|\), where \(|x|\) is the geodesic distance from \(x\) to a fixed point \(O\) in the compact core. As opposed to \(\mathbb{R}^n\), in a general Riemannian manifold \(|x|^2\) is not smooth, so there is no advantage in using \((1 + |x|^2)^{1/2}\) instead of for example \(1 + |x|\). In the literature the norms \(\|f\|_{C^0_\delta(M)} := \sup_{x \in \Omega(M)} (1 + |x|)^{1/2} - \delta |f|\) and \(\|f\|_{L^\infty_\delta(M)} = \|(1 + |x|)^{1/2} - \delta f\|_\infty\) have also been used for \(C^0_\delta(M)\) and \(L^\infty_\delta(M)\), respectively. Clearly these norms are equivalent to the original ones.

Here we just show two of the previously stated facts for weighted spaces on \(\mathbb{R}^n\) are also true for weighted spaces on AF manifolds. The other items in Theorem 3.16 and Remark 3.17 can be proved for AF manifolds in a similar way.

- **Continuous Embedding**: For \(s \geq s'\) and \(\delta \leq \delta'\) the inclusion \(W_s^{p,M} \subseteq W_s^{p,M}\) is continuous:

\[
\|u\|_{W_s^{p,M}} = \sum_{i=1}^m \|\phi_i^{-1} \ast (\chi_i u)\|_{W_s^{p,M}(\mathbb{R}^n)} + \sum_{i=m+1}^k \|\phi_i^{-1} \ast (\chi_i u)\|_{W_s^{p,B_1}} \leq \sum_{i=1}^m \|\phi_i^{-1} \ast (\chi_i u)\|_{W_s^{p,R^n}} + \sum_{i=m+1}^k \|\phi_i^{-1} \ast (\chi_i u)\|_{W_s^{p,B_1}} = \|u\|_{W_s^{p,M}}.
\]

- **Compact Embedding**: For \(s > s'\) and \(\delta < \delta'\) the inclusion \(W_s^{p,M} \subseteq W_s^{p,M}\) is compact:

Let \(\{u_j\}\) be a bounded sequence in \(W_s^{p,M}\): \(\|u_j\|_{W_s^{p,M}} \leq \tilde{M}\). We must prove that there
exists a subsequence of \( \{u_j\} \) that is Cauchy in \( W^{s',p}_\delta \) (recall that \( W^{s',p}_\delta \) is complete).

\[
\tilde{M} \geq \|u_j\|_{W^{s,p}_\delta} = \sum_{i=1}^{m} \|(\phi_i^{-1})^*(\chi_i u_j)\|_{W^{s,p}_\delta(\mathbb{R}^n)} + \sum_{i=m+1}^{k} \|(\phi_i^{-1})^*(\chi_i u_j)\|_{W^{s,p}(B_1)}.
\]

Therefore

\[
\begin{cases}
\forall 1 \leq i \leq m & \forall j \quad \|(\phi_i^{-1})^*(\chi_i u_j)\|_{W^{s,p}_\delta(\mathbb{R}^n)} \leq \tilde{M}, \\
\forall m + 1 \leq i \leq k & \forall j \quad \|(\phi_i^{-1})^*(\chi_i u_j)\|_{W^{s,p}(B_1)} \leq \tilde{M}.
\end{cases}
\]

Since \( W^{s,p}_\delta(\mathbb{R}^n) \hookrightarrow W^{s',p}_\delta(\mathbb{R}^n) \) and \( W^{s,p}(B_1) \hookrightarrow W^{s',p}(B_1) \) are compact (by Theorem 3.16 and Rellich-Kondrachov theorem, respectively), we can conclude that

\[
\begin{cases}
\forall 1 \leq i \leq m, \exists \text{ a subsequence of } \{(\phi_i^{-1})^*(\chi_i u_j)\}_{j=1}^\infty \text{ that converges in } W^{s',p}_\delta(\mathbb{R}^n), \\
\forall m + 1 \leq i \leq k, \exists \text{ a subsequence of } \{(\phi_i^{-1})^*(\chi_i u_j)\}_{j=1}^\infty \text{ that converges in } W^{s',p}(B_1).
\end{cases}
\]

In fact, by a diagonalization argument we can construct a subsequence \( \{v_j\} \) that converges in the corresponding space for all \( 1 \leq i \leq k \) (Start with \( i = 1 \) and find a subsequence that converges. Then for \( i = 2 \) find a subsequence from the preceding subsequence that converges and so on. At each step we find a subsequence of the preceding subsequence). So

\[
\begin{cases}
\forall 1 \leq i \leq m & \{(\phi_i^{-1})^*(\chi_i v_j)\}_{j=1}^\infty \text{ converges in } W^{s',p}_\delta(\mathbb{R}^n), \\
\forall m + 1 \leq i \leq k & \{(\phi_i^{-1})^*(\chi_i v_j)\}_{j=1}^\infty \text{ converges in } W^{s',p}(B_1).
\end{cases}
\]

We claim that \( \{v_j\} \) is Cauchy in \( W^{s',p}_\delta(M) \). Let \( \epsilon > 0 \) be given. For each \( 1 \leq i \leq m \), let
Let $N_i$ be such that if $l, \tilde{l} > N_i$ then

$$
\| (\phi_i^{-1})^* (\chi_l v_l) - (\phi_i^{-1})^* (\chi_l v_l) \|_{W_{\delta'}^{s',p}(\mathbb{R}^n)} < \frac{\epsilon}{k}.
$$

Also for each $m + 1 \leq i \leq k$, let $N_i$ be such that if $l, \tilde{l} > N_i$ then

$$
\| (\phi_i^{-1})^* (\chi_l v_l) - (\phi_i^{-1})^* (\chi_l v_l) \|_{W_{\delta'}^{s',p}(B_1)} < \frac{\epsilon}{k}.
$$

Now let $N = \max\{N_1, \ldots, N_k\}$. Clearly for all $l, \tilde{l} > N$ we have

$$
\| v_l - v_{\tilde{l}} \|_{W_{\delta'}^{s',p}(M)} = \sum_{i=1}^{m} \| (\phi_i^{-1})^* (\chi_i (v_l - v_{\tilde{l}})) \|_{W_{\delta'}^{s',p}(\mathbb{R}^n)}
+ \sum_{i=m+1}^{k} \| (\phi_i^{-1})^* (\chi_i (v_l - v_{\tilde{l}})) \|_{W_{\delta'}^{s',p}(B_1)}
< k \frac{\epsilon}{k} = \epsilon.
$$

This proves that $\{v_j\}$ is Cauchy in $W_{\delta'}^{s',p}(M)$.

**Lemma 3.19.** Let $k \in \mathbb{N}_0$, $\delta \in \mathbb{R}$ and $p \in (1, \infty)$. Then

$$
u \in W_{\delta}^{k,p}(\mathbb{R}^n) \iff \partial^\alpha u \in L_{\delta - |\alpha|}^p(\mathbb{R}^n) \quad \forall |\alpha| \leq k.
$$

**Proof:** (Lemma 3.19) The case $k = 0$ is obvious. In general we have

$$
u \in W_{\delta}^{k,p} \iff \|\nu\|_{W_{\delta}^{k,p}} < \infty \iff \forall |\alpha| \leq k \quad \|x|^{-\delta - \frac{n}{p} |\alpha|} \partial^\alpha u\|_{L^p} < \infty
\iff \forall |\alpha| \leq k \quad \|x|^{-(\delta - |\alpha|) - \frac{n}{p}} \partial^\alpha u\|_{L^p} < \infty
\iff \forall |\alpha| \leq k \quad \partial^\alpha u \in L_{\delta - |\alpha|}^p.
$$
**Lemma 3.20.** Let \( s \in \mathbb{R}, p, q \in (1, \infty) \). If \( p \geq q \) and \( \delta' < \delta \), then \( W^{s,p}_{\delta'}(\mathbb{R}^n) \hookrightarrow W^{s,q}_{\delta}(\mathbb{R}^n) \).

**Proof.** (Lemma 3.20) We consider three cases:

- **Case 1:** \( s = k \in \mathbb{N}_0 \).
  
  \[
  u \in W^{k,p}_{\delta'} \Rightarrow \forall |\alpha| \leq k \quad \partial^\alpha u \in L^p_{\delta'-|\alpha|} \\
  \Rightarrow \forall |\alpha| \leq k \quad \partial^\alpha u \in L^q_{\delta-|\alpha|} \quad \text{(by item 1. of Theorem 3.16)} \\
  \Rightarrow u \in W^{k,q}_{\delta}.
  \]

  In fact,
  
  \[
  \| u \|_{k,q,\delta} = \sum_{|\beta| \leq k} \| \langle x \rangle^{-\delta - \frac{n+1}{p}|\beta|} \partial^\beta u \|_{L^q(\mathbb{R}^n)} = \sum_{|\beta| \leq k} \| \langle x \rangle^{-\delta - \frac{n}{p}|\beta|} \partial^\beta u \|_{L^q(\mathbb{R}^n)}
  \]
  
  \[
  = \sum_{|\beta| \leq k} \| \partial^\beta u \|_{L^q_{\delta-|\beta|}(\mathbb{R}^n)} \leq \sum_{|\beta| \leq k} \| \partial^\beta u \|_{L^p_{\delta'-|\beta|}(\mathbb{R}^n)} (L^p_{\delta'-|\beta|} \hookrightarrow L^q_{\delta-|\beta|}) \\
  = \sum_{|\beta| \leq k} \| \langle x \rangle^{-\delta' - \frac{n+1}{p}|\beta|} \partial^\beta u \|_{L^p(\mathbb{R}^n)} = \| u \|_{k,p,\delta'}.
  \]

- **Case 2:** \( s \geq 0, s \not\in \mathbb{N}_0 \).

  Let \( k = \lfloor s \rfloor, \theta = s - k \). By what was proved in the previous case

  \[
  W^{k,p}_{\delta'} \hookrightarrow W^{k,q}_{\delta}, \quad W^{k+1,p}_{\delta'} \hookrightarrow W^{k+1,q}_{\delta}.
  \]

  Since \( s = (1 - \theta)k + \theta(k + 1) \), the claim follows from real interpolation.

- **Case 3:** \( s < 0 \).

  By assumption \( p \geq q \) and \( \delta' < \delta \), therefore

  \[
  p' \leq q', \quad -n - \delta' > -n - \delta.
  \]
Here \( p' \) and \( q' \) are the conjugates of \( p \) and \( q \), respectively. Thus by what was proved in the previous cases we have

\[
W^{-s,q'}_{-n-\delta} \hookrightarrow W^{-s,p'}_{-n-\delta}.
\]

The result follows by taking the dual.

\( \square \)

**Lemma 3.21.** Let the following assumptions hold:

(i) \( 1 < p \leq r < \infty \),

(ii) \( t, s \in \mathbb{R} \) with \( 0 \leq t \leq s \),

(iii) \( s - \frac{n}{p} \geq t - \frac{n}{r} \).

Then: For all \( \delta \in \mathbb{R} \) \( W^{s,p}_\delta \hookrightarrow W^{t,r}_\delta \).

**Proof.** (Lemma 3.21) In the proof we use the fact that if \( 1 \leq \alpha \leq \beta \), then \( l\alpha \hookrightarrow l\beta \) (\( l\alpha \) denotes the space of \( \alpha \)-power summable sequences); in fact for any sequence \( a = \{a_j\} \), \( \|a\|_{l\beta} \leq \|a\|_{l\alpha} \). From the assumption it follows that \( W^{s,p}_\delta \hookrightarrow W^{t,r}_\delta \) (see Theorem 3.44) and so

\[
\|u\|_{t,r,\delta} = \left[ \sum_{j=0}^{\infty} 2^{-r\delta j} \|S_{2^j}(\varphi_j u)\|_{t,r}\bigg]^\frac{1}{r} \leq \left[ \sum_{j=0}^{\infty} 2^{-r\delta j} \|S_{2^j}(\varphi_j u)\|_{s,p}\bigg]^\frac{1}{r} \leq \left[ \sum_{j=0}^{\infty} (2^{-\delta j} \|S_{2^j}(\varphi_j u)\|_{s,p})^p\bigg]^\frac{1}{p} \leq \|u\|_{s,p,\delta} \quad (\text{Note that } p \leq r \text{ and so } \|\cdot\|_{l\beta} \leq \|\cdot\|_{l\alpha}).
\]

\( \square \)

**Theorem 3.22** (Embedding Theorem I, Weighted Spaces). Let the following assumptions hold:
(i) $1 < p \leq r < \infty$,

(ii) $t, s \in \mathbb{R}$ with $t \leq s$,

(iii) $s - \frac{n}{p} \geq t - \frac{n}{r}$.

Then: If $\delta' \leq \delta$ then $W_{\delta'}^{s, p} \rightsquigarrow W_{\delta}^{t, r}$.

Proof. (Theorem 3.22) Note that, since $\delta' \leq \delta$, $W_{\delta'}^{s, p} \rightsquigarrow W_{\delta}^{s, p}$, so we just need to show that $W_{\delta}^{s, p} \rightsquigarrow W_{\delta}^{t, r}$. By Lemma 3.21 we know that the claim is true for the case $0 \leq t$. So we just need to consider the case where $t < 0$.

- **Case 1:** $t < 0, s \leq 0$

  It is enough to show that $(W_{\delta}^{t, r})^\ast \rightsquigarrow (W_{\delta}^{s, p})^\ast$, that is, we need to prove that

  \[ W_{-n-\delta}^{s-t, r} \rightsquigarrow W_{-n-\delta}^{s-p}. \]

  Note that $-t$ and $-s$ are nonnegative so we just need to check that the assumptions of Lemma 3.21 hold true:

  \[ t \leq s \leq 0 \Rightarrow 0 \leq -s \leq -t \]

  \[ 1 < p \leq r \Rightarrow 1 < r' \leq p' \]

  \[ s - \frac{n}{p} \geq t - \frac{n}{r} \Rightarrow s + \frac{n}{r} - n \geq t + \frac{n}{r} - n \Rightarrow s - \frac{n}{r'} \geq t - \frac{n}{p'} \Rightarrow -t - \frac{n}{r'} \geq -s - \frac{n}{p'}. \]

- **Case 2:** $t < 0, s > 0$

  In this case we will prove that there exists $q \geq 1$ such that

  \[ W_{\delta}^{s, p} \rightsquigarrow L_{\delta}^{q} \rightsquigarrow W_{\delta}^{t, r}. \]
By what was proved previously, in order to make sure that the above inclusions hold true it is enough to find \( q \) such that

\[
\frac{t - n}{r} \leq \frac{s - n}{q} \leq \frac{s}{p} \quad (\Leftrightarrow \frac{s}{n} - \frac{1}{q} \leq \frac{1}{r} - \frac{1}{t})
\]

\[p \leq q \leq r \quad (\Leftrightarrow \frac{1}{r} \leq \frac{1}{q} \leq \frac{1}{p})\]

Note that by assumption \(-\frac{s}{n} + \frac{1}{p} \leq -\frac{t}{n} + \frac{1}{r}\). If \(-\frac{s}{n} + \frac{1}{p} = -\frac{t}{n} + \frac{1}{r}\), then \( q \) defined by \( \frac{1}{q} = -\frac{s}{n} + \frac{1}{p} \) clearly satisfies the desired conditions. So it remains to consider the case where \(-\frac{s}{n} + \frac{1}{p} < -\frac{t}{n} + \frac{1}{r}\). The inequalities in the first line are satisfied if and only if

\[
\frac{1}{q} = -\frac{s}{n} + \frac{1}{p} + \sigma\left(\frac{s - t}{n} + \frac{1}{r} - \frac{1}{p}\right)
\]

for some \( \sigma \in [0, 1] \). The question is “can we choose \( \sigma \) so that the above expression lies between \( \frac{1}{r} \) and \( \frac{1}{p} \)?” We want to find \( \sigma \in [0, 1] \) such that

\[
\frac{1}{r} \leq -\frac{s}{n} + \frac{1}{p} + \sigma\left(\frac{s - t}{n} + \frac{1}{r} - \frac{1}{p}\right) \leq \frac{1}{p}
\]

That is we want to find \( \sigma \in [0, 1] \) such that

\[
\frac{\frac{1}{r} - \frac{1}{p} + \frac{s}{n}}{\frac{s - t}{n} + \frac{1}{r} - \frac{1}{p}} \leq \sigma \leq \frac{s}{n}
\]

Note that since \( \frac{1}{r} \leq \frac{1}{p} \) clearly

\[
\frac{\frac{1}{r} - \frac{1}{p} + \frac{s}{n}}{\frac{s - t}{n} + \frac{1}{r} - \frac{1}{p}} \leq \frac{s}{n}
\]
So it would be possible to find $\sigma$ if and only if

\[
\frac{s}{n} \geq 0 \quad \text{and} \quad \frac{1}{r} - \frac{1}{p} + \frac{s}{n} \leq 1.
\]

The first inequality is true because by assumption $s > 0$ and $s - \frac{n}{p} \geq t - \frac{n}{r}$. The second inequality is true because by assumption $t < 0$ and

\[
\frac{1}{r} - \frac{1}{p} + \frac{s}{n} \leq 1 \Leftrightarrow \frac{s - t}{n} + \frac{1}{r} - \frac{1}{p} \leq 0.
\]

\[\square\]

**Theorem 3.23** (Embedding Theorem II, Weighted Spaces). *Let the following assumptions hold:*

(i) $1 < p, r < \infty$,

(ii) $t, s \in \mathbb{R}$ with $t \leq s$,

(iii) $s - \frac{n}{p} \geq t - \frac{n}{r}$,

(iv) $\delta'$ is strictly less than $\delta$.

*Then: $W^{s,p}_{\delta'} \hookrightarrow W^{t,r}_{\delta}$. (Note that if $p > r$, then the third assumption follows from the second assumption.)*

**Proof. (Theorem 3.23)** If $p \leq r$, then the claim follows from Theorem 3.22. Let’s assume $p > r$. Then by Lemma 3.20 we have $W^{s,p}_{\delta'} \hookrightarrow W^{s,r}_{\delta}$ and by Theorem 3.16 we have $W^{s,r}_{\delta} \hookrightarrow W^{t,r}_{\delta}$. Consequently $W^{s,p}_{\delta'} \hookrightarrow W^{t,r}_{\delta}$.

\[\square\]

**Lemma 3.24** (Multiplication by bounded smooth functions). *Let $\sigma \in \mathbb{R}$, $q \in [1, \infty)$ (if $\sigma < 0, q \neq 1$). Let $N = \lfloor |\sigma| \rfloor$. If $f \in BC^N(\mathbb{R}^n)$ and $u \in W^{\sigma,q}(\mathbb{R}^n)$, then $fu \in W^{\sigma,q}(\mathbb{R}^n)$ and*
moreover \( \| f u \|_{\sigma,q} \preceq \| u \|_{\sigma,q} \) (the implicit constant depends on \( f \) but it does not depend on \( u \)).

**Proof.** (Lemma 3.24) The proof consists of four steps:

- **Step 1:** \( \sigma = k \in \mathbb{N}_0 \). The claim is proved in [27].

- **Step 2:** \( 0 < \sigma < 1 \). The claim has been proved in [62] for the case where \( \sigma \in (0,1) \), \( f \) is Lipschitz continuous and \( 0 \leq f \leq 1 \). With an obvious modification that proof also works for the case where \( f \in BC^1(\mathbb{R}^n) \).

- **Step 3:** \( 1 < \sigma \notin \mathbb{N} \). In this case we can proceed as follows: Let \( k = \lfloor \sigma \rfloor \), \( \theta = \sigma - k \).

  \[
  \| f u \|_{\sigma,q} = \| f u \|_{k,q} + \sum_{|v| = k} \| \partial^v (f u) \|_{\theta,q} \\
  \leq \| f u \|_{k,q} + \sum_{|v| = k} \sum_{\beta \leq v} \| \partial^{\nu-\beta} f \partial^\beta u \|_{\theta,q} \\
  \leq \| u \|_{k,q} + \sum_{|v| = k} \sum_{\beta \leq v} \| \partial^\beta u \|_{\theta,q} \quad (\text{by Step 1 and Step 2}) \\
  = \| u \|_{\sigma,q} + \sum_{|v| = k} \sum_{\beta \leq v} \| \partial^\beta u \|_{\theta,q} \\
  \leq \| u \|_{\sigma,q} + \sum_{|v| = k} \sum_{\beta \leq v} \| u \|_{\theta + |\beta|,q} \quad (\partial^\beta : W^{\theta + |\beta|,q} \to W^{\theta,q} \text{ is continuous}) \\
  \leq \| u \|_{\sigma,q} + \sum_{|v| = k} \sum_{\beta \leq v} \| u \|_{\sigma,q} \quad (\theta + |\beta| < \sigma \Rightarrow W^{\sigma,q} \to W^{\theta + |\beta|,q}) \\
  \leq \| u \|_{\sigma,q}.
  \]

- **Step 4:** \( \sigma < 0 \). For this case we use a duality argument:

  \[
  \| f u \|_{\sigma,q} = \sup_{v \in W^{-\sigma,q'}} \frac{|\langle f u, v \rangle|}{\| v \|_{-\sigma,q'}} = \sup_{v \in W^{-\sigma,q'}} \frac{|\langle u, f v \rangle|}{\| v \|_{-\sigma,q'}} \\
  \leq \sup_{v \in W^{-\sigma,q'}} \frac{\| u \|_{\sigma,q} \| f v \|_{-\sigma,q'}}{\| v \|_{-\sigma,q'}} \leq \sup_{v \in W^{-\sigma,q'}} \frac{\| u \|_{\sigma,q} \| v \|_{-\sigma,q'}}{\| v \|_{-\sigma,q'}} = \| u \|_{\sigma,q}.
  \]
Lemma 3.25. Let $\sigma, \delta \in \mathbb{R}$, $q \in (1, \infty)$. Let $N = ||\sigma|| + 1$. Suppose $f \in C^N(\mathbb{R}^n)$ is such that for all multi-indices $\nu$ with $|\nu| \leq N$

$$|\partial^\nu f(x)| \leq b(\nu)|x|^{-|\nu|},$$

where $b(\nu)$ are appropriate numbers independent of $x$. If $u \in W^{\sigma,q}_\delta(\mathbb{R}^n)$, then $fu \in W^{\sigma,q}_\delta(\mathbb{R}^n)$ and moreover $\|fu\|_{\sigma,q,\delta} \leq \|u\|_{\sigma,q,\delta}$ where the implicit constant depends on $b(\nu)$.

Proof. (Lemma 3.25) The case $\sigma \geq 0$ is a special case of Lemma 3 in [70]. For the case $\sigma < 0$ we may use a duality argument exactly similar to the proof of Lemma 3.24. \(\square\)

Lemma 3.26 (Multiplication Lemma, Unweighted spaces). Let $s_i \geq s$ with $s_1 + s_2 \geq 0$, and $1 < p, p_i < \infty$ ($i = 1, 2$) be real numbers satisfying

$$s_i - s \geq n\left(1 - \frac{1}{p_i} - \frac{1}{p}\right), \quad (\text{if } s_i = s \notin \mathbb{Z}, \text{then let } p_i \leq p)$$

$$s_1 + s_2 - s > n\left(\frac{1}{p_1} + \frac{1}{p_2} - 1\right) \geq 0.$$  

In case $s < 0$, in addition let

$$s_1 + s_2 > n\left(\frac{1}{p_1} + \frac{1}{p_2} - 1\right) \quad (\text{equality is allowed if } \min(s_1, s_2) < 0).$$

Also in case where $s_1 + s_2 = 0$ and $\min(s_1, s_2) \notin \mathbb{Z}$, in addition let $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$. Then the pointwise multiplication of functions extends uniquely to a continuous bilinear map

$$W^{s_1,p_1}(\mathbb{R}^n) \times W^{s_2,p_2}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n).$$

Proof. (Lemma 3.26) In this proof we use the notations introduced in the beginning
of Section 3.4.2. We may consider three cases:

- **Case 1**: \( s \geq 0 \).
- **Case 2**: \( s < 0 \) and \( \min(s_1, s_2) < 0 \).
- **Case 3**: \( s < 0 \) and \( \min(s_1, s_2) \geq 0 \).

In what follows we study each of the above cases separately.

- **Case 1**: See Theorem 3.53 for the case where \( s \in \mathbb{N}_0 \); see Theorem 3.57 for the case where \( p_1, p_2 \leq p \). It remains to prove the claim in the following cases:
  
  i. \( s_1 > s, s_2 = s, s \not\in \mathbb{N}_0 \)
  
  \[ p_1 > p, p_2 \leq p \]

  ii. \( s_1 = s, s_2 > s, s \not\in \mathbb{N}_0 \)

  \[ p_1 \leq p, p_2 > p \]

  iii. \( s_1 > s, s_2 > s, s \not\in \mathbb{N}_0 \)

  \[ p_1 > p, p_2 > p \]

Proofs of [i] and [ii] are completely similar. Here we only prove item [i] and item [iii].

**Proof of [i]:** Let

\[
\epsilon := \frac{1}{4} \min\{s_1 - s, s_1 - s - n\left(\frac{1}{p_1} - \frac{1}{p}\right), s_1 + s_2 - s - n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right)\}.
\]

We have

\[
W^{s_1, p_1} \times W^{s_2, p_2} \hookrightarrow B^{s_1 - \frac{s}{2}}_{p_1, p_1} \times W^{s_2, p_2} \hookrightarrow B^{s_1 - \epsilon}_{p_1, p} \times W^{s_2, p_2}
\]

\[
= B^{s_1 - \epsilon}_{p_1, p} \times B^{s_2}_{p_2, p} \hookrightarrow B^\epsilon_{p, p} = W^{s, p}.
\]
Proof of [iii]: Let

\[ \epsilon := \frac{1}{4} \min\{s_1 - s, s_2 - s, s_1 - s - n\left(\frac{1}{p_1} - \frac{1}{p}\right), s_2 - s - n\left(\frac{1}{p_2} - \frac{1}{p}\right), s_1 + s_2 - s - n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right)\}. \]

Let \( \tilde{q}_1, \tilde{q}_2, \) and \( \tilde{q} \) be numbers in \((1, \infty)\) such that \( \tilde{q}_1, \tilde{q}_2 \leq \tilde{q} \). We have

\[ W^{s_1, p_1} \times W^{s_2, p_2} \hookrightarrow B^{s_1 - \frac{\epsilon}{2}}_{p_1, \tilde{q}_1} \times B^{s_2 - \frac{\epsilon}{2}}_{p_2, \tilde{q}_2} \hookrightarrow B^{s_1 + \epsilon}_{p_1, \tilde{q}_1} \times B^{s_2 + \epsilon}_{p_2, \tilde{q}_2} \hookrightarrow B^s_{p, \tilde{q}} \hookrightarrow B^s_{p, p} = W^{s, p} . \]

In the above we have used the well-known embedding theorems for Besov spaces together with the following multiplication theorem that is proved in [75]:

Let \( 0 \leq s \leq s_i, 1 < p_i, p < \infty, 1 < q_i \leq q < \infty (i = 1, 2) \) be such that

\[ s_i - s \geq n\left(\frac{1}{p_i} - \frac{1}{p}\right), \quad i = 1, 2, \]

\[ s_1 + s_2 - s > n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) . \]

Then for \( s \notin \mathbb{N} \) one has \( B^{s_1}_{p_1, q_1} \times B^{s_2}_{p_2, q_2} \hookrightarrow B^s_{p, q} \).

- **Case 2:** It follows from the argument given in the proof of Theorem 3.60 that without loss of generality we may assume \( s_1 < 0 \) and \( s_2 > 0 \). According to the same argument it is enough to prove that

\[ W^{s_2, p} \times W^{-s, p'} \hookrightarrow W^{-s_1, p'} . \quad (3.1) \]

Since \( s_2, -s, \) and \( -s_1 \) are all nonnegative numbers, we can use what was shown in **Case 1** to prove the above embedding. We have

- \( s_1 + s_2 \geq 0 \implies s_2 \geq -s_1 . \)
• $s_1 \geq s \implies -s \geq -s_1$.

• If $s_2 = -s_1$ (that is, if $s_1 + s_2 = 0$) and $-s_1 \not\in \mathbb{N}$, we must have $p_2 \leq p_1'$, i.e., $1 \leq \frac{1}{p_1} + \frac{1}{p_2}$.

(holds true by assumption)

• If $-s = -s_1$ and $-s_1 \not\in \mathbb{N}$, we must have $p' \leq p_1'$, i.e., $p_1 \leq p$. (holds true by assumption)

• $s_2 + s_1 \geq n(\frac{1}{p_1} + \frac{1}{p_2} - 1) = n(\frac{1}{p_2} - \frac{1}{p_1})$.

• $-s + s_1 \geq n(\frac{1}{p_1} - \frac{1}{p}) = n(\frac{1}{p} - \frac{1}{p_1})$.

• $s_2 - s + s_1 > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) = n(\frac{1}{p_2} + \frac{1}{p'} - \frac{1}{p_1})$.

So according to what was proved in Case 1, the embedding (3.1) holds true.

• For Case 3, see Theorem 3.62.

\[\square\]

\textbf{Remark 3.27.} Note that in case $s_i = s \not\in \mathbb{Z}$, the condition $p_i \leq p$ together with $s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p})$ in fact implies that we must have $p_i = p$.

\textbf{Corollary 3.28.} Let $s_i \geq s$ with $s_1 + s_2 > 0$, and $2 \leq p < \infty$ ($i = 1, 2$) be real numbers satisfying

$$s_1 + s_2 - s > \frac{n}{p}.$$ 

Then the pointwise multiplication of functions extends uniquely to a continuous bilinear map

$$\mathcal{W}^{s_1,p}(\mathbb{R}^n) \times \mathcal{W}^{s_2,p}(\mathbb{R}^n) \to \mathcal{W}^{s,p}(\mathbb{R}^n).$$

\textbf{Corollary 3.29.} As a direct consequence of the multiplication lemma we have:

• If $p \in (1, \infty)$ and $s \in \left(\frac{n}{p}, \infty\right)$, then $\mathcal{W}^{s,p}(\mathbb{R}^n)$ is a Banach algebra.
- Let $p \in (1, \infty)$ and $s \in \left(\frac{n}{p}, \infty\right)$. Suppose $q \in (1, \infty)$ and $\sigma \in [-s, s]$ satisfy $\sigma - \frac{n}{q} \in [-n - s + \frac{n}{p}, s - \frac{n}{p}]$; in case $s \notin \mathbb{N}_0$, assume $\sigma \neq -s$; in case $s \notin \mathbb{N}_0$, $q < p$, in addition assume $\sigma \neq s$. Then the pointwise multiplication is bounded as a map $W^{s, p}(\mathbb{R}^n) \times W^{\sigma, q}(\mathbb{R}^n) \rightarrow W^{\sigma, q}(\mathbb{R}^n)$.

Note: In the statement of the second item of the above corollary, the case $\sigma = -s \notin \mathbb{Z}$ has been excluded. However, it follows from the multiplication lemma that the claim holds true even if $\sigma = -s \notin \mathbb{Z}$ provided we additionally assume $\frac{1}{p} + \frac{1}{q} \geq 1$. Of course, if $\sigma = -s$, the assumption $\frac{1}{p} + \frac{1}{q} \geq 1$ together with $\sigma - \frac{n}{q} \in [-n - s + \frac{n}{p}, s - \frac{n}{p}]$ implies that $\frac{1}{p} + \frac{1}{q} = 1$.

**Lemma 3.30** (Multiplication Lemma, Weighted spaces). Assume that $s, s_1, s_2$ and that $1 < p, p_1, p_2 < \infty$ are real numbers satisfying

(i) $s_i \geq s$ \quad $(i = 1, 2)$ (if $s_i = s \notin \mathbb{Z}$, then let $p_i \leq p$),

(ii) $s_1 + s_2 \geq 0$ \quad $(if s_1 + s_2 = 0 and \min(s_1, s_2) \notin \mathbb{Z}$, then let $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$),

(iii) $s_i - s \geq n\left(\frac{1}{p_i} - \frac{1}{p}\right)$ \quad $(i = 1, 2)$,

(iv) $s_1 + s_2 - s > n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) \geq 0$.

In case $\min(s_1, s_2) < 0$, in addition let

(v) $s_1 + s_2 \geq n\left(\frac{1}{p_1} + \frac{1}{p_2} - 1\right)$.

In case $s < 0$ and $\min(s_1, s_2) \geq 0$, we assume the above inequality is strict ($s_1 + s_2 > n\left(\frac{1}{p_1} + \frac{1}{p_2} - 1\right)$). Then for all $\delta_1, \delta_2 \in \mathbb{R}$, the pointwise multiplication of functions extends uniquely to a continuous bilinear map

$$W^{s_1, p_1}(\mathbb{R}^n) \times W^{s_2, p_2}(\mathbb{R}^n) \rightarrow W^{s, p}(\mathbb{R}^n).$$
Proof. (Lemma 3.30) A proof for the case \( p_1 = p_2 = p = 2 \) is given in [52]. In what follows we use Lemma 3.26 to extend that proof to our general setting. In the proof we will make use of the following facts:

- **Fact 1**: If \( f \) is a smooth function with compact support and \( u \in W^{l,q} \) then \( fu \in W^{l,q} \) and \( \|fu\|_{W^{l,q}} \leq \|u\|_{W^{l,q}} \) (this is in fact a special case of Lemma 3.24).

- **Fact 2**: For all \( j \geq 1 \), \( S_{2j}\varphi_j = S_{2j-1}\varphi = \varphi \). So \( S_{2j}\varphi_j \) is zero if \( x \not\in B_2 \setminus B_{\frac{1}{2}} \). Also for \( j = 0 \), \( S_2\varphi_0 = \varphi_0 \) is zero if \( x \not\in B_2 \).

- **Fact 3**: Let \( \{a_j\}_{j=1}^m \) be positive numbers. Define \( f : (0, \infty) \to \mathbb{R} \) as follows:
  
  \[
  f(r) = \left( \sum_{j=1}^m a_j^r \right)^{\frac{1}{r}}.
  \]

  \( f(r) \) is a decreasing function. The reason is as follows: Suppose \( t \geq r > 0 \). We want to show \( \left( \sum_{j=1}^m a_j^t \right)^{\frac{1}{t}} \leq \left( \sum_{j=1}^m a_j^r \right)^{\frac{1}{r}} \). Since \( g(x) = x^t \) is an increasing function over \( (0, \infty) \), it is enough to show \( \left( \sum_{j=1}^m a_j^t \right) \leq \left( \sum_{j=1}^m a_j^r \right)^{\frac{1}{r}} \). Letting \( b_j = a_j^r \), \( \beta = \frac{t}{r} \), we want to prove \( \left( \sum_{j=1}^m b_j^\beta \right) \leq \left( \sum_{j=1}^m b_j \right)^\beta \). To this end we just need to show that

  \[
  \sum_{j=1}^m \left( \frac{b_j}{\sum_{j=1}^m b_j} \right)^\beta \leq 1.
  \]

  Set \( e_j = \frac{b_j}{\sum_{j=1}^m b_j} \). Clearly \( \sum_{j=1}^m e_j = 1 \). Since \( \beta \geq 1 \), \( 0 \leq e_j \leq 1 \), we have \( e_j^\beta \leq e_j \).

  Therefore

  \[
  \sum_{j=1}^m e_j^\beta \leq \sum_{j=1}^m e_j = 1.
  \]

- **Fact 4**: For \( a_k > 0 \) we have \( \sum_{k=1}^m a_k^p \sim (\sum_{k=1}^m a_k)^p \) (that is \( \sum_{k=1}^m a_k^p \leq (\sum_{k=1}^m a_k)^p \leq \sum_{k=1}^m a_k^p \)).

- **Fact 5**: \( \|S_r u\|_{W^{s,p}} \leq C(r, s, p, n) \|u\|_{W^{s,p}} \).
Now let's start proving the lemma. Suppose \( u_i \in W^{s_i, p_i}_{\delta_i} \). Let \( \varphi_j = 0 \) for \( j < 0 \). We have

\[
S_{2j}(\varphi_j u_1 u_2) = S_{2j}(\varphi_j) S_{2j} u_1 S_{2j} u_2.
\]

By Fact 2, for \( j \geq 1 \), \( S_{2j} \varphi_j \) is zero if \( x \not\in B_2 \setminus B_{\frac{1}{2}} \). Also it is easy to see that for \( x \in B_2 \setminus B_{\frac{1}{2}} \), \( \varphi_k(2^j x) = 0 \) if \( k \not\in \{j - 1, j, j + 1\} \). Since for all \( x \), \( \sum_{k=0}^{\infty} \varphi_k(2^j x) = 1 \), we can conclude that for \( x \in B_2 \setminus B_{\frac{1}{2}} \)

\[
\sum_{k=j-1}^{j+1} \varphi_k(2^j x) = 1.
\]

For \( j = 0 \), \( S_{2j} \varphi_j \) is zero if \( x \not\in B_2 \); one can easily check that if \( j = 0 \), the above equality holds true for all \( x \in B_2 \). Therefore for all \( x \)

\[
S_{2j}(\varphi_j u_1 u_2) = S_{2j}(\varphi_j) \sum_{k=j-1}^{j+1} S_{2j}(\varphi_k u_1) \sum_{l=j-1}^{j+1} S_{2j}(\varphi_l u_2).
\]

Now by Fact 1 and Fact 4 we have

\[
\| S_{2j}(\varphi_j u_1 u_2) \|^p_{W^{s, p}} \leq \sum_{k,l=j-1}^{j+1} \| S_{2j}(\varphi_k u_1) S_{2j}(\varphi_l u_2) \|^p_{W^{s, p}},
\]

and by the multiplication lemma for the corresponding unweighted Sobolev spaces we get

\[
\| S_{2j}(\varphi_j u_1 u_2) \|^p_{W^{s, p}} \leq \sum_{k,l=j-1}^{j+1} \| S_{2j+k} S_{2k}(\varphi_k u_1) \|^p_{W^{s_1, p_1}} \| S_{2j+l} S_{2l}(\varphi_l u_2) \|^p_{W^{s_2, p_2}}
\]

\[
\leq \sum_{k,l=j-1}^{j+1} \| S_{2j+k} S_{2k} \|^p_{W^{s_1, p_1}} \| S_{2j+l} S_{2l} \|^p_{W^{s_2, p_2}}.
\]
$S_{2^j - k}$ is one of $S_{2^-1}$, $S_0$, or $S_2$. So, by Fact 5

\[
\sum_{k=j-1}^{j+1} \| S_{2^j-k} S_{2^k} (\varphi_k u_1) \|_{W^{s_1,p_1}}^p \leq \sum_{k=j-1}^{j+1} (\| S_{2^-1} S_{2^k} (\varphi_k u_1) \|_{W^{s_1,p_1}}^p + \| S_0 S_{2^k} (\varphi_k u_1) \|_{W^{s_1,p_1}}^p) \\
\leq \sum_{k=j-1}^{j+1} \| S_{2^k} (\varphi_k u_1) \|_{W^{s_1,p_1}}^p
\]

and the similar result is true for $\sum_{l=j-1}^{j+1} \| S_{2^j-l} S_{2^l} (\varphi_l u_2) \|_{W^{s_2,p_2}}^p$. Consequently

\[
\| S_{2^j} (\varphi_j u_1 u_2) \|_{W^{s,p}}^p \leq \sum_{k,l=j-1}^{j+1} \| S_{2^k} (\varphi_k u_1) \|_{W^{s_1,p_1}}^p \| S_{2^l} (\varphi_l u_2) \|_{W^{s_2,p_2}}^p \\
\leq (\sum_{k=j-1}^{j+1} \| S_{2^k} (\varphi_k u_1) \|_{W^{s_1,p_1}}^p \sum_{l=j-1}^{j+1} \| S_{2^l} (\varphi_l u_2) \|_{W^{s_2,p_2}}^p)^p
\]

Therefore

\[
\sum_{j=0}^{\infty} 2^{-p(\delta_1 + \delta_2)j} \| S_{2^j} (\varphi_j u_1 u_2) \|_{W^{s,p}}^p \\
\leq \sum_{j=0}^{\infty} \left[ 2^{-p\delta_1 j} \sum_{k=j-1}^{j+1} \| S_{2^k} (\varphi_k u_1) \|_{W^{s_1,p_1}}^p \right] 2^{-p\delta_2 j} \left[ \sum_{l=j-1}^{j+1} \| S_{2^l} (\varphi_l u_2) \|_{W^{s_2,p_2}}^p \right]^p.
\]

Let

\[
a_j = 2^{-\delta_1 j} \sum_{k=j-1}^{j+1} \| S_{2^k} (\varphi_k u_1) \|_{W^{s_1,p_1}}^p \\
b_j = 2^{-\delta_2 j} \sum_{l=j-1}^{j+1} \| S_{2^l} (\varphi_l u_2) \|_{W^{s_2,p_2}}^p.
\]

So we have

\[
\| u_1 u_2 \|_{W^{s,p}}^p = \left[ \sum_{j=0}^{\infty} 2^{-p(\delta_1 + \delta_2)j} \| S_{2^j} (\varphi_j u_1 u_2) \|_{W^{s,p}}^p \right]^{\frac{1}{p}} \leq \left[ \sum_{j=0}^{\infty} (a_j b_j)^p \right]^{\frac{1}{p}}.
\]
Now let \( r \) be such that \( \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} \). By assumption \( \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} \geq 0 \) and so \( r \leq p \). Thus by Fact 3 and Holder’s inequality we get

\[
\left[ \sum_{j=0}^{\infty} (a_j b_j)^p \right]^{\frac{1}{p}} \leq \left[ \sum_{j=0}^{\infty} (a_j)^{p_1} \right]^{\frac{1}{p_1}} \left[ \sum_{j=0}^{\infty} (b_j)^{p_2} \right]^{\frac{1}{p_2}}
\]

\[
\leq \left[ \sum_{j=0}^{\infty} 2^{-p_1 \delta_1 j} \sum_{k=j}^{j+1} \| S_{2^k} (\varphi_k u_1) \|_{W^{s_1, p_1}} \right]^{\frac{1}{p_1}} \left[ \sum_{j=0}^{\infty} 2^{-p_2 \delta_2 j} \sum_{l=j}^{j+1} \| S_{2^l} (\varphi_l u_2) \|_{W^{s_2, p_2}} \right]^{\frac{1}{p_2}}
\]

\[
\leq \left[ \sum_{j=0}^{\infty} 2^{-p_2 \delta_2 j} \sum_{l=j}^{j+1} \| S_{2^l} (\varphi_l u_2) \|_{W^{s_2, p_2}} \right]^{\frac{1}{p_2}} \left[ \sum_{j=0}^{\infty} 2^{-p_1 \delta_1 j} \| S_{2^j} (\varphi_j u_1) \|_{W^{s_1, p_1}} \right]^{\frac{1}{p_1}}
\]

\[
= \| u_1 \|_{W^{s_1, p_1}} \| u_2 \|_{W^{s_2, p_2}}
\]

This proves \( \| u_1 u_2 \|_{W^{s_1, p_1}} \| u_2 \|_{W^{s_2, p_2}} \).  

**Remark 3.31.** By using partition of unity and charts one can show that the above lemma also holds for AF manifolds.

**Corollary 3.32** (The case where \( p_1 = p_2 = p \)). Assume \( s \leq \min\{s_1, s_2\} \), \( s_1 + s_2 > s + \frac{n}{p} \), \( s_1 + s_2 > 0 \), \( s_1 + s_2 > n(\frac{2}{p} - 1) \) and \( \delta_1 + \delta_2 \leq \delta \), then the multiplication

\[
W_{\delta_1}^{s_1, p} \times W_{\delta_2}^{s_2, p} \to W_{\delta}^{s, p},
\]

is continuous.

**Corollary 3.33.** Let \( p \in (1, \infty) \), \( s \in (\frac{n}{p}, \infty) \), and \( \delta < 0 \), then the space \( W_{\delta}^{s, p} \) is an algebra.

**Lemma 3.34.** Let the following assumptions hold:

- \( f : \mathbb{R} \to \mathbb{R} \) is smooth,
\[ u \in W^{s,p}_\rho(\mathbb{R}^n), \text{ where } s > \frac{n}{p}, \rho < 0, \text{ and } p \in (1, \infty), \]

\[ v \in W^{\sigma,q}_\delta(\mathbb{R}^n), \text{ where } \delta \in \mathbb{R}, \sigma \in (1, \infty) \text{ and (i) } \sigma \in [-s,s] (\sigma \neq -s \text{ if } s \not\in \mathbb{N}_0; \sigma \neq s \text{ if } s \not\in \mathbb{N}_0 \text{ and } q < p), (ii) \sigma - \frac{n}{q} \in [-n-s + \frac{n}{p}, s - \frac{n}{p}]. \]

Then: \( f(u)v \in W^{\sigma,q}_\delta(\mathbb{R}^n) \) and moreover the map taking \((u, v) \to f(u)v\) is continuous.

Note: The claim of the above lemma holds true even if \( \sigma = -s \not\in \mathbb{Z} \) provided we additionally assume \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** (Lemma 3.34) A proof for the case \( p = q = 2 \) is given in [52]. Here we use the multiplication lemma to extend that proof to our general setting. In the proof we make use of the following facts:

- **Fact 1**: If \( \eta \) is a smooth function with compact support, \( f \) is as in the statement of lemma, and \( u \in W^{t,q} \) with \( tq > n \), then \( \eta f(u) \in W^{t,q} \) and the map taking \( u \) to \( \eta f(u) \) is continuous from \( W^{t,q} \) to \( W^{t,q} \).

- **Fact 2**: For all \( j \geq 1 \), \( S_2^{j}\varphi_j = S_2^{j}S_2^{-j}\varphi = \varphi \). So \( S_2^{j}\varphi_j \) is zero if \( x \not\in B_2 \setminus B_{\frac{1}{2}} \). Also it is easy to see that for \( x \in B_2 \setminus B_{\frac{1}{2}} \), \( \varphi_k(2^j x) = 0 \) if \( k \not\in \{ j-1, j, j+1 \} \). Since for all \( x \), \( \sum_{k=0}^{\infty} \varphi_k(2^j x) = 1 \), we can conclude that for \( x \in B_2 \setminus B_{\frac{1}{2}} \)

\[
\sum_{k=j-1}^{j+1} \varphi_k(2^j x) = 1.
\]

For \( j = 0 \), \( S_2^{j}\varphi_j \) is zero if \( x \not\in B_2 \); one can easily check that if \( j = 0 \), the above equality holds true for all \( x \in B_2 \).

- **Fact 3**: \( \| S_r u \|_{W^{t,e}} \leq C(r, t, e, n) \| u \|_{W^{t,e}} \).

We prove the lemma in six steps:

**Step 1**: Suppose \( u \) and \( v \) satisfy the hypotheses of the lemma. Then considering Fact
Step 2 and the fact that $W^{s,p} \times W^{\sigma,q} \hookrightarrow W^{\sigma,q}$, we can write

$$\|f(u)v\|_{W^{\sigma,q}}^q = \sum_{j=0}^{\infty} 2^{-q\delta j} \|S_{2j}(\varphi_j f(u)v)\|_{W^{\sigma,q}}^q$$

$$= \sum_{j=0}^{\infty} 2^{-q\delta j} \sum_{k=j-1}^{j+1} (S_{2j}\varphi_k)f(\sum_{i=j-1}^{j+1} S_{2j-i}S_{2i}(\varphi_i u))S_{2j}(\varphi_j v)\|_{W^{\sigma,q}}^q$$

$$\leq \sum_{j=0}^{\infty} 2^{-q\delta j} \sum_{k=j-1}^{j+1} (S_{2j}\varphi_k)f(\sum_{i=j-1}^{j+1} S_{2j-i}S_{2i}(\varphi_i u))\|_{W^{s,p}}^q \|S_{2j}(\varphi_j v)\|_{W^{\sigma,q}}^q.$$
Moreover it follows from $2^{-p\rho^i} \geq 1$ that if $g \in W^{s,p}_\rho$ with $\rho < 0$ then it holds that $\|S_{2i}(\varphi_i g)\|_{W^{s,p}} \leq \|g\|_{W^{s,p}_\rho}$ for all $i \geq 0$. Also we have

$$\|R_j g - 0\|_{W^{s,p}} = \sum_{i=j-1}^{j+1} \|S_{2j-i} S_{2i}(\varphi_i g)\|_{W^{s,p}} \leq \sum_{i=j-1}^{j+1} \|S_{2j-i} S_{2i}(\varphi_i g)\|_{W^{s,p}}$$

$$\leq \sum_{i=j-1}^{j+1} (\|S_{2j-i} S_{2i}(\varphi_i g)\|_{W^{s,p}} + \|S_{2j} S_{2i}(\varphi_i g)\|_{W^{s,p}}$$

$$+ \|S_{2j} S_{2i}(\varphi_i g)\|_{W^{s,p}})$$

$$\leq \sum_{i=j-1}^{j+1} \|S_{2j}(\varphi_i g)\|_{W^{s,p}} \rightarrow 0.$$

**Step 3:** Let $\eta_j := \sum_{k=j-1}^{j+1} (S_{2j} \varphi_k)$. For $j > 1$ we may write

$$\sum_{k=j-1}^{j+1} (S_{2j} \varphi_k) = \sum_{k=j-1}^{j+1} S_{2j} S_{2j-k} \varphi = \sum_{k=j-1}^{j+1} S_{2j} S_{2j-k} S_2 \varphi_1 = \sum_{k=j-1}^{j+1} S_{2j-k+1} \varphi_1$$

$$= \sum_{i=0}^{2} S_{2i} \varphi_1 =: \eta.$$

That is for $j > 1$, $\eta_j$ does not depend on $j$. Now, by **Step 2**, we know that $R_j u \rightarrow 0$ in $W^{s,p}$. So it follows from **Fact 1** that $\eta f(R_j u) \rightarrow \eta f(0)$ in $W^{s,p}$. Consequently $\{\|\eta f(R_j u)\|_{W^{s,p}}\}_{j=2}^{\infty}$ is a bounded sequence:

$$\exists M_1 \text{ such that } \forall j \geq 2 \quad \|\eta f(R_j u)\|_{W^{s,p}} < M_1.$$

Let

$$M = \max\{M_1, \|\eta_1 f(R_1 u)\|_{W^{s,p}}, \|\eta_0 f(R_0 u)\|_{W^{s,p}}\}$$

(M is independent of $j$ but it may depend on $u$).
So by what was proved in **Step 1** we have

\[
\|f(u)v\|_{W^\sigma,q_\delta}^q \leq \sum_{j=0}^{\infty} 2^{-q\delta_j} M^q \|S_{2j}(\varphi_j v)\|_{W^\sigma,q_\delta}^q = M^q \|v\|_{W^\sigma,q_\delta}^q.
\]

This shows that \(f(u)v\) is in \(W^\sigma,q_\delta\). Now it remains to prove the continuity.

**Step 4:** Let \((u_k, v_k)\) be a sequence in \(W^{s,p}_\rho \times W^\sigma,q_\delta\) that converges to \((u, v) \in W^{s,p}_\rho \times W^\sigma,q_\delta\).

We must show that \(f(u_k)v_k \to f(u)v\) in \(W^\sigma,q_\delta\). Note that

\[
f(u)v - f(u_k)v_k = f(u)(v - v_k) + (f(u) - f(u_k))v_k.
\]

By what was proved in **Step 3**, we have

\[
\|f(u)(v - v_k)\|_{W^\sigma,q_\delta} \leq \|v - v_k\|_{W^\sigma,q_\delta} \to 0.
\]

So it remains to show that \(\|(f(u) - f(u_k))v_k\|_{W^\sigma,q_\delta} \to 0\).

**Step 5:** By calculations similar to what was done in **Step 1** we have

\[
\|f(u) - f(u_k)\|_{W^\sigma,q_\delta}^q \leq \sum_{j=0}^{\infty} 2^{-q\delta_j} \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}}^q \|S_{2j}(\varphi_j v_k)\|_{W^\sigma,q_\delta}^q
\]

\[
\leq \|v_k\|_{W^\sigma,q_\delta}^q \sup_{j \geq 0} \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}}^q.
\]

Note that \(\{v_k\}\) is convergent and so \(\{v_k\}\) is bounded in \(W^\sigma,q_\delta\). Thus it is enough to show that \(\sup_{j \geq 0} \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} \to 0\) as \(k \to \infty\).

**Step 6:** We need to show

\[
\forall \epsilon > 0 \exists N \text{ s.t. } \forall k \geq N \sup_{j \geq 0} \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} < \epsilon.
\]
Let $\epsilon > 0$ be given. Note that

$$
\|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} \leq \|\eta_j(f(R_j u) - f(0))\|_{W^{s,p}} + \|\eta_j(f(0) - f(R_j u_k))\|_{W^{s,p}}.
$$

(3.2)

Let’s start by considering the first term on RHS. By Fact 1, there exists $\alpha > 0$ such that if $\|g\|_{W^{s,p}} < \alpha$ then $\|\eta_j(f(g) - f(0))\|_{W^{s,p}} < \frac{\epsilon}{4}$. Note that for $j > 1$, $\eta_j$ does not depend on $j$ and so $\alpha$ can be chosen independent of $j$. By Step 2 we know that $R_j u \to 0$ in $W^{s,p}$ and so there exists a number $P \geq 2$ such that for $j \geq P$, $\|R_j u\|_{W^{s,p}} < \frac{\alpha}{2}$. It follows that

$$
\forall j \geq P \quad \|\eta_j(f(R_j u) - f(0))\|_{W^{s,p}} < \frac{\epsilon}{4}
$$

So

$$
\sup_{j \geq P} \|\eta_j(f(R_j u) - f(0))\|_{W^{s,p}} \leq \frac{\epsilon}{4} \quad (3.3)
$$

Now we show that there exists $N_1$ such that if $k \geq N_1$ then it holds that $\sup_{j \geq P} \|\eta(f(0) - f(R_j u_k))\|_{W^{s,p}} \leq \frac{\epsilon}{4}$. (Note that since $P \geq 2$ we have $\eta_j = \eta$.)

- **Claim:** For all $j$, $R_j u_k \to R_j u$ in $W^{s,p}$ uniformly with respect to $j$ as $k \to \infty$.

- **Proof of the claim:** By what was stated in Step 2, since we have that $\rho < 0$, $\|S_{2i}(\varphi_i(u_k - u))\|_{W^{s,p}} \leq \|u_k - u\|_{W^{s,p}}$ for all $i$, and we have

$$
\|R_j(u_k - u)\|_{W^{s,p}} \leq \sum_{i=j-1}^{j+1} \|S_{2i}(\varphi_i(u_k - u))\|_{W^{s,p}} \leq \sum_{i=j-1}^{j+1} \|u_k - u\|_{W^{s,p}}
$$

$$
= 3\|u_k - u\|_{W^{s,p}} \to 0 \quad \text{uniform in } j \text{ as } k \to \infty
$$

Therefore

$$
\exists N_1 \text{ s.t. } \forall j \quad \forall k \geq N_1 \quad \|R_j(u_k - u)\|_{W^{s,p}} < \frac{\alpha}{2}.
$$
In particular, for all \( j \geq P \) and \( k \geq N_1 \) we have

\[
\|R_j u_k\|_{W^{s,p}} \leq \|R_j (u_k - u)\|_{W^{s,p}} + \|R_j u\|_{W^{s,p}} < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.
\]

Consequently for all \( j \geq P \) and \( k \geq N_1 \) we have

\[
\|\eta(f(0) - f(R_j u_k))\|_{W^{s,p}} < \frac{\epsilon}{4},
\]

which implies

\[
\forall \ k \geq N_1 \sup_{j \geq P} \|\eta(f(0) - f(R_j u_k))\|_{W^{s,p}} \leq \frac{\epsilon}{4}. \tag{3.4}
\]

From (3.2), (3.3), and (3.4) we get

\[
\forall \ k \geq N_1 \sup_{j \geq P} \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} \leq \frac{\epsilon}{2}.
\]

Now note that by the claim that was proved above, we know that \( R_j u_k \rightarrow R_j u \) in \( W^{s,p} \).

So by Fact 1, \( \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} \rightarrow 0 \) for any fixed \( j \) as \( k \rightarrow \infty \). In particular for \( 0 \leq j \leq P - 1 \),

\[
\exists M_j \ s.t. \ \forall \ k \geq M_j \ \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} < \frac{\epsilon}{2}.
\]

So if we let \( N = \max\{N_1, M_0, M_1, \ldots, M_{P-1}\} \), then for all \( k \geq N \)

\[
\sup_{j \geq P} \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} \leq \frac{\epsilon}{2}
\]

and

\[
\sup_{0 \leq j \leq P-1} \|\eta_j(f(R_j u) - f(R_j u_k))\|_{W^{s,p}} \leq \frac{\epsilon}{2}.
\]
That is
\[ \forall k \geq N \sup_{j \geq 0} \| \eta_j (f(R_j u) - f(R_j u_k)) \|_{W^{s,p}} \leq \frac{\epsilon}{2} < \epsilon, \]
which is exactly what we wanted to prove.

**Remark 3.35.** Obviously the above result also holds true if \( f \) is only smooth on an open interval containing the range of \( u \). By using partition of unity and charts one can show that the claim also holds for AF manifolds (of any class).

**Corollary 3.36.** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is smooth and \( f(0) = 0 \). If \( u \in W^{s,p}_\rho \) where \( sp > n, \rho < 0 \) then \( f(u) \in W^{s,p}_\rho \) and the map taking \( u \) to \( f(u) \) is continuous from \( W^{s,p}_\rho \) to \( W^{s,p}_\rho \).

**Proof.** (Corollary 3.36) \( f(0) = 0 \), so by Taylor’s theorem we have \( f(x) = xF(x) \) where \( F \) is smooth. Therefore by the previous lemma, \( f(u) = uF(u) \in W^{s,p}_\rho \) and moreover the map taking \( u \) to \( f(u) = uF(u) \) is continuous from \( W^{s,p}_\rho \) to \( W^{s,p}_\rho \).

