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Current and Potential Distributions on a Cylinder Electrode

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Current and Potential Distributions on a Cylinder Electrode

By

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SECTION 1: INTRODUCTION

This work presents the numerical solution for the current and potential distributions of a cylindrical electrode. In particular, it investigates the primary current distribution, the secondary current distribution with linearized kinetics, and the potential distribution due to constant current density on the working electrode. Figure 1 illustrates the idealized electrochemical cell geometry and its geometric parameters: the working electrode of length $2l_0$ and radius $r_0$ is imbedded in an infinite insulating cylinder, while the counterelectrode is at infinity.

1.1 THE PRIMARY AND SECONDARY CURRENT DISTRIBUTIONS

Laplace's equation describes the potential variation in dilute electrolytic solutions with negligible concentration gradients.

$$\nabla^2 \Phi = 0 \quad 1.1-1$$

Ohm's law gives the current density under these conditions.

$$i = -\kappa \nabla \Phi \quad 1.1-2$$

Equations 1.1-1 and 1.1-2 approximate behavior in electrochemical systems with negligible mass transfer resistance. In particular, they apply to the bulk solution for electrochemical systems in which convection dominates mass transport in the bulk and concentration gradients are confined to a thin boundary layer near the electrode surface.

1.1.1 Electrode Kinetics

The Butler-Volmer equation describes many electrochemical reactions at an electrode. For constant reactant concentration, this equation becomes:
In this equation, $i_n$ is the current density at the electrode due to the reaction, $i_o$ is a kinetic parameter called the exchange current density, $\alpha_a$ and $\alpha_c$ are kinetic parameters called the apparent transfer coefficients, $F$ is Faraday's constant, and $R$ is the universal gas constant. The surface overpotential, $\eta_s$, is the potential difference between the electrode potential, $V$, and the potential in solution adjacent to the electrode, $\Phi_0$, measured with a reference electrode identical to the working electrode.

For small surface overpotentials, one can linearize equation 1.1-3 around $\eta_s=0$, yielding:

$$i_n = i_o \left( \exp \left( \frac{\alpha_a F}{RT} \eta_s \right) - \exp \left( - \frac{\alpha_c F}{RT} \eta_s \right) \right), \quad \eta_s = V - \Phi_0. \tag{1.1-3, 1.1-4}$$

For large surface overpotentials one of the terms in equation 1.1-3 becomes negligible, depending upon the sign of $\eta_s$, resulting in the so-called Tafel approximation.

1.1.2 The Primary Current Distribution

Negligible electrode kinetic resistance implies negligible surface overpotential. Assuming no ohmic resistance in the electrode, the boundary condition at the electrode becomes:

$$\Phi_0 = V, \quad 1.1-6$$

where $V$ is a constant. The current distribution governed by Laplace's equation in the electrolytic solution and constant $\Phi_0$ at the electrode is called the primary current distribution. The primary distribution is limited only by ohmic resistance in the solution. It is characterized solely by geometric parameter ratios (1). Ohmic resistance, calculated from the primary distribution, gives the relationship between potential drop and total current in a cell limited by electrolyte ohmic resistance.
Near an electrode-insulator edge, the following equation describes the primary current distribution:

\[
\frac{i_n}{P_o} = x^{(\gamma/2\alpha - 1)} , \ x \rightarrow 0 ,
\]

where \( x \) is the distance from the electrode-insulator edge, \( \alpha \) is the electrode-insulator intersection angle, and \( P_o \) is a function of the system geometry. Current density is zero at the edge for acute angles, a constant for right angles, and infinite for obtuse angles.

1.1.3 The Secondary Current Distribution

The current distribution governed by Laplace's equation in solution and finite electrode kinetics is called the secondary current distribution. This distribution depends not only upon geometric parameter ratios, but also upon the absolute size of the system.

If one uses the full Butler-Volmer equation to describe electrode kinetics, the secondary current distribution depends upon three parameters in addition to the geometric parameters of the primary distribution (2). Due to the large number of parameters, this system is difficult to analyze. The linear, or Tafel kinetics, approximation simplifies the problem. The secondary current distribution with linear electrode kinetics introduces only one additional parameter, \( J_h \):

\[
J_h = (\alpha + \chi) \frac{i_0 F h}{kRT} ,
\]

where \( h \) is an important geometric length in the system. \( J_h \) represents the ratio of ohmic resistance to kinetic resistance. As \( J_h \) approaches zero, kinetic resistance predominates, rendering the current distribution uniform on the electrode. Conversely, as \( J_h \) approaches infinity, ohmic resistance predominates, and the current distribution approaches the primary current distribution. Tafel kinetics also involves only one additional parameter, which is proportional to the average current density on the electrode instead of the exchange current density.
1.2 POTENTIAL DISTRIBUTION DUE TO CONSTANT ELECTRODE CURRENT DENSITY

The potential due to uniform $i_n$ provides an estimate of the maximum possible potential variation on an electrode. Oxygen mass transport generally limits corrosion processes. Consequently, these constant current density calculations are relevant to cathodic protection technology, in which sacrificial anodes protect metal objects from corroding. In these systems, the largest potential difference on the protected metal object, $\Delta \Phi_o$, must be kept within a finite range to avoid both corrosion and hydrogen evolution. This restriction dictates, for example, the maximum size of the object which a counterelectrode at infinity can protect.

1.3 PREVIOUS WORK

Most nontrivial, analytic solutions to current distribution on finite cylindrical electrodes have been obtained for concentric electrodes with symmetric insulators at right angles to the axis of symmetry. Weisselberg (3) solved the primary current distribution analytically in this geometry for an inner electrode with finite ohmic resistance. Waber (4) followed with an analytic solution to the secondary current distribution with linearized kinetics. Finally, Alkire and Varjian (5) solved the problem numerically, with full Butler-Volmer kinetics, for a thin resistive wire inner electrode.

More closely related to the geometry under consideration, Strommen and Rodland (6) developed a fairly sophisticated finite difference method for calculating potential distributions on underwater pipelines protected by exterior sacrificial anodes.
SECTION 2: MATHEMATICAL METHOD

2.1 RING SOURCE IMBEDDED IN AN INSULATING CYLINDER

The following expression gives the potential at \( z \) and \( r \), due to a differentially thin ring at \( z' \) and \( r_0 \), emitting total current \( 2\pi(2i_n)r_0dz' \) (7).

\[
d\Phi = \frac{2r_0}{\pi \kappa} \frac{K(m) i_n dz'}{\left((z - z')^2 + (r + r_0)^2\right)^{1/2}} \tag{2.1-1}
\]

Here \( K \) is the elliptic integral of the first kind and \( m \) is defined as:

\[
m = \frac{4r r_0}{(z - z')^2 + (r + r_0)^2} \tag{2.1-2}
\]

Figure 2a shows the ring source described by equation 2.1-1.

One can nondimensionalize the differential potential as follows:

\[
G(R^*, Z) = \frac{2K(m)}{\left(Z^2 + (R^* + 1)^2\right)^{1/2}} \tag{2.1-3}
\]

\( G \), \( R^* \), and \( Z \) are the following groupings:

\[
G(R^*, Z) = \frac{\pi \kappa}{i_n dz'} d\Phi, \quad Z = \frac{z - z'}{r_0}, \quad R^* = \frac{r}{r_0} \tag{2.1-4,5,6}
\]

To obtain the potential due to a ring source imbedded in an infinite insulating cylinder, \( G^* \), as shown in figure 2b, Mak (8) added the following integrated sum of orthogonal functions to \( G \):

\[\text{5}\]
\[ G^* = G + F, \quad F(R^*,Z) = \int_0^\infty B(\zeta) K_0(\zeta R^*) \cos(\zeta Z) \, d\zeta. \]

Here \( K_0 \) is the modified Bessel function of the second kind of order 0. Appendix A provides a detailed derivation of equation 2.1-8 and Mak's calculation of \( F \) and \( B(\zeta) \).

The following equation approximates \( F \)'s behavior on the cylinder surface:

\[ F = -\frac{1}{2 \pi} \frac{1}{\sqrt{1 + Z^2}}. \quad 2.1-9 \]

\( G^* \) behaves in three fairly distinct fashions on the cylinder, depending on distance from the ring source. For small \( Z \), \( m \) approaches one, and \( K \) approaches infinity logarithmically.

\[ K(m) = -\ln \left( \frac{16}{1-m} \right), \quad m \to 1 \quad 2.1-10 \]

In this region, the ring source appears as a line source imbedded in an infinite insulating plane, and \( G^* \) behaves accordingly:

\[ G^* = \frac{1}{2} \ln(64) - \frac{\pi}{2} - \frac{1}{2} \ln|Z|^2, \quad Z \to 0. \]

\[ 2.1-11 \]

For distances of one radius to a finite number of radii, the ring appears as a three dimensional object and requires the full form of equation 2.1-7 to describe the potential variation. Finally, as \( Z \) approaches infinity, \( K(m) \) approaches \( \pi/2 \), and the ring behaves as a point source. Equation 2.1-7 becomes:

\[ G^* = \frac{1}{2} \pi \frac{1}{|Z|}, \quad |Z| \to \infty. \]

\[ 2.1-12 \]
2.2 SUPERPOSITION OF RING SOURCES

Ring sources described by equation 2.1-7 can be superimposed, giving the potential as a function of current density on a cylinder electrode.

\[
\Phi\left(\frac{r, z}{r_0, z_0}\right) = \frac{1}{\pi \kappa} \int_{-l_o}^{l_o} G^*(R^*, Z) i_n(z') dz'
\]

2.2-1

We can then adjust \(i_n\) to satisfy electrode boundary conditions.

