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SEPARATION EFFECTS IN GAS-PARTICLE FLOWS AT HIGH REYNOLDS NUMBERS

Jonathan A. Laitone
(Ph.D. thesis)

October 1979

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Separation Effects in Gas-Particle Flows at High Reynolds Numbers

by

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Abstract

Predicting the fluid mechanical characteristics of a gas-solid two-phase flow is crucial for the successful design and operation of coal gasification systems, coal fired turbines, rocket nozzles, and other energy conversion systems. The difficulties associated with the analysis of gas-particle flows have precluded general solutions, with most research being applied to simple geometries and often to flow conditions that are not particularly useful.

This work presents a general numerical solution which extends a numerical scheme for gas flow developed by Chorin to a solution suitable for dilute gas-solid particle flows over a much wider range of conditions and geometries. The proposed method is designed to solve the time dependent equations but may be used in the steady state case as well. An exact solution for the inviscid two-phase flow near a stagnation point is also presented. A transformation is used that reduces the partial differential governing equation to a second order linear differential equation.

The numerical method is applied to the flow of gas and particles
about a cylinder. A cut-off value is found in particle size which determines particles that do not impact with the cylinder but are deflected around it. Good agreement is found between the numerical method and experiment.

Maurice Holt.
Chairman of Committee
Car c'est chose divine
D'aimer, lorsq'on devine,
Rêve, invente, imagine
  A peine...
Le seul rêve intéresse;
Vivre sans rêve, qu'est-ce?
Et moi, j'aime la princesse
  Lointaine.

Rostand, La Princesse Lointaine.
Acknowledgment

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Lastly, I am grateful to my family, to my father, Edmund, for the endless discussions about potential theory, to my mother, Dorothy, for her continual encouragement, and to my sister, Vicki, for her delightful questions.

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Jonathan A. Laitone
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List of Symbols

Capital Letters

E Relative error in gas velocity.

E_p Relative error in particle velocity.

J Number density flux.

L Characteristic length of system.

M Number of gas vortices diffused from surface.

M Particle mass flow rate.

R Cylinder radius.

X Particle vortex blob initial x-coordinate.

Y Particle vortex blob initial y-coordinate.

X_o Particle vortex blob impact x-coordinate.

Y_o Particle vortex blob impact y-coordinate.

Lower Case Letters

d Particle diameter.

h Discretization length of boundary layer segment.

k Time step.

m Time step counter.

m_p Mass of a particle.

n_p Particle number density.

r Radius vector, \( \mathbf{r} = (x, y) \).

u Gas velocity, \( \mathbf{u} = (u, v) \).

u_p Particle velocity, \( \mathbf{u}_p = (u_p, v_p) \).

z Transformation variable, \( z = t/\tau \)
Greek Letters

\( \alpha(q) \) Single layer source potential function.

\( \alpha \) Particle volume fraction.

\( \alpha' \) Particle impact angle (measured from surface).

\( \Gamma \) Particle volume.

\( \delta \) Boundary layer thickness.

\( \delta_1 \) Relative density, \( \delta_1 = \frac{\rho_p}{\rho} \).

\( \bar{\delta} \) Average displacement thickness.

\( \Delta_\varepsilon \) Displacement error in gas vortex location.

\( \eta_1 \) Random variable (distributed: \( 0 \leq x \leq 1 \)).

\( \eta_2 \) Random variable (distributed: \( 0 \leq y \leq 1 \)).

\( \theta \) Cylinder angle (measured from front stagnation point).

\( \kappa \) Particle density ratio, \( \kappa = \frac{\rho_p}{\rho} \).

\( \mu \) Gas viscosity.

\( \nu \) Gas kinematic viscosity.

\( \xi \) Vorticity.

\( \bar{\xi} \) Vorticity strength.

\( \rho \) Gas density.

\( \rho_p \) Particle material density.

\( \bar{\rho_p} \) Particle phase density (mass particles/unit volume mixture).

\( \sigma \) Particle radius.

\( \sigma_1 \) Cut-off length, \( \sigma_1 = h/\pi \).

\( \tau \) Momentum equilibration time \( \tau = \frac{2}{9} \frac{\rho_p \sigma^2}{\mu} \).

\( \phi \) Potential function.

\( \psi \) Stream function.

\( \nabla \) Gradient operator.

\( \Delta \) Laplacian operator, \( \Delta = \nabla^2 \).
Parametric Groups

AR Aspect ratio, \( \sigma/L \) or \( \sigma/R \).

\( C_D \) Drag coefficient of particle.

\( \text{Fr}_p \) Particle Froude number, \( U^2/2g\sigma \).

\( \text{Re} \) Flow Reynolds number, \( UL/\nu \) or \( 2UR/\nu \).

\( \text{Re}_p \) Particle Reynolds number.

\( \lambda \) Momentum equilibration number, \( \tau U/L \) or \( \tau U/R \).

\( \psi \) Phase coupling number, \( \lambda/\omega \delta_1 \).

Subscripts

c Continuous phase.

d Dispersed phase.

D Flow due to potential source distribution.

m Mixture.

o Conditions at the wall.

p Particle phase.

s Flow due to vortex field.

\( \infty \) Free stream conditions.
1. Introduction

Within the rich experience of nature we find a uniformity of flow; the winds carry the dust, the sea, the sand; and hence each element is mirrored in motion. Of course, we account for the change in motion of any object as being caused by an external agent; however, in the case of wind sculpted sand dunes or a desert dust storm, the causation of motion is not as tractable to our minds as when solid bodies like billiard balls interact and consequently change their initial motion.

Gazing at the delicate patterns formed by wind carved snow does not tell us about the cause of the pattern; but by our senses we perceive the wind's motion and deduce that the snow's downward fall is influenced by the wind. Even though the idea of the wind influencing snow motion seems confirmed by the naive realism of common sense, care must be exercised in projecting our sensations onto the physical world. And so to gain a deeper understanding of these natural events, in the hope of extracting predictive information, we turn to the abstraction of science and superimpose a formal description upon the common sense view.

The description developed in this work aims to colligate many common flows where solid discrete elements are entrained in a gas flow. We will call this mixture a two-phase, gas-solid flow, one phase consisting of the solid discrete elements, the other phase comprising the gas. This work, then, will present a formalized predictive description of a two-phase flow which adheres to certain mathematically defined physical constraints.

The bibliography of gas-solids suspension flow is extensive and scattered in a wide variety of journals. The books of Boothroyd (1971), Govier and Aziz (1972), Soo (1967) and Zenz and Othmer (1960) cite most
of the important literature. The erosive wear of material surfaces exposed to a gas-particle flow has led to many investigations of specific systems. These include the combustion of pulverized coal, gas-particle separation by mechanical and other means, pneumatic transport, and rocket nozzle performance studies. Much of the early work is limited to simple flow geometries. In the early 1950's the National Advisory Committee for Aeronautics (NACA) conducted a comprehensive aircraft ice-protection research program. In these studies (Brun and Mergler, 1953) the raindrops impinging on aircraft surfaces were treated as solid particles.

More recently the motion of solid particles in rocket motor exhaust has increased scientific interest in the field. Much of the important research in rocket nozzle performance has been reviewed by Hoglund (1962). The laminar gas-particle flow over a flat plate has been investigated by Marble (1963). Soo (1965) studied both the laminar and separated flow of a jet and the cavity problem. The effects of electrostatic and gravity forces was analyzed by Soo and Tung (1971). The compressible boundary layer of a gas-particle flow has been considered by Singleton (1965).

The current interest in energy conversion systems has focused attention upon gas-particle flows again. The Department of Energy (DOE) has an extensive research program aimed at developing materials suitable for containing the gas-coal particle mixture found in a coal-gasification process. Developing erosion resistant materials of this type requires a thorough understanding of the two-phase flow characteristics that the material experiences at its surface. Many applications of gas-solid suspension technique may be expected to become more commonplace in the future; fluidized transport chemical reactors offer the benefits of a fluidized bed along with improved temperature control; particle laden gas is an efficient absorber of radiation in proposed solar thermal electric
conversion (STEC) facilities.

To the present time, the dynamics of gas-particle systems has remained closely associated with particular detailed problems and has not found a place in the general discipline of fluid mechanics. This is primarily due to the inherent complexity and unique difficulties of each particular system and to the inadequacy of many standard analytical techniques to deal with them.

It is the purpose of this work to show how gas-particle flow phenomena fit into and extend the pattern of fluid mechanics research. First, general statements are developed of the equations governing solid-liquid and solid-gas systems. Second, the significant dimensionless parameters are introduced and their general physical effect on the solution is discussed. Thirdly, a numerical solution is presented for gas-solids which is general in nature and is applicable to an arbitrarily shaped geometry (closed or open domain). Finally, the solution technique is applied to the gas-solid flow about a right circular cylinder. The solution at the front stagnation point is compared to an exact solution and the effect of the dimensionless parameters is illustrated through a variety of numerical experiments.
2. **Description of problem**

The type of two-phase model chosen for a particular analysis depends upon the detail desired in the flow pattern solution. For instance, if it is desired to compute the drag on a body immersed in a dilute suspension flow, then it suffices to use a single fluid model where the effect of the particulate phase is included as a non-Newtonian viscosity term (Einstein, 1906). However, if information is required concerning the nature of the flow near the body, it is necessary to find solutions to the exact flow equations.

In most industrial two-phase systems where erosion occurs, the distribution of erosive wear around the surface of the body must be determined. This requires a solution giving the particle velocity and position history; with this information an erosion model may be applied to determine the erosion distribution about the body. Generally, the collective motion of the particles is a desired solution as well as the gas motion; thus the two phases are treated as separate mechanically interacting mediums.

The vast majority of problems involves two-dimensional subsonic flow. In this case the governing equations for the particle and gas phase are similar in appearance to the well-known Navier-Stokes equations of fluid mechanics. Although several calculation schemes have been developed for a general one-dimensional flow (Crowe, 1965; Kliegel, 1960; Wallis, 1969), much less research has been conducted for two-dimensional flow. The only general two-dimensional model that does not use an inviscid flow field simplification is the "tank-and-tube" cellular approach developed by Spalding et al. (Gosman, 1969). In this method the flow field is subdivided into a series of "tanks" connected to
adjacent tanks by "tubes." Finite difference equations are derived and solved with the appropriate boundary conditions using Gauss-Seidel successive substitution. This technique has been successfully applied to isothermal flow fields in cyclone separators (Crowe, 1973) and electrostatic precipitators (Stock, 1973).

However, because of the high Reynolds number found in most industrial flows, the influence of viscosity is confined to narrow regions close to the surface of bodies. These small regions, which are initially invisible to a finite difference grid, grow larger—particularly if under the influence of an adverse pressure gradient. Qualitatively, the effects of these small regions of boundary layer back-flow are quite pronounced; the flow can become separated from the body by a region of reversed or recirculating flow. Finite difference schemes produce unreliable results in this situation since the computer cannot store enough grid points falling within the boundary layer to predict boundary layer growth and subsequent separation satisfactorily. Furthermore, it is often observed that in a boundary layer large truncation errors lead to the formation of an artificial numerical viscosity (Dorodnicyn, 1973).

A numerical scheme developed by Chorin (1973) for gas flow only, circumvents these difficulties. The scheme is grid free in that the vorticity within the fluid is partitioned into vortex "blobs" which are moved according to two components. One component is a random displacement of the vortex blob position; in this way the effect of viscous diffusion is modeled. The other component is a deterministic displacement found by moving the blobs according to their mutual interaction effects. This interaction is determined in a way similar to that in which the motion of point vortices interacting in an inviscid fluid is determined, according to the governing equations of classical hydrodynamics.
This vortex method has been further developed and successfully applied to several problems (Ashurst, 1975, 1977; Shestakov, 1975). Chorin (1977) has introduced a similar vortex method which solves Prandtl's boundary layer equations. Shestakov (1975) developed a hybrid technique in which the vortex method is used near boundaries, where vorticity is created; away from boundaries the vorticity is coalesced onto a finite difference grid. This technique reduces the computational time required by the vortex method. This can become extensive since in each time step the \( n \) vortex blobs must be moved according to their mutual interaction, requiring \( O(n^2) \) interaction computations. Detailed theoretical analyses of the vortex method have been considered by Marsden (1974).