**Lemma 3.37.** Let the following assumptions hold:

- \( p \in (1, \infty), s \in \left( \frac{n}{p}, \infty \right), \delta < 0 \) and \( u \in W^{s,p}_\delta \),
- \( \nu \in \mathbb{R}, \sigma \in [-1, 1], \theta = \frac{1}{p} - \frac{s-1}{n}, \frac{1}{q} \in (\frac{1+\sigma}{2}\theta, 1 - \frac{1-\sigma}{2}\theta) \) and \( v \in W^{\sigma,q}_\nu \),
- \( f : [\inf u, \sup u] \to \mathbb{R} \) is a smooth function. (Note that \( W^{s,p}_\delta \to C^0_\delta \to L^\infty \) and therefore \( \inf u \) and \( \sup u \) are finite numbers.)

Then:
\[ \| v f(u) \|_{\sigma,q,v} \leq \| v \|_{\sigma,q,v} (\| f(u) \|_\infty + \| f'(u) \|_\infty \| u \|_{s,p,\delta}). \]
Proof. (Lemma 3.37) First we prove the claim for the case $\sigma = 1$. We have

$$
\| v f(u) \|_{1,q,v} \leq \| \langle x \rangle^{-\frac{\eta}{q}} v f(u) \|_{L^q} + \| \langle x \rangle^{-\frac{\eta}{q} + 1} \text{grad}(v f(u)) \|_{L^q} + \| \langle x \rangle^{-\frac{\eta}{q} + 1} v f'(u) \|_{L^q} + \| \langle x \rangle^{-\frac{\eta}{q} + 1} \text{grad}(v) f(u) \|_{L^q}
$$

(note that $f$ is smooth on $[\inf u, \sup u]$ so $f(u) \in L^\infty$, $f'(u) \in L^\infty$)

$$
\leq \| v \|_{1,q,v} \| f(u) \|_{L^\infty} + \| v \|_{1,q,v} \| \text{grad}(u) \|_{L^\infty} + \| v \|_{1,q,v} \| \text{grad}(u) \|_{L^\infty} + \| v \|_{1,q,v} \| \text{grad}(u) \|_{L^\infty}
$$

$$
\leq \| v \|_{1,q,v} \| f(u) \|_{L^\infty} + \| v \|_{1,q,v} \| f(u) \|_{L^\infty} + \| v \|_{1,q,v} \| u \|_{L^\infty} + \| v \|_{1,q,v} \| u \|_{L^\infty} + \| v \|_{1,q,v} \| u \|_{L^\infty} + \| v \|_{1,q,v} \| u \|_{L^\infty}
$$

Now we prove the case $\sigma = -1$ by a duality argument. Note that

$$
\| v f(u) \|_{-1,q,v} = \sup_{\eta \in C_c^\infty} \frac{\| \langle v f(u), \eta \rangle \|_{W^{-1,q}_v \times W^{1,q'}_{-n-v}}}{\| \eta \|_{1,q',-n-v}}.
$$

We have

$$
\frac{\| \langle v f(u), \eta \rangle \|_{W^{-1,q}_v \times W^{1,q'}_{-n-v}}}{\| \eta \|_{1,q',-n-v}} \leq \frac{\| \langle v, f(u) \eta \rangle \|_{W^{-1,q}_v \times W^{1,q'}_{-n-v}}}{\| \eta \|_{1,q',-n-v}} \leq \frac{\| v \|_{-1,q,v} \| f(u) \|_{1,q',-n-v}}{\| \eta \|_{1,q',-n-v}}.
$$

By assumption $\frac{1}{q} < 1 - \theta$, so $\frac{1}{q'} > \theta$ and thus we can apply what was proved for the case
\[ \sigma = 1 \text{ to } \| f(u) \eta \|_{1,q',-n-v}; \]
\[
\frac{\| v \|_{-1,q,v} f(u) \|_{1,q',-n-v} \|}{\eta \|_{1,q',-n-v}} \leq \frac{\| v \|_{-1,q,v} (\| \eta \|_{1,q',-n-v} (\| f(u) \|_{L^\infty} + \| f'(u) \|_{L^\infty} u_{s,p} \)))}{\eta \|_{1,q',-n-v}} = \| v \|_{-1,q,v} (\| f(u) \|_{L^\infty} + \| f'(u) \|_{L^\infty} u_{s,p} ).
\]

Therefore
\[
\| v f(u) \|_{-1,q,v} \leq \| v \|_{-1,q,v} (\| f(u) \|_{L^\infty} + \| f'(u) \|_{L^\infty} u_{s,p} ).
\]

Now we prove the case where \( \sigma \in (-1, 1) \) by interpolation. According to what was proved we have
\[
\| v f(u) \|_{1,q_1,v} \leq \| v \|_{1,q_1,v} (\| f(u) \|_{L^\infty} + \| f'(u) \|_{L^\infty} u_{s,p} ), \tag{3.5}
\]
\[
\| v f(u) \|_{-1,q_2,v} \leq \| v \|_{-1,q_2,v} (\| f(u) \|_{L^\infty} + \| f'(u) \|_{L^\infty} u_{s,p} ), \tag{3.6}
\]

where \( q_1 \) and \( q_2 \) are any two numbers that satisfy \( \theta < \frac{1}{q_1} < 1 \) and \( 0 < \frac{1}{q_2} < 1 - \theta \). Let \( t = \frac{1 - \sigma}{2} \). Clearly \( t \in (0, 1) \). Also note that if we set \( \frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2} \) then
\[
\frac{1}{q_1} > \theta, \ \frac{1}{q_2} > 0 \Rightarrow \frac{1}{q} > (1-t)\theta = \frac{1+\sigma}{2}-\theta.
\]
\[
\frac{1}{q_2} < 1-\delta, \ \frac{1}{q_1} < 1 \Rightarrow \frac{1}{q} < 1-t\theta = 1-\frac{1-\sigma}{2}\theta.
\]

So by choosing appropriate \( q_1 \) and \( q_2 \) we can get any \( q \) with the property that \( \frac{1}{q} \in (\frac{1+\sigma}{2}\theta, 1-\frac{1-\sigma}{2}\theta) \). This implies if \( \frac{1}{q} \in (\frac{1+\sigma}{2}\theta, 1-\frac{1-\sigma}{2}\theta) \) then we may find \( q_1 \) and \( q_2 \) for which inequalities 3.5, 3.6 hold true and moreover
\[
(W^{-1,q_1}_v, W^{-1,q_2}_v)_{t,q} = W^{\sigma,q}_v \quad \text{if } \sigma \neq 0 \quad \text{(real interpolation)}
\]
\[
(W^{-1,q_1}_v, W^{-1,q_2}_v)_{t,q} = W^{\sigma,q}_v \quad \text{if } \sigma = 0 \quad \text{(complex interpolation)}.
\]
So by interpolation we get

\[
\| v f(u) \|_{\sigma, q, \nu} \leq \| v \|_{\sigma, q, \nu} (\| f(u) \|_{\infty} + \| f'(u) \|_{\infty}) \| u \|_{s, p, \delta}).
\]

\[\square\]

### 3.3 More on Duality Pairing

Let \( \hat{h} \) denote the Euclidean metric on \( \mathbb{R}^n \). Let \( \sigma, \delta \in \mathbb{R} \) and \( q \in (1, \infty) \). We denote the duality pairing \( W^{-\sigma, q'}(\mathbb{R}^n) \times W^{\sigma, q}(\mathbb{R}^n) \to \mathbb{R} \) by \( \langle \cdot, \cdot \rangle_{W^{-\sigma, q'}(\mathbb{R}^n) \times W^{\sigma, q}(\mathbb{R}^n)} \) or just \( \langle \cdot, \cdot \rangle_{(\mathbb{R}^n, \hat{h})} \) if the spaces are clear from the context. Clearly the duality pairing is a continuous bilinear map. The restriction of this map to \( C^\infty_c(\mathbb{R}^n) \times C^\infty_c(\mathbb{R}^n) \) is the \( L^2 \) inner product:

\[
\forall u, v \in C^\infty_c(\mathbb{R}^n) \quad \langle u, v \rangle_{(\mathbb{R}^n, \hat{h})} = \int_{\mathbb{R}^n} u v dx.
\]

Now suppose \((M, h)\) is an \( n \)-dimensional AF manifold of class \( W^{\alpha, \gamma}_\rho \) where \( \rho < 0 \) and \( \gamma \in (1, \infty) \). Our claim is that \((W^{\sigma, q}_\delta(M))^*\) can be identified with \( W^{-\sigma, q'}(M) \). This identification can be done in at least two ways which we describe below:

- **First Method:** By using the corresponding AF atlas and the subordinate partition of unity that was used in the Definition 3.12 one can construct a smooth metric \( \hat{h} \) such that \((M, \hat{h})\) is of class \( W^{\alpha, \gamma}_\rho \). Recall that our definition of Sobolev spaces on \( M \) is independent of the underlying metric. The bilinear map \( \langle \cdot, \cdot \rangle_{(M, \hat{h})} : C^\infty_c(M) \times C^\infty_c(M) \to \mathbb{R} \) which is defined by

\[
\langle u, v \rangle_{(M, \hat{h})} = \int_M u v dV_{\hat{h}}
\]
can be uniquely extended to a continuous bilinear form

\[ \langle \cdot, \cdot \rangle_{(M, \hat{h})} : W_{-n-\delta}^{-\sigma, q'}(M) \times W_{\delta}^{\sigma, q}(M) \to \mathbb{R}. \]

The above bilinear map induces a topological isomorphism \((W_{\delta}^{\sigma, q}(M))^* = W_{-n-\delta}^{-\sigma, q'}(M)\);

if \( u, v \) are smooth and \( v \) has compact support in \( U_j \) (domain of a coordinate chart in the AF atlas), then

\[ \langle u, v \rangle_{(M, \hat{h})} = \langle u \circ \phi_j^{-1}, \sqrt{\det \hat{h} v \circ \phi_j^{-1}} \rangle_{(\mathbb{R}^n, \bar{h})}. \]

Note that in the above, \( u \circ \phi_j^{-1} \) represents any extension of \( u \circ \phi_j^{-1} \) from \( W_{-n-\delta}^{-\sigma, q'}(\phi_j(U_j)) \)
to \( W_{-n-\delta}^{-\sigma, q'}(\mathbb{R}^n) \). Also \( v \circ \phi_j^{-1} \) represents the extension of \( v \circ \phi_j^{-1} \in W_{\delta}^{\sigma, q}(\phi_j(U_j)) \) by
zero. Since \( v \) has compact support, we know that \( \sqrt{\det \hat{h} v \circ \phi_j^{-1}} \in W_{\delta}^{\sigma, q}(\mathbb{R}^n) \).

Similarly there exists a continuous bilinear form \( \langle \cdot, \cdot \rangle_{(M, \hat{h})} : W_{-n-\delta}^{-\sigma, q'}(TM) \times W_{\delta}^{\sigma, q}(TM) \to \mathbb{R} \)
whose restriction to \( C_c^\infty(TM) \times C_c^\infty(TM) \) is

\[ (Y, X) \rightarrow \int_M \hat{h}(Y, X) dV_{\hat{h}}. \]

This map induces an isomorphism \((W_{\delta}^{\sigma, q}(TM))^* = W_{-n-\delta}^{-\sigma, q'}(TM)\); if \( X \in W_{\delta}^{\sigma, q}(TM), \)
\( Y \in W_{-n-\delta}^{-\sigma, q'}(TM) \) are smooth and \( X \) has compact support in \( U_j \) then

\[ \langle Y, X \rangle_{(M, \hat{h})} = \sum_{l, p} \langle Y_l \circ \phi_j^{-1}, \sqrt{\det \hat{h} \hat{h}^{lp} X_p \circ \phi_j^{-1}} \rangle_{(\mathbb{R}^n, \bar{h})}. \]

The disadvantage of this method is that the restriction of the bilinear form that was
constructed above to \( C_c^\infty \) is \( \int_M uv dV_{\hat{h}} \) instead of \( \int_M uv dV_{\hat{h}}. \) We prefer to construct
the isomorphism using the rough metric instead of \( \hat{h} \). It turns out that this can be
done for a limited range of \( \sigma \) and \( q \).
**Second Method:** Suppose $\alpha \gamma > n$. Then there exists a continuous function $f$ such that $dV_h = f dV_{\hat{h}}$ and $f - \zeta \in W^{\alpha,\gamma}_{\rho}$ for some constant $\zeta > 0$ [52, 11]. Formally we can write

$$
\langle u, v \rangle_{(M, \hat{h})} = \int_M u v dV_h = \int_M u f v dV_{\hat{h}} = \int_M u f v dV_{\hat{h}} = \langle u, f v \rangle_{(M, \hat{h})}.
$$

This motivates the following definition:

$$
\forall u \in W^{-\sigma, q'}_{-n-\delta} \forall v \in W^{\alpha, q}_{\delta} \langle u, v \rangle_{(M, h)} := \langle u, f v \rangle_{(M, \hat{h})}.
$$

Of course for the above definition to make sense we need to make sure that $f v \in W^{\sigma, q}_{\delta}$. Note that $f - \zeta \in W^{\alpha, \gamma}_{\rho}$ and so by Lemma 3.34 this holds provided

$$
\sigma \in [-\alpha, \alpha] \quad (\sigma \neq -\alpha \text{ if } \alpha \notin \mathbb{N}_0; \sigma \neq \alpha \text{ if } \alpha \notin \mathbb{N}_0 \text{ and } q < \gamma)
$$

$$
\sigma - \frac{n}{q} \in [-n - \alpha + \frac{n}{\gamma}, \alpha - \frac{n}{\gamma}].
$$

It is easy to see that (since $\alpha \gamma > n$) if $\sigma \in [0, \alpha]$ and $q = \gamma$ then the above conditions hold true. Clearly the restriction of $\langle \cdot, \cdot \rangle_{(M, h)}$ to $C^\infty_c \times C^\infty_c$ is given by $\langle u, v \rangle_{(M, h)} = \int_M u v dV_h$. This shows that this bilinear form does not depend on the choice of $\hat{h}$.

The above pairing makes sense even if $u \in W^{-\sigma, q'}_{loc}$ and $v \in W^{\sigma, q}_{loc}$ provided at least one of $u$ or $v$ has compact support.

Similarly for vector fields $X$ and $Y$ formally we may write

$$
\langle Y, X \rangle_{(M, \hat{h})} = \int_M h(Y, X) dV_h = \int_M h_{bc} X^c Y^b f dV_{\hat{h}}
$$

$$
= \int_M \hat{h}_{ad}(f \hat{h}^{ab} h_{bc} X^c) Y^d dV_{\hat{h}} \quad (Y^b = \delta^b_d Y^d = \hat{h}_{ad} \hat{h}^{ab} Y^d)
$$

$$
= \int_M \hat{h}(Y, X_*) dV_{\hat{h}} = \langle Y, X_* \rangle_{(M, \hat{h})} \quad (X_*^a := f \hat{h}^{ab} h_{bc} X^c).\]
This motivates the following definition:

\[ \forall Y \in W_{-n-\delta}^{\alpha,q} \forall X \in W_{\delta}^{\alpha,q} \langle Y, X \rangle_{(M,h)} := \langle Y, X_\ast \rangle_{(M,h)} \quad (W_{\delta}^{\alpha,q} := W_{\delta}^{\alpha,q}(TM)) \]

where \( X_\ast := f \tilde{f}^{ab} h_{bc} X^c \). Again one can check that the above definition makes sense provided \( \sigma \in [-\alpha, \alpha] \) \((\sigma \neq -\alpha \text{ if } \alpha \not\in \mathbb{N}_0; \sigma \neq \alpha \text{ if } \alpha \not\in \mathbb{N}_0 \text{ and } q < \gamma)\), \( \sigma - \frac{n}{q} \in [-n - \alpha + \frac{n}{\gamma}, \alpha - \frac{n}{\gamma}] \). As an example, if \( n = 3 \) and \( \alpha > 1 \) (and of course \( \alpha > \frac{3}{\gamma} \)), then the duality pairing of \( W_{-3-\delta}^{1,2} \) and \( W_{\delta}^{1,2} \) is well-defined:

\[ 1 \in (-\alpha, \alpha), \quad 1 - \frac{3}{2} \in \left[-\alpha + \frac{3}{\gamma}, \alpha - \frac{3}{\gamma} \right] \quad \text{(because } \alpha > \frac{3}{\gamma} \text{)} \]

**Remark 3.38. Order on** \( W_{\delta}^{1,2}(M) \) **for** \( \sigma \in (-\infty, \alpha] \)

As before suppose \((M, h)\) is an \( n \)-dimensional AF manifold of class \( W_{\rho,\gamma}^{\alpha,\gamma} \) where \( \rho < 0, \gamma \in (1, \infty), \) and \( \alpha \gamma > n \).

- If \( \sigma \leq 0 \), then \( W_{\delta}^{\sigma,q} \leftarrow L_{\delta}^q \) and so the elements of \( W_{\delta}^{\sigma,q} \) are ordinary functions (or more precisely, equivalence classes of ordinary functions). In this case we define an order on \( W_{\delta}^{\sigma,q} \) as follows: the functions \( u, v \in W_{\delta}^{\sigma,q} \) satisfy \( u \geq v \) if and only if \( u(x) - v(x) \geq 0 \) for almost all \( x \in M \).

- If \( \sigma \in (0, \alpha) \) \((\sigma \neq \alpha \text{ if } \alpha \not\in \mathbb{N}_0)\), then it is easy to check that the duality pairing \( \langle \cdot, \cdot \rangle_{(M,h)} : W_{-n-\delta}^{\sigma,\gamma}(M) \times W_{\delta}^{\sigma,\gamma}(M) \to \mathbb{R} \) is well-defined. We define an order on \( W_{\delta}^{\sigma,\gamma} \) as follows: the functions \( u, v \in W_{\delta}^{\sigma,\gamma} \) satisfy \( u \geq v \) if and only if \( \langle u - v, \xi \rangle_{(M,h)} \geq 0 \) for all \( \xi \in C_0^{\infty}(M) \text{ with } \xi \geq 0 \). Notice that if \( u \) and \( v \) are ordinary functions in \( W_{\delta}^{\sigma,\gamma}(M) \), then it follows from the definition that \( u \geq v \) if and only if \( u(x) \geq v(x) \) a.e.

According to the above items, if \( \alpha \geq 1 \) we have a well-defined order on \( W_{\delta}^{\alpha-2,\gamma}(M) \) and in particular if \( u \) is an ordinary function in \( W_{\delta}^{\alpha-2,\gamma}(M) \), then \( u \geq 0 \) if and only if \( u(x) \geq 0 \) for almost all \( x \).
3.4 More on Multiplication Properties of Sobolev Spaces

The method that was used to prove Lemma 3.30 can be employed to translate any multiplication property of unweighted Sobolev spaces to a corresponding multiplication property for weighted Sobolev spaces. So in this section we focus on the pointwise multiplication in unweighted Sobolev spaces.

Let $f \in W^{s_1,p_1}$ and $g \in W^{s_2,p_2}$. What can be said about $fg$? To which Sobolev spaces the product $fg$ belongs? This is the question that we want to answer. Why do we care about this question? One of the main applications of such results is in the theory of partial differential equations (PDEs) and in particular elliptic PDEs. As it was pointed out in the Introduction, in the theory of partial differential equations, PDEs are interpreted as equations of the form $Au = f$ where $A$ is an operator between suitable function spaces. In this view, the existence of a unique solution for all right hand sides is equivalent to $A$ being bijective. A main difficulty is in choosing the domain of realization of the operator $A$, that is, choosing appropriate function spaces $X$ and $Y$ such that

1. $A$ can be considered as an operator from $X$ to $Y$ and $f \in Y$, i.e., we need to ensure that the equation makes sense if we consider $X$ and $Y$ as the domain and codomain of $A$.

2. $A$ (or a family of approximations of $A$) has “nice” properties as an operator (or a family of operators) from $X$ to $Y$. Here “nice properties” may refer to any of the following properties: $A$ is continuous, $A$ is compact, $A$ is Fredholm, $A$ is injective, $A$ is surjective, $A$ satisfies a maximum principle, etc.

As it was pointed out in the beginning of this chapter, for elliptic equations, using Sobolev spaces (or weighted Sobolev spaces) as domain and codomain of $A$ help us to ensure that $A$ has “nice” properties. But how to determine appropriate Sobolev spaces
to make sure that the equation makes sense? This is one of the places where pointwise multiplication theorems come in handy. The best way to see this is by looking at a very simple example. Consider the equation $-\Delta u + V u = f$ on $\Omega \subseteq \mathbb{R}^n$. Suppose we want to seek the unknown function $u$ in the Sobolev space $W^{s,p}$. Having this assumption, what restrictions do we need to impose on the data $V$ and $f$? The assumption $u \in W^{s,p}$ implies that $-\Delta u \in W^{s-2,p}$. Therefore for the equation to make sense (as an equality in $W^{s-2,p}$), $f$ and $V u$ must belong to $W^{s-2,p}$. So now we need to find those Sobolev spaces $W^{r,q}$ such that if $V \in W^{r,q}$, then $V u \in W^{s-2,p}$. That is we need to find those exponents $r$ and $q$ for which the product of a function in $W^{r,q}$ and a function in $W^{s,p}$ belongs to $W^{s-2,p}$.

There are a number of articles and book chapters that are devoted to the study of pointwise multiplication in function spaces, e.g. [75, 67]. Unfortunately most references study the question in the general setting of Triebel-Lizorkin spaces and use technical tools from Littlewood-Paley theory and theory of Besov spaces to prove the results. The main feature of this section is that the key results are proved without any direct reference to Littlewood-Paley theory and Besov spaces which makes it accessible to a wider range of readers. In particular, we give alternative proofs for a number of results first stated in [75] for Sobolev spaces with nonnegative exponents. Also we extend those results to Sobolev spaces with negative exponents. We clearly distinguish between the case of Sobolev spaces defined on the entire $\mathbb{R}^n$ and the case where Sobolev spaces are defined on a bounded domain.

In Section 3.4.1 we will go over some of the basic well-known facts about Sobolev spaces and interpolation theory. The very short discussion that we will have about interpolation theory in Section 3.4.1 is enough to completely understand the proofs. In Section 3.4.2 we state and prove the main theorems.
3.4.1 Important Properties

We begin with reviewing the basic definitions of interpolation theory in Banach spaces. A detailed discussion can be found in [72].

A pair \( \{A_0, A_1\} \) of two Banach spaces is said to be an interpolation couple, if both spaces are continuously embedded in a common Hausdorff topological vector space \( A \). We may consider the following two subspaces:

- \( A_0 \cap A_1 \), and
- \( A_0 + A_1 := \{a \in A : \exists a_0 \in A_0, \exists a_1 \in A_1, a = a_0 + a_1\} \).

Equipped with the norms

\[
\|a\|_{A_0 \cap A_1} := \min\{\|a\|_{A_0}, \|a\|_{A_1}\}
\]
\[
\|a\|_{A_0 + A_1} := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\} \quad (\text{here } 0 < t < \infty)
\]

\( A_0 \cap A_1 \) and \( A_0 + A_1 \) become Banach spaces. Real interpolation and complex interpolation are two, generally nonequivalent, methods for constructing intermediate spaces between \( A_0 \) and \( A_1 \) in the sense that the new space lies between \( A_0 \cap A_1 \) and \( A_0 + A_1 \) (with continuous injections).

- Given a pair \((\theta, p)\) with \( 0 < \theta < 1 \) and \( 1 < p < \infty \), the real interpolation functor constructs an intermediate Banach space denoted by \((A_0, A_1)_{\theta, p}\).

- Given \( 0 < \theta < 1 \), the complex interpolation functor constructs an intermediate Banach space denoted by \([A_0, A_1]_{\theta}\).

**Theorem 3.39.** [72][Real Interpolation] Let \( \Omega \) be a bounded open set with smooth boundary in \( \mathbb{R}^n \) or \( \Omega = \mathbb{R}^n \). Suppose \( \theta \in (0, 1) \), \( 0 \leq s_0, s_1 < \infty \), and \( 1 < p_0, p_1 < \infty \). Additionally assume one of the following cases holds:
• $s_0, s_1, s$ are nonintegers.

• $s_0 \in \mathbb{R}, s_1 \in \mathbb{Z},$ and $s \in \mathbb{R} \setminus \mathbb{Z}$.

If

$$s = (1 - \theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},$$

then $W^{s,p}(\Omega) = (W^{s_0,p_0}(\Omega), W^{s_1,p_1}(\Omega))_{\theta,p}$.

**Theorem 3.40.** [72] [Complex Interpolation] Let $\Omega$ be a bounded open set with smooth boundary in $\mathbb{R}^n$ or $\Omega = \mathbb{R}^n$. Suppose $\theta \in (0, 1)$, $0 \leq s_0, s_1 < \infty$, and $1 < p_0, p_1 < \infty$. If

$$s = (1 - \theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},$$

then

• $H^{s,p}(\Omega) = [H^{s_0,p_0}(\Omega), H^{s_1,p_1}(\Omega)]_{\theta}$.

• $W^{s,p}(\Omega) = [W^{s_0,p_0}(\Omega), W^{s_1,p_1}(\Omega)]_{\theta}$ provided $s_0, s_1, s > 0$ are nonintegers.

• $W^{s,p}(\Omega) \hookrightarrow [W^{s_0,p_0}(\Omega), W^{s_1,p_1}(\Omega)]_{\theta}$ provided $s_0$ and $s_1$ are not integers and $p \geq 2$.

(This is a consequence of the fact that for $s_0, s_1 \not\in \mathbb{Z}$, $[W^{s_0,p_0}(\Omega), W^{s_1,p_1}(\Omega)]_{\theta} = B_{p,p}^s$.

If $s \not\in \mathbb{Z}$, then $B_{p,p}^s = W^{s,p}$; if $s \in \mathbb{Z}$, then $W^{s,p} \hookrightarrow B_{p,p}^s$ provided $p \geq 2$.)

**Remark 3.41.** According to [72], the above interpolation facts remain true even if we only assume the bounded open set $\Omega$ is of cone-type. According to [1] if $\Omega$ is a bounded open set with Lipschitz continuous boundary, then it is of cone-type.

**Theorem 3.42** (Interpolation Properties of Bilinear Forms). [72] Let $A_0 \subseteq A_1$, $B_0 \subseteq B_1$, and $C_0 \subseteq C_1$ be couples of Banach spaces. If $T_1 : A_1 \times B_1 \to C_1$ is a continuous bilinear map that restricts to a continuous bilinear map $T_0 : A_0 \times B_0 \to C_0$, then $T_1$ also restricts to a continuous bilinear map.
• (complex interpolation) from \([A_0, A_1]_\theta \times [B_0, B_1]_\theta\) into \([C_0, C_1]_\theta\), and

• (real interpolation) from \((A_0, A_1)_\theta, p \times (B_0, B_1)_\theta, q\) into \((C_0, C_1)_\theta, r\)

where \(0 < \theta < 1\) and \(\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0\).

**Theorem 3.43** (Extension Property). [8] Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set with Lipschitz continuous boundary. Then for all \(s > 0\) and for \(1 \leq p < \infty\), there exists a continuous linear extension operator \(P : W^{s,p}(\Omega) \hookrightarrow W^{s,\infty}(\mathbb{R}^n)\) such that \((Pu)|_\Omega = u\) and \(\|Pu\|_{W^{s,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{s,p}(\Omega)}\) for some constant \(C\) that may depend on \(s, p,\) and \(\Omega\) but is independent of \(u\).

**Theorem 3.44** (Embedding Theorem I). [8, 72, 75] Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^n\) with Lipschitz continuous boundary or \(\Omega = \mathbb{R}^n\). Suppose \(1 \leq p \leq q < \infty\) and \(0 \leq t \leq s\) satisfy \(s - \frac{n}{p} \geq t - \frac{n}{q}\). Then

- \(W^{s,p}(\Omega) \hookrightarrow W^{t,q}(\Omega)\),

- \(H^{s,p}(\mathbb{R}^n) \hookrightarrow H^{t,q}(\mathbb{R}^n)\) provided we assume \(p > 1\).

**Theorem 3.45** (Embedding Theorem II). [33] Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^n\) with Lipschitz continuous boundary or \(\Omega = \mathbb{R}^n\).

i. If \(sp > n\), then \(W^{s,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \cap C^0(\Omega)\) and \(W^{s,p}(\Omega)\) is a Banach algebra.

ii. If \(sp = n\), then \(W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)\) for \(p \leq q < \infty\).

iii. If \(0 \leq sp < n\), then \(W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)\) for \(p \leq q \leq \frac{np}{n-sp}\).

(Items (ii) and (iii) are direct consequences of Theorem 3.44.)

**Theorem 3.46** (Embedding Theorem III). Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^n\) with Lipschitz continuous boundary. Suppose \(1 \leq p, q < \infty\) (\(p\) does NOT need to be less than \(q\)) and \(0 \leq t \leq s\) satisfy \(s - \frac{n}{p} \geq t - \frac{n}{q}\). Then \(W^{s,p}(\Omega) \hookrightarrow W^{t,q}(\Omega)\).
Proof. (Theorem 3.46) If $p \leq q$, the claim follows from Theorem 3.44. So we may assume $p > q$. We consider three cases:

- **Case 1** $s = t = k \in \mathbb{N}_0$: Note that since $\Omega$ is a bounded open set, $L^p(\Omega) \hookrightarrow L^q(\Omega)$.

  We can write

  $$
  \| u \|_{W^{k,q}(\Omega)} = \sum_{|\beta| \leq k} \| \partial^\beta u \|_{L^q(\Omega)} \leq \sum_{|\beta| \leq k} \| \partial^\beta u \|_{L^p(\Omega)} = \| u \|_{W^{k,p}(\Omega)},
  $$

  which precisely means that $W^{k,p}(\Omega) \hookrightarrow W^{k,q}(\Omega)$.

- **Case 2** $s = t \not\in \mathbb{N}_0$: Let $k = \lfloor s \rfloor$, $\theta = s - k$. By what was shown in the previous case

  $$
  W^{k,p}(\Omega) \hookrightarrow W^{k,q}(\Omega), \quad W^{k+1,p}(\Omega) \hookrightarrow W^{k+1,q}(\Omega).
  $$

  Since $s = (1 - \theta)k + \theta(k + 1)$, the claim follows from real interpolation.

- **Case 3 General case (of course $p > q$)**: By what was shown in the previous steps we know that $W^{s,p}(\Omega) \hookrightarrow W^{s,q}(\Omega)$ and by Theorem 3.44 $W^{s,q}(\Omega) \hookrightarrow W^{t,q}(\Omega)$.

  Consequently $W^{s,p}(\Omega) \hookrightarrow W^{t,q}(\Omega)$.

\[ \square \]

### 3.4.2 Main Theorems

Before stating the main theorems, we discuss a simple case which demonstrates that multiplication properties of Sobolev-Slobodeckij spaces can be quite counterintuitive.

**Notation**: Let $A_i$ and $B_i$ ($i = 1, 2$) and $C$ be sobolev spaces.

- By writing $A_1 \times A_2 \hookrightarrow B_1 \times B_2$ we merely mean that $A_1 \times A_2 \subseteq B_1 \times B_2$ and if $u \in A_1$ and $v \in A_2$, then $\| u \|_{B_1} \| v \|_{B_2} \leq \| u \|_{A_1} \| v \|_{A_2}$. ($A_1 \times A_2 = \{a_1a_2 : a_1 \in A_1, a_2 \in A_2\}$)
• By writing \( B_1 \times B_2 \hookrightarrow C \) we mean that \( B_1 \times B_2 \subseteq C \) and if \( u \in B_1 \) and \( v \in B_2 \), then 
\[ ||uv||_C \leq ||u||_{B_1} ||v||_{B_2}. \]

**Theorem 3.47.** Suppose \( k \in \mathbb{N}_0 \), and \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \). Then

\[ W^{k,p_1}(\mathbb{R}^n) \times W^{k,p_2}(\mathbb{R}^n) \hookrightarrow W^{k,p}(\mathbb{R}^n). \]

More generally, if \( s \geq 0 \), then

\[ H^{s,p_1}(\mathbb{R}^n) \times H^{s,p_2}(\mathbb{R}^n) \hookrightarrow H^{s,p}(\mathbb{R}^n). \]

**Proof.** (Theorem 3.47) For \( k \in \mathbb{N}_0 \) the claim is a direct consequence of the definition of Sobolev norm, Leibniz formula \( (\partial^\alpha(uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha - \beta} u \partial^\beta v) \), and the Holder's inequality for Lebesgue spaces.

If \( s \not\in \mathbb{N}_0 \), then let \( k = \lfloor s \rfloor \) and \( \theta = s - k \). We have

\[ H^{k,p_1}(\mathbb{R}^n) \times H^{k,p_2}(\mathbb{R}^n) \hookrightarrow H^{k,p}(\mathbb{R}^n), \]
\[ H^{k+1,p_1}(\mathbb{R}^n) \times H^{k+1,p_2}(\mathbb{R}^n) \hookrightarrow H^{k+1,p}(\mathbb{R}^n). \]

Since

\[ H^{s,p} = [H^{k,p}, H^{k+1,p}]_0, \quad H^{s,p_1} = [H^{k,p_1}, H^{k+1,p_1}]_0, \quad H^{s,p_2} = [H^{k,p_2}, H^{k+1,p_2}]_0 \]

the claim follows from complex interpolation.

Now we ask the following question: does the claim of Theorem 3.47 hold true for Sobolev-Slobodeckij spaces? More specifically, suppose \( s > 0, s \not\in \mathbb{Z} \), and \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \). Can we conclude that \( W^{s,p_1}(\mathbb{R}^n) \times W^{s,p_2}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n) \)? Surprisingly, the answer is **NO**! In what follows we will prove that if \( s \not\in \mathbb{Z} \) (and of course \( s > 0 \)) for \( W^{s,p_1}(\mathbb{R}^n) \times
\( W^{s_2, p_2}(\mathbb{R}^n) \hookrightarrow W^{s, p}(\mathbb{R}^n) \) to be true it is necessary to have \( p_1 \leq p \).

**Lemma 3.48.** Suppose \( s > 0 \) is given. Let \( f \in S(\mathbb{R}^n) \) be a function such that

\[
\text{supp } \mathcal{F} f \subseteq \{ \xi : |\xi| < \epsilon \}, \quad f \neq 0.
\]

If \( \epsilon \) is sufficiently small, then there exists a sequence of functions \( \{g_N\}_{N=1}^\infty \) (each \( g_N \) depends on \( s \)) such that for any \( p, q > 1 \)

\[
\| g_N \|_{F^s_{p, q}} = N^{\frac{1}{q}} \| f \|_p \quad \text{and} \quad \| g_N f \|_{F^s_{p, q}} = N^{\frac{1}{q}} \| f^2 \|_p.
\]

The construction of \( g_N \)'s is based on the Littlewood-Paley characterization of Triebel-Lizorkin spaces and can be found in [67].

**Proposition 3.49.** Suppose \( s, s_2 \geq 0, s \not\in \mathbb{Z} \) and \( p_1, p_2, p > 1 \). If \( W^{s, p_1}(\mathbb{R}^n) \times W^{s_2, p_2}(\mathbb{R}^n) \hookrightarrow W^{s, p}(\mathbb{R}^n) \), then \( p_1 \leq p \).

**Proof.** (Proposition 3.49) Note that, since \( s \not\in \mathbb{Z} \), we have \( W^{s, p} = F^s_{p, p} \). Consider the product of \( f \) and \( g_N \); by assumption we must have

\[
\| g_N f \|_{W^{s, p_2}} \leq \| g_N \|_{W^{s, p_1}} \| f \|_{W^{s_2, p_2}}
\]

where the implicit constant is independent of \( N \). Therefore

\[
N^{\frac{1}{p}} \| f^2 \|_p \leq N^{\frac{1}{p_1}} \| f \|_{p_1} \| f \|_{W^{s_2, p_2}}.
\]

So for all \( N \in \mathbb{N} \)

\[
0 < \frac{\| f^2 \|_p}{\| f \|_{p_1} \| f \|_{W^{s_2, p_2}}} \leq N^{\frac{1}{p_1} - \frac{1}{p}},
\]

Which implies that \( p_1 \leq p \). \( \square \)
We start our main theorems by a theorem on multiplication in spaces $H^{s,p}(\mathbb{R}^n)$ with $s \geq 0$. The reason that we begin with a theorem on Bessel potential spaces is that although for these spaces the situation is considerably simpler (comparing to Sobolev-Slobodeckij spaces), it showcases the main ideas without encountering technical difficulties. The aforementioned simplicity is due to the fact that we have a uniform formula for $[H^{s_0,p_0}, H^{s_1,p_1}]_\theta$ regardless of whether each of $s_0$, $s_1$, or $(1-\theta)s_0 + \theta s_1$ is an integer or not.

**Theorem 3.50** (Pointwise multiplication in spaces $H^{s,p}(\mathbb{R}^n)$ with $s \geq 0$). Assume $s_i, s$ and $1 < p_i \leq p < \infty$ ($i = 1, 2$) are real numbers satisfying

i. $s_i \geq s$

ii. $s \geq 0$,

iii. $s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p})$

iv. $s_1 + s_2 - s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p})$.

Claim: If $u \in H^{s_1,p_1}(\mathbb{R}^n)$ and $v \in H^{s_2,p_2}(\mathbb{R}^n)$, then $uv \in H^{s,p}(\mathbb{R}^n)$ and moreover the pointwise multiplication of functions is a continuous bilinear map

$$H^{s_1,p_1}(\mathbb{R}^n) \times H^{s_2,p_2}(\mathbb{R}^n) \to H^{s,p}(\mathbb{R}^n).$$

**Proof.** *(Theorem 3.50)* Our proof consists of two steps. In the first step we consider the special case $p_1 = p_2 = p$ and then in the second step we prove the general case based on the special case that is proved in Step 1.

- **Step 1:** Here we want to prove the theorem for the special case $p = p_1 = p_2$. In this
case the assumptions can be rewritten as follows:

\[ s_1, s_2 \geq s \geq 0, \quad s_1 + s_2 - s > \frac{n}{p}. \]

In order to proceed, we state and prove a simple lemma.

**Lemma 3.51.**

\[
\forall \epsilon > 0 \quad \forall t \in \left[0, \frac{n}{p}\right] \quad H^{l,p}(\mathbb{R}^n) \times H^{n+\epsilon,p}(\mathbb{R}^n) \hookrightarrow H^{l,p}(\mathbb{R}^n). 
\]

\[
\forall \epsilon > 0 \quad \forall t \in \left[0, \frac{n}{p}\right] \quad H^{n+\epsilon,p}(\mathbb{R}^n) \times H^{l,p}(\mathbb{R}^n) \hookrightarrow H^{l,p}(\mathbb{R}^n). 
\]

**Proof of the Lemma** Clearly it is enough to prove the first statement. Let \( \epsilon > 0 \) be given. Since \( \frac{n}{p} + \epsilon > \frac{n}{p} \), \( H^{n+\epsilon,p}(\mathbb{R}^n) \) is an algebra and

\[
H^{n+\epsilon,p}(\mathbb{R}^n) \times H^{n+\epsilon,p}(\mathbb{R}^n) \hookrightarrow H^{n+\epsilon,p}(\mathbb{R}^n). \tag{3.7}
\]

Also \( H^{n+\epsilon,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \). Hence

\[
H^{n+\epsilon,p}(\mathbb{R}^n) \times H^{0,p}(\mathbb{R}^n) \hookrightarrow H^{0,p}(\mathbb{R}^n) \quad (L^\infty \times L^p \hookrightarrow L^p). \tag{3.8}
\]

By complex interpolation between (3.7) and (3.8) we get

\[
\forall \theta \in [0, 1] \quad H^{n+\epsilon,p}(\mathbb{R}^n) \times H^{0(\frac{n}{p}+\epsilon),p}(\mathbb{R}^n) \hookrightarrow H^{\theta(\frac{n}{p}+\epsilon),p}(\mathbb{R}^n)
\]

which clearly implies the claim.

Now using the above lemma we can prove the theorem for the special case \( p = p_1 = p_2 \).

To this end we consider two cases:
Case 1 \( s > \frac{n}{p} \): If \( s > \frac{n}{p} \), then \( H^{s,p}(\mathbb{R}^n) \) is an algebra and we can write

\[
H^{s_1,p_1}(\mathbb{R}^n) \times H^{s_2,p_2}(\mathbb{R}^n) \hookrightarrow H^{s,p}(\mathbb{R}^n) \quad \text{(by assumption } s_1, s_2 \geq s) \\
\hookrightarrow H^{s,p}(\mathbb{R}^n).
\]

Case 2 \( s \leq \frac{n}{p} \): Let \( \epsilon = s_1 + s_2 - s - \frac{n}{p} > 0 \). By Lemma 3.51 we have

\[
H^{s,p}(\mathbb{R}^n) \times H^{n+\epsilon,p}(\mathbb{R}^n) \hookrightarrow H^{s,p}(\mathbb{R}^n). \tag{3.9}
\]

\[
H^{n+\epsilon,p}(\mathbb{R}^n) \times H^{s,p}(\mathbb{R}^n) \hookrightarrow H^{s,p}(\mathbb{R}^n). \tag{3.10}
\]

Note that

\[
s \leq s_2 \implies s_1 \leq s_1 + s_2 - s \implies s_1 \leq \frac{n}{p} + \epsilon.
\]

So there exists \( \theta \in [0,1] \) such that \((1-\theta)s + \theta\left(\frac{n}{p} + \epsilon\right) = s_1\). Clearly

\[
[(1-\theta)s + \theta\left(\frac{n}{p} + \epsilon\right)] + [(1-\theta)(\frac{n}{p} + \epsilon) + \theta s] = s + \frac{n}{p} + \epsilon = s_1 + s_2.
\]

That is, \( s_1 + [(1-\theta)(\frac{n}{p} + \epsilon) + \theta s] = s_1 + s_2 \) which means that \((1-\theta)(\frac{n}{p} + \epsilon) + \theta s = s_2\).

Consequently

\[
[H^{s,p}(\mathbb{R}^n), H^{n+\epsilon,p}(\mathbb{R}^n)]_\theta = H^{s_1,p}(\mathbb{R}^n), \quad [H^{n+\epsilon,p}(\mathbb{R}^n), H^{s,p}(\mathbb{R}^n)]_\theta = H^{s_2,p}(\mathbb{R}^n).
\]

So using complex interpolation and (3.9), (3.10) we get

\[
H^{s_1,p_1}(\mathbb{R}^n) \times H^{s_2,p_2}(\mathbb{R}^n) \rightarrow H^{s,p}(\mathbb{R}^n).
\]
• Step 2: Now we are in the position to prove the general case. Let

\[ \tilde{s}_1 = s_1 - \frac{n}{p_1} + \frac{n}{p}, \quad \tilde{s}_2 = s_2 - \frac{n}{p_2} + \frac{n}{p}. \]

We just need to prove the following claim:

Claim:

i. \( H^{\tilde{s}_1, p}(\mathbb{R}^n) \times H^{\tilde{s}_2, p}(\mathbb{R}^n) \hookrightarrow H^{s, p}(\mathbb{R}^n). \)

ii. \( H^{s_1, p_1}(\mathbb{R}^n) \hookrightarrow H^{\tilde{s}_1, p}(\mathbb{R}^n). \)

iii. \( H^{s_2, p_2}(\mathbb{R}^n) \hookrightarrow H^{\tilde{s}_2, p}(\mathbb{R}^n). \)

Indeed, if we prove the above claim, then

\[ H^{s_1, p_1}(\mathbb{R}^n) \times H^{s_2, p_2}(\mathbb{R}^n) \hookrightarrow H^{\tilde{s}_1, p}(\mathbb{R}^n) \times H^{\tilde{s}_2, p}(\mathbb{R}^n) \hookrightarrow H^{s, p}(\mathbb{R}^n). \]

• Proof of (i): By step 1 we need to check the following items:

\[ \tilde{s}_1 \geq s \quad (\text{true because } s_1 - s \geq n(\frac{1}{p_1} - \frac{1}{p})) \]
\[ \tilde{s}_2 \geq s \quad (\text{true because } s_2 - s \geq n(\frac{1}{p_2} - \frac{1}{p})) \]
\[ \tilde{s}_1 + \tilde{s}_2 - s > \frac{n}{p} \]

The last item is true because

\[ s_1 + s_2 - s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \implies (s_1 - \frac{n}{p_1} + \frac{n}{p}) + (s_2 - \frac{n}{p_2} + \frac{n}{p}) - s > \frac{n}{p} \implies \tilde{s}_1 + \tilde{s}_2 - s > \frac{n}{p}. \]

• Proof of (ii): According to the embedding theorem we must check the following
items:

\[ p_1 \leq p \quad \text{(true by assumption)} \]

\[ s_1 \geq \tilde{s}_1 \quad \text{(true because } p \geq p_1 \Rightarrow \frac{n}{p_1} \geq \frac{n}{p} \Rightarrow s_1 \geq \tilde{s}_1 - \frac{n}{p_1} + \frac{n}{p} ) \]

\[ s_1 - \frac{n}{p_1} \geq \tilde{s}_1 - \frac{n}{p} \quad \text{(true because } \tilde{s}_1 - \frac{n}{p} = s_1 - \frac{n}{p_1} + \frac{n}{p} = s_1 - \frac{n}{p_1} ) \]

- **Proof of (iii):** Completely analogous to the proof of the previous item!

\[ \square \]

**Remark 3.52.** Theorem 3.50 remains true if we replace \( H^{s,p} \) with \( B^s_{p,p} \).

**Theorem 3.53** (Pointwise multiplication in spaces \( W^{s,p}(\mathbb{R}^n) \) with \( s \in \mathbb{N}_0 \)). Let \( s_i, s \) and \( 1 \leq p, p_i < \infty \) \((i = 1, 2)\) be real numbers satisfying

1. \( s_i \geq s \geq 0 \)

2. \( s \in \mathbb{N}_0, \)

3. \( s_i - s \geq n \left( \frac{1}{p_i} - \frac{1}{p} \right), \)

4. \( s_1 + s_2 - s > n \left( \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} \right) \geq 0. \)

where the strictness of the inequalities in items (iii) and (iv) can be interchanged.

**Claim:** If \( u \in W^{s_1,p_1}(\mathbb{R}^n) \) and \( v \in W^{s_2,p_2}(\mathbb{R}^n) \), then \( uv \in W^{s,p}(\mathbb{R}^n) \) and moreover the pointwise multiplication of functions is a continuous bilinear map

\[ W^{s_1,p_1}(\mathbb{R}^n) \times W^{s_2,p_2}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n). \]

Note: We do not require \( p_i \) to be less than or equal to \( p \) in the statement of the theorem.
Proof. (Theorem 3.53) Let \( u \in W^{s_1, p_1}(\mathbb{R}^n) \) and \( v \in W^{s_2, p_2}(\mathbb{R}^n) \). Our goal is to prove that \( \| uv \|_{s, p} \leq \| u \|_{s_1, p_1} \| v \|_{s_2, p_2} \). We have

\[
\| uv \|_{s, p} = \sum_{|\alpha| \leq s} \| \partial^{\alpha} (uv) \|_p .
\]

So it is enough to prove that for all \(|\alpha| \leq s\), \( \| \partial^{\alpha} (uv) \|_p \leq \| u \|_{s_1, p_1} \| v \|_{s_2, p_2} \). For now let's assume \( v \in C_\infty(\mathbb{R}^n) \). So we are allowed to use the Leibniz formula [1] to write

\[
\partial^{\alpha} (uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha - \beta} u \partial^\beta v .
\]

Thus we just need to show that

\[
\forall |\alpha| \leq s \quad \forall \beta \leq \alpha \quad \| \partial^{\alpha - \beta} u \partial^\beta v \|_p \leq \| u \|_{s_1, p_1} \| v \|_{s_2, p_2} .
\]

Fix \( \alpha, \beta \in \mathbb{N}_0^n \) such that \(|\alpha| \leq s\) and \( \beta \leq \alpha \). In what follows we will prove the following claim:

Claim: There exist \( r \in [1, \infty] \) and \( q \in [1, \infty] \) such that

\[
\frac{1}{r} + \frac{1}{q} = \frac{1}{p}, \quad W^{s_1 - |\alpha - \beta|, p_1}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n), \quad W^{s_2 - |\beta|, p_2}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n). \quad (3.11)
\]

For the moment, let's assume the above claim is true. Then

\[
\begin{align*}
            u & \in W^{s_1, p_1}(\mathbb{R}^n) \implies \partial^{\alpha - \beta} u \in W^{s_1 - |\alpha - \beta|, p_1}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n), \\
v & \in W^{s_2, p_2}(\mathbb{R}^n) \implies \partial^\beta v \in W^{s_2 - |\beta|, p_2}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n),
\end{align*}
\]
and therefore

\[
\| \partial^{a-\beta} u \partial^\beta v \|_{p} \leq \| \partial^{a-\beta} u \|_{r} \| \partial^\beta v \|_{q} \leq \| \partial^{a-\beta} u \|_{s_1-|\alpha-\beta|, p_1} \| \partial^\beta v \|_{s_2-|\beta|, p_2} \\
\leq \| u \|_{s_1, p_1} \| v \|_{s_2, p_2}.
\]

So it is enough to prove the above claim. We consider two cases separately:

**Case 1:** \( s_i - s > n \left( \frac{1}{p_i} - \frac{1}{p} \right) \) \((i = 1, 2)\) and \( s_1 + s_2 - s \geq n \left( \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} \right)\).

As a direct consequence of assumptions we have

\[
\frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n} \leq \frac{1}{p_1} - \frac{s_1 - s}{n} < \frac{1}{p}, \quad \text{(3.12)}
\]

\[
\frac{1}{p_2} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n} \geq \frac{1}{p_2} - \frac{s_2 - s}{n} > 0. \quad \text{(3.13)}
\]

In what follows we will show that there exist \( r \in [1, \infty) \) and \( q \in [1, \infty) \) that satisfy (3.11).

According to Theorem 3.44 it is enough to show that there exist \( r \) and \( q \) that satisfy the following conditions:

\[
0 < \frac{1}{r} \leq 1, \quad 0 < \frac{1}{q} \leq 1, \quad \frac{1}{r} + \frac{1}{q} = \frac{1}{p},
\]

\[
\frac{1}{r} \leq \frac{1}{p_1}, \quad \frac{1}{q} \leq \frac{1}{p_2}, \quad s_1 - |\alpha - \beta| - \frac{n}{p_1} \geq 0 - \frac{n}{r}, \quad s_2 - |\beta| - \frac{n}{p_2} \geq 0 - \frac{n}{q}.
\]

In fact if we let \( R = \frac{1}{r} \) and \( Q = \frac{1}{q} \), then our goal is to show that there exist \( 0 < R \leq 1 \) and \( 0 < Q \leq 1 \) such that

\[
R + Q = \frac{1}{p}, \quad \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n} \leq R \leq \frac{1}{p_1}, \quad \frac{1}{p_2} - \frac{s_2 - |\beta|}{n} \leq Q \leq \frac{1}{p_2}.
\]

Note that since \( \frac{1}{p_1} \leq 1 \) and \( \frac{1}{p_2} \leq 1 \), conditions \( R \leq 1 \) and \( Q \leq 1 \) are superfluous. So we
need to show that there exists $0 < R < \frac{1}{p}$ such that

$$
\frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n} \leq R \leq \frac{1}{p_1},
$$

$$
\frac{1}{p_2} - \frac{s_2 - |\beta|}{n} \leq -\frac{1}{p} - R \leq \frac{1}{p_2} \quad (\iff \frac{1}{p} - \frac{1}{p_2} \leq R \leq \frac{1}{p} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n}).
$$

Consequently it is enough to show that the following intersection is nonempty:

$$
(0, \frac{1}{p}) \cap \left[ \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n}, \frac{1}{p_1} \right] \cap \left[ \frac{1}{p} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n}, \frac{1}{p_2} \right].
$$

Note that by (3.12), $\frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n} < \frac{1}{p}$ and so the first intersection is nonempty. We may consider four cases:

i) $\frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n} \leq 0, \frac{1}{p_1} < \frac{1}{p}$,

$$(0, \frac{1}{p}) \cap \left[ \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n}, \frac{1}{p_1} \right] = (0, \frac{1}{p_1}).$$

Now note that by assumption $\frac{1}{p_1} \geq \frac{1}{p} - \frac{1}{p_2}$ and also by (3.13), $\frac{1}{p} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n} > 0$.

Hence

$$(0, \frac{1}{p_1}) \cap \left[ \frac{1}{p} - \frac{1}{p_2}, \frac{1}{p} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n} \right] \neq \emptyset$$

ii) $\frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n} \leq 0, \frac{1}{p_1} \geq \frac{1}{p}$,

$$(0, \frac{1}{p}) \cap \left[ \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n}, \frac{1}{p_1} \right] = (0, \frac{1}{p}).$$

Clearly $\frac{1}{p} > \frac{1}{p} - \frac{1}{p_2}$ and also by (3.13), $\frac{1}{p} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n} > 0$. Hence

$$(0, \frac{1}{p}) \cap \left[ \frac{1}{p} - \frac{1}{p_2}, \frac{1}{p} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n} \right] \neq \emptyset$$
iii \( \frac{1}{p_1} - \frac{1}{n} \frac{s_1 - |\alpha - \beta|}{n} > 0, \frac{1}{p} \leq \frac{1}{p_1} \):

\[
(0, \frac{1}{p}) \cap \left[ \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n}, \frac{1}{p_1} \right] = \left[ \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n}, \frac{1}{p_1} \right].
\]

Clearly \( \frac{1}{p} > \frac{1}{p_1} - \frac{1}{p_2} \) and also by assumption \( s_1 + s_2 - s \geq n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \) and so \( \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n} \leq \frac{1}{p} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n} \). Consequently

\[
\left[ \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n}, \frac{1}{p_1} \right] \cap \left[ \frac{1}{p} - \frac{1}{p_2}, \frac{1}{p} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n} \right] \neq \emptyset
\]

iv \( \frac{1}{p_1} - \frac{1}{n} \frac{s_1 - |\alpha - \beta|}{n} > 0, \frac{1}{p} < \frac{1}{p_1} \):

\[
(0, \frac{1}{p}) \cap \left[ \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n}, \frac{1}{p_1} \right] = \left[ \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n}, \frac{1}{p_1} \right].
\]

By assumption \( \frac{1}{p_1} \geq \frac{1}{p} - \frac{1}{p_2} \) and also (exactly the same as the previous item) \( \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n} \leq \frac{1}{p} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n} \). Consequently

\[
\left[ \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n}, \frac{1}{p_1} \right] \cap \left[ \frac{1}{p} - \frac{1}{p_2}, \frac{1}{p} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n} \right] \neq \emptyset
\]

Case 2: \( s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p}) (i = 1, 2) \) and \( s_1 + s_2 - s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \).

If \( s_i - s > n(\frac{1}{p_i} - \frac{1}{p}) (i = 1, 2) \), then the proof of previous case works. So we just need to consider the following cases:

i. \( s_1 - s = n(\frac{1}{p_1} - \frac{1}{p}), s_2 - s \neq n(\frac{1}{p_2} - \frac{1}{p}) \): If \( |\alpha - \beta| < s \), then the proof of Case 1 works.

In fact note that the proof of Case 1 was based on the inequalities \( \frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n} < \frac{1}{p} \) and \( \frac{1}{p} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n} > 0 \) ((3.12) and (3.13)) and both inequalities hold true in this case: the second inequality is true because as in Case 1 \( s_2 - s > n(\frac{1}{p_2} - \frac{1}{p}) \), and the
first inequality is true because
\[
\frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n} < \frac{1}{p_1} - \frac{s_1 - s}{n} \leq \frac{1}{p}.
\]

So we may assume $|\alpha - \beta| = s$. Since $|\alpha| \leq s$ and $\beta \leq \alpha$, this is possible only if $|\alpha| = s$ and $|\beta| = 0$.

By assumption $s_1 + s_2 - s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p})$, so $s_2 > \frac{n}{p_2}$. Also $s_1 - s \geq 0$ and therefore $p_1 \leq p$. Consequently
\[
W^{s_1-s,p_1}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n), \quad W^{s_2,p_2}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n).
\]

That is, (3.11) is satisfied with $r = p$ and $q = \infty$. (Note that $|\alpha - \beta| = s$ and $|\beta| = 0$)

ii. $s_2 - s = n(\frac{1}{p_2} - \frac{1}{p})$, $s_1 - s \neq n(\frac{1}{p_1} - \frac{1}{p})$: If $|\beta| < s$, then the proof of Case 1 works (again because inequalities $\frac{1}{p_1} - \frac{s_1 - |\alpha - \beta|}{n} < \frac{1}{p}$ and $\frac{1}{p} - \frac{1}{p_2} + \frac{s_2 - |\beta|}{n} > 0$ hold true). So we may assume $|\beta| = s$. Since $|\alpha| \leq s$ and $\beta \leq \alpha$, this is possible only if $|\alpha| = s$ and $\beta = \alpha$.

Exactly similar to [i.], one can show that $r = \infty$ and $q = p$ satisfy (3.11).

iii. $s_1 - s = n(\frac{1}{p_1} - \frac{1}{p})$, $s_2 - s = n(\frac{1}{p_2} - \frac{1}{p})$: If $|\alpha - \beta| < s$, $|\beta| < s$, then the proof of Case 1 works. If $|\alpha - \beta| = s$ and $|\beta| < s$, then the argument given in item [i.] works. If $|\alpha - \beta| < s$ and $|\beta| = s$, then the argument given in item [ii.] works. Also note that since $|\alpha| \leq s$ and $\beta \leq \alpha$, it is not possible to have $|\alpha - \beta| = |\beta| = s$.