With the following dimensionless variables,

\[
\Phi^* = \frac{\Phi}{V}, \quad Z^* = \frac{z}{r_0}, \quad Z' = \frac{z'}{r_0}
\]

2.2-2,3,4

equation 2.2-1 becomes:

\[
\Phi^* = \frac{1}{\pi} \int_{-l_o}^{l_o} \frac{1}{r_0} G^*(R^*, Z^*-Z') \left( -\frac{\partial \Phi^*}{\partial R^*} \right) dZ'.
\]

2.2-5

Boundary conditions for the primary distribution and linear kinetics (equations 1.1-6 and 1.1-5) become:

\[
1 = \Phi^* \text{ at } R^* = 1, \quad 1 = \Phi^* + \frac{1}{J_r} \left( -\frac{\partial \Phi^*}{\partial R^*} \right) \text{ at } R^* = 1,
\]

2.2-6,7

respectively. One can propose a linear combination of electrode surface current density functions to satisfy these boundary conditions:
where the $a_j$'s are the adjustable constants. Substituting 2.2-8 into 2.2-5 provides an expression for the potential as a linear combination of potential functions.

$$
- \frac{\partial \Phi^*}{\partial R^*} = \sum_j a_j \left( \frac{\partial \Phi^*}{\partial R^*} \right)_j \quad 2.2-8
$$

$$
\Phi^*(R^*, Z^*) = \sum_j a_j \frac{1}{\pi} \int_{-r_o}^{r_o} \frac{1}{r'_o} \left( \frac{\partial \Phi^*}{\partial R^*} \right)_j dZ' = \sum_j a_j \Phi^*_j \quad 2.2-9
$$

When equation 2.2-9 is substituted into equation 2.2-6, the boundary condition for the primary distribution becomes:

$$
1 = \sum_j a_j \Phi^*_j \text{ at } R^*=1. \quad 2.2-10
$$

Inserting 2.2-8 and 2.2-9 into 2.2-7, the boundary condition for the linear kinetics becomes:

$$
1 = \sum_j a_j \left( \Phi^*_j + \frac{1}{j} \left( \frac{\partial \Phi^*}{\partial R^*} \right)_j \right) \text{ at } R^*=1. \quad 2.2-11
$$

The first current density function in equation 2.2-8 is chosen to approximate the expected current distribution behavior. The rest of the current functions are even Legendre polynomials.

$$
- \frac{\partial \Phi^*}{\partial R^*} = a_0 \left( \frac{\partial \Phi^*}{\partial R^*} \right)_0 + \sum_{j=0}^{19} a_j P_{2j} \left( \frac{r_o}{r'_o} \right) \quad 2.2-12
$$
For the primary distribution, the first current density function simulates the current singularity at the edge. Wagner (9) has given the primary current distribution for an infinite strip imbedded in an infinite insulating plane. This expression is used as the first function for small $l_0/r_0$.

\[
\left( -\frac{\partial \Phi^*}{\partial R^*} \right)_{-1} = \frac{1}{\sqrt{1 - Z'^2}} \tag{2.2-13}
\]

For large $l_0/r_0$, the following function is used to approximate the primary current distribution:

\[
\left( -\frac{\partial \Phi^*}{\partial R^*} \right)_{-1} = \left( \frac{r_{o_{\text{center}}}}{\sqrt{\kappa}} \right)^b - \left( \frac{P_r \sqrt{2}}{\sqrt{1/r_0}} \right)^b + \left( \frac{P_r \sqrt{2 l_0/r_0}}{\sqrt{\left(1/r_0\right)^2 - Z'^2}} \right)^b \tag{2.2-14}
\]

where $b$ is a constant between one and two. The parameters $b$, $r_{o_{\text{center}}}/\sqrt{\kappa}$, and $P_r$ are fit using the results from the primary distribution employing equation 2.2-13. At the center of the electrode, equation 2.2-14 approaches:

\[
\left( -\frac{\partial \Phi^*}{\partial R^*} \right)_{-1} = \frac{r_{o_{\text{center}}}}{\sqrt{\kappa}} \tag{2.2-15}
\]

At the electrode edge equation 2.2-14 becomes:

\[
\left( -\frac{\partial \Phi^*}{\partial R^*} \right)_{-1} = \frac{P_r}{\sqrt{l_0/r_0 - Z'}} \tag{2.2-16}
\]

The first function in equation 2.2-12 is also tailored to the expected results for the secondary current distribution. The secondary current distribution becomes uniform over the electrode as the polarization parameter becomes small. Consequently, in this
limit, the first Legendre polynomial, \( P_0 = 1 \), approximates current distribution behavior well, and the first function was set to zero.

\[
\left( - \frac{\partial \Phi^*}{\partial R^*} \right) = 0 \quad 2.2-17
\]

Nisancioglu and Newman (10) solved the secondary current distribution governed by linear kinetics, in the electrode-insulator edge region, for high polarization parameters and an electrode-insulator intersection angle of \( \pi \). When their results are couched in terms of the cylinder-electrode problem, the following grouping is a universal function of stretched edge distance in the electrode-insulator edge region:

\[
\frac{-\frac{\partial \Phi^*}{\partial R^*}}{P_r J_r^{1/2}} = f_{\text{univ}}(x^*/2), \quad J_r \to \infty, \quad J_1 \to \infty, \quad x^* = J_r \frac{l_0 - z}{r_0} \quad 2.2-18,19
\]

In this equation \( f_{\text{univ}} \) equals 1.75 at \( z = l_0 \) while approaching the primary distribution in the edge region, \( x^* \cdot l/2 \), as \( x^* \) approaches infinity. Both \( J_r \) and \( J_1 \) must be large for equation 2.2-18 to be valid. Equation 2.2-14 and equation 2.2-18 form the basis for the following expression, valid for high polarization parameters, and used for all \( l_0/r_0 \).

\[
\left( - \frac{\partial \Phi^*}{\partial R^*} \right) = \left( \frac{r_0 \text{center}}{V \kappa} \right)^b \left( P_r J_r^{1/2} f_{\text{univ}}(\|J_r^{1/2}\|) \right)^b + \left( P_r J_r^{1/2} f_{\text{univ}}(\|y^{1/2}\|) \right)^b \quad 2.2-20
\]

\[
y' = J_r \left( 1 - \left( \frac{r_0 - Z}{l_0} \right)^2 \right) \quad 2.2-21
\]

Equation 2.2-20 behaves like equation 2.2-18 in the small edge region, where kinetic limitations are significant for high-polarization parameters. It behaves like equation
2.2-14 away from the edge region, where kinetic limitations are not significant for high polarization parameters.

To solve for the $a_j$'s in equation 2.2-8, a collocation technique was employed. For $n$ current functions, equation 2.2-10 or 2.2-11 was evaluated at $n$ points on the electrode, generating the following $n \times n$ matrix for the primary and secondary current distributions, respectively:

\[
[1] = [\Phi_{ij}] [a_j], \quad 2.2-22
\]

\[
[1] = \left[ \frac{1}{J_r} \left( -\frac{\partial \Phi^*}{\partial R^*} \right) + \Phi^*_{ij} \right] [a_j]. \quad 2.2-23
\]

The $a_j$'s were obtained by inverting matrix 2.2-22 or 2.2-23. Using the zero's of the Legendre polynomials and the electrode edge as collocation points provided the best results. For $n$ current functions, the zeros of the $2(n-1)$ order Legendre polynomials were used along with the edge point.

Current functions were integrated in equation 2.2-9 using Simpson's rule. The methods for integrating singularities inherent in equations 2.2-13, 2.2-14, 2.2-20, and $G^*$ are given in Appendix B.

2.2.1 Convergence Criterion

Results were considered accurate when boundary conditions 2.2-10 or 2.2-11 were satisfied to within 0.01% at each sampled electrode boundary point. In addition, the primary current distributions, using equations 2.2-13 and 2.2-14 as first functions, were compared for $l_0/r_0=10$ and $l_0/r_0=100$. For both aspect ratios, the difference between the results for the two functions was within 0.1%. Secondary current distribution results, using 2.2-17 and 2.2-20 as first functions, for $l_0/r_0=1$ and $J_r=10$ were also compared. These results also differed by less than 0.1%.
3.1 PARAMETER RANGE OF RESULTS

Both the primary current distribution and the potential distribution due to constant current density on the electrode were calculated for $l_0/r_0 = 0$ to $l_0/r_0 = 1000$. Beyond $l_0/r_0 = 1000$ the method broke down. This breakdown probably occurred because the Legendre polynomial correction functions were unable to satisfy boundary condition in the edge region of $O(r_0)$ where the distributions vary greatly as $l_0/r_0$ becomes large.