In this work Chorin's vortex technique is extended to a two-phase mixture. Apart from the capacity of the vortex method to simulate the physics of viscous fluids and the process of vorticity injection, the scheme overcomes a major difficulty in modeling the particle phase boundary conditions. Particles striking a surface boundary can either adhere to the surface, leading to particle attrition, or form a bed that slides along the surface, or rebound from the wall. In the vortex method the particles vorticity field is coagulated into "blobs" which are then followed throughout the flow field. This Lagrangian description of particle vortex blobs allows precise mathematical consistency with the appropriate physical boundary condition; in an Eulerian formulation the appropriate conditions at a physical boundary are extremely complicated (Yeung, 1978a).

In this work the particles are assumed to adhere to the surface upon first impact. This is consistent with typical industrial processes where the particles act as an erosive material, cutting and imbedding into the surface. It should be mentioned that this work can be easily extended.
to include rebound phenomena; it is only necessary to specify rebound angles and velocity coefficients of restitution for an averaged collection of particles.

The scheme presented in section 7 is applicable to an arbitrary body as is heuristically depicted in Fig. 1. The particles are assumed to be travelling with the gas and with a uniform constant phase density at some initial position far from the body \((X,Y)\) where free stream conditions exist. The unsteady equations of motion are solved outside the body with the "no slip" boundary conditions for the gas and particle phase:

For the gas

\[ u \text{ (at the boundary)} = \begin{cases} \text{velocity of the boundary,} \\ 0 \text{ if boundary at rest} \end{cases} \quad (2.1) \]

For the particles

\[ u_p \text{ (at the boundary)} = \begin{cases} \text{velocity of the boundary,} \\ 0 \text{ if boundary at rest} \end{cases} \quad (2.2) \]

In section 9 this scheme is applied to the two-phase flow around a cylinder. This problem has been investigated by Glauert (1940), Tilly (1969) and Pettit (1977) under the simplifying assumption of inviscid gas flow. Since many investigations have used potential theory to determine the gas flow, this work will compare the inviscid approximation and the viscous solution given by the vortex method. We can then assess the magnitude of error introduced in the particle phase solution as a result of the inviscid approximation.

It is hoped that this work will serve as another contribution to the study of two-phase, gas-solid flow about a cylinder, and more saliently, introduce a technique useful in solving two-phase flow problems.
3. Equations of motion

In this section the equations of motion of a two phase mixture are presented in their most general form. These equations represent the unsteady flow of a compressible continuous phase within which resides a dispersed phase. The continuous phase is either a gas or liquid and the dispersed phase consists of discrete particles which are liquid droplets or suspended solids. In applying these equations to model a physical system many terms are usually neglected. In the next section the conditions where certain terms of the equations are neglected are described in full.

The constitutive equations have been derived by Drew (1976) and are presented here in the following form.

For the fluid or continuous phase:

Conservation of Mass

\[
\frac{\partial (1-\alpha)}{\partial t} + \nabla \cdot (1-\alpha) u_c = 0 .
\]  

Conservation of Momentum

\[
(1-\alpha) \rho_c \left[ \frac{\partial u_c}{\partial t} + u_c \cdot \nabla u_c \right] = - (1-\alpha) \nabla p
\]

\[+ \alpha F (u_d - u_c) \]

\[+ \alpha L (u_d - u_c) \cdot D \]

\[+ K_c \nabla (1-\alpha) + (1-\alpha) \rho_c f - \alpha B \gamma^2 u_c \]

\[+ \nabla \cdot (1-\alpha) \mu D(\alpha) \cdot D .
\]  

And for the dispersed phase

Conservation of Mass

\[
\frac{\partial \alpha}{\partial t} + \nabla \cdot (\alpha u_d) = 0 .
\]
Conservation of Momentum

\[ \alpha \rho_d \left[ \frac{\partial u_d}{\partial t} + u_d \cdot \nabla u_d \right] = - \alpha \nabla p - \alpha F(u_d - u_c) - \alpha L_\| (u_d - u_c) \cdot \nabla \alpha - K_d \nabla \alpha + \alpha \rho_d f + \alpha \beta \gamma^2 u_c \]

where \( u_c = (u_c, v_c) \) is the velocity vector for the continuous phase,
\( u_d = (u_d, v_d) \) is the velocity vector for the dispersed phase, and
\( \alpha \) is the volume fraction of the dispersed phase in the continuous phase;
\( D(\alpha) \) is a functional of the volume fraction;
\( D \) is the continuous phase rate of deformation tensor
\( (D = \frac{1}{2} (\nabla \mathbf{u} + \mathbf{u} \otimes \nabla)) \);
\( f \) is the external acceleration imposed on the system;
\( F \) is the drag force function;
\( B, K_c \) and \( K_d \) are constant (\( K_c \) and \( K_d \) are proportional to the kinetic energy of particle's random motion);
\( L_\| \) is the lift force function;
\( p \) is the continuous phase static pressure;
\( \rho_c, \rho_d \) are the densities of the continuous and dispersed phase, respectively;
\( \mu_m \) is the mixture viscosity.

Equations (3.2) and (3.4) are essentially Newton's Second Law written for a control volume analysis of a continuous and dispersed continuum. The forces acting to accelerate the continuous phase as written on the right-hand side of (3.2) are the pressure force, drag
force, lift force, Brownian motion force, gravity force, viscous force, and viscous diffusion force, respectively. The forces acting to accelerate the dispersed phase as written on the right-hand side of (3.4) are the pressure force, drag force, lift force, Brownian motion force, gravity force, and viscous force, respectively.

The Eulerian formulation used here assumes the dispersed phase is a continuum (A1)* and the resulting equations are similar in character to the Navier-Stokes equations. Fortunately, most two-phase systems encountered in practice satisfy the continuum assumption. For a detailed discussion of the continuum assumption the reader is referred to Soo (1967). The Eulerian formulation includes implicit information concerning interaction between the dispersed elements, and basically averages the velocity and density effects of a small volume (small compared to a characteristic length of the system) which contains many dispersed elements.

Generally, the interaction between the elements of the dispersed phases is insignificant when compared to the interaction of the dispersed and continuous phase. Thus, the dispersed phase has an insignificant viscosity, diffusivity and there is no shear stress term due to interaction between the dispersed elements. However, the dispersed phase can transfer momentum throughout its domain through coupling with the continuous phase. This effect gives rise to the drag force term appearing in (3.2) and (3.4).

The pressure scalar, \( p \), appearing in (3.2) and (3.4) is due solely to the continuous phase (A2). This is because the root mean square speed of the dispersed elements is very small compared to that of the molecules.

*All assumptions used to simplify (3.1)-(3.4) are denoted by A1, A2, A3, etc.
comprising the continuous phase.

We have assumed that the Basset force (A3), virtual mass (A4) effects, and the Magnus force (A5) are negligible.

The lift force arises when elements of the dispersed phase lie in a shear flow of the continuous phase. See Saffman (1965) for a discussion of this effect on small solid spherical particles.

We now make one important additional assumption that the dispersed phase is sufficiently dilute in the continuous phase (A6) so that the low concentration limits in the governing equations may be used. The "principle of correct low concentration limits" (Drew, 1976) states that when the dispersed phase is sufficiently dilute, the mixture behaves as if it were made up of the continuous phase alone. Moreover, in sufficient dilution (i.e., $\alpha \ll 1$) the dispersed phase behaves like individual elements suspended in the continuous phase. Thus, when $\alpha \to 0$, the equations of motion for the continuous phase must reduce to the appropriate equations for a compressible viscous fluid. A further implication of the principle of correct low concentration limits is that the forces acting on the dispersed phase must reduce to the expression for the force acting on a single element of the dispersed phase as $\alpha \to 0$.

In this analysis the mixtures under investigation are dilute since the volume fraction is much less than unity ($O[10^{-3}]$ or less). Two phase systems such as sand and dust ingestion in jet aircraft, raindrops impinging on iced aircraft wings, char particles and product gases in a coal gasification system and many other industrial flows are examples of dilute suspensions.

The two phase flow we are modeling consists of a viscous gas continuous phase and particulate dispersed phase. Hence, we shall refer to
the gas flow (no subscript) and the particle flow (subscript p).

For low concentrations we have the following (Drew, 1976):

\[ B = \frac{3}{4} \mu \quad , \tag{3.5} \]

\[ L_1 = L_1 (\rho \mu)^{\frac{1}{2}} \sigma^{-1} |D|^{-\frac{1}{2}} \quad , \quad L_1 = 0.771 \quad , \tag{3.6} \]

\[ D(\alpha) = 1 \quad , \tag{3.7} \]

\[ \mu_m = \mu (1 + \frac{5}{2} \alpha + 7.6 \alpha^2) \sim \mu (1 + \frac{5}{2} \alpha) \quad . \tag{3.8} \]

It is readily shown for \( \alpha \ll 1 \) that

\[ \rho_m = \bar{\rho}_p + \rho \sim \rho \quad , \tag{3.9} \]

and

\[ \alpha \sim \bar{\rho}_p / \rho_p \quad , \tag{3.10} \]

where \( \bar{\rho}_p \) is the particle phase density, that is the mass of particles per unit volume of gas.

Equations (3.2) and (3.4) now become:

Gas Momentum Equation

\[ (1-\alpha) \rho \left[ \frac{\partial u}{\partial t} + u \cdot \nabla u \right] = -(1-\alpha) \nabla p + \alpha \nabla (\bar{\rho}_p \cdot u) \]

\[ + \alpha L_1 (\rho \mu)^{\frac{1}{2}} \sigma^{-1} |D|^{-\frac{1}{2}} (u_p \cdot u) \cdot D \]

\[ - K \nu \alpha + \alpha \rho_f \nu - \alpha \frac{3}{4} \mu \nu^2 u \]

\[ + \nu \cdot \mu (1 + \frac{5}{2} \alpha) (1-\alpha) \cdot D \quad . \tag{3.11} \]
Particle Momentum Equation

\[ \alpha p_p \left( \frac{\partial u_p}{\partial t} + u_p \cdot \nabla u_p \right) = -\alpha \rho_p - \alpha F(u_p - u) \]

\[ - \alpha l \frac{1}{2} \sigma^{-1} \left| D \right|^{\frac{1}{2}} (u_p - u) \cdot D \]

\[ - K_p \rho + \alpha \rho_p \frac{f}{p} \]

\[ + \alpha \frac{3}{4} \mu \nabla^2 u \]

(3.12)

Assuming the gas to be incompressible (A7), the equations of continuity become, using (3.10):

Gas Phase

\[ \nabla \cdot u = 0 \]  \hspace{1cm} (3.13)

Particle Phase

\[ \frac{\partial \alpha}{\partial t} + \nabla \cdot (\alpha u_p) = 0 \]  \hspace{1cm} (3.14)
4. **Scaling and Approximations**

In practice there exist conditions where certain terms of the momentum equations can be neglected. Scaling the pertinent variables tests the validity of omitting various terms in a given physical situation.

The existence of velocity and length scales $U$ and $L$ is assumed. Scaled vorticity, velocities, spatial coordinates and time are defined by

\[
\xi = UL^{-1}\xi' , \quad \xi_p = UL^{-1}\xi_p' , \quad \xi = U\xi' , \quad \xi_p = U\xi_p' ,
\]

\[
x = Lx' , \quad t = LU^{-1}t' ,
\]

where "prime" denotes the scaled variable. The pressure is scaled by

\[
p = \rho U^2p' ,
\]

the particle volume fraction is scaled by

\[
\alpha = \alpha' ,
\]

the external acceleration force, $f$, is scaled by the acceleration due to gravity, $g$

\[
f = gf'
\]
and the rate of deformation tensor is scaled by
\[ \mathbf{D} = UL^{-1} \mathbf{D}' \]  

The following dimensionless groups are relevant:

\[ \text{Re} = \rho UL^{-1} \equiv \text{Reynolds Number} \]  
\[ \lambda = \frac{2 \rho \sigma^2 U}{\eta \mu L} \equiv \text{Momentum Equilibration Number} \]  
\[ \gamma = \frac{\lambda}{\alpha \delta_1} \equiv \text{Phase Coupling Number} \]  
\[ \text{Fr}_p = \frac{u_p^2}{2 g \sigma} \equiv \text{Particle Froude Number} \]  
\[ \text{AR} = \frac{\sigma}{L} \equiv \text{Aspect Ratio} \]  
\[ \delta_1 = \frac{\rho_p}{\rho} \equiv \text{Relative Density} \]  
\[ \gamma_1 = \frac{K_p}{\rho_p U^2} \]  

K and \( K_p \) are constants proportional to the kinetic energy of random particle motions.