So we proved $\|uv\|_{s,p} \leq \|u\|_{s_1,p_1} \|v\|_{s_2,p_2}$ for $v \in C_c^\infty(\mathbb{R}^n)$ and $u \in W^{s_1,p_1}(\mathbb{R}^n)$. Now suppose $v$ is an arbitrary element of $W^{s_2,p_2}(\mathbb{R}^n)$. There exists a sequence $v_j \in C_c^\infty(\mathbb{R}^n)$ such that $v_j \rightharpoonup v$ in $W^{s_2,p_2}(\mathbb{R}^n)$. We have
\[
\|uv - uv_j\|_{s,p} \leq \|v - v_j\|_{s_2,p_2} \|u\|_{s_1,p_1}
\]
Therefore $uv_j$ is a Cauchy sequence in $W^{s,p}(\mathbb{R}^n)$ and so $uv_j$ converges to an element $w \in W^{s,p}(\mathbb{R}^n)$. Since $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$, $uv_j \to w$ in $L^p(\mathbb{R}^n)$. Hence there exists a subsequence $uv_j$ that converges to $w$ almost everywhere. On the other hand,

$$
\tilde{v}_j \to v \quad \text{in} \quad W^{s_2,p_2}(\mathbb{R}^n) \implies \tilde{v}_j \to v \quad \text{in} \quad L^{p_2}(\mathbb{R}^n)
$$

$$
\implies \exists \text{a subsequence } \tilde{v}_j \text{ such that } \tilde{v}_j \to v \ a.e.
$$

Consequently $u\tilde{v}_j \to uv \ a.e.$ and $u\tilde{v}_j \to w \ a.e.$ and so $uv = w \ a.e.$ Therefore $uv \in W^{s,p}(\mathbb{R}^n)$ and

$$
\| uv \|_{s,p} = \lim_{j \to \infty} \| uv_j \|_{s,p} = \lim_{j \to \infty} \| (uv_j) \|_{s,p} \leq \lim_{j \to \infty} \| v_j \|_{s_2,p_2} \| u \|_{s_1,p_1} \leq \| v \|_{s_2,p_2} \| u \|_{s_1,p_1}.
$$

\[\Box\]

**Corollary 3.54.** Using extension operators, one can easily show that the above result holds also for Sobolev spaces on any bounded domain with Lipschitz continuous boundary. Indeed, if $P_1 : W^{s_1,p_1}(\Omega) \to W^{s_1,p_1}(\mathbb{R}^n)$ and $P_2 : W^{s_2,p_2}(\Omega) \to W^{s_2,p_2}(\mathbb{R}^n)$ are extension operators, then $(P_1 u)(P_2 v)|_\Omega = uv$ and therefore

$$
\| uv \|_{W^{s,p}(\Omega)} \leq \| (P_1 u)(P_2 v) \|_{W^{s,p}(\mathbb{R}^n)} \leq \| P_1 u \|_{W^{s_1,p_1}(\mathbb{R}^n)} \| P_2 v \|_{W^{s_2,p_2}(\mathbb{R}^n)} \leq \| u \|_{W^{s_1,p_1}(\Omega)} \| v \|_{W^{s_2,p_2}(\Omega)}.
$$

Before proceeding any further, first we need to state two lemmas:

**Lemma 3.55.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with Lipschitz continuous bound-
ary, or $\Omega = \mathbb{R}^n$.

$$\forall \epsilon > 0 \quad \forall m \in \left[0, \frac{n}{p}\right] \cap \mathbb{Z} \quad W^{m,p}(\Omega) \times W^{n+c,p}(\Omega) \hookrightarrow W^{m,p}(\Omega).$$

$$\forall \epsilon > 0 \quad \forall m \in \left[0, \frac{n}{p}\right] \cap \mathbb{Z} \quad W^{n+c,p}(\Omega) \times W^{m,p}(\Omega) \hookrightarrow W^{m,p}(\Omega).$$

**Proof.** (Lemma 3.55) This is a direct consequence of the previous theorem. \qed

**Lemma 3.56.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary, or $\Omega = \mathbb{R}^n$.

$$\forall \epsilon > 0 \quad \forall s \in \left[0, \frac{n}{p}\right] \quad W^{s,p}(\Omega) \times W^{n+c,p}(\Omega) \hookrightarrow W^{s,p}(\Omega).$$

$$\forall \epsilon > 0 \quad \forall s \in \left[0, \frac{n}{p}\right] \quad W^{n+c,p}(\Omega) \times W^{s,p}(\Omega) \hookrightarrow W^{s,p}(\Omega).$$

**Proof.** (Lemma 3.56) Clearly we just need to prove the first statement. Let $\epsilon > 0$ and $s \in \left[0, \frac{n}{p}\right]$ be given. By Lemma 3.55 if $s \in \mathbb{Z}$ the claim holds true. So we may assume $s \notin \mathbb{Z}$. since $\frac{n}{p} + \epsilon > \frac{n}{p}$, $W_{\frac{n}{p}+c,p}(\Omega)$ is an algebra and

$$W^{\frac{n}{p}+c,p}(\Omega) \times W^{\frac{n}{p}+c,p}(\Omega) \hookrightarrow W^{\frac{n}{p}+c,p}(\Omega). \quad (3.14)$$

Also $W^{\frac{n}{p}+c,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. Hence

$$W^{\frac{n}{p}+c,p}(\Omega) \times W^{0,p}(\Omega) \hookrightarrow W^{0,p}(\Omega). \quad (L^\infty \times L^p \hookrightarrow L^p) \quad (3.15)$$

Let $\theta = \frac{s}{\frac{n}{p}+c}$; clearly $0 < \theta < 1$. Let $p_1 = 1$ (so if we let $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p} - 1$, then $r = p$). We want to use real interpolation between (3.14) and (3.15). By Theorem 3.42 we have

$$(W^{\frac{n}{p}+c,p}(\Omega), W^{\frac{n}{p}+c,p}(\Omega))_{\theta, p_1} \times (W^{0,p}(\Omega), W^{n+c,p}(\Omega))_{\theta, p} \hookrightarrow (W^{0,p}(\Omega), W^{n+c,p}(\Omega))_{\theta, r}$$
By Theorem 3.39 we have

$$(W^{n+\epsilon,p}(\Omega), W^{n+\epsilon,p}(\Omega))_{\theta,p_1} = W^{n+\epsilon,p}, \quad (W^{0,p}(\Omega), W^{n+\epsilon,p}(\Omega))_{\theta,p} = W^{s,p}. \quad (s \not\in \mathbb{Z})$$

Hence

$$W^{n+\epsilon,p}(\Omega) \times W^{s,p}(\Omega) \hookrightarrow W^{s,p}(\Omega).$$

\[\Box\]

**Theorem 3.57** (Pointwise multiplication in spaces $W^{s,p}(\mathbb{R}^n)$ with $s \geq 0$). Assume $s_i, s$ and $1 \leq p_i \leq p < \infty$ ($i = 1, 2$) are real numbers satisfying

1. $s_i \geq s$
2. $s \geq 0$,
3. $s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p})$,
4. $s_1 + s_2 - s > n\frac{1}{p_1} + n\frac{1}{p_2} - \frac{1}{p}$.

**Claim:** If $u \in W^{s_1,p_1}(\mathbb{R}^n)$ and $v \in W^{s_2,p_2}(\mathbb{R}^n)$, then $uv \in W^{s,p}(\mathbb{R}^n)$ and moreover the pointwise multiplication of functions is a continuous bilinear map

$$W^{s_1,p_1}(\mathbb{R}^n) \times W^{s_2,p_2}(\mathbb{R}^n) \rightarrow W^{s,p}(\mathbb{R}^n).$$

**Proof.** (Theorem 3.57) First we consider the special case where $p_1 = p_2 = p$ and then we will prove the general case.

- **Step 1** $p_1 = p_2 = p$: In this case the assumptions can be rewritten as follows:

  $$s_1, s_2 \geq s \geq 0, \quad s_1 + s_2 - s > \frac{n}{p}. $$
Case 1 $s > \frac{n}{p}$: If $s > \frac{n}{p}$, then $W^{s,p}(\mathbb{R}^n)$ is an algebra and therefore we can write

$$W^{s_1,p_1}(\mathbb{R}^n) \times W^{s_2,p_2}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n) \times W^{s,p}(\mathbb{R}^n) \quad \text{(by assumption $s_1, s_2 \geq s$)}$$

$$\hookrightarrow W^{s,p}(\mathbb{R}^n).$$

Case 2 $s \leq \frac{n}{p}$: By Lemma 3.56 for all $\epsilon > 0$

$$W^{s,p}(\mathbb{R}^n) \times W^{n+\epsilon,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n),$$

$$W^{n+\epsilon,p}(\mathbb{R}^n) \times W^{s,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n).$$

In particular for $\epsilon = s_1 + s_2 - s - \frac{n}{p} > 0$ we have

$$W^{s,p}(\mathbb{R}^n) \times W^{s_1+s_2-s,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n),$$

$$W^{s_1+s_2-s,p}(\mathbb{R}^n) \times W^{s,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n).$$

We may consider the following cases:

i. $p < 2, s_1, s_2 \not\in \mathbb{Z}$: Let $\frac{1}{r} = \frac{1}{p} + \frac{1}{p} - 1 > 0$. Let $\theta$ be such that $(1-\theta)s + \theta(s_1 + s_2 - s) = s_1$.

As it was discussed in the proof of Theorem 3.50, for this $\theta$, $(1-\theta)(s_1 + s_2 - s) + \theta s = s_2$. By Theorem 3.42 we have

$$(W^{s,p}(\mathbb{R}^n), W^{s_1+s_2-s,p}(\mathbb{R}^n))_{\theta,p} \times (W^{s_1+s_2-s,p}(\mathbb{R}^n), W^{s,p}(\mathbb{R}^n))_{\theta,p}$$

$$\hookrightarrow (W^{s,p}(\mathbb{R}^n), W^{s,p}(\mathbb{R}^n))_{\theta,r}.$$
ii  $p < 2, s_1 \in \mathbb{Z}, s_2 \not\in \mathbb{Z}$: If $s_1 = s$, then from $s_1 + s_2 - s > \frac{n}{p}$ it follows that $s_2 > \frac{n}{p}$. So in this case the claim reduces to what was proved in Lemma 3.56. If $s_1 \neq s$, let $\tilde{s}_1 = s_1 - \varepsilon$ where

$$\varepsilon = \frac{1}{2} \min(s_1 - \lfloor s_1 \rfloor, s_1 - s, s_1 + s_2 - s - \frac{n}{p}) > 0.$$ 

Clearly

$$\tilde{s}_1 \not\in \mathbb{Z}, \quad \tilde{s}_1 \geq s, \quad s_2 \geq s, \quad \tilde{s}_1 + s_2 - s > \frac{n}{p}.$$ 

Therefore by what was proved in the previous case

$$W^{\tilde{s}_1, p} \times W^{s_2, p} \hookrightarrow W^{s, p}.$$ 

Now the claim follows from the fact that $W^{\tilde{s}_1, p} \hookrightarrow W^{s, p}$.

iii  $p < 2, s_1 \not\in \mathbb{Z}, s_2 \in \mathbb{Z}$: Just switch the roles of $s_1$ and $s_2$ in the previous case.

iv  $p < 2, s_1 \in \mathbb{Z}, s_2 \in \mathbb{Z}$: Note that both of $s_1$ and $s_2$ cannot be equal to $s$ because $s_1 + s_2 - s > \frac{n}{p}$ but $s \leq \frac{n}{p}$. Because of the symmetry in the roles of $s_1$ and $s_2$, without loss of generality we may assume that $s_1 \neq s$. Let $\tilde{s}_1 = s_1 - \varepsilon$ where

$$\varepsilon = \frac{1}{2} \min(s_1 - \lfloor s_1 \rfloor, s_1 - s, s_1 + s_2 - s - \frac{n}{p}) > 0.$$ 

Clearly

$$\tilde{s}_1 \not\in \mathbb{Z}, \quad \tilde{s}_1 \geq s, \quad s_2 \geq s, \quad \tilde{s}_1 + s_2 - s > \frac{n}{p}.$$ 

and so the problem reduces to the previous case.

At this point we are done with the case $p < 2$.

v  $p \geq 2, s \not\in \mathbb{Z}, s_1 + s_2 - s \not\in \mathbb{Z}$: This time we use complex interpolation. Define $\theta$ as
before. By Theorem 3.42 we have

\[ [W^{s,p}(\mathbb{R}^n), W^{s_1+s_2-s,p}(\mathbb{R}^n)]_\theta \times [W^{s_1+s_2-s,p}(\mathbb{R}^n), W^{s,p}(\mathbb{R}^n)]_\theta \]

\[ \leftarrow [W^{s,p}(\mathbb{R}^n), W^{s,p}(\mathbb{R}^n)]_\theta. \]

Since \( s \) and \( s_1 + s_2 - s \) are not integers and \( p \geq 2 \) (see Theorem 3.40),

\[ W^{s_1,p}(\mathbb{R}^n) \leftarrow [W^{s,p}(\mathbb{R}^n), W^{s_1+s_2-s,p}(\mathbb{R}^n)]_\theta, \]

\[ W^{s_2,p}(\mathbb{R}^n) \leftarrow [W^{s_1+s_2-s,p}(\mathbb{R}^n), W^{s,p}(\mathbb{R}^n)]_\theta. \]

Consequently

\[ W^{s_1,p}(\mathbb{R}^n) \times W^{s_2,p}(\mathbb{R}^n) \leftarrow W^{s,p}(\mathbb{R}^n). \]

vi \( p \geq 2, s \not\in \mathbb{Z}, s_1 + s_2 - s \in \mathbb{Z} \): Both of \( s_1 \) and \( s_2 \) cannot be equal to \( s \) because \( s_1 + s_2 - s > \frac{n}{p} \) but \( s \leq \frac{n}{p} \). Because of the symmetry in the roles of \( s_1 \) and \( s_2 \), without loss of generality we may assume that \( s_1 \neq s \). Let \( \bar{s}_1 = s_1 - \epsilon \) where

\[ \epsilon = \frac{1}{2} \min(1, s_1 - s, s_1 + s_2 - s - \frac{n}{p}). \]

Clearly

\[ \bar{s}_1 \geq s, \quad \bar{s}_1 + s_2 - s = s_1 + s_2 - s - \epsilon > \frac{n}{p}, \]

\[ \bar{s}_1 + s_2 - s = s_1 + s_2 - s - \epsilon \not\in \mathbb{Z} \quad \text{(because } \epsilon \leq \frac{1}{2}). \]

So by what was proved in the previous case we have

\[ W^{\bar{s}_1,p}(\mathbb{R}^n) \times W^{s_2,p}(\mathbb{R}^n) \leftarrow W^{s,p}(\mathbb{R}^n). \]
and since $W^{s_1,p} \hookrightarrow W^{\tilde{s}_1,p}$

$$W^{s_1,p}(\mathbb{R}^n) \times W^{s_2,p}(\mathbb{R}^n) \hookrightarrow W^{\tilde{s},p}(\mathbb{R}^n).$$

vii $p \geq 2, s \in \mathbb{Z}, s_1 + s_2 - s \notin \mathbb{Z}$: If $s_1 = s$ or $s_2 = s$, the claim follows from Lemma 3.56.

So we may assume $s_1, s_2 > s$. Let $\bar{s} = s + \epsilon$ where

$$\epsilon = \frac{1}{2} \min(s_1 - s, s_2 - s, s_1 + s_2 - s - [s_1 + s_2 - s], s_1 + s_2 - s - \frac{n}{p})$$

(note that $s_1 + s_2 - s - [s_1 + s_2 - s] < 1$)

Clearly $\bar{s}$ and $s_1 + s_2 - \bar{s}$ are not integers and

$$s_1 \geq \tilde{s}, \quad s_2 \geq \tilde{s}, \quad s_1 + s_2 - \tilde{s} > \frac{n}{p}.$$ 

So by what was proved in previous cases

$$W^{s_1,p}(\mathbb{R}^n) \times W^{s_2,p}(\mathbb{R}^n) \hookrightarrow W^{\tilde{s},p}(\mathbb{R}^n) = W^{s_1+p+\epsilon,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n).$$

viii $p \geq 2, s \in \mathbb{Z}, s_1 + s_2 - s \in \mathbb{Z}$: If $s_1 = s$ or $s_2 = s$, the claim follows from Lemma 3.56.

So we may assume $s_1, s_2 > s$. Let $\bar{s} = s + \epsilon$ where

$$\epsilon = \frac{1}{2}(1, s_1 - s, s_2 - s, s_1 + s_2 - s - \frac{n}{p}).$$

We have $\epsilon \leq \frac{1}{2}$, so $\bar{s}$ and $s_1 + s_2 - \bar{s}$ are not integers. Also clearly

$$s_1 \geq \tilde{s}, \quad s_2 \geq \tilde{s}, \quad s_1 + s_2 - \tilde{s} > \frac{n}{p}.$$ 

So by what was proved in previous cases

\[ W^{s_1,p}(\mathbb{R}^n) \times W^{s_2,p}(\mathbb{R}^n) \hookrightarrow W^{\tilde{s},p}(\mathbb{R}^n) = W^{s+e,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n). \]

- **Step2: General Case** This step is exactly the same as step 2 in the proof of Theorem 3.50. We just need to replace every occurrence of \( H^{r,q}(\mathbb{R}^n) \) with \( W^{r,q}(\mathbb{R}^n) \).

\[ \square \]

**Remark 3.58.** Proposition 3.49 shows that the claim of Theorem 3.57 does not necessarily hold if one removes the assumption \( p_i \leq p \). Of course, the next theorem shows that the assumption \( p_i \leq p \) is not necessary on bounded domains.

**Theorem 3.59** (Multiplication on bounded domains, nonnegative exponents). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with Lipschitz continuous boundary. Assume \( s_i, s \) and \( 1 \leq p_i, p < \infty \) \((i = 1, 2)\) are real numbers satisfying

1. \( s_i \geq s \)
2. \( s \geq 0 \),
3. \( s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p}) \),
4. \( s_1 + s_2 - s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \).

In the case where \( \max\{p_1, p_2\} > p \) instead of (iv) assume that \( s_1 + s_2 - s > \frac{n}{\min\{p_1, p_2\}} \).

**Claim:** If \( u \in W^{s_1,p_1}(\Omega) \) and \( v \in W^{s_2,p_2}(\Omega) \), then \( uv \in W^{s,p}(\Omega) \) and moreover the pointwise multiplication of functions is a continuous bilinear map

\[ W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \rightarrow W^{s,p}(\Omega). \]

**Proof.** (Theorem 3.59)
• **Step 1** $p_1 = p_2 = p$: By the exact same proof as the one given in step 1 of the proof of Theorem 3.57 we have

\[ W^{s_1:p} (\Omega) \times W^{s_2:p} (\Omega) \hookrightarrow W^{s:p} (\Omega), \]

provided $s_1, s_2 \geq s$ and $s_1 + s_2 - s > \frac{n}{p}$.

• **Step 2:** Now we prove the general case. Because of the symmetry in the roles of $p_1$ and $p_2$ without loss of generality we may assume $p_2 \leq p_1$. We may consider three cases:

  ◦ **Case 1** $p_1, p_2 \leq p$: The proof is exactly the same as the one presented in Step 2 of Theorem 3.57.

  ◦ **Case 2** $p_1 > p, p_2 \leq p$: Let $\tilde{p}_1 = p$. It is easy to see that the tuple $(s_1, s_2, s, \tilde{p}_1, p_2, p)$ also satisfies all the assumptions of the theorem; in particular note that $s_1 + s_2 - s > n(\frac{1}{\tilde{p}_1} + \frac{1}{p_2} - \frac{1}{p})$ because by assumption $s_1 + s_2 - s > \frac{n}{p_2}$. So by what was proved in the previous case we have

\[ W^{s_1:\tilde{p}_1} (\Omega) \times W^{s_2:p_2} (\Omega) \hookrightarrow W^{s:p} (\Omega). \]

By the third embedding theorem (Theorem 3.46) $W^{s_1:p_1} (\Omega) \hookrightarrow W^{s_1,\tilde{p}_1} (\Omega)$ (because $p_1 > \tilde{p}_1 = p$). Hence

\[ W^{s_1:p_1} (\Omega) \times W^{s_2:p_2} (\Omega) \hookrightarrow W^{s:p} (\Omega). \]

  ◦ **Case 3** $p < p_2 \leq p_1$: Since $s_1 + s_2 - s > \frac{n}{p_2}$ by what was proved in Step 1 we have

\[ W^{s_1:p_2} (\Omega) \times W^{s_2:p_2} (\Omega) \hookrightarrow W^{s:p_2} (\Omega). \]
Now note that $p_1 \geq p_2$ and $p_2 > p$, so by the third embedding theorem (Theorem 3.46) $W^{s_1,p_1}(\Omega) \hookrightarrow W^{s_1,p_2}(\Omega)$ and $W^{s_2,p_2}(\Omega) \hookrightarrow W^{s,p}(\Omega)$. Therefore

$$W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \hookrightarrow W^{s,p}(\Omega).$$

\[ \blacklozenge \]

**Theorem 3.60** (Multiplication theorem for Sobolev spaces on the whole space, negative exponents I). Assume $s_i, s$ and $1 < p_i \leq p < \infty (i = 1, 2)$ are real numbers satisfying

i. $s_i \geq s$

ii. $\min\{s_1, s_2\} < 0$,

iii. $s_i - s \geq n\left(\frac{1}{p_i} - \frac{1}{p}\right)$,

iv. $s_1 + s_2 - s > n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right)$,

v. $s_1 + s_2 \geq n\left(\frac{1}{p_1} + \frac{1}{p_2} - 1\right) \geq 0$.

Then the pointwise multiplication of functions extends uniquely to a continuous bilinear map

$$W^{s_1,p_1}(\mathbb{R}^n) \times W^{s_2,p_2}(\mathbb{R}^n) \rightarrow W^{s,p}(\mathbb{R}^n).$$

**Proof.** (Theorem 3.60) Since by assumption $s_1 + s_2 \geq 0$, $s_1$ and $s_2$ cannot both be negative. WLOG we can assume $s_1$ is negative and $s_2$ is positive. Also note that by assumption $s \leq s_1$ so $s$ is also negative.

Note that $C_c^\infty$ is dense in all Sobolev spaces on $\mathbb{R}^n$. Considering this, first we prove that for $u \in W^{s_1,p_1}, \varphi \in C_c^\infty$

$$||u\varphi||_{s,p} \leq ||u||_{s_1,p_1} ||\varphi||_{s_2,p_2},$$

(3.16)
Note that 
\[ \|f\|_{s,p} = \sup_{\psi \in C_\infty^\infty} \frac{|\langle f, \psi \rangle_{W^{s,p} \times W^{-s,p'}}|}{\|\psi\|_{-s,p'}}. \]

Thus we just need to show that
\[ |\langle u \varphi, \psi \rangle_{W^{s,p} \times W^{-s,p'}}| \leq \|u\|_{s_1,p_1} \|\varphi\|_{s_2,p_2} \|\psi\|_{-s,p'}. \]

We have
\[ |\langle u \varphi, \psi \rangle_{W^{s,p} \times W^{-s,p'}}| = |\langle u, \varphi \psi \rangle_{W^{s_1,p_1} \times W^{-s_1,p_1'}}| \leq \|u\|_{s_1,p_1} \|\varphi \psi\|_{-s_1,p_1'}. \]

So it is enough to prove that
\[ \|\varphi \psi\|_{-s_1,p_1'} \leq \|\varphi\|_{s_2,p_2} \|\psi\|_{-s,p'}. \]

\(-s_1, s_2, -s\) are all nonnegative. So, by Theorem 3.57, in order to ensure that the above inequality is true we just need to check the followings:

\[ p' \leq p_1' \quad (\text{true because } p_1 \leq p), \quad p_2 \leq p_1' \quad (\text{true because } \frac{1}{p_1} + \frac{1}{p_2} \geq 1), \]
\[ -s_1 \leq -s \quad (\text{true because } s \leq s_1), \quad s_2 \geq -s_1 \quad (\text{true because } s_1 + s_2 \geq 0), \]
\[ s_2 + (-s) \geq 0. \quad (\text{true because } s \leq s_2), \]
\[ s_2 - (-s_1) \geq n\left(\frac{1}{p_2} - \frac{1}{p_1'}\right) \quad (\text{true because } s_2 + s_1 \geq n\left(\frac{1}{p_2} + \frac{1}{p_1} - 1\right)), \]
\[ -s - (-s_1) \geq n\left(\frac{1}{p_1'} - \frac{1}{p_1}\right) \quad (\text{true because } s_1 - s \geq n\left(\frac{1}{p_1} - 1\right)), \]
\[ s_2 + (-s) - (-s_1) > n\left(\frac{1}{p_2} + \frac{1}{p'} - \frac{1}{p_1'}\right) \quad (\text{true because } s_1 + s_2 - s > n\left(\frac{1}{p_2} + \frac{1}{p_1} - 1\right)). \]

Therefore the inequality (3.16) holds for \( u \in W^{s_1,p_1} \) and \( \varphi \in C_\infty^\infty \). To prove the general case we proceed as follows: Suppose \( v \in W^{s_2,p_2} \). There exists a sequence \( \varphi_k \in C_\infty^\infty \)
such that \( \varphi_k \to v \) in \( W^{s_2, p_2} \). Since \( s_2 \geq 0, W^{s_2, p_2} \hookrightarrow L^{p_2} \) and therefore \( \varphi_k \to v \) in \( L^{p_2} \). Consequently \( \varphi_k \to v \) a.e. which implies that \( u \varphi_k \to u v \) a.e..

On the other hand we have

\[
||u(\varphi_i - \varphi_j)||_{s, p} \leq ||u||_{s_1, p_1} ||\varphi_i - \varphi_j||_{s_2, p_2}.
\]

It follows that \( u \varphi_k \) is a Cauchy sequence in \( W^{s, p} \) and thus it is convergent to some function \( w \in W^{s, p} \). Since \( u \varphi_k \to u v \) a.e. we can conclude that \( w = u v \), that is, \( u \varphi_k \to u v \) in \( W^{s, p} \). Finally

\[
||u \varphi_k||_{s, p} \leq ||u||_{s_1, p_1} ||\varphi_k||_{s_2, p_2} \ \forall \ k
\]

and so by passing to the limit as \( k \to \infty \)

\[
||uv||_{s, p} \leq ||u||_{s_1, p_1} ||v||_{s_2, p_2}.
\]

Remark 3.61. A similar proof shows that the above theorem holds true for any bounded domain with Lipschitz continuous boundary as well. Of course in the case of bounded domains we can drop the assumption \( \frac{1}{p_1} + \frac{1}{p_2} \geq 1 \).

Theorem 3.62 (Multiplication theorem for Sobolev spaces on the whole space, negative exponents II). Assume \( s_i, s \) and \( 1 < p, p_i < \infty (i = 1, 2) \) are real numbers satisfying

i. \( s_i \geq s \)

ii. \( \min\{s_1, s_2\} \geq 0 \) and \( s < 0 \),

iii. \( s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p}) \),

iv. \( s_1 + s_2 - s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \geq 0 \).
v. \( s_1 + s_2 > n\left(\frac{1}{p_1} + \frac{1}{p_2} - 1\right) \). \( \text{(the inequality is strict)} \)

Then the pointwise multiplication of functions extends uniquely to a continuous bilinear map

\[ W^{s_1,p_1}(\mathbb{R}^n) \times W^{s_2,p_2}(\mathbb{R}^n) \rightarrow W^{s,p}(\mathbb{R}^n). \]

**Proof.** (Theorem 3.62) Let \( \epsilon > 0 \) be such that

\[ \epsilon < \frac{1}{n} \min\{s_1 + s_2 - s - \left(\frac{n}{p_1} + \frac{n}{p_2} - n\right), s_1 + s_2 - \left(\frac{n}{p_1} + \frac{n}{p_2} - n\right)\}. \]

Let

\[ \frac{1}{r} = \max\left\{\frac{1}{p_1} - \frac{s_1}{n}, \frac{1}{p_2} - \frac{s_2}{n}, \frac{1}{p_1} - \frac{s_1}{n} + \frac{1}{p_2} - \frac{s_2}{n} + \epsilon, 1\right\}. \]

Note that \( r > 0 \) because \( \frac{1}{r} > \frac{1}{p_1} > 0 \). Also \( \frac{1}{r} < 1 \) because each element in the set over which we are taking the maximum is strictly less than 1:

\[ \frac{1}{p_1} - \frac{s_1}{n} \leq 1 < 1, \quad \frac{1}{p_2} - \frac{s_2}{n} \leq 1 < 1, \quad \frac{1}{p} < 1. \]

\[ \epsilon < \frac{1}{n} \left[ s_1 + s_2 - \left(\frac{n}{p_1} + \frac{n}{p_2} - n\right)\right] \Rightarrow \frac{1}{p_1} - \frac{s_1}{n} + \frac{1}{p_2} - \frac{s_2}{n} + \epsilon < 1. \]

- **Claim 1:** \( W^{s_1,p_1}(\mathbb{R}^n) \times W^{s_2,p_2}(\mathbb{R}^n) \rightarrow L^r(\mathbb{R}^n). \)

- **Claim 2:** \( L^r(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n). \)

Clearly if we prove **Claim 1** and **Claim 2**, then we are done.

**Proof of Claim 1:** All the exponents are nonnegative, so it is enough to check the
assumptions of Theorem 3.53.

\[
s_1 - 0 \geq n\left(\frac{1}{p_1} - \frac{1}{r}\right) \quad \text{(true because } \frac{1}{r} \geq \frac{1}{p_1} - \frac{s_1}{n}\text{)}
\]

\[
s_2 - 0 \geq n\left(\frac{1}{p_2} - \frac{1}{r}\right) \quad \text{(true because } \frac{1}{r} \geq \frac{1}{p_2} - \frac{s_2}{n}\text{)}
\]

\[
s_1 + s_2 > n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{r}\right) \quad \text{(true because } \frac{1}{r} > \frac{1}{p_1} - \frac{s_1}{n} + \frac{1}{p_2} - \frac{s_2}{n}\text{)}.
\]

**Proof of Claim 2**: We have \((L^r(\mathbb{R}^n))^* = L^{r'}(\mathbb{R}^n)\) and \((W^{s,p}(\mathbb{R}^n))^* = W^{-s,p'}(\mathbb{R}^n)\). In what follows we will show that \(W^{-s,p'}(\mathbb{R}^n) \hookrightarrow L^{r'}(\mathbb{R}^n)\); then since \(W^{-s,p'}(\mathbb{R}^n)\) is dense in \(L^{r'}(\mathbb{R}^n)\) \((C^\infty_c(\mathbb{R}^n) \subseteq W^{-s,p'}(\mathbb{R}^n)\) and \(C^\infty_c(\mathbb{R}^n)\) is dense in \(L^{r'}(\mathbb{R}^n)\)), we are allowed to take the dual of both sides and it immediately follows that the claim is true.

Note that by the definition of \(r\), we have \(\frac{1}{p} \leq \frac{1}{r}\) and therefore \(p' \leq r'\). So, according to Theorem 3.44, in order to show that \(W^{-s,p'}(\mathbb{R}^n) \hookrightarrow L^{r'}(\mathbb{R}^n)\), it is enough to prove that \(-s - \frac{n}{p} \geq 0 - \frac{n}{r}\), that is we need to prove that \(\frac{1}{p} - \frac{s}{n} \geq \frac{1}{r}\). This is true because each element in the set over which we are taking the maximum in the definition of \(\frac{1}{r}\) is less than or equal to \(\frac{1}{p} - \frac{s}{n}\):

\[
\frac{1}{p} - \frac{s}{n} \geq \frac{1}{p_1} - \frac{s_1}{n} \quad \text{(true because } s_1 - s \geq n\left(\frac{1}{p_1} - \frac{1}{p}\right)\text{)}
\]

\[
\frac{1}{p} - \frac{s}{n} \geq \frac{1}{p_2} - \frac{s_2}{n} \quad \text{(true because } s_2 - s \geq n\left(\frac{1}{p_2} - \frac{1}{p}\right)\text{)}
\]

\[
\frac{1}{p} - \frac{s}{n} \geq \frac{1}{p_1} + \frac{1}{p_2} - \frac{s_1}{n} + \frac{s_2}{n} + \epsilon \quad \text{(true because } s_1 + s_2 - s \geq n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) + n\epsilon\text{)}
\]

\[
\frac{1}{p} - \frac{s}{n} \geq \frac{1}{p} \quad \text{(true because } s < 0\text{)}
\]

\[
\square
\]

**Remark 3.63.** A similar argument can be used to prove the above theorem for any bounded domain whose boundary is Lipschitz continuous.
pear as A. Behzadan and M. Holst, *Rough Solutions of the Einstein Constraint Equations on Asymptotically Flat Manifolds Without Near-CMC Conditions*. The dissertation author was the primary investigator and author of this paper.

Another part of Chapter 3 is currently being prepared for submission for publication. The material may appear as A. Behzadan and M. Holst, *Multiplication in Sobolev-Slobodeckij Spaces, Revisited*. The dissertation author was the primary investigator and author of this paper.
Chapter 4

Differential Operators in Weighted Spaces

The goal of this chapter is to assemble some results we need for differential operators in Weighted spaces. In particular, we are interested in the properties of the Laplacian and the vector Laplacian. We begin with introducing a general class of linear differential operators \( D^{\alpha,\gamma}_{m,\rho} \).

Let \( M \) be an \( n \)-dimensional AF manifold and let \( E \) be a smooth vector bundle over \( M \) with fiber dimension \( k \). Consider the linear differential operator \( A : \Gamma(E) \to \Gamma(E) \) of order \( m \) where \( \Gamma(E) \) denotes the space of smooth sections of \( E \). By definition, we know that in any local coordinates (trivializing \( E \)) \( A \) can be written as \( A = \sum_{|\nu| \leq m} a_\nu \partial^\nu \) where \( a_\nu \) is a \( \mathbb{R}^{k \times k} \) valued function.

Definition 4.1. Let \( \alpha \in \mathbb{R}, \gamma \in (1, \infty), \) and \( \rho < 0 \).

- We say \( A \) belongs to the class \( D^{\alpha,\gamma}_m(E) \) if and only if \( a_\nu \in W^{\alpha-m+|\nu|,\gamma} \) for \( |\nu| \leq m \).

- We say \( A \) belongs to the class \( D^{\alpha,\gamma}_{m,\rho}(E) \) if and only if \( a_\nu \in W^{\alpha-m+|\nu|,\gamma}_{\rho-m+|\nu|} \) for \( |\nu| < m \) and there are constants \( a_\nu^\infty \) such that \( a_\nu^\infty - a_\nu \in W^{\alpha,\gamma}_{\rho} \) for all \( |\nu| = m \). We call \( A^\infty = \sum_{|\nu| = m} a_\nu^\infty \partial^\nu \) the principal part of \( A \) at infinity.
If \( \alpha \gamma > n \), then the highest order coefficients of \( A \in D^{\alpha,\gamma}_m(E) \) are continuous and so it makes sense to talk about their pointwise values. We say \( A \) is elliptic if for each \( x \), the constant coefficient operator \( \sum_{|\nu| = m} a_{\nu}(x) \partial^\nu \) is elliptic.

If \( \alpha \gamma > n \), then the highest order coefficients of \( A \in D^{\alpha,\gamma}_{m,\rho}(E) \) are continuous and so it makes sense to talk about their pointwise values. We say \( A \) is elliptic if \( A_\infty \) is elliptic and moreover for each \( x \), the constant coefficient operator \( \sum_{|\nu| = m} a_{\nu}(x) \partial^\nu \) is elliptic.

**Theorem 4.2.** If \( \delta \in \mathbb{R} \), \( \rho < 0 \) and if \( A \in D^{\alpha,\gamma}_{m,\rho}(E) \) then \( A \) can be viewed as a bounded linear map

\[
A : W^{s,q}_\delta(E) \to W^{\sigma,q}_{\delta - m}(E),
\]

provided

(i) \( \gamma, q \in (1, \infty) \),

(ii) \( s \geq m - \alpha \) (let \( \frac{1}{q} + \frac{1}{\gamma} \geq 1 \) if \( s = m - \alpha \notin \mathbb{Z} \)),

(iii) \( \sigma \leq \min(s, \alpha) - m \) (let \( \gamma \leq q \) if \( \alpha - m = \sigma \notin \mathbb{Z} \))

(iv) \( \sigma < s - m + \alpha - \frac{n}{\gamma} \),

(v) \( \sigma - \frac{n}{q} \leq \alpha - \frac{n}{\gamma} - m \),

(vi) \( s - n/q > m - n - \alpha + n/\gamma \).

If moreover \( A_\infty = 0 \), then \( A \) is a continuous map

\[
A : W^{s,q}_\delta(E) \to W^{\sigma,q}_{\delta - m + \rho}(E)
\]

**Proof.** (Theorem 4.2) First let’s consider the case where \( A_\infty \neq 0 \). The goal is to find sufficient conditions to make sure that \( A = \sum_{|\nu| \leq m} a_{\nu} \partial^\nu \) is a continuous operator from
\( W_{\delta}^{s,q} \rightarrow W_{\beta}^{\sigma,q} \). Clearly this will be true provided

1. For all \(|v| < m\)

\[
W_{\rho-m+|v|}^{\alpha-m+|v|, \gamma} \times W_{\delta-|v|}^{s-|v|, q} \rightarrow W_{\beta}^{\sigma, q}, \quad \text{(note that } a_v \in W_{\rho-m+|v|}^{\alpha-m+|v|, \gamma}, \partial^v u \in W_{\delta-|v|}^{s-|v|, q})
\]

It follows from the multiplication lemma and previously mentioned embedding theorems that the above embedding holds true provided (the numbering of the items corresponds to the numbering of the assumptions in multiplication lemma)

\[
(i) \; \sigma \leq \alpha - m \quad (\gamma \leq q \text{ if } \alpha - m = \sigma \notin \mathbb{Z}),
\]

\[
(ii) \; s \geq m - \alpha, \quad (\frac{1}{q} + \frac{1}{\gamma} \geq 1 \text{ if } s = m - \alpha \notin \mathbb{Z})
\]

\[
(i, iii) \; \sigma \leq s - (m - 1),
\]

\[
(i iv) \; \sigma < s - m + \alpha - \frac{n}{\gamma},
\]

\[
(iii) \; \sigma - \frac{n}{q} \leq \alpha - \frac{n}{\gamma} - m,
\]

\[
(v) \; s - \frac{n}{q} > m - n - \alpha + \frac{n}{\gamma},
\]

and of course we need \((\rho - m + |v|) + (\delta - |v|)\) to be less than or equal to \(\beta\), that is, \(\rho - m + \delta \leq \beta\).

2. For \(|v| = m\)

\[
W_{\rho}^{\alpha, \gamma} \times W_{\delta-m}^{s-m,q} \hookrightarrow W_{\beta}^{\sigma, q},
\]

\[
W_{\delta-m}^{s-m,q} \hookrightarrow W_{\beta}^{\sigma, q}.
\]

Note that, \(a_v \partial^v = (a_v - a_v^\infty) \partial^v + a_v^\infty \partial^v\). \(a_v^\infty\) is constant and \((a_v - a_v^\infty) \in W_{\rho}^{\alpha, \gamma}, \)
so it should be clear why we need the above embeddings to be true. By using the multiplication lemma it turns out that the only extra assumption that we need for the first embedding to be true is that \( \sigma \leq s - m \) and then the only extra assumption that we need for the second embedding to be true is that \( \beta \geq \delta - m \).

To complete the proof we just need to note that if \( A_\infty = 0 \) then we do not need to have the embedding \( W^{s-m,q}_{\delta-m} \hookrightarrow W^{\sigma,q}_{\beta} \) and so \( \beta \) can be any number larger than or equal to \( \delta - m + \rho \).

Remark 4.3. In the above proof we implicitly assumed that the following statement is true: If \( A : \Gamma(E) \to \Gamma(E) \) is a linear partial differential operator whose representation in each local chart is continuous from \( W^{s,q}_{\delta} \) to \( W^{\sigma,q}_{\beta} \), then \( A \) is a continuous operator from \( W^{s,q}_{\delta}(E) \) to \( W^{\sigma,q}_{\beta}(E) \).

Example: It can be easily checked with the help of the multiplication lemma (Lemma (3.30)) and its consequences (Lemma (3.34) and Corollary (3.36)) that if \((M, g)\) is an asymptotically flat manifold of class \( W^{\alpha,\gamma}_{\rho} \) with \( \alpha \gamma > n \) and \( \rho < 0 \), then the Laplacian and conformal Laplacian are elliptic operators in class \( D^{\alpha,\gamma}_{2,\rho}(M \times \mathbb{R}) \); vector Laplacian is an elliptic operator in the class \( D^{\alpha,\gamma}_{2,\rho}(TM) \). Some details about ellipticity can be found in Appendix B.

Our next goal is to prove \textit{a priori} estimates for elliptic operators in \( D^{\alpha,\gamma}_{m,\rho} \). In what follows, we state and prove Lemma 4.4, Lemma 4.5, Proposition 4.6, and Lemma 4.8 for \( \mathbb{R}^n \). Using partition of unity arguments, the main results can be easily extended to AF manifolds. Although the situations that we study here are more complicated, the main ideas which are employed to prove the statements, mostly follow those which have been used in [52] for weighted \( H^s \) spaces.

Lemma 4.4. Let the following assumptions hold:

- \( A \in D^{\alpha,\gamma}_m \) where \( \gamma \in (1, \infty) \) and \( \alpha - \frac{n}{\gamma} > \max\{0, \frac{m-n}{2}\} \); \( A \) is elliptic.
\[ q \in (1, \infty), s \in (m - \alpha, \alpha) \quad \text{(if } s = \alpha \not\in \mathbb{N}_0, \text{ then let } q \in [\gamma, \infty)). \]

\[ s - \frac{n}{q} \in (m - n - \alpha + \frac{n}{\gamma}, \alpha - \frac{n}{\gamma}). \]

Then: If \( U \) and \( V \) are bounded open sets with \( \bar{U} \subseteq V \), then there exists \( \hat{s} < s \) such that for all \( u \in W^{s,q} \)
\[
\| u \|_{s,q,U} \leq \| Au \|_{s-m,q,V} + \| u \|_{s,q,V}. 
\] (4.1)

Note: The assumptions in the statement of the lemma are to ensure that \( A \in D_m^{\alpha,\gamma} \) sends elements of \( W^{s,q} \) to elements of \( W^{s-m,q} \). In fact the conditions in Theorem 4.2 work for unweighted spaces too and the restrictions in the statement of the above lemma agree with the conditions in Theorem 4.2. The assumption \( \alpha - \frac{n}{\gamma} > \frac{m-n}{2} \) is to ensure that the interval \( (m - n - \alpha + \frac{n}{\gamma}, \alpha - \frac{n}{\gamma}) \) is nonempty.

**Proof. (Lemma 4.4)** The proof of the interior regularity lemma in [38] (Lemma A.25), with obvious changes, goes through for the above claim as well. The approach of the proof is similar to our proof for Proposition 4.6. Since the claim is about unweighted Sobolev spaces we do not repeat that argument here. \( \square \)

**Lemma 4.5.** Suppose \( A \) is a constant coefficient elliptic operator that has only derivatives of order \( m \) with \( m < n \) on \( \mathbb{R}^n \) (clearly \( A \in D_m^{\alpha,\gamma} \) for all possible \( \alpha, \gamma, \) and \( \rho < 0 \) because one can take \( A_\infty = A \) and so \( A_\infty - A = 0 \)). Then for \( s \in \mathbb{R}, \rho \in (1, \infty), \) and \( \delta \in (m - n, 0), A: W^{s,p}_\delta \to W^{s-m,p}_\delta \) is an isomorphism.

**Proof. (Lemma 4.5)** Let \( A_{s,p,\delta} \) denote the operator \( A \) acting on \( W^{s,p}_\delta \). We consider three cases \( s \geq m, s \in (-\infty, 0], \) and \( s \in (0, m). \)

- **Case 1:** \( s \geq m. \)

For \( s \in \mathbb{N} \) and \( s \geq m, \) the claim follows from the argument in [51]. If \( s \not\in \mathbb{N}, \) let \( k = [s], \)
\( \theta = s - k \). We know that \( A_{k,p,\delta} \) and \( A_{k+1,p,\delta} \) have inverses and in fact

\[
A_{k,p,\delta}^{-1} : W_{\delta-m}^{k-m,p} \to W_{\delta}^{k,p},
\]

\[
A_{k+1,p,\delta}^{-1} : W_{\delta-m}^{k+1-m,p} \to W_{\delta}^{k+1,p},
\]

are continuous maps. Note that

\[
W_{\delta}^{k+1,p} \to W_{\delta}^{k,p}, \quad W_{\delta-m}^{k+1-m,p} \to W_{\delta-m}^{k-m,p},
\]

\[
(W_{\delta}^{k,p}, W_{\delta}^{k+1,p})_{\theta,p} = W_{\delta}^{s,p}, \quad (W_{\delta-m}^{k-m,p}, W_{\delta-m}^{k+1-m,p})_{\theta,p} = W_{\delta-m}^{s-m,p}.
\]

So by interpolation we get a continuous operator \( T : W_{\delta-m}^{s-m,p} \to W_{\delta}^{s,p} \) which must be the restriction of \( A_{k,p,\delta}^{-1} \) to \( W_{\delta-m}^{s-m,p} \). Now for all \( u \in W_{\delta}^{s,p} \) we have

\[
u \in W_{\delta}^{s,p} \to W_{\delta}^{k,p} \Rightarrow A_{s,p,\delta}u = A_{k,p,\delta}u \Rightarrow T(A_{s,p,\delta}u) = T(A_{k,p,\delta}u) = A_{k,p,\delta}^{-1}(A_{k,p,\delta}u) = u.
\]

Similarly \( A_{s,p,\delta}Tu = u \). It follows that \( T = A_{s,p,\delta}^{-1} \).

- **Case 2: \( s \leq 0 \).**

We want to show that \( A_{s,p,\delta} : W_{\delta}^{s,p} \to W_{\delta-m}^{s-m,p} \) is an isomorphism. We note that since \( A_{s,p,\delta} \) is a homogeneous constant coefficient elliptic operator, its adjoint \( (A_{s,p,\delta})^* : W_{-s-n+m}^{-s+m,p'} \to W_{-s-n}^{-s,p'} \) is also a homogeneous constant coefficient elliptic operator. So by what was proved in the previous case we know that if \(-s + m \geq m\) and \(-\delta - n + m \in (m - n, 0)\) (which are true because by assumption \( s \leq 0 \) and \( \delta \in (m - n, 0) \)) then \( (A_{s,p,\delta})^* \) is an isomorphism. Now for \( u \in W_{\delta-m}^{s-m,p} \) define the
distribution $Tu$ by

$$
\langle Tu, \varphi \rangle = \langle u, ((A_{s,p,\delta})^*)^{-1} \varphi \rangle_{W_{s-m,p}^{s-m,p}(W_{s-m,p}^{s-m,p})^*}
$$

(note that $((A_{s,p,\delta})^*)^{-1} : W_{s-m,p}^{-s,p} \rightarrow (W_{s-m,p}^{s-m,p})^*$)

for all $\varphi \in C_c$. We claim that $T$ is the inverse of $A_{s,p,\delta}$. To this end first we show that $T$ sends $W_{s-m,p}^{s-m,p}$ to $W_{s}^{s,p}$ and then we show that the composition of $T$ and $A_{s,p,\delta}$ is the identity map.

Suppose $u \in W_{s-m,p}^{s-m,p}$. In order to prove that $Tu \in W_{s}^{s,p}$ it is enough to show that

$$
\| Tu \|_{s,p,\delta} = \sup_{\varphi \in C_c} \frac{|\langle Tu, \varphi \rangle|}{\| \varphi \|_{-s,p',-\delta-n}} < \infty
$$

(we are interpreting $W_{s}^{s,p}$ as $(W_{s-m,p}^{s-m,p})^*$)

We have

$$
|\langle Tu, \varphi \rangle| \leq \| u \|_{s-m,p,\delta-m} \langle (A_{s,p,\delta})^* \rangle_{(W_{s-m,p}^{s-m,p})^*}^{-1} \| \varphi \|_{W_{s-m,p}^{s-m,p}}
$$

$$
\leq \| u \|_{s-m,p,\delta-m} \langle (A_{s,p,\delta})^* \rangle_{op}^{-1} \| \varphi \|_{-s,p',-\delta-n}
$$

Therefore

$$
\| Tu \|_{s,p,\delta} \leq \| u \|_{s-m,p,\delta-m} \langle (A_{s,p,\delta})^* \rangle_{op}^{-1} < \infty.
$$

This implies that $T$ sends $W_{s-m,p}^{s-m,p}$ to $W_{s}^{s,p}$. Now note that for all $u \in W_{s}^{s,p}$, $\varphi \in C_c$

$$
\langle TA_{s,p,\delta} u, \varphi \rangle = \langle A_{s,p,\delta} u, ((A_{s,p,\delta})^*)^{-1} \varphi \rangle = \langle u, (A_{s,p,\delta})^* ((A_{s,p,\delta})^*)^{-1} \varphi \rangle = \langle u, \varphi \rangle.
$$

This means $TA_{s,p,\delta} u = u$. Similarly $A_{s,p,\delta} Tu = u$.

- **Case 3:** $s \in (0, m)$.

By what was proved in the previous cases we know that $A_{0,p,\delta}$ and $A_{m,p,\delta}$ are
invertible. In fact

\[ A^{-1}_{0,p,\delta} : W^{-m,p}_{\delta-m} \to W^{0,p}_{\delta}, \]
\[ A^{-1}_{m,p,\delta} : W^{0,p}_{\delta-m} \to W^{m,p}_{\delta}, \]

are continuous maps. Let \( \theta = \frac{s}{m} \). Note that

\[
W^{m,p}_\delta \hookrightarrow W^{0,p}_\delta, \quad W^{0,p}_{\delta-m} \hookrightarrow W^{-m,p}_{\delta-m},
\]
\[
(W^{0,p}_\delta, W^{m,p}_\delta)_{0,p} = W^{s,p}_\delta, \quad (W^{-m,p}_{\delta-m}, W^{0,p}_{\delta-m})_{0,p} = W^{s-m,p}_{\delta-m} \quad \text{if } s \not\in \mathbb{N},
\]
\[
[W^{0,p}_\delta, W^{m,p}_\delta]_0 = W^{s,p}_\delta, \quad [W^{-m,p}_{\delta-m}, W^{0,p}_{\delta-m}]_0 = W^{s-m,p}_{\delta-m} \quad \text{if } s \in \mathbb{N}.
\]

So by interpolation we get a continuous operator \( T : W^{s-m,p}_{\delta-m} \to W^{s,p}_\delta \) which must be the restriction of \( A^{-1}_{0,p,\delta} \) to \( W^{s-m,p}_{\delta-m} \). Now for all \( u \in W^{s,p}_\delta \) we have

\[
u \in W^{s,p}_\delta \hookrightarrow W^{0,p}_\delta \Rightarrow A_{s,p,\delta} u = A_{0,p,\delta} u \Rightarrow T(A_{s,p,\delta} u) = T(A_{0,p,\delta} u) = A^{-1}_{0,p,\delta}(A_{0,p,\delta} u) = u.
\]

Similarly \( A_{s,p,\delta} Tu = u \). It follows that \( T = A^{-1}_{s,p,\delta} \).

\[ \square \]

**Proposition 4.6.** Let the following assumptions hold:

- \( A \in D_{m,\rho}^{\alpha,\gamma} \) where \( \gamma \in (1, \infty) \), \( \alpha > \frac{n}{\gamma}, \rho < 0 \), and \( m < n \); \( A \) is elliptic.
- \( q \in (1, \infty), s \in (m - \alpha, \alpha] \) (if \( s = \alpha \not\in \mathbb{N}_0 \), then let \( q \in [\gamma, \infty) \)).
- \( s - \frac{n}{q} \in (m - n - \alpha + \frac{n}{\gamma}, \alpha - \frac{n}{\gamma}) \).
- \( \delta \in (m - n, 0) \).
In particular note that if the elliptic operator $A \in D_{m,\rho}^{\alpha,\gamma}$ is given, then $s := \alpha$ and $q := \gamma$ satisfy the desired conditions.

Then: If $t < s$ and $\delta' > \delta$, then for every $u \in W^{s,q}_\delta$ we have

$$\|u\|_{s,q,\delta} \leq \|Au\|_{s-m,q,\delta-m} + \|u\|_{t,q,\delta'}$$

Moreover $A : W^{s,q}_\delta \to W^{s-m,q}_{\delta-m}$ is semi-Fredholm.

**Proof.** (Proposition 4.6) The approach of the proof is standard. In the proof we use the following facts (for these facts $s, \delta \in \mathbb{R}$ and $p \in (1, \infty)$):

- **Fact 1:** (see Lemma 3.24) If $f \in C^\infty_c(\mathbb{R}^n)$ and $u \in W^{s,p}(\mathbb{R}^n)$, then $fu \in W^{s,p}(\mathbb{R}^n)$ and moreover $\|fu\|_{s,p} \leq \|u\|_{s,p}$ (the implicit constant may depend on $f$ but is independent of $u$).

- **Fact 2:** (see Lemma 3.25) Let $\chi \in C^\infty_c(\mathbb{R}^n)$ be a cutoff function equal to 1 on $B_1$ and equal to 0 outside of $B_2$. Let $\tilde{\chi}(x) = 1 - \chi(x)$ and for all $\epsilon > 0$ define $\chi_\epsilon(x) = \chi(\frac{x}{\epsilon}), \tilde{\chi}_\epsilon(x) = \tilde{\chi}(\frac{x}{\epsilon})$. Then we have $\|\chi_\epsilon u\|_{s,p,\delta} \leq \|u\|_{s,p,\delta}$ and $\|\tilde{\chi}_\epsilon u\|_{s,p,\delta} \leq \|u\|_{s,p,\delta}$.

- **Fact 3:** Let $u \in W^{s,p}_\delta(\mathbb{R}^n)$. Also let $\Omega$ be an open bounded subset of $\mathbb{R}^n$. Then
  
  $\circ$ $u \in W^{s,p}(\Omega)$ and $\|u\|_{W^{s,p}(\Omega)} = \|u\|_{W^{s,p}_\delta(\mathbb{R}^n)}$.
  
  $\circ$ If $\text{supp } u \subseteq \Omega$, then $u \in W^{s,p}(\mathbb{R}^n)$ and $\|u\|_{W^{s,p}(\Omega)} \approx \|u\|_{W^{s,p}_\delta(\mathbb{R}^n)}$.

If $s \in \mathbb{N}_0$, the above items follow from the fact that weights are of the form $\langle x \rangle^a$ and so they attain their maximum and minimum on any compact subset of $\mathbb{R}^n$. If $s$ is not an integer, they can be proved by interpolation.

- **Fact 4:** Suppose $f \in W^{s,p}_\delta$ with $\delta > 0$ and $f$ vanishes in a neighborhood of the origin. Then
  
  $$\lim_{i \to \infty} \|S^{-i}f\|_{s,p,\delta} = 0.$$
The reason is as follows: by assumption there exists \( l \in \mathbb{N} \) such that \( f = 0 \) on \( B_{2^{-l}} \). So if \( \hat{l} \in \mathbb{Z} \) and \( \hat{l} < -l - 1 \) then \( S_{\hat{l}} f = 0 \) on \( B_2 \). Indeed,
\[
x \in B_2 \Rightarrow |2^l x| < 2^{l+1} < 2^{-l} \Rightarrow f(2^l x) = 0 \Rightarrow S_{\hat{l}} f(x) = 0.
\]

So for \( i > l + 2 \) we can write
\[
\| S_{2^{-i}} f \|_{s,p,\delta}^p = \sum_{j=0}^{\infty} 2^{-p\delta j} \| S_{2^{j}} (\phi_j S_{2^{-i}} f) \|_{W^s,p(\mathbb{R}^n)}^p
\]
\[
= \sum_{j=0}^{\infty} 2^{-p\delta j} \| \phi S_{2^{j-i}} f \|_{W^s,p(B_2)}^p \quad (S_{2^{j}} \phi_j = \phi, \text{ supp } \phi \subseteq B_2)
\]
\[
= \sum_{j=i-l-1}^{\infty} 2^{-p\delta j} \| \phi S_{2^{j-i}} f \|_{W^s,p(B_2)}^p \quad (S_{2^{j-i}} f = 0 \text{ on } B_2 \text{ if } j - i < -l - 1)
\]
\[
= \sum_{j=1}^{\infty} 2^{-p\delta (\hat{j} + i - l - 1)} \| \phi S_{2^{j-i-2}} f \|_{W^s,p(B_2)}^p \quad (\hat{j} := j - (i - l) + 2)
\]
\[
= 2^{-p\delta (i-l-2)} \sum_{j=1}^{\infty} 2^{-p\delta j} \| S_{2^{j}} (\phi_j S_{2^{-i-2}} f) \|_{W^s,p(\mathbb{R}^n)}^p
\]
\[
= 2^{-p\delta (i-l-2)} \| S_{2^{-l-2}} f \|_{s,p,\delta}^p \leq 2^{-p\delta (i-l-2)} \| f \|_{s,p,\delta}^p.
\]

It follows that \( \lim_{i \to \infty} \| S_{2^{-i}} f \|_{s,p,\delta} = 0 \).

**Fact 5: [Equivalence Lemma]**[69] Let \( E_1 \) be a Banach space, \( E_2, E_3 \) normed spaces, and let \( A \in L(E_1, E_2), B \in L(E_1, E_3) \) such that one has:
- \( \| u \|_1 \overset{\ll}{=} \| Au \|_2 + \| Bu \|_3 \).
- \( B \) is compact.

Then \( \ker A \) is finite dimensional and the range of \( A \) is closed, i.e., \( A \) is semi-Fredholm.