Figure 3 shows the parameter space for which the numerical method accurately calculated the secondary current distribution. For aspect ratios from zero to ten, current distributions could be calculated for the entire polarization parameter range. Beyond $l_0/r_0 = 10$, results could only be obtained for large $J_r$ using equation 2.2-20, and small $J_r$ using equation 2.2-17. When used as first current density functions in this range, neither equation yielded accurate results for intermediate polarization parameters.

3.2 PRIMARY CURRENT DISTRIBUTION

Ohmic resistance in solution solely determines the primary current distribution. As a result, primary distributions are a unique function of cell geometry (11). In the cylinder electrode case, the ratio $l_0/r_0$ uniquely determines the primary current distribution.

Figure 4 shows the results for the primary current distribution. For all $l_0/r_0$, the current density approaches infinity inversely proportional to the square root of distance from electrode-insulator edge, as described by equation 1.1-7.

Here we examine the primary distribution's behavior for asymptotically large and small $l_0/r_0$. In the limit of small $l_0/r_0$, with finite $l_0$, the cylinder loses its curvature, appearing as an infinite strip imbedded in an infinite plane. The primary distribution for an infinite strip has been treated by Wagner (12) giving:
\[ \frac{i_n}{i_{n,\text{avg}}} = \frac{2}{\pi} \frac{1}{\left(1 - \left|\frac{z}{l_0}\right|^2\right)^{1/2}}. \]

3.2-1

In the opposite extreme, as \( l_0/r_0 \) becomes large, the current distribution appears to approach infinity in an edge region which grows increasingly small when compared with \( l_0 \). As \( l_0/r_0 \) approaches infinity, the mid-region carries all the current, and the singularity at the edge appears as a spike of current. The current distributions for finite \( l_0/r_0 \) fall between these two cases.

To understand further the singular behavior in the edge region, one can examine the asymptotic coefficient of equation 1.1-7 as a function of \( l_0/r_0 \). Equation 3.2-1 describes the primary distribution as \( l_0/r_0 \) approaches zero. So in this limit it is reasonable to examine the behavior of \( P_{\text{avg},l} \) given by:

\[ \frac{i_n}{i_{n,\text{avg}}} = P_{\text{avg},l} \frac{1}{\left(1 - \left|\frac{z}{l_0}\right|^2\right)^{1/2}}. \]

3.2-2

As \( l_0/r_0 \) approaches infinity with finite \( r_0 \), the electrode appears as a semi-infinite cylinder. It is reasonable to expect that the singular edge region has a length corresponding to a finite number of radii. Thus, in this limit, we examine \( P_{\text{avg},r} \) given by:

\[ \frac{i_n}{i_{n,\text{avg}}} = P_{\text{avg},r} \frac{1}{\left(\frac{l_0}{r_0} - \left|\frac{z}{r_0}\right|\right)^{1/2}}. \]

3.2-3

Figure 5 shows \( P_{\text{avg},l} \) and \( P_{\text{avg},r} \) as functions of \( l_0/r_0 \). As \( l_0/r_0 \) approaches zero, the thin strip limit, \( P_{\text{avg},l} \) approaches \( 2^{1/2}/\pi \), as evidenced by equation 3.2-1. \( P_{\text{avg},r} \) on the other hand continues to increase as \( l_0/r_0 \) approaches infinity. \( P_{\text{avg},r} \)'s functional behavior at large aspect ratios could not be determined.
3.2.1 Ohmic Resistance

The primary distribution ohmic resistance determines the relationship between potential difference and total current in a cell limited by ohmic resistance. It is convenient to use two dimensionless resistances for the cylinder electrode system, $R_{\kappa r_0}$ and $R_{kl_0}$, related by $l_0/r_0$. As $l_0/r_0$ approaches zero, for finite $r_0$, the cylinder electrode shrinks to a differentially thin ring. As a result, $R_{\kappa r_0}$ approaches infinity in this limit. At the other extreme, as $l_0/r_0$ approaches infinity, for finite $l_0$, the cylinder shrinks to a differentially thin needle. In this limit $R_{kl_0}$ approaches infinity.

$R_{\kappa r_0}$ approaches infinity logarithmically as the geometric aspect ratio shrinks according to the following expression derived in Appendix C.

$$R_{\kappa r_0} = \frac{1}{2} \left( \frac{\ln(16)}{2\pi} - \frac{\pi}{2} \ln \left( \frac{l_0}{r_0} \right) \right), \quad \frac{l_0}{r_0} \to 0$$

3.2.4

For asymptotically large $l_0/r_0$, it has already been shown that the current density is essentially constant over the electrode except for a negligible spike of current at the edges. Thus, for resistance calculations, it is reasonable to assume constant current density on the electrode. At a distance from the electrode much greater than $l_0$, the electrode behaves as a point source. In Appendix C, the potential for a cylinder source and a point source are assumed equal at a distance of $O(1/0)$, and the following estimate is derived for $R_{kl_0}$, as $l_0/r_0$ approaches infinity.

$$R_{kl_0} \approx \frac{1}{4\pi} \left( 1 + \ln \left( \frac{l_0}{r_0} \right) \right), \quad \frac{l_0}{r_0} \to \infty$$

3.2.5

Here $R_{kl_0}$ approaches infinity logarithmically as $l_0/r_0$ becomes large.

Figure 6 is a graph of the ohmic resistance, along with asymptotic equations 3.2.4 and 3.2.5, as functions of $l_0/r_0$. 

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3.3 SECONDARY CURRENT DISTRIBUTION

The secondary current distribution takes into account finite electrode kinetics and ohmic resistance in solution. Thus, in addition to geometric parameters which govern the primary distribution on an electrode, additional kinetic parameters determine the secondary current distribution. For small surface overpotentials, the linear kinetics approximation can be invoked introducing the polarization parameter, $J_h$, as a governing parameter.

The secondary current distribution on the cylinder electrode for linear kinetics is dictated by the geometric ratio $l_0/r_0$ and either $J_r$ or $J_1$. $J_r$ and $J_1$ are polarization parameters containing lengths $r_0$ and $l_0$, respectively. They are related by $l_0/r_0$. Figure 7 illustrates the thirteen distinct asymptotic regions resulting from these governing parameters.

Figure 8 portrays the results for the secondary current distribution for $l_0/r_0=1$, while figure 9 shows the location of the curves in figure 8 in parameter space. $J_1$ (or $J_r$) represents the ratio of ohmic resistance to kinetic resistance. As $J_1$ approaches infinity, ohmic resistance predominates, and the current distribution approaches the primary distribution. Conversely, as $J_1$ approaches zero, electrode kinetic resistance predominates, and the current density becomes uniform over the entire electrode. Current distributions for finite $J_1$ fall between these two extremes.

3.3.1 Secondary Current Distribution as the Primary Current Distribution is Approached

It is of interest to examine the way the current distribution approaches the primary distribution as the polarization parameters approach infinity (regions two, three and four in figure 7). Nicancioglu and Newman (13) and West and Newman (14) examined the secondary current distribution with linear kinetics, for high polarization parameters. Nicancioglu and Newman solved the current distribution problem in the electrode-insulator edge region with intersection angle $\pi$. West and Newman solved the problem for an arbitrary intersection angle. These results showed that when ohmic effects dominate the secondary current distribution governed by linear kinetics, the current distribution near an electrode-insulator interface deviates from the primary current distribution only in a small edge region of order $\sigma$, where
\[ \sigma = ((\alpha_a + \alpha_c) F \alpha / RT \kappa)^{-1}. \]

In this edge region, the current distribution is relatively independent of the details of the cell's geometry. "Relatively independent" will be quantified shortly. Away from this edge region, toward the center of the electrode, the current distribution approaches the primary distribution. This edge region can be expressed in terms of a dimensionless distance by dividing by \( h \), one of the important geometric lengths. This gives \( \sigma / h = 1 / J_h \).

In the cylinder electrode problem, the edge region can be expressed in two dimensionless forms, \( \sigma / r_0 = 1 / J_r \) and \( \sigma / l_0 = 1 / J_l \). The current distribution will be relatively independent of cell geometry in the cylinder edge region only if both \( \sigma / r_0 \) and \( \sigma / l_0 \) are small, so that the edge region is small relative to both \( r_0 \) and \( l_0 \). Figure 10 illustrates the secondary current distribution at high \( J_1 \) (or \( J_r \)) and \( l_0 / r_0 = 1 \), confirming deviation from the primary current distribution at \( (l_0 - z) / l_0 = O(1 / J_1) \). Figure 9 shows the location of the curves in figure 10 in parameter space.