We must specify the drag force function \( F \). We assume the particles to be spherical. Irregular size particles are used with the diameter of an equivalent sphere, \( d \), as suggested by Bagnold (1960), where \( d \) equals 75\% of the mean sieve diameter. The aerodynamic drag function per unit volume is given by Stokes' Law (Schlichting, 1960)

\[ F(u_p-u) = 3 \mu d (u-u_p)/\Gamma \]  

which is found by solving the Navier-Stokes equations for a slow flow (i.e., no inertia terms). This expression is correct only when the particle Reynolds number is less than unity, i.e.,
The particle volume is

\[ \Gamma = \frac{4}{3} \pi \sigma^3 . \]

Now as the two-phase mixture approaches a boundary or is strongly accelerated a particle will experience fluid moving away from it. For example, as a particle approaches a large body the fluid begins to move smoothly about the body tangential to the surface. The particle, however, moves perpendicular to the surface towards the body through fluid. Thus the particle "sees" fluid moving away from it. In this case the slip velocity of the particle through the fluid can become large and consequently the particle Reynolds number can become much greater than unity. The total drag force per unit volume acting on the particles may be written as follows (Hussein, 1972)

\[ F(u - u_p) = 3 \eta d (u - u) g(Re_p) / \Gamma \]

where

\[ g(Re_p) = Re_p C_D / 24 . \]

As discussed previously, in the case of slow flow

\[ C_D = 24 / Re_p , \quad 0 < Re_p \leq 1.0 \]

and \( g(Re_p) = 1 \) yielding Eq. (4.18). For a slightly higher particle Reynolds number the motion of the particle produces a wake and Oseen's solution of the Navier-Stokes equations with first order inertia terms yields (Schlichting, 1960)

\[ C_D = \frac{24}{Re_p} \left( 1 + \frac{3}{16} Re_p \right) , \quad 1.0 < Re_p \leq 4.0 . \]
Now by fitting experimental data (Boothroyd, 1971) for the drag coefficient of spherical particles, the following formulas are obtained

$$C_D = 24(1 + 0.15 \frac{Re_{p}^{0.687}}{Re_p}) \quad 0 < Re_p \leq 200 ,$$  \hspace{1cm} (4.21)

$$C_D = 21.9416 \frac{Re_p^{-0.718} + 0.324}{200 < Re_p \leq 2500 ,}$$  \hspace{1cm} (4.22)

$$C_D = 0.4 \quad Re_p > 2500 .$$  \hspace{1cm} (4.23)

These will be used in the following numerical solution.

Substituting (4.3)-(4.17) and (4.19) into (3.1)-(3.4) yields after algebraic manipulation

Gas Equations of Motion

$$\nabla' \cdot \mathbf{u'} = 0 ,$$  \hspace{1cm} (4.24)

$$\frac{\partial \mathbf{u'}}{\partial t'} + \mathbf{u'} \cdot \nabla' \mathbf{u'} = - \nabla' p' + \frac{A \alpha' \delta_g(Re_p)}{(1-A \alpha') \lambda} (u'_p - u')$$

$$+ \frac{\sqrt{2 A \alpha' \lambda^{-\frac{1}{2}} g_1}}{\sqrt{9(1-A \alpha')}} |D'|^{-\frac{1}{2}} (u'_p - u') \cdot D'$$

$$+ \frac{\beta_1 \gamma \alpha'}{(1-A \alpha')} \nabla' \alpha' + \frac{A \alpha' Fr^{-1} AR^{-1}}{(1-A \alpha')2} f'$$

$$- \frac{A \alpha'}{(1-A \alpha')} \frac{3}{4} Re^{-1}(\nabla')^2 u'$$

$$+ Re^{-1} \nabla' \cdot \frac{(1 + \frac{5}{2} A \alpha')}{(1-A \alpha')} D' .$$  \hspace{1cm} (4.25)

Particle Equations of Motion

$$\frac{\partial \alpha'}{\partial t'} + \nabla' \cdot (\alpha' \mathbf{u'}_p) = 0 ,$$  \hspace{1cm} (4.26)
\[ \frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla \mathbf{u}' = -\delta_1 \nabla' \mathbf{p}' - \frac{g(Re)}{\lambda} (\mathbf{u}' - \mathbf{u}) \]

\[ -\delta_1 \sqrt{\frac{2}{g}} \lambda^{-\frac{1}{2}} |D'|^{-\frac{1}{2}} (\mathbf{u}' - \mathbf{u}) \cdot D' \]

\[ -\gamma_1 \mathbf{v}' \mathbf{a}' + \frac{1}{2} \mathbf{Fr}^{-1} \mathbf{AR}^{-1} f' \]

\[ + \delta_1 \mathbf{Re}^{-1} (\mathbf{v}')^2 \mathbf{u}' \]  \hspace{1cm} (4.27)

Several terms in the non-dimensional gas momentum equation can be simplified by expanding the expression \((1-Aa')^{-1}\) in a series. Letting

\[(1-Aa')^{-1} = 1 + (Aa') + (Aa')^2 + \ldots\]

and noting that \(Aa' \ll 1\) by the dilute suspension assumption (A6) we have

\[(1-Aa')^{-1} \approx 1 \] \hspace{1cm} (4.28)

Thus equation (4.25) becomes

\[ \frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla \mathbf{u}' = -\nabla' \mathbf{p}' + \frac{g(Re)}{\mathbf{v}} (\mathbf{u}' - \mathbf{u}) \]

\[ -\frac{2}{9} \frac{Aa' \delta_1}{\lambda} |D'|^{-\frac{1}{2}} (\mathbf{u}' - \mathbf{u}) \cdot D' \]

\[ + \delta_1 \mathbf{Re}^{-1} (\mathbf{v}')^2 \mathbf{u}' \]

\[ + Aa' \mathbf{Re}^{-1} (\mathbf{v}')^2 \mathbf{u}' \] \hspace{1cm} (4.29)

The physical systems of interest have flow Reynolds numbers ranging from \(10^3\) to \(10^6\), the relative density for gas-solid particle and gas-liquid droplet flows range from 800 to 3000. For these types of flows, \(\lambda\), the momentum equilibration number, is usually of order unity; however, it can vary from as little as \(10^{-1}\) to as much as \(10^4\). It will be shown
that values of the momentum equilibration number between 0.1 and 10.0 span the limits of approximately straight line trajectories to drag dominated trajectories where all the particles follow the gas streamlines around objects in the flow. Physically the momentum equilibration number indicates the non-dimensional distance required for the particle to reduce its initial slip velocity by exp(-1). Typically the phase coupling number, \( \psi \), lies between 10 and \( 10^{14} \). The particles corresponding to the range given for \( \lambda \) and \( \psi \) have diameters between five and several hundred microns and a specific gravity between one and four.

Using these physical data the order of magnitude of the various terms in equations (4.27) and (4.29) can be determined. The left-hand side of the equations, consisting of the local and convective acceleration terms, are of order unity. The force terms on the right-hand side are all of order \( 10^{-3} \) or less except for the second terms representing the drag force, and the first and last terms in (4.29) (the viscous diffusion term is not omitted, otherwise the order of the equation would be reduced).

In the gas momentum equation it is seen that the phase coupling number, \( \psi \), determines the momentum coupling from the particles to the gas. This parameter determines to what extent the gas flow is affected by the presence of the particles. For the dilute suspension under consideration the particle phase density is very small compared to the gas density, i.e., \( \alpha \ll 1 \). And even though the relative density, \( \delta_1 \), is large, the phase coupling term in the gas momentum equation can be neglected as a first order approximation (A8 follows from A6). It would appear in equation (4.29) that very small values of \( \lambda \) would lead to a large phase coupling term; since we have

\[
\frac{1}{\psi} (u'_p-u') = \frac{\alpha \delta_1}{\lambda} (u'_p-u') .
\]
However, if $\lambda$ is very small then equation (4.27) predicts that the particle acceleration will match the gas acceleration; hence at all times the particle slip velocity $(u' - u_p')$ is very small and consequently the phase coupling term is again negligible.

This type of one way momentum coupling is common in dilute gas-solid flows. In a flow where the fluid phase has high viscosity and density we find very small values of $\lambda$ and $\Psi$. The flow of a liquid-solid particle mixture is an example of such a flow; here the phases are both coupled together and the particle phase affects the fluid flow. In this case the second term on the right-hand side of (4.29) is included.

Consider for a moment a liquid-solid system. Each scaled term in equations (4.27) and (4.29) is of order unity except terms with $Re^{-1}$, $\Psi^{-1}$ and $\lambda^{-1}$ as coefficients. Thus the three dynamically similar variables for this type of liquid-solid two phase flow are:

$$Re, \lambda \text{ and } \Psi.$$

By holding these variables constant, experiments can be performed on a small scale that will accurately reproduce the dynamics of a full scale system.

The cases corresponding to the limiting values of the parameters $\lambda$ and $\Psi$ are of particular interest since the equations of motion can be greatly simplified in several cases. A large value of $\lambda$ indicates that the particle enters and leaves the region of interest before there is opportunity to alter its state appreciably. Hence for $\lambda \gg 1$, the particle motion depends largely on its initial conditions, that is on its state at the time it enters the system. When $\lambda \ll 1$, the particle has time to adjust to the local gas or liquid motion before it has moved appreciably. If the fluid motion were uniform, the particle would achieve this identical uniform motion after traveling a small distance. If the fluid motion
is accelerated, the particle adjusts itself to this situation by taking on a slip velocity that provides the force to accelerate the particle at nearly the local rate; the important feature here is that the particle motion depends on the local flow acceleration and is relatively independent of its previous history. Thus for \( \lambda \ll 1 \) the particle motion depends on the local fluid acceleration. For values of \( \lambda \) that are neither very large nor very small, the local particle motion is dependent on its entire history. In other words, when \( \lambda \sim 1 \), the particles have a memory of events that took place throughout a region of dimension \( \lambda L \).

The effect of the particulate motion on the fluid motion is, in addition to the criteria mentioned, directly dependent on the mass of particulate material in a given region. Thus for particles of mass \( m_p \), number \( n_p \) per unit volume of space, the particle density ratio (also referred to as the particle loading ratio)

\[
\frac{n_p}{\rho} = \frac{\rho_p}{\rho} = \kappa
\]  

(4.30)
is a measure of the interaction force per unit mass of fluid. This force as well as the drag force exerted on the gas by the particles is quantified by the non-dimensional parameter \( \Upsilon \) in the gas momentum equation (4.29). We can rewrite (4.13) as

\[
\Upsilon = \frac{\lambda}{\alpha \delta} = \frac{\lambda}{\rho_p \rho_p} = \frac{\lambda}{\rho \rho} = \frac{\lambda}{\kappa}.
\]  

(4.31)

For \( \Upsilon \gg 1 \) - an extreme example is a single small particle moving through a fluid - the gas motion is affected very little. For \( \Upsilon \ll 1 \), the inverse is true. When the particle density and size is such that \( \Upsilon \sim 1 \), both fluid and particle motion are affected by the interaction.

A common example of these two flows, one way and two way coupled,
is found at any sandy beach. A small round rock placed in the sand as
the surf rushes back down the beach will deflect sand and water in a "V"
formation, clearly visible in the sand even after the water has receded.
The sand curves around the rock at the vertex of the "V" and gains enough
kinetic energy to overcome the fluid trying to drag it around to the rear
stagnation point. As it moves off it carries fluid with it, altering
the fluid streamlines that would exist if no sand were present, and
leaving a "V" depression in the beach.

Farther up the beach away from the surf we find an example of a one
way momentum coupling gas-solid particle flow. By placing the same rock
part way in the sand in a wind blown area we find an entirely different
result. Here the air flow is not affected by the sand motion and is not
nearly as effective as water in dragging a bit of sand large distances.
As the air flows around the rock, the sand picks up kinetic energy as
before, but not as quickly and gravity pulls it down out of the air flow
just behind the rock. After a period of time there will be a built up
area of sand behind the rock.

The only change in the two flows is in the relative density and
the carrier fluid viscosity. This simple example demonstrates how
drastically the characteristics of a two-phase flow change with changes
in the physical variables.

Returning to the equations governing the flow of a gas-solid, gas-
liquid droplet two phase flow, we now eliminate from equations (4.27)
and (4.29) the negligibly small terms (i.e., essentially deleting terms
of order α and higher). In the gas momentum equation we neglect the
drag force (A8), lift force (A9), Brownian motion force (A10), gravity
force (A11), and viscous force (A12), but not the viscous diffusion
force which is the random transfer of molecular momentum. In the
particle momentum equation we neglect the pressure force (A2), lift force (A9), Brownian motion force (A10), gravity force (A11), and viscous force (A12). We have the following governing equations (note: the primes denoting non-dimensional quantities are dropped henceforth)

**Gas Governing Equations**

\[
\nabla \cdot \mathbf{u} = 0 , \tag{4.32}
\]

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} . \tag{4.33}
\]

**Particle Governing Equation**

\[
\frac{\partial \alpha}{\partial t} + \mathbf{u}_p \cdot \nabla (\alpha \mathbf{u}_p) = 0 , \tag{4.34}
\]

\[
\frac{\partial \mathbf{u}_p}{\partial t} + \mathbf{u}_p \cdot \nabla \mathbf{u}_p = g \frac{(\mathbf{u} - \mathbf{u}_p)}{\lambda} , \tag{4.35}
\]

where \( g = g(Re_p) \).