Now let’s start the proof. Let \( A = A_\infty + R \) where \( A_\infty \) is the principal part of \( A \) at infinity.
Let \( r \) be a fixed dyadic integer to be selected later and let \( u_r = \tilde{\chi}_r u \). By Lemma 4.5 we have

\[
\| u_r \|_{s,q,\delta} \leq \| A_{\infty} u_r \|_{s-m,q,\delta-m} \leq \| A u_r \|_{s-m,q,\delta-m} + \| R u_r \|_{s-m,q,\delta-m}.
\]

The implicit constant in the above inequality does not depend on \( r \). Now \( R \in D^{a,\gamma}_{m,\rho} \) has vanishing principal part at infinity. Therefore by Theorem 4.2 we can consider \( R \) as a continuous operator from \( W^{s,q}_{\delta-\rho} \) to \( W^{s-m,q}_{\delta-\rho-m+\rho} = W^{s-m,q}_{\delta} \). Also since \( \rho < 0 \) we have \( u_r \in W^{s,q}_{\delta} \hookrightarrow W^{s,q}_{\delta-\rho} \). Consequently

\[
\| R u_r \|_{s-m,q,\delta-m} \leq \| R \|_{op} \| u_r \|_{s,q,\delta-\rho} = \| R \|_{op} \| \tilde{\chi}_r u_r \|_{s,q,\delta-\rho} \quad \text{(note that} \quad \tilde{\chi}_r u_r = u_r \text{)}
\]

Now it is easy to check that \( W^{a,q}_{-\rho} \times W^{s,q}_{\delta} \hookrightarrow W^{s,q}_{\delta-\rho} \). Therefore

\[
\| R u_r \|_{s-m,q,\delta-m} \leq \| R \|_{op} \| \tilde{\chi}_r \|_{a,q,-\rho} \| u_r \|_{s,q,\delta}.
\]

By Fact 4, \( \lim_{i \to \infty} \| \tilde{\chi}_{2^i} \|_{a,q,-\rho} \to 0 \). Thus we can choose the fixed dyadic number \( r \) large enough so that

\[
\| R \|_{op} \| \tilde{\chi}_r \|_{a,q,-\rho} < \frac{1}{2},
\]

and so we get

\[
\| u_r \|_{s,q,\delta} \leq \| A u_r \|_{s-m,q,\delta-m} + \frac{1}{2} \| u_r \|_{s,q,\delta}.
\]

Hence

\[
\| u_r \|_{s,q,\delta} \leq \| A u_r \|_{s-m,q,\delta-m} \leq \| \tilde{\chi}_r A u \|_{s-m,q,\delta-m} + \| [A, \tilde{\chi}_r] u \|_{s-m,q,\delta-m}.
\]

By Fact 2, \( \| \tilde{\chi}_r A u \|_{s-m,q,\delta-m} \leq \| A u \|_{s-m,q,\delta-m} \). Also one can easily show that \( [A, \tilde{\chi}_r] u \) has support in \( B_{2r} \) and so by Fact 3, \( \| [A, \tilde{\chi}_r] u \|_{s-m,q,\delta-m} \leq \| [A, \tilde{\chi}_r] u \|_{W^{s-m,q}(B_{2r})} \). On the bounded domain \( B_{2r} \), \( [A, \tilde{\chi}_r] \in D^{a,\gamma}_{m-1} \), so \( [A, \tilde{\chi}_r] : W^{s-1,q}(B_{2r}) \to W^{s-m,q}(B_{2r}) \) is
continuous. Consequently

\[ \| [A, \tilde{\chi}_r] u \|_{W^{s-m,q}(B_{2r})} \lesssim \| u \|_{W^{s-1,q}(B_{2r})} \lesssim \| u \|_{W^{s,q}(B_{2r})}. \]

Thus

\[ \| u_r \|_{s,q,\delta} \lesssim \| Au \|_{s-m,q,\delta-m} + \| u \|_{W^{s,q}(B_{2r})}. \]

Now we can write

\[ \| u \|_{s,q,\delta} = \| u_r + \chi_r u \|_{s,q,\delta} \leq \| u_r \|_{s,q,\delta} + \| \chi_r u \|_{s,q,\delta} \]

\[ \leq \| u_r \|_{s,q,\delta} + \| \chi_r u \|_{W^{s,q}(B_{2r})} \quad (\chi_r u \text{ has support in } B_{2r}, \text{ Fact 3}) \]

\[ \leq \| u_r \|_{s,q,\delta} + \| u \|_{W^{s,q}(B_{2r})} \quad (\text{Fact 1}) \]

\[ \leq \| Au \|_{s-m,q,\delta-m} + \| u \|_{W^{s,q}(B_{2r})}. \]

From interior regularity estimate for elliptic operators on unweighted Sobolev spaces (Lemma 4.4) we know there exists \( \tilde{s} < s \) such that

\[ \| u \|_{W^{s,q}(B_{2r})} \leq \| Au \|_{W^{s-m,q}(B_{3r})} + \| u \|_{W^{s,q}(B_{3r})}, \]

(4.2)

and by Fact 3

\[ \| Au \|_{W^{s-m,q}(B_{3r})} \leq \| Au \|_{s-m,q,\delta-m}. \]

It follows that

\[ \| u \|_{s,q,\delta} \leq \| Au \|_{s-m,q,\delta-m} + \| u \|_{W^{s,q}(B_{3r})}. \]

But by Fact 3, for any \( \delta' \in \mathbb{R} \) we have \( \| u \|_{W^{s,q}(B_{3r})} \leq \| u \|_{s,q,\delta'} \). This implies

\[ \| u \|_{s,q,\delta} \leq \| Au \|_{s-m,q,\delta-m} + \| u \|_{s,q,\delta'}. \]

(4.3)
Now, if \( t < s \) then either \( t \geq \tilde{s} \) or \( t < \tilde{s} \). If \( t \geq \tilde{s} \) then \( W^{s,q}_{\delta} \hookrightarrow W^{s,q}_{\delta'} \) and so \( \|u\|_{s,q,\delta'} \leq \|u\|_{t,q,\delta'} \). If \( t < \tilde{s} \), then for \( \delta' > \delta \) we have \( W^{s,q}_{\delta} \hookrightarrow W^{s,q}_{\delta'} \hookrightarrow W^{r,q}_{\delta} \) where the first embedding is compact and the second is continuous. Therefore, by Ehrling’s lemma, for all \( \epsilon > 0 \) there exists \( C(\epsilon) \) such that

\[
\|u\|_{s,q,\delta'} \leq \epsilon \|u\|_{s,q,\delta} + C(\epsilon) \|u\|_{t,q,\delta'}
\]

In particular the above inequality holds for \( \epsilon = \frac{1}{2} \). Combining this with (4.3) we can conclude that for all \( t < s \) and \( \delta' > \delta \)

\[
\|u\|_{s,q,\delta} \leq \|Au\|_{s-m,q,\delta-m} + \|u\|_{t,q,\delta'}
\]

It remains to show that \( A : W^{s,q}_{\delta} \rightarrow W^{s-m,q}_{\delta-m} \) is semi-Fredholm. Pick any \( \delta' \) strictly larger than \( \delta \). By assumption \( s > m - \alpha \), so we have \( \|u\|_{s,q,\delta} \leq \|Au\|_{s-m,q,\delta-m} + \|u\|_{m-a,q,\delta'} \).

Also \( W^{s,q}_{\delta} \hookrightarrow W^{m-a,q}_{\delta'} \) is compact. Hence by the estimate that was proved above and

\[\text{Fact 5}, \ A : W^{s,q}_{\delta} \rightarrow W^{s-m,q}_{\delta-m} \text{ is semi-Fredholm.} \]

\[\square\]

\textbf{Remark 4.7.} The proof of Proposition 4.6 in fact shows that if \( u \in W^{s,q}_{\delta'} \) for some \( t < s \) and \( \delta' > \delta \) and if \( Au \in W^{s-m,q}_{\delta-m} \) then \( u \in W^{s,q}_{\delta} \).

\textbf{Lemma 4.8.} Let the following assumptions hold:

- \( A \in D^{\alpha,\gamma}_{m,\rho}, \gamma \in (1,\infty), \alpha > \frac{n}{\gamma}, \rho < 0, \) and \( m < n \). \( A \) is elliptic.

- \( e \in (m - \alpha, \alpha) \) (if \( e = \alpha \not\in \mathbb{N}_0 \), then let \( q \in (\gamma, \infty) \)).

- \( e - \frac{n}{q} \in (m - n - \alpha + \frac{n}{\gamma}, \alpha - \frac{n}{\gamma}] \).

Then: If \( u \in W^{e,q}_{\beta} \) for some \( \beta < 0 \) satisfies \( Au = 0 \), then \( u \in W^{e,q}_{\beta'} \) for all \( \beta' \in (m - n, 0) \).
Proof. (Lemma 4.8) Let $A = A_\infty + R$ where $A_\infty$ is the principal part of $A$ at infinity. $R$ has vanishing principal part at infinity and therefore by Theorem 4.2, $Ru \in W^{e-m,q}_{\beta-m+\rho}$. Consequently $A_\infty u = -Ru \in W^{e-m,q}_{\beta-m+\rho}$. Now we may consider two cases:

- If $\beta + \rho \leq m - n$, then $\beta - m + \rho \leq -n$ and so $W^{e-m,q}_{\beta-m+\rho} \hookrightarrow W^{e-m,q}_\eta$ for all $\eta \geq -n$. Consequently $A_\infty u \in W^{e-m,q}_\eta$ for all $\eta \geq -n$. Since $A_\infty : W^{s,q}_\beta \rightarrow W^{e-m,q}_\beta$ is an isomorphism for all $\beta' \in (m-n,0)$ we conclude that $u \in W^{e,q}_\beta$ for all $\beta' \in (m-n,0)$.

- If $\beta + \rho > m - n$ then $A_\infty : W^{e,q}_{\beta + \rho} \rightarrow W^{e-m,q}_{\beta-m+\rho}$ is an isomorphism and therefore $u \in W^{e,q}_{\beta + \rho}$ which implies $u \in W^{e,q}_{\beta'}$ for all $\beta' \in (\beta + \rho,0)$

Combining the above observations, we can conclude that $u \in W^{e,q}_{\beta'}$ for every $\beta' \in (\max(m-n,\beta + \rho),0)$.

Now clearly for some $N \in \mathbb{N}$ we have $\beta + N\rho < m - n$ and therefore by iteration we get $u \in W^{e,q}_{\beta'}$ for every $\beta' \in (m-n,0)$. \qed

Lemma 4.9 (Maximum principle). Suppose $(M, h)$ is an $n$-dimensional AF manifold of class $W^{s,p}_\delta$ where $s \in (\frac{1}{p}, \infty) \cap [1, \infty), p \in (1, \infty)$, and $\delta < 0$. Also suppose $a \in W^{s-2,p}_{\eta-2}$, $\eta \in \mathbb{R}$, $\eta < 0$. Suppose that $a \geq 0$.

- (a) If $u \in W^{s,p}_\rho$ for some $\rho < 0$ satisfies

$$-\Delta_h u + au \leq 0$$

then $u \leq 0$. In particular, if $-\Delta_h u + au = 0$, then applying this result to $u$ and $-u$ shows that $u = 0$.

- (b) Suppose that $u \in W^{s,p}_\rho$ is nonpositive and satisfies

$$-\Delta_h u \leq 0.$$
If \( u(x) = 0 \) at some point \( x \in M \), then \( u \) vanishes identically.

**Proof.** (Lemma 4.9) For (a), we combine the proof that is given in [38] for the case of closed manifolds and the proof that is given in [52] for the case where \( p = 2 \). Fix \( \epsilon > 0 \). By assumptions \( u \in W^{s,p}_p \hookrightarrow C^0_p \) and therefore \( u \) goes to zero at infinity. Therefore if we let \( v = (u - \epsilon)^+ := \max\{u - \epsilon, 0\} \), then \( v \) is compactly supported. Note that if \( f \in W^{1,q}_{\text{loc}} \), then \( f + \in W^{1,q}_{\text{loc}} \) and so we have

\[
\begin{align*}
  u \in W^{s,p}_p & \hookrightarrow W^{1,n}_p \Rightarrow u \in W^{1,n}_{\text{loc}} \Rightarrow u - \epsilon \in W^{1,n}_{\text{loc}} \Rightarrow v \in W^{1,n}_{\text{loc}} \Rightarrow v \in W^{1,n},
\end{align*}
\]

since \( v \) has compact support. Now \( u \in W^{s,p}_{\text{loc}} \) and so \( u \in W^{s,p} \) in the support of \( v \). By the multiplication lemma \( W^{s,p} \times W^{1,n} \hookrightarrow W^{1,n} \), therefore \( uv \) is a nonnegative, compactly supported element of \( W^{1,n} \). Since \( W^{1,n} \hookrightarrow (W^{s-2,p})^* \) and \( a \in W^{s-2,p}_{\eta-2} \subseteq W^{s-2,p}_{\text{loc}} \), we can apply \( a \) to \( uv \) and since \( a \geq 0 \) and \( uv \geq 0 \) we have \( \langle a, uv \rangle_{(M,h)} \geq 0 \). Hence

\[
0 \geq -\langle a, uv \rangle \geq \langle -\Delta_h u, v \rangle = \langle \text{grad} u, \text{grad} v \rangle = \langle \text{grad} v, \text{grad} v \rangle.
\]

It follows that \( v \) is constant with compact support which means \( v \equiv 0 \). Note that \( v \equiv 0 \) if and only if \( u - \epsilon \leq 0 \). So \( u \leq \epsilon \) for all \( \epsilon > 0 \). This shows \( u \leq 0 \).

For (b), the proof is based on the weak Harnack inequality. The exact same proof as the one that is given in [38] for closed manifolds, works for the above setting as well. \( \square \)

**Lemma 4.10** (Elliptic estimate for Laplacian). Suppose \((M,h)\) is an \( n \)-dimensional \((n > 2)\) AF manifold of class \( W^{\alpha,\gamma}_p \), \( \alpha \geq 1, \alpha > \frac{n}{2} \), \( \rho < 0 \). If \( \alpha > 1 \), let \( \sigma \in (2 - \alpha, \alpha] \) be such that \( (\sigma - \frac{n}{2}) + (\alpha - \frac{n}{2}) > 2 - n \). If \( \alpha = 1 \), let \( \sigma = 1 \). Then

1. \(-\Delta_h \in D^{\alpha,\gamma}_{2,\rho} \).

2. For all \( \delta \in \mathbb{R} \), \(-\Delta_h : W^{\sigma,\gamma}_\delta \rightarrow W^{\sigma - 2,\gamma}_{\delta - 2} \) is a continuous elliptic operator.
3. For all $\delta \in (2 - n, 0)$, $-\Delta_h : W_\delta^{\sigma,\gamma} \to W_{\delta-2}^{\sigma-2,\gamma}$ is semi-Fredholm and satisfies the following elliptic estimate:

$$\|u\|_{W_\delta^{\sigma,\gamma}} \lesssim \|-\Delta_h u\|_{W_{\delta-2}^{\sigma-2,\gamma}} + \|u\|_{W_{\delta-2}^{\sigma-2,\gamma}}.$$

where $\delta'$ can be any real number larger than $\delta$.

4. For all $\delta \in (2 - n, 0)$, $-\Delta_h : W_\delta^{\alpha,\gamma} \to W_{\delta-2}^{\alpha-2,\gamma}$ is an isomorphism. In particular

$$\|u\|_{W_\delta^{\alpha,\gamma}} \lesssim \|-\Delta_h u\|_{W_{\delta-2}^{\alpha-2,\gamma}}.$$

**Proof. (Lemma 4.10)** Item 1 is a direct consequence of the multiplication lemma and the expression of Laplacian in local coordinates. Item 2 is a direct consequence of item 1 and Theorem 4.2. Item 3 is a direct consequence of item 1 and Proposition 4.6. For the last item we can proceed as follows:

By item 3, $-\Delta_h : W_\delta^{\alpha,p} \to W_{\delta-2}^{\alpha-2,p}$ is semi-Fredholm. On the other hand, Laplacian of the rough metric can be approximated by the Laplacian of smooth metrics and it is well known that Laplacian of a smooth metric is Fredholm of index zero. Therefore since the index of a semi-Fredholm map is locally constant, it follows that $-\Delta_h$ is Fredholm with index zero. Now maximum principle (Lemma 4.9) implies that the kernel of $-\Delta_h : W_\delta^{\alpha,p} \to W_{\delta-2}^{\alpha-2,p}$ is trivial. An injective operator of index zero is surjective as well. Consequently $-\Delta_h : W_\delta^{\alpha,p} \to W_{\delta-2}^{\alpha-2,p}$ is a continuous bijective operator. Therefore by the open mapping theorem, $-\Delta_h : W_\delta^{\alpha,p} \to W_{\delta-2}^{\alpha-2,p}$ is an isomorphism of Banach spaces. In particular the inverse is continuous and so $\|u\|_{W_\delta^{\alpha,\gamma}} \lesssim \|-\Delta_h u\|_{W_{\delta-2}^{\alpha-2,\gamma}}$.  

As it is discussed in Appendix A, compact perturbations of Fredholm operators remain Fredholm. The following lemma comes handy in identifying a useful collection of compact operators.
Lemma 4.11. Let the following assumptions hold:

- \( \eta \in \mathbb{R}, \delta \in (-\infty, 0) \).
- \( p \in (1, \infty), \alpha \in \left( \frac{n}{p}, \infty \right) \cap (1, \infty) \).
- \( \sigma \in (2 - \alpha, \alpha] \cap \left( \frac{2n}{p} - n + 2 - \alpha, \infty \right) \).
- \( a(x) \in W^{\alpha - 2, p}_{\eta - 2} \).

Then: For all \( \nu > \delta + \eta - 2 \), the operator \( K : W^{\alpha, p}_\delta (\mathbb{R}^n) \to W^{\sigma - 2, p}_\nu (\mathbb{R}^n) \) defined by \( K(\psi) = a \psi \) is compact. (In particular, we can set \( \nu = \eta - 2 \) and for \( n \geq 3 \) we can set \( \sigma = \alpha \).)

Proof. (Lemma 4.11) \( W^{\alpha, p}_\delta \) is a reflexive Banach space; so in order to show that \( K \) is a compact operator, we just need to prove that it is completely continuous, that is, we need to show if \( \psi_n \to \psi \) weakly in \( W^{\alpha, p}_\delta \), then \( K\psi_n \to K\psi \) strongly in \( W^{\sigma - 2, p}_\nu \). Let

\[
\beta = \min\{\alpha - \frac{n}{p}, \sigma - (2 - \alpha), 1, \sigma - n\left(\frac{2}{p} - 1\right) - 2 + \alpha\}.
\]

\[
\theta = \sigma - \frac{1}{2} \beta.
\]

\[
\delta' = \delta + \frac{1}{2} [\nu - (\delta + \eta - 2)].
\]

**Step 1:** It follows from the assumptions that \( \beta > 0 \) and so \( \theta < \sigma \). Also clearly \( \delta' > \delta \).

Thus we can conclude that \( W^{\alpha, p}_\delta \hookrightarrow W^{\theta, p}_{\delta'} \) is compact. Therefore \( \psi_n \to \psi \) strongly in \( W^{\theta, p}_{\delta'} \).

**Step 2:** Now we prove that

\[
W^{\alpha - 2, p}_{\eta - 2} \times W^{\theta, p}_{\delta'} \hookrightarrow W^{\sigma - 2, p}_\nu.
\]

According to the multiplication lemma we need to check the following conditions
(i) \( \alpha - 2 \geq \sigma - 2 \). True because \( \alpha \geq \sigma \).

\[ \theta \geq \sigma - 2. \] True because \( \beta \leq 1 \) and so

\[ \theta = \sigma - \frac{1}{2} \beta \geq \sigma - \frac{1}{2} \geq \sigma - 2. \]

(ii) \( \alpha - 2 + \theta > 0 \). True because \( \beta \leq \sigma - (2 - \alpha) \) and so

\[ \theta = \sigma - \frac{1}{2} \beta > \sigma - \beta \geq \sigma - (2 - \alpha) = 2 - \alpha. \]

(iii) \( (\alpha - 2) - (\sigma - 2) \geq 0 \). True because \( \alpha \geq \sigma \).

\[ \theta - (\sigma - 2) \geq 0. \] This one was shown above.

(iv) \( (\alpha - 2) + \theta - (\sigma - 2) > n(\frac{1}{p} + \frac{1}{p} - \frac{1}{p}) \). That is, we need to show \( \theta > \sigma - (\alpha - \frac{n}{p}) \).

This is true because \( \beta \leq \alpha - \frac{n}{p} \) and so

\[ \theta = \sigma - \frac{1}{2} \beta \geq \sigma - \frac{1}{2}(\alpha - \frac{n}{p}) > \sigma - (\alpha - \frac{n}{p}). \]

(v) \( (\alpha - 2) + \theta > n(\frac{1}{p} + \frac{1}{p} - 1) \). This is true because \( \beta \leq \sigma - n(\frac{2}{p} - 1) - 2 + \alpha \) and so

\[ \theta = \sigma - \frac{1}{2} \beta > \sigma - \beta \geq \sigma - [\sigma - n(\frac{2}{p} - 1) - 2 + \alpha] = n(\frac{2}{p} - 1) + 2 - \alpha. \]

The numbering of the above items agrees with the numbering of the conditions in the multiplication lemma. Also note that \( (\eta - 2) + \delta' \leq \nu \) by the definition of \( \delta' \).

**Step 3:** By what was proved in **Step 2** we have

\[ \| a(\psi_n - \psi) \|_{W_{\sigma-2, p}} \leq \| a \|_{W_{\eta-2, p}} \| \psi_n - \psi \|_{W_{\theta, p}}. \]

But by **Step 1**, the right hand side goes to zero, which means \( a\psi_n \to a\psi \) strongly in
Chapter 4, in part, has been submitted for publication. The material may appear as A. Behzadan and M. Holst, *Rough Solutions of the Einstein Constraint Equations on Asymptotically Flat Manifolds Without Near-CMC Conditions*. The dissertation author was the primary investigator and author of this paper.
Chapter 5

LCBY Equations on AF Manifolds

In order to give a concise mathematical formulation of the LCBY equations (i.e. equations (2.8) and (2.9)), we set

\[ F(\phi, W) = a_R \phi + a_\tau \phi^5 - a_W \phi^{-7} - a_\rho \phi^{-3}, \quad F(\phi) = b_\tau \phi^6 + b_J, \]

where

\[ b^b_\tau = (2/3)\nabla^b \tau, \quad b^b_J = \kappa J^b, \quad a_R = R/8, \quad a_\tau = \tau^2/12, \]
\[ a_\rho = \kappa \rho/4, \quad a_W = [\sigma_{ab} + (\mathcal{L} W)_{ab}][\sigma^{ab} + (\mathcal{L} W)^{ab}]/8. \]

The classical formulation of the LCBY equations can be stated as follows.

**Classical Formulation.** *Given smooth functions \( \tau \) and \( \rho \), rank 2 transverse-traceless tensor field \( \sigma \), and vector field \( J \) on the smooth 3-dimensional Riemannian manifold*
\((M, h)\), find a smooth scalar field \(\phi > 0\) and a vector field \(W\) in \(\chi(M)\) such that

\[
\begin{align*}
-\Delta \phi + F(\phi, W) &= 0, \\
-\Delta_L W + F(\phi) &= 0.
\end{align*}
\]

Our goal in this chapter is to provide an answer to the question of existence of solutions in the case of AF manifolds with very low regularity assumptions on the data. The first step to achieve this goal is to come up with a weak formulation of the LCBY equations in order to accommodate nonsmooth data on an AF manifold \((M, h)\) of class \(W^{s, p}_{\delta}\). Then we combine a priori estimates for the individual Hamiltonian and momentum constraints, sub- and supersolution construction for the Hamiltonian constraint, together with a topological fixed-point argument for the coupled system to establish existence of non-CMC weak solutions of the LCBY equations.

### 5.1 Weak Formulation on Asymptotically Flat Manifolds

There are at least two different general settings where the constraint equations are well-defined with rough data; one of them is described in this chapter and the other is discussed in Appendix H. In both settings it is assumed that the AF manifold is of class \(W^{s, p}_{\delta}\) where \(s > \frac{3}{p}\) (and of course \(p \in (1, \infty)\), \(\delta < 0\)). So by Corollary 3.33, \(W^{s, p}_{\delta}\) is a Banach algebra and \(W^{s, p}_{\delta} \hookrightarrow C^0_{\delta} \hookrightarrow L^\infty_{\delta}\). The framework that is described in Appendix H (which we refer to as Weak Formulation 2) only works for \(s \leq 2\), but the framework that is described in this chapter (which we refer to as Weak Formulation 1) works for all \(s > 3/p\) with \(p \in (1, \infty)\), even when \(s > 2\). (However, as we explain at some length in Appendix H, Weak Formulation 2 is not simply a special case of Weak Formulation 1.)
Note that if \((M, h)\) is a 3-dimensional AF manifold of class \(W_\delta^{s,p}\) and if \(u \in W_\delta^{s,p}\) is a positive function, then \((M, u^4h)\) is not asymptotically flat of class \(W_\delta^{s,p}\) (item (4) in Definition 3.11 is not satisfied). However, if \(u\) is a positive function such that \(u - \mu \in W_\delta^{s,p}(M)\) for some positive constant \(\mu\), then \((M, u^4h)\) is also AF of class \(W_\delta^{s,p}\).

Indeed,

\[
u - \mu \in W_\delta^{s,p} \Rightarrow u - \mu \in W_{loc}^{s,p} \Rightarrow u \in W_{loc}^{s,p} \Rightarrow u^4h \in W_{loc}^{s,p} \quad (W_{loc}^{s,p} \text{ is an algebra}).\]

In addition, \(u - \mu \in W_\delta^{s,p}\) implies that \(u\) is bounded and \(\inf u > 0\) (see Remark 5.12; note that \(u\) is a positive function). Thus, there exists a positive number \(\zeta\) such that \(\zeta^{-1} < u^4 < \zeta\). Consequently for each \(i\), \((\phi_i^{-1})^* u^4 = u^4 \circ \phi_i^{-1}\) is between \(\zeta^{-1}\) and \(\zeta\) which subsequently implies that

\[
\forall x \in E_1 \forall y \in \mathbb{R}^n \quad (\zeta \theta)^{-1} |y|^2 \leq ((\phi_i^{-1})^* (u^4h))_{rs}(x) y^r y^s \leq (\zeta \theta) |y|^2.
\]

\(\{(\phi_i)_{i=1}^m\}\) are the diffeomorphisms that were introduced in item 2. of the definition of asymptotically flat manifolds and \(\theta\) is the constant in item 3. of that definition (Definition 3.11).)

Finally, since \((M, h)\) is AF of class \(W_\delta^{s,p}\), there exists a constant \(\omega\) such that \((\phi_i^{-1})^* h - \omega^4 \tilde{h} \in W_\delta^{s,p}(E_1)\); if we let \(v = u - \mu\) and \(f(x) = (\mu + x)^4\), then for each \(1 \leq i \leq m\) (end coordinates) we have

\[
(\phi_i^{-1})^* (u^4h) - (\mu \omega)^4 \tilde{h} = (u^4 \circ \phi_i^{-1})(\phi_i^{-1})^* h - (\mu \omega)^4 \tilde{h}
\]

\[
= (\mu + v \circ \phi_i^{-1})^4 (\phi_i^{-1})^* h - (\mu \omega)^4 \tilde{h}
\]

\[
= f(v \circ \phi_i^{-1}) (\phi_i^{-1})^* h - (\mu \omega)^4 \tilde{h}
\]

\[
= f(v \circ \phi_i^{-1}) ((\phi_i^{-1})^* h - \omega^4 \tilde{h}) + (\omega^4 f(v \circ \phi_i^{-1}) - (\mu \omega)^4) \tilde{h}.
\]
Since \((\phi_i^{-1})^* h - \omega^A \bar{h} \in W^{s,p}_\delta (E_1)\), by Lemma 3.34 the first term on the right is in \(W^{s,p}_\delta (E_1)\).

Also as a direct consequence of Corollary 3.36, the second term on the right is in \(W^{s,p}_\delta (E_1)\).

In the LCBY equations, \(\phi > 0\) is the conformal factor, so assuming \((M, h)\) is a 3-dimensional AF manifold of class \(W^{s,p}_\delta\), by what was mentioned above, it seems reasonable to let \(\phi = \psi + \mu\) (so \(\psi > -\mu\)) where \(\psi \in W^{s,p}_\delta\) and \(\mu\) is an arbitrary but fixed positive constant (we have freedom in choosing the constant \(\mu\)).

We can write the Hamiltonian constraint in terms of \(\psi\) as:

\[-\Delta \psi + f(\psi, W) = 0,\]

where

\[f(\psi, W) = F(\phi, W) = a_R \phi + a_\tau \phi^5 - a_W \phi^{-7} - a_\rho \phi^{-3}\]

\[= a_R (\psi + \mu) + a_\tau (\psi + \mu)^5 - a_W (\psi + \mu)^{-7} - a_\rho (\psi + \mu)^{-3}.\]

Since \(\psi \in W^{s,p}_\delta\), we want to be able to extend \(-\Delta : C^\infty \rightarrow C^\infty\) to an operator \(A_L : W^{s,p}_\delta \rightarrow W^{s-2,p}_{\delta-2}\). Since \(-\Delta \in D^{s,p}_{2,\delta}\) (See Definition 4.1), by the extension theorem (Theorem 4.2), the only extra assumption needed to ensure the above extension is possible is \(s \geq 1\). Indeed, according to Theorem 4.2, we must check the following conditions (following the numbering used in Theorem 4.2):
(i) $p \in (1, \infty)$, \hspace{1cm} \text{(true by assumption)}

(ii) $s \geq 2 - s$, \hspace{1cm} \text{(so need to assume $s \geq 1$)}

(iii) $s - 2 \leq s - 2$, \hspace{1cm} \text{(trivially true)}

(iv) $s - 2 < s - 2 + s - \frac{3}{p}$, \hspace{1cm} \text{(since $s > \frac{3}{p}$)}

(v) $s - 2 - \frac{3}{p} \leq s - \frac{3}{p} - 2$, \hspace{1cm} \text{(trivially true)}

(vi) $s - \frac{3}{p} > 2 - 3 - s + \frac{3}{p}$, \hspace{1cm} \text{(since $s > \frac{3}{p}$)}

\textbf{Framework 1:}

In this framework we look for $W$ in $W^{s,q}_{\beta}$ where $\beta < 0$. For the momentum constraint to be well-defined, we need to ensure that

The operator: $- \Delta_L : C^\infty \to C^\infty$ can be extended to $A_L : W^{s,q}_{\beta} \to W^{e-2,q}_{\beta-2}$, \hspace{1cm} (5.1)

It holds that: $b_t (\psi + \mu)^6 + b_f \in W^{e-2,q}_{\beta-2}$. \hspace{1cm} (5.2)

The vector Laplacian belongs to the class $D^{s,p}_{2,\delta}$ (See Definition 4.1). Therefore, by Theorem 4.2, in order to ensure that condition (5.1) holds true, it is enough to require $e$ and $q$ satisfy the following conditions (again, the numbering below corresponds to numbering in Theorem 4.2):

(i) $q \in (1, \infty)$,

(ii) $e > 2 - s$,

(iii) $e - 2 \leq \min\{e, s\} - 2$, $p \leq q$ if $e = s \not\in \mathbb{N}_0$, \hspace{1cm} \text{(in particular, need $e \leq s$)}

(iv) $e - 2 < e - 2 + s - \frac{3}{p}$, \hspace{1cm} \text{(holds by assumption $s > \frac{3}{p}$)}

(v) $e - 2 - \frac{3}{q} \leq s - \frac{3}{p} - 2$, \hspace{1cm} \text{(must assume $e \leq s + \frac{3}{q} - \frac{3}{p}$)}

(vi) $e - \frac{3}{q} > 2 - 3 - s + \frac{3}{p}$, \hspace{1cm} \text{(must assume $e > -s + \frac{3}{p} - 1 + \frac{3}{q}$)}
Combining these constraints, we see it is enough to have

\[ q \in (1, \infty), \]
\[ e \in (2 - s, s] \cap (-s + \frac{3}{p} - 1 + \frac{3}{q}, s + \frac{3}{q} - \frac{3}{p}). \quad (p = q \text{ if } e = s \not\in \mathbb{N}_0) \]

Note that in case \( e = s \not\in \mathbb{N}_0 \) we need to assume \( p \leq q \), which together with the inequality \( s = e \leq s + \frac{3}{q} - \frac{3}{p} \) justifies the assumption \( p = q \) in this case.

In order to ensure that condition (5.2) holds true, it is enough to make the extra assumptions that \( \tau \) is given in \( W^{e-1,q}_{\beta-1} \) and \( J \) is given in \( W^{e-2,q}_{\beta-2} \). Indeed, note that \( \tau \in W^{e-1,q}_{\beta-1} \) implies \( b_{\tau} \in W^{e-2,q}_{\beta-2} \). Since \( \psi \in W^{s,p}_{\delta} \), it follows from Lemma 3.34 that \( b_{\tau}(\psi + \mu)^6 \in W^{e-2,q}_{\beta-2} \); Lemma 3.34 can be applied since (numbering corresponds to the numbering of conditions in Lemma 3.34):

(i) \( e - 2 \in (-s, s] \), \quad (since \( e \in (2 - s, s] \))

(ii) \( e - 2 - \frac{3}{q} \leq s - \frac{3}{p} \), \quad (since \( e \leq s + \frac{3}{q} - \frac{3}{p} \))

\[ -3 - s + \frac{3}{p} \leq e - 2 - \frac{3}{q}. \quad (since \( e > -s + \frac{3}{p} - 1 + \frac{3}{q} \)) \]

In summary, for the momentum constraint to be well-defined, it is enough to make the following additional assumptions:

\[ q \in (1, \infty), \quad e \in (2 - s, s] \cap (-s + \frac{3}{p} - 1 + \frac{3}{q}, s + \frac{3}{q} - \frac{3}{p}), \quad \tau \in W^{e-1,q}_{\beta-1}, \quad J \in W^{e-2,q}_{\beta-2}. \]

Of course, we let \( p = q \) if \( e = s \not\in \mathbb{N}_0 \), and the base assumptions hold as well \((s \geq 1, p \in (1, \infty), \delta, \beta < 0, s > \frac{3}{p}\)\). Note that for \((2 - s, s] \) to be nonempty, in fact we need \( s > 1 \).

Finally, we now consider the Hamiltonian constraint. Note that \( W \in W^{e,q}_{\beta} \) and so that \( \mathcal{L}W \in W^{e-1,q}_{\beta-1} \). For \( aW = \frac{1}{2} |\sigma + \mathcal{L}W|^2 \) to be well-defined, it is enough to assume \( \sigma \in W^{e-1,q}_{\beta-1} \). Recall that \( A_{\ell} \) is a well-defined operator from \( W^{s,p}_{\delta} \) to \( W^{s-2,p}_{\delta-2} \).

If we set \( \eta = \max(\beta, \delta) \), then \( W^{s-2,p}_{\delta-2} \hookrightarrow W^{s-2,p}_{\eta-2} \). In fact, \( A_{\ell} \) can be considered as an operator from \( W^{s,p}_{\delta} \) to \( W^{s-2,p}_{\eta-2} \) where \( \eta = \max(\beta, \delta) \). Consequently, for the Hamiltonian
constraint to be well-defined, we need to have

\[ f(\psi, W) = a_R(\psi + \mu) + a^5_R(\psi + \mu)^5 - a_W(\psi + \mu)^7 - a^3_R(\psi + \mu)^3 \in W^{s-2,p}_{\eta-2}. \]

One way to guarantee that the above statement holds true is to ensure that

\[ a^\tau, a^\rho, a^W \in W^{s-2,p}_{\beta-2}, \quad a^R \in W^{s-2,p}_{\delta-2}, \]

and then show that if \( f \) is smooth on \((-\mu, \infty), u \in W^{s,p}_\delta, \) and \( v \in W^{s-2,p}_{\eta-2} \) then \( f(u)v \in W^{s-2,p}_{\eta-2}. \) We claim that for above statements to be true it is enough to make the following extra assumptions:

\[ e > 1 + \frac{3}{q}, \quad e \geq s - 1, \quad e \geq \frac{3}{q} + s - \frac{3}{p} - 1, \quad \rho \in W^{s-2,p}_{\beta-2}. \]

The details are as follows:

1. If \( f \) is smooth and \( u \in W^{s,p}_\delta, \) \( v \in W^{s-2,p}_{\eta-2} \) then \( f(u)v \in W^{s-2,p}_{\eta-2}. \)

   By Lemma 3.34, we just need to check the following (the numbering matches that of the conditions in Lemma 3.34):

   (i) \( s - 2 \in (-s, s), \) \quad (since \( s > 1 \))

   (ii) \( s - 2 - \frac{3}{p} \in [-3 - s + \frac{3}{p}, s - \frac{3}{p}], \) \quad (since \( s > \frac{3}{p} \))

   This shows that \( f(u)v \in W^{s-2,p}_{\eta-2}. \)

2. \( a^\tau = \frac{1}{12} \tau^2. \)

   We want to ensure \( a^\tau \in W^{s-2,p}_{\beta-2}. \) Note that \( \tau \in W^{e-1,q}_{\beta-1}; \) since \( e - 1 > \frac{3}{q} \), \( W^{e-1,q}_{\beta-1} \times W^{e-1,q}_{\beta-1} \rightarrow W^{e-1,q}_{2\beta-2} \) (see Corollary 3.32). Therefore \( \tau^2 \in W^{e-1,q}_{2\beta-2}. \) Thus we want to have \( W^{e-1,q}_{2\beta-2} \rightarrow W^{s-2,p}_{\beta-2}. \) We will see that because of the assumptions \( e \geq s - 1 \)
and \( e \geq \frac{3}{q} + s - \frac{3}{p} - 1 \) this embedding holds true. We just need to check that the assumptions of Theorem 3.23 are satisfied (numbering follows assumptions of Theorem 3.23):

(ii) \( e - 1 \geq s - 2 \), (since \( e \geq s - 1 \))

(iii) \( e - 1 - \frac{3}{q} \geq s - 2 - \frac{3}{p} \), (since \( e \geq \frac{3}{q} + s - \frac{3}{p} - 1 \))

(iv) \( 2\beta - 2 < \beta - 2 \). (since \( \beta < 0 \))

3. \( a_R = \frac{\beta}{8} \).

We want to ensure \( a_R \in W_{s-2,p}^{s-2,p} \). Note that \( h \) is an AF metric of class \( W_{\delta}^{s,p} \) and \( R \) involves the second derivatives of \( h \), so \( R \in W_{s-2}^{s-2,p} \). We do not need to impose any extra restrictions for this one.

4. \( a_\rho = \kappa \rho / 4 \).

Clearly \( a_\rho \in W_{\beta-2}^{s-2,p} \) iff \( \rho \in W_{\beta-2}^{s-2,p} \).

5. \( a_W = [\sigma_{ab} + (L W)_{ab}][\sigma^{ab} + (L W)^{ab}] / 8 \).

We want to ensure that \( a_W \in W_{\beta-2}^{s-2,p} \). Note that \( L W, \sigma \in W_{\beta-1}^{e-1,q} \) and by our restrictions on \( e \), \( W_{\beta-1}^{e-1,q} \times W_{\beta-1}^{e-1,q} \hookrightarrow W_{2\beta-2}^{e-1,q} \) and \( W_{2\beta-2}^{e-1,q} \hookrightarrow W_{\beta-2}^{s-2,p} \). Thus, we have \( a_W = \frac{1}{8} |\sigma + L W|^2 \in W_{2\beta-2}^{e-1,q} \hookrightarrow W_{\beta-2}^{s-2,p} \).

We are finally in a position to give a well-defined weak formulation of the Einstein constraint equations on AF manifolds with rough data, through the use of Framework 1. (In Appendix H, we show how Framework 2 leads to an alternative weak formulation.)

**Weak Formulation 1.** Let \((M, h)\) be a 3-dimensional AF Riemannian manifold of class...
$W^{s,p}_\delta$ where $p \in (1, \infty)$, $\beta, \delta < 0$ and $s \in (1 + \frac{3}{p}, \infty)$. Select $q$ and $e$ to satisfy

\[
\frac{1}{q} \in (0, 1) \cap \left(0, \frac{s-1}{3}\right) \cap \left[\frac{3-p}{3p}, \frac{3+p}{3p}\right],
\]

\[
e \in (1 + \frac{3}{q}, \infty) \cap [s-1, s] \cap \left[\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}\right].
\]

Let $q = p$ if $e = s \not\in \mathbb{N}_0$. Fix source functions:

\[
\tau \in W^{e-1,q}_\beta, \quad \sigma \in W^{e-1,q}_\beta, \quad \rho \in W^{s-2,p}_\beta (\rho \geq 0), \quad J \in W^{e-2,q}_\beta.
\]

Let $\eta = \max\{\beta, \delta\}$. Define $f : W^{s,p}_\delta \times W^{e,q}_\beta \to W^{s-2,p}_\eta$ and $f : W^{s,p}_\delta \to W^{e-2,q}_\beta$ by

\[
f(\psi, W) = a_R (\psi + \mu) + a_t (\psi + \mu)^5 - a_W (\psi + \mu)^{-7} - a_p (\psi + \mu)^{-3},
\]

\[
f(\psi) = b_t (\psi + \mu)^6 + b_J.
\]

Find $(\psi, W) \in W^{s,p}_\delta \times W^{e,q}_\beta$ such that

\[
A_L \psi + f(\psi, W) = 0, \quad (5.3)
\]

\[
\mathcal{A}_L W + f(\psi) = 0. \quad (5.4)
\]

**Remark 5.1.** We make the following observations regarding **Weak Formulation 1**.

- Since $s \geq 1$, the condition $e > 1$ implies $e > 2 - s$. Therefore, we did not explicitly state the condition $e > 2 - s$ in the above formulation.

- The condition $e > \frac{3}{q} + 1$ together with $s > \frac{3}{p}$ imply that $e > -s + \frac{3}{p} - 1 + \frac{3}{q}$. Therefore, we did not explicitly state the condition $e > -s + \frac{3}{p} - 1 + \frac{3}{q}$ in the above formulation.

- For $(1 + \frac{3}{q}, \infty) \cap [s-1, s]$ to be nonempty we need to have $1 + \frac{3}{q} < s$. This is why we have $\frac{1}{q} \in (0, \frac{s-1}{3})$ in the weak formulation.
○ For \((1 + \frac{3}{q}, \infty) \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}]\) to be nonempty we need to have \(1 + \frac{3}{q} < \frac{3}{q} + s - \frac{3}{p}\).

That is, \(s > 1 + \frac{3}{p}\) (therefore, we did not need to explicitly state \(s \geq 1\)).

○ For \([s - 1, s] \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}]\) to be nonempty we need to have \(\frac{1}{p} - \frac{1}{q} \leq \frac{1}{q} \leq \frac{1}{p} + \frac{1}{3}\).

That is, \(\frac{1}{q} \in [\frac{3 - p}{3p}, \frac{3 + p}{3p}]\).

**Remark 5.2.** *Our analysis in this chapter is based on the weak formulation described above.* In some of the theorems that follow, for the claimed estimates to be true in the case \(e \leq 2\) we will need to restrict the admissible space of \(\tau\). In those cases we will assume \(\tau \in W_{\beta - 1}^{1, z}\) where \(z = \frac{3q}{3q + (2 - e)q}\). We note that \(z\) has been chosen in this form to ensure that \(W_{\beta - 1}^{1, z} \hookrightarrow W_{\beta - 1}^{e - 1, q}\) (and so \(L_{\beta - 2}^{z} \hookrightarrow W_{\beta - 2}^{e - 2, q}\)). Indeed, by Theorem 3.22, for \(W_{\beta - 1}^{1, z} \hookrightarrow W_{\beta - 1}^{e - 1, q}\) to hold true we need to have (the numbering follows the assumptions in Theorem 3.22):

(i) \(z \leq q\),

(ii) \(1 \geq e - 1\), \hspace{1cm} (true for \(e \leq 2\))

(iii) \(1 - \frac{3}{z} \geq e - 1 - \frac{3}{q}\).

Now note that if we set \(z = \frac{3q}{(2 - e)q + 3}\), then the first condition and the third condition are both satisfied (for \(e \leq 2\)):

\[
\begin{align*}
z \leq q & \iff \frac{3q}{(2 - e)q + 3} \leq q \iff \frac{3}{(2 - e)q + 3} \leq 1 \iff 2 - e \geq 0, \\
1 - \frac{3}{z} & \geq e - 1 - \frac{3}{q} \iff \frac{3}{z} \leq (2 - e) + \frac{3}{q} \iff z \geq \frac{3q}{(2 - e)q + 3}.
\end{align*}
\]

### 5.2 Results for the Momentum Constraint

We now develop the main results will need for the momentum constraint operator on AF manifolds with rough data.

**Theorem 5.3.** *Let \((M, h)\) be a 3-dimensional AF Riemannian manifold of class \(W_{\delta}^{s, p}\)*
with \( p \in (1, \infty) \), \( \delta < 0 \) and \( s \in (\frac{3}{p}, \infty) \cap (1, \infty) \). Select \( q, e \) to satisfy:

\[
q \in (1, \infty), \quad e \in (2 - s, s] \cap (-s + \frac{3}{p} - 1 + \frac{3}{q}, s - \frac{3}{p} + \frac{3}{q}).
\]  

(5.5)

In case \( e = s \notin \mathbb{N}_0 \), assume \( q = p \). In case \( e = s \in \mathbb{N}_0, p > 2, q < 2 \), assume \( e > \frac{3}{q} - \frac{1}{2} \). In case \( e = s - \frac{3}{p} + \frac{3}{q} \), \( p < 2, q > 2 \) assume \( e > \frac{1}{2} \). Suppose \( \beta \in (-1, 0) \) and \( b_1 \) and \( b_\tau \) and \( \psi \) are such that \( f(\psi) \in W^{e-2,q}_{\beta-2} \) (in particular, we know that if we fix the source terms \( b_j \) and \( b_\tau \) in \( W^{e-2,q}_{\beta-2} \) and \( \psi \in W^{s,p}_\delta \) then \( f(\psi) \in W^{e-2,q}_{\beta-2} \)). Then \( \mathcal{A}_L : W^{e,q}_\beta \rightarrow W^{e-2,q}_{\beta-2} \) is Fredholm of index zero. Moreover if \( h \) has no nontrivial conformal Killing fields, then the momentum constraint \( \mathcal{A}_L W + f(\psi) = 0 \) has a unique solution \( W \in W^{e,q}_\beta \) with

\[
\|W\|_{W^{e,q}_\beta} \leq C \|f(\psi)\|_{W^{e-2,q}_{\beta-2}},
\]

where \( C > 0 \) is a constant.

**Remark 5.4.** In the above theorem the ranges for \( e \) and \( q \) are chosen so that the momentum constraint is well-defined. Also note that for \((2 - s, s] \) to be a nonempty interval we had to assume that \( s \) is strictly larger than 1.

**Remark 5.5.** There are important cases where the assumption that “\( h \) has no nontrivial conformal Killing fields” is automatically satisfied. For instance in [55] it is proved that if \( (M, h) \) is AF of class \( W^{s,2}_\delta \) with \( s > \frac{3}{2} \) (and of course \( \delta < 0 \)) and if \( X \in W^{s,2}_\rho \) with \( \rho < 0 \) is a conformal Killing field, then \( X \) vanishes identically. We do not pursue this issue here, but interested readers may find more information in [55] and [54].

**Proof.** (Theorem 5.3) The proof will involve three main steps.

- **Step 1: Establish that \( \mathcal{A}_L \) is Fredholm of index zero.**

  \( \mathcal{A}_L \) is of class \( D^{s,p}_{2,\delta} \). Therefore by Proposition 4.6, \( \mathcal{A}_L : W^{e,q}_\beta \rightarrow W^{e-2,q}_{\beta-2} \) is semi-Fredholm (this is exactly why it is assumed \( \beta \in (-1, 0) \)). On the other hand, vector
Laplacian of the rough metric can be approximated by the vector Laplacian of smooth metrics and it is well known that vector Laplacian of a smooth metric is Fredholm of index zero. Therefore since the index of a semi-Fredholm map is locally constant, it follows that $\mathcal{A}_L : W^{e,q}_\beta \to W^{e-2,q}_{\beta-2}$ is Fredholm with index 0.

- **Step 2: Show that if $\text{Ker } \mathcal{L} = \{0\}$, then $\text{Ker } \mathcal{A}_L = \{0\}$.**

The proof of this step involves considering six distinct cases. In each case, we denote the operator $\mathcal{A}_L$ acting on $W^{e,q}_\beta$ by $(\mathcal{A}_L)_{e,q,\beta}$. In order to best organize the arguments for these six cases, we make the following definitions:

- **nice triple:** A triple $(e, q, \beta)$ where $-1 < \beta < 0$ and $e, q$ satisfy (5.5).
- **super nice triple:** A nice triple $(e, q, \beta)$ where $e \neq s$ and $e \neq s - \frac{3}{p} + \frac{3}{q}$.

We now make three observations about relationships between these definitions.

- **Observation 1:** For any $-1 < \beta < 0$, $(e = 1, q = 2, \beta)$ is super nice and $(e = s, q = p, \beta)$ is nice. Indeed,

  
  
  \[
  1 \in (2 - s, s), \quad \text{(since } s > 1) \\
  1 > -s + \frac{3}{p} - 1 + \frac{3}{2}, \quad \text{(since } s > \frac{3}{p}) \\
  1 < s - \frac{3}{p} + \frac{3}{2}, \quad \text{(since } s > \frac{3}{p}) \\
  s \in (2 - s, s], \quad \text{(trivially true; note } s > 1) \\
  s > -s + \frac{3}{p} - 1 + \frac{3}{p}, \quad \text{(since } s > \frac{3}{p}) \\
  s \leq s - \frac{3}{p} + \frac{3}{p}, \quad \text{(trivially true)}
  \]

- **Observation 2:** If $(e, q, \beta)$ is super nice, then $(2 - e, q', -1 - \beta)$ is also super nice. Indeed,
\[ q \in (1, \infty) \Rightarrow q' \in (1, \infty), \]
\[ \beta \in (-1, 0) \Rightarrow -1 - \beta \in (-1, 0), \]
\[ e \in (2 - s, s) \Rightarrow 2 - e \in (2 - s, s), \]
\[ e < s - \frac{3}{p} + \frac{3}{q} \Rightarrow 2 - e > 2 - s + \frac{3}{p} - \frac{3}{q} = -s + \frac{3}{p} - 1 + \frac{3}{q}, \]
\[ e > -s + \frac{3}{p} - 1 + \frac{3}{q} \Rightarrow 2 - e < 2 + s - \frac{3}{p} + 1 - \frac{3}{q} = s - \frac{3}{p} + \frac{3}{q}. \]

- **Observation 3:** Suppose \((e_1, q_1, \beta_1)\) and \((e_2, q_2, \beta_2)\) are *nice* triples. If we have \(W_{\beta_2}^{e_2, q_2} \hookrightarrow W_{\beta_1}^{e_1, q_1}\), then \((\mathcal{A}_L)_{e_2, q_2, \beta_2}\) is the restriction of \((\mathcal{A}_L)_{e_1, q_1, \beta_1}\) to \(W_{\beta_2}^{e_2, q_2}\) and so \(\text{Ker}(\mathcal{A}_L)_{e_2, q_2, \beta_2} \subseteq \text{Ker}(\mathcal{A}_L)_{e_1, q_1, \beta_1}\). In particular, if \(\text{Ker}(\mathcal{A}_L)_{e_1, q_1, \beta_1} = \{0\}\) holds, then \(\text{Ker}(\mathcal{A}_L)_{e_2, q_2, \beta_2} = \{0\}\).

Now let \((e, q, \beta)\) be a *nice* triple. We consider the following six cases:

- **Case 1:** \(e = 1, q = 2\)

  In order to prove the claim first we show that if \(\beta' \in (-1, -\frac{1}{2})\) then
  \[
  \forall X, Y \in W_{\beta'}^{1, 2} \quad \langle \mathcal{A}_L X, Y \rangle_{(M, h)} = \frac{1}{2} \langle \mathcal{L} X, \mathcal{L} Y \rangle_{L^2}.
  \]

First let us ensure that both sides are well-defined. Note that \(\mathcal{A}_L : W_{\beta'}^{1, 2} \rightarrow W_{-1 - \beta'}^{1, 2}\) and so \(\mathcal{A}_L X \in W_{-1 - \beta'}^{1, 2}\). According to our discussion on duality pairing in Chapter 3, we know that the duality pairing of \(W_{-1 - \beta'}^{1, 2}\) and \(W_{-1 - \beta'}^{1, 2}\) is well-defined. So for the LHS to be well-defined, we just need to ensure that \(Y \in W_{-1 - \beta'}^{1, 2}\) that is we need to have \(W_{\beta'}^{1, 2} \hookrightarrow W_{-1 - \beta'}^{1, 2}\). But clearly this is true because by assumption \(\beta' < -\frac{1}{2}\). Also note that \(\mathcal{L} X, \mathcal{L} Y \in L^2_{\beta' - 1}\); since \(\beta' - 1 < -\frac{3}{2}\), by Remark 3.7 we have \(L^2_{\beta' - 1} \hookrightarrow L^2\) and so the RHS makes sense. Now, as we saw in Section 1.1.6, the claimed equality holds true for \(X, Y \in C^\infty_c\) and so by density it holds true for \(X, Y \in W_{\beta'}^{1, 2}\).

Let \(X \in \text{Ker}(\mathcal{A}_L)_{e=1, q=2, \beta}\). Since \((e = 1, q = 2, \beta)\) is a *nice* triple, by Lemma 4.8
there exists $\beta' \in (-1, \frac{-1}{2})$ such that $X \in W^{1,2}_{\beta'}$. So by what was proved above we can conclude that $\langle \mathcal{L} X, \mathcal{L} X \rangle_{L^2} = 0$ which implies that $X$ is a conformal Killing field and so $X = 0$.

- **Case 2:** $e \neq 1, q = 2$

  If $e > 1$, then $W^{e, q}_{\beta} \hookrightarrow W^{1, q}_{\beta}$ and hence the claim follows from Observation 3. Suppose $e < 1$. So in particular $e \neq s$ and $e \neq s - \frac{3}{p} + \frac{3}{2}$ (because both $s$ and $s - \frac{3}{p} + \frac{3}{2}$ are larger than 1) and therefore $(e, q = 2, \beta)$ is **super nice**. Consequently $(2 - e, q' = 2, -1 - \beta)$ is also **super nice**. Since $2 - e > 1$ we know that $\text{Ker}(\mathscr{A}_L)_{2-e, q' = 2, -1-\beta} = \{0\}$. But $\mathscr{A}_L$ is formally self adjoint and so

  $$\text{Ker}(\mathscr{A}_L)_{e, q = 2, \beta})^* = \text{Ker}(\mathscr{A}_L)_{2-e, q' = 2, -1-\beta} = \{0\}.$$ 

  Finally $(\mathscr{A}_L)_{e, q = 2, \beta}$ is Fredholm of index zero, so $\text{Ker}(\mathscr{A}_L)_{e, q = 2, \beta} = \{0\}$.

- **Case 3:** $(p \leq 2, q < 2)$ or $(e > \frac{3}{q} - \frac{1}{2}, q < 2)$

  It is enough to show that there exists $\tilde{e}$ such that $W^{e,q}_{\beta} \hookrightarrow W^{\tilde{e},2}_{\beta}$ where $(\tilde{e}, 2, \beta)$ is **super nice**. That is, we need to find $\tilde{e}$ that satisfies

  $$e \geq \tilde{e},$$
  $$\frac{e - 3}{q} \geq \frac{\tilde{e} - 3}{2} \quad (\Leftrightarrow \tilde{e} \leq e + \frac{3}{2} - \frac{3}{q}),$$
  $$\tilde{e} \in (2 - s, s),$$
  $$\tilde{e} \in (-s + \frac{3}{p} - 1 + \frac{3}{2}, s - \frac{3}{p} + \frac{3}{2}).$$

  Since $\frac{3}{q} > \frac{3}{2}$, the second condition is stronger than the first condition. Also $s > \frac{3}{p}$
so \((-s + \frac{3}{p} + \frac{1}{2}, s - \frac{3}{p} + \frac{3}{2})\) is nonempty. So such an \(\tilde{e}\) exists if

\[
(-\infty, e + \frac{3}{q} - \frac{3}{2}) \cap (2-s, s) \cap (-s + \frac{3}{p} - 1 + \frac{3}{2}, s - \frac{3}{p} + \frac{3}{2}) \neq \emptyset.
\]

Now note that

- If \(p \leq 2\) then \((-s + \frac{3}{p} - 1 + \frac{3}{2}, s - \frac{3}{p} + \frac{3}{2}) \subseteq (2-s, s),\)
- If \(p > 2\) then \((2-s, s) \subseteq (-s + \frac{3}{p} - 1 + \frac{3}{2}, s - \frac{3}{p} + \frac{3}{2}).\)

Therefore in order to ensure such an \(\tilde{e}\) exists it is enough to have

\[
2-s < e + \frac{3}{2} - \frac{3}{q} \quad \text{if} \quad p > 2,
\]
\[
-s + \frac{3}{p} + \frac{1}{2} < e + \frac{3}{2} - \frac{3}{q} \quad \text{if} \quad p \leq 2.
\]

The second inequality is true because \((e, q, \beta)\) is a **nice** triple. Moreover, for all values of \(p\), if \(e > \frac{3}{q} - \frac{1}{2}\), then the first inequality holds true (note that \(s > 1\)).

- **Case 4:** \((p \geq 2, q > 2)\) or \((e > \frac{1}{2}, q > 2)\)

First we consider the case where \(e \neq s\) or \(e = s \in \mathbb{N}_0\). Let \(\beta' \in (\beta, 0)\). By Theorem 3.23, \(W^{e,q}_{\beta} \hookrightarrow W^{e,2}_{\beta'}\). So it is enough to show that under the assumption of this case, \((e, 2, \beta')\) is a **nice** triple. Note that since we have assumed \(e \neq s\) or \(e = s \in \mathbb{N}_0\), we do not require \(p\) to be equal to 2. Since \((e, q, \beta)\) is **nice**, we know \(e \in (2-s, s]\).