Nisancioglu and Newman solved the aforementioned problem in the edge region. When their results are cast in terms of the cylinder electrode problem (see Appendix D), one obtains the following grouping as a universal function of stretched distance from the edge:

\[
\frac{1}{J_1^{1/2} \rho_{avg,l}} \frac{i_n}{i_{n,avg}} = f_{uni}(\frac{l_0 - z}{\sigma})^{1/2} = f_{uni}(\left(\frac{1}{l_0} \frac{z}{J_1}\right)^{1/2}).
\]

or equivalently,

\[
\frac{1}{J_r^{1/2} \rho_{avg,r}} \frac{i_n}{i_{n,avg}} = f_{uni}(\frac{l_0 - z}{\sigma})^{1/2} = f_{uni}(\left(\frac{l_0}{r_0} \frac{z}{J_r}\right)^{1/2}).
\]

Again, these relationships hold only when both \( J_1 \) and \( J_r \) are large. At the electrode edge, \( f_{uni} \) equals 1.75; as the stretched variable approaches infinity, \( f_{uni} \) approaches the edge region primary distribution, \( (l_0 - z) / \sigma \)\(^{-1/2} \). Figure 11 shows the secondary current distribution for \( l_0 / r_0 = 1 \) and high \( J_1 \), plotted with the variables given in equation 3.3-1.
Since equations 3.3-1 and 3.3-2 equal 1.75 at the electrode edge, they can be used along with $P_{\text{avg},r}$ and $P_{\text{avg},l}$ (given in figure 5) to estimate the edge current for high polarization parameters:

$$\frac{i_{\text{edge}}}{i_{\text{avg}}} = 1.75 P_{\text{avg},l} J_{1}^{1/2}, \quad J_{1} \to \infty, J_{r} \to \infty.$$  \hspace{1cm} 3.3-3

and equivalently,

$$\frac{i_{\text{edge}}}{i_{\text{avg}}} = 1.75 P_{\text{avg},r} J_{r}^{1/2}, \quad J_{r} \to \infty, J_{1} \to \infty.$$  \hspace{1cm} 3.3-4

### 3.3.2 Secondary Current Distribution as $l_{0}/r_{0}$ Approaches zero

As $l_{0}/r_{0}$ nears zero (regions four, five, six, seven and eight in figure 7), the secondary current distribution approaches the infinite strip limit, leaving $J_{1}$ as the only governing parameter. Figure 12 illustrates how the current distribution becomes a sole function of $J_{1}$ in the limit of small $l_{0}/r_{0}$. Figure 9 shows the location of the curves in figure 12 in parameter space. Equation 3.3-3 and equation 3.2-1 can be combined to give an expression for the current density at the electrode edge for high $J_{1}$ and vanishing $l_{0}/r_{0}$:

$$\frac{i_{\text{edge}}}{i_{\text{avg}}} = 1.75 \sqrt{2} J_{1}^{1/2}, \quad J_{1} \to \infty, \frac{l_{0}}{r_{0}} \to 0.$$  \hspace{1cm} 3.3-5

One can insert the result for the small $l_{0}/r_{0}$, uniform $i_{n}$, potential distribution, equation 3.4-6 (derived in section 3.4.1), into the linear kinetics boundary condition to obtain an estimate for edge current density in the limit of small $l_{0}/r_{0}$:

$$\frac{i_{\text{edge}}}{i_{\text{avg}}} = 1 + \frac{2 \ln 2}{\pi} J_{1}, \quad J_{1} \to 0, \frac{l_{0}}{r_{0}} \to 0.$$  \hspace{1cm} 3.3-6

Thus, when $J_{1}$ approaches 0, in the limit of small $l_{0}/r_{0}$, the current distribution becomes uniform.
Wagner (15) numerically solved the secondary current distribution governed by linear kinetics on a thin strip for finite $J_l$. His results apply for vanishing $I_0/r_0$.

A current distribution can be considered uniform if the current density varies by less than one percent from the average current density over the entire electrode. In the cylinder electrode case this implies, $i_{n,edge}/i_{n,avg}<1.01$. Equation 3.3-6 shows that in the small $l_0/r_0$ limit, $J_l$ must be less than roughly 0.01 to achieve a uniform current distribution. Thus, in figure 7, the line separating uniform current distribution parameter space from non uniform current distribution parameter space, as $l_0/r_0$ approaches 0, is given by $J_l=O(10^{-2})$.

3.3.3 Secondary Current Distribution as $l_0/r_0$ Approaches Infinity

As $l_0/r_0$ approaches infinity (regions two, ten, eleven, twelve and thirteen in figure 7), one might suspect that only $J_r$ governs the secondary current distribution in the electrode edge region. However, numerical results show that $l_0/r_0$ also influences the current distribution. Figure 13 illustrates the current distribution for large $l_0/r_0$ and $J_r=10$, while figure 9 portrays the location of the curves in figure 13 in parameter space. Figure 13 shows the edge current density increasing with $l_0/r_0$. Equation 3.3-4 further illustrates this point, for large $J_r$. Figure 5 shows $P_{avg,r}$ increasing as $l_0/r_0$ approaches infinity. Thus, for large $J_r$, $i_{edge}/i_{avg}$ increases as $l_0/r_0$ approaches infinity. Unfortunately, the numerical method can not produce accurate results for intermediate $J_r$ in the limit of large $l_0/r_0$ (regions twelve and thirteen in figure 7). This hampers one's understanding of the current distribution for large $l_0/r_0$.

One can estimate the boundary between uniform and non-uniform current distribution parameter space for large $l_0/r_0$ in the same manner as the small $l_0/r_0$ boundary is estimated. Inserting equation 3.4-15 (developed in section 3.4.2.4) into the linear kinetics boundary condition, one obtains the following expression for large $l_0/r_0$ and vanishing $J_r$:

$$
\frac{i_{n,edge}}{i_{n,avg}} = 1 + \frac{1}{2} \ln \left( \frac{l_0}{2r_0} \right) J_r, \quad J_r \to 0, \quad \frac{l_0}{r_0} \to \infty.
$$

3.3-7
where $\beta = O(1)$. Equations 3.3-6 and 3.3-7, along with computer generated results for $l_0/r_0 = O(1)$, give the boundary between uniform and non-uniform current distribution parameter space illustrated in figure 7.

3.4 POTENTIAL DISTRIBUTION DUE TO CONSTANT ELECTRODE CURRENT DENSITY

The potential distribution due to constant $i_0$ gives an estimate of the maximum possible potential variation on an electrode. In the cylinder electrode system, $l_0/r_0$ solely governs this potential distribution. Here, we examine behavior in the extreme limits of this ratio.

3.4.1 Potential Distribution Due to Constant $i_0$ as $l_0/r_0$ Approaches Zero

As $l_0/r_0$ approaches zero, one can imagine four regions in the solution in which the potential due to constant $i_0$ varies in distinctly different fashions. In a region between the electrode surface and distances of $O(l_0)$ from the electrode surface, the electrode appears to have no curvature and finite width. When one moves further out into solution--to distances between $O(l_0)$ and $O(r_0)$--the electrode still appears to have no curvature, but now appears as a line source. Here, the potential varies logarithmically according to equation 2.1-11. When one moves still further out--to distances from order $r_0$ to a finite number of radii--the electrode appears as a ring source. Consequently, the full ring source expression, equation 2.1-7, determines the potential distribution. Finally, at distances approaching infinity, the electrode appears as a point source, and equation 2.1-12 governs the potential distribution.

The following equations (developed in Appendix E) express the potential variation on the electrode and insulator, up to distances of $O(l_0)$, for small $l_0/r_0$. On the electrode one obtains:

$$\frac{\Phi(x)}{i_0} = \frac{1}{\pi} \left[ \ln(64) + 2 - \pi - 2\ln\left(\frac{l_0}{r_0}\right) \right] \left(1 - \frac{z}{l_0}\right) \ln\left(1 - \frac{z}{l_0}\right) - \left(1 + \frac{z}{l_0}\right) \ln\left(1 + \frac{z}{l_0}\right) + O\left(\frac{l_0}{r_0}\right)$$

3.4-1

and on the insulator,
\[
\frac{\Phi_o \kappa}{i_0^2} = \frac{1}{\pi} \left[ \ln(64) + 2 - \pi - 2 \ln \left( \frac{1}{r_o} \right) + \left( z - 1 \right) \ln \left( 1 + \frac{z}{l_o} \right) - \left( 1 + \frac{z}{l_o} \right) \ln \left( 1 + \frac{z}{l_o} \right) \right] + O \left( \frac{l_o}{r_o} \right) \quad 3.4-2
\]

where \( i_c \) is the constant current density on the electrode. These equations approach infinity as \( l_0/r_o \) approaches zero in the same manner as equations 3.2-4. Equation 3.2-4 grossly approximates equations 3.4-1 and 3.4-2. It gives a rough estimate of the relationship between potential drop and total current. Subtracting equation 3.2-4 from equation 3.4-1 and 3.4-2, one obtains finite dimensionless potential expressions:

\[
\frac{\Phi_o \kappa}{i_0^2} - 4 \pi R \kappa r_o = \frac{1}{\pi} \left[ \left( 1 - \frac{z}{l_o} \right) \ln \left( 1 + \frac{z}{l_o} \right) - \left( 1 + \frac{z}{l_o} \right) \ln \left( 1 + \frac{z}{l_o} \right) \right], \quad \frac{l_o}{r_o} \to 0, \quad 3.4-3
\]

and,

\[
\frac{\Phi_o \kappa}{i_0^2} - 4 \pi R \kappa r_o = \frac{1}{\pi} \left[ 2 - 2 \ln \left( \frac{z}{l_o} - 1 \right) \ln \left( 1 + \frac{z}{l_o} \right) - \left( 1 + \frac{z}{l_o} \right) \ln \left( 1 + \frac{z}{l_o} \right) \right], \quad \frac{l_o}{r_o} \to 0, \quad 3.4-4
\]

respectively. Subtracting equation 3.2-4 from the asymptotic ring source equation (2.1-11) yields:

\[
\frac{d \Phi \kappa}{i_0^2} - 4 \pi R \kappa r_o = - \frac{2 \ln (2z)}{\pi} \quad 3.4-5
\]

In the limit as \( z/l_o \) approaches infinity, equation 3.4-4 approaches equation 3.4-5.