It is more convenient to rewrite the non-dimensional governing equations in terms of vorticity. Taking the curl of (4.33) and (4.35) yields (again deleting primes):

**Gas Vorticity Transport**

\[
\nabla \cdot \mathbf{u} = 0 , \tag{4.36}
\]

\[
\frac{\partial \mathbf{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{\omega} = \frac{1}{Re} \nabla^2 \mathbf{\omega} . \tag{4.37}
\]

**Particle Vorticity Transport**

\[
\frac{\partial \mathbf{\omega}_p}{\partial t} + \mathbf{u}_p \cdot \nabla \mathbf{\omega}_p = \frac{g(\mathbf{\omega} - \mathbf{\omega}_p)}{\lambda} . \tag{4.38}
\]

where \( \mathbf{\omega} \) and \( \mathbf{\omega}_p \) are scalars representing the fluid phase vorticity,
\( \xi = \text{curl } \mathbf{u} \), and the particle phase vorticity, \( \xi_p = \text{curl } \mathbf{u}_p \), respectively.

We can write \( \lambda \) in a more tractable form as

\[
\lambda = \frac{2}{9} \frac{\rho_p}{\rho} \left( \frac{\rho}{L} \right)^2 \left( \frac{UL}{v} \right)
= \frac{2}{9} \delta_1 (AR)^2 \text{Re} \quad \tag{4.40}
\]

Thus for gas-solids experiments on different scales to be dynamically similar we require \( \lambda \) to be constant and \( \text{Re} \) to be constant between experiments.
5. **Principle Method of Solution**

Equations (4.36) to (4.39) are solved by integrating forward in time. At time step $m$ we assume the vorticity is known for the gas flow field and the particle flow field. We want to determine the gas and particle vorticity distribution at the time step $(m+1)$. This is done as follows. First consider a flow field without boundaries present; the vorticity is partitioned into a sum of blobs

$$
\mathbf{\xi} = \sum_{j=1}^{N} \mathbf{\xi}_j ,
$$

$$
\mathbf{\xi}_p = \sum_{i=1}^{N_p} \mathbf{\xi}_{pi} ,
$$

where the gas vortex blobs, $\mathbf{\xi}_j$, and the particle vortex blobs, $\mathbf{\xi}_{pi}$, each have small support, i.e., the function vanishes uniformly outside a small but finite region (or blob) around a point $\mathbf{r}_j$ and $\mathbf{r}_{pi}$.

Now in the case of the fluid the vortex field is advanced using Chorin's (1973) scheme, as described in section 6.

In the case of the particles we must first discuss the characterization of the particle continuum as a set of discrete vortex blobs and then describe the technique by which these blobs are advanced to the $m+1$ time step.

As described in section 3, the Eulerian formulation of the particle motion includes implicit information concerning interaction between the dispersed particles, and basically averages the velocity and density effects of a small volume which contains many dispersed elements. It is pointed out that the interaction between a given particle and another particle is insignificant when compared to its interaction with the dispersed phase. This assumption is justified by the scaling analysis in
in section 4 (see A12).

Clearly then, partitioning the particle vorticity distribution into blobs of small support is in physical agreement with the spatial averaging found in the Eulerian description. Also, we are only interested in mean values of the dependent variables, since these are the only ones sampled experimentally. Thus the information carried by the vortex blobs associated with the particle determines the mean motion of the particle.

Now we write the particle vorticity transport equation (4.39) in Lagrangian form yielding

\[
\frac{D \xi_{pi}}{Dt} = \frac{g(\bar{\xi} - \xi_{pi})}{\lambda} .
\]

(5.3)

We propose to solve this equation by following the motion of the particle vortex blobs, \( \xi_{pj} \). A question arises as to the interaction effects between neighboring vortex blobs. Even in a dilute suspension there is some particle-particle collision, in our formulation this would imply an interaction between particle vortex blobs determined by the mechanics of particle collision within the union of the supports of the two interacting blobs. To overcome this difficulty, we associate with the \( i^{th} \) particle vortex blob the notion of an ensemble average used in statistical mechanics (Hill, 1960).

The motion of the blob can be represented as

\[
\lambda \frac{D \xi_{pi}}{g Dt} = F(t) ,
\]

(5.4)

where \( F(t) \) represents the effect of molecular collision on all the particles within the support of \( \xi_{pi} \) and the collisions caused by particles from another blob. (This representation is much the same as that of a single particle experiencing Brownian motion; see Yeung (1978b).)
Assuming that \( F(t) \) can be written as the sum of an averaged part which is the viscous drag and a time fluctuating part (Al3), equation (5.4) becomes

\[
\lambda \frac{D\xi_{pl}}{Dt} = (\xi - \xi_{pl}) + f(t) .
\]  

(5.5)

Consider an ensemble average of a very large number, \( N_p \), of particle vortex blobs. Taking the time average of (5.5) gives

\[
\lambda \frac{D<\xi_{pl}>}{Dt} = <\xi> - <\xi_{pl}> + <f(t)> .
\]  

(5.6)

In laminar flow if the particles are all of uniform size \( <f(t)> = 0 \) (Boothroyd, 1971, p. 30); if the flow becomes turbulent or is polydisperse (i.e., contains particles of many sizes; see Marble (1964) for a simple analysis), then we can assume \( <f(t)> = 0 \) because the interaction is random. However, it is well known (Boothroyd, 1971) that the frequency of collision is proportional to the square of the number density. For a dilute suspension the number density is very small, thus by the dilute suspension assumption we assume \( <f(t)> = 0 \) ((A14) follows from (A6)).

We write (5.6) as

\[
\frac{D\xi_{pl}}{Dt} = \frac{g(\xi - \xi_{pl})}{\lambda} ,
\]  

(5.7)

where the brackets are dropped. Henceforth only the mean (spatial and time) values of \( \xi_{pl} \) are calculated.

The motion of the particle vortex blob, then, is the time space average of the motion of particles near it. The idea here is analogous to the motion of the gas vortex blobs far from any boundaries. Since the fluid is inviscid their motion is along fluid streamlines. This follows from Kelvin's circulation theorem which states that vorticity moves...
with the fluid elements in an inviscid fluid. For the particles, which are essentially an inviscid continuum, the particle vortex blobs move according to the mean particle trajectories. However, because the mutual interaction between particle vortex blobs has been shown to be negligible, we need only solve for a small number of blobs, $N_p$. This greatly reduces the computational time, which would be prohibitive if the entire particle vortex field needed to be solved.

Now the motion of the particle vortex blobs is described by

$$\frac{dx_{p(i_p)}}{dt} = u_{p(i_p)} \quad i_p = 1, \ldots, N_p,$$

$$\frac{dy_{p(i_p)}}{dt} = v_{p(i_p)} \quad i_p = 1, \ldots, N_p,$$

where $N_p$ is arbitrary and $u_{p(i_p)}, v_{p(i_p)}$ are found by using a scheme presented in section 7.
6. Chorin's Vortex Scheme

This section presents an outline of Chorin's (1973) vortex method as it pertains to the problem under investigation. Since the flow is incompressible and two-dimensional, there exists a stream function \( \psi \); physically \( \psi \) is a measure of the two-dimensional fluid flow rate and is related to the velocity as follows:

\[
  u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}.
\]  

(6.1)

We first consider the flow of an inviscid fluid (i.e., \( \text{Re} = \infty \)). Equation (4.37) becomes

\[
  \frac{D\psi}{Dt} = 0
\]

with

\[
  \Delta \psi = -\xi
\]

(6.2)

with the vorticity partitioned into blobs as described in (5.1), \( \psi_j \) then has the form

\[
  \psi = \sum_{j=1}^{N} \psi_j \quad \text{with} \quad \Delta \psi_j = -\xi_j.
\]

When the distance between some arbitrary point and the \( j^{th} \) vortex blob is large (i.e., \( |x-x_j| \) large), \( \psi_j \) will have a form

\[
  \psi_j \approx \frac{-\xi_j}{2\pi} \log|x-x_j|, \quad \xi_j = \iint \xi_j \, dx \, dy.
\]

The equation above is the expression for a point vortex stream function. In the neighborhood of a point vortex the fluid's tangential velocity varies inversely with the radius. However, the velocity field created by the vortex blobs is made bounded as opposed to the infinite velocity at the center of a point vortex. This is done by constructing a basic
blob of the form

\[ \psi^o(r) = \begin{cases} 
\frac{1}{2\pi} \log r & r \geq \sigma_1 \\
\frac{1}{2\pi} \frac{r}{\sigma_1} & r < \sigma_1
\end{cases}, \quad (6.3) \]

where \( r = |r| \) and \( \sigma_1 \) is a cut-off length which will be discussed shortly.

Since the blobs are small it is assumed that their total vorticity \( \xi_j \) is small and hence their interaction effect with neighboring blobs is small.

Now the stream function is written as

\[ \psi = \sum_{j=1}^{N} \xi_j \psi^o(r-r_j) \]

and

\[ \xi = \sum_{j=1}^{N} \xi_j \xi_j^o, \quad (6.4) \]

where each basic blob satisfies

\[ \xi_j^o = -\Delta \psi^o(r-r_j). \]

The motion of the vortex blobs is then described by

\[ \frac{dx_i}{dt} = -\sum_{j \neq i} \xi_j \frac{\partial \psi^o}{\partial y} (r-r_j) \quad i = 1, \ldots, N, \]

\[ \frac{dy_i}{dt} = \sum_{j \neq i} \xi_j \frac{\partial \psi^o}{\partial x} (r-r_j) \quad i = 1, \ldots, N \]

the components of the radius vector \( r \) are \((x_i, y_i)\).

These equations can be approximated by

\[ x_i^{n+1} = x_i^n + ku^n \frac{1}{2}, \quad (6.5) \]

\[ y_i^{n+1} = y_i^n + kv^n \frac{1}{2}, \quad (6.6) \]

where \( k \) is the time step.
Now consider the case when $\text{Re} \neq \infty$. The diffusion equation for the fluid is

\[
\frac{\partial \xi}{\partial t} = \frac{1}{\text{Re}} \Delta \xi
\]

with initial data $\xi(0) = \xi(x,y,t=0)$. A solution to this equation using a random walk is obtained as follows. Assume for the moment that $\xi$ is a known function in space and time, then distribute over the $x,y$ plane points of masses $\xi_i$ with locations $r_i = (x_i, y_i)$, $i=1, \ldots, N$, $N$ large. This is done so that the mass density approximates the initial condition $\xi(0)$. Then the points are moved by the following equations:

\[
\begin{align*}
    x_i^{n+1} &= x_i^n + \eta_1^n, \\
    y_i^{n+1} &= y_i^n + \eta_2^n,
\end{align*}
\]

(6.7) 
(6.8)

where $\eta_1$ and $\eta_2$ are Gaussianly distributed random variables with zero mean and variance $2k/\text{Re}$, $k$ being the time step.

The vorticity density generated by their mutual interaction and random walk is given by

\[
\begin{align*}
    x_i^{n+1} &= x_i^n + ku^n \frac{1}{2} + \eta_1^n, \\
    y_i^{n+1} &= y_i^n + kv^n \frac{1}{2} + \eta_2^n,
\end{align*}
\]

which approximates the solution to (4.37), (6.1) and (6.2).