Therefore we just need to check that \(e \in (-s + \frac{3}{p} - 1 + \frac{3}{2}, s - \frac{3}{p} + \frac{3}{2}]\).

\[
q > 2 \Rightarrow \frac{3}{q} < \frac{3}{2} \Rightarrow e \leq s - \frac{3}{p} + \frac{3}{q} < s - \frac{3}{p} + \frac{3}{2}.
\]

Also if \(p \geq 2\), then \(-s + \frac{3}{p} + \frac{1}{2} \leq -s + 2 < e\). Moreover, for all values of \(p\), if \(e > \frac{1}{2}\),
then \( e > -s + \frac{3}{p} + \frac{1}{2} \).

Now let's consider the case where \( e = s \not\in \mathbb{N}_0 \). By the statement of the theorem and the assumptions of this case, we must have \( p = q > 2 \). It is enough to show that there exists \( \tilde{e} \) and \( \tilde{\beta} \) such that \( W_{\tilde{e},q}^\beta \hookrightarrow W_{\tilde{e},2}^\beta \) where \( (\tilde{e}, 2, \tilde{\beta}) \) is \textit{super nice}. Let \( \tilde{\beta} \in (\beta, 0) \). We need to find \( \tilde{e} \) that satisfies

\[
\begin{align*}
\tilde{e} &\leq e = s, \\
\tilde{e} &\in (2 - s, s), \\
\tilde{e} &\in (-s + \frac{3}{p} - \frac{3}{2}, s - \frac{3}{p} + \frac{3}{2}).
\end{align*}
\]

Such an \( \tilde{e} \) exists if

\[
(2 - s, s) \cap (-s + \frac{3}{p} - \frac{3}{2}, s - \frac{3}{p} + \frac{3}{2}) \neq \emptyset.
\]

Since \( p > 2 \), the above intersection is equal to \( (2 - s, s) \) which is clearly nonempty (since \( s > 1 \)).

**Case 5: \( p < 2, q > 2 \)**

First note that \( e \) cannot be equal to \( s \). Otherwise we would have \( s = e \leq s - \frac{3}{p} + \frac{3}{q} \) and so \( p \geq q \) which contradicts the assumption of this case.

If \( e = s - \frac{3}{p} + \frac{3}{q} \), then the claim follows from case 4 (because by assumption \( e > \frac{1}{2} \)).

So WLOG we can assume that \( (e, q > 2, \beta) \) is a \textit{super nice} triple. This implies that \( (2 - e, q', < 2, -1 - \beta) \) is also a \textit{super nice} triple. Since \( q' < 2 \) by what was proved in case 3, \( \text{Ker}(\mathcal{A}_L)_{2-e,q',-1-\beta} = \{0\} \). So by an argument exactly the same as the one given in case 2 we can conclude that \( \text{Ker}(\mathcal{A}_L)_{e,q,\beta} = \{0\} \).
Case 6: $p > 2, q < 2$

First note that $e$ cannot be equal to $s - \frac{3}{p} + \frac{3}{q}$. Otherwise we would have $s - \frac{3}{p} + \frac{3}{q} = e \leq s$ and so $p \leq q$ which contradicts the assumption of this case.

Since $p \neq q$, if $e = s$, then we must have $e = s \in \mathbb{N}_0$. If $e = s \in \mathbb{N}_0$, then the claim follows from case 3 (because by assumption $e > \frac{3}{q} - \frac{1}{2}$). So WLOG we can assume $(e, q < 2, \beta)$ is a super nice triple. Therefore $(2 - e, q' > 2, -1 - \beta)$ is also a super nice triple. Since $q' > 2$ by what was proved in case 4, $\text{Ker}(\mathcal{A}_L)_{2-e,q',-1-\beta} = \{0\}$.

So by an argument exactly the same as the one given in case 2 we can conclude that $\text{Ker}(\mathcal{A}_L)_{e,q,\beta} = \{0\}$.

Step 3: Show that if $\text{Ker} \mathcal{A}_L = \{0\}$, then $\mathcal{A}_L$ is an isomorphism.

By the previous steps we know that $\mathcal{A}_L$ is Fredholm of index zero and also it is injective. It follows that $\mathcal{A}_L$ is a bijective continuous operator and so according to the open mapping theorem it is an isomorphism. In particular $(\mathcal{A}_L)^{-1}$ is continuous and so $\|W\|_{W^{e,q}_\beta} \leq C \|f(\psi)\|_{W^{e-2,q}_{\beta-2}}$.

Corollary 5.6. Let the following assumptions hold:

- $(M, h)$ is a 3-dimensional AF Riemannian manifold of class $W^{s,p}_\delta$.
- $p \in (1, \infty), s \in (1 + \frac{3}{p}, \infty), \delta < 0$.
- $q \in (3, \infty), e \in (1, s] \cap (1 + \frac{3}{q}, s - \frac{3}{p} + \frac{3}{q}] \cap (1, 2]$. $(q = p$ if $e = s \notin \mathbb{N}_0)$
- $-1 < \beta < 0, z = \frac{3q}{3+(2-e)q}, b_r \in L^z_{\beta-2}$.
- $h$ has no conformal Killing fields.
- $W \in W^{e,q}_\beta$ uniquely solves the momentum constraint with source $\psi \in W^{s,p}_\delta$. 
Then:

\[ \| \mathcal{L} W \|_{L^\infty_{\beta-1}} \leq \| b_t (\mu + \psi)^6 \|_{L^z_{\beta-2}} + \| b_j \|_{W^{e-2,q}_{\beta-2}} \leq (\mu + \| \psi \|_{L^\infty})^6 \| b_t \|_{L^z_{\beta-2}} + \| b_j \|_{W^{e-2,q}_{\beta-2}}. \]

Moreover, \( \| W \|_{W^{e,q}_{\beta}} \) can be bounded by the same expressions. The implicit constants in the above inequalities do not depend on \( \mu, W, \) or \( \psi. \)

**Remark 5.7.** In this theorem, the restrictions on \( e \) and \( q \) serve the following purposes:

- \( L^z_{\beta-2} \hookrightarrow W^{e-2,q}_{\beta-2}. \) (note that \( e \leq 2 \))

- \( \mathcal{A}_L : W^{e,q}_{\beta} \rightarrow W^{e-2,q}_{\beta} \) is well-defined.

- \( e > 1 + \frac{3}{q} \) and so \( W^{e,q}_{\beta} \hookrightarrow L^\infty_{\beta} \) and also \( W^{e-1,q}_{\beta-1} \hookrightarrow L^\infty_{\beta-1}. \)

Also note that

- If \( e > 1 + \frac{3}{q} \) then \( e > -s + \frac{3}{p} - 1 + \frac{3}{q} \) is automatically satisfied.

- For \( (1 + \frac{3}{q}, s - \frac{3}{p} + \frac{3}{q}) \) to be nonempty we must have \( s > 1 + \frac{3}{p}. \)

- If \( s > 1 \) then \( e > 2 - s \) follows from \( e > 1. \)

- \( 2 \geq e > 1 + \frac{3}{q} \) and so we must have \( q > 3. \)

**Proof.** (Corollary 5.6) First note that \( e > 1 + \frac{3}{q} \) and so \( W^{e,q}_{\beta} \hookrightarrow L^\infty_{\beta} \) and also \( W^{e-1,q}_{\beta-1} \hookrightarrow L^\infty_{\beta-1}. \) That is, \( W^{e,q}_{\beta} \hookrightarrow W^{1,q}_{\beta} \). Also \( \mathcal{L} : W^{1,q}_{\beta} \rightarrow L^{\infty}_{\beta-1} \) is continuous (\( \mathcal{L} \) is a differential operator of order 1) and so we have

\[
\| \mathcal{L} W \|_{L^\infty_{\beta-1}} \leq \| W \|_{W^{1,q}_{\beta}} \leq \| W \|_{W^{e,q}_{\beta}} \leq \| f(\psi) \|_{W^{e-2,q}_{\beta-2}}
\]

\[
= \| b_t (\mu + \psi)^6 + b_j \|_{W^{e-2,q}_{\beta-2}} \leq \| b_t (\mu + \psi)^6 \|_{W^{e-2,q}_{\beta-2}} + \| b_j \|_{W^{e-2,q}_{\beta-2}}
\]

\[
\leq \| b_t (\mu + \psi)^6 \|_{L^z_{\beta-2}} + \| b_j \|_{W^{e-2,q}_{\beta-2}} \text{ (note that } L^z_{\beta-2} \hookrightarrow W^{e-2,q}_{\beta-2})
\]
Now note that

\[
\| b_t (\mu + \psi)^6 \|_{L^2_{-\beta}} \leq \sum_{k=0}^{6} \binom{6}{k} \mu^{6-k} \| b_t \psi^k \|_{L^2_{-\beta}} \leq \sum_{k=0}^{6} \binom{6}{k} \mu^{6-k} \| b_t \psi^k \|_{L^2_{-\beta}}
\]

(note that \( L^\infty_{\delta} \times L^2_{-\beta} \hookrightarrow L^2_{-\beta+\delta-2} \))

\[
\leq \sum_{k=0}^{6} \binom{6}{k} \mu^{6-k} \| \psi \|_{L^\infty_{\delta}} b_t \|_{L^2_{-\beta}}
\]

(note that \( L^\infty_{\delta} \times L^\infty_{\delta} \hookrightarrow L^\infty_{2\delta} \hookrightarrow L^\infty_{\delta} \))

\[
= (\mu + \| \psi \|_{L^\infty_{\delta}})^6 \| b_t \|_{L^2_{-\beta}}.
\]

Hence

\[
\| \mathcal{L} W \|_{L^\infty_{-\beta-1}} \leq \| W \|_{W^{e,q}_\beta}
\]

\[
\leq \| b_t (\mu + \psi)^6 \|_{L^2_{-\beta}} + \| b_j \|_{W^{e-2,q}_{-\beta}}
\]

\[
\leq (\mu + \| \psi \|_{L^\infty_{\delta}})^6 \| b_t \|_{L^2_{-\beta}} + \| b_j \|_{W^{e-2,q}_{-\beta}}.
\]

\[
\square
\]

**Lemma 5.8.** All the assumptions in corollary 5.6 hold. In particular, \( W \) is the solution to the momentum constraint with source \( \psi \). Then

\[
a_W \leq r^{2\beta-2} (k_1 \| \mu + \psi \|_{L^\infty}^{12} + k_2)
\]

where

1. \( r = (1 + |x|^2)^{\frac{1}{2}} \) and \( |x| \) is the geodesic distance from a fixed point \( O \) in the compact core (see Remark 3.18),
The implicit constant in the above inequality does not depend on $\mu$, $W$ or $\psi$.

**Proof.** *(Lemma 5.8)* By Corollary 5.6 we have

$$\| L W \|_{L^\infty} \leq \| b_\tau (\mu + \psi) \|_{L^z_{\beta-2}} + \| b_J \|_{W^{e-2,q}_{\beta-2}}$$

$$\leq \| b_\tau \|_{L^z_{\beta-2}} \| \mu + \psi \|_{L^6} + \| b_J \|_{W^{e-2,q}_{\beta-2}}, \quad \text{(here we used Remark 3.7)}$$

and considering Remark 3.17 we get the following pointwise bound for $L W$:

$$| L W | \leq r^{\beta-1} (\| b_\tau \|_{L^z_{\beta-2}} \| \mu + \psi \|_{L^6} + \| b_J \|_{W^{e-2,q}_{\beta-2}}).$$

Note that $L W$ has a continuous version and so the above inequality holds everywhere (not just “almost everywhere”). Now we can write

$$a_W = \frac{1}{8} | \sigma + L W |^2 \leq | \sigma |^2 + | L W |^2$$

$$\leq r^{2\beta-2} \| \sigma \|_{L^\infty}^2 + | L W |^2$$

$$\leq r^{2\beta-2} \| \sigma \|_{L^\infty}^2 + r^{2\beta-2} (\| b_\tau \|_{L^z_{\beta-2}} \| \mu + \psi \|_{L^6} + \| b_J \|_{W^{e-2,q}_{\beta-2}})^2$$

$$\leq r^{2\beta-2} (k_1 \| \mu + \psi \|_{L^\infty}^2 + k_2).$$

**Remark 5.9.** We make the following important remark concerning notation. Consider the space $W^{a,\gamma}_\delta (M)$ where $a \gamma > 3$. An order on $W^{a,\gamma}_\delta (M)$ can be defined as follows: the functions $\chi_1, \chi_2 \in W^{a,\gamma}_\delta (M)$ satisfy $\chi_2 \geq \chi_1$ if and only if the continuous versions of $\chi_1, \chi_2$ satisfy $\chi_2 (x) \geq \chi_1 (x)$ for all $x \in M$ (clearly this definition agrees with the one that
is described in Remark 3.38). Equipped with this order, $W_0^{\alpha, \gamma}(M)$ becomes an ordered Banach space. By the interval $[\chi_1, \chi_2]_{\alpha, \gamma, \delta}$ we mean the set of all functions $\chi \in W_0^{\alpha, \gamma}(M)$ such that $\chi_1 \leq \chi \leq \chi_2$.

**Lemma 5.10.** Let the following assumptions hold:

- All the assumptions in corollary 5.6 hold.
- $\tilde{s} \in \left(\frac{3}{p}, s\right]$ and $\tilde{\delta} \in [\delta, 0)$ are such that $W_{\beta - 2}^{e, 2, q} \times W_{\tilde{s}, p}^{\tilde{\delta}, p} \rightarrow W_{\beta - 2}^{e, 2, q}$ . For example, using multiplication lemma, one can easily check that for $\tilde{s} = s$ and $\tilde{\delta} = \delta$ these inclusions hold true.
- $\psi_-, \psi_+ \in W_{\delta}^{\tilde{s}, p}, \psi_+ \geq \psi_- > -\mu, \; \psi_1, \psi_2 \in [\psi_-, \psi_+]_{\tilde{s}, p, \delta}$;
- $W_1$ and $W_2$ are solutions to the momentum constraint corresponding to $\psi_1$ and $\psi_2$, respectively.

Then:

$$\|W_1 - W_2\|_{e, q, \beta} \leq (1 + \max\{\|\psi_-\|_{L^\infty}, \|\psi_+\|_{L^\infty}\})^5 \|b_1\|_{L^2_{\beta - 2}} \|\psi_2 - \psi_1\|_{\tilde{s}, p, \delta}.$$ 

The implicit constant in the above inequality depends on $\mu$ but it is independent of $\psi_1, \psi_2, W_1, W_2$.

**Proof.** (Lemma 5.10) The momentum equation is linear and so $W_1 - W_2$ is the solution to the momentum constraint with right hand side $f(\psi_1) - f(\psi_2)$.

$$\|W_1 - W_2\|_{e, q, \beta} \leq \|f(\psi_1) - f(\psi_2)\|_{e - 2, q, \beta - 2} = \|b_1[(\mu + \psi_1)^6 - (\mu + \psi_2)^6]\|_{e - 2, q, \beta - 2}$$

$$= \|b_1 \sum_{j=0}^5 (\mu + \psi_2)^j (\mu + \psi_1)^{5-j}(\psi_2 - \psi_1)\|_{e - 2, q, \beta - 2} \leq \sum_{j=0}^5 \|b_1 (\mu + \psi_2)^j (\mu + \psi_1)^{5-j}(\psi_2 - \psi_1)\|_{e - 2, q, \beta - 2}$$
\begin{align*}
\sum_{j=0}^{5} & \|b_{r}(\mu + \psi_{2})^{j}(\mu + \psi_{1})^{5-j}\|_{e^{-2,q},\beta^{-2}}\|\psi_{2} - \psi_{1}\|_{\tilde{s},p,\tilde{\delta}} \\
& \leq \sum_{j=0}^{5} \|b_{r}(\mu + \psi_{2})^{j}(\mu + \psi_{1})^{5-j}\|_{L_{\beta^{-2}}^{\tilde{z}}} \|\psi_{2} - \psi_{1}\|_{\tilde{s},p,\tilde{\delta}} \\
& \leq \sum_{j=0}^{5} \sum_{m=0}^{j} \sum_{l=0}^{5-j} \binom{j}{m} \psi_{2}^{m} \psi_{1}^{l} \|L_{\beta^{-2}}^{\tilde{z}}\|_{L_{\tilde{\delta}}} \|\psi_{2} - \psi_{1}\|_{\tilde{s},p,\tilde{\delta}} \\
& \leq \sum_{j=0}^{5} \sum_{m=0}^{j} \sum_{l=0}^{5-j} \binom{j}{m} \|b_{r}\|_{L_{\beta^{-2}}^{\tilde{z}}} \|\psi_{2}^{m}\|_{L_{\tilde{\delta}}} \|\psi_{1}^{l}\|_{L_{\tilde{\delta}}} \|\psi_{2} - \psi_{1}\|_{\tilde{s},p,\tilde{\delta}} \\
& \leq \sum_{j=0}^{5} \sum_{m=0}^{j} \sum_{l=0}^{5-j} \binom{j}{m} \|b_{r}\|_{L_{\beta^{-2}}^{\tilde{z}}} \|\psi_{2}^{m}\|_{L_{\tilde{\delta}}} \|\psi_{1}^{l}\|_{L_{\tilde{\delta}}} \|\psi_{2} - \psi_{1}\|_{\tilde{s},p,\tilde{\delta}} \\
& \leq 5 \|b_{r}\|_{L_{\beta^{-2}}^{\tilde{z}}} (1 + \|\psi_{2}\|_{L_{\tilde{\delta}}}^{j}) (1 + \|\psi_{1}\|_{L_{\tilde{\delta}}}^{5-j}) \|\psi_{2} - \psi_{1}\|_{\tilde{s},p,\tilde{\delta}} \\
& \leq 5 \|b_{r}\|_{L_{\beta^{-2}}^{\tilde{z}}} (1 + \max(\|\psi_{-}\|_{L_{\tilde{\delta}}}, \|\psi_{+}\|_{L_{\tilde{\delta}}}))^{5} \|\psi_{2} - \psi_{1}\|_{\tilde{s},p,\tilde{\delta}} \\
& = 6(1 + \max(\|\psi_{-}\|_{L_{\tilde{\delta}}}, \|\psi_{+}\|_{L_{\tilde{\delta}}}))^{5} \|b_{r}\|_{L_{\beta^{-2}}^{\tilde{z}}} \|\psi_{2} - \psi_{1}\|_{\tilde{s},p,\tilde{\delta}},
\end{align*}

where we have used \(|\psi_{1}| \leq \max(|\psi_{+}|, |\psi_{-}|)|, so that \|\psi_{1}\|_{L_{\tilde{\delta}}} \leq \max(\|\psi_{-}\|_{L_{\tilde{\delta}}}, \|\psi_{+}\|_{L_{\tilde{\delta}}}).
5.3 Results for the Hamiltonian Constraint

We now develop the main results we need for the Hamiltonian constraint on AF manifolds with rough data. We study primarily the “shifted” Hamiltonian constraint; the reason for introducing a shift (the function $a_s$ in the following lemma) is briefly discussed in Remark 5.15.

**Lemma 5.11.** Let the following assumptions hold:

- $(M, h)$ is a 3-dimensional AF Riemannian manifold of class $W^{s,p}_0$.
- $p \in (\frac{3}{2}, \infty), s \in (\frac{3}{p}, \infty) \cap [1, 3]$.
- $\beta < 0, -1 < \delta < 0$, and $\eta = \max[\delta, \beta]$.
- $a_\tau, a_\rho, a_W \in W^{s-2,p}_{\beta-2}, a_R \in W^{s-2,p}_{\delta-2}$.
- $a_0 \in W^{s-2,p}_{\eta-2}$, $a_0 \neq 0$, and $a_0 \geq 0$ (see Remark 3.38).
- $\tilde{s} \in (\frac{3}{p}, s] \cap [1, 1 + \frac{3}{p}), \delta \leq \tilde{\delta} < 0$.
- $t \in (\frac{3}{p}, \tilde{s}] \cap [1, 1 + \frac{3}{p}), \tilde{\delta} \leq \gamma < 0$.
- $\psi_-, \psi_+ \in W^{\tilde{s},p}_0$ and $-\mu < \psi_- \leq \psi_+$.
- $V \in W^{\tilde{s},p}_{loc}, V > 0$ is such that $a_W V \in W^{s-2,p}_{\eta-2}$, and
  \[ \|a_W V\|_{s-2,p,\eta-2} \leq C(\psi_+, \psi_-) \|a_W\|_{s-2,p,\eta-2} \] where $C(\psi_+, \psi_-)$ is a constant independent of $V$.
- $a_s = a_0 + a_W V \in W^{s-2,p}_{\eta-2}$.
- $A^{shifted}_L : W^{s,p}_0 \rightarrow W^{s-2,p}_{\eta-2}$ is defined by $A^{shifted}_L \psi = A_L \psi + a_s \psi$. 

- $f^{shifted}_W: [\psi_-, \psi_+]_{\tilde{s}, p, \tilde{\delta}} \to W^{s-2,p}_{\eta-2}$ is defined by $f^{shifted}_W(\psi) = f_W(\psi) - a_s \psi$ where

$$f_W(\psi) = a_t (\mu + \psi)^5 + a_R (\mu + \psi) - a_p (\mu + \psi)^3 - a_{W}(\mu + \psi)^7.$$  

Then:

1. Suppose $A^{shifted}_L : W^{s,p}_\delta \to W^{s-2,p}_{\eta-2}$ is an isomorphism. If we define $T^{shifted} : [\psi_-^s, \psi_+]_{\tilde{s}, p, \tilde{\delta}} \times W^{s-2,p}_{\tilde{\beta}-2} \to W^{s,p}_\delta$ by $T^{shifted}(\psi, a_W) = -(A^{shifted}_L)^{-1} f^{shifted}_W(\psi)$, then $T^{shifted}$ is continuous in both arguments and moreover

$$\| T^{shifted}(\psi, a_W) \|_{s, p, \tilde{\delta}} \leq (1 + \| a_W \|_{s-2, p, \eta-2}) (1 + \| \psi \|_{l, p, \gamma}).$$

The implicit constant in the above inequality depends on $\mu$ but it is independent of $\psi$ and $a_W$.

2. If $\beta \leq \delta$, (that is if $\eta = \delta$), then $A^{shifted}_L : W^{s,p}_\delta \to W^{s-2,p}_{\eta-2}$ is an isomorphism.

**Proof.** (Lemma 5.11) The proof will involve six main steps.

- **Step 1**: We first check that the assumptions actually make sense. To this end, we need to check that both $A^{shifted}_L$ and $f^{shifted}_W(\psi)$ are well-defined.

We first verify that $A^{shifted}_L$ is well-defined, that it sends elements of $W^{s,p}_\delta$ to elements in $W^{s-2,p}_{\eta-2}$. Since we know this is true for $A_L$, we just need to show that if $\psi \in W^{s,p}_\delta \hookrightarrow W^{s-2,p}_{\eta-2}$ then $a_s \psi \in W^{s-2,p}_{\eta-2}$ (note that $a_s \in W^{s-2,p}_{\eta-2}$). To this end we use the multiplication lemma (Lemma 3.30) to prove that $W^{s-2,p}_{\eta-2} \times W^{s,p}_\delta \hookrightarrow W^{s-2,p}_{\eta-2}$. To use the lemma, we need the following conditions (the numbering follows the numbering in Lemma 3.30):
(i) \( s - 2 \geq s - 2 \), \hspace{1cm} \text{(trivially true)}

\( t \geq s - 2 \), \hspace{1cm} \text{(since \( t \geq 1 \geq s - 2 \))}

(ii) \( s - 2 + t \geq 0 \), \hspace{1cm} \text{(since \( s - 2 \geq -1, \ t \geq 1 \))}

(note that \( s - 2 + t = 0 \) if and only if \( s = t = 1 \in \mathbb{N}_0 \))

(iii) \((s - 2) - (s - 2) \geq 3(\frac{1}{p} - \frac{1}{p}), \hspace{1cm} \text{(trivially true)}\)

\( t - (s - 2) \geq 3(\frac{1}{p} - \frac{1}{p}), \hspace{1cm} \text{(since \( t \geq 1 \geq s - 2 \))}\)

(iv) \((s - 2) + t - (s - 2) > 3(\frac{1}{p} + \frac{1}{p} - \frac{1}{p}), \hspace{1cm} \text{(since \( t > \frac{3}{p} \))}\)

(v) Case \( s - 2 < 0 \): \((s - 2) + t > 3(\frac{1}{p} + \frac{1}{p} - 1), \hspace{1cm} \)

where the last item holds since \((s - 2) + t > \frac{3}{p} - 2 + \frac{3}{p} > 3(\frac{1}{p} + \frac{1}{p} - 1)\). Therefore, we can conclude that \( W_{\eta-2}^s \times W_{\gamma}^t \rightarrow W_{\eta-2}^{s-2} \rightarrow W_{\eta-2}^{s-2} \).

We now confirm that \( f_{W_{\psi}}^{shifted} \) is well-defined. To this end, we just need to show \( f_{W_{\psi}} \) sends \( W_{\theta}^{s,p} \) to \( W_{\eta-2}^{s-2,p} \). Note that previously by using Lemma 3.34 we showed that \( f_{W_{\psi}} \) sends \( W_{\theta}^{s,p} \) to \( W_{\eta-2}^{s-2,p} \). By the same argument the above claim can be proved.

- **Step 2:** As a direct consequence of Lemma 3.34 and the multiplication lemma, \( f_{W_{\psi}}^{shifted} \) is a continuous function from \( W_{\theta}^{s,p} \) to \( W_{\eta-2}^{s-2,p} \) (note that \( a_W, a_t, a_R, a_\rho \) are fixed). The continuity of \( a_W \rightarrow f_W(\psi) \) for a fixed \( \psi \in W_{\theta}^{s,p} \) also follows from Lemma 3.34.

- **Step 3:** According to Step 2 and the assumption that \( A_L^{shifted} \) is an isomorphism, \( T^{shifted} \) is a composition of continuous maps with respect to each of its arguments. Therefore \( T^{shifted} \) is continuous in both arguments.

- **Step 4:** Let \( \theta = \frac{1}{p} - \frac{t - 1}{3} \); note that by assumption \( t < 1 + \frac{3}{p} \) and so \( \theta > 0 \). We claim
that $\frac{1}{p} \in (\frac{s-1}{2}, 1 - \frac{3-s}{2} \theta)$. Indeed, $\frac{1}{p} < 1 - \frac{3-s}{2} \theta$ because

$$t > \frac{3}{p} \Rightarrow \theta = \frac{1}{3} + \frac{1}{p} - \frac{t}{3} < \frac{1}{3},$$

$$s \geq 1 \Rightarrow \frac{3-s}{2} \leq 1 \Rightarrow 1 - \frac{3-s}{2} \theta > 1 - \theta > \frac{2}{3}.$$  

Consequently, since $p > \frac{3}{2}$, we have $\frac{1}{p} < \frac{2}{3} < 1 - \frac{3-s}{2} \theta$. It remains to show that $\frac{1}{p} > \frac{s-1}{2} \theta$. Note that

$$\frac{s-1}{2} \theta = \frac{s-1}{2} \left(\frac{1}{p} - \frac{t-1}{3}\right) = \frac{s-1}{2p} - \frac{(s-1)(t-1)}{6},$$

and so

$$\frac{1}{p} > \frac{s-1}{2} \theta \Leftrightarrow \frac{(s-1)(t-1)}{6} > \frac{s-1}{2p} - \frac{1}{p} = \frac{s-3}{2p}.$$  

The latter inequality is obviously true: if $s = 1$ then LHS is zero but RHS is negative. If $s > 1$ then LHS is positive but RHS is less than or equal to zero (recall that by assumption $s \leq 3$).

**Step 5:** Since $s - 2 \in [-1, 1]$ and $\frac{1}{p} \in (\frac{s-1}{2}, 1 - \frac{3-s}{2} \theta)$, we may use Lemma 3.37 to estimate $\|f_{\tilde{s},p}^\text{shifted}(\psi)\|_{s-2,p,\eta-2}$. (Lemma 3.37 is used for estimating similar quantities in later arguments as well, so we give the justification for use of Lemma 3.37 as Remark 5.12 following this proof.)

For all $\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$ we have (note that $W_{\tilde{s},p}^\tilde{\delta} \hookrightarrow W_{\gamma}^{t,p}$ so $\psi \in W_{\gamma}^{t,p}$)

$$\|f_{\tilde{s},p}^\text{shifted}(\psi)\|_{s-2,p,\eta-2}$$

$$= \|a_\gamma(\mu + \psi)^5 + a_R(\mu + \psi) - a_\rho(\mu + \psi)^{-3} - a_W(\mu + \psi)^{-7} - a_\gamma \psi\|_{s-2,p,\eta-2}$$

$$\leq \|a_\gamma(\mu + \psi)^5\|_{s-2,p,\eta-2} + \|a_R(\mu + \psi)\|_{s-2,p,\eta-2} + \|a_\rho(\mu + \psi)^{-3}\|_{s-2,p,\eta-2}$$

$$+ \|a_W(\mu + \psi)^{-7}\|_{s-2,p,\eta-2} + \|a_\gamma \psi\|_{s-2,p,\eta-2}$$
\[ \| a_r \|_{s-2,p,\eta-2}((\mu + \psi)^5)_{L^\infty} + \| 5(\mu + \psi)^4 \|_{L^\infty} \| \psi \|_{t,\mu,\eta} \]
\[ + \| a_R \|_{s-2,p,\eta-2}(\| \mu + \psi \|_{L^\infty} + \| 1 \|_{L^\infty} \| \psi \|_{t,\mu,\eta}) \]
\[ + \| a_0 \|_{s-2,p,\eta-2} + \| a_W V \|_{s-2,p,\eta-2} \]
\[ + \| a_p \|_{s-2,p,\eta-2}((\mu + \psi)^{-3})_{L^\infty} + \| -3(\mu + \psi)^{-4} \|_{L^\infty} \| \psi \|_{t,\mu,\eta} \]
\[ + \| a_W \|_{s-2,p,\eta-2}(\| \mu + \psi \|^{-7})_{L^\infty} + \| -7(\mu + \psi)^{-8} \|_{L^\infty} \| \psi \|_{t,\mu,\eta} \]
\[ \leq \| a_r \|_{s-2,p,\eta-2}((\mu + \psi)^4)_{L^\infty} \| \mu + \psi \|_{L^\infty} + \| 5(\mu + \psi)^4 \|_{L^\infty} \| \psi \|_{t,\mu,\eta} \]
\[ + \| a_R \|_{s-2,p,\eta-2}(\| \mu + \psi \|_{L^\infty} + \| \psi \|_{t,\mu,\eta}) \]
\[ + \| a_0 \|_{s-2,p,\eta-2} \| \psi \|_{t,\mu,\eta} + \| a_W V \|_{s-2,p,\eta-2} \| \psi \|_{t,\mu,\eta} \]

(note that \( a_W V \in W^{-2,p}_{s-2,p} \), \( W^{-2,p}_{s-2,p} \rightarrow W^{-2,p}_{s-2,p} \))
\[ + \| a_p \|_{s-2,p,\eta-2}((\mu + \psi)^{-4})_{L^\infty} \| \mu + \psi \|_{L^\infty} + \| -3(\mu + \psi)^{-4} \|_{L^\infty} \| \psi \|_{t,\mu,\eta} \]
\[ + \| a_W \|_{s-2,p,\eta-2}((\mu + \psi)^{-8})_{L^\infty} \| \mu + \psi \|_{L^\infty} + \| -7(\mu + \psi)^{-8} \|_{L^\infty} \| \psi \|_{t,\mu,\eta} \].

Now note that \( W^{-2,p}_{t,\mu,\eta} \rightarrow L^\infty \rightarrow L^\infty \), so
\[ \| \mu + \psi \|_{L^\infty} \leq \mu + \| \psi \|_{L^\infty} \leq \mu + \| \psi \|_{L^\infty} \leq 1 + \| \psi \|_{t,\mu,\eta}. \]

Hence
\[ \| f_{W}^{shifted}(\psi) \|_{s-2,p,\eta-2} \leq \| a_r \|_{s-2,p,\eta-2}((\mu + \psi)^4)_{L^\infty} (1 + \| \psi \|_{t,\mu,\eta}) \]
\[ + \| a_R \|_{s-2,p,\eta-2}(1 + \| \psi \|_{t,\mu,\eta}) \]
\[ + \| a_0 \|_{s-2,p,\eta-2}(1 + \| \psi \|_{t,\mu,\eta}) \]
\[ + \| a_W \|_{s-2,p,\eta-2} C(\psi_+, \psi_-)(1 + \| \psi \|_{t,\mu,\eta}) \]
\[ + \| a_p \|_{s-2,p,\eta-2}((\mu + \psi)^{-4})_{L^\infty} (1 + \| \psi \|_{t,\mu,\eta}) \]
Consequently

$$
\| f_{\text{shifted}} W(\psi) \|_{s^{-2}, p, \eta^{-2}} \leq \left[ \| a_\tau \|_{s^{-2}, p, \eta^{-2}} (\mu + \psi_+)^4 \|_{L^\infty} + \| a_R \|_{s^{-2}, p, \eta^{-2}} \\
+ \| a_\rho \|_{s^{-2}, p, \eta^{-2}} (\mu + \psi_-)^{-4} \|_{L^\infty} + \| a_0 \|_{s^{-2}, p, \eta^{-2}} \\
+ \| a_W \|_{s^{-2}, p, \eta^{-2}} (\| \mu + \psi_- \|^{-8} \|_{L^\infty} + C(\psi_+, \psi_-)) (1 + \| \psi \|_{l, p, \gamma}) \right] (1 + \| \psi \|_{l, p, \gamma}).
$$

Finally note that by assumption $A_L^{\text{shifted}} : W_{\delta^{-2}, p} \rightarrow W_{\delta}^{s, p} \rightarrow W_{\delta}^{s, p}$ is continuous and therefore

$$
\| T_{\text{shifted}} (\psi, a_W) \|_{s, p, \delta} = \| (A_L^{\text{shifted}})^{-1} f_{\text{shifted}} W(\psi) \|_{s, p, \delta} \\
\leq \| f_{\text{shifted}} W(\psi) \|_{s^{-2}, p, \eta^{-2}} \\
\leq [1 + \| a_W \|_{s^{-2}, p, \eta^{-2}}] (1 + \| \psi \|_{l, p, \gamma}).
$$

**Step 6:** In this step we prove the second claim. By the last item in Lemma 4.10, $A_L : W_{\delta}^{s, p} \rightarrow W_{\delta^{-2}, p}^{s, p}$ is Fredholm of index zero. By Lemma 4.11, $A_L^{\text{shifted}} : W_{\delta}^{s, p} \rightarrow W_{\delta^{-2}, p}^{s, p}$ is a compact perturbation of $A_L$. Since $A_L$ is Fredholm of index zero we can conclude that $A_L^{\text{shifted}}$ is also Fredholm of index zero. Now maximum principle (Lemma 4.9) implies that the kernel of $A_L^{\text{shifted}} : W_{\delta}^{s, p} \rightarrow W_{\delta^{-2}, p}^{s, p}$ is trivial. An injective operator of index zero is surjective as well. Consequently $A_L^{\text{shifted}} : W_{\delta}^{s, p} \rightarrow W_{\delta^{-2}, p}^{s, p}$ is a continuous bijective operator. Therefore by the open mapping theorem, $A_L^{\text{shifted}} : W_{\delta}^{s, p} \rightarrow W_{\delta^{-2}, p}^{s, p}$ is an isomorphism of Banach spaces. In particular the inverse is continuous and so $\| u \|_{s, p, \delta} \leq \| A_L^{\text{shifted}} u \|_{s^{-2}, p, \eta^{-2}}$. 

$$
+ \| a_W \|_{s^{-2}, p, \eta^{-2}} \| (\mu + \psi)^{-8} \|_{L^\infty} (1 + \| \psi \|_{l, p, \gamma}).
$$
Remark 5.12. In the above proof we used Lemma 3.37 to estimate $\|f_{\text{shifted}}(\psi)\|_{s-2,p,\eta-2}$. Note that since $\psi \in W_{\delta}^{s,p} \hookrightarrow C_{\delta}^{0}$, and $\delta < 0$ we can conclude that $\psi \to 0$ as $|x| \to \infty$ (in the asymptotic ends). Therefore there exists a compact set $B$ such that outside of $B$, $|\psi| < \frac{\mu}{2}$. On the compact set $B$, the continuous function $\psi$ attains its minimum which by assumption must be larger than $-\mu$. Consequently $\inf \psi > \min \{-\mu, \min_{x \in B} \psi(x)\} > -\mu$. Because of this functions of the form $f(x) = (\mu + x)^{-m}$ where $m \in \mathbb{N}$ are smooth on $[\inf \psi, \sup \psi]$ as it is required by Lemma 3.37.

Lemma 5.13. In addition to the conditions of Lemma 5.11 (including $\beta \leq \delta$), assume $a_{R} \geq 0$ (see Remark 3.38) and define the shift function $a_{s}$ by

$$a_{s} = a_{R} + 3 \frac{(\mu + \psi_{+})^{2}}{(\mu + \psi_{-})^{6}} a_{\rho} + 5(\mu + \psi_{+})^{4} a_{t} + \frac{7}{14} \frac{a_{W}}{(\mu + \psi_{-})^{14}}.$$ 

Then for any fixed $a_{W} \in W_{\beta-2}^{s-2,p}$, the map $T_{\text{shifted}}: [\psi_{-}, \psi_{+}]_{s,p,\delta} \to W_{\delta}^{s,p}$ is monotone increasing.

Proof. (Lemma 5.13) First note that the above definition of $a_{s}$ satisfies the assumptions that we had for $a_{s}$ in Lemma 5.11. Note that

$$a_{0} = a_{R} + 3 \frac{(\mu + \psi_{+})^{2}}{(\mu + \psi_{-})^{6}} a_{\rho} + 5(\mu + \psi_{+})^{4} a_{t},$$

$$V = \frac{7}{14} \frac{(\mu + \psi_{+})^{6}}{(\mu + \psi_{-})^{14}}.$$

We first must check $a_{0} \in W_{\eta-2}^{s-2,p}$ and $\|a_{W} V\|_{s-2,p,\eta-2} \leq C(\psi_{+}, \psi_{-}) \|a_{W}\|_{s-2,p,\eta-2}$. We first check that $a_{0} \in W_{\eta-2}^{s-2,p}$. By assumption $a_{R} \in W_{\delta-2}^{s-2,p} = W_{\eta-2}^{s-2,p}$. The fact that $\frac{(\mu + \psi_{+})^{2}}{(\mu + \psi_{-})^{6}} a_{\rho}$ and $(\mu + \psi_{+})^{4} a_{t}$ are in $W_{\eta-2}^{s-2,p}$ follows directly from Lemma 3.34. Therefore $a_{0} \in W_{\eta-2}^{s-2,p}$.
We now check that \( \| a_W V \|_{s-2,p,\eta-2} \leq C(\psi_+,\psi_-) \| a_W \|_{s-2,p,\eta-2} \). By Lemma 3.37 we have

\[
\| a_W (\mu + \psi_+)^6 \|_{s-2,p,\eta-2} \leq \| a_W \|_{s-2,p,\eta-2} (\| (\mu + \psi_+)^6 \|_{L^\infty} + \| 6(\mu + \psi_+)^5 \|_{L^\infty} \| \psi_+ \|_{s,p,\delta})
\]

\[
= C_1(\psi_+) \| a_W \|_{s-2,p,\eta-2},
\]

and so (recall Remark 5.12)

\[
\| a_W V \|_{s-2,p,\eta-2} \leq \| a_W (\mu + \psi_+)^6 (\mu + \psi_-)^{-14} \|_{s-2,p,\eta-2}
\]

\[
\leq \| a_W (\mu + \psi_+)^6 \|_{s-2,p,\eta-2} (\| (\mu + \psi_-)^{-14} \|_{L^\infty}
\]

\[
+ \|- 14(\mu + \psi_-)^{-15} \|_{L^\infty} \| \psi_- \|_{s,p,\delta})
\]

\[
= C_2(\psi_-) \| a_W (\mu + \psi_+)^6 \|_{s-2,p,\eta-2}
\]

\[
\leq C_1(\psi_+) C_2(\psi_-) \| a_W \|_{s-2,p,\eta-2}
\]

\[
= C(\psi_+,\psi_-) \| a_W \|_{s-2,p,\eta-2}.
\]

Now that we have confirmed the two conditions we can proceed. For all \( \psi_1, \psi_2 \in [\psi_-,\psi_+]_{s,p,\delta} \) with \( \psi_1 \leq \psi_2 \) we have

\[
f_W^{shifted}(\psi_2) - f_W^{shifted}(\psi_1) = f_W(\psi_2) - f_W(\psi_1) - a_s(\psi_2 - \psi_1)
\]

\[
= a_t[(\mu + \psi_2)^5 - (\mu + \psi_1)^5] + a_R[\psi_2 - \psi_1]
\]

\[
- a_p[(\mu + \psi_2)^{-3} - (\mu + \psi_1)^{-3}]
\]

\[
- a_W[(\mu + \psi_2)^{-7} - (\mu + \psi_1)^{-7}] - a_s(\psi_2 - \psi_1).
\]
Note that for all \( m \in \mathbb{N} \)

\[
(\mu + \psi_2)^m - (\mu + \psi_1)^m = \left( \sum_{j=0}^{m-1} (\mu + \psi_2)^j (\mu + \psi_1)^{m-1-j} \right) (\psi_2 - \psi_1)
\]

\[
\leq m(\mu + \psi_+)^{m-1}(\psi_2 - \psi_1) - [(\mu + \psi_2)^{-m} - (\mu + \psi_1)^{-m}]
\]

\[
= \frac{(\mu + \psi_2)^m - (\mu + \psi_1)^m}{[(\mu + \psi_2)(\mu + \psi_1)]^m}
\]

\[
\leq m(\mu + \psi_+)^{m-1} - (\mu + \psi_-)^{2m}(\psi_2 - \psi_1).
\]

Therefore

\[
f_W^{\text{shifted}}(\psi_2) - f_W^{\text{shifted}}(\psi_1) \leq 5(\mu + \psi_+)^4 a_+ + a_R + 3(\mu + \psi_+)^2 a_R
\]

\[
+ \frac{(\mu + \psi_+)^6}{(\mu + \psi_-)^{14}} a_W - a_s |(\psi_2 - \psi_1)|
\]

\[
= 0.
\]

So \( f_W^{\text{shifted}} \) is decreasing over \([\psi_-, \psi_+]_{\bar{s}, \bar{p}, \bar{\delta}}\). Also \( A_L^{\text{shifted}} : W^{s,p}_{\delta} \rightarrow W^{s-2,p}_{\eta-2} \) satisfies the maximum principle, hence the inverse \( (A_L^{\text{shifted}})^{-1} \) is monotone increasing (see Appendix A). Thus \( T^{\text{shifted}} : [\psi_-, \psi_+]_{\bar{s}, \bar{p}, \bar{\delta}} \rightarrow W^{s,p}_{\delta} \) defined by \( -(A_L^{\text{shifted}})^{-1} f_W^{\text{shifted}} \) is monotone increasing. \( \square \)

**Lemma 5.14.** Let the conditions of Lemma 5.13 hold, with \( \psi_- \) and \( \psi_+ \) sub- and supersolutions of the Hamiltonian constraint (equation (5.3)), respectively (with \( a_W \) as source). Then, we have \( T^{\text{shifted}}(\psi_+, a_W) \leq \psi_+ \) and \( T^{\text{shifted}}(\psi_-, a_W) \geq \psi_- \). In particular, since \( T^{\text{shifted}} \) is monotone increasing in its first variable, \( T^{\text{shifted}} \) is invariant on \( U = [\psi_-, \psi_+]_{\bar{s}, \bar{p}, \bar{\delta}} \), that is, if \( \psi \in [\psi_-, \psi_+]_{\bar{s}, \bar{p}, \bar{\delta}} \), then \( T^{\text{shifted}}(\psi, a_W) \in [\psi_-, \psi_+]_{\bar{s}, \bar{p}, \bar{\delta}} \).

**Proof.** (Lemma 5.14) Since \( \psi_+ \) is a supersolution, by definition (which can be found in the next section), \( A_L \psi_+ + f_W(\psi_+) \geq 0 \) with respect to the order of \( W^{s-2,p}_{\delta-2} \) (see Remark
3.38). We have

\[ \psi_+ - T^{\text{shifted}}(\psi_+, a_W) = (A_L^{\text{shifted}})^{-1} [A_L^{\text{shifted}} \psi_+ + f^W_{\text{shifted}}(\psi_+)] \]

\[ = (A_L^{\text{shifted}})^{-1} [A_L \psi_+ + f_W(\psi_+)], \]

which is nonnegative since \( \psi_+ \) is supersolution and \( (A_L^{\text{shifted}})^{-1} \) is linear and monotone increasing. The proof of the other inequality is completely analogous.

**Remark 5.15.** As seen in the proof of the above lemmas, the introduction of the shift function \( a_s \) into \( f^W_{\text{shifted}} \) ensures it is a decreasing function on \([\psi_-, \psi_+]_{\tilde{s}, \tilde{p}, \tilde{\delta}}\), which subsequently implies that \( T^{\text{shifted}} \) is invariant on \( U = [\psi_-, \psi_+]_{\tilde{s}, \tilde{p}, \tilde{\delta}} \). This property of \( T^{\text{shifted}} \) plays an important role in the fixed point framework we use for our existence theorem for the coupled system.

### 5.4 Global Sub- and Supersolution Constructions

In this section, based on a combination of ideas employed in [38, 55, 26], we introduce a new method for constructing global sub- and supersolutions for the Hamiltonian constraint on AF manifolds. We begin with giving the precise definitions of local and global sub- and supersolutions.

Consider the Hamiltonian constraint (equation 5.3):

\[ A_L \psi + f(\psi, W) = 0. \]

- **A local subsolution** of (5.3) is a function \( \psi_- \in W^s_{\delta, p, \xi} \), \( \psi_- > -\mu \) such that

\[ A_L \psi_- + f(\psi_-, W) \leq 0 \]
for at least one $W \in W_{\beta}^{e,q}$. Note that the inequality is with respect to the order of $W_{\delta-2}^{s,2,p}$ (see Remark 3.38).

- A **local supersolution** of (5.3) is a function $\psi_+ \in W_{\delta}^{s,p}, \psi_+ > -\mu$ such that

$$A_L \psi_+ + f(\psi_+, W) \geq 0$$

for at least one $W \in W_{\beta}^{e,q}$.

- A **global subsolution** of (5.3) is a function $\psi_- \in W_{\delta}^{s,p}, \psi_- > -\mu$ such that

$$A_L \psi_- + f(\psi_-, W) \leq 0$$

for all vector fields $W_\psi$ solution of (5.4) (momentum constraint) with source $\psi \in W_{\delta}^{s,p}$ and $\psi \geq \psi_-$.

- A **global supersolution** of (5.3) is a function $\psi_+ \in W_{\delta}^{s,p}, \psi_+ > -\mu$ such that

$$A_L \psi_+ + f(\psi_+, W) \geq 0$$

for all vector fields $W_\psi$ solution of (5.4) (momentum constraint) with source $\psi \in W_{\delta}^{s,p}$ and $-\mu < \psi \leq \psi_+$.

- We say a pair of a subsolution and a supersolution, $\psi_-$ and $\psi_+$, is **compatible** if $-\mu < \psi_- \leq \psi_+ < \infty$ (so $[\psi_-, \psi_+]_{s,p,\delta}$ is nonempty).

For our main existence theorem we need to have compatible global subsolution and supersolution. The goal of this section is to prove the existence of such compatible global barriers. In what follows we use the following notation: Given any scalar function $v \in L^\infty$, let $v^\wedge = \text{ess sup}_M v$, and $v^\vee = \text{ess inf}_M v$. 
Proposition 5.16. Assume all the conditions of Weak Formulation 1 and Corollary 5.6 hold true. Additionally assume that $h$ belongs to the positive Yamabe class, $-1 < \beta \leq \delta < 0$, and $\|\sigma\|_{L^\infty_{\beta-1}}$, $\|\rho\|_{L^\infty_{2\beta-2}}$, $\|J\|_{W^{2,q}_{\delta-2}}$ are sufficiently small. Moreover, suppose that there exists a positive continuous function $\Lambda \in W^{s-2,p}_{\delta-2}$ and a number $\delta' \in (2\beta, \delta)$ such that $\Lambda \sim r^{\delta' - 2}$ (that is, $r^{\delta' - 2} \leq \Lambda \leq r^{\delta' - 2}$) for sufficiently large $r = (1 + |x|^2)^{\frac{1}{2}}$ (see Remark 5.17). If $\mu > 0$ is chosen to be sufficiently small, then there exists a global supersolution $\psi_+ \in W^{s,p}_\delta$ to the Hamiltonian constraint.

Proof. (Proposition 5.16) Since $h$ belongs to the positive Yamabe class, there exists a function $\xi \in W^{s,p}_\delta$, $\xi > -1$ such that if we set $\tilde{h} = (1 + \xi)^4 h$, then $R_{\tilde{h}} = 0$. Let $H(\psi, a_W, a_t, a_\rho)$ and $\tilde{H}(\psi, a_W, a_t, a_\rho)$ be as in Appendix D. In what follows we will show that there exists $\tilde{\psi}_+ > 0$ such that

$$\forall \varphi \in (-\mu, (\xi + 1)\tilde{\psi}_+ + \mu \xi]_{s,p,\delta}, \quad \tilde{H}(\tilde{\psi}_+, a_W, a_t, a_\rho) \geq 0. \quad (5.6)$$

Here $W_\varphi$ is the solution of the momentum constraint with source $\varphi$. Let's assume we find such a function. Then if we define $\psi_+ = (\xi + 1)\tilde{\psi}_+ + \mu \xi$, we have $\psi_+ \in W^{s,p}_\delta$, $\psi_+ > -\mu$ and it follows from Corollary D.2 that

$$\forall \varphi \in (-\mu, \psi_+]_{s,p,\delta}, \quad H(\psi_+, a_W, a_t, a_\rho) \geq 0$$

which precisely means that $\psi_+$ is a global supersolution of the Hamiltonian constraint. So it is enough to prove the existence of $\tilde{\psi}_+$.

Let $\Lambda \in W^{s-2,p}_{\delta-2}$ be a positive continuous function such that $\Lambda \sim r^{\delta' - 2}$ for sufficiently large $|x|$; here $\delta'$ is a fixed but arbitrary number in the interval $(2\beta, \delta)$. By Lemma 4.10 there exists a unique function $u \in W^{s,p}_\delta$ such that $-\Delta_{\tilde{h}} u = \Lambda$. By the maximum principle (Lemma 4.9) $u$ is positive ($u > 0$). Recall that $\mu$ is a fixed nonzero number but we have freedom in choosing $\mu$. We claim that if $\mu > 0$ is sufficiently small,
then $\tilde{\psi} := \mu u$ satisfies (5.6). Indeed, for all $\varphi \in (-\mu, (\xi + 1)\tilde{\psi} + \mu \xi]_{s,p,\delta}$ we have

$$
\begin{align*}
\tilde{H}(\tilde{\psi} +, a \psi, a r, a p) &= -\Delta \tilde{\psi} + a r (\tilde{\psi} + \mu)^5 - (1 + \xi)^{-12} a \psi (\tilde{\psi} + \mu)^{-7} \\
&\quad - (1 + \xi)^{-8} a p (\tilde{\psi} + \mu)^{-3} \quad (R_{ij} = 0) \\
&= \mu \Lambda + a r (\tilde{\psi} + \mu)^5 - (1 + \xi)^{-12} a \psi (\tilde{\psi} + \mu)^{-7} \\
&\quad - (1 + \xi)^{-8} a p (\tilde{\psi} + \mu)^{-3} \\
&\geq \mu \Lambda - (1 + \xi)^{-12} a \psi (\tilde{\psi} + \mu)^{-7} - (1 + \xi)^{-8} a p (\tilde{\psi} + \mu)^{-3}.
\end{align*}
$$

The argument in Remark 5.12 shows that $(\inf \xi) > -1$ and so $\inf (1 + \xi) > 0$. Therefore if we let $\tilde{C} = \max\{(1 + \xi)^{-12}, (1 + \xi)^{-8}\}$, then

$$
\begin{align*}
\tilde{H}(\tilde{\psi} +, a \psi, a r, a p) &\geq \mu \Lambda - \tilde{C} a \psi (\tilde{\psi} + \mu)^{-7} - \tilde{C} a p (\tilde{\psi} + \mu)^{-3} \\
&= \mu \Lambda - \tilde{C} \mu^{-7} a \psi (u + 1)^{-7} - \tilde{C} \mu^{-3} a p (u + 1)^{-3} \\
&\geq \mu \Lambda - C \mu^{-7} r^{2\beta-2} (k_1 \|\mu + \varphi\|_\infty^{12} + k_2) (u + 1)^{-7} \\
&\quad - \tilde{C} \mu^{-3} a p (u + 1)^{-3},
\end{align*}
$$

where we have used Lemma 5.8. Recall that $C$ (the implicit constant in Lemma 5.8) does not depend on $\mu$. Now note that $\forall \varphi \in (-\mu, (\xi + 1)\tilde{\psi} + \mu \xi]_{s,p,\delta}$ we have $0 \leq \mu + \varphi \leq (\xi + 1)(\mu + \tilde{\psi})$ and so

$$
\|\mu + \varphi\|_\infty^{12} \leq [(\xi + 1)^\gamma]^{12}[(\mu + \tilde{\psi})^\gamma]^{12}.
$$

Let $k_3 = (\xi + 1)^\gamma (1 + u)^\gamma$. We can write

$$
\begin{align*}
\|\mu + \varphi\|_\infty^{12} &\leq [(\xi + 1)^\gamma]^{12}[(\mu + \tilde{\psi} + \gamma)]^{12} = [(\xi + 1)^\gamma]^{12} \mu^{12}[(1 + u)^\gamma]^{12} \\
&= k_3^{12} \mu^{12}[(1 + u)^\gamma]^{12} \leq k_3^{12} \mu^{12} (u + 1)^{12}.
\end{align*}
$$
Consequently

\[ \tilde{H}(\tilde{\psi}_+, a_{W_\phi}, a_r, a_\rho) \geq \mu \Lambda - C\mu^{-7} r^{2\beta - 2} (k_1 k_3^{12} \mu^{12} (u + 1)^{12} + k_2) (u + 1)^{-7} \]

\[ - \tilde{C} \mu^{-3} a_\rho (u + 1)^{-3} \]

\[ = \mu \Lambda - C\mu^5 r^{2\beta - 2} k_1 k_3^{12} (u + 1)^5 - C\mu^{-7} r^{2\beta - 2} k_2 (u + 1)^{-7} \]

\[ - \tilde{C} \mu^{-3} a_\rho (u + 1)^{-3} \]

\[ \geq \mu \Lambda - C\mu^5 r^{2\beta - 2} k_1 k_3^{12} ((u + 1)^{\gamma})^5 - C\mu^{-7} r^{2\beta - 2} k_2 ((u + 1)^{\gamma})^{-7} \]

\[ - \tilde{C} \mu^{-3} a_\rho ((u + 1)^{\gamma})^{-3}. \]

Note that \( \Lambda \sim r^{\delta' - 2} \) for sufficiently large \( r \) and \( 2\beta - 2 < \delta' - 2 < 0 \). We claim that this allows one to choose \( \mu \) small enough so that

\[ \frac{\Lambda}{2} > C\mu^4 r^{2\beta - 2} k_1 k_3^{12} ((u + 1)^{\gamma})^5. \] (5.7)

The justification of this claim is as follows. There exists a constant \( C_1 \) and a number \( r_1 > 0 \) such that for \( r > r_1 \), we have \( \Lambda \geq C_1 r^{\delta' - 2} \). Also since \( 2\beta - 2 < \delta' - 2 < 0 \), there exists \( r_2 > 0 \) such that for all \( r > r_2 \)

\[ \frac{C_1}{2} r^{\delta' - 2} > r^{2\beta - 2} [C k_1 k_3^{12} ((u + 1)^{\gamma})^5]. \]

Consequently for all \( 0 < \mu \leq 1 \) and \( r > \max\{r_1, r_2\} \)

\[ \frac{\Lambda}{2} > r^{2\beta - 2} [C k_1 k_3^{12} ((u + 1)^{\gamma})^5] \geq C\mu^4 r^{2\beta - 2} k_1 k_3^{12} ((u + 1)^{\gamma})^5. \]

Also the positive continuous function \( \Lambda \) attains its minimum \( \Lambda^\vee > 0 \) on the compact
set \( r \leq \max\{r_1, r_2\} \). We choose \( \mu \leq 1 \) small enough such that

\[
\frac{\Lambda^\gamma}{2} > C\mu^4 k_1 k_3^2 ((u + 1)^\gamma)^5. \tag{5.8}
\]

Since \( r^{2\beta-2} \leq 1 \) the above inequality implies that (5.7) holds even if \( r \leq \max\{r_1, r_2\} \).