Figure 15 shows the potential distribution on the cylinder, due to constant \( i_n \), for vanishing \( l_0/r_0 \). Equation 3.4-3 and 3.4-4 predict the potential distribution well to about \( z=2l_0 \), as expected. As \( z/l_o \) increases further, the potential distribution begins to approach the ring source asymptote described by equation 3.4-5. Thus, at distances from the electrode greater than order \( l_0 \), the potential distribution is described by the ring source equation 2.1-7.
The potential difference between the center and edge of the electrode follows from equation 3.4-1. As \( l_0/r_0 \) approaches zero, this difference becomes \( (2\ln 2)/\pi \).

\[
\Delta \Phi_0 \propto \frac{2\ln 2 + O\left(\frac{l_0}{r_0}\right)}{\pi} \quad 3.4-6
\]

Figure 16 shows the center-edge potential difference as a function of \( l_0/r_0 \).

### 3.4.2 Potential Distribution Due to Constant \( \pi \) Approaches Infinity

For the large \( l_0/r_0 \) potential distribution, as with the thin ring potential distribution, we can identify four fairly distinct regions, on the cylinder and insulator, where potential varies in different fashions. From the middle of the cylinder to an edge distance of \( O(r_0) \), one would expect the potential distribution to be fairly flat, behaving like an infinitely long cylinder electrode. For distances from the edge of \( O(r_0) \), the electrode appears as a semi-infinite electrode, and the potential should behave correspondingly. On the insulator, for distances from the edge between \( O(r_0) \) and a finite number of electrode lengths, the electrode appears as a three dimensional object. Finally, at an infinite distance, the cylinder behaves as a point source.

As mentioned previously, the elliptic integral \( K(m) \) in equation 2.1-1 approaches infinity logarithmically as \( m \) approaches one. Consequently, at large \( l_0/r_0 \), the portion of the integrand where \( z'' \gg r_0 \) provides the major contribution to the integral in equation 2.2-1. To obtain an estimate for the potential on the electrode surface and insulator, due to constant \( \pi \), for large \( l_0/r_0 \), one can approximate this integral as:

\[
\Phi_0 \equiv \int_{-l_0}^{l_0} \frac{2\pi r_0 \kappa}{4\pi \kappa \sqrt{(z - z')^2 + 4(\beta r_0)^2}} \, dz'.
\]

Here the ring-like behavior at small to finite \( z/r_0 \) is lumped into the constant \( \beta \). One can derive an integrated expression for the above integral (as outlined in Appendix E) yielding:
\[ \frac{\Phi_i}{i} = \frac{1}{2} \left( \ln \left( \frac{l_o - z}{2 \beta r_o} + \sqrt{\left( \frac{l_o - z}{2 \beta r_o} \right)^2 + 1} \right) + \frac{1}{2} \ln \left( \frac{l_o + z}{2 \beta r_o} + \sqrt{\left( \frac{l_o + z}{2 \beta r_o} \right)^2 + 1} \right) \right) \]

This equation equals \( \ln(l_o/\beta r_o) \) at the middle of the electrode, \( 1/2 \ln(2l_o/\beta r_o) \) at the electrode edge, and approaches \( 1/(z/l_o) \) at an infinite distance from the electrode.

3.4.2.1 Potential Distribution Due to Constant \( i_n \), for Large \( l_o/r_o \), on the Electrode Mid-Region

For \( l_o - z = O(l_o) \) on the electrode, equation 3.4-8 reduces to:

\[ \frac{\Phi_i}{i} = \ln \left( \frac{l_o}{\beta r_o} \right) \approx \frac{1}{2} \ln \left( 1 - \left( \frac{z}{l_o} \right)^2 \right) \] 3.4-9

(See Appendix E.) Approaching the electrode edge, equation 3.4-9 becomes:

\[ \frac{\Phi_i}{i} - \ln \left( \frac{l_o}{\beta r_o} \right) \approx \frac{1}{2} \ln \left( 2 \left( 1 - \frac{z}{l_o} \right) \right) \] 3.4-10

This equation approaches negative infinity logarithmically as \( z/l_o \) approaches one.

Figure 17 shows the computer-generated potential and equation 3.4-9, on the electrode mid-region. Computer-generated results for potentials at the center of the electrode were used to fit a \( \beta \)-value of 0.44. Equation 3.4-9 and the numerical results agreed, as expected, away from the edge region.

3.4.2.2 Potential Distribution Due to Constant \( i_n \), for Large \( l_o/r_o \), on the Electrode Edge Region

For \( l_o - z = O(r_o) \), equation 3.4-8 reduces to:
\[
\frac{\Phi_o \kappa}{i \sigma_o} - \frac{1}{2} \ln \left( \frac{2l_o}{\beta r_o} \right) \equiv \frac{1}{2} \ln \left( \frac{l_o - z}{2\beta r_o} + \sqrt{\left( \frac{l_o - z}{2\beta r_o} \right)^2 + 1} \right)
\]

3.4-11

(See Appendix E.). As \((l_o - z)/2\beta r_o\) approaches infinity (the electrode mid-region), equation 3.4-11 approaches equation 3.4-10:

\[
\frac{\Phi_o \kappa}{i \sigma_o} - \ln \left( \frac{l_o}{\beta r_o} \right) \equiv \frac{1}{2} \ln \left(2 \left(1 - \frac{z}{l_o}\right)\right).
\]

3.4-10

As \((l_o - z)/2\beta r_o\) approaches negative infinity (away from the insulator-edge region), equation 3.4-11 approaches minus infinity logarithmically:

\[
\frac{\Phi_o \kappa}{i \sigma_o} \equiv - \frac{1}{2} \ln \left(\frac{1}{2 \left| \frac{z}{l_o} \right|} \right).
\]

3.4-12

Figure 18 portrays the computer generated potential distributions and equation 3.4-11 in the edge region. Equation 3.4-11 poorly predicts the numerically generated potentials for edge distances of \(O(r_o)\). \(\partial \Phi/\partial z\) approaches infinity logarithmically as the electrode insulator edge region is approached (See Appendix E.). However equation 3.4-11 predicts finite \(\partial \Phi/\partial z\) at the electrode edge. This discrepancy occurs because equation 3.4-7 poorly approximates equation 2.2-1 for \(l_o - z = O(r_o)\). Equation 3.4-11 describes the potential distribution more accurately for edge distances several radii away because equation 3.4-7 is accurate for \(z\) at distances of \(O(l_o)\) or greater from the electrode edge.

3.4.2.3 Potential Due to a Constant \(i_n\), for Large \(l_o/r_o\), on the Insulator, away from the Edge Region

If \(z - l_o = O(l_o)\), on the insulator, equation 3.4-8 reduces to:
\[ \frac{\Phi_0 \kappa}{i \tau_o} \equiv \frac{1}{2} \ln \left( \frac{1 + \frac{z}{\tau_o}}{\frac{z}{\tau_o} - 1} \right) \] 3.4-13

(See Appendix E.) As \( z/\tau_o \) approaches one (the electrode edge region), equation 3.4-13 approaches equation 3.4-12.

\[ \frac{\Phi_0 \kappa}{i \tau_o} \equiv \frac{1}{2} \ln \left( \frac{1}{2} \frac{z}{\tau_o} - 1 \right) \] 3.4-12

As \( z/\tau_o \) approaches infinity, equation 3.4-13 assumes the form of a point source potential.

\[ \frac{\Phi_0 \kappa}{i \tau_o} = \frac{1}{2} \ln \left( \frac{z}{\tau_o} \right), \quad z \to \infty \] 3.4-14

Figure 19 shows the potential variation on the insulator. Equation 3.4-13 agrees with the computer-generated potential variation away from the electrode insulator edge region.

3.4.2.4 Difference In Potential between Electrode Center and Edge, for Constant \( i_n \), as \( l_o/\tau_o \) Approaches Infinity

The potential difference between the electrode center and edge can be estimated from equation 3.4-8.

\[ \frac{\Phi_0 \kappa}{i \tau_o} \equiv \frac{1}{2} \ln \left( \frac{l_o}{2 \beta r_o} \right), \quad l_o \to \infty \] 3.4-15

Figure 16 shows this asymptote along with numerically generated potential differences.