This analysis has neglected the effect of boundaries. We must satisfy the no slip condition and create a potential flow that will exactly cancel the normal component of flow. The normal component is developed in the next section. To satisfy the no slip condition (tangential component) the vorticity necessary to create a velocity exactly cancelling the flow velocity in the tangential direction must be
determined. Integrating the vorticity in the boundary layer will yield the desired result. The total vorticity in a boundary layer of thickness $\delta$ and length $h$ is (see Fig. 2)

$$\bar{\xi} = \int_{-h/2}^{h/2} \int_{0}^{\delta} (\nabla \times \mathbf{u}) \, dy \, dx = \int_{-h/2}^{h/2} \int_{0}^{\delta} \left(-\frac{\partial u}{\partial y}\right) dy \, dx \approx -U(0, \delta) h \ , \ (6.9)$$

where the integral has been approximated using the midpoint rule, and $U(0, \delta)$ is the free stream speed. This total vorticity is assigned to the blob which has a constant velocity field inside a cut-off length $\sigma_1$ that exactly annihilates the tangential velocity and gives the appropriate value of $\sigma_1$ as $\sigma_1 = h/2\pi$ from (6.1), (6.3) and (6.9).
7. **Numerical Details**

We must find a potential flow $u_D$, such that $u_D \cdot n = -u_s \cdot n$ for each point on the boundary of an obstacle. In this way $u_D + u_s$ will satisfy the normal boundary condition. We require a solution to

$$\Delta \Psi = 0$$

(7.1)

subject to the boundary condition

$$u \cdot n = -u_s \cdot n \text{ on } \partial D.$$  

(7.2)

Laplace's equation can be satisfied by a flow of the form

$$u = \Psi \phi,$$

(7.3)

where $\phi$ is given by

$$\phi(r) = \frac{1}{2\pi} \int_{\partial D} \sigma(q) \log R(q) dq$$  

(7.4)

here, $q = (x_q, y_q)$ is a point on the boundary (see Fig. 3), and

$$R(q) = [(x-x_q)^2 + (y-y_q)^2]^{1/2}.$$  

(7.5)

This construction results from linear superposition of logarithmic potential functions (i.e., sources). It is readily shown (Sneddon, 1957) that if $\partial D$ has continuous curvature and $\sigma(q')$ is bounded and integrable, $\phi(r)$ is continuous for all finite points $r$ including passage through $\partial D$. If $\sigma(q')$ is continuous on $\partial D$ which itself has continuous curvature, then in the notation of Fig. 4

$$\left[ \frac{\partial \Psi}{\partial n} - \frac{\partial \phi}{\partial n} \right]_q = -\sigma(q),$$

(7.6)
\[
\left[ \frac{\partial \Phi}{\partial n} + \frac{\partial \phi}{\partial n} \right]_q = \frac{1}{\pi} \int_{\partial D} \alpha(q')[\frac{\partial}{\partial n} (\log R(q'))] dq' .
\] (7.7)

This follows from the Green's function solution to the corresponding Dirichlet problem, namely

\[
u(q) = \frac{1}{2\pi} \int_{\partial D} \alpha(q')[\frac{\partial}{\partial n} (\log R(q'))] dq' .
\] (7.8)

Now adding Eq. (7.6) and (7.7) we have

\[
2 \frac{\partial \Phi}{\partial n} \bigg|_q = -\alpha(q) + \frac{1}{\pi} \int_{\partial D} \alpha(q')[\frac{\partial}{\partial n} \log R(q')] dq' .
\]

Thus the single layer source function \( \alpha(q) \) satisfies the integral equation

\[
\alpha(q) = \frac{1}{\pi} \int_{\partial D} \alpha(q')[\frac{\partial}{\partial n} \log R(q')] dq' = -2u_q \cdot n .
\] (7.9)

For a full discussion of the applications of the theory of integral equations to Dirichlet's problem see Muskhelishvili (1953).

We approximate (7.9) by a system of linear equations. A source of strength \( \alpha(q)_{i} \) at \( Q_i \) induces at \( Q_j \), \( i \neq j \), a velocity field with components

\[
U_1(ij) = -\frac{1}{2\pi} \frac{(X_j - X_i)}{R_{ij}^2} ,
\] (7.10)

\[
U_2(ij) = -\frac{1}{2\pi} \frac{(Y_j - Y_i)}{R_{ij}^2} ,
\] (7.11)

\[
R_{ij}^2 = (X_j - X_i)^2 + (Y_j - Y_i)^2 .
\]

Now \( \alpha(q) \) is approximated by the M component vector \( \alpha = (\alpha(Q_1), \ldots, \alpha(Q_M)) \), which must in turn satisfy the matrix equation

\[
A \alpha = b ,
\] (7.12)
where $b$ has components that are the values of $-u_\xi \cdot n$ computed at the points $Q_i$. The components of the matrix $A$ are given by

$$a_{ij} = U_1(ij)n_1 + U_2(ij)n_2 \quad (i \neq j),$$

$$a_{ii} = \frac{1}{2h} \quad i = 1, \ldots, N.$$  

The velocity due to the distribution of sources is found by summing up the contribution of each component:

$$u_D(x) = \sum_{i=1}^{M} u_D(i),$$

(7.13)

where

$$u_D(i) = \begin{cases} 
\frac{1}{2\pi} a(Q_i) \frac{r(Q_i)}{r^2(Q_i)} & \text{if } r(Q_i) \geq \frac{1}{2}h \\
\frac{1}{2h} a(Q_i) n(Q_i) & \text{if } r(Q_i) < \frac{1}{2}h 
\end{cases}$$

(7.14)

where $r(Q_i) = |r(Q_i)|$, and $n$ is a unit normal vector (see Fig. 4 for notation).

Summing Eq. (7.13) with (7.14) we obtain the velocity at some arbitrary point $P(x_p(i_p),y_p(i_p))$. Writing this in component form we find

$$u(i_p) = \frac{1}{2\pi} \sum_i a(Q_i) \left( \frac{(x_p(i_p) - x_i)}{r^2(Q_i)} \right)$$

$$+ \frac{1}{2\pi} \sum_i 2a(Q_i)X_i + u_\xi(i_p) + U$$

(7.15)

$$v(i_p) = \frac{1}{2\pi} \sum_i a(Q_i) \left( \frac{(y_p(i_p) - y_i)}{r^2(Q_i)} \right)$$

$$+ \frac{1}{2\pi} \sum_i 2a(Q_i)Y_i + v_\xi(i_p) + V$$

(7.16)

where
\[ r^2(Q_i) = [(x_p(i_p)-x_i)^2 + (y_p(i_p)-y_i)^2] , \]

\( \Sigma_1 \) is for \( r(Q_i) \geq \frac{1}{3}h \)

\( \Sigma_2 \) is for \( r(Q_i) < \frac{1}{3}h \)

\( i \) indicates points on \( \partial D \)

\( U \) is the gas free stream velocity in the x-direction, \( (U=1) \)

\( V \) is the gas free stream velocity in the y-direction, \( (V=0) \).

We now have at hand the means to find \( u_p(i_p) \) and \( v_p(i_p) \) and hence advance the particle vortex blob \( \xi_{p1} \). First we rewrite (5.7) in terms of the \( i_p \) particle vortex blob velocity

\[
\frac{Du_{\xi_p(i_p)}}{dt} = \frac{g}{\lambda} (u(i_p)-u_{\xi_p(i_p)}) , \quad (7.17)
\]

\[
\frac{Dv_{\xi_p(i_p)}}{dt} = \frac{g}{\lambda} (v(i_p)-v_{\xi_p(i_p)}) . \quad (7.18)
\]

This follows from Kelvin's circulation theorem as described in section 5.

Equations (7.17) and (7.18) are integrated using a Runge-Kutta fifth order integration scheme with variable step size to preserve accuracy near boundaries. We assume over a time step \( k \) that \( g = g(Re_p) \) is constant. A test case of \( \lambda = 0.0 \) and \( Re = 100,000 \) showed this to be true even when approaching a boundary in a normal direction.

The Runge-Kutta integration scheme determines the blob velocity through one time step; once the blob velocity is known the position after the time step is found by approximating (5.8) and (5.9) with
or more accurately by integrating (5.8) and (5.9) directly using the Runge-Kutta scheme. In the computer program* equations (7.17), (7.18), (5.8) and (5.9) are integrated simultaneously yielding the approximate solution to (4.39).

Boundaries are handled by keeping track of gas and particle vortex blobs; once a blob crosses the boundary it is destroyed. In this way the boundary conditions (2.1) and (2.2) are satisfied.

The particle continuity equation (4.38) is satisfied by analyzing the rate at which mass accumulates on the boundary or any other arbitrary boundary. If the particle vortex blobs moved with the velocity of the undisturbed gas flow and were not deflected by the disturbance due to the presence of an object, the rate at which mass would accumulate on the boundary would be \( LU_p \), where \( L \) is the maximum dimension of the object measured perpendicular to the gas free streamlines. If \( \dot{M} \) is the rate at which mass actually accumulates, the rate of accumulation can be found by tracing the paths (i.e., following the streamline) of all the particle vortex blobs from the points \((X,Y)\) where the initial conditions are assumed, to the points \((X_o,Y_o)\) where the paths cross the boundary of the object \((\partial D)\). We have

\[
\frac{\dot{M}}{U_p \alpha} = \frac{\dot{M}}{U_p \alpha} = \frac{dY}{ds} = \frac{dY_o}{ds} \cdot \frac{dY}{dY_o} \cdot \frac{dY_o}{ds},
\]

(7.21)

where \( s \) is the non-dimensional distance around the contour from some fixed origin. For the circular cylinder which will be considered in section 9, we have \( s = \theta \). Then the rate at which mass accumulates on

*The computer program is available from the author.
the arc from \( \theta = 0 \) to \( \theta \) is

\[
\int_0^\theta \mathbf{M} \mathbf{r} \, \mathrm{d}\theta = \mathbf{U}_p \mathbf{R} \mathbf{Y}
\]

since from equation (7.21)

\[
\mathbf{M} = \frac{\mathbf{U}_p}{\mathbf{R}} \frac{\mathrm{d}Y}{\mathrm{d}\theta}.
\]

The relative number flux (non-dimensional) is

\[
\frac{J_0}{J_\infty} = \frac{1}{R} \frac{\mathrm{d}Y}{\mathrm{d}\theta},
\]

where

\[
J = \text{Number of Particles per unit area-unit time}.
\]

We are now in a position to estimate the order of magnitude of the error associated with the approximate solution. Chorin (1973) has conjectured that the mean error in the gas flow is \( O(k) + O(Re^{-\frac{1}{2}}) \), where the first term is the error in the deterministic technique used to solve Euler's equations. The second term arises in the random walk solution of the diffusion equation. The standard deviation of \( \eta_1 \) and \( \eta_2 \) is \( (2k/Re)^{\frac{1}{2}} \). After \( m \) steps the random motion of the vortex blobs will displace the location of the vortex by an amount of order

\[
(m2k/Re)^{\frac{1}{2}} \sim O(Re^{-\frac{1}{2}}).
\]

The error in the particle vortex blob may now be estimated. Far from a boundary the flow may be considered inviscid; here, the particle vortex blob is advanced by integrating between each time step taken in advancing the gas vortex blobs. In this way the errors introduced can always be made negligible compared to the error of \( O(k) + O(Re^{-\frac{1}{2}}) \) of the gas vortex blobs. Thus we estimate, since particle vortex motion is determined by gas vortex motion, that the error far from boundaries
is order $O(k) + O(Re^{-\frac{1}{2}})$ or less.

Now, close to a boundary the randomly positioned gas vortex blobs give a randomness to the gas flow field, and convergence on the solution can only be expected by averaging over a large area. Furthermore, with each time step we expect roughly half of the newly created gas vortices to jump randomly over the boundary to be destroyed. Thus we expect the boundary layer to be noisy with convergence occurring over long time averages.

Although this analysis forebodes a horrific picture of particle vortex blobs drifting into a boundary layer abyss of random perturbations, in reality the particle blobs fare quite well due to their high "inertial" effect.

As described in section 4, for values of the momentum equilibration number of order unity, the vortex motion of the particle depends upon its entire history. In this way the particle vortex blobs move through the gas blobs with a motion determined largely by their previous history. In essence, the particle vortex blobs sample a space-time average of the gas vortex blobs. A simple error analysis will make this notion more precise.