(Note that on the entire \( M \), \( \Lambda^\gamma = 0 \), so we could not use (5.8) on whole \( M \) to determine \( \mu \); this is exactly why first we needed to study what happens for large \( r \).) For such \( \mu \) by requiring that \( \|\sigma\|_{L^\infty_{\beta-1}}, \|\rho\|_{L^\infty_{2\beta-2}}, \|J\|_{W^{s-2,p}_{\delta-2}} \) are sufficiently small (note that according to Remark 3.17, \( a_\rho \leq r^{2\beta-2}\|a_\rho\|_{L^\infty_{2\beta-2}} \) a.e.) we can ensure that

\[
\frac{\Lambda}{2} \geq C\mu^{-8} r^{2\beta-2} k_2 ((u + 1)^\gamma)^{-7} + \tilde{C}\mu^{-4} a_\rho ((u + 1)^\gamma)^{-3}.
\]

\[
\text{Remark 5.17.} \text{ Pick an arbitrary number } \delta' \in (2\beta, \delta). \text{ If } s \leq 2, \text{ then } \Lambda = r^{\delta'-2} \text{ satisfies the desired conditions: clearly } \Lambda \text{ is positive, continuous, and } \Lambda \sim r^{\delta'-2}. \text{ Also obviously } \Lambda \in L^\infty_{\delta'-2} \text{ and }
\]

\[
L^\infty_{s-2} \hookrightarrow L^p_{\delta-2} \hookrightarrow W^{s-2,p}_{\delta-2} \quad (\implies \Lambda \in W^{s-2,p}_{\delta-2})
\]

The first inclusion is true because \( \delta' \) is strictly less than \( \delta \); the second inclusion is true because \( s - 2 \leq 0 \).

\[
\text{Proposition 5.18.} \text{ Assume all the conditions of Weak Formulation 1. Additionally assume that } h \text{ belongs to the positive Yamabe class and } -1 < \beta \leq \delta < 0. \text{ If } \mu > 0 \text{ is chosen to be sufficiently small, then there exists a global subsolution } \psi_- \in W^{s,p}_{\delta} \text{ to the Hamiltonian constraint which is compatible with the global supersolution that was constructed in Proposition 5.16 (provided the extra assumptions of that proposition hold true).}
\]

Proof. (Proposition 5.18) Since \( h \) belongs to the positive Yamabe class, there ex-
exists a function $\xi \in W^s_{\delta, p}$, $\xi > -1$ such that if we set $\hat{h} = (1 + \xi)^4 h$, then $\hat{R}_{\hat{h}} = 0$. Let $H(\psi, a_W, a_t, a_\rho)$ and $\hat{H}(\psi, a_W, a_t, a_\rho)$ be as in Appendix D. In what follows we will show that there exists $-\mu < \hat{\psi}_- < 0$ such that

$$\forall \varphi \in W^s_{\delta, p}, \quad \hat{H}(\hat{\psi}_-, a_W, a_t, a_\rho) \leq 0. \quad (5.9)$$

Here $W_\varphi$ is the solution of the momentum constraint with source $\varphi$. Note that since $\hat{\psi}_+ > 0$ ($\hat{\psi}_+$ is the function that was introduced in the proof of the previous proposition), clearly $\hat{\psi}_- \leq 0 < \hat{\psi}_+$. Let’s assume we find such a function. Then if we define $\psi_- = (\xi + 1)\hat{\psi}_- + \mu \xi$, we have $\psi_- \in W^s_{\delta, p}$, and

$$\hat{\psi}_- > -\mu \implies (\xi + 1)(\hat{\psi}_- + \mu) > 0 \implies (\xi + 1)\hat{\psi}_- + \mu \xi > -\mu \implies \psi_- > -\mu$$

$$\hat{\psi}_- \leq \hat{\psi}_+ \implies \psi_- \leq \psi_+$$

Moreover, it follows from Corollary D.2 that

$$\forall \varphi \in W^s_{\delta, p}, \quad H(\psi_-, a_W, a_t, a_\rho) \leq 0,$$

which clearly implies that $\psi_-$ is a global subsolution of the Hamiltonian constraint. So it is enough to prove the existence of $\hat{\psi}_-$.

We may consider two cases:

**Case 1:** $a_t \equiv 0$

In this case $\hat{\psi}_- \equiv 0$ satisfies the desired conditions; Indeed,

$$\hat{H}(\hat{\psi}_- \equiv 0, a_W, a_t, a_\rho) = -(1 + \xi)^{-12} a_W \mu^{-7} - (1 + \xi)^{-8} a_\rho \mu^{-3} \leq 0.$$
By Lemma 4.10 there exists a unique function \( u \in W^{s,p}_\delta \) such that \(-\Delta \tilde{h} u = -a_\tau\). By the maximum principle (Lemma 4.9) \( u \leq 0 \) and clearly \( u \not\equiv 0 \) (because \( a_\tau \not\equiv 0 \)). Note that 
\[
W^{s,p}_\delta \hookrightarrow L^\infty_\delta \hookrightarrow L^\infty \text{ (the latter embedding is true because } \delta < 0) \text{. Let } m = \| u \|_\infty + 1; \text{ so in particular } -m < \inf u < 0. \text{ Recall that we have freedom in choosing the fixed number } \mu \text{ as small as we want. We claim that if } \mu > 0 \text{ is sufficiently small, then } \tilde{\psi} := \frac{1}{m} \mu u \text{ satisfies (5.9). Clearly } \tilde{\psi} \leq 0; \text{ also }
\]
\[
\mu > -m \implies \mu(u + m) > 0 \implies \mu\left(\frac{u + m}{m}\right) > 0 \implies \frac{1}{m} \mu u > -\mu \implies \tilde{\psi} > -\mu. \]

Moreover, for all \( \varphi \in W^{s,p}_\delta \) we have
\[
\tilde{H}(\tilde{\psi}, aW_\varphi, a_\tau, a_\rho) = -\Delta \tilde{h} \tilde{\psi} - a_\tau (\tilde{\psi} + \mu)^5 - (1 + \xi)^{-12} aW_\varphi (\tilde{\psi} + \mu)^{-7}
\]
\[
- (1 + \xi)^{-8} a_\rho (\tilde{\psi} + \mu)^{-3}
\]
\[
\leq -\Delta \tilde{h} \left( \frac{1}{m} \mu u \right) + a_\tau \left( \frac{1}{m} \mu u + \mu \right)^5
\]
\[
= -\frac{1}{m} \mu a_\tau + \mu^5 a_\tau \left( \frac{1}{m} u + 1 \right)^5
\]
\[
= \mu a_\tau \left[ -\frac{1}{m} + \mu^4 \left( \frac{1}{m} u + 1 \right)^5 \right].
\]

Now note that \(-m < u < m\) and so \(0 < 1 + \frac{1}{m} u < 2\), therefore
\[
\tilde{H}(\tilde{\psi}, aW_\varphi, a_\tau, a_\rho) \leq \mu a_\tau \left[ -\frac{1}{m} + 32 \mu^4 \right].
\]

Thus if we choose \( \mu \) so that \( \mu^4 < \frac{1}{32m} \), then \( \tilde{H}(\tilde{\psi}, aW_\varphi, a_\tau, a_\rho) \leq 0. \)

**Remark 5.19.** The compatible global barrier constructions in [26] and [38] both make critical use of the fact that the conformal factor \( \phi \), which is the primary unknown in their formulations, is positive. When the subsolution and supersolution are both positive, then one can scale the subsolution to make it smaller than the supersolution. In
the formulation presented in this chapter, which is designed to allow very low regularity assumptions on the data on AF manifolds, the primary unknown is a shifted version of the conformal factor ($\psi$). $\psi$ can be negative and so in particular the scaling argument cannot be directly applied here. Due to the nonlinear nature of the Hamiltonian constraint, this situation cannot be resolved simply by finding compatible barriers for the original positive unknown $\phi$ and then shifting those to obtain compatible barriers for $\psi$.

5.5 The Main Existence Result

We now establish existence of coupled weak solutions for AF manifolds by combining the results for the individual Hamiltonian and momentum constraints developed in Sections 5.3 and 5.2, the barrier constructions developed in Section 5.4, together with the following topological fixed-point theorem for the coupled system from [38]:

**Theorem 5.20** (Coupled Schauder Theorem). Let $X$ and $Y$ be Banach spaces, and let $Z$ be an ordered Banach space with compact embedding $X \hookrightarrow Z$. Let $[\psi_-, \psi_+] \subset Z$ be a nonempty interval, and set $U = [\psi_-, \psi_+] \cap \bar{B}_M \subset Z$ where $\bar{B}_M$ is a closed ball of finite radius $M > 0$ in $Z$. Assume $U$ is nonempty and let $S : U \rightarrow \mathcal{R}(S) \subset Y$ and $T : U \times \mathcal{R}(S) \rightarrow U \cap X$ be continuous maps. Then, there exist $w \in \mathcal{R}(S)$ and $\psi \in U \cap X$ such that

$$\psi = T(\psi, w) \quad \text{and} \quad w = S(\psi).$$

**Remark 5.21.** The proof in [38] is based on Theorem A.20. In [38] the above theorem is stated with the extra assumption that $\bar{B}_M$ is a ball of radius $M$ **about the origin** but the same proof works even if $\bar{B}_M$ is not centered at the origin.
With all of the supporting results we need now in place, we state and prove our main result.

**Theorem 5.22.** Let \((M, h)\) be a 3-dimensional AF Riemannian manifold of class \(W^{p,q}_\delta\) where \(p \in (1, \infty), s \in (1+\frac{3}{p}, \infty)\) and \(-1 < \delta < 0\) are given. Suppose \(h\) admits no nontrivial conformal Killing field (see Remark 5.5) and is in the positive Yamabe class. Let \(\beta \in (-1, \delta]\). Select \(q\) and \(e\) to satisfy:

\[
\frac{1}{q} \in \left(0, 1\right) \cap \left(0, \frac{s-1}{3}\right) \cap \left[\frac{3-p}{3p}, \frac{3+p}{3p}\right],
\]

\[
e \in \left(1 + \frac{3}{q}, \infty\right) \cap \left[s-1, s\right] \cap \left[\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}\right].
\]

Let \(q = p\) if \(e = s \notin \mathbb{N}_0\). Moreover if \(s > 2, s \notin \mathbb{N}_0\), assume \(e < s\).

Assume that the data satisfies:

- \(\tau \in W^{e-1,q}_{\beta-1}\) if \(e \geq 2\) and \(\tau \in W^{1,z}_{\beta-1}\) otherwise, where \(z = \frac{3q}{3 + (2-e)q}\) (note that if \(e = 2\), then \(W^{e-1,q}_{\beta-1} = W^{1,z}_{\beta-1}\)),

- \(\sigma \in W^{e-1,q}_{\beta-1}\),

- \(\rho \in W^{s-2,p}_{\beta-2} \cap L^\infty, \rho \geq 0\) (\(\rho\) can be identically zero),

- \(J \in W^{e-2,q}_{\beta-2}\).

Recall that we have freedom in choosing the positive constant \(\mu\) in equations (5.3) and (5.4). If \(\mu\) is chosen to be sufficiently small and if \(\|\sigma\|_{L^\infty,\beta-1}, \|\rho\|_{L^\infty,\beta-2}^\infty\), and \(\|J\|_{W^{e-2,q}_{\beta-2}}\) are sufficiently small, then there exists \(\psi \in W^{3,p}_\delta\) with \(\psi > -\mu\) and \(W \in W^{p,q}_{\beta}\) solving (5.3) and (5.4).

**Remark 5.23.** As discussed in Appendix G, the assumptions “\(p = q\) if \(e = s \notin \mathbb{N}_0\)” and “\(e < s\) if \(s > 2, s \notin \mathbb{N}_0\)” can be removed if we replace weighted Sobolev-Slobodeckij spaces with weighted Bessel potential spaces.
Proof. (Theorem 5.22) First we prove the claim for the case $s \leq 2$ and then we extend the proof for $s > 2$ by bootstrapping.

**Case 1: $s \leq 2$**

Note that by assumption $e \leq s$, so $e$ is also less than or equal to 2. Also since $2 \geq s > 1 + \frac{3}{p}$, $p$ is larger than 3.

Since $s \leq 2$, it follows from Proposition 5.16, Remark 5.17 and Proposition 5.18 that if $\mu$ is chosen to be sufficiently small, then for $\|\sigma\|_{L_{\beta-1}^\infty}, \|\rho\|_{L_{2\beta-2}^\infty}$, and $\|J\|_{W^{e-2,q}_{\beta-2}}$ sufficiently small, there exists a compatible pair of global subsolution and supersolution. We fix such $\mu$ and assume that $\|\sigma\|_{L_{\beta-1}^\infty}, \|\rho\|_{L_{2\beta-2}^\infty}$, and $\|J\|_{W^{e-2,q}_{\beta-2}}$ are sufficiently small (according to Proposition 5.16).

**Step 1: The choice of function spaces.**

- $X = W^{s,p}_\delta$, with $s$ and $p$ as given in the theorem statement.
- $Y = W^{e,q}_\beta$, with $e, q$ as given in the theorem statement.
- $Z = W^{\bar{s},p}_\delta$, $\bar{s} \in (1, 1 + \frac{3}{p})$ and $\bar{\delta} > \delta$, so that $X = W^{\bar{s},p}_\delta \hookrightarrow W^{\bar{s},p}_\bar{\delta} = Z$ is compact. Note that $\bar{s} \in (1, 1 + \frac{3}{p})$ implies that $\bar{s} \in (\frac{3}{p}, s)$ (because $p > 3$ and $s > 1 + \frac{3}{p}$).
- $U = [\psi_-, \psi_+]_{W^{\bar{s},p}_\delta} \cap \bar{B}_M \subset W^{\bar{s},p}_\delta = Z$, with $\psi_-$ and $\psi_+$ compatible global barriers constructed in the previous section and with sufficiently large $M$ to be determined below.

**Step 2: Construction of the mapping $S$.** Using Lemma 3.34, it can be easily checked that for any $\psi \in [\psi_-, \psi_+]_{\bar{s},p,\bar{\delta}}$, $f(\psi) = b_r(\psi + \mu)^6 + b_f \in W^{e-2,q}_{\beta-2}$. Therefore, since the metric admits no nontrivial conformal Killing field, by Theorem 5.3, the momentum constraint is uniquely solvable for any “source” $\psi \in [\psi_-, \psi_+]_{\bar{s},p,\bar{\delta}}$ (it is easy to see that the assumptions of Theorem 5.3 are satisfied; see Remark 5.1). The ranges for the
exponents ensure that the momentum constraint solution map

\[ S : [\psi_-, \psi_+]_{s, p, \delta} \to W_{\beta-2}^{s, q} = Y, \quad S(\psi) = -L_{\beta}^{-1}f(\psi) \]

is continuous. Indeed, by Lemma 3.34, \( \psi \to f(\psi) \) is a continuous map from \( W_{\delta}^{s, p} \) to \( W_{\beta-2}^{s-2, q} \) and by Theorem 5.3, \( L_{\beta}^{-1} : W_{\beta-2}^{s-2, q} \to W_{\beta}^{s, q} \) is continuous.

**Step 3: Construction of the mapping \( T \).** Our construction of the mapping \( T \) makes use of Lemmas 5.13, and 5.14 where one of the assumptions is that \( a_R \geq 0 \). To satisfy this assumption, first we need to make a conformal transformation. To this end, we proceed as follows: By assumption \( h \) belongs to the positive Yamabe class. In particular, there exists \( \xi \in W_{\delta}^{s, p}, \xi > -1 \) such that \( R_\tilde{h} = 0 \) where \( \tilde{h} = (1 + \xi)^4h \). Let \( \psi_+ \) and \( \psi_- \) be the functions that were constructed in the proofs of Proposition 5.16 and Proposition 5.18. Also let

\[ \tilde{a}_{r} := a_{r}, \quad \tilde{a}_{p} := (1 + \xi)^{-9}a_{p}, \quad \tilde{a}_{W} := (1 + \xi)^{-12}a_{W}, \quad \tilde{a}_{R} := a_{R_\tilde{h}} = 0. \]

Notice that the above notations agree with the ones that are introduced in Appendix D. Using Lemma 3.34 it is easy to see that \( \tilde{a}_{p}, \tilde{a}_{W} \) remain in \( W_{\beta-2}^{s-2, p} \). So we may use \( (\tilde{h}, \tilde{a}_{r}, \tilde{a}_{p}, \tilde{a}_{W}, \tilde{a}_{R} = 0, \psi_+, \psi_-) \) as data in Lemmas 5.11, 5.13, and 5.14. That is, if we define

\[ \tilde{a}_s := \tilde{a}_R + 3\frac{(\mu + \tilde{\psi}_+)^2}{(\mu + \tilde{\psi}_-)^6} \tilde{a}_p + 5(\mu + \tilde{\psi}_+)^4 \tilde{a}_r + 7\frac{(\mu + \tilde{\psi}_+)^6}{(\mu + \tilde{\psi}_-)^14} \tilde{a}_W, \]

\[ \tilde{L}_{\beta}^{shifted} : W_{\delta}^{s, p} \to W_{\delta-2}^{s-2, p}, \quad \tilde{L}_{\beta}^{shifted} \psi = -\Delta_{\tilde{h}}\psi + \tilde{a}_s\psi, \]

\[ \tilde{f}_{\tilde{W}}^{shifted}(\psi) = \tilde{a}_{r}(\mu + \psi)^5 + \tilde{a}_{R}(\mu + \psi) - \tilde{a}_{p}(\mu + \psi)^{-3} - \tilde{a}_{W}(\mu + \psi)^{-7} - \tilde{a}_s\psi, \]

\[ \tilde{T}^{shifted} : [\psi_-, \psi_+]_{s, p, \delta} \times W_{\beta-2}^{s-2, p} \to W_{\delta}^{s, p}, \quad \tilde{T}^{shifted}(\psi, \tilde{a}_W) = -(\tilde{L}_{\beta}^{shifted})^{-1}\tilde{f}_{\tilde{W}}^{shifted}(\psi). \]
then, according to the aforementioned lemmas, \( \tilde{T}^{shifted} \) is continuous with respect to both of its arguments and it is invariant on \([\tilde{\psi}_-, \tilde{\psi}_+]_{\tilde{s}, p, \tilde{\delta}}\). Notice that if we define the **scaled Hamiltonian constraint** as in Appendix D, that is, if we let

\[
\tilde{H}(\psi, a_W, a_r, a_\rho) = -\Delta_{\tilde{\eta}}\psi + \tilde{a}_R(\psi + \mu) + \tilde{a}_r(\psi + \mu)^5 - \tilde{a}_W(\psi + \mu)^{-7} - \tilde{a}_\rho(\psi + \mu)^{-3}
\]

then \( \tilde{\psi}_- \) and \( \tilde{\psi}_+ \) are subsolution and supersolution of \( \tilde{H} = 0 \) and moreover

\[
\tilde{H}(\psi, a_W, a_r, a_\rho) = 0 \iff \tilde{T}^{shifted}(\psi, \tilde{a}_W) = \psi.
\]

Now we define the mapping \( T: [\psi_-, \psi_+]_{\tilde{s}, p, \tilde{\delta}} \times W^\alpha_q \rightarrow W^{s,p} \) as follows:

\[
T(\psi, W) = (\xi + 1) \tilde{T}^{shifted}(\frac{\psi - \mu \xi}{\xi + 1}, (\xi + 1)^{-12} a_W) + \mu \xi.
\]

Here \( \psi_+ \) and \( \psi_- \) are the supersolution and subsolution that were constructed in the proofs of Proposition 5.16 and Proposition 5.18. Recall that by our construction

\[
\tilde{\psi}_- = \frac{\psi_- - \mu \xi}{\xi + 1}, \quad \tilde{\psi}_+ = \frac{\psi_+ - \mu \xi}{\xi + 1}
\]

so for \( \psi \in [\psi_-, \psi_+]_{\tilde{s}, p, \tilde{\delta}} \), we have \( \tilde{\psi}_- \leq \frac{\psi_- - \mu \xi}{\xi + 1} \leq \tilde{\psi}_+ \). In fact using Lemma 3.34 one can easily show that \( T \) is well-defined. That is, \( \frac{\psi_- - \mu \xi}{\xi + 1} \) is in \( [\tilde{\psi}_-, \tilde{\psi}_+]_{\tilde{s}, p, \tilde{\delta}} \) and \( (\xi + 1) \tilde{T}^{shifted}(\cdot, \cdot) + \mu \xi \) is in \( W^{s,p} \). Continuity of \( T \) follows from the continuity of \( \tilde{T}^{shifted} \) and Lemma 3.34.

Considering the coupled Schauder theorem, in order to complete the proof for the case \( s \leq 2 \), it is enough to prove the following claim:

**Claim**: There exists \( M > 0 \) such that if we set \( U = [\psi_-, \psi_+]_{\tilde{s}, p, \tilde{\delta}} \cap \tilde{B}_M(\mu \xi) \), then...
$U$ is nonempty and
\[
(\psi, W) \in U \times S(U) \implies T(\psi, a_W) \in U,
\]
where $\tilde{B}_M(\mu \xi)$ is the ball of radius $M$ in $W^{s,p}_\delta$ centered at $\mu \xi \in W^{s,p}_\delta \hookrightarrow W^{\tilde{s}}_{\tilde{p}}$.

**Proof of Claim.** First, as mentioned above, note that $T(\psi, a_W)$ certainly belongs to $X = W^{s,p}_\delta$, so instead of $T(\psi, a_W) \in U$ on the right hand side we could write $T(\psi, a_W) \in U \cap X$. We now prove that if $\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$, then for all $a_W \in W^{s-2,p}_\beta$ (and so for all $W \in W^{s,q}_\beta$), $T(\psi, W) \in [\tilde{\psi}_-, \tilde{\psi}_+]_{\tilde{s},p,\tilde{\delta}}$:

\[
\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}} \implies \frac{\psi - \mu \xi}{\zeta + 1} \in [\tilde{\psi}_-, \tilde{\psi}_+]_{\tilde{s},p,\tilde{\delta}}.
\]

But we know that $\tilde{T}^{shifted}$ is invariant on $[\tilde{\psi}_-, \tilde{\psi}_+]_{\tilde{s},p,\tilde{\delta}}$ and so

\[
\forall a_W \in W^{s-2,p}_\beta \quad \tilde{T}^{shifted}\left(\frac{\psi - \mu \xi}{\zeta + 1}, (1 + \zeta)^{-1} a_W\right) \in [\tilde{\psi}_-, \tilde{\psi}_+]_{\tilde{s},p,\tilde{\delta}}.
\]

Therefore for all $W \in W^{s,q}_\beta$

\[
(1 + \zeta) \tilde{\psi}_- + \mu \xi \leq (1 + \zeta) \tilde{T}^{shifted}\left(\frac{\psi - \mu \xi}{\zeta + 1}, (1 + \zeta)^{-1} a_W\right) + \mu \xi \leq (1 + \zeta) \tilde{\psi}_+ + \mu \xi
\]

\[
\psi_- \leq T(\psi, W) \leq \psi_+
\]

Thus $T(\psi, W) \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}}$ (note that as it was already mentioned $T(\psi, W) \in W^{s,p}_\delta \hookrightarrow W^{\tilde{s}}_{\tilde{p}}$).

Now to complete the proof of the claim above, it is enough to show that the following auxiliary claim holds true:

**Auxiliary Claim:** There exists $\hat{M} > 0$ such that for all $M \geq \hat{M}$ the following holds:

If $\psi \in [\psi_-, \psi_+]_{\tilde{s},p,\tilde{\delta}} \cap \tilde{B}_M(\mu \xi)$
Then \( \forall W \in S([\psi_-, \psi_+]_{\tilde{s}, p, \delta}), \quad T(\psi, a_W) \in \tilde{B}_M(\mu \xi) \). \hspace{1cm} (5.10)

**Remark 5.24.** We make two remarks before we continue.

1. In order to prove the main claim, it is enough to prove the auxiliary claim for \( W \in S([\psi_-, \psi_+]_{\tilde{s}, p, \delta} \cap \tilde{B}_M(\mu \xi)) \) not \( W \in S([\psi_-, \psi_+]_{\tilde{s}, p, \delta}) \). So what we will prove here is slightly stronger than what we need.

2. Since we will prove (5.10) is true for all \( M \geq \hat{M} \), we can certainly choose an \( M \) such that \( [\psi_-, \psi_+]_{\tilde{s}, p, \delta} \cap \tilde{B}_M(\mu \xi) \neq \emptyset \).

**Proof of Auxiliary Claim.** We will rely on two supporting results (Lemma 5.26 and 5.27), which will be stated and proved following the completion of our proof of the main result here.

To begin, let \( t \in (\frac{3}{p}, \varepsilon) \cap [1, 1 + \frac{3}{p}) \) and let \( \gamma \in (\tilde{s}, 0) \); also for all \( \psi \in [\psi_-, \psi_+]_{\tilde{s}, p, \delta} \) let \( \tilde{\psi} := \frac{\psi - \mu \xi}{\xi + 1} \). It follows from Lemma 5.11 that there exists \( K > 0 \) such that for all \( \psi \in [\psi_-, \psi_+]_{\tilde{s}, p, \delta} \) and for all \( W \in S([\psi_-, \psi_+]_{\tilde{s}, p, \delta}) \)

\[
\| \tilde{T}^{shifted}(\tilde{\psi}, a_W) \|_{\tilde{s}, p, \delta} \leq \tilde{C} \| \tilde{T}^{shifted}(\tilde{\psi}, a_W) \|_{s, p, \delta} \leq K[1 + \| a_W \|_{s-2, p, \delta-2}](1 + \| \tilde{\psi} \|_{t, p, \gamma}).
\]

Now note that \( W_{\tilde{s}, p}^{\tilde{\delta}} \hookrightarrow W_{\gamma}^{l, p} \) is compact and \( W_{\gamma}^{l, p} \hookrightarrow L_{\gamma}^{p} \) is continuous. Therefore by Ehrling's lemma (Lemma A.3) for any \( \epsilon > 0 \) there exists \( \tilde{C}(\epsilon) > 0 \) such that

\[
\| \tilde{\psi} \|_{t, p, \gamma} \leq \epsilon \| \tilde{\psi} \|_{\tilde{s}, p, \delta} + \tilde{C}(\epsilon) \| \tilde{\psi} \|_{L_{\gamma}^{p}}.
\]

Since \( -\mu < \tilde{\psi}_- \leq \tilde{\psi} \leq \tilde{\psi}_+, \| \tilde{\psi} \|_{L_{\gamma}^{p}} \) is bounded uniformly with a constant \( P \) which we absorb into \( \tilde{C}(\epsilon) \). Making use of Lemma 5.27 below, we have

\[
\| \tilde{T}^{shifted}(\tilde{\psi}, a_W) \|_{\tilde{s}, p, \delta} \leq K[1 + C](1 + \epsilon \| \tilde{\psi} \|_{\tilde{s}, p, \delta} + \tilde{C}(\epsilon))
\]
Therefore we can write $\forall \psi \in [\psi_-, \psi_+]_{s,p,\tilde{\delta}}$ and $\forall W \in S([\psi_-, \psi_+]_{s,p,\tilde{\delta}})$,

$$\| T(\psi, W) - \mu \xi \|_{s,p,\tilde{\delta}} = \| (1 + \xi) \tilde{T}^{\text{shifted}}(\tilde{\psi}, \tilde{\alpha}_W) \|_{s,p,\tilde{\delta}}$$

$$\leq C_4(\| \xi \|_{s,p,\tilde{\delta}} + 1) \| \tilde{T}^{\text{shifted}}(\tilde{\psi}, \tilde{\alpha}_W) \|_{s,p,\tilde{\delta}}$$

(note that $W^{s,p}_\delta \times W^{s,p}_\delta \rightarrow W^{s,p}_\delta$)

$$\leq C_4(\| \xi \|_{s,p,\tilde{\delta}} + 1) K[1 + C] (1 + \epsilon \| \tilde{\psi} \|_{s,p,\tilde{\delta}} + \tilde{C}(\epsilon)).$$

Now let $A := C_4(\| \xi \|_{s,p,\tilde{\delta}} + 1) K[1 + C]$, so for all $\psi \in [\psi_-, \psi_+]_{s,p,\tilde{\delta}}$ and $W \in S([\psi_-, \psi_+]_{s,p,\tilde{\delta}})$

$$\| T(\psi, W) - \mu \xi \|_{s,p,\tilde{\delta}} \leq A(1 + \epsilon \| \psi - \mu \xi \|_{s,p,\tilde{\delta}} + \tilde{C}(\epsilon))$$

Using the argument in Lemma 5.26 below, one can show that for $f \in W^{s,p}_\delta$

$$\| \frac{1}{\xi + 1} f \|_{s,p,\tilde{\delta}} \leq C_5(1 + \| \frac{\xi}{\xi + 1} \|_{s,p,\tilde{\delta}}) \| f \|_{s,p,\tilde{\delta}}$$

so if we let $\alpha := C_5(1 + \| \frac{\xi}{\xi + 1} \|_{s,p,\tilde{\delta}})$, then $\| \frac{\psi - \mu \xi}{\xi + 1} \|_{s,p,\tilde{\delta}} \leq \alpha \| \psi - \mu \xi \|_{s,p,\tilde{\delta}}$ and therefore

$$\| T(\psi, W) - \mu \xi \|_{s,p,\tilde{\delta}} \leq A(1 + \epsilon \alpha \| \psi - \mu \xi \|_{s,p,\tilde{\delta}} + \tilde{C}(\epsilon)).$$
Let $\epsilon = \frac{1}{2\alpha A}$ and define $\hat{M} := 2A + 2A\tilde{C}(\epsilon)$. For all $M \geq \hat{M}$ we have

$$\forall \psi \in [\psi_-, \psi_+]_{\tilde{s}, \tilde{p}, \tilde{\delta}} \cap \tilde{B}_M(\mu \xi) \quad \forall W \in S([\psi_-, \psi_+]_{\tilde{s}, \tilde{p}, \tilde{\delta}})$$

$$\|T(\psi, W) - \mu \xi\|_{\tilde{s}, \tilde{p}, \tilde{\delta}} \leq A(1 + \epsilon \alpha M + \tilde{C}(\epsilon)) \quad \text{(note that } \|\psi - \mu \xi\|_{\tilde{s}, \tilde{p}, \tilde{\delta}} \leq M)$$

$$= A + (\epsilon \alpha A)M + A\tilde{C}(\epsilon)$$

$$= A + \frac{1}{2}M + \frac{\hat{M} - 2A}{2}$$

$$= \frac{1}{2}M + \frac{1}{2}\hat{M} \leq M.$$ 

Therefore $T(\psi, W) \in \tilde{B}_M(\mu \xi)$. This completes the proof of the auxiliary claim. Clearly the claim of the theorem now follows from the coupled Schauder theorem.

**Case 2: $s > 2$**

We say the 10-tuple $A = (s, p, e, q, \delta, \beta, \tau, \sigma, \rho, J)$ is **beautiful** if it satisfies the hypotheses of the theorem, that is, if

$$p \in (1, \infty), \quad s \in (1 + \frac{3}{p}, \infty), \quad -1 < \beta \leq \delta < 0,$$

$$\frac{1}{q} \in (0, 1), \quad e \in (1 + \frac{3}{q}, \infty) \cap [s - 1, s] \cap \left[\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}\right]$$

$$p = q \text{ if } e \not\in \mathbb{N}_0, \quad e < s \text{ if } s > 2 \text{ and } s \not\in \mathbb{N}_0$$

and

- $\tau \in W^{e-1,q}_{\beta-1}$ if $e \geq 2$ and $\tau \in W^{1,z}_{\beta-1}$ otherwise, where $z = \frac{3q}{3 + (2-e)q},$
- $\sigma \in W^{e-1,q}_{\beta-1},$
- $\rho \in W^{s-2,p}_{\beta-2} \cap L^\infty_{2\beta-2}, \rho \geq 0,$
- $J \in W^{e-2,q}_{\beta-2}.$
Note that the condition \( \frac{1}{q} \in \mathcal{N}(0, \frac{s-1}{3}) \cap [\frac{3-p}{3p} \cdot \frac{3+p}{3p}] \) in the statement of the theorem was to ensure that the intersection for the admissible intervals for \( e \) is nonempty. Here since we start by the assumption that \( e \) exists, we do not need to explicitly state that condition.

We say that a 10-tuple \( \tilde{A} = (\tilde{s}, \tilde{p}, \tilde{e}, \tilde{q}, \tilde{\delta}, \tilde{\beta}, \tilde{\tau}, \tilde{\sigma}, \tilde{\rho}, \tilde{J}) \) is **faithful** to the 10-tuple \( A = (s, p, e, q, \delta, \beta, \tau, \sigma, \rho, J) \) if

\[
\tilde{s} = s + \frac{|\delta|}{2}, \quad \tilde{p} = p + \frac{|\delta|}{2}, \quad \tilde{e} = e, \quad \tilde{q} = q, \quad \tilde{\delta} = \delta, \quad \tilde{\beta} = \beta, \quad \tilde{\tau} = \tau, \quad \tilde{\sigma} = \sigma, \quad \tilde{\rho} = \rho, \quad \tilde{J} = J,
\]

\[
\tilde{\tau} = \max(2, e - 2), \quad \tilde{\sigma} = \max(2, s - 2), \quad \tilde{\rho} = \max(2, s - 2),
\]

\[
\frac{1}{\tilde{p}} \leq \frac{1}{p}, \quad 1 < \frac{\tilde{e} - 3}{\tilde{q}} \leq e - 3, \quad 1 < \frac{\tilde{s} - 3}{\tilde{p}} \leq s - 3.
\]

We say that \( \tilde{A} \) is **extremely faithful** to \( A \) if \( \tilde{A} \) is both **beautiful** and **faithful** to \( A \).

**Remark 5.25.** Note that if \( \tilde{A} \) is **faithful** to \( A \), then

\[
L^\infty_{\tilde{p}^{-1}} \hookrightarrow L^\infty_{\tilde{p}^{-1}}, \quad L^\infty_{2\tilde{p}^{-2}} \hookrightarrow L^\infty_{2\tilde{p}^{-2}}, \quad W^{\tilde{e}-2, \tilde{q}}_{\tilde{p}^{-2}, \tilde{q}} \hookrightarrow W^{e-2, q}_{p^{-2}, p}.
\]

So, in particular, \( \| \cdot \|_{L^\infty_{\tilde{p}^{-1}}, \| \cdot \|_{L^\infty_{2\tilde{p}^{-2}}, \| \cdot \|_{W^{\tilde{e}-2, \tilde{q}}_{\tilde{p}^{-2}}}} \) can be controlled by \( \| \cdot \|_{L^\infty_{\tilde{p}^{-1}}, \| \cdot \|_{L^\infty_{2\tilde{p}^{-2}}, \| \cdot \|_{W^{e-2, q}_{p^{-2}, p}}} \),

\( \| \cdot \|_{W^{e-2, q}_{p^{-2}, p}} \), respectively.

We now complete the proof of the theorem for \( s > 2 \), under the condition that the following two claims hold. We will then proceed to prove both claims.

**Claim 1:** Suppose the 10-tuple \( \tilde{A} \) is **faithful** to the **beautiful** 10-tuple \( A \). If \( (\psi, W) \in W^{s,p}_{\tilde{\delta}} \times W^{\tilde{e}, \tilde{q}}_{\tilde{\beta}} \) is a solution of the constraint equations with data \( (\tau, \sigma, \rho, J) \) (which is the same as \((\tilde{\tau}, \tilde{\sigma}, \tilde{\rho}, \tilde{J})\)) , then \( (\psi, W) \in W^{s,p}_{\tilde{\delta}} \times W^{e, q}_{\tilde{\beta}} \).

**Claim 2:** If \( A \) is a **beautiful** 10-tuple with \( s > 2 \), then there exists a 10-tuple \( \tilde{A} \) that is **extremely faithful** to \( A \).

**Proof of the Theorem under Claims 1 and 2.** The argument to complete the
proof in the case \( s > 2 \) based on these two claims holding is as follows. Let \( A \) denote the 10-tuple associated to the given data in the statement of the theorem. By **Claim 2**, there exists a finite chain

\[
A = A_0 \to A_1 = (s_1, p_1, \ldots) \to A_2 = (s_2, p_2, \ldots) \to \ldots \to A_m = (s_m, p_m, \ldots)
\]

of 10-tuples such that \( s_m = 2 \) and each \( A_i \) is extremely faithful to \( A_{i-1} \). Now since \( A_m \) is beautiful and \( s_m = 2 \), by what was proved in the previous case we can choose \( \mu \) small enough so that (5.3) and (5.4) have a solution \((\psi, W) \in W_{\delta_m}^{s = 2, p_m} \times W_{\beta_m}^{e = q_m}\) (note that according to Remark 5.25 by assuming \( \|\sigma\|_{L_{\beta-1}^\infty}, \|\rho\|_{L_{2\beta-2}^\infty}, \|J\|_{W_{e-2,q}^{e - 2, q}} \) are sufficiently small, we can ensure that \( \|\sigma\|_{L_{\beta-1}^\infty}, \|\rho\|_{L_{2\beta-2}^\infty}, \|J\|_{W_{e-2,q}^{e - 2, q}} \) are as small as needed). By **Claim 1**, since each \( A_i \) is faithful to \( A_{i-1} \), we can conclude that \((\psi, W) \in W_{\delta}^{s, p} \times W_{\beta}^{e, q}\).

The main claim of the theorem in the case of \( s > 2 \) now follows.

Therefore, in the case \( s > 2 \) it is enough to prove **Claim 1** and **Claim 2**, which we now proceed to do. Before we begin, note that since in both claims \( A \) is assumed to be beautiful, we have \(-1 < \beta \leq \delta < 0 \) and so clearly \(-1 < \beta \leq \delta < 0 \); moreover \( \beta < \beta \) and \( \delta < \delta \).

**Proof of Claim 1.**

**Step 1:**

\[
b_t(\psi + \mu)^6 + b_f \in W_{e-2, q}^{\beta - 2, q}.
\]

Note that \( b_t, b_f \in W_{e-2, q}^{\beta - 2, q} \) and \( \psi \in W_{\delta}^{s, \tilde{p}}. \) By Lemma 3.34 in order to show that \( b_t(\psi + \mu)^6 \in W_{e-2, q}^{\beta - 2, q} \) it is enough to prove the following:

(i) \( e - 2 \in [-\tilde{s}, \tilde{s}] \ (e - 2 \in (-\tilde{s}, \tilde{s}) \text{ if } \tilde{s} \not\in \mathbb{N}_0), \) (ii) \( e - 2 - \frac{3}{q} \in [-3 - \tilde{s} + \frac{3}{p}, \tilde{s} - \frac{3}{p}] \).
For (ii) we have

\[
e^{-\frac{3}{q}} > 1 \Rightarrow e^{-\frac{3}{q}} - 2 > -1 > -3 - (\tilde{s} - \frac{3}{p}) \quad \text{(note that } \tilde{s} > \frac{3}{p}),
\]

\[
e \leq s + \frac{3}{q} - \frac{3}{p} \leq \tilde{s} + 2 + \frac{3}{q} - \frac{3}{p} \Rightarrow e - 2 - \frac{3}{q} \leq \tilde{s} - \frac{3}{p} \quad \text{(note } s \leq \tilde{s} + 2 \text{ and } \frac{1}{p} \leq \frac{1}{p}).
\]

In order to prove (i) we consider two cases:

**Case 1:** \(0 < s - 2 \leq 2\). In this case \(\tilde{s} = 2\) and therefore

\[
e - 2 \in [-\tilde{s}, \tilde{s}] \Leftrightarrow e - 2 \in [-2, 2] \Leftrightarrow e \in [0, 4] \text{ (clearly true since } e \in [s - 1, s]).
\]

**Case 2:** \(s - 2 > 2\). In this case \(\tilde{s} = s - 2\). Therefore

\[
e - 2 \in [-\tilde{s}, \tilde{s}] \Leftrightarrow e - 2 \in [-s + 2, s - 2] \Leftrightarrow e \in [4 - s, s].
\]

\(e \leq s\) is true by assumption. Also by assumption \(e \geq s - 1\) and since \(s > 4\) we have \(s - 1 > 4 - s\). It follows that \(e > 4 - s\). Note that if \(\tilde{s} = s - 2 > 0\) is not in \(\mathbb{N}_{0}\), then \(s \not\in \mathbb{N}_{0}\) and so since \(A\) is **beautiful** and \(s > 2\) we can conclude that \(e < s\). That is, in this case we have \(e - 2 \in (-\tilde{s}, \tilde{s})\) exactly as desired.

**Step 2:** \(W \in W_{\beta}^{e,q}\).

By what was shown in the previous step we know that \(A_{L}W = -(b_{r}(\psi + \mu)^{6} + b_{f}) \in W_{\beta-2,q}^{s-2,q}\). It follows from Remark 4.7 that \(W \in W_{\beta}^{e,q}\).

**Step 3:** \(\psi \in W_{\delta}^{s,p}\).

Since \(W \in W_{\beta}^{e,q}\) according to the argument that we had in deriving **Weak Formulation 1** we have \(a_{W} \in W_{\delta-2,p}^{s-2,p}\). It follows that \(A_{L}\psi \in W_{\delta-2,p}^{s-2,p}\). So again by Remark 4.7, we can conclude that \(\psi \in W_{\delta}^{s,p}\).

Therefore, we have shown that **Claim 1** holds. We now proceed to **Claim 2**.
Proof of Claim 2. We want to find a 10-tuple $\tilde{A}$ that is extremely faithful to $A$.

Note that all the components of $\tilde{A}$, except $\tilde{p}$ and $\tilde{q}$, are automatically determined by $A$. So we need to find $\tilde{p}$ and $\tilde{q}$ so that $\tilde{A}$ becomes extremely faithful to $A$. We must consider three cases:

Case 1: $0 < s - 2 \leq 2, e - 2 \leq 2$ (so $\tilde{s} = \tilde{e} = 2$)

Select $\tilde{p}$ and $\tilde{q}$ to satisfy

$$\frac{1}{\tilde{p}} \in \left[ \frac{1}{p} - \frac{s - 2}{3}, \frac{1}{3} \right) \cap \left( 0, \frac{1}{p} \right), \quad \frac{1}{\tilde{q}} \in \left[ \frac{1}{q} - \frac{e - 2}{3}, \frac{1}{3} \right) \cap \left( \frac{1}{p}, \infty \right).$$

Our claim is that the 10-tuple $\tilde{A} = (\tilde{s} = 2, \tilde{p}, \tilde{e} = 2, \tilde{q}, \tilde{d} = \delta + \frac{|\delta|}{2}, \tilde{\beta} = \beta + \frac{|\delta|}{2}, \tilde{t} = \tau, \tilde{\sigma} = \sigma, \tilde{\rho} = \rho, \tilde{J} = f)$ is extremely faithful to $A$.

First note that it is possible to pick such $\tilde{p}$ and $\tilde{q}$. Indeed,

$$\left[ \frac{1}{p} - \frac{s - 2}{3}, \frac{1}{3} \right) \neq \emptyset, \quad \text{since } s > 1 + \frac{3}{p},$$

$$\left[ \frac{1}{p} - \frac{s - 2}{3}, \frac{1}{3} \right) \cap \left( 0, \frac{1}{p} \right) \neq \emptyset, \quad \text{since for } s > 2, \text{ we have } \frac{1}{p} - \frac{s - 2}{3} < \frac{1}{p},$$

$$\left[ \frac{1}{q} - \frac{e - 2}{3}, \frac{1}{3} \right) \neq \emptyset, \quad \text{since } e > 1 + \frac{3}{q},$$

$$\left[ \frac{1}{q} - \frac{e - 2}{3}, \frac{1}{3} \right) \cap \left( \frac{1}{p}, \infty \right) \neq \emptyset, \quad \text{since } \frac{1}{p} < \frac{1}{3}.$$

In order to show that $\tilde{A}$ is extremely faithful to $A$ we need to show that 1) $\tilde{A}$ is faithful to $A$ and 2) $\tilde{A}$ is beautiful.
1) $\bar{A}$ is faithful to $A$:

$(i)$ By definition of $\bar{p}$ we have $\frac{1}{\bar{p}} \leq \frac{1}{p}$.

$(ii)$ Clearly $\bar{e} = 2 = \max\{2, e - 2\}$, $\bar{s} = 2 = \max\{2, s - 2\}$.

$(iii)$ $\frac{1}{p} - \frac{s - 2}{3} \leq \frac{1}{\bar{p}} < -\frac{3}{3} \Rightarrow -1 < -\frac{3}{\bar{p}} \leq s - \frac{3}{p} - 2$

$\Rightarrow 1 < 2 - \frac{3}{\bar{p}} \leq s - \frac{3}{p} \Rightarrow 1 < \bar{s} - \frac{3}{\bar{p}} \leq s - \frac{3}{p}$ ($\bar{s} = 2$).

$(iv)$ Similarly $\frac{1}{q} - \frac{e - 2}{3} \leq \frac{1}{\bar{q}} < -\frac{3}{3} \Rightarrow 1 < \bar{e} - \frac{3}{\bar{q}} \leq e - \frac{3}{q}$.

2) $\bar{A}$ is beautiful:

$(i)$ Clearly $\bar{p}, \bar{q} \in (1, \infty)$.

In addition, by what was proved above, $\bar{s} > 1 + \frac{3}{\bar{p}}$ and $\bar{e} > 1 + \frac{3}{\bar{q}}$.

$(ii)$ $\bar{e} \in [\bar{s} - 1, \bar{s}] \Rightarrow 2 \in [2 - 1, 2]$ (which is clearly true).

$(iii)$ $\bar{e} \in \left[ \frac{3}{\bar{q}} + \bar{s} - \frac{3}{\bar{p}} - 1, \frac{3}{\bar{q}} + \bar{s} - \frac{3}{\bar{p}} \right] \Rightarrow 2 \in \left[ \frac{3}{\bar{q}} - \frac{3}{\bar{p}} + 1, \frac{3}{\bar{q}} - \frac{3}{\bar{p}} + 2 \right]$

$\Rightarrow 0 \leq \frac{3}{\bar{q}} - \frac{3}{\bar{p}} \leq 1$ ($\bar{s} = \bar{e} = 2$)

$\Rightarrow \frac{1}{\bar{p}} \leq \frac{1}{\bar{q}}$ and $\frac{1}{\bar{q}} \leq \frac{1}{3} + \frac{1}{\bar{p}}$ (since we know $\frac{1}{3} > \frac{1}{\bar{q}} > \frac{1}{\bar{p}}$).

Also since $\bar{s} - \frac{3}{\bar{p}} \leq s - \frac{3}{p}$, $\bar{e} - \frac{3}{\bar{q}} \leq e - \frac{3}{q}$, $\bar{e} < \bar{\beta}$ and $\delta < \bar{\delta}$, it follows from the embedding theorem that

$$W^{s,p}_{\bar{s}, \bar{p}} \hookrightarrow W^{s,\bar{s}, \bar{p}}_{\bar{s}, \bar{p}}$$

$$W^{s-2,p}_{\bar{s}-2, \bar{p}} \hookrightarrow W^{s-2,\bar{s}, \bar{p}}_{\bar{s}-2, \bar{p}}$$

$$W^{e,q}_{\bar{e}, \bar{q}} \hookrightarrow W^{e,\bar{e}, \bar{q}}_{\bar{e}, \bar{q}}$$

$$W^{e-1,q}_{\bar{e}-1, \bar{q}} \hookrightarrow W^{e-1,\bar{e}, \bar{q}}_{\bar{e}-1, \bar{q}}$$

Therefore $\tau, \sigma, \rho$ and $J$ are in the correct spaces.
Case 2: \( s - 2 > 2, e - 2 \leq 2 \) (so \( \bar{s} = s - 2, \bar{e} = 2 \))

Select \( \bar{q} \) such that \( \frac{1}{q} \in \left[ \frac{1}{q} - \frac{e - 2}{3}, \frac{1}{3} \right] \cap \left[ \frac{1}{p} - \frac{2}{3}, \frac{1}{p} \right] \). Let \( \bar{p} := \bar{q} \). Our claim is that the 10-tuple \( \bar{A} = (\bar{s} = s - 2, \bar{p}, \bar{e} = 2, \bar{q}, \bar{\delta} = \delta + \frac{|\delta|}{2}, \bar{\beta} = \beta + \frac{|\delta|}{2}, \bar{\tau} = \tau, \bar{\sigma} = \sigma, \bar{p} = p, \bar{J} = J) \) is extremely faithful to \( A \).

First note that it is possible to pick such \( \bar{q} \). Indeed, \( \left[ \frac{1}{q} - \frac{e - 2}{3}, \frac{1}{3} \right] \neq \emptyset \) because \( e > 1 + \frac{3}{q} \). For the intersection to be nonempty we need to check \( \frac{1}{p} - \frac{2}{3} < \frac{1}{3} \) and \( \frac{1}{q} - \frac{e - 2}{3} < \frac{1}{p} \). The first inequality is clearly true. The second inequality is also true because

\[
eq \frac{3}{q} - (e - 2) < \frac{3}{p} \Rightarrow \frac{1}{q} - \frac{e - 2}{3} < \frac{1}{p}.
\]

1) \( \bar{A} \) is faithful to \( A \):

\[
(i) \quad \bar{p} = \bar{q}, \text{ and } \frac{1}{q} < \frac{1}{p} \Rightarrow \frac{1}{\bar{p}} \leq \frac{1}{p}.
\]

\[
(ii) \quad \frac{1}{q} - \frac{e - 2}{3} \leq \frac{1}{\bar{q}} < \frac{1}{3} \Rightarrow 1 < \bar{e} - \frac{3}{q} \leq e - \frac{3}{q}. \quad \left( \bar{e} = 2 \right)
\]

\[
(iii) \quad \bar{s} = s - 2 > 2 \Rightarrow \bar{s} - \frac{3}{q} > 2 - \frac{3}{q} > 1 \Rightarrow \bar{s} - \frac{3}{p} > 1. \quad \left( \text{note } \frac{1}{q} < \frac{1}{3} \text{ and } \bar{q} = \bar{p} \right)
\]

\[
(iv) \quad \frac{1}{q} \geq \frac{1}{p} - \frac{2}{3} \Rightarrow \frac{3}{p} \leq 2 + \frac{3}{q} \Rightarrow \frac{s - 2}{q} \leq s - \frac{3}{p} \Rightarrow \bar{s} - \frac{3}{p} \leq s - \frac{3}{p}. \quad \left( \text{note } \bar{s} = s - 2 \text{ and } \bar{q} = \bar{p} \right)
\]
2) \( \tilde{A} \) is beautiful:

(i) Clearly \( \tilde{p}, \tilde{q} \in (1, \infty) \). By what was proved above \( \tilde{s} > 1 + \frac{3}{\tilde{p}} \) and \( \tilde{e} > 1 + \frac{3}{\tilde{q}} \).

(ii) \( \tilde{e} \in [\tilde{s} - 1, \tilde{s}] \Rightarrow 2 \in [s - 3, s - 2] \Rightarrow 4 \leq s \leq 5. \)

(by assumption \( s > 4 \); also \( s - 1 \leq e \leq 4 \) and so \( s \leq 5 \)).

(iii) \( \tilde{e} \in \left[ \frac{3}{\tilde{q}} + \tilde{s} - \frac{3}{\tilde{p}}, \frac{3}{\tilde{q}} + \tilde{s} - \frac{3}{\tilde{p}} \right] \Rightarrow 2 \in \left[ \frac{3}{\tilde{q}} + s - 3 - \frac{3}{\tilde{q}} p, \frac{3}{\tilde{q}} + s - 2 - \frac{3}{\tilde{q}} \right] \)

\[ \Rightarrow 4 \leq s + \frac{3}{\tilde{q}} - \frac{3}{\tilde{p}} \leq 5 \Leftrightarrow 4 \leq s \leq 5. \]

(which is true; note that \( \tilde{s} = s - 2, \tilde{e} = 2, \tilde{q} = \tilde{p} \))

The proof that \( \tau, \sigma, \rho \) and \( J \) belong to the correct spaces is exactly the same as Case 1.

Case 3: \( s - 2 > 2, e - 2 > 2 \) (so \( \tilde{s} = s - 2, \tilde{e} = e - 2 \)).

Select \( \tilde{q} \) to satisfy

\[
\frac{1}{\tilde{q}} \in \left[ \frac{1}{\tilde{q}} - \frac{2}{\tilde{q}} \frac{e}{\tilde{q}}, \frac{1}{\tilde{q}} - \frac{1}{\tilde{q}} \frac{3}{\tilde{q}} - 1 \right] \cap \left( 0, \frac{1}{\tilde{q}} \right) \cap \left( \frac{1}{\tilde{q}} - \frac{1}{\tilde{q}} \frac{3}{\tilde{q}}, \infty \right).
\]

Define \( \tilde{p} \) by \( \frac{1}{\tilde{p}} := \frac{1}{\tilde{q}} - \frac{1}{\tilde{q}} + \frac{1}{\tilde{p}} \). Our claim is that the 10-tuple \( \tilde{A} = (\tilde{s} = s - 2, \tilde{p}, \tilde{e} = e - 2, \tilde{q}, \tilde{\delta} = \delta + \frac{|\delta|}{2}, \tilde{\beta} = \beta + \frac{|\delta|}{2}, \tilde{\tau} = \tau, \tilde{\sigma} = \sigma, \tilde{\rho} = \rho, \tilde{J} = J) \) is extremely faithful to \( A \).

First note that it is possible to pick such \( \tilde{q} \). Indeed, \( [\frac{1}{\tilde{q}} - \frac{2}{\tilde{q}}, \frac{e}{\tilde{q}} - 1] \neq \emptyset \) because \( e > 1 + \frac{3}{\tilde{q}} \). In order to show that the intersection of the three intervals is nonempty we consider two possibilities:

- Possibility 1: \( \frac{1}{\tilde{q}} - \frac{1}{\tilde{p}} > 0 \). In this case

\[
(0, \frac{1}{\tilde{q}}) \cap \left( \frac{1}{\tilde{q}} - \frac{1}{\tilde{p}}, \infty \right) = \left( \frac{1}{\tilde{q}} - \frac{1}{\tilde{p}}, \frac{1}{\tilde{q}} \right).
\]
and so it is enough to show that
\[
\left[ \frac{1}{q} - \frac{2}{3}, \frac{e}{3} - 1 \right) \cap \left( \frac{1}{p}, \frac{1}{q} \right) \neq \emptyset.
\]

This is true because

(i) Clearly \( \frac{1}{q} - \frac{2}{3} < \frac{1}{q} \),

(ii) \( e \geq \frac{3}{q} - \frac{3}{p} + s - 1 > \frac{3}{q} - \frac{3}{p} + 3 \Rightarrow \frac{e}{3} - 1 > \frac{1}{q} - \frac{1}{p} \).

(note that \( s > 4 \))

○ Possibility 2: \( \frac{1}{q} - \frac{1}{p} \leq 0 \). In this case

\[
(0, \frac{1}{q}) \cap (\frac{1}{q} - \frac{1}{p}, \infty) = (0, \frac{1}{q}),
\]

and so it is enough to show that

\[
\left[ \frac{1}{q} - \frac{2}{3}, \frac{e}{3} - 1 \right) \cap (0, \frac{1}{q}) \neq \emptyset.
\]

This is true because

(i) Clearly \( \frac{1}{q} - \frac{2}{3} < \frac{1}{q} \), and (ii) \( e > 3 \Rightarrow \frac{e}{3} - 1 > 0 \).
1) $\tilde{A}$ is faithful to $A$:

(i) $\frac{1}{\tilde{q}} < \frac{1}{q} \Rightarrow \frac{1}{\tilde{q}} - \frac{1}{q} + \frac{1}{p} < \frac{1}{p} \Rightarrow \frac{1}{\tilde{q}} < \frac{1}{p}$.

(ii) $\frac{1}{\tilde{q}} < \frac{e}{3} - 1 \Rightarrow e - 2 - \frac{3}{\tilde{q}} > 1 \Rightarrow \tilde{e} - \frac{3}{\tilde{q}} > 1$. ($\tilde{e} = e - 2$)

(iii) $\frac{1}{\tilde{q}} - \frac{2}{3} \leq \frac{1}{q} \Rightarrow \frac{3}{\tilde{q}} - \frac{3}{q} \leq \frac{e - 3}{q} \Rightarrow \tilde{e} - \frac{3}{\tilde{q}} \leq e - \frac{3}{q}$. ($\tilde{e} = e - 2$)

(iv) $3 + \frac{3}{\tilde{q}} < e < s + \frac{3}{q} \Rightarrow 3 + \frac{3}{\tilde{q}} \leq \frac{s - 3}{q} \Rightarrow \frac{1}{\tilde{q}} - \frac{3}{q} + \frac{3}{p} \Rightarrow \frac{3}{\tilde{q}} - \frac{3}{q} + \frac{3}{p} \leq \frac{e - 3}{q}$

$\Rightarrow 1 < s - 2 - \frac{3}{p} \Rightarrow 1 < \tilde{s} - \frac{3}{p}$. (note that $\frac{1}{p} := \frac{1}{q} - \frac{1}{\tilde{q}}$ and $\tilde{s} = s - 2$)

(v) $\frac{1}{\tilde{q}} - \frac{2}{3} \leq \frac{1}{q} \Rightarrow 0 \leq \frac{3}{\tilde{q}} - 3 + \frac{2}{q} \Rightarrow \frac{3}{\tilde{q}} - 3 + \frac{3}{p} \Rightarrow \frac{3}{\tilde{q}} - \frac{3}{p} + \frac{3}{p} \Rightarrow \frac{3}{\tilde{q}} - \frac{3}{p} + \frac{3}{p} \leq \frac{3}{p} + 2$

$\Rightarrow s - 2 - \frac{3}{p} \leq \tilde{s} - \frac{3}{p} \leq s - \frac{3}{p}$.

2) $\tilde{A}$ is beautiful:

(i) Clearly $\tilde{p}, \tilde{q} \in (1, \infty)$. By what was proved above, $\tilde{s} > 1 + \frac{3}{\tilde{p}}$ and $\tilde{e} > 1 + \frac{3}{\tilde{q}}$.

(ii) $\tilde{e} \in [\tilde{s} - 1, \tilde{s}] \Leftrightarrow e - 2 \in [s - 3, s - 2] \Leftrightarrow e \in [s - 1, s]$. (which is clearly true)

(iii) $\tilde{e} \in [\tilde{s} - 1, \tilde{s}] \Leftrightarrow e - 2 \in [\frac{3}{q} + s - 3, \frac{3}{p} + s - 2]$.

(Note that $\frac{1}{p} := \frac{1}{q} - \frac{1}{\tilde{q}}$).

The last inclusion is true because $A$ is beautiful. The proof of the fact that $\tau, \sigma, \rho$ and $J$ belong to the correct spaces is exactly the same as Case 1.

Note that $e \leq s$, so if $s - 2 \leq 2$ then $e - 2 \leq 2$ and therefore the case where $s - 2 \leq 2$, $e - 2 > 2$ does not happen.