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SECTION 4: SUMMARY

The primary current distribution, secondary current distribution with linearized kinetics, and the potential distribution due to constant electrode current density, were calculated numerically for a cylindrical electrode by superimposing ring current sources to satisfy relevant boundary conditions. Results were obtained for length-to-diameter ratios from zero to one thousand for both the primary current distribution and the potential distribution calculations. The secondary current distribution was solved for the full range of polarization parameters, for aspect ratios ranging from zero to ten. For aspect ratios greater than ten the numerical method could only yield solutions for very high and low $J_r$.

For small aspect ratios, the primary distribution on the cylinder electrode approached that of a thin strip embedded in an insulating wall; for large aspect ratios the primary distribution's functional form could not be deduced. The primary ohmic resistances were calculated and asymptotic forms given for large and small aspect ratios.

The secondary current distribution also approached the infinite strip limit in the small aspect ratio limit, while its functional form at large aspect ratios remained unclear. Asymptotic expressions were developed for the electrode edge current density in the high and low polarization parameter limit.

Asymptotic expressions for the potential on the electrode and insulator, due to constant electrode current density, were derived for high and low aspect ratios. In addition, correlations describing the maximum potential difference on the electrode were produced for this boundary condition.
fig.1: Electrode Geometry

infinitely long insulating cylinder

electrode
fig. 2a: (above) Differential Ring Current Source

fig. 2b: (below) Differential Ring Current Source Imbedded in an Infinite Insulating Cylinder
fig. 3: Current Distribution Parameter Range of Results

- 1 \( \log(\theta_0) = 1000.0 \)
- 2 \( \log(\theta_0) = 10.0 \)
- 3 \( \log(\theta_0) = 1.0 \)
- 4 \( \log(\theta_0) = 0.1 \)
- 5 \( \log(\theta_0) = 0.001 \)

Results obtained for parameter space below this line.

\[ \log(J_r) \]
\[ \log(J_p) \]
fig. 4: Primary Current Distribution

\[ \frac{1}{\nu/\nu_0} = \infty \]

- \[ \nu/\nu_0 = 1000.0 \]
- \[ \nu/\nu_0 = 100.0 \]
- \[ \nu/\nu_0 = 10.0 \]
- \[ \nu/\nu_0 = 1.0 \]
- \[ \nu/\nu_0 = 0.1 \]
- \[ \nu/\nu_0 = 0, \text{ equation 3.2-1} \]
fig. 5: Primary Current Distribution Asymptotic Coefficients
fig. 6: Ohmic Resistance

\[
R_{k\sigma_0}, \quad R_{kl_0}
\]

- equation 3.2-4
- equation 3.2-5
fig. 7: Asymptotic Regions in Current Distribution Parameter Space

- \( \frac{I}{I_0} \rightarrow \infty \) when \( \frac{I}{I_0} \rightarrow \infty \) and \( I_r \rightarrow 0 \)
- \( \frac{I}{I_0} = O(1) \) when \( I_r = O(1) \) and \( I_r \rightarrow 0 \)
- \( \frac{I}{I_0} \rightarrow 0 \) when \( I_r \rightarrow 0 \) and \( I_r \rightarrow 0 \)

- Small \( \frac{I}{I_0} \) asymptote
- Large \( \frac{I}{I_0} \) asymptote

\( \log(J_r) \) vs. \( \log(J_\tau) \)

○ denotes separation between uniform and non-uniform current distribution parameter space.
fig. 8: Secondary Current Distribution, \( l_0/r_0 = 1.0 \)
fig. 9: Current Distribution Parameter Space

- 1 \( \frac{J}{J_0} = 1000.0 \)
- 2 \( \frac{J}{J_0} = 1.0 \)
- 3 \( \frac{J}{J_0} = 0.001 \)

- parameters in fig. 8, 10, 11
- parameters in fig. 12
- parameters in fig. 13
- parameters in fig. 14

Log scale for \( J \) and \( J_0 \)
fig. 10: Secondary Current Distribution for Large \( J_f, l_\alpha/l_0 = 1.0 \)
fig. 11: Secondary Current Distribution for Large $J_1$, $l_0/r_0 = 1.0$
fig. 12: Secondary Current Distribution for Small $l_0/r_0$, $J_l = 10.0$
fig. 13: Secondary Current Distribution for Large $l_0/r_0$, $J_T = 10.0$
fig. 14: Secondary Current Distribution for Small $J_r$, $J_l = 1.0$

$\frac{i_n}{i_{n,\text{avg}}}$ vs. $1 - \frac{z}{l_o}$

- 1 $J_r = 1.0$
- 2 $J_r = 0.1$
- 3 $J_r = 0.01$
- 4 $J_r = 0.001$
fig. 15: Potential Distribution for Uniform $i_n$ and Small $l_0/r_0$
fig. 16: Maximum Potential Difference on the Electrode for Uniform $i_n$
fig 17: Potential Distribution on the Electrode for Uniform $i_n$ and Large $l_0/r_0$
fig 18: Potential Distribution on the Edge Region for Uniform $i_n$ and Large $l_0/r_0$. 

\[ \Phi_{\omega} = \frac{1}{2m(2l_0/\beta r_0)} \]

- --- $\frac{l_0}{r_0}=10.0$
- --- $\frac{l_0}{r_0}=100.0$
- --- $\frac{l_0}{r_0}=1000.0$
- equation 3.4-11.
fig 19: Potential Distribution on the Insulator for Uniform $i_n$ and Large $l_0/r_0$

\[ \frac{\Phi_0}{\delta^2_0} \]

- - - - -

\[ \frac{b}{R} = 1.0 \]

\[ \frac{b}{R} = 10.0 \]

\[ \frac{b}{R} = 100.0 \]

---

equation 3.4-13

\[ \frac{z}{l_0} \]

1 2 3 4

\[ \phi_0 \delta_0^2 \]
NOMENCLATURE

a_j  dimensionless current function coefficient
b    dimensionless exponent
B    dimensionless ring source correction coefficient
F    dimensionless ring source correction function
F    Faraday's constant, 96,487 coulombs/equivalent
G    dimensionless ring source potential
G*   dimensionless ring source potential, includes F
h    important geometric length, cm
i    current density, A/cm^2
i_c  constant current density on the electrode surface, A/cm^2
i_{center}  current density at the electrode center, A/cm^2
i_n  current density on the electrode surface, A/cm^2
i_{n,avg}  average current density on the electrode surface, A/cm^2
i_{n,edge}  current density at the electrode edge, A/cm^2
i_0  exchange current density, A/cm^2
J_h  dimensionless polarization parameter containing geometric length h
J_l  dimensionless polarization parameter containing geometric length l_0
J_r  dimensionless polarization parameter containing geometric length r_0
K(m)  elliptic integral of the second kind
K_{0}  modified Bessel function of the second kind of order 0
l_0  electrode center to edge distance, cm
m    dimensionless elliptic integral variable
P_{avg,l}  dimensionless primary distribution asymptotic edge coefficient, when the edge region is stretched with l_0
P_{avg,r}  dimensionless primary distribution asymptotic edge region coefficient, when the edge region is stretched with r_0
P_{2j}  even Legendre Polynomial of order 2j
P_0  primary distribution asymptotic edge region coefficient, A/cm^{(1+\pi/2\alpha)}
P_r  dimensionless primary distribution asymptotic edge region coefficient, when the edge region is stretched with r_0
r    radial distance coordinate, cm
r_0  electrode radius, cm
R \text{ electrolyte ohmic resistance, ohm}
\[ R \text{ universal gas constant, } 8.3143 \text{ J/mol-K} \]
\[ R^* \text{ dimensionless radial distance coordinate} \]
T \text{ absolute temperature, K}
V \text{ electrode potential, V}
x \text{ distance from the electrode edge, cm}
x^* \text{ dimensionless stretched distance from the electrode edge}
y' \text{ dimensionless stretched distance variable in equation 2.2-20}
z \text{ axial distance coordinate, cm}
z' \text{ axial location of ring source, cm}
Z \text{ dimensionless axial distance from ring source}
Z' \text{ dimensionless axial ring source location}
Z^* \text{ dimensionless axial distance}
\alpha \text{ insulator-electrode intersection angle, radians}
\alpha_a, \alpha_c \text{ kinetic transfer coefficient for the anodic or cathodic reaction}
\beta \text{ dimensionless parameter characterizing ring source behavior near the ring}
\zeta \text{ separation of variables constant}
\eta_s \text{ surface overpotential, V}
\kappa \text{ electrolyte conductivity, } \text{ohm}^{-1}\text{cm}^{-1}
\sigma \text{ edge region where finite electrode kinetics limit the electrode current density for high polarization parameters, cm}
d\Phi \text{ differential electric potential, V}
\Phi \text{ electric potential, V}
\Phi^* \text{ dimensionless electric potential}
\Phi_0 \text{ electric potential on the electrode or insulator, V}
REFERENCES

(2) J. Newman, ibid.
(4) A. Weisselberg, ibid.
(8) C. Y. Mak, ibid.
(12) C. Wagner, op. cit.
(15) C. Wagner, op. cit.
APPENDIX A: CALCULATION OF F

F must satisfy Laplace's equation in cylindrical coordinates.