Consider the effect of one gas vortex displaced from its correct position by an amount $\Delta E$. We consider then two cases: first the displacement is in line with the particle vortex blob's path (see Fig. 5); and second the displacement is perpendicular to the approaching particle vortex blob (see Fig. 6). We now write the particle vortex velocity and gas velocity in polar coordinates. Equation (4.35) can be integrated directly if we assume the gas velocity to be a constant over one time step and $g(Re_p) = 1$ yielding:
This equation clearly indicates the importance of the initial conditions; for $\lambda \to \infty$ the motion of the particle vortex blob is determined exactly by its initial conditions and consequently any errors in the gas velocity have no effect on the particle phase solution. For smaller values of $\lambda$ we consider the two cases discussed above. Since the velocity induced by one gas vortex is in polar coordinates

$$v_\theta = \frac{\xi}{2\pi r} ,$$

the relative error in the gas velocity is given by

$$E = \frac{\frac{\xi}{2\pi r} - \frac{\xi}{2\pi (r+\Delta_E)}}{\frac{\xi}{2\pi r}} = 1 - \frac{r}{r+\Delta_E} .$$

Now in the first case (Fig. 5) the particle vortex blob has an initial velocity in the $\theta$-direction of zero, since it is moving directly towards the gas vortex. We expect the induced error to be largest in this case. The error in the particle vortex velocity is found using (7.24)

$$E_p = \frac{v_p \theta - v_p \theta (\text{error})}{v_p \theta} = \frac{\left[\frac{\xi}{2\pi r} - \frac{\xi}{2\pi (r+\Delta_E)}\right][1-e^{-k/\lambda}]}{\frac{\xi}{2\pi r} [1-e^{-k/\lambda}]} = E .$$

Thus the error is identical to the error in the gas velocity.

In the second case (Fig. 6) the particle vortex blob has a high initial $\theta$-velocity (non-dimensional). Assuming the initial velocity is of order unity we have
\[
E_p = \frac{\left[ \frac{\bar{g}}{2\pi} - \frac{\bar{g}}{2\pi(r+\Delta E)} \right] \left[ 1 - e^{-k/\lambda} \right] \left[ 1 - e^{-k/\lambda} \right] + e^{-k/\lambda}}{\frac{\bar{g}}{2\pi} \left[ 1 - e^{-k/\lambda} \right] + e^{-k/\lambda}}
\]

As discussed previously if \( \lambda \to \infty \), then \( E_p \to 0 \). Thus these simple error analyses indicate that the error introduced in the particle vortex blobs is always less than or equal to the error present in the gas vortex blob. As \( \lambda \to 0 \) we expect the particles to follow fluid streamlines and hence the error in their position will be identical to the error in fluid element locations, which is of order \( O(k) + O(Re^{-\frac{1}{2}}) \). For example, if \( \lambda = 1 \) the error in the particle vortex blobs is 18% of the gas vortex blob's error. However, since the particles deviate from fluid streamlines near boundaries, we expect the particle vortex blobs to sample an ensemble space average of the gas vortex blobs. Thus this simple analysis based on one displaced vortex is a worst case example. In practice, the displacement error is randomly dispersed in all directions among the many gas blobs; thus a particle blob samples the mean spatial error which can vary from zero to its largest value of \( O(k) + O(Re^{-\frac{1}{2}}) \) depending on the value of the momentum equilibration parameter, \( \lambda \).
8. **Exact solution near a stagnation point**

There are few exact solutions of two phase gas-solid particle flows because of the non-linearity of the governing equations. See Soo (1967) for a description of several exact solutions, Boothroyd (1971) for a solution of particle trajectories in a fluid vortex, and Neilson and Gilchrist (1968) for a gas-particle nozzle flow solution. Laitone (1979a) presented an exact solution for inviscid gas-particle flow into a plane stagnation point.

We are interested in applying the vortex numerical method to flow about a circular cylinder. Therefore, an exact solution for limiting values of the pertinent parameters, namely the Reynolds number, is desirable to compare with the numerical experiments.

When the flow Reynolds number about the cylinder is high enough so that potential theory may be used to solve for the gas flow, it will be shown that a transformation exists that reduces the particle partial differential governing equation to a second order linear ordinary differential equation. The reduction from P.D.E. to O.D.E. occurs only in the neighborhood of the forward stagnation point of the cylinder.

The equations of particle vortex motion in a steady gas flow (gas velocity: \((u,v)\)) have been derived in Eq. (5.7) and are presented here again in component velocity form and with dimensional variables*

\[
\frac{Du}{Dt} = \frac{u-u_p}{\tau}, \quad (8.1) \\
\frac{Dv}{Dt} = \frac{v-v_p}{\tau}, \quad (8.2)
\]

where

*In this section (§8) all variables are dimensional unless otherwise specified.*
\[ \tau = \frac{2p_{\text{p}} \sigma^2}{9 \mu} \]

Since \( u_p = \frac{Dx}{Dt} \) and \( v_p = \frac{Dy}{Dt} \), substitution into equations (8.1) and (8.2) yields

\[ \tau \frac{d^2 x}{dt^2} + \frac{dx}{dt} = u = 0 \] \hspace{1cm} (8.3)

\[ \tau \frac{d^2 y}{dt^2} + \frac{dy}{dt} = v = 0 \] \hspace{1cm} (8.4)

Now consider the inviscid flow near the front stagnation point. The complex potential is given outside the thin boundary layer by:

\[ W = -\frac{1}{2} \frac{2U}{R} z^2 \]

where \( R \) is the cylinder radius. The velocity components corresponding to this flow are

\[ u = -\frac{2U}{R} x \] \hspace{1cm} (8.5)

\[ v = \frac{2U}{R} y \] \hspace{1cm} (8.6)

Make the following transformation to non-dimensional quantities; let

\[ z = \frac{t}{\tau} \quad \lambda = \frac{\tau U}{R} = \frac{2}{9} \frac{p\ U R \sigma^2}{\rho \ v R^2} \]

where \( \lambda \) is the momentum equilibration length number based on cylinder radius (\( \lambda \) is a similarity variable for this particular two-phase flow). Physically, \( \tau \) is the time required for a particle to reduce its initial slip velocity by \( e^{-1} \).

Equations (8.3) and (8.4) with (8.5) and (8.6) become
\[
\frac{d^2 x}{dz^2} + \frac{dx}{dz} + 2\lambda x = 0 , \quad (8.7)
\]
\[
\frac{d^2 y}{dz^2} + \frac{dy}{dz} - 2\lambda y = 0 . \quad (8.8)
\]

**SOLUTION PROCEDURE**

Equation (8.8) has solution

\[
y = A_1 e^{r_1 z} + A_2 e^{r_2 z} , \quad (8.9)
\]

\[
r_1 = -\frac{1}{2} + \frac{1}{2}(1+8\lambda)^{\frac{1}{2}} , \quad r_2 = -\frac{1}{2} - \frac{1}{2}(1+8\lambda)^{\frac{1}{2}} . \quad (8.10)
\]

Equation (8.7) has solutions of different form depending on the value of \( \lambda \).

**Case \( \lambda < \frac{1}{2} \)**

Here

\[
x = A_3 e^{r_3 z} + A_4 e^{r_4 z} , \quad (8.11)
\]

\[
r_3 = -\frac{1}{2} + \frac{1}{2}(1-8\lambda)^{\frac{1}{2}} , \quad r_4 = -\frac{1}{2} - \frac{1}{2}(1-8\lambda)^{\frac{1}{2}} . \quad (8.12)
\]

To solve for \( A_3 \) and \( A_4 \), assume an initial condition that at the plane \( x = X \) the particles have the gas stream velocity

\[
u_p = -U = \frac{dx}{dt} = \frac{1}{r} \frac{dx}{dz} . \quad (8.13)
\]

Now if \( t = 0 \) (\( z = 0 \)) when \( x = X \), then from (8.11) \( X = A_3 + A_4 \) and Eq. (8.13) gives:

\[
(r_3 + 2\lambda)A_3 + (r_4 + 2\lambda)A_4 = 0 .
\]

Thus

\[
x = \left[ \frac{r_3 + 2\lambda}{r_4 - r_3} \right] e^{r_3 z} + \left[ \frac{r_3 + 2\lambda}{r_3 - r_4} \right] e^{r_4 z} . \quad (8.14)
\]
The particles that are at some initial position $X$ traveling with the gas will hit the plate, provided that $x = 0$ for some finite $z$. Otherwise the particles will only approach the plate asymptotically. Thus

$$z_s = \frac{1}{r_3 - r_4} \ln \left[ \frac{r_3 + 2\lambda}{r_4 + 2\lambda} \right].$$

Now $r_3 - r_4 = (1 - 8\lambda_m)^{\frac{1}{2}} > 0$ since $0 \leq \lambda_m < \frac{1}{6}$. The following equation is obtained by substituting for $r_3$ and $r_4$:

$$\frac{r_3 + 2\lambda}{r_4 + 2\lambda} = 1 + \frac{(1 - 8\lambda)^{\frac{1}{2}}}{\lambda^{\frac{1}{2}} - \frac{1}{2} (1 - 8\lambda)^{\frac{1}{2}}}.$$

This shows that

$$\frac{r_3 + 2\lambda}{r_4 + 2\lambda} < 1$$

and therefore

$$z_s = \ln \frac{r_3 + 2\lambda}{r_4 + 2\lambda} < 0.$$

Thus, $z_s < 0$ and at no time will the particles strike the plate.

**Case $\lambda > \frac{1}{6}$**

Here

$$x = A_6 e^{-\frac{1}{2}z} \cos (2\lambda - \frac{1}{4})^{\frac{1}{2}} z + A_6 e^{-\frac{1}{2}z} \sin (2\lambda - \frac{1}{4})^{\frac{1}{2}} z. \quad (8.15)$$

Assume an initial condition at $t = 0$ ($z = 0$) $x = X$ and

$$u_p = \frac{dx}{dt} = -U = \frac{1}{\tau} \frac{dx}{dz}. \quad (8.16)$$

Then Eq. (8.15) gives $A_6 = X$

$$A_6 = \left[ (2\lambda - \frac{1}{4})^{\frac{1}{2}} \right] X.$$
Thus

\[
\frac{X}{x} = e^{-\frac{3}{2}z} \left[ \cos(2\lambda - \frac{1}{4})^z_2 + \frac{\frac{1}{2} - 2\lambda}{(2\lambda - \frac{1}{4})^z_2} \sin(2\lambda - \frac{1}{4})^z_2 \right].
\]  

(8.17)

When \( x = 0 \)

\[
\tan(2\lambda - \frac{1}{4})^z_2 = \frac{(2\lambda - \frac{1}{4})^z_2}{(2\lambda - \frac{1}{4})^z_2}.
\]  

(8.18)

As \( \lambda \) increases from \( \frac{1}{6} \) to \( \frac{1}{4} \), \( z_s \) decreases from \( \pi \) to \( \pi/2 \). As \( \lambda \) increases from \( \frac{1}{4} \) to \( \infty \), \( z_s \) decreases from \( \pi \) to \( 0 \).

Thus if \( \lambda > \frac{1}{6} \), the particles will always strike the plate, provided they are at some \( x = X \) moving with the gas.

**Case \( \lambda = \frac{1}{6} \)**

Here

\[
x = A_7 e^{-\frac{3}{2}z} + A_8 2e^{-\frac{1}{2}z}.
\]  

(8.19)

Assume an initial condition at \( t = 0 \)

\[
(z = 0) \ x = X \ and \\
\ u_p = \frac{dx}{dt} = -U = \frac{1}{\tau} \frac{dx}{dz}.
\]

Then Eq. (8.19) gives \( A_7 = X \), \( A_8 = \frac{1}{4}X \). Then

\[
\frac{X}{x} = e^{-\frac{3}{2}z(1+\frac{1}{4})}.
\]  

(8.20)

The solution to Eq. (8.9) gives the y path.

Assume the initial condition \( y = Y \) when \( x = X \) at \( z = 0 \). Then, from Eq. (8.9)

\[
Y = A_1 + A_2.
\]  

(8.21)
For the second condition we consider two cases:

**Case A)** \( v_p = 0 \)

Then

\[
0 = A_1r_1 + A_2r_2 .
\]

(8.22)

Thus

\[
A_2 = \frac{y r_1}{r_1 - r_2} ,
\]

\[
A_1 = Y \left[ \frac{-r_2}{r_1 - r_2} \right] .
\]

Thus

\[
\frac{r_1}{Y} = \left( -r_2 e^{-r_1} + r_1 e^{-r_2} \right)/r_1 - r_2 .
\]

(8.23)

See Fig. 7.