This establishes Claim 2, and by earlier arguments the main claim of the Theorem now follows. □
5.6 Two Auxiliary Lemmas

We now state and prove two auxiliary lemmas that were used in the proof of Theorem 5.22.

**Lemma 5.26.** Let $\chi \in W^s_\delta$, $\chi > -1$ and let $f \in W^{s-2,p}_{\delta-2}$. Then $\frac{1}{1+\chi} f \in W^{s-2,p}_{\delta-2}$ and

$$\| \frac{1}{1+\chi} f \|_{s-2,p,\delta-2} \leq (1 + \| \frac{\chi}{\chi+1} \|_{s,p,\delta}) \| f \|_{s-2,p,\delta-2}.$$ 

In particular, for a fixed $\chi$, the mapping $f \mapsto \frac{1}{1+\chi} f$ (from $W^{s-2,p}_{\delta-2}$ to $W^{s-2,p}_{\delta-2}$) sends bounded sets to bounded sets.

**Proof.** (Lemma 5.26) By Lemma 3.34 $\frac{1}{1+\chi} f \in W^{s-2,p}_{\delta-2}$. Moreover

$$\| \frac{1}{1+\chi} f \|_{s-2,p,\delta-2} = \| (\frac{1}{1+\chi} - 1 + 1) f \|_{s-2,p,\delta-2} = \| \frac{-\chi}{\chi+1} f + f \|_{s-2,p,\delta-2}.$$ 

It follows from Lemma 3.34 that $\frac{-\chi}{\chi+1} \in W^s_\delta$. Also by Lemma 3.30 $W^s_\delta \times W^{s-2,p}_{\delta-2} \rightarrow W^{s-2,p}_{\delta-2}$. Thus

$$\| \frac{-\chi}{\chi+1} f + f \|_{s-2,p,\delta-2} \leq \| \frac{-\chi}{\chi+1} f \|_{s-2,p,\delta-2} + \| f \|_{s-2,p,\delta-2}$$

$$\leq \| \frac{-\chi}{\chi+1} \|_{s,p,\delta} \| f \|_{s-2,p,\delta-2} + \| f \|_{s-2,p,\delta-2}$$

$$= (1 + \| \frac{\chi}{\chi+1} \|_{s,p,\delta}) \| f \|_{s-2,p,\delta-2}.$$ 

□

**Lemma 5.27.** There exists a constant $C$ independent of $W$ such that

$$\forall W \in S([\psi_-, \psi_+]_{s,p,\delta}), \quad \| a_W \|_{s-2,p,\delta-2} \leq C.$$
Proof. (Lemma 5.27) By Corollary 5.6 if $W \in S([\psi_-, \psi_+]_{s,p,\delta})$, that is, if $W$ is the solution to the momentum constraint with some source $\psi \in [\psi_-, \psi_+]_{s,p,\delta}$, then

$$\|W\|_{e,q,\beta} \leq C_1 \left[ (\mu + \|\psi\|_{L_\infty})^6 \|b_I\|_{L_{\beta-2}^2} + \|b_I\|_{W_{\beta-2}^{s-2,q}} \right]$$

$$\leq C_1 \left[ (\mu + \max(\|\psi\|_{L_\infty}, \|\psi_+\|_{L_\infty}, \|\psi_-\|_{L_\infty}))^6 \|b_I\|_{L_{\beta-2}^2} + \|b_I\|_{W_{\beta-2}^{s-2,q}} \right].$$

Here we used the fact that $|\psi| \leq \max(|\psi_+|, |\psi_-|)$ and so $\|\psi\|_{L_\infty} \leq \max(\|\psi_+\|_{L_\infty}, \|\psi_-\|_{L_\infty})$. Consequently there exists a constant $C_2$ such that for all $W \in S([\psi_-, \psi_+]_{s,p,\delta})$ we have $\|W\|_{e,q,\beta} \leq C_2$.

Considering the restrictions on the exponents $s, p, \delta, e, q, \beta$ and using our embedding theorem and multiplication lemma, it is easy to check $W_{\beta-2}^{s-2,p} \hookrightarrow W_{\delta-2}^{s-2,p}$, $W_{\beta-2}^{e-1,q} \hookrightarrow W_{\delta-2}^{s-2,p}$, and $W_{\beta-1}^{e-1,q} \times W_{\beta-1}^{e-1,q} \hookrightarrow W_{\beta-2}^{s-2,p}$. Therefore we can write

$$\|aW\|_{s-2,p,\delta-2} \leq \|aW\|_{s-2,p,\beta-2} \leq \|aW\|_{e-1,q,2,\beta-2}$$

$$\leq \|\sigma + L W\|_{e-1,q,\beta-1}^2 \leq (\|\sigma\|_{e-1,q,\beta-1} + \|L W\|_{e-1,q,\beta-1})^2$$

$$\leq \|\sigma\|_{e-1,q,\beta-1}^2 + \|L W\|_{e-1,q,\beta-1}^2 \leq \|\sigma\|_{e-1,q,\beta-1}^2 + \|W\|_{e,q,\beta}^2$$

$$\leq \|\sigma\|_{e-1,q,\beta-1}^2 + C_2.$$ 

Hence there is a constant $C_3$ such that for all $W \in S([\psi_-, \psi_+]_{s,p,\delta})$, $\|aW\|_{s-2,p,\delta-2} \leq C_3$. Now notice that $\tilde{a}_W = (1 + \xi)^{-12} a_W$, that is, $\tilde{a}_W$ is obtained from $a_W$ by applying the mapping $f \mapsto \frac{1}{1+\xi} f$ twelve times. But by Lemma 5.26 the mapping $f \mapsto \frac{1}{1+\xi} f$ sends bounded sets in $W_{\delta-2}^{s-2,p}$ to bounded sets in $W_{\delta-2}^{s-2,p}$. Consequently there exists a constant $C$ such that

$$\forall W \in S([\psi_-, \psi_+]_{s,p,\delta}), \quad \|\tilde{a}_W\|_{s-2,p,\delta-2} \leq C.$$
Chapter 5, in part, has been submitted for publication. The material may appear as A. Behzadan and M. Holst, *Rough Solutions of the Einstein Constraint Equations on Asymptotically Flat Manifolds Without Near-CMC Conditions*. The dissertation author was the primary investigator and author of this paper.
Appendices
Appendix A

Review of Real Analysis

Here we just summarize some of the basic definitions and results from real analysis that are needed in the study of Einstein constraint equations. The main references for this appendix are [24], [34], [59], [61], [64], and [74].

We begin with the definition of ordered Banach spaces. Let $X$ be a (real) Banach space and $\mathbb{R}_+$ be the nonnegative real numbers. A subset $X_+ \subset X$ is said to be an order cone if and only if the following properties hold:

- The set $X_+$ is nonempty, closed, and $X_+ \neq \{0\}$.
- If $\alpha_1, \alpha_2 \in \mathbb{R}_+$ and $x_1, x_2 \in X_+$, then $\alpha_1 x_1 + \alpha_2 x_2 \in X_+$.
- If $x \in X_+$ and $-x \in X_+$, then $x = 0$.

A pair $X, X_+$ is called an ordered Banach space if and only if $X$ is a Banach space and $X_+ \subset X$ is an order cone. We write $u \geq v$ if and only if $u - v \in X_+$. We write $u > v$ if and only if $u \geq v$ and $u \neq v$. Let $X, X_+$ and $Y, Y_+$ be two ordered Banach spaces. We say that a linear operator $A : X \to Y$ satisfies the maximum principle if for all $u \in X$ such that $Au \in Y_+$ holds that $u \in X_+$. 

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Theorem A.1. Let $X, X_+, Y, Y_+$ be two ordered Banach spaces. Let $A : X \rightarrow Y$ be a linear, invertible operator satisfying the maximum principle. Then the inverse operator $A^{-1} : Y \rightarrow X$ is monotone increasing, that is, if $u \geq v$, then $Au \geq Av$ (note that as a consequence of the invertibility of $A$, if $u \neq v$, then $Au \neq Av$).

Next we state the open mapping theorem. The open mapping theorem is a powerful tool in proving the continuity of the inverse operator and subsequently in obtaining estimates for differential operators. In particular, this theorem plays an important role in the study of the momentum constraint (Theorem 5.3).

Theorem A.2 (Open mapping theorem). Let $X$ and $Y$ be Banach spaces. If $A : X \rightarrow Y$ is linear and surjective, then $A$ is an open map (i.e., $A$ sends open sets to open sets).

The following lemma, which is known as Ehrling’s lemma, is used in the proof of our main existence theorem for the LCBY equations on asymptotically flat manifolds.

Lemma A.3 (Ehrling’s lemma). [64] Let $X, Y$ and $Z$ be Banach spaces. Assume that $X$ is compactly embedded in $Y$ and $Y$ is continuously embedded in $Z$. Then for every $\epsilon > 0$ there exists a constant $c(\epsilon)$ such that

$$\|x\|_Y \leq \epsilon \|x\|_X + c(\epsilon) \|x\|_Z$$

Next we recall the concepts of dual space and adjoint operator and we discuss some of the corresponding notions and properties. Let $X$ be a Banach space. The collection of all continuous linear functions from $X$ to $\mathbb{R}$ is called the dual space of $X$ and is denoted by $X^*$. Let

$$T : X \rightarrow X^{**} := (X^*)^*$$

$$x \mapsto G_x, \quad G_x(h) := h(x).$$
One can easily show that $T$ is linear, one-to-one, and an isometry. However, $T$ is not necessarily surjective. We say that $X$ is *reflexive* if the above map is also surjective. Hilbert spaces are all reflexive. If $\Omega$ is an open subset of $\mathbb{R}^n$, then $W^{m,p}(\Omega)$ and $L^p(\Omega)$ are reflexive for $1 < p < \infty$. $X$ is said to be *separable* if it has a countable dense subset. $W^{m,p}(\Omega)$ and $L^p(\Omega)$ are separable Banach spaces for $1 \leq p < \infty$.

**Definition A.4.** Let $X$ be a Banach space.

- A sequence $x_n$ in $X$ converges weakly to $x \in X$ if $f(x_n) \to f(x)$ for all $f \in X^*$.
- A sequence $f_n$ in $X^*$ converges weakly* to $f \in X^*$ if $f_n(x) \to f(x)$ for all $x \in X$.

**Theorem A.5.** Let $X$ be a Banach space.

- If $X$ is reflexive, then every bounded sequence in $X$ has a weakly convergent subsequence.
- If $X$ is separable, then every bounded sequence in $X^*$ has a weakly* convergent subsequence.

**Theorem A.6** (Riesz representation). Let $(H, \langle \cdot, \cdot \rangle)$ be a (real) Hilbert space. The map $T_H$ defined by

$$T_H : H \to H^*$$

$$x \mapsto l_x, \quad l_x(y) := \langle x, y \rangle,$$

is linear, bijective and an isometry ($\| T_H(x) \|_{H^*} = \| x \|_H$).

We denote the inverse of $T_H$ by $A_H$. As a direct consequence of the above theorem, we can identify $H$ and $H^*$. 
Definition A.7 (Adjoint). Let $X$ and $Y$ be Banach spaces. Let $A : X \to Y$ be a bounded linear operator. The adjoint $A^* : Y^* \to X^*$ is defined as follows:

$$\langle A^* v, u \rangle_{X^* \times X} = \langle v, Au \rangle_{Y^* \times Y} \quad \forall \ v \in Y^* \forall \ u \in X.$$ 

Theorem A.8. $A^*$ (defined above) is linear and bounded. Moreover

$$\| A^* \|_{L(Y^*,X^*)} = \| A \|_{L(X,Y)}.$$ 

($L(X,Y)$ denotes the space of all continuous (bounded) linear operators from $X$ to $Y$.)

Definition A.9 (Hilbert adjoint). Let $H_1$ and $H_2$ be two Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. The Hilbert adjoint $A^\times : H_2 \to H_1$ is defined as follows:

$$\langle A^\times v, u \rangle_{H_1} = \langle v, Au \rangle_{H_2} \quad \forall \ v \in H_2 \forall \ u \in H_1.$$ 

Note that for all $v \in H_2$, the map $u \mapsto \langle v, Au \rangle_{H_2}$ is in $H_1^*$ and therefore by the Riesz representation theorem, there exists a unique element $z \in H_1$ such that for all $u \in H_1$, $\langle z, u \rangle_{H_1} = \langle v, Au \rangle_{H_2}$. $A^\times v$ is defined as $z$.

Theorem A.10. $A^\times$ (defined above) is linear and bounded. Moreover

$$\| A^\times \|_{L(H_2,H_1)} = \| A \|_{L(H_1,H_2)}.$$ 

Remark A.11 (Relationship between the adjoint and the Hilbert adjoint). Suppose $H_1$ and $H_2$ are Hilbert spaces. By Riesz representation theorem, every map $F : H_2^* \to H_1^*$ induces a corresponding map $A_{H_1} \circ F \circ A_{H_2}^{-1}$ from $H_2$ to $H_1$. We call this map the Riesz copy of $F$. 

Now suppose \( A \in L(H_1, H_2) \). The Riesz copy of \( A^* \in L(H_2^*, H_1^*) \) is

\[
A_{H_1} \circ A^* \circ A_{H_2}^{-1} : H_2 \to H_1.
\]

This map is in fact the Hilbert adjoint of \( A \), that is \( A^* = A_{H_1} \circ A^* \circ A_{H_2}^{-1} \). Indeed,

\[
\langle A_{H_1} \circ A^* \circ A_{H_2}^{-1} v, u \rangle_{H_1} = \langle A^* \circ A_{H_2}^{-1} v, u \rangle_{H_1^* \times H_2} = \langle A_{H_2}^{-1} v, Au \rangle_{H_2^* \times H_2} = \langle v, Au \rangle_{H_2}.
\]

So under the identification made by the Riesz representation theorem, \( A^* \) and \( A^* \) agree with each other.

**Definition A.12.** Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) and \( L \) be a linear differential operator. A differential operator \( L^{\text{formal}} \) is said to be the formal adjoint of \( L \) if

\[
\langle L^{\text{formal}} \phi, \psi \rangle_{L^2(\Omega)} = \langle \phi, L \psi \rangle_{L^2(\Omega)} \quad \forall \phi, \psi \in C^\infty_\text{c}(\Omega).
\]

\( L \) is said to be formally self-adjoint if \( L = L^{\text{formal}} \).

Next we recall the concepts of compact operators and Fredholm operators.

**Theorem A.13.** Let \( X \) and \( Y \) be normed linear spaces and \( A : X \to Y \) be a linear operator. Then the following are equivalent:

- The closure of the set \( \{ Ax : \| x \| \leq 1 \} \) is compact in \( Y \).
- The closure of the set \( \{ Ax : \| x \| < 1 \} \) is compact in \( Y \).
- For every bounded subset \( M \) of \( X \), the closure of \( A(M) \) is compact in \( Y \).
- For every bounded sequence \( x_n \) in \( X \), the sequence \( Ax_n \) has a convergent subsequence in \( Y \).
The linear operator $A : X \to Y$ is said to be compact if it satisfies any of the above equivalent conditions. We denote the collection of all compact operators from $X$ to $Y$ by $K(X, Y)$. If $X \subseteq Y$ and the inclusion map is compact, we write "$X \hookrightarrow Y$ is compact".

**Theorem A.14.** Let $X$, $Y$, and $Z$ be Banach spaces. Then

- $K(X, Y)$ is a closed subspace of $L(X, Y)$.
- If $A \in L(X, Y)$ and $B \in L(Y, Z)$ and if one of them is compact, then $B \circ A \in K(X, Z)$.
- If $A \in K(X, Y)$, then for every sequence $x_n$ in $X$,
  $$x_n \to x \quad \text{weakly} \quad \implies \quad Ax_n \to Ax.$$  
- Suppose $X$ is a reflexive Banach space, and $A : X \to Y$ is a linear operator such that for every sequence $x_n$ in $X$,
  $$x_n \to x \quad \text{weakly} \quad \implies \quad Ax_n \to Ax.$$  

Then $A \in K(X, Y)$.

**Definition A.15.** Let $X$ and $Y$ be two Banach spaces. $T \in L(X, Y)$ is called Fredholm if $\ker(T)$ and $\coker(T) := \frac{Y}{\text{Im} T}$ are finite dimensional. We denote the space of all Fredholm operators from $X$ to $Y$ by $\mathcal{F}(X, Y)$.

The index of a Fredholm operator is defined by

$$\text{index}(T) = \dim(\ker(T)) - \dim(\coker(T)).$$

**Remark A.16.**
• If \( T \in L(X, Y) \) is Fredholm then \( R(T) = \text{Im}(T) \) is a closed subset of \( Y \).

• A linear operator \( T : X \rightarrow Y \) is said to be semi-Fredholm if \( \dim(\ker T) < \infty \) and \( R(T) \) is closed.

• \( T \) is Fredholm if both \( T \) and \( T^* \) are semi-Fredholm (so self adjoint semi-Fredholm operators are Fredholm).

Compact perturbations do not change Fredholmness and do not change the index, and zero index is achieved only by compact perturbations of invertible operators. More precisely we have:

**Theorem A.17.** Let \( X \) and \( Y \) be Banach spaces.

- If \( T : X \rightarrow Y \) is Fredholm, then so is \( T^* : Y^* \rightarrow X^* \) and \( \text{index}(T^*) = -\text{index}(T) \).

- \( \mathcal{F}(X, Y) \) is an open subset of \( L(X, Y) \) and \( \text{index} : \mathcal{F}(X, Y) \rightarrow \mathbb{Z} \) is locally constant.

- If \( K : X \rightarrow Y \) is a compact operator and \( T : X \rightarrow Y \) is Fredholm, then \( T + K \) is Fredholm and \( \text{index}(T + k) = \text{index}(T) \).

- If \( T : X \rightarrow Y \) is Fredholm then \( \text{index}(T) = 0 \) iff \( T = T_0 + K \) for some invertible operator \( T_0 \) and some compact operator \( K \).

For many linear PDEs it is much easier to prove uniqueness than existence. For operators in a finite dimensional vector space, it is well-known that uniqueness and existence are in fact equivalent. It is important to consider those operators in infinite dimensions for which a similar property holds.

**Theorem A.18** (Fredholm’s theorem). Suppose \( T : X \rightarrow Y \) is Fredholm and \( \text{index} T = 0 \). Then one and only one of the following cases happen:

**Case 1:** The homogeneous equation \( Tu = 0 \) has the unique solution \( u = 0 \). In this case:
• ∀ f ∈ Y, the inhomogeneous equation Tu = f has a unique solution.

• ∀ g ∈ X* the inhomogeneous equation T* v = g has a unique solution.

**Case 2:** The homogeneous equation Tu = 0 has exactly p linearly independent solutions u₁, ..., uₚ. In this case:

• T* v = 0 has exactly p linearly independent solutions v₁, ..., vₚ

• ∀ f ∈ Y, the inhomogeneous equation Tu = f has solution iff ∀ j ⟨v_j, f⟩ = 0.

• ∀ g ∈ X* the inhomogeneous equation T* v = g has solution iff ∀ i ⟨g, u_i⟩ = 0.

In particular, if T : X → Y is Fredholm of index 0, then \( \dim \ker T = \dim \ker (T^*) \); furthermore, if T is injective, then T is also surjective.

Fixed-point theorems play an important role in the analysis of nonlinear partial differential equations. Here we state two of the most well-known fixed-point theorems. A variant of the Schauder theorem plays an important role in our study of the LCBY equations.

**Theorem A.19** (Contraction Mapping Theorem). Let \((X, d)\) be a complete metric space and suppose \(G : X → X\) is a contraction mapping on \(X\), i.e., there exists \(α ∈ [0, 1)\) such that \(d(G(x), G(y)) ≤ αd(x, y)\). Then there exists a unique fixed point \(x ∈ X\) such that \(G(x) = x\).

**Theorem A.20** (Schauder Theorem). Let \(X\) be a Banach space, and let \(U ⊂ X\) be a nonempty, convex, closed, bounded subset. If the continuous map \(G : U → X\) is such that

• \(G(U) ⊆ U\),

• the closure of \(G(B)\) is compact whenever \(B\) is a subset of \(U\) (or equivalently, the closure of \(G(U)\) is compact),
then there exists a fixed-point \( u \in U \) such that \( u = G(u) \). (Note that uniqueness is not claimed in Schauder theorem.)

A nonlinear mapping \( G : U \subseteq X \to X \) is said to be compact if \( G \) is continuous and the closure of \( G(B) \) is compact whenever \( B \) is a bounded subset of \( U \). So the above theorem states that if \( U \) is a nonempty, convex, closed, bounded subset of a Banach space \( X \) and if \( G : U \to X \) is a compact map such that \( G(U) \subseteq U \), then \( G \) has a fixed point.

Finally we review some of the basic definitions of distribution theory. A distribution is a continuous linear operator on a set of test functions. Here we consider two cases: first, the case where \( C_\infty^c(\mathbb{R}^n) \) is used as the collection of test functions and, second, the case where \( S(\mathbb{R}^n) \) is considered as the set of test functions. What follows is just a low-key summary of the basic definitions; in particular we do not mention anything about topological vector spaces. In each case, in order to describe what is meant by "continuity", we introduce a notion of convergence of sequences in the space of test functions.

Consider the space of smooth functions with compact support on \( \mathbb{R}^n, C_\infty^c(\mathbb{R}^n) \). We say that a sequence \( \phi_i \) in \( C_\infty^c(\mathbb{R}^n) \) converges to \( \phi \in C_\infty^c(\mathbb{R}^n) \), if the following two conditions are satisfied:

1. There exists a bounded set \( K \subset \mathbb{R}^n \), such that \( \phi \) and all the functions in the sequence vanish outside \( K \).

2. For every multi-index \( \alpha \in \mathbb{N}_0^n \), \( \partial^\alpha \phi_i \to \partial^\alpha \phi \) uniformly on \( \mathbb{R}^n \).

A linear function \( F : C_\infty^c(\mathbb{R}^n) \to \mathbb{R} \) is said to be continuous if for every sequence \( \phi_i \) in \( C_\infty^c(\mathbb{R}^n) \), the condition \( \phi_i \to \phi \) in \( C_\infty^c(\mathbb{R}^n) \) results in \( F(\phi_i) \to F(\phi) \) in \( \mathbb{R} \).

**Definition A.21.** A continuous linear function \( F : C_\infty^c(\mathbb{R}^n) \to \mathbb{R} \) is called a distribution
on \( \mathbb{R}^n \). The collection of all distributions on \( \mathbb{R}^n \) is a vector space and is denoted by \( D'(\mathbb{R}^n) \).

If \( F \) is a distribution and \( \phi \) is a test function, then \( F(\phi) \) is sometimes denoted by \( \langle F, \phi \rangle \). As an example, any locally integrable function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) defines a distribution as follows:

\[
\forall \phi \in C_c^\infty(\mathbb{R}^n) \quad \langle F, \phi \rangle := \int_{\mathbb{R}^n} f \phi \, dx.
\]

Another important example is the Dirac delta which is defined as follows:

\[
\forall \phi \in C_c^\infty(\mathbb{R}^n) \quad \langle \delta, \phi \rangle := \phi(0).
\]

**Definition A.22** (Distributional derivative). *Let \( F \) be a distribution. For every \( \alpha \in \mathbb{N}_0^n \) we define \( \partial^{\alpha} F : C_c^\infty(\mathbb{R}^n) \to \mathbb{R} \) as follows

\[
\forall \phi \in C_c^\infty(\mathbb{R}^n) \quad \langle \partial^{\alpha} F, \phi \rangle := (-1)^{\vert \alpha \vert} \langle F, \partial^{\alpha} \phi \rangle
\]

It can be easily shown that \( \partial^{\alpha} F \) is linear and continuous, that is, \( \partial^{\alpha} F \) is another distribution.

A sequence \( F_i \) in \( D'(\mathbb{R}^n) \) is said to be convergent to a distribution \( F \in D'(\mathbb{R}^n) \) if

\[
\forall \phi \in C_c^\infty(\mathbb{R}^n) \quad \langle F_i, \phi \rangle \to \langle F, \phi \rangle.
\]

**Definition A.23** (Schwartz space). *The Schwartz space, \( S(\mathbb{R}^n) \), consists of all functions \( \phi : \mathbb{R}^n \to \mathbb{R} \) such that

\[
\forall \alpha, \beta \in \mathbb{N}_0^n \quad p_{\alpha,\beta}(\phi) := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)| < \infty.
\]

That is, the Schwartz space consists of functions that together with all their derivatives
We say that a sequence \( \phi_i \) in \( S(\mathbb{R}^n) \) converges to \( \phi \in S(\mathbb{R}^n) \), if \( p_{\alpha,\beta}(\phi_i - \phi) \to 0 \) for all \( \alpha, \beta \in \mathbb{N}_0^n \). A linear function \( F : S(\mathbb{R}^n) \to \mathbb{R} \) is said to be continuous if for every sequence \( \phi_i \) in \( S(\mathbb{R}^n) \), the condition \( \phi_i \to \phi \) in \( S(\mathbb{R}^n) \) results in \( F(\phi_i) \to F(\phi) \) in \( \mathbb{R} \).

**Definition A.24.** A continuous linear function \( F : S(\mathbb{R}^n) \to \mathbb{R} \) is called a tempered distribution. The collection of all tempered distributions is a vector space and is denoted by \( S'(\mathbb{R}^n) \).

As an important example, if \( f \in L^p(\mathbb{R}^n), 1 \leq p < \infty \), then the function \( F : S(\mathbb{R}^n) \to \mathbb{R} \) defined by \( \langle F, \phi \rangle := \int_{\mathbb{R}^n} f \phi \, dx \) is a tempered distribution. We may identify \( f \) with the corresponding tempered distribution and subsequently consider \( L^p(\mathbb{R}^n) \) as a subset of \( S'(\mathbb{R}^n) \). In particular, each element of \( S(\mathbb{R}^n) \) can be viewed as a tempered distribution.

Distributional derivative can be defined completely analogous to the case of \( D'(\mathbb{R}^n) \). Furthermore we can define the Fourier transform of a tempered distribution. Recall that the Fourier transform of \( \phi \in S(\mathbb{R}^n) \), denoted by \( \mathcal{F} \phi \), is defined by

\[
\mathcal{F} \phi(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \phi(x) e^{-i\xi \cdot x} \, dx.
\]

One can show that \( \mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n) \) is a bijective map. The Fourier transform can be extended to a bijective map \( \mathcal{F} : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n) \) as follows:

\[
\forall \phi \in S(\mathbb{R}^n) \quad \forall u \in S'(\mathbb{R}^n) \quad \langle \mathcal{F} u, \phi \rangle := \langle u, \mathcal{F} \phi \rangle.
\]
Appendix B

Ellipticity of Laplacian and Vector Laplacian

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold and let \(\pi : E \to M\) be a smooth vector bundle over \(M\). In this appendix we denote the space of smooth sections of \(E\) by \(C^\infty(E)\). As usual the fiber over \(q \in M\) is denoted by \(E_q\). Suppose \(L : C^\infty(E) \to C^\infty(E)\) is a linear differential operator. We denote the principal symbol of \(L\) at \(q \in M\) with respect to \(\xi \in T^*_qM\) by \(L^{pr}(q, \xi) : E_q \to E_q\). We will discuss how this linear map is constructed through specific examples. For a general definition of the principal symbol see for example \([34]\). \(L\) is said to be elliptic at \(q\), if for all \(\xi \neq 0\) in \(T^*_qM\), \(L^{pr}(q, \xi) : E_q \to E_q\) is an isomorphism.

- **Laplacian**: Recall that if \(f\) is a real-valued function on \(M\), then \(\Delta f = \text{div}(\text{grad} f)\). In any local coordinate we have

\[
\Delta f = \frac{1}{\sqrt{\det g}} \partial_j (\sqrt{\det g} g^{ij} \partial_i f).
\]

The first step in finding the principal symbol is to keep all the terms involving the
highest derivatives in the local representation and get rid of the other terms. Clearly

\[ \Delta f = g^{i j} \partial_j \partial_i f + \text{terms involving derivatives of } f \text{ of order at most 1}. \]

Next we replace each occurrence of \( \partial_j \) by the component \( \xi_j \) of the linear form \( \xi \) to obtain the principal symbol.

\[ \forall \, q \in M \quad \forall \xi \in T_q^* M \setminus \{0\} \quad L^{pr}(q, \xi) : \mathbb{R} \to \mathbb{R} \]

\[ \alpha \mapsto g^{i j}(q) \xi_j \xi_i \alpha \]

Note that in this example \( E_q = \mathbb{R} \) for all \( q \in M \). Also since \( g^{i j} \xi_j \xi_i = g(\xi^i, \xi^i) \) and \( g \) is positive definite, we have \( g^{i j} \xi_j \xi_i \neq 0 \) for all \( \xi \in T_q^* M \setminus \{0\} \). Consequently \( L^{pr}(q, \xi) \) is an isomorphism for all \( \xi \in T_q^* M \setminus \{0\} \).

Similarly one can show that the conformal Laplacian is an elliptic operator.

- **Vector Laplacian**: Recall that the vector Laplacian \( \Delta_L := \text{div} \mathcal{L} \) sends vector fields to vector fields. In any local coordinates we have

\[ (\mathcal{L} X)^{ij} = \nabla^i X^j + \nabla^j X^i - \frac{2}{n} \nabla^k X^k g^{ij}. \]
Therefore

\[(\Delta L)_i = (\text{div}\mathcal{L}X)^i = \nabla_j(\mathcal{L}X)^{ij} = \nabla_j \nabla^i X^j + \nabla_j \nabla^j X^i - \frac{2}{n} \nabla_j \nabla_k X^k g^{ij} \]

\[= \nabla_j \nabla^i X^j + \nabla_j \nabla^j X^i - \frac{2}{n} \nabla^i \nabla_k X^k \]

\[= \nabla^i \nabla_j X^j + R^i_j X^j + \nabla_j \nabla^j X^i - \frac{2}{n} \nabla^i \nabla_j X^j \quad (\nabla_j \nabla^i - \nabla^i \nabla_j = R^i_j) \]

\[= \nabla_j \nabla^j X^i + (1 - \frac{2}{n}) \nabla^i \nabla_j X^j + R^i_j X^j \]

\[= g^{ik} \nabla_j \nabla_k X^i + (1 - \frac{2}{n}) g^{ik} \nabla_k \nabla_j X^j + R^i_j X^j. \]

Recall that for any two vector fields \(Y\) and \(Z\):

\[\nabla_Z Y = (ZY^s + Z^i Y^j \Gamma^s_{ij}) \partial_s\]

In particular, if \(Z = \partial_r\), then

\[\nabla_r Y = (\partial_r Y^s + Y^j \Gamma^s_{jr}) \partial_s.\]

We have

\[\nabla_j \nabla_k X^i = (\nabla_j(\nabla_k X))^i = \partial_j(\nabla_k X)^i + (\nabla_k X)^m \Gamma^i_{jm} \]

\[= \partial_j[\partial_k X^i + X^m \Gamma^i_{km}] + (\nabla_k X)^m \Gamma^i_{jm} \]

\[= \partial_j[\partial_k X^i + X^m \Gamma^i_{km}] + (\partial_k X^m + X^p \Gamma^m_{kp}) \Gamma^i_{jm} \]

\[= \partial_j \partial_k X^i + \text{terms that involve derivatives of at most order 1}.\]
Similarly,

\[ \nabla_k \nabla_j X^j = \partial_k \partial_j X^j + \text{terms that involve derivatives of at most order 1}. \]

Consequently

\[ (\Delta_L X)^i = g^{jk} \partial_j \partial_k X^i + (1 - \frac{2}{n}) g^{ik} \partial_k \partial_j X^i \]

\[ + \text{terms that involve derivatives of at most order 1 of } X. \]

So the principal part of \( \Delta_L \) is

\[ X^i \mapsto g^{jk} \partial_j \partial_k X^i + (1 - \frac{2}{n}) g^{ik} \partial_k \partial_j X^i. \]

Now we replace each occurrence of \( \partial_j \) by the component \( \xi_j \) of the linear form \( \xi \) to obtain the principal symbol \( L^{pr}(q, \xi) : T_q M \to T_q M \) it sends the vector \( X \in T_q M \) to the vector whose \( i^{th} \) component is

\[ g^{jk}(q) \xi_j \xi_k X^i + (1 - \frac{2}{n}) g^{ik}(q) \xi_k \xi_j X^i. \]

The \( j^{th} \) column of the associated matrix with respect to the coordinate basis is \( L^{pr}(q, \xi)(\partial_j) \); so the \( (i, j) \)-entry of this matrix is

\[ (L^{pr})^i_j = g^{sk}(q) \xi_s \xi_k \delta^i_j + (1 - \frac{2}{n}) g^{sk}(q) \xi_k \xi_s \delta^i_j = g^{sk}(q) \xi_s \xi_k \delta^i_j + (1 - \frac{2}{n}) \xi_i \xi_j. \]

In order to show that \( L^{pr}(q, \xi) \) is an isomorphism for \( \xi \in T_q^* M \setminus \{0\} \) we consider the associated bilinear form \( \hat{L}^{pr}(X, Y) := g(X, L^{pr}(Y)) \) whose components are

\[ (L^{pr})_{rj} = g_{ri}(q)(L^{pr})^i_j = g^{sk}(q) \xi_s \xi_k g_{rj}(q) + (1 - \frac{2}{n}) \xi_r \xi_j. \]
Thus clearly for all $X \in T_q M \setminus \{0\}$ and $\xi \in T^*_q M \setminus \{0\}$

$$\tilde{L}^p(X, X) = (L^p)_{r j} X^r X^j = g^{sk}(q)\xi_s \xi_k g_{r j}(q) X^r X^j + (1 - \frac{2}{n})\xi_r \xi_j X^r X^j$$

$$= g_q(\xi, \xi) g_q(X, X) + (1 - \frac{2}{n})(\xi_r X^r)^2 > 0$$

Consequently $L^p(q, \xi) : T_q M \to T_q M$ is an isomorphism for $\xi \in T^*_q M \setminus \{0\}$. 
Appendix C

Conformal Rescaling of Constraint Equations

The purpose of this appendix is to discuss the details of the conformal method which was briefly reviewed in Section 2.3. The focus will be on the semi-decoupling split. The calculations that follow are based on and expand upon the corresponding presentation in [38]. Here we consider the Einstein constraint equations on an \( n \)-dimensional smooth manifold \( M \). Of course \( n = 3 \) is the case that we are interested in for the classical general theory of relativity; nevertheless, in what follows we work with a general integer \( n \geq 3 \). Recall that the statement of the problem is as follows:

Find a Riemannian metric \( \hat{h} \) and a symmetric rank 2 tensor \( \hat{k} \), such that the triple \( (M, \hat{h}, \hat{k}) \) forms an initial data set for the Einstein constraint equations, i.e., such that \( (\hat{h}, \hat{k}) \) satisfies the constraint equations. Using any local frame we may write the constraint equations as follows:

\[
\hat{H} = 2\kappa \hat{\rho},
\]
\[
\hat{M}^b = -\kappa \hat{j}^b, \quad 1 \leq b \leq n,
\]
where

\[
\hat{H} := \hat{R} + (\hat{h}^{ab}\hat{k}_{ab})^2 - \hat{k}_{ab}\hat{k}^{ab},
\]

\[
\hat{M}^b := \hat{\nabla}^b (\hat{h}^{ac}\hat{k}_{ac}) - \hat{\nabla}_a\hat{k}^{ab}, \quad 1 \leq b \leq n.
\]

Clearly the number of equations is \(n + 1\). \(\hat{h}\) and \(\hat{k}\) each has \(\frac{n^2 + n}{2}\) distinct components, so the total number of unknown functions is \(n^2 + n\). The conformal method proceeds as follows:

- **Step 1:** We decompose \(\hat{k}\) into the trace-free and the pure trace parts:

\[
\hat{k}^{ab} = \hat{s}^{ab} + \frac{1}{n}\text{tr}_{\hat{h}}\hat{k}\hat{h}^{ab}.
\]

Clearly \(\text{tr}_{\hat{h}}\hat{s} = 0\).

- **Step 2:** Let

\[
\hat{h}_{ab} = \phi^r h_{ab}, \quad \hat{s}^{ab} = \phi^ss^{ab}, \quad \text{tr}_{\hat{h}}\hat{k} = \phi^t \tau,
\]

where \(r, s,\) and \(t\) are fixed but arbitrary integers. We denote the Levi-Civita connection for \(h\) by \(\nabla\). We will assume \(h\) and \(\tau\) are given.

- **Step 3:** We write \(\hat{H}\) and \(\hat{M}^b\) in terms of the new variables. In what follows indices of hatted tensors are raised and lowered with \(\hat{h}\) and indices of unhatted tensors are raised and lowered with \(h\). To this end we make use of the following lemma:

**Lemma C.1.** Under the equalities (C.1) we have

1. \(\hat{h}^{ab} = \phi^{-r} h^{ab}\).

2. \(\hat{s}^{ab} = \phi^{2r + s} s^{ab}\).

3. \(\hat{k}^{ab} = \phi^s s^{ab} + \frac{1}{n}\phi^{t-r} \tau h^{ab}\).
4. \( \hat{k}_{ab} = \phi^{2r+s}s_{ab} + \frac{1}{n} \phi^{t+r} \tau h_{ab}. \)

**Proof.** (Lemma C.1)

1. This is true because the matrix \( h^a_b \) is the inverse of \( h^a_b \). Alternatively, we may proceed as follows

\[
\hat{h}_{cd} = \hat{h}_{ca} \hat{h}_{db} \hat{h}^{ab} \implies \phi^r h_{cd} = \phi^{2r} h_{ca} h_{db} \hat{h}^{ab} \implies h_{cd} = \phi^r h_{ca} h_{db} \hat{h}^{ab}
\]

\[
\implies \frac{h_{cd}}{\delta^c_d} h^{cs} h^{dt} = \phi^r \left( h_{ca} h^{cs} \right) \left( h_{db} h^{dt} \right) \hat{h}^{ab} \implies h^{st} = \phi^r \hat{h}^{st}.
\]

2. \( \hat{s}_{ab} = \hat{h}_{ma} \hat{h}_{nb} s^{mn} = \phi^{2r} h_{ma} h_{nb} \phi^{s} s^{mn} = \phi^{(2r+s)} s_{ab}. \)

3. \( \hat{k}^{ab} = \hat{s}^{ab} + \frac{1}{n} (\text{tr} \hat{k}) \hat{h}^{ab} = \phi^s s^{ab} + \frac{1}{n} (\phi^t \tau) (\phi^{-r} h^{ab}) = \phi^s s^{ab} + \frac{1}{n} \phi^{t-r} \tau h^{ab}. \)

4. \( \hat{k}_{ab} = \hat{s}_{ab} + \frac{1}{n} (\text{tr} \hat{k}) \hat{h}_{ab} = \phi^{2r+s}s_{ab} + \frac{1}{n} (\phi^t \tau) (\phi^r h_{ab}) = \phi^{2r+s}s_{ab} + \frac{1}{n} \phi^{t+r} \tau h_{ab}. \)
Using the above lemma we can write $\hat{H}$ in terms of the new variables as follows:

$$\hat{H} = \hat{R} + (\hat{h}^{ab} \hat{k}_{ab})^2 - \hat{k}_{ab} \hat{k}^{ab}$$

$$= \hat{R} + \phi^{2t} \tau^2 - \left[ \phi^{2r+s} s_{ab} + \frac{1}{n} \phi^{t+r} \tau h_{ab} \right] \left[ \phi^{s} s_{ab} + \frac{1}{n} \phi^{t-r} \tau h^{ab} \right]$$

$$= \hat{R} + \phi^{2t} \tau^2 - \left[ \phi^{2r+2s} s_{ab} s^{ab} + \frac{1}{n^2} \phi^{2t} \tau^2 h_{ab} h^{ab} \right.$$  

$$\left. + \text{ (terms involving tr } h_s) \right]$$

$$= \hat{R} + \phi^{2t} \tau^2 - \phi^{2r+s} s_{ab} s^{ab} - \frac{1}{n} \phi^{2t} \tau^2$$

$$= \hat{R} + \frac{n-1}{n} \phi^{2t} \tau^2 - \phi^{2(r+s)} s_{ab} s^{ab}.$$  

Therefore considering equation (1.11), we get

$$\hat{H} = \phi^{-r-1} \left[ \phi R - r(n-1)\Delta \phi + \frac{r(n-1)}{4\phi} (4 - (n-2)r) \| \text{grad } \phi \|^2 \right]$$

$$+ \frac{n-1}{n} \phi^{2t} \tau^2 - \phi^{2(r+s)} s_{ab} s^{ab}. \quad (C.2)$$

It remains to write $\hat{M}^b = \hat{\nabla}^b (\hat{h}_{ac} \hat{k}^{ac}) - \hat{\nabla}_a \hat{k}^{ab}$ in terms of the new variables. In order to avoid any confusion, in what follows we denote the difference tensor which was introduced in Section 1.2.2 by $\hat{S}_{jk}^i$ instead of $\hat{S}_{jk}^i$. We have

$$\hat{\nabla}_a \hat{k}^{ab} = \nabla_a \hat{k}^{ab} + \hat{S}^a_{ac} \hat{k}^{cb} + \hat{S}^b_{ac} \hat{k}^{ac} \quad \text{(by Proposition 1.34)}$$

$$\hat{\nabla}^b (\hat{h}_{ac} \hat{k}^{ac}) = \phi^{-r} \psi^b (\hat{h}_{ac} \hat{k}^{ac}) \quad \text{(by Proposition 1.35)}$$
1. \[
\n\nabla_a \hat{k}^{ab} = \nabla_a (\phi^s s^{ab} + \frac{1}{n} \phi^{t-r} \tau h^{ab})
\]
\[
= \nabla_a (\phi^s s^{ab}) + \frac{1}{n} \nabla_a (\phi^{t-r} \tau h^{ab})
\]
\[
= s \phi^{s-1} (\nabla_a \phi) s^{ab} + \phi^s \nabla_a s^{ab} + \frac{1}{n} h^{ab} \nabla_a (\phi^{t-r} \tau)
\]
\[
= s \phi^s (\nabla_a (\ln \phi)) s^{ab} + \phi^s \nabla_a s^{ab} + \frac{1}{n} h^{ab} [(t-r) \phi^{t-r} \tau (\nabla_a (\ln \phi)) + \phi^{t-r} \nabla_a \tau]
\]
\[
= s \phi^s s^{ab} \nabla_a (\ln \phi) + \phi^s \nabla_a s^{ab} + \frac{1}{n} (t-r) \tau \phi^{t-r} \nabla^b (\ln \phi) + \frac{1}{n} \phi^{t-r} \nabla^b \tau.
\]

2. By equation (1.6) we have
\[
\tilde{S}^a_{ac} \hat{k}^{cb} = r \phi^s s^{ab} \nabla_a (\ln \phi) \left[ \phi^s s^{cb} + \frac{1}{n} \phi^{t-r} \tau h^{cb} \right] = \frac{nr}{2} \phi^s s^{ab} \nabla_a (\ln \phi) + \frac{r}{n} \phi^{t-r} \nabla^b (\ln \phi).
\]

Therefore
\[
\tilde{S}^a_{ac} \hat{k}^{cb} = \frac{nr}{2} \nabla_c (\ln \phi) [\phi^s s^{cb} + \frac{1}{n} \phi^{t-r} \tau h^{cb}] = \frac{nr}{2} \phi^s s^{ab} \nabla_a (\ln \phi) + \frac{r}{n} \phi^{t-r} \nabla^b (\ln \phi).
\]

Similar calculations show that
\[
\tilde{S}^b_{ac} \hat{k}^{ac} = r \phi^s s^{ab} \nabla_a (\ln \phi) + \left( \frac{r}{n} - \frac{r}{2} \right) \phi^{t-r} \nabla^b (\ln \phi).
\]
3.

\[
\phi^{-r} \nabla^b (\hat{h}_{ac} \hat{k}^{ac}) = \phi^{-r} \nabla^b [\phi^r h_{ac} (\phi^s s^{ac} + \frac{1}{n} \phi^{t-r} \tau h^{ac})] \\
= \phi^{-r} \nabla^b [\phi^r h_{ac} \frac{1}{n} \phi^{t-r} \tau h^{ac}] \quad (h_{ac} s^{ac} = 0) \\
= \phi^{-r} \nabla^b [\phi^t \tau] \quad (h_{ac} h^{ac} = n) \\
= \phi^{-r} [t \phi^t \tau \nabla^b (\ln \phi) + \phi^b \nabla^b \tau] \\
= t \phi^{t-r} \tau \nabla^b (\ln \phi) + \phi^{t-r} \nabla^b \tau.
\]

In conclusion

\[
\hat{M}^b = \nabla^b (\hat{h}_{ac} \hat{k}^{ac}) - \nabla_a \hat{k}^{ab} \\
= -\phi^s \nabla_a s^a_b + \frac{n-1}{n} \phi^{t-r} \tau \nabla^b \tau - (\frac{n r}{2} + r + s) \phi^s s^{ab} \nabla_a (\ln \phi) \\
+ \frac{n-1}{n} t \phi^{t-r} \tau \nabla^b (\ln \phi).
\] (C.3)

Note that \( \hat{M}_b = \hat{h}_{bc} \hat{M}^c = \phi^r h_{bc} \hat{M}^c \), therefore

\[
\hat{M}_b = -\phi^{r+s} \nabla_a s^a_b + \frac{n-1}{n} \phi^t \nabla_b \tau - (\frac{n r}{2} + r + s) \phi^{r+s} s^a_b \nabla_a (\ln \phi) \\
+ \frac{n-1}{n} t \phi^t \tau \nabla_b (\ln \phi).
\] (C.4)

By rearranging terms, we may rewrite equations (C.2), (C.3), and (C.4) as follows:

\[1\] \(- r(n-1) \Delta \phi + \frac{r(n-1)}{4 \phi} (4 - (n - 2) r)|\text{grad} \phi|^2 + R \phi \\
+ \frac{n-1}{n} \phi^{2t+r+1} \tau^2 - \phi^{3r+2s+1} s_{ab} s^{ab} = \phi^{r+1} \hat{H}.
\]
\[ (2) \quad -\nabla_a s_{ab} - \left( \frac{n+2}{2} r + s \right) s_{ab} \nabla_a (\ln \phi) + \frac{n-1}{n} \phi^{-r-s} \nabla^b \tau \\
\quad \quad \quad + \frac{n-1}{n} t \phi^{-r-s-1} \tau \nabla^b \phi = \phi^{-s} \hat{M}^b. \]

\[ (3) \quad -\nabla_a s_{ab} - \left( \frac{n+2}{2} r + s \right) s_{ab} \nabla_a (\ln \phi) + \frac{n-1}{n} \phi^{-r-s} \nabla_b \tau \\
\quad \quad \quad + \frac{n-1}{n} t \phi^{-r-s-1} \tau \nabla_b \phi = \phi^{-s} \hat{M}^b. \]

• **Step 4: The semi-decoupling split.** We notice that if we set

\[ r = \frac{4}{n-2}, \quad s = -\frac{n+2}{2} r, \quad t = 0, \]

then the terms involving the coefficients \( t, [4 - r(n-2)], \) and \( [\frac{n+2}{2} r + s] \) in \( \hat{H} \) and \( \hat{M} \) disappear and we get the following simplified expressions:

\[ -\frac{4(n-1)}{n-2} \Delta \phi + R \phi + \frac{n-1}{n} \tau^2 \phi^{\frac{n+2}{n-2}} - s_{ab} \phi^{-\frac{3n-2}{n-2}} = \phi^{\frac{n+2}{n-2}} \hat{H}. \quad \text{(C.5)} \]

\[ -\nabla_a s_{ab} + \frac{n-1}{n} \phi^{\frac{2n}{n-2}} \tau = \phi^{\frac{n+2}{n-2}} \hat{M}^b. \quad \text{(C.6)} \]

• **Step 5: York decomposition.** According to York splitting, we decompose \( s_{ab} \) as follows:

\[ s_{ab} = \sigma_{ab} + (\mathcal{L}W)_{ab}, \]

where \( \sigma_{ab} \) is divergence-free and trace-free tensor field and \( W \in \chi(M) \). Substituting
this into (C.5) and (C.6) we get

\[-\frac{4(n-1)}{n-2} \Delta \phi + R \phi + \frac{n-1}{n} \tau^2 \phi^{\frac{n+2}{n-2}} - [\sigma_{ab} + (\mathcal{L} W)_{ab}] [\sigma^{ab} + (\mathcal{L} W)^{ab}] \phi^{\frac{3n-2}{n-2}} = \phi^{\frac{n+2}{n-2}} \hat{H}.
\]
\[-\nabla_a (\mathcal{L} W)^{ab} + \frac{n-1}{n} \phi^{\frac{2n}{n-2}} \nabla^b \tau = \phi^{2\frac{n+2}{n-2}} \hat{M}^b.
\]

In particular, if \(n = 3\), then

\[-8 \Delta \phi + R \phi + \frac{2}{3} \tau^2 \phi^5 - [\sigma_{ab} + (\mathcal{L} W)_{ab}] [\sigma^{ab} + (\mathcal{L} W)^{ab}] \phi^{-7} = \phi^5 \hat{H}.
\]
\[-\nabla_a (\mathcal{L} W)^{ab} + \frac{2}{3} \phi^6 \nabla^b \tau = \phi^{10} \hat{M}^b.
\]

Recall that \(\hat{H} = 2 \kappa \hat{\rho}\) and \(\hat{M}^b = -\kappa \hat{j}^b\), so if we let \(\rho := \phi^8 \hat{\rho}\) and \(j^b := \phi^{10} \hat{j}^b\), we get

\[-8 \Delta \phi + R \phi + \frac{2}{3} \tau^2 \phi^5 - [\sigma_{ab} + (\mathcal{L} W)_{ab}] [\sigma^{ab} + (\mathcal{L} W)^{ab}] \phi^{-7} = 2 \kappa \rho \phi^{-3}.
\]
\[-\nabla_a (\mathcal{L} W)^{ab} + \frac{2}{3} \phi^6 \nabla^b \tau = -\kappa j^b.
\]
Appendix D

Conformal Covariance of the Hamiltonian Constraint

Here we develop several results we need involving properties of the Hamiltonian constraint under a conformal change.

Let $(M, h)$ be a 3-dimensional AF manifold of class $W^{s,p}_\delta$ where $p \in (1, \infty)$, $s \in (\frac{3}{p}, \infty) \cap [1, \infty)$, and $\delta < 0$. Suppose $\beta < 0$. For $\psi \in W^{s,p}_\delta$ and $a_\tau, a_\rho, a_W \in W^{s-2,-p}_{\beta-2}$, let

$$H(\psi, a_W, a_\tau, a_\rho) := -\Delta h \psi + a_{R_h} (\psi + \mu) + a_\tau (\psi + \mu)^5 - a_W (\psi + \mu)^7 - a_\rho (\psi + \mu)^3$$

where $\mu$ is a fixed positive constant, $a_{R_h} = \frac{R_h}{\delta}$, and $R_h \in W^{s-2,-p}_{\delta-2}$ is the scalar curvature of the metric $h$. Note that the Hamiltonian constraint can be represented by the equation $H = 0$.

Now let $\tilde{h} = (\xi + 1)^4 h$ where $\xi \in W^{s,p}_\delta$ is a fixed function with $\xi > -1$. According to the discussion in the beginning of Section 5.1 we know that $(M, \tilde{h})$ is also AF of class $W^{s,p}_\delta$. Define

$$\tilde{H}(\psi, a_W, a_\tau, a_\rho) := -\Delta \tilde{h} \psi + a_{R_{\tilde{h}}} (\psi + \mu) + a_\tau (\psi + \mu)^5 - \tilde{a}_W (\psi + \mu)^7 - \tilde{a}_\rho (\psi + \mu)^3$$
where \( \tilde{a}_W := (\xi + 1)^{-12} a_W \) and \( \tilde{a}_\rho := (\xi + 1)^{-8} a_\rho \). Note that it follows from Lemma 3.34 that \( \tilde{a}_W \) and \( \tilde{a}_\rho \) are in \( W^{s-2, p}_{\beta-2} \).

**Proposition D.1.** For all \( \psi \in W^{s, p}_\delta \)

\[
\tilde{H}(\psi, a_W, a_r, a_\rho) = (\xi + 1)^{-5} H((\xi + 1) \psi + \mu \xi, a_W, a_r, a_\rho).
\]

**Proof.** *(Proposition D.1)* Let \( \theta = \xi + 1 \). Then we have

\[
R^\theta_h = (-8 \Delta^\theta_h + R_h \theta) \theta^{-5},
\]

\[
\Delta^\theta_h (\theta \psi + \mu (\theta - 1)) = \Delta^\theta_h (\theta \psi) + \mu \Delta^\theta_h \theta = (\Delta^\theta_h \psi + \theta \Delta^\theta_h \psi + 2 \langle \text{grad} \psi, \text{grad} \theta \rangle_h + \mu \Delta^\theta_h \theta,
\]

\[
\Delta^\theta_h \psi = \theta^{-4} \Delta^\theta_h \psi + 2 \theta^{-5} \langle \text{grad} \psi, \text{grad} \theta \rangle_h. \quad \text{(see Section 1.2.6)}
\]

Therefore we can write

\[
(\xi + 1)^{-5} H((\xi + 1) \psi + \mu \xi, a_W, a_r, a_\rho) = \theta^{-5} H(\theta \psi + \mu \theta - \mu, a_W, a_r, a_\rho)
\]

\[
= \theta^{-5} [-\Delta^\theta_h (\theta \psi + \mu \theta - \mu) + \frac{1}{8} R^\theta_h (\theta \psi + \theta \mu) + a_r (\theta \psi + \theta \mu)^5
\]

\[
- a_W (\theta \psi + \theta \mu)^{-7} - a_\rho (\theta \psi + \theta \mu)^{-3}]
\]

\[
= \theta^{-5} [-\Delta^\theta_h (\theta \psi - \theta \Delta^\theta_h \psi - 2 \langle \text{grad} \psi, \text{grad} \theta \rangle_h - \mu \Delta^\theta_h \theta + \frac{1}{8} R^\theta_h (\psi + \mu)
\]

\[
+ a_r \theta^5 (\psi + \mu)^5 - a_W \theta^{-7} (\psi + \mu)^{-7} - a_\rho \theta^{-3} (\psi + \mu)^{-3}]
\]

\[
= [- \theta^{-4} \Delta^\theta_h \psi - 2 \theta^{-5} \langle \text{grad} \psi, \text{grad} \theta \rangle_h] + [- \theta^{-5} (\Delta^\theta_h \psi - \mu \theta^{-5} \Delta^\theta_h \theta
\]

\[
+ \frac{1}{8} R^\theta_h \theta^{-4} (\psi + \mu)] + a_r (\psi + \mu)^5 - a_W \theta^{-12} (\psi + \mu)^{-7} - a_\rho \theta^{-8} (\psi + \mu)^{-3}
\]

\[
= - \Delta^\theta_h \psi + \frac{1}{8} R^\theta_h (\psi + \mu) + a_r (\psi + \mu)^5 - \tilde{a}_W (\psi + \mu)^{-7} - \tilde{a}_\rho (\psi + \mu)^{-3}
\]

\[
= \tilde{H}(\psi, a_W, a_r, a_\rho).
\]
We have the following important corollary:

**Corollary D.2.** Assume the above setting. Then we have

\[
\begin{align*}
\tilde{H}(\tilde{\psi}, a_W, a_\tau, a_\rho) &= 0 \iff H((\xi + 1)\tilde{\psi} + \mu \xi, a_W, a_\tau, a_\rho) = 0, \\
\tilde{H}(\tilde{\psi}, a_W, a_\tau, a_\rho) &\geq 0 \iff H((\xi + 1)\tilde{\psi} + \mu \xi, a_W, a_\tau, a_\rho) \geq 0, \\
\tilde{H}(\tilde{\psi}, a_W, a_\tau, a_\rho) &\leq 0 \iff H((\xi + 1)\tilde{\psi} + \mu \xi, a_W, a_\tau, a_\rho) \leq 0.
\end{align*}
\]

In particular, if \( \tilde{\psi}_+ \) and \( \tilde{\psi}_- \) are sub and supersolutions for the equation \( \tilde{H} = 0 \), then \( \psi_+ := (\xi + 1)\tilde{\psi}_+ + \mu \xi \) and \( \psi_- := (\xi + 1)\tilde{\psi}_- + \mu \xi \) are sub and supersolutions for the equation \( H = 0 \).

Appendix D, in full, has been submitted for publication. The material may appear as A. Behzadan and M. Holst, *Rough Solutions of the Einstein Constraint Equations on Asymptotically Flat Manifolds Without Near-CMC Conditions*. The dissertation author was the primary investigator and author of this paper.
Appendix E

The Yamabe Problem on Compact Manifolds

Yamabe classes play an important role in the study of the conformal formulation of the Einstein constraint equations. In this appendix we review some of the important results on the Yamabe classification of metrics on compact manifolds. In Section E.1 we briefly go over the main ideas for the case where the manifold is endowed with a smooth metric; Section E.1 closely follows the paper by Lee and Parker [47]; In Section E.2 we consider the case where the metric is not necessarily smooth; the main reference for Section E.2 is [53]. In particular we reorganize and expand upon the argument given in [53] for the case of rough metrics.

Notation: Throughout this appendix, we denote the gradient of a real-valued function $u$ by $\nabla u$ instead of $\text{grad} u$. 


E.1 Smooth Metrics

We start with the statement of the Yamabe conjecture. Let $M$ be a smooth connected compact manifold with a smooth Riemannian metric $g$. We say $\tilde{g}$ and $g$ are \textit{conformally equivalent} if there exists a \textbf{positive} smooth function $\varphi$ such that $\tilde{g} = \varphi g$. This defines an equivalence relation on the space of smooth metrics on $M$. We denote the equivalence class (conformal class) containing $g$ by $[g]$.

- \textbf{The Yamabe conjecture}: Given a compact smooth connected Riemannian manifold $(M, g)$ of dimension $n \geq 3$ there exists a metric $\tilde{g} \in [g]$ with constant scalar curvature.