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial F}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 F}{\partial z^2} = 0. \tag{A-1}
\]

Or equivalently, in dimensionless coordinates:

\[
\frac{1}{R^*} \frac{\partial}{\partial R^*} \left( R^* \frac{\partial F}{\partial R^*} \right) + \frac{1}{R^*} \frac{\partial^2 F}{\partial Z^2} = 0. \tag{A-2}
\]

Using the separation of variables method, one assumes \( F = X(R^*)Y(Z) \). Equation A-2 becomes:

\[
\frac{1}{R^*} \frac{\partial}{\partial R^*} \left( R^* \frac{\partial X}{\partial R^*} \right) = -\frac{1}{Y} \frac{\partial^2 Y}{\partial Z^2} = \zeta^2. \tag{A-3}
\]

Where \( \zeta \) must be a constant.

The solution to the differential equation for \( Y \) is:

\[
Y = c_1 \cos(\zeta Z) + c_2 \sin(\zeta Z). \tag{A-4}
\]

\( Y \) must be symmetrical about \( Z=0 \) so \( c_2 \) must be 0. The solution for \( X \) is:

\[
X = c_3 K_0(\zeta R^*) + c_4 I_0(\zeta R^*). \tag{A-5}
\]

\( K_0 \) is the modified Bessel function of the first kind of order zero and \( I_0 \) is the modified Bessel function of the second kind of order zero. As \( R \) approaches infinity, \( I_0 \)
approaches infinity exponentially and $K_0$ approaches zero exponentially. The potential must be zero at infinity so $c_4$ is zero.

Next the solutions to equation A-3 are integrated to create a general solution.

$$F(R^*, Z) = \int_0^\infty B(\zeta) K_0(\zeta R^*) \cos(\zeta Z) \, d\zeta$$  \hspace{1cm} 2.1-8

The insulator boundary condition determines $B(\zeta)$.

$$\frac{\partial G^*}{\partial R^*} = \frac{\partial G}{\partial R^*} + \frac{\partial F}{\partial R^*} = 0 \text{ at } R^* = 1, \ Z \neq 0$$  \hspace{1cm} A-6

Using equations 2.1-8, A-6, 2.1-3 and the following identity:

$$\frac{\partial K(m)}{\partial m} = -\frac{K(m)}{2m} + \frac{E(m)}{2m(1 - m)},$$  \hspace{1cm} A-7

where $E(m)$ is the elliptical integral of the second kind, one finds:

$$\frac{\partial F}{\partial R^*} = -\int_0^\infty B(\zeta)\zeta K_1(\zeta) \cos(\zeta Z) \, d\zeta = -\frac{\partial G}{\partial R^*} = \frac{K\left(\frac{4}{4 + Z^2}\right) - E\left(\frac{4}{4 + Z^2}\right)}{(4 + Z^2)^{1/2}} \text{ at } R^* = 1.$$  \hspace{1cm} A-8

Where $K_1(\zeta)$ is the modified Bessel function of the second kind of order 1. Equation A-8 is in the form of a Fourier cosine integral which can be inverted to obtain $B(\zeta)$.

$$B(\zeta) = \frac{2}{\pi \zeta K_1(\zeta)} \int_0^\infty \frac{E\left(\frac{4}{t^2+4}\right) - K\left(\frac{4}{t^2+4}\right)}{(t^2+4)^{1/2}} \cos(\zeta t) \, dt$$  \hspace{1cm} A-9
Mak integrated A-9 and 2.1-8 numerically to obtain F values on the surface of the cylinder. Their results were fit with the following function:

\[
F = -\frac{0.5\pi}{1 + |Z|} \pm 0.0001 \quad 0.003 > Z > 20.0
\]

\[
F = \frac{1}{\pi} \left( -\frac{0.5}{1 + |Z|} -1.1083 + 0.29872\Theta - 0.63683\Theta^2 
+ 1.5519\Theta^3 - 2.4429\Theta^4 + 1.2281\Theta^5 \right) \pm 0.0001 , \quad 0.003 \leq Z \leq 20
\]

where;

\[
\Theta = \frac{|Z|}{1 - |Z|}
\]
APPENDIX B: INTEGRATING SINGULARITIES

To integrate equations 2.2-13, 2.2-14, 2.2-20, and Legendre polynomials as current functions in equation 2.2-9, one must deal with the electrode edge singularities in equations 2.2-13, 2.2-14, 2.2-20, and the logarithmic singularity in K(m) at Z' = Z*. The following change of variables renders equations 2.2-13, 2.2-14 and 2.2-20 numerically integrable:

\[ U' = \sqrt{\frac{1}{(2 - \frac{1}{\sqrt{r_0^2 - (\frac{1}{r_0} - Z')^2}}) \frac{1}{2}}) \frac{1}{2}} \]

B-1

In general, the logarithmic singularity inherent in K(m) was subtracted out and integrated analytically.

Substituting equation 2.2-13 in equation 2.2-5, we have:

\[ \Phi^* = \frac{1}{\pi} \int_{\frac{1}{r_o}}^{1} G*(R* = 1, Z* - Z') \frac{1}{r_0 \sqrt{\frac{r_o^2}{1 - \frac{1}{r_o^2} \cdot Z'^2}}} dZ' . \]

B-2

Substituting equation B-1 into equation B-2, subtracting out the limiting behavior of K(m) embodied in equation 2.1-11, and using the identity

\[ \int_{-1}^{1} \frac{\ln|x - x'|^2}{\sqrt{1 - x'^2}} dx' = -2\pi \ln 2 , \]

B-3

one obtains the following numerically integrable expression:
\[ \Phi^-_1 = \frac{4}{\pi} \int_0^{r_o} \left( \frac{1}{2} \right)^{\frac{1}{4}} \left( \frac{1}{2-r_o} \right)^{\frac{1}{4}} \left( G^* + \frac{1}{2} \ln(Z^*-Z')^2 \right) \frac{1}{r_o} \sqrt{2 \left( \frac{1}{r_o} \right)^2 - U'^2} \ dU' - \frac{1}{2} \left( \ln \left( \frac{1}{r_o} \right) - 2 \ln 2 \right) \]

Substituting equation 2.2-14 into equation 2.2-5 yields:

\[ \Phi^* = \frac{1}{\pi} \int \left( \frac{1}{2} \right)^{\frac{1}{4}} \left( \frac{1}{2-r_o} \right)^{\frac{1}{4}} G^* (R^*=1, Z^*-Z') \left( i_{\text{const}} + \left( \frac{P_r \sqrt{2} / r_o}{\sqrt{\left( \frac{1}{r_o} \right)^2 - Z'^2}} \right)^b \right) dZ' \]

where \( i_{\text{const}} \) is

\[ i_{\text{const}} = \left( \frac{r_0 \text{center}}{V_k} \right)^b - \left( \frac{P_r \sqrt{2}}{\sqrt{\left( \frac{1}{r_o} \right)^2}} \right)^b \]

By substituting equation B-1 into equation B-5, then subtracting out equation 2.1-11, multiplied by the now numerically integrable current expression evaluated at \( Z^* \), one obtains the following numerically integrable expression for equation B-5.

\[ \Phi^-_1 = \frac{4}{\pi} \int_0^{r_o} \left( \frac{1}{2} \right)^{\frac{1}{4}} \left( \frac{1}{2-r_o} \right)^{\frac{1}{4}} G^* i_U \left( \frac{3 \ln 2 - \pi}{2} - \frac{1}{2} \ln(Z^* - Z)^2 \right) i_{U^*} dU' \]

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In this equation, $U^*$ is given by equation B-1 with $Z^*$ replacing $Z'$, $i_{U'}$ is expressed as:

\[
i_{U'} = \left( \frac{1}{2} \right)^{\frac{1}{b}} (2 \frac{l_0}{r_0} - U')_b U'^{b_{\text{const}}} + \left( \frac{P_l (2 \frac{l_0}{r_0})^{\frac{1}{b}}}{\sqrt{2 \left( \frac{2 l_0}{r_0} \right)^{\frac{1}{2}} - U'^{2}}} \right)^{\frac{1}{b}}\right),
\]

B-8

while equation B-8 with $U'$ replaced by $U^*$ gives $i_{U^*}$.