**Case B)** The particles follow the streamline

Initially

\[
v_p = v = \frac{2UY}{R} = \frac{dy}{dt} = \frac{1}{\tau} \frac{dy}{dz} .
\]

Combining this with Eq. (8.21) we obtain

\[
A_1 = Y \left( \frac{2\lambda - r_2}{r_1 - r_2} \right) ,
\]

\[
A_2 = Y \left[ 1 - \frac{2\lambda - r_2}{r_1 - r_2} \right] .
\]

The y trajectories are therefore given by

\[
\frac{Y}{Y} = \frac{2\lambda - r_2}{r_1 - r_2} e^{-1/r_2} + \frac{1}{r_1 - r_2} e^{2\lambda} .
\]

(8.24)

**METHOD OF DETERMINING IMPACT DENSITY, SPEED AND ANGLE**

As the momentum equilibration time \( \tau \) increases (i.e., as \( \lambda \) increases), \( z \rightarrow 0 \). In the limit \( \tau \rightarrow \infty \), if \( Y_0 \) is the y value for the particle striking the plane \( x = 0 \), we find that in case A) \( Y_0 / Y \rightarrow 1 \), and in case B) \( Y_0 / Y \rightarrow 1 \).
From Eqs. (8.23) and (8.24) we see that $Y_0/Y$ is a function of $\lambda$ only. Thus the number density of particles along the wall ($x=0$) is constant for a given $\lambda$. Since all particles for which $y/Y<1$ will strike the plane at a distance less than $Y_0$, we arrive at the result

$$J_0 = \frac{Y}{Y_0} N_{p\infty} U = \frac{Y}{Y_0} J_\infty,$$  \hspace{1cm} (8.25)

where

$$N_{p\infty} = \frac{\text{Number of particles in free stream}}{\text{unit volume}},$$

$$U = \text{Free stream gas velocity},$$

$$J_0 = \frac{\text{Number of particles striking wall}}{\text{unit area unit time}}.$$  \hspace{1cm} (8.26)

The variation of $J_0/J_\infty$ with $\lambda$ is shown in Fig. 8. For particles for which initially $y/Y = \beta$ ($0 \leq \beta \leq 1$) and $\lambda > \frac{1}{6}$ the trajectories in case A) are given by

$$\frac{x}{X} \Big|_{\beta=\beta} = e^{\frac{1}{\beta^2} \left\{ \cos(2\lambda-\frac{1}{6}) \frac{1}{\beta^2} + \left[ \frac{1}{2} - 2\lambda \left( \frac{1}{6} \right) \frac{1}{\beta^2} \right] \sin(2\lambda-\frac{1}{6}) \right\}} = \frac{x}{X} \Big|_{\beta=1},$$  \hspace{1cm} (8.27)

$$\frac{v}{Y} \Big|_{\beta=\beta} = \beta \left[ \frac{1}{2} + \frac{1}{2} \left( 1 + 8\lambda \right) \frac{1}{\beta^2} \right] e^{\left[ \frac{1}{2} - \frac{1}{2} \left( 1 + 8\lambda \right) \frac{1}{\beta^2} \right] z} + \left[ -\frac{1}{2} + \frac{1}{2} \left( 1 + 8\lambda \right) \frac{1}{\beta^2} \right] e^{\left[ -\frac{1}{2} + \frac{1}{2} \left( 1 + 8\lambda \right) \frac{1}{\beta^2} \right] z} / (1 + 8\lambda)^{\frac{1}{2}} = \beta \frac{v}{Y} \Big|_{\beta=1}. $$  \hspace{1cm} (8.28)

Furthermore, the speed on impact $q$ (letting $\gamma = X/Y$) is given by (see Fig. 9)

$$q = \frac{1}{2\lambda} \left\{ \left[ \frac{1}{X} \frac{dx}{dz} \right]_{z=z_S} \right\}^2 + \frac{\beta^2}{\gamma^2} \left[ \frac{1}{Y} \frac{dy}{dz} \right]_{z=z_S} \right\} z^{2\frac{1}{2}},$$  \hspace{1cm} (8.28)

where $z_S$ is given in Eq. (8.18).

When $x=0$, $z=z_S$ and $y=Y_0$, and
\[
\frac{1}{X} \frac{dx}{dz} = e^{-\frac{1}{2}\beta z} \left\{ \frac{-\lambda}{(2\lambda - \frac{1}{2})^2} \sin \left( (2\lambda - \frac{1}{4})^2 z \right) - 2\lambda \cos \left( (2\lambda - \frac{1}{4})^2 z \right) \right\},
\]

\[
\frac{1}{Y} \frac{dy}{dz} = \frac{r_1 r_2}{r_1 - r_2} \left\{ e^{-\beta z} - e^{-\beta z} \right\}.
\]

The angle of impact is

\[
\alpha' = \tan^{-1} \left( \frac{\frac{1}{X} \left| \frac{dx}{dz} \right|}{\frac{1}{Y} \left| \frac{dy}{dz} \right|} \right)_{z=z_s}.
\]

By assigning successive values to \( \beta \) over the interval \( 0 < y / Y \leq 1 \) and solving for a fixed \( \lambda \), the particle trajectories Eqs. (8.26) and (8.27), the impact speed Eq. (8.28), and the impact angle Eq. (8.29) are obtained for a distribution of similar particles. Since the particles are all of the same size and density, for a given distribution no particle-particle collision will occur before impact with the wall. However, this model does not take into account particles that rebound from the wall.

The relative impact speed is plotted in Fig. 9 as a function of the momentum equilibration parameter.

The intriguing shape of the relative impact speed curve shown in Fig. 9 can be explained by examining the velocity components of the particles. As is expected, particles with large \( \lambda \) are relatively unaffected by the continuous phase, and travel in straight lines with trajectories determined by the initial conditions. As \( \lambda \) decreases to a value of 1.0, the particles experience a very slight deceleration through the continuous phase in the \( x \) direction, along with an acceleration in the \( y \) direction due to the continuous phase accelerating away from the stagnation point. The overall increase in speed and erosion then is due to the magnitude of the vector sum of these two relatively large components.
Now for $\lambda$ decreasing from 1.0 to 0.2, the particles experience a large deceleration in the $x$ direction due to the increasing effect of the gas viscosity, leading to a small $u$ velocity component and very low impact speeds.

This trend is suddenly reversed as $\lambda$ decreases further toward 0.125. Here the particles are entrained in the continuous phase over a long time interval, and consequently are accelerated with the continuous phase in the $y$ direction. The impacts occur far from the stagnation point with the large $v$ velocity component dominating the impact speed, and giving rise to the increased relative impact speed. As noted previously, if $\lambda \leq \frac{1}{8}$ the particles will not impact.

To compare the stagnation point flow approximation to the solution given about the cylinder by potential theory, the particle trajectories were calculated using Eqs. (8.1) and (8.2) with the gas velocity given by

$$u = 1 - \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad (8.30)$$

$$v = \frac{-2xy}{(x^2 + y^2)^2}. \quad (8.31)$$

The equations were integrated using a Runge-Kutta fifth order scheme with variable step size. The values of velocity, angle and position of impact were recorded for values of $\lambda$ from 0.1 to 6. Particles with $\lambda < \frac{1}{8}$ did not impact and were deflected over the cylinder. This confirms the analytical prediction (8.15)-(8.18) that for $\lambda < \frac{1}{8}$ no particles would meet the surface.

Figure 7 shows the numerical data points for $\lambda = 1.0$; the gas velocity given by the stagnation point flow differs from the potential flow in (8.30) and (8.31) by less than 1% here. The $y$ impact coordinate
for $\lambda = 1.0$ from the stagnation point flow analytic solution differs from the $y$ impact coordinate for the potential flow numerical solution by a distance of $\Delta y/R = 10^{-4}$, which is an error of $13\%$. For particle paths closer to the stagnation point the stagnation flow approximation improves.

It should be pointed out that these conditions arise in many two-phase flows. For instance in the domain of the pressure vessel of a coal gasifier the gas velocity is high enough so that the inviscid flow approximation is valid. However, the particle's size and slip velocity through the gas is small enough so that the Reynolds number based on particle diameter is less than unity. In this case the particles experience a highly viscous flow locally, while the bulk gas flow can be treated as inviscid.
9. Application to flow about a circular cylinder

The origin is taken at the center of a fixed cylinder with a non-dimensional radius of 1. The negative x axis is parallel to the undisturbed stream. The flow is from left to right; at time $t = 0$ the flow is started with constant non-dimensional velocity of magnitude 1 in the x direction. Thus the velocity at position $(-\infty,0)$ is $(1,0)$. The boundary of the domain, $\partial D$, is the circumference of the cylinder.

The circumference is divided into $M = 20$ pieces of length $h = 2\pi/M$. The time step is $k = 0.2$. The value of $k$ is chosen so that a decrease in $k$ does not affect the flow. The time step must also be small enough so that the particle equations can be integrated without an excessive number of derivative evaluations. Furthermore, since information concerning the particle vortex position and velocity is only printed at the beginning and end of each time step, the particle path between steps must be small enough to be approximated by a straight line. This allows the impact angle and impact speed (computed vectorially) to be computed accurately. The value of $k = 0.2$ proved accurate for the range of Reynolds numbers of interest.

Once $k$ is chosen, $M$ must be selected large enough so that any increase in $M$ does not change the solution. $M$ must be increased for decreasing values of $k$ because decreases in $k$ give the gas vortices a higher probability of crossing the boundary of the cylinder and being eliminated. We require a minimum number of gas vortices present beyond the boundary; thus more gas vortices need be created on the boundary as some vanish.

The average drag coefficient (averaged over 120 time steps) was calculated using a scheme outlined by Chorin (1973). The gas vorticity
near the boundary is sampled and from this the skin friction and pressure drag contribution to the total drag are evaluated. At Re = 100 the average drag is \( C_D = 2.02 \), the experimental value is 1.9 (Schlichting 1960, p. 16); at Re = 1,000, \( C_D = 1.04 \), the experimental value is 1.00; at Re = 10,000, \( C_D = 0.87 \), the experimental value is 1.05. Chorin (1973) conjectures that the discrete number of vortices roughly representing a smooth boundary layer trips prematurely the drag crisis, much like a rough wall does. The conjecture is apparently confirmed because at Re = 100,000, \( C_D = 0.29 \), the experimental value is 0.28 beyond the drag crisis.

For values of \( \lambda \geq 1 \), it was assumed that at points along the line \( x/R = -X/R \) the particles were traveling with the gas and therefore \( u_p = (1,0) \). For \( \lambda < 1 \) it was assumed that at points along the line \( x/R = -X/R \) the particles were traveling with the gas at that point, i.e., \( u_p = u = (u,v) \). A test case with \( X/R = 3 \) and \( \lambda = \frac{1}{3} \) showed after a small interval of time the paths of the particles on either assumption were the same.

Figure 10 shows the computed data points for Re = 10,000 and \( \lambda = 0.5, 2 \) and 4. The cylinder has been expanded in the y direction to clearly show the particle deflection. The particles, due to the difference in their inertia, are driven away from the gas streamlines and impact with the cylinder. The higher values of \( \lambda \), corresponding to bigger or heavier particles, yield particle paths affected less by the gas flow acceleration away from the stagnation point. These larger particles follow nearly straight line trajectories.

In Fig. 11 the particle paths are shown for different \( y/R = Y/R \) values of the particle initial position. The upper particle path, which starts at \( y/R = 0.5 \), does not impact with the cylinder, but passes over the top.
By distributing a large number of particle vortex blobs between $y/R = 0$ and $y/R = 1$, the angle of impact and particle phase density can be calculated as a function of distance along the surface or cylinder angle, $\theta$. Figure 12 shows the difference between true and apparent impact angles which result from the non-linear particle paths. In the limiting case of $\lambda \to \infty$, Re $\to \infty$ (i.e., straight line paths), the true impact angle ($\alpha'$) is the same as the apparent impingement angle ($90 - \theta$) between the tangent to the cylinder at the point of impact and the gas flow direction. Particles deflected by the fluid, however, have true impact angles between the tangent to the cylinder and the tangent to the particle path at the point of impact. For all except straight line impacts the true impact angle is less than the apparent impingement angle.

Near the stagnation point the vortex method and potential theory yield almost the same impact angles. However, the effect of viscosity becomes apparent for particles impacting at cylinder angles, $\theta$, greater than $10^\circ$. The effect of viscosity is to produce a boundary layer and displacement thickness outside of which potential theory adequately describes the flow. But the displacement thickness acts to increase the effective radius of the cylinder. This effect can be seen more clearly in Fig. 13, where a particle with $\lambda = 1.0$ is started from the same position ($y/R = 0.46$) in a Re $= 1,000$ and inviscid flow. Using potential theory we compute numerically that the particle impacts at $\theta = 55^\circ$. Using the vortex method we find the particle is deflected completely around the cylinder with no impact. A particle started in a flow with Re $= 1,000$ but an initial $y/R$ position slightly less than 0.46 will impact at a value of $\theta$ no larger than $44^\circ$, as indicated in Fig. 13. It should be noted that although we refer to a particle path,
the trajectory represents the collective average motion of the many particles within a small neighborhood of the particle vortex blob.

Using a distribution of particles as in Fig. 11, the relative number density flux at the cylinder surface is found using (7.23). This information (see Fig. 14) is useful in determining the distribution of erosion about the cylinder.