- \textbf{The Yamabe conjecture, weak form}: Given a compact smooth connected Riemannian manifold $(M, g)$ of dimension $n \geq 3$ there exists a metric $\tilde{g} \in [g]$ whose scalar curvature has constant sign.

Note that if $(M, g)$ is an $n$-dimensional smooth Riemannian manifold, then any metric conformal to $g$ can be written in the form $\tilde{g} = \varphi^{\frac{4}{n-2}} g$ where $\varphi$ is a positive smooth real valued function on $M$. As it is discussed in Chapter 1, by using the exponent $\frac{4}{n-2}$, the transformation formula for the scalar curvature will be considerably simplified: if $R_g$ and $R_{\tilde{g}}$ denote the scalar curvatures of $g$ and $\tilde{g}$, respectively, then

$$R_{\tilde{g}} = \varphi^{1-p}(-a\Delta_g \varphi + R_g \varphi)$$

where

$$p = \frac{2n}{n-2}, \quad a = \frac{4}{n-2}.$$ 

Note that $p - 2 = \frac{4}{n-2}$. If $n = 3$, then $p = 6$ and $a = 8$. The differential operator $\Box := -a\Delta_g + R_g$ is called the \textit{conformal Laplacian}.
Therefore \([g]\) contains a metric with constant scalar curvature \(\lambda\) if and only if the Yamabe equation

\[
\Box \varphi = \lambda \varphi^{p-1}
\]

has a \textbf{positive} solution \(\varphi\).

This is a sort of nonlinear eigenvalue problem. The analytic properties of the equation \(\Box \varphi = \lambda \varphi^q\) depend on the value of the exponent \(q\):

- \(q = 1 \leadsto\) the equation is the linear eigenvalue problem for conformal Laplacian.
- \(q \text{ close to } 1 \leadsto\) the analytic behaviour of the equation is similar to that of the linear case and the problem can be easily solved.
- \(q = p - 1 \leadsto\) this is the \textit{critical exponent} below which the equation is easy to solve and above which it may be impossible.

It can be shown that the Yamabe equation is the Euler-Lagrange equation for the functional

\[
Q_p(\varphi) = \frac{\int_M a|\nabla \varphi|^2 + R_g \varphi^2 dV_g}{\| \varphi \|_p^2} = : \frac{E(\varphi)}{\| \varphi \|_p^2}
\]

The above functional is well-defined on \(H^1(M) \setminus \{0\}\). In particular if \(\varphi\) is a smooth positive function on \(M\) then

\(\varphi\) is a \textbf{critical point} of \(Q_p \Leftrightarrow \varphi\) satisfies the Yamabe equation with \(\lambda = \frac{E(\varphi)}{\| \varphi \|_p^2}\).

It is important to notice that

1. \(Q_p\) is bounded below and so

\[
\lambda_p := \inf\{Q_p(\varphi) : \varphi \text{ a smooth, positive function on } M\}
\]

\[
= \inf\{Q_p(\varphi) : \varphi \in H^1(M) \setminus \{0\}\}
\]
is a finite number which is called the Yamabe invariant of \((M, g)\). Clearly if the smooth function \(\phi\) is a minimizer of \(Q_p\), i.e, \(Q_p(\phi) = \lambda_p\), then \(\phi\) is a critical point of \(Q_p\) and so satisfies the Yamabe equation.

2. \(Q_p\) is scale invariant, that is \(Q_p(\alpha \phi) = Q_p(\phi)\) for all nonzero real numbers \(\alpha\). So if \(\phi\) is a minimizer of \(Q_p\), \(\frac{q}{p} \|\phi\|_p\) will also be a minimizer.

In order to show that \(Q_p\) has a critical point (and therefore Yamabe problem has a solution) it is enough to show that \(Q_p\) attains its infimum. The most direct approach to minimizing \(Q_p\) is to construct a sequence of functions for which the functional \(Q_p\) approaches its infimum (a minimizing sequence), and hope that some subsequence converges to an actual minimizing function.

Unfortunately because of the exponent \(q = p - 1 = n + 2 - \frac{2}{n-2}\) that occurs in the equation, this direct approach does not work. Why does the exponent \(q = p - 1 = n + 2 - \frac{2}{n-2}\) make things difficult? The short answer is that because according to Rellich-Kondrachov theorem the embedding \(H^1 \hookrightarrow L^b\) is compact exactly for \(b < p\) and not for \(b = p\):

**Theorem E.1.** Suppose \(M\) is a closed Riemannian manifold of dimension \(n \geq 3\), \(k \in \mathbb{N}_0\), and \(r, q \in (1, \infty)\).

- If \(\frac{1}{r} \geq \frac{1}{q} - \frac{k}{n}\) then \(W^{k,q}(M)\) is continuously embedded in \(L^r(M)\).
- If \(\frac{1}{r} > \frac{1}{q} - \frac{k}{n}\) then the inclusion \(W^{k,q}(M) \hookrightarrow L^r(M)\) is compact. In particular, if \(k = 1\) and \(q = 2\), then \(r\) must be strictly less than \(p = \frac{2n}{n-2}\).
Indeed, suppose \( \{u_i\} \) is a sequence of smooth functions such that \( Q_p(u_i) \to \lambda_p \).

Since \( Q_p \) is scale invariant we can assume that \( \|u_i\|_p = 1 \) for each \( i \). Then

\[
\|u_i\|_{H^1}^2 = \int_M (|\nabla u_i|^2 + u_i^2) \, dV_g = \frac{1}{a} Q_p(u_i) + \int_M (1 - \frac{R_g}{a}) u_i^2 \, dV_g \\
\leq \frac{1}{a} Q_p(u_i) + C \|u_i\|_p^2,
\]

by Holder’s inequality. Therefore, \( \{u_i\} \) is bounded in \( H^1 \). \( H^1 \) is a Hilbert space, so \( H^1 \)

is reflexive and therefore there exists a subsequence of \( \{u_i\} \) that converges weakly to

a function \( u \in H^1 \). Since \( H^1 \hookrightarrow L^p \) is NOT compact, we cannot conclude that \( u_i \to u \)

strongly in \( L^p \). Hence we have no guarantee that the constraint \( \|u_i\|_p = 1 \) is preserved

in the limit. In particular, the limit function \( u \) may be identically zero.

Yamabe’s main idea was to consider first the perturbed functionals \( Q_s(\varphi) = \frac{E(\varphi)}{\|\varphi\|_s^2} \)

for \( 2 \leq s < p \) instead of starting with the functional \( Q_p \), and then take the limit as

\( s \to p \). Yamabe showed that for \( s < p \), the functional \( Q_s \) always has a smooth, positive

minimizer \( \varphi_s \) with \( \|\varphi_s\|_s = 1 \). Later it was shown by Aubin, based on earlier work by

Trudinger, that if \( \lambda_p < \lambda_p(S^n, \bar{g}) \) (where \( \bar{g} \) is the standard metric on \( S^n \)), then as \( s \to p \)

a subsequence of \( \{\varphi_s\} \) converges uniformly to a positive function \( \varphi \in C^\infty(M) \) which

satisfies \( Q_p(\varphi) = \lambda_p \) and \( \Box \varphi = \lambda_p \varphi^{p-1} \). Thus the metric \( \tilde{g} = \varphi^{p-2} g \)

has constant scalar curvature \( \lambda_p \). Details can be found in \([47]\). Here we just list the contributions that

resulted in the proof of the Yamabe conjecture:

- **1960** Yamabe attempted to solve the problem using techniques of calculus of variations and elliptic PDEs.

- **1968** Trudinger discovered an error in Yamabe’s proof. He showed that there is a positive constant \( \alpha \) such that the Yamabe’s proof works when \( \lambda_p < \alpha \).
• **1976** Aubin proved $\alpha = \lambda_p(S^n, \tilde{g})$. So if $\lambda_p < \lambda_p(S^n, \tilde{g})$ then the Yamabe problem can be solved.

• **1976** Aubin showed that if $n \geq 6$ and if $M$ is not locally conformally flat then $\lambda_p < \lambda_p(S^n, \tilde{g})$.

• **1984** Schoen proved that if $M$ has dimension 3, 4, or 5, or if $M$ is locally conformally flat, then $\lambda_p < \lambda_p(S^n, \tilde{g})$ unless $M$ is conformal to the standard sphere.

### E.2 Nonsmooth Metrics

The goal of this section is to show that the weak version of the Yamabe conjecture holds true for rough metrics. In fact, we have the following theorem:

**Theorem E.2.** [53] Let $(M, g)$ be an $n$-dimensional ($n \geq 3$) compact connected Riemannian manifold with $g \in H^s$ and $s > \frac{n}{2}$. Then the following are equivalent:

- 1- There exists a positive function $\varphi \in H^s$ such that $\tilde{g} = \varphi^{\frac{4}{n-2}}g$ satisfies $R_{\tilde{g}}$ is continuous and positive (resp. zero, resp. negative).

- 2- $\lambda_p$ is positive (resp. zero, resp. negative).

- 3- $\lambda_2$ is positive (resp. zero, resp. negative).

Recall that for smooth metrics and $2 \leq r \leq p = \frac{2n}{n-2}$

$$Q_r(\varphi) = \frac{\int_M a|\nabla \varphi|^2 + R_g \varphi^2 dV_g}{\| \varphi \|^2_{L^r}} = \frac{E(\varphi)}{\| \varphi \|^2_{L^r}}, \quad \lambda_r = \inf_{\varphi \in H^1 \setminus \{0\}} Q_r(\varphi),$$

and $\lambda_p$ is the Yamabe invariant. Can we use the same expression to define the Yamabe invariant for rough metrics? Assuming $g \in H^s$, $s > \frac{n}{2}$, and $\varphi \in H^1$, $\int_M a|\nabla \varphi|^2 dV_g$ poses no difficulty, but $R_g \varphi^2$ is not necessarily integrable. That is, $R_g \in H^{s-2}$ and $\varphi \in H^1$.
do not imply that $R_g \varphi^2 \in L^1$. So we need to somehow give a meaning to the term $\int_M R_g \varphi^2 dV_g$.

In order to make this section as self-contained as possible, here we list the main background facts that we need to know in order to make sense of the term $\int_M R_g \varphi^2 dV_g$ and prove Theorem E.2.

**Fact 1: Multiplication lemma** Suppose $M$ is an $n$-dimensional ($n \geq 3$) compact manifold, $\sigma \leq \min(s_1, s_2)$, $s_1 + s_2 \geq 0$ and $\sigma < s_1 + s_2 - \frac{n}{2}$. Then pointwise multiplication extends to a continuous bilinear map

$$H^{s_1}(M) \times H^{s_2}(M) \to H^\sigma(M)$$

**Fact 2: Compact embedding** Suppose $M$ is an $n$-dimensional compact manifold. If $-\infty < s < t < \infty$, then the inclusion $H^t(M) \subseteq H^s(M)$ is **compact**.

**Fact 3: Ehrling’s lemma** Let $X$, $Y$ and $Z$ be Banach spaces. Assume that $X$ is compactly embedded in $Y$ and $Y$ is continuously embedded in $Z$. Then for every $\epsilon > 0$ there exists a constant $c(\epsilon)$ such that

$$\|x\|_Y \leq \epsilon \|x\|_X + c(\epsilon) \|x\|_Z$$

**Fact 4: Regularity** Suppose $(M, g)$ is an $n$-dimensional ($n \geq 3$) compact Riemannian manifold with $g \in H^s$ and $s > \frac{n}{2}$. Suppose also that $V \in H^{s-2}$ and let $\mathcal{L} = -\Delta_g + V$. Then

- $\mathcal{L}$ sends $H^\sigma$ to $H^{\sigma-2}$ for any $\sigma \in (2 - s, s]$.
- If $u \in H^\sigma$ for some $\sigma \in (2 - s, s]$, and if $\mathcal{L} u \in H^{\tau-2}$ for some $\tau \in [\sigma, s]$, then $u \in H^\tau$.

**Fact 5: Strong maximum principle** Suppose $(M, g)$ is an $n$-dimensional ($n \geq 3$) compact Riemannian manifold with $g \in H^s$ and $s > \frac{n}{2}$. Suppose that $u \in H^s$ is nonnegative
and that $V \in H^{s-2}$ satisfies

$$-\Delta_g u + Vu \geq 0.$$ 

If $u(x) = 0$ at some point $x \in M$, then $u$ vanishes identically.

**Fact 6: Stampacchia's Theorem** Suppose $M$ is an $n$-dimensional ($n \geq 3$) compact manifold. Let $G$ be a Lipschitz continuous function of $\mathbb{R}$ into itself such that $G(0) = 0$. Then if $u \in H^1$, we have $G \circ u \in H^1$. In particular, if $u \in H^1$, then $|u| \in H^1$.

**Remark E.3** (Interpretation of the scalar curvature term).

Note that by the multiplication lemma $H^1 \times H^1 \hookrightarrow H^{2-s}$; we interpret $\int_M R_g \varphi^2$ as the duality pairing $\langle R_g, \varphi^2 \rangle_{H^{1-\eta} \times H^{1-\eta}}$. Consequently we have

$$|\langle R_g, \varphi^2 \rangle| \lesssim \| R_g \|_{H^{1-\eta}} \| \varphi^2 \|_{H^{2-s}} \lesssim \| R_g \|_{H^{s-2}} \| \varphi \|_{H^1}^2.$$ 

*In fact, we can say more than this. Pick $0 < \eta < 1$ such that $2\eta < s - \frac{n}{2}$. Then we can conclude from the multiplication lemma that $H^{1-\eta} \times H^{1-\eta} \hookrightarrow H^{2-s}$ and so*

$$|\langle R_g, \varphi^2 \rangle| \lesssim \| R_g \|_{H^{1-\eta}} \| \varphi \|_{H^{1-\eta}}^2.$$ 

Now we are ready to prove Theorem E.2. In the proof we will make use of the following four lemmas. We will prove these lemmas at the end.

Suppose $(M, g)$ is an $n$-dimensional ($n \geq 3$) compact Riemannian manifold with $g \in H^s$ and $s > \frac{n}{2}$.

- **Lemma 1:** There are positive constants $C_1$ and $C_2$ such that for every $\varphi \in H^1$

  $$E(\varphi) \geq C_1 \| \varphi \|_{H^1}^2 - C_2 \| \varphi \|_{L^2}^2.$$ 

- **Lemma 2:** $\lambda_p$ and $\lambda_2$ are both finite.
• **Lemma 3**: \( Q_2 : H^1 \setminus \{0\} \to \mathbb{R} \) is sequentially weakly lower semicontinuous (wlsc), that is, if \( u_n \) converges weakly to \( u \) in \( H^1 \setminus \{0\} \), then

\[
Q_2(u) \leq \liminf_{n \to \infty} Q_2(u_n).
\]

• **Lemma 4**: If \( \varphi \in H^1 \) is a minimizer of \( Q_2 \), then it satisfies

\[
-a \Delta_g \varphi + R_g \varphi = \lambda_2 \varphi
\]

in \( H^{-1} \).

**Proof. (Theorem E.2)** We will show that (3) \( \Rightarrow \) (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3).

**Step 1**: (3) \( \Rightarrow \) (1)

**Assumption**: \( \lambda_2 \) is positive (resp. zero, resp. negative).

**Claim**: There exists a positive function \( \varphi \in H^s \) such that \( \tilde{g} = \varphi^{\frac{1}{n-2}} g \) satisfies \( R_g \) is continuous and positive (resp. zero, resp. negative).

We start by proving that there is a function \( f \) in \( H^1 \), not identically zero, such that \( Q_2(f) = \lambda_2 \). Let \( \{f_k\} \) be a sequence of functions such that \( Q_2(f_k) \) converges to \( \lambda_2 \). Since \( Q_2 \) is scale invariant, we can assume WLOG that \( \|f_k\|_{H^1} = 1 \). Now since the sequence is bounded in the Hilbert space \( H^1 \), by passing to a subsequence if necessary, we can assume it converges weakly to some \( f \in H^1 \). The inclusion \( H^1 \subseteq L^2 \) is compact so \( f_k \to f \) strongly in \( L^2 \). From Lemma 1 we have

\[
Q_2(f_k) \geq \frac{C_1 \|f_k\|_{H^1}^2 - C_2 \|f_k\|_{L^2}^2}{\|f_k\|_{L^2}^2} = \frac{C_1}{\|f_k\|_{L^2}^2} - C_2.
\]

Therefore since \( f_k \to f \) in \( L^2 \), and since the sequence \( Q_2(f_k) \) is bounded (because by Lemma 2 it converges to the finite number \( \lambda_2 \)), we conclude that \( f \neq 0 \). By Lemma 3, \( Q_2(f) \leq \liminf_{k \to \infty} Q_2(f_k) = \lambda_2 \). By definition of \( \lambda_2 \), \( Q_2(f) \geq \lambda_2 \). So \( Q_2(f) = \lambda_2 \). Since \( f \)
is a minimizer in $H^1$ of $Q_2(f)$, from Lemma 4 we conclude that

$$-a\Delta_g f + R_g f = \lambda_2 f \quad \text{in} \quad H^{-1}.$$

Now it follows from Fact 4 that $f \in H^3$. By Fact 6 (Stampacchia’s theorem), $|f| \in H^1$. Also clearly $Q_2(f) = Q_2(|f|)$, so we can assume WLOG that $f \geq 0$. Now from the strong maximum principle we can conclude that $f > 0$ everywhere. Let $\varphi = f$ and $\bar{g} = \varphi^{\frac{4}{n-2}} g$.

It follows that

$$R_{\bar{g}} = \varphi^{1-p}(-a\Delta_g \varphi + R_g \varphi) = \varphi^{2-p} \lambda_2.$$

Since $\varphi$ is continuous and positive, we conclude that $R_{\bar{g}}$ is continuous and everywhere has the same sign as $\lambda_2$.

**Step 2: (1) \Rightarrow (2)**

**Assumption:** There exists a positive function $\varphi \in H^4$ such that $\bar{g} = \varphi^{\frac{4}{n-2}} g$ satisfies $R_{\bar{g}}$ is continuous and positive (resp. zero, resp. negative).

**Claim:** $\lambda_p$ is positive (resp. zero, resp. negative).

$\lambda_p$ is a conformal invariant, therefore we can assume WLOG that $R_{\bar{g}}$ is continuous and has constant sign. Recall that

$$\lambda_p = \inf_{\varphi \in H^1 \setminus \{0\}} Q_p(\varphi) = \inf_{\varphi \in H^1 \setminus \{0\}} \frac{\int_M a|\nabla \varphi|^2 + R_g \varphi^2 dV_g}{\|\varphi\|_p^2}.$$

So

- If $R_{\bar{g}} < 0$, then $Q_p(1) < 0$ and hence $\lambda_p < 0$.
- If $R_{\bar{g}} = 0$, then clearly $Q_p \geq 0$, also $Q_p(1) = 0$ and so $\lambda_p = 0$.
- If $R_{\bar{g}} > 0$, then for all $\varphi \in H^1 \setminus \{0\}$

$$E(\varphi) \geq C \|\varphi\|_{H^1}^2 \geq \tilde{C} \|\varphi\|_{L^p}^2,$$
for some positive constants $C$ and $\tilde{C}$ independent of $\psi$. Note that the last inequality is true because the inclusion $H^1 \subseteq L^p$ is continuous (of course we know that it is not compact). Dividing by $\| \psi \|_{L^p}^2$ we can conclude that $\lambda_p \geq \tilde{C} > 0$.

**Step 3:** $(2) \Rightarrow (3)$

**Assumption:** $\lambda_p$ is positive (resp. zero, resp. negative).

**Claim:** $\lambda_2$ is positive (resp. zero, resp. negative).

We have

$$\lambda_r = \inf_{\psi \in H^1 \setminus \{0\}} Q_r(\psi) = \inf_{\psi \in H^1 \setminus \{0\}} \frac{\int_M a|\nabla \psi|^2 + R_g \psi^2 dV_g}{\| \psi \|_{L^r}^2}.$$

- If $\lambda_p < 0$, then $E(\psi) < 0$ for some function $\psi \in H^1 \setminus \{0\}$, so $\lambda_2 < 0$.

- If $\lambda_p > 0$, then $E(\psi) > 0$ for all $\psi \in H^1 \setminus \{0\}$. Since $p = \frac{2n}{n-2} > 2$, $\| \psi \|_{L^2} \lesssim \| \psi \|_{L^p}^2$. Consequently $Q_p(\psi) \lesssim Q_2(\psi)$ for all $\psi \in H^1 \setminus \{0\}$. Hence if $\lambda_p > 0$ then $\lambda_2 > 0$.

- If $\lambda_p = 0$ then $E(\psi) \geq 0$ for all $\psi \in H^1 \setminus \{0\}$. Therefore we must have $\lambda_2 \geq 0$. But by what was proved in **Step 1** and **Step 2** we know that $\lambda_2 > 0$ implies $\lambda_p > 0$. So if $\lambda_p = 0$ then $\lambda_2 = 0$.

Now we prove the previously stated lemmas.

**Proof.** (**Lemma 1**) Pick $0 < \eta < 1$ such that $2\eta < s - \frac{n}{2}$. We have

- $H^1 \hookrightarrow H^{1-\eta}$ is compact and $H^{1-\eta} \hookrightarrow L^2$ is continuous, so by Fact 3 (Ehrling’s lemma) for any $\varepsilon > 0$

$$\| \varphi \|_{H^{1-\eta}} \leq C(\eta, \varepsilon) \| \varphi \|_{L^2} + \varepsilon \| \varphi \|_{H^1}.$$
\( H^{1-\eta} \times H^{1-\eta} \to H^{2-s} \) is continuous, so

\[
|\langle R_g, \varphi^2 \rangle| \lesssim \| R_g \|_{H^{s-2}} \| \varphi \|_{H^{1-\eta}}^2.
\]

Combining the above inequalities we get

\[
|\langle R_g, \varphi^2 \rangle| \lesssim \tilde{C}(\eta, \epsilon, \| R_g \|_{H^{s-2}}) \| \varphi \|_{L^2}^2 + \epsilon^2 \| R_g \|_{H^{s-2}} \| \varphi \|_{H^1}^2.
\]

Now note that

\[
E(\varphi) = \int_M a|\nabla \varphi|^2 \, dV + \langle R_g, \varphi^2 \rangle
\]

\[
\geq \int_M a|\nabla \varphi|^2 \, dV - |\langle R_g, \varphi^2 \rangle|\]

\[
\gtrless \| \varphi \|_{H^1}^2 - \| \varphi \|_{L^2}^2 - |\langle R_g, \varphi^2 \rangle|\]

\[
\gtrsim (1 - \| R_g \|_{H^{s-2}} \epsilon^2) \| \varphi \|_{H^1}^2 - (1 + \tilde{C}(\eta, \epsilon, \| R_g \|_{H^{s-2}})) \| \varphi \|_{L^2}^2.
\]

Taking \( \epsilon \) sufficiently small in the above inequality proves the claim.

Proof. (Lemma 2) Recall that:

\[
\lambda_2 = \inf_{\varphi \in H^1 \setminus \{0\}} \frac{E(\varphi)}{\| \varphi \|_2^2}, \quad \lambda_p = \inf_{\varphi \in H^1 \setminus \{0\}} \frac{E(\varphi)}{\| \varphi \|_p^2}
\]

Clearly \( p = \frac{2n}{n-2} > 2 \), so the inclusion \( L^p \subseteq L^2 \) is continuous and \( \| \varphi \|_{L^2} \lesssim \| \varphi \|_{L^p} \). It follows that if \( \lambda_p = -\infty \) then \( \lambda_2 = -\infty \). So it is enough to show that \( \lambda_2 \) is finite. From Lemma 1 we have

\[
E(\varphi) \geq C_1 \| \varphi \|_{H^1}^2 - C_2 \| \varphi \|_{L^2}^2 \geq -C_2 \| \varphi \|_{L^2}^2.
\]

Dividing by \( \| \varphi \|_{L^2}^2 \) we conclude that \( \lambda_2 \geq -C_2 \).
Proof. (Lemma 3) Suppose $u_k \to u$ weakly in $H^1$ and $u \neq 0$. We want to show that $Q_2(u) \leq \liminf_{k \to \infty} Q_2(u_k)$.

- **Step 1:** Let $v_k = \frac{u_k}{\|u_k\|_{L^2}}$ and $v = \frac{u}{\|u\|_{L^2}}$.

Note that $u \neq 0$ in $H^1$ and so $u \neq 0$ in $L^2$. Since $H^1$ is compactly embedded in $L^2$, $u_k$ converges to $u$ in $L^2$. So WLOG we may assume that $u_k \neq 0$ in $L^2$. Therefore $v_k$ and $v$ are well-defined.

- **Step 2:** $Q_2$ is scale invariant, so $Q_2(v) = Q_2(u)$ and $Q_2(v_k) = Q_2(u_k)$. Moreover since $\|v_k\|_{L^2} = \|v\|_{L^2} = 1$ we have $Q_2(v) = E(v)$ and $Q_2(v_k) = E(v_k)$. So it is enough to show that

$$E(v) \leq \liminf_{k \to \infty} E(v_k)$$

- **Step 3:** $v_k \to v$ weakly in $H^1$. Indeed, for all $\varphi \in H^1$

$$\lim_{k \to \infty} \langle \varphi, v_k \rangle_{H^1} = \lim_{k \to \infty} \frac{1}{\|u_k\|_{L^2}} \langle \varphi, u_k \rangle_{H^1} = \frac{1}{\|u\|_{L^2}} \langle \varphi, u \rangle_{H^1} = \langle \varphi, v \rangle_{H^1}$$

Therefore

$$\|v\|_{H^1} \leq \liminf_{k \to \infty} \|v_k\|_{H^1}.$$

- **Step 4:** Pick $0 < \eta < 1$ as in the proof of Lemma 1. Thus the map that sends $\varphi$ to $\langle R_g, \varphi^2 \rangle_{H^{1-\eta} \times H^{1-\eta}}$ from $H^{1-\eta}$ to $\mathbb{R}$ is continuous. In particular, since the inclusion $H^1 \subseteq H^{1-\eta}$ is compact, we conclude that $v_k \to v$ w.r.t. the norm of $H^{1-\eta}$ and consequently

$$\lim_{k \to \infty} \langle R_g, v_k^2 \rangle = \langle R_g, v^2 \rangle$$
• Step 5:

\[
E(v) = a \int_M |\nabla v|^2 dV_g + \langle R_g, v^2 \rangle_{H^{s-2} \times H^{s-1}}
\]

\[
= a \| v \|_{H^1}^2 - a \| v \|_{L^2}^2 + \langle R_g, v^2 \rangle
\]

\[
= a \| v \|_{H^1}^2 - a \lim_{k \to \infty} \| v_k \|_{L^2}^2 + \lim_{k \to \infty} \langle R_g, v_k^2 \rangle
\]

\[
\leq a \liminf_{k \to \infty} \| v_k \|_{H^1}^2 - a \liminf_{k \to \infty} \| v_k \|_{L^2}^2 + \liminf \langle R_g, v_k^2 \rangle
\]

\[
= \liminf_{k \to \infty} \int_M |\nabla v_k|^2 dV_g + \langle R_g, v_k^2 \rangle = \liminf_{k \to \infty} E(v_k).
\]

\[
\square
\]

**Proof. (Lemma 4)** Define the bilinear form \( A : H^1 \times H^1 \to \mathbb{R} \) as follows:

\[
A(f, h) = \int_M a(\nabla f, \nabla h) g dV_g + \langle R_g, f h \rangle_{H^{s-2} \times H^{s-1}}.
\]

Clearly \( E(\varphi) = A(\varphi, \varphi) \) and \( Q_2(\varphi) = \frac{A(\varphi, \varphi)}{\| \varphi \|_{L^2}^2} \).

• Step 1: Let \( f = \frac{\varphi}{\| \varphi \|_{L^2}} \). It is enough to show that \( f \) satisfies the equation. Note that if \( \psi \in H^1 \) is such that \( \| \psi \|_{L^2} = 1 \) then

\[
A(\psi, \psi) = Q_2(\psi) \geq Q_2(\varphi) = A\left(\frac{\varphi}{\| \varphi \|_{L^2}}, \frac{\varphi}{\| \varphi \|_{L^2}}\right) = A(f, f).
\]

It follows that \( f \) is a solution to the following minimization problem:

**Minimize** \( E(\psi) = A(\psi, \psi) \) on \( H^1 \) subject to the constraint \( G(\psi) = \| \psi \|_{L^2}^2 = 1 \).

• Step 2: \( A \) and the \( L^2 \) inner product are both continuous bilinear forms on \( H^1 \times H^1 \).

Using the general formula for the Frechet derivative of continuous bilinear forms,
one can easily show that: \( (E(\psi) = A(\psi, \psi)) \)

\[
E'(\psi)v = 2A(\psi, v), \quad G'(\psi)v = 2\langle \psi, v \rangle_{L^2},
\]
for all \( v \in H^1 \).

**Step 3:** By the Lagrange multiplier theorem, either \( G'(f)v = 0 \) for all \( v \in H^1 \) or there exists \( \mu \in \mathbb{R} \) so that \( E'(f)v = \mu G'(f)v \) for all \( v \in H^1 \).

Since \( f \neq 0 \) in \( L^2 \) the first case cannot happen and therefore we must have:

\[
2A(f, v) = 2\mu \langle f, v \rangle_{L^2} \quad \text{for all } v \in H^1.
\]

That is, for all \( v \in H^1 \)

\[
\int_M a\langle \nabla f, \nabla v \rangle_g dV_g + \langle R_g, f v \rangle_{H^{s-2} \times H^{2-s}} = \mu \langle f, v \rangle_{L^2}.
\]

Consequently (using integration by parts)

\[
\langle a\Delta_g f + \mu f, v \rangle_{L^2} = \langle R_g, f v \rangle_{H^{s-2} \times H^{2-s}} \quad \text{for all } v \in H^1.
\]

**Step 4:** Since \( s > \frac{n}{2} \) we have \( H^{s-2} \times H^1 \hookrightarrow H^{-1} \). In particular, \( R_g f \in H^{-1} \) and so

\[
\langle R_g, f v \rangle_{H^{s-2} \times H^{2-s}} = \langle R_g f, v \rangle_{H^{-1} \times H^1}.
\]

Also clearly

\[
\langle a\Delta_g f + \mu f, v \rangle_{L^2} = \langle a\Delta_g f + \mu f, v \rangle_{H^{-1} \times H^1}.
\]
It follows that for all $v \in H^1$

$$\langle a\Delta_g f + \mu f, v \rangle_{H^{-1} \times H^1} = \langle R_g f, v \rangle_{H^{-1} \times H^1}.$$

Therefore

$$\langle a\Delta_g f + \mu f - R_g f, v \rangle_{H^{-1} \times H^1} = 0 \text{ for all } v \in H^1.$$

By the definition of the zero element of the dual space, the above equality precisely means that

$$-a\Delta_g f + R_g f = \mu f \text{ in } H^{-1}.$$

- **Step 5:** By setting $v = f$ in the Lagrange equation we get $A(f, f) = \mu \langle f, f \rangle_{L^2} = \mu$. Thus

$$\mu = A(f, f) = \frac{A(\varphi, \varphi)}{\| \varphi \|^2_{L^2}} = Q_2(\varphi).$$

But $\varphi$ is the minimizer of $Q_2$, so

$$\mu = Q_2(\varphi) = \inf_{\psi \in H^1 \backslash \{0\}} Q_2(\psi) = \lambda_2.$$
Appendix F

The Positive Yamabe Class in AF Manifolds

Here we collect some facts regarding the Yamabe invariant in the case of AF manifolds.

Let \((M, h)\) be a 3-dimensional AF manifold of class \(W^{s,p}_\delta\) where \(p \in (1, \infty)\), \(s \in \left(\frac{3}{p}, \infty\right) \cap (1, \infty)\), and \(-1 < \delta < 0\). We define the Yamabe invariant as follows: [11, 55]

\[
\lambda_h = \inf_{f \in C^\infty_c(M), f \neq 0} \frac{\int_M |\text{grad} f|^2 dV_h + \langle R_h, f^2 \rangle_{(M,h)}}{\|f\|^2_{L^6}}
\]

We say \(h\) is in the positive Yamabe class if and only if \(\lambda_h > 0\). Contrary to what we have for compact manifolds, one can show that [11, 55]:

\(\lambda_h > 0\) if and only if there exists a conformal factor \(\eta > 0\) such that \(\eta - 1 \in W^{s,p}_\delta\) and \((M, \eta^4 h)\) is scalar flat.

It is interesting to notice that if \(\lambda_h > 0\), then \(h\) is also conformal to a metric with continuous positive scalar curvature.

**Proposition F.1.** Let \((M, h)\) be a 3-dimensional AF manifold of class \(W^{s,p}_\delta\) where \(p \in\)
If \( h \) belongs to the positive Yamabe class, then there exist \( \chi \in W^{s,p}_\delta \) such that if we set \( \hat{h} = (1 + \chi)^4 h \), then \( R_{\hat{h}} \) is continuous and positive.

**Proof.** (Proposition F.1) If \( h \) is in the positive Yamabe class, then there exists \( \eta \in W^{s,p}_\delta \), \( \eta > -1 \) such that \( R_{\tilde{h}} = 0 \) where \( \tilde{h} = (1 + \eta)^4 h \). Let \( f \) be a smooth positive function in \( W^{s-2,p}_{\delta-2} \). By Lemma 4.10 there exists a unique function \( v \in W^{s,p}_\delta \) such that \( -8\Delta_{\tilde{h}} v = f \).

By the maximum principle (Lemma 4.9) \( v \) is positive. Now define \( \hat{h} = (1 + v)^4 \tilde{h} \). We have

\[
R_{\hat{h}} = (-8\Delta_{\hat{h}} v + R_{\hat{h}}(1 + v)) (1 + v)^{-5} = 8 f (1 + v)^{-5}.
\]

Since \( f \) and \( v \) are both continuous and positive we can conclude that \( R_{\hat{h}} \) is continuous and positive. If we set \( \chi = v + \eta + \eta v \), then \( \chi \in W^{s,p}_\delta \) and

\[
\hat{h} = (1 + v)^4 (1 + \eta)^4 h = (1 + \chi)^4 h.
\]

Note that since \( v > 0 \) and \( \eta > -1 \) we have \( \chi > -1 \).

Appendix F, in full, has been submitted for publication. The material may appear as A. Behzadan and M. Holst, *Rough Solutions of the Einstein Constraint Equations on Asymptotically Flat Manifolds Without Near-CMC Conditions*. The dissertation author was the primary investigator and author of this paper.
Appendix G

The LCBY Equations in Bessel Potential Spaces

As it was pointed out in Chapter 3, Sobolev-Slobodeckij spaces are not the only option that we have if we wish to work with noninteger order Sobolev spaces. Another option is to consider the Bessel potential spaces $H^{s,p}(\mathbb{R}^n)$ and then define the weighted spaces based on Bessel potential spaces. Bessel potential spaces agree with Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ when $s$ is an integer and therefore they can be considered as an extension of integer order Sobolev spaces. There are two main advantages in working with Bessel potential spaces (and the corresponding weighted versions) in comparison with Sobolev-Slobodeckij spaces: First, Bessel potential spaces have better interpolation properties; second we have a better (stronger) multiplication lemma for Bessel potential spaces.

**Theorem G.1** (Complex Interpolation). [72] Suppose $\theta \in (0, 1)$, $0 \leq s_0, s_1 < \infty$, and $1 < p_0, p_1 < \infty$. If

$$s = (1 - \theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},$$

then $H^{s,p}(\mathbb{R}^n) = [H^{s_0,p_0}(\mathbb{R}^n), H^{s_1,p_1}(\mathbb{R}^n)]_{\theta}$. 

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Lemma G.2. Let $s_i \geq s$ with $s_1 + s_2 \geq 0$, and $1 < p, p_i < \infty$ ($i = 1, 2$) be real numbers satisfying

$$s_i - s \geq n\left(\frac{1}{p_i} - \frac{1}{p}\right), \quad s_1 + s_2 - s > n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) \geq 0,$$

In case $s < 0$ let

$$s_1 + s_2 > n\left(\frac{1}{p_1} + \frac{1}{p_2} - 1\right) \quad (equality\ is\ allowed\ if\ min(s_1, s_2) < 0).$$

Then the pointwise multiplication of functions extends uniquely to a continuous bilinear map

$$H^{s_1,p_1}(\mathbb{R}^n) \times H^{s_2,p_2}(\mathbb{R}^n) \to H^{s,p}(\mathbb{R}^n).$$

We will prove the above lemma later in this Appendix.

Remark G.3. We make the following observations.

- Note that in the above multiplication lemma there is no restriction on the values of $p_1$ and $p_2$ with respect to $p$. That is, it is allowed for $p_1$ or $p_2$ to be greater than $p$.

- Note that contrary to what we had for Sobolev-Slobodeckij spaces, the complex interpolation works regardless of whether exponents are integer or noninteger. This feature is crucial because complex interpolation works much better for interpolation of bilinear forms. This is one of the reasons that we have a stronger multiplication lemma for Bessel potential spaces.

Let us denote the weighted spaces based on $H^{s,p}$ by $H^{s,p}_\delta$ (rather than $W^{s,p}_\delta$). Our spaces $H^{s,p}_\delta(\mathbb{R}^n)$ correspond with the spaces $h^{s,p}_\delta, p^\delta - p, p\delta - n(\mathbb{R}^n)$ in [70, 71].

Theorem G.4 (Complex Interpolation, Weighted Spaces). [70, 71] Suppose $\theta \in (0, 1)$. If

$$s = (1 - \theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \delta = (1 - \theta)\delta_0 + \theta \delta_1$$
then \( H^{s,p}_\delta(\mathbb{R}^n) = [H^{s_0,p_0}_\delta(\mathbb{R}^n), H^{s_1,p_1}_\delta(\mathbb{R}^n)]_\theta \).

The corresponding weighted version of the multiplication lemma can be proved using the exact same argument as the one that we used for weighted Sobolev-Slobodeckij spaces.

**Lemma G.5** (Multiplication Lemma, Weighted Bessel potential spaces). Assume \( s, s_1, s_2 \) and \( 1 < p, p_1, p_2 < \infty \) are real numbers satisfying

(i) \( s_i \geq s \quad (i = 1, 2) \),

(ii) \( s_1 + s_2 \geq 0 \),

(iii) \( s_i - s \geq n(\frac{1}{p_i} - \frac{1}{p}) \quad (i = 1, 2) \),

(iv) \( s_1 + s_2 - s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \geq 0 \).

In case \( s < 0 \), in addition let

(v) \( s_1 + s_2 > n(\frac{1}{p_1} + \frac{1}{p_2} - 1) \quad (equality \ is \ allowed \ if \ \min(s_1, s_2) < 0) \).

Then for all \( \delta_1, \delta_2 \in \mathbb{R} \), the pointwise multiplication of functions extends uniquely to a continuous bilinear map

\[
H^{s_1,p_1}_\delta(\mathbb{R}^n) \times H^{s_2,p_2}_\delta(\mathbb{R}^n) \rightarrow H^{s,p}_{\delta_1+\delta_2}(\mathbb{R}^n).
\]

Again notice that \( p_1 \) and \( p_2 \) do NOT need to be less than or equal to \( p \). This extra degree of freedom that we have for multiplication in Bessel potential spaces allows us to remove the restrictions of the type “\( p = q \) if \( e = s \notin \mathbb{N}_0 \)” in all the statements of the main text. Consequently we will have a stronger existence theorem as follows:

**Theorem G.6.** Let \( (M, h) \) be a 3-dimensional AF Riemannian manifold of class \( H^{s,p}_\delta \) where \( p \in (1, \infty) , \ s \in (1 + \frac{3}{p}, \infty) \) and \( -1 < \delta < 0 \) are given. Suppose \( h \) admits no nontrivial
conformal Killing field and is in the positive Yamabe class. Let $\beta \in (-1, \delta]$. Select $q$ and $e$ to satisfy:

$$
\frac{1}{q} \in (0, 1) \cap (0, \frac{s-1}{3}) \cap \left[ \frac{3-p}{3p}, \frac{3+p}{3p} \right],
$$

$$
e \in (1 + \frac{3}{q}, \infty) \cap [s-1, s] \cap \left[ \frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p} \right].
$$

Assume that the data satisfies:

- $\tau \in H^{e-1,q}_{\beta-1}$ if $e \geq 2$ and $\tau \in H^{1,z}_{\beta-1}$ otherwise, where $z = \frac{3q}{3 + (2 - e)q}$ (note that if $e = 2$, then $H^{e-1,q}_{\beta-1} = H^{1,z}_{\beta-1}$),

- $\sigma \in H^{e-1,q}_{\beta-1}$,

- $\rho \in H^{s-2,p}_{\beta-2} \cap L^{\infty}_{2\beta-2}$, $\rho \geq 0$ ($\rho$ can be identically zero),

- $J \in H^{e-2,q}_{\beta-2}$.

If $\mu$ is chosen to be sufficiently small and if $\|\sigma\|_{L^{\infty}_{\beta-1}}$, $\|\rho\|_{L^{\infty}_{2\beta-2}}$, and $\|J\|_{H^{e-2,q}_{\beta-2}}$ are sufficiently small, then there exists $\psi \in H^{s,p}_{\delta}$ with $\psi > -\mu$ and $W \in H^{e,q}_{\beta}$ solving (5.3) and (5.4).

Our plan for the remainder of this appendix is to discuss the proof of the stronger version of multiplication lemma that was stated for Bessel potential spaces. In our proof we will make use of some of the well-known results for pointwise multiplication in Triebel-Lizorkin spaces that can be found in [67].

Proof. (Lemma G.2) We prove the lemma for the case $s \geq 0$. The case $s < 0$ can be proved by using a duality argument similar to the one presented in Section 3.4.2. We may consider three cases:

- **Case 1**: $p_1, p_2 \leq p$: This case is proved in Section 3.4.2.
• **Case 2:** \( p \leq \min\{p_1, p_2\} \): In what follows we will discuss the proof of this case. For now let’s assume the lemma holds true in this case.

• **Case 3:** \( p_1 > p, p_2 \leq p \) or \( p_2 > p, p_1 \leq p \): Here we prove the case where \( p_1 > p, p_2 \leq p \). The proof of the other case is completely analogous. We have

\[
H^{s_1,p_1} \times H^{s_2,p_2} \hookrightarrow H^{s_1,p_1} \times H^{s_2-n/p_2+n/p,p} \hookrightarrow H^{s,p}.
\]

Note that by assumption \( s_2 - \frac{n}{p_2} \geq s - \frac{n}{p} \) and so \( s_2 - \frac{n}{p_2} + \frac{n}{p} \geq s \geq 0 \). The first embedding is true because \( H^{s_2,p_2} \hookrightarrow H^{s_2-n/p_2+n/p,p} \) (one can easily check that the conditions of Theorem 3.44 are satisfied). Also as a direct consequence of the claim of **Case 2**, the second embedding holds true (note that \( p \leq \min\{p, p_1\} \)).

So it remains to prove the claim of **Case 2**, that is the case where \( p \leq \min\{p_1, p_2\} \). Of course if both \( p_1 \) and \( p_2 \) are equal to \( p \), then the claim follows from case 1; so we may assume at least one of \( p_1 \) or \( p_2 \) is strictly larger than \( p \). To prove **Case 2** we proceed as follows:

• **Step 1:** In this step we consider the case where \( s_1 = s_2 = s \). Note that by assumption

\[
\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} \geq 0. \text{ If } \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \text{ then let } k = [s]. \text{ We have }
\]

\[
H^{k+1,p_1} \times H^{k+1,p_2} \hookrightarrow H^{k+1,p},
\]

\[
H^{k,p_1} \times H^{k,p_2} \hookrightarrow H^{k,p},
\]

so by complex interpolation we get

\[
H^{s,p_1} \times H^{s,p_2} \hookrightarrow H^{s,p}.
\]

As a direct consequence of Theorem 2 in page 239 of [67], the above embedding
remains valid if \( \frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{p} \) and \( p_1, p_2 > p \). What if \( p_2 = p \) or \( p_1 = p \)? Here we consider the case where \( p_2 = p \) (and so \( p_1 > p \)). The proof of the other case is completely analogous. Note that by assumption \( s > n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) = \frac{n}{p_1} \); under this assumption we need to prove the following:

\[
H^{s,p_1} \times H^{s,p_2} \hookrightarrow H^{s,p}.
\]

If \( s \neq \frac{n}{p} \), the above embedding follows from Theorem 1 in page 176 and Theorem 2 in page 177 of [67]. Now if \( s = \frac{n}{p} \), we set \( \epsilon = \frac{n}{p} - \frac{n}{p_1} \) and then since the claim is true for \( s \neq \frac{n}{p} \) we have

\[
H^{n - \frac{\epsilon}{2},p_1} \times H^{n - \frac{\epsilon}{2},p_2} \hookrightarrow H^{\frac{n}{p} + \frac{\epsilon}{2},p},
\]

\[
H^{n + \frac{\epsilon}{2},p_1} \times H^{n + \frac{\epsilon}{2},p_2} \hookrightarrow H^{\frac{n}{p} - \frac{\epsilon}{2},p},
\]

so the result follows from complex interpolation.

- **Step 2:** Let \( t_1, t_2 \in \left[0, \frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p}\right] \) and suppose \( \epsilon > 0 \) is such that \( t_1 + t_2 - \left(\frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p}\right) - \epsilon \geq 0 \). Then as a direct consequence of the Corollary that is stated in page 189 of [67] we have

\[
H^{t_1,p_1} \times H^{t_2,p_2} \hookrightarrow H^{t_1 + t_2 - \left(\frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p}\right) - \epsilon, p},
\]

- **Step 3:** Note that by **Step 1**, if \( b > \frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p} \), then

\[
H^{b,p_1} \times H^{b,p_2} \hookrightarrow H^{b,p}.
\]

Also if we let \( \frac{1}{r} = \frac{1}{p} - \frac{1}{p_2} \), then \( H^{b,p_1} \hookrightarrow L^r \) and so by Holder's inequality

\[
H^{b,p_1} \times H^{0,p_2} \hookrightarrow H^{0,p}.
\]
By complex interpolation we get

\[ \forall t \in [0, b] \quad H^{b,p_1} \times H^{t,p_2} \hookrightarrow H^{t,p}. \]

Therefore

\[ \forall \epsilon > 0 \forall t \in [0, \frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p}] \quad H^{\frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p} + \epsilon, p_1} \times H^{t,p_2} \hookrightarrow H^{t,p}. \]

**Step 4:** In this step we prove the claim of **Case 2** in its general form. Without loss of generality we may assume \( s_1 = \max\{s_1, s_2\} \), so \( s_2 \in [0, s_1] \). If \( s_1 > \frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p} \), then by what was proved in **Step 3** we have

\[ H^{s_1, p_1} \times H^{s_2, p_2} \hookrightarrow H^{s_2, p} \hookrightarrow H^{s, p}. \]

In case \( s_1 \leq \frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p} \) (that is, if \( s_1, s_2 \in [0, \frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p}] \)), choose \( \epsilon > 0 \) such that \( s_1 + s_2 - \left( \frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p} \right) - \epsilon > s \geq 0 \). Then by **Step 2** we have:

\[ H^{s_1, p_1} \times H^{s_2, p_2} \hookrightarrow H^{s_1 + s_2 - \left( \frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p} \right) - \epsilon, p} \hookrightarrow H^{s, p}. \]

\[ \square \]

Appendix G, in full, has been submitted for publication. The material may appear as A. Behzadan and M. Holst, *Rough Solutions of the Einstein Constraint Equations on Asymptotically Flat Manifolds Without Near-CMC Conditions*. The dissertation author was the primary investigator and author of this paper.
Appendix H

An Alternative Weak Formulation

In Section 5.1 we described a setting where the constraint equations make sense with rough data. Here we describe a second framework in which rough data is allowed. Recall that according to our preliminary discussion in Section 5.1, we have already imposed the following restrictions:

\[ p \in (1, \infty), \quad s \in \left( \frac{3}{p}, \infty \right) \cap [1, \infty), \quad \delta < 0. \]

**Framework 2:**

In this framework we seek \( W \) in \( W_{\beta}^{1,2r} \) where \( r \geq 1 \) and \( \beta < 0 \). For the momentum constraint to make sense we need to ensure that

1. it is possible to extend the operator \( -\Delta_L : C^\infty \to C^\infty \) to an operator \( \mathcal{A}_L : W_{\beta}^{1,2r} \to W_{\beta-2}^{1,2r} \).
2. \( b_I(\psi + \mu)^6 + b_J \in W_{\beta-2}^{-1,2r} \).

Note that \( \Delta_L \) belongs to the class \( D_{2,\delta}^{s,p} \). Therefore by Theorem 4.2 we just need to check the following conditions (numbering corresponds to conditions in Theorem 4.2):
(i) \(2r \in (1, \infty)\), (since \(r \geq 1\))

(ii) \(1 > 2 - s\), (enough to assume \(s > 1\))

(iii) \(-1 < \min(1, s) - 2\), (enough to assume \(s > 1\))

(iv) \(-1 < 1 - 2 + s - \frac{3}{p}\), (since \(\frac{3}{p} < s\))

(v) \(-1 - \frac{3}{2r} \leq s - \frac{3}{p} - 2\), (so need to assume \(1 - \frac{3}{2r} \leq s - \frac{3}{p}\))

(vi) \(1 - \frac{3}{2r} > 2 - 3 - s + \frac{3}{p}\), (since \(r \geq 1\) and so \(1 - \frac{3}{2r} \geq -\frac{1}{2} > -1 - (s - \frac{3}{p})\))

So the only extra assumptions that we need to make is that \(1 - \frac{3}{2r} \leq s - \frac{3}{p}\) and \(s > 1\).

Also in order to ensure that the second condition holds true it is enough to assume \(\tau\) is given in \(L_{\beta-1}^{2r}\) and \(J\) is given in \(W_{\beta-1}^{1,2r}\). Indeed, note that \(\tau \in L_{\beta-1}^{2r}\) implies \(b_\tau \in W_{\beta-1}^{1,2r}\).

Since \(\psi \in W_\delta^{s,p}\), it follows from Lemma 3.34 that \(b_\tau (\psi + \mu)^6 \in W_{\beta-2}^{1,2r}\). Lemma 3.34 can be applied because clearly \(2r \in (1, \infty)\) and moreover

(i) \(-1 \in (-s, s)\), (since \(s > 1\))

(ii) \(-1 - \frac{3}{2r} \leq s - \frac{3}{p}\), (since we assumed \(1 - \frac{3}{2r} \leq s - \frac{3}{p}\))

\(-3 - s + \frac{3}{p} \leq -1 - \frac{3}{2r}\). (the same as item (vi) above)

The numbering of the above items corresponds to the numbering of the conditions in Lemma 3.34.

Now let’s consider the Hamiltonian constraint. Note that \(W \in W_{\beta}^{1,2r}\) and so \(\mathcal{L}W \in L_{\beta-1}^{2r}\). So for \(a_W\) to be well-defined it is enough to assume \(\sigma \in L_{\beta-1}^{2r}\). Exactly similar to our discussion for weak formulation 1, for Hamiltonian constraint to make sense it is enough to ensure that

\[f(\psi, W) = a_R(\psi + \mu) + a_\tau(\psi + \mu)^5 - a_W(\psi + \mu)^7 - a_\rho(\psi + \mu)^{-3} \in W_{\eta-2}^{s-2,p},\]

where \(\eta = \max{\delta, \beta}\). One way to guarantee that the above statement holds true is to ensure that

\[a_\tau, a_\rho, a_W \in W_{\beta-2}^{s-2,p}, \quad a_R \in W_{\delta-2}^{s-2,p},\]
We claim that for the above statements to be true it is enough to make the following extra assumptions:

\[
s \leq 2, \quad 1 - \frac{3}{2r} \geq \frac{1}{2}(s - \frac{3}{p}), \quad \rho \in W^{s-2,p}_{\beta-2}.
\]

The details are as follows:

1. \(a_r = \frac{1}{12} r^2\).

We want to ensure \(a_r \in W^{s-2,p}_{\beta-2}\). Note that \(\tau \in L^{2r}_{\beta-1}\), so \(\tau^2 \in L^{r}_{2\beta-2}\). Thus we want to have \(L^{r}_{2\beta-2} \hookrightarrow W^{s-2,p}_{\beta-2}\). We will see that this embedding becomes true provided \(s \leq 2\) and \(1 - \frac{3}{2r} \geq \frac{1}{2}(s - \frac{3}{p})\).

We just need to check that the assumptions of Theorem 3.23 are satisfied (numbering corresponds to the assumptions in Theorem 3.23)

\[
(iii) \quad 0 \geq s - 2 \quad \text{(equivalent to } s \leq 2),
\]

\[
(iii) \quad 0 - \frac{3}{r} \geq s - 2 - \frac{3}{p} \quad \text{(equivalent to } 1 - \frac{3}{2r} \geq \frac{1}{2}(s - \frac{3}{p})),
\]

\[
(iv) \quad 2\beta - 2 < \beta - 2 \quad \text{(true because } \beta < 0).
\]

2. \(a_R = \frac{R}{8}\).

We want to ensure \(a_R \in W^{s-2,p}_{\delta-2}\). Note that \(h\) is an AF metric of class \(W^{s,p}_{\delta}\) and \(R\) involves the second derivatives of \(h\), so \(R \in W^{s-2,p}_{\delta-2}\). We do not need to impose any extra restrictions for this one.

3. \(a_\rho = \kappa \rho / 4\).

Clearly \(a_\rho \in W^{s-2,p}_{\beta-2}\) iff \(\rho \in W^{s-2,p}_{\beta-2}\).

4. \(a_W = [\sigma_{ab} + (\mathcal{L} W)_{ab}][\sigma_{ab} + (\mathcal{L} W)^{ab}] / 8\).
We want to ensure that $a_W \in W^{s-2,p}_{\beta-2}$. Note that $\mathcal{L}W, \sigma \in L^2_{\beta-1}$ and as discussed above, $L^r_{2\beta-2} \hookrightarrow W^{s-2,p}_{\beta-2}$. So $a_W = \frac{1}{6}|\sigma| + \mathcal{L}W|^2 \in L^r_{2\beta-2} \hookrightarrow W^{s-2,p}_{\beta-2}$.

**Remark H.1.** According to the above discussion we need $r \geq 1$ satisfy

$$\frac{1}{2} \left( s - \frac{3}{p} \right) \leq 1 - \frac{3}{2r} \leq s - \frac{3}{p}. $$

In particular, if we choose $r$ such that $\frac{1}{2} (s - \frac{3}{p}) = 1 - \frac{3}{2r}$, that is, if we set $r = \frac{3p}{3 + (2 - s)p}$, then clearly $r$ satisfies the above inequalities and moreover since $s \leq 2$, we have $r \geq 1$.

**Weak Formulation 2.** Let $(M, h)$ be a 3D AF Riemannian manifold of class $W^{s,p}_{\delta}$ where $p \in (1, \infty)$, $\beta, \delta < 0$ and $s \in (\frac{3}{p}, \infty) \cap (1, 2]$. Let $r = \frac{3p}{3 + (2 - s)p}$. Fix source functions:

$$\tau \in L^2_{\beta-1}, \quad \sigma \in L^2_{\beta-1}, \quad \rho \in W^{s-2,p}_{\beta-2} (\rho \geq 0), \quad J \in W^{1,2r}_{\beta-2}. $$

Let $\eta = \max\{\beta, \delta\}$. Define $f : W^{s,p}_{\delta} \times W^{1,2r}_{\beta} \rightarrow W^{s-2,p}_{\eta-2}$ and $f : W^{s,p}_{\delta} \rightarrow W^{1,2r}_{\beta}$ as

$$f(\psi, W) = a_R(\psi + \mu) + a_T(\psi + \mu)^5 - a_W(\psi + \mu)^7 - a_\rho(\psi + \mu)^3, $$

$$f(\psi) = b_T(\psi + \mu)^6 + b_J.$$

Find $(\psi, W) \in W^{s,p}_{\delta} \times W^{1,2r}_{\beta}$ such that

$$A_L \psi + f(\psi, W) = 0, \quad (H.1) $$

$$\mathcal{A}_L W + f(\psi) = 0. \quad (H.2) $$

**Remark H.2.** Consider Weak Formulation 1. In the case where $s \leq 2$ and $\frac{1}{q} \geq \frac{2-d}{6}$ where $d = s - \frac{3}{p}$, this formulation becomes a special case of Weak Formulation 2. Indeed, we just need to check that in this case $W^{e,q}_{\beta} \hookrightarrow W^{1,2r}_{\beta}$. By Theorem 3.22 we need to make
sure that the followings hold true:

(i) $q \leq 2r$, (true because $\frac{1}{q} \geq \frac{2 - d}{6} = \frac{3 + (2 - s)p}{6p} = \frac{1}{2r}$)

(ii) $e \geq 1$, (true because $e > 1 + \frac{3}{q}$)

(iii) $e - \frac{3}{q} \geq 1 - \frac{3}{2r}$.

(The numbering of the above items agrees with the numbering of the assumptions in Theorem 3.22.) The third condition is true because

$$e - \frac{3}{q} \geq 1 - \frac{3}{2r} \iff e \geq 1 + \frac{3}{q} - \frac{3 + (2 - s)p}{2p} \iff e \geq \frac{3}{q} + \frac{d}{2},$$

and

- if $d > 2$, then $d - 1 > \frac{d}{2}$ and so $e \geq \frac{3}{q} + d - 1 > \frac{3}{q} + \frac{d}{2}$,
- if $d \leq 2$, then $1 \geq \frac{d}{2}$ and so $e > 1 + \frac{3}{q} \geq \frac{3}{q} + \frac{d}{2}$.

Appendix H, in full, has been submitted for publication. The material may appear as A. Behzadan and M. Holst, *Rough Solutions of the Einstein Constraint Equations on Asymptotically Flat Manifolds Without Near-CMC Conditions*. The dissertation author was the primary investigator and author of this paper.
Bibliography