The effect of the displacement thickness is more clearly presented in Fig. 15 for \( \lambda = 0.1 \) and \( Re = 1,000 \). In the viscous case the particle path is seen to be displaced away from the cylinder in a uniform way compared to the inviscid case. This analysis indicates a more correct method for utilizing the simplifying assumption of inviscid flow for a particular problem. First an average displacement thickness, \( \overline{\delta}^* \), should be calculated from boundary layer theory for the particular geometry and Reynolds number under consideration. Next the characteristic dimension of the system, \( L \), should be increased by \( \overline{\delta}^* \). Finally the inviscid problem should be solved using an equivalent value of the momentum equilibration parameter, namely

\[
\lambda = \frac{2}{9} \frac{\rho \sigma^2}{\mu} \left( \frac{U}{L + \overline{\delta}^*} \right)
\]  

(9.1)

This will give particle paths corresponding more closely to the viscous case.

Other researchers have ignored displacement thickness effects, assuming inviscid flow (Tilly 1969 and Pettit 1977). One can argue that because the particle residence time in the thin boundary layer is negligibly small that the trajectories will not be substantially affected by its presence. This is because the work done on a particle by the boundary layer depends on the distance traversed which is very small, i.e., \( \delta \sim O(Re^{-1}) \). However, this overlooks the obvious fact that the
boundary layer creates a displacement thickness altering the apparent size of the object and consequently the inviscid flow. Even more significant is the separation that may occur. This drastically alters the flow profile and particle trajectories.

In Fig. 16 the collection efficiency of a cylinder is given for various values of the momentum equilibration number for the viscous and inviscid case. The collection efficiency, a number quoted frequently in the industrial literature, is the ratio of the number of particles impacting with the object to the number which would impact if they followed straight line trajectories without deflection by the gas. Clearly the effect of viscosity is to reduce the collection efficiency for a given value of $\lambda$. This is directly a result of the increased apparent size of the object deflecting more particles. As indicated in Fig. 16 the discrepancy between the viscous and inviscid case is negligible for larger particles where $\lambda > 2.0$.

We now turn attention towards the back shoulder of the cylinder and the wake region. Shortly after the start of fluid motion the external pressure field causes fluid transversing the rear shoulder to reverse its direction. The reverse motion moves forward and the boundary layer thickens. This motion gives rise to a vortex which increases in size, until it separates from the cylinder and moves downstream. At a distance from the cylinder a regular pattern of vortices moving alternately clockwise and counterclockwise is apparent. This is known as a Kármán vortex street. When viewed in a frame traveling with the vortex street system, the streamlines between the vortices have a sinusoidal appearance.

This same effect is visible in the particle paths in the wake region, as calculated numerically in Fig. 17 and 18 for $Re = 1,000$ using the vortex method. The larger values of $\lambda$ are affected less dramatically
as are the smaller values of \( \lambda \); however, the sinusoidal particle streamlines are obvious. Potential theory is of course entirely inadequate in predicting the particle behavior in this region behind the cylinder (see Fig. 19). The vortex method predicts separation occurring asymmetrically, and produces the vortex street behind the cylinder. It requires about 20 seconds of CDC 7600 computer time to follow ten particle vortices to the cylinder and about 17 minutes to follow the evolution from \( t = 0 \) to \( t = 50 \). In Fig. 19 the path of a particle completely entrained in the fluid flow is indicated. Here \( \lambda = 0.1 \) and the motion is identical to fluid elements. This type of sub-micron size particle is used in laser-doppler anemometry techniques to determine the fluid velocity.

By numerical means, the particle velocity was actually observed to decelerate to rest at a point behind the cylinder. The gas velocity was checked at this point and also found to be zero, thus indicating a rear detached stagnation point. Experiments filmed by Prandtl at Göttingen with pollen dusting the surface of fluid flow about a cylinder are available through Encyclopedia Britannica (1969). The film provides a dramatic visualization of vortex growth and shedding. A close inspection shows small pollen particles coming to rest at a rear detached stagnation point. Much work is being done with computerized flow visualization using Chorin's vortex method (Ashurst 1977). By following the gas vortices in real time the flow of fluid elements is reproduced. The benefits of inexpensive computer modeling compared with the expense of experiments is obvious.

It is difficult to compare the numerical results with experiments since to date two-phase, gas-solid experiments have measured only secondary effects such as erosion of surfaces. With the recent advances in laser-doppler techniques for gas-solid flow it is hoped in future
work to measure particle and gas velocity components at any point in the flow domain.

Some experimental work suitable for comparison has been conducted on the velocity dependence of erosion. These experiments indicate that erosion varies with high exponent (typically 2 to 4) values of the gas free stream velocity. Previous quantitative erosion models do not predict these high exponent values. It was shown by Laitone (1979b) that the high exponent values are the result of aerodynamic effects and are not due to particle-surface material interaction mechanisms.

A plot of the numerically determined impact speed $q$ as a function of free stream speed $U$ shows $q \propto U^m$ where $m$ varies from 1.15 to 1.23. This is shown in Fig. 20 for 10 micron quartz particles in air.

Finnie (1972) developed a theoretical erosion model which gives excellent agreement with shallow angle impact experiments. The model gives a relationship between the impact speed of a particle (which must be deduced by solving the fluid mechanical system) and the resulting volume of surface material removed or erosion of a ductile metal. The model assumes the particles act as cutting tools with the cutting depth a function of the surface material hardness. The erosion, $E_r$, is predicted to vary with impact speed squared, i.e., $E_r \propto q^2$. The results in Fig. 20 are the solution to the fluid mechanical system for particles impacting high on the front shoulder, i.e., impact angles with $\alpha \leq 20^\circ$. The vortex method predicts $q \propto U^{1.23}$. Combining this aerodynamic effect with the surface interaction effect predicted by Finnie, we arrive at
In erosion experiments the impact speed is not measured, however the gas velocity, \( U \), and hence particle velocity, \( u \), is measured far from the body. Grant and Tabakoff (1975) have conducted experiments with flat plates at shallow angles of attack \( (\alpha = 20^\circ) \) entrained in a gas-solid flow. They find for quartz particles in air

\[
\eta \propto U^{2.8} \quad \text{Experiment} \tag{9.3}
\]

The agreement between that predicted by the vortex method (9.2) and experiment (9.3) is quite good, however more importantly it points out the importance in solving the fluid mechanical system first, before applying an erosion model. Researchers have proposed explanations for exponent values above 2.0 based on particle fragmentation (Tilly 1970) and based on indentation hardness theory (Sheldon 1972). This analysis shows the exponent values above 2.0 are due to aerodynamics alone. The vortex method presented in this study can provide researchers with a numerical solution technique suitable for a wide class of two-phase, gas-solid flows about various types of bodies.
10. **Concluding remarks**

It has been the three-fold aim of this work (1) to show the significance of the parameters and similarity groups that determine the characteristics of two-phase flow systems, and (2) to present a general solution technique applicable to a wide variety of commonly encountered geometries in dilute gas-solid flow problems, and (3) to apply the solution technique to a specific geometry and thereby indicate the type of information obtainable which may prove useful to scientists and engineers working with specific industrial systems.

The application to the cylinder demonstrates the discrepancy between the viscous and inviscid solution. At Reynolds numbers of 1,000 and higher the inviscid approximation provides an accurate solution only for values of the momentum equilibration number greater than 2.0. By applying an erosion model to the results predicted by the vortex method a good agreement was gained with the erosion found in experiments of particles impacting surfaces at shallow angles of attack.

The viscous and inviscid numerical solutions were compared to the inviscid exact solution near the stagnation point. The numerical and exact inviscid solutions agree well, however the viscous solution predicts that particles are deflected to a greater degree as a result of the displacement thickness effect. The viscous and inviscid exact solution yield the same impact speed distribution. Although the boundary layer effects the particle path, it is so thin it does not alter the particle speed, thus we expect the viscous and inviscid exact solutions to agree well near the stagnation point.

One of the problems that must be faced in the course of developing this technique to a wider class of flows is the inclusion of a two-way
momentum coupling effects. This would extend the capability of the method to include non-dilute liquid-solid flows.

The author believes the developing field of gas-particle, two-phase flows is one in which research workers in fluid mechanics may contribute significant advances by utilizing standard techniques and applying experience to the understanding of fundamental theory.
11. References


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12. Figures
Initial conditions

\[ t = 0 \]

\[ u_p(i_p) = u_p(l, 0) \]

Boundary condition

\[ \partial D: u_f = (0, 0) \]
Fig. 2

XBL 799-2782
Fig. 3
Fig. 4

Vortex blob $P(x_p(i_p), y_p(i_p))$

Particle vortex blob position
Gas vortex blob position

Displaced vortex blob position

Boundary of object

Particle vortex blob

Fig. 5

XBL 799-2777
Fig. 6

$u_p$, particle velocity

Particle vortex blob

$\Delta E$

Displaced vortex blob

Gas vortex blob position

Boundary of object

$\partial D$
Second quadrant cylinder outline

- Exact solution
  \( \text{Re} \to \infty \)
  Stagnation pt. flow approx.

- Numerical solution
  \( \text{Re} \to \infty \)
  Potential theory

- Numerical solution, vortex method
  \( \text{Re} = 100,000 \)
  \( \lambda = 1.0 \)

- Particle path when momentum equilibration number is
  \( \lambda = 0.0625 \)

- Streamlines potential theory

- Stagnation point

\( x/R \)

\( y/R \)

\( \text{Cylinder angle, } \theta (\text{deg}) \)

Fig. 7
Fig. 8

Relative number density flux, $J_0/J_0\omega$ vs. Momentum equilibration number, $\lambda$

$\beta = 1$
Eqn. (A):

\[ q = \left( u_p^2 + v_p^2 \right)^{1/2} \]

\[ C_o = (2\lambda - 1/4)^{1/2} \]

\[ u_p = -U e^{-1/2} Z / C_o \left\{ \frac{1}{2} \sin(C_0 Z) + C_0 \cos(C_0 Z) \right\} \]

\[ v_p = U e^{-1/2Z/2\lambda} \sinh(C_0 Z) / C_o \]

\[ \tan(C_0 Z) = C_o / (2\lambda - 1/2) \]

Fig. 9
Re = 10,000    K = 0.2
\( \lambda = 0.5 \) •
\( \lambda = 2.0 \) △
\( \lambda = 4.0 \) ○

Cylinder angle \( \theta \) (deg.)

Cylinder second quadrant outline

One time step between data points

Fig. 10
Re = 10,000
\( \lambda = 1.0 \)

Cylinder
angle, \( \theta \)
(deg)

Cylinder second
quadrant outline

[ORDINATE EXPANDED
TWO TIMES ABSCISSA]
Fig. 12

- \( \beta \)
- \( 90 - \theta \)
- \( \theta \)

Graph showing impact angle, \( \alpha' \) (deg) vs. cylinder angle, \( \theta \) (deg).

- \( \{ \text{Re} = 1000 \}
- \( \lambda = 1.0 \)
- \( \Delta \{ \text{Re} \to \infty \}
- \( \lambda = 1.0 \)

Limit for \( \lambda \to \infty \) and \( \text{Re} \to \infty \).
$\lambda = 1.0$

$\text{Re} = 1000$

---Potential theory
Fig. 14

$\text{Relative number density flux, } J_0 / J_\infty$

Limit for $\lambda \to \infty$

$\begin{cases} \text{Re} = 10,000 \\ \lambda = 1.0 \end{cases}$
Fig. 1.5

\[ \lambda = 0.1 \]

\[ \text{Re} = 1000 \]

--- Potential theory

\[ y/R \]

\[ x/R \]

Cylinder second quadrant outline

[ORDINATE EXPANDED TWO TIMES ABSCISSA]
Fig. 16

Momentum equilibration number, $\lambda$

Collection efficiency, $\eta_c$

Re = 1000

Potential theory

XBL 799-2769
\[ \lambda = 2.0 \]
\[ \lambda = 0.5 \]
\[ \lambda = 0.125 \]

\( R = 1000 \)

\( x/R \)

\( y/R \)
\( \lambda = 0.1 \)

Potential theory

Particle decelerates to zero velocity; gas velocity is zero.

\( \text{Re} = 100,000 \)
Impact speed, q, (cm/sec) vs. Free stream speed, U, (cm/sec)

$q \propto U^m$

Quartz particles in air
Particle diameter 10 \( \mu \)m
Cylinder radius 1 cm

Slope, m

- \( m = 1.15 \)
- \( m = 1.23 \)

Fig. 20
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