We can deal with the singularity in equation 2.2-20 in the same manner as the singularities in equation 2.2-14. When equation 2.2-20 is substituted into equation 2.2-5, one obtains:
In this equation,

$$i_{\text{const}, J} = \left( \frac{r_0^{j_{\text{center}}}}{V_k} \right)^b \left( P_r J_{\text{uni}} \right)^b \left( J/2 \right)^b.$$  \hspace{1cm} B-10

Substituting equation B-1 into equation B-9, then subtracting equation 2.1-11, multiplied by the (now well-behaved) current function evaluated at $Z^*$, results in the following expression, which we can easily integrate numerically.

$$\Phi_* = \frac{4}{\pi} \int_0^{\frac{1}{r_o}} \frac{1}{r_o^2} \int_0^{\frac{1}{r_o}} G^* \left( R^* = 1, Z^* - Z' \right) \left( i_{\text{const}, J} + \left( P_r J_{\text{uni}} \right)^b \right) dZ'. \hspace{1cm} B-9$$

$$\Phi_{-1} = \frac{4}{\pi} \int_0^{\frac{1}{r_o}} \frac{1}{r_o^2} \int_0^{\frac{1}{r_o}} G^* \left( R^* = 1, Z^* - Z' \right) \left( \frac{3}{2} \ln 2 - \frac{\pi}{2} - \frac{1}{2} \ln (Z^* - Z')^2 \right) i_{y^*} dU'$$

$$+ \frac{4}{\pi} i_{y^*} \left( \frac{3}{2} \ln 2 \frac{2}{r_0} \frac{2}{r_0} \right)^{1/4} + \left( \frac{2}{r_0} \right)^{1/4} - \left( \frac{2}{r_0} \right)^{1/4} \ln \left( \frac{2}{r_0} \right) + U^*$$

$$- \left( \frac{2}{r_0} \right)^{1/4} + U^* \ln \left( \frac{2}{r_0} \right)^{1/4} + U^*$$

$$+ \sqrt{2} \left( \frac{2}{r_0} \right)^{1/2} - U^* \ln \left( \sqrt{2} \left( \frac{2}{r_0} \right)^{1/2} - U^* \right) - \left( \frac{1}{r_0} \right)^{1/4}$$

$$- \left( \frac{2}{r_0} \right)^{1/4} + \sqrt{2} \left( \frac{2}{r_0} \right)^{1/2} - U^* \ln \left( \frac{2}{r_0} \right)^{1/4} + \sqrt{2} \left( \frac{2}{r_0} \right)^{1/2} - U^*$$

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Here, $y^*$ is the same as $y'$ (equation 2.2-21) if $Z'$ is replaced with $Z^*$. In addition, $i_y'$ is:

$$i_y' = \left[ i_{\text{const},J} + (P_j f_{\text{uni}}(y')) \right] \left[ \frac{1}{\sqrt{2r_0}} \right] \left( \frac{2r_0}{r_0} \right)^{1/2} U', \quad \text{B-12}$$

while $i_y^*$ is given by equation B-12, with $y'$ and $U'$ replaced by $y^*$ and $U^*$, respectively.

If we substitute a Legendre polynomial into equation 2.1-5, we obtain:

$$\Phi^* = \frac{1}{\pi} \int_{-\frac{l_o}{r_0}}^{\frac{l_o}{r_0}} G^* \left[ R^*=1, Z^* - Z' \right] P_{2j} \left[ \frac{r_0^*}{r_0} Z' \right] dZ'. \quad \text{B-13}$$

Since Legendre polynomials are well-behaved, we only have to deal with the $G^*$'s logarithmic singularity in this case. If we subtract out equation 2.1-11, multiplied by $P_{2j}$ evaluated at $Z^*$, we obtain the following numerically integrable expression.

$$\Phi_{2j}^* = \frac{1}{\pi} \int_{-\frac{l_o}{r_0}}^{\frac{l_o}{r_0}} \left[ G P_{2j} \left( \frac{r_0^*}{r_0} Z' \right) + \frac{1}{2} \ln(Z^* - Z') P_{2j} \left( \frac{r_0^*}{r_0} Z^* \right) \right] dZ'$$

$$- \frac{1}{\pi} P_{2j} \left( \frac{r_0^*}{l_0^*} \right) \left[ \ln \left( \frac{r_0}{l_0} - Z^* \right) + \left( \frac{r_0}{l_0} + Z^* \right) \ln \left( \frac{r_0}{l_0} + Z^* \right) \right] \quad \text{B-14}$$
C.1 RESISTANCE FOR SMALL \( \frac{l_0}{r_0} \)

If we take equation 2.1-2, and make the following substitutions,

\[
x = \frac{z}{l_0}, \quad x' = \frac{z'}{l_0}, \quad \varepsilon = \frac{l_0}{r_0},
\]

we find:

\[
1 - m = \frac{\varepsilon^2 (x - x')^2}{\varepsilon^2 (x - x')^2 + 4}.
\]

Assuming equation 2.2-13, the thin ring limit, approximates the current distribution, we can substitute it into equation 2.2-1 to obtain:

\[
\Phi_o = \frac{l_0}{\pi k} \int_{-1}^{1} \left( \frac{2K(m)}{\varepsilon^2 (x - x')^2 + 4} + F \right) \frac{1}{\sqrt{1 - x'^2}} dx'.
\]

As \( \varepsilon \) approaches zero, \( K(m) \) behaves as follows (16):

\[
K(m) = -\frac{1}{2} \ln \left( \frac{1-m}{16} \right) + \mathcal{O}(\varepsilon^2 \ln(\varepsilon)),
\]

or,

\[
K(m) = \ln \left( \frac{4}{\varepsilon} \right) + \frac{1}{2} \ln(\varepsilon (x - x')^2 + 4) - \ln|x - x'| + \mathcal{O}(\varepsilon^2 \ln(\varepsilon)).
\]
Substituting equations C.1-1a,b,c into equation 2.1-9, we find that the following equation roughly describes $F$:

$$F = \frac{\pi}{2} \frac{1}{\sqrt{1 + \varepsilon (x - x')^2}} .$$  \hspace{1cm} \text{C.1-6}$$

Therefore, it is reasonable to state, for small $\varepsilon$,

$$F = -\frac{1}{2\pi} + \mathcal{O}(\varepsilon). \hspace{1cm} \text{C.1-7}$$

We can make two further simplifications for small $\varepsilon$:

$$\left( \varepsilon \left( x - x' \right)^2 + 4 \right)^{-\frac{1}{2}} = \frac{1}{2} + \mathcal{O}(\varepsilon^2), \hspace{0.5cm} \ln \left( \left( x - x' \right)^2 + 4 \right) = \ln(4) + \mathcal{O}(\varepsilon^2) .$$ \hspace{1cm} \text{C.1-8,9}$$

Substituting equations C.1-5, C.1-7 and equations C.1.8,9 into equation C.1-3 and taking advantage of equation B-3, we find:

$$\Phi_0 = \frac{2l_0}{\kappa} \left( \frac{1}{2} \ln \left( \frac{4}{\varepsilon} \right) + \ln 2 - \frac{\pi}{4} \right) + \mathcal{O}(\varepsilon). \hspace{1cm} \text{C.1-10}$$

It should be noted that equation C.1-10 is not strictly valid because equation 2.2-13 applies only for the limit as $l_0/r_0$ approaches zero.

Integrating over the cylinder surface, we can solve for the total current:

$$i_{\text{total}} = 2\pi r_0 \int_{-l_0}^{l_0} \frac{1}{\sqrt{1 - x^2}} dz = 4\pi l_0 r_0 \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} dx = 2\pi l_0 r_0 .$$ \hspace{1cm} \text{C.1-11}$$

Dividing equation C.1-10 by C.1-11 results in the following expression of resistance.
C.2 RESISTANCE FOR LARGE \( l_o/r_o \)

For large \( l_o/r_o \), we postulate a constant current on the electrode for the primary distribution, and solve Laplace's equation for an infinitely long cylinder.

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{d\Phi}{dr} \right) = 0 \quad \Phi = \Phi_o, \quad i = i_{avg} \quad \text{at} \quad r = r_o
\]

C.2-1

The solution to equation C.2-1 is:

\[
\Phi = \Phi_o - \frac{i_{avg} r_o}{\kappa} \ln \left( \frac{r}{r_o} \right).
\]

C.2-2

For distances in solution much greater than \( l_o \), the electrode appears as a point source.

\[
\Phi = \frac{I}{4\pi \kappa \sqrt{r_o^2 + z^2}}, \quad I=\text{total current} = 4\pi l_o r_o i_{avg}
\]

C.2-3

Setting equation C.2-2 equal to equation C.2-3 at a distance of \( l_o \) from the middle of the electrode, one obtains:

\[
\frac{i_{avg} r_o}{\kappa \Phi_o} = \frac{1}{1 + \ln \left( \frac{l_o}{r_o} \right)}
\]

C.2-4

A resistance approximation for large \( l_o/r_o \) follows from this equation.
\[
\frac{4\pi \Phi_0 k l_o}{4\pi l_o r_o i_{avg}} = 4\pi R k l_o = \frac{1}{1 + \ln \left( \frac{l_o}{r_o} \right)} , \quad \frac{l_o}{r_o} \to \infty
\]

3.2-5
APPENDIX D: EQUATIONS DESCRIBING THE SECONDARY CURRENT DISTRIBUTION FOR LARGE POLARIZATION PARAMETERS

West and Newman solved the secondary current distribution in the electrode-insulator edge region for large polarization parameters. They showed that current in the edge region is described by:

\[ \frac{i_n(x)}{P_o} = -\left(\frac{1}{\sigma}\right)^{\frac{1}{2\alpha}} \frac{\pi}{2\alpha} \phi \]  \hspace{1cm} (D-1)

where \(\alpha\) is the electrode-insulator intersection angle, \(P_o\) is the constant in equation 1.1-7, \(\sigma = ((\alpha + \alpha_c) F_0 / RT \kappa)^{-1}\), and the stretched potential is

\[ -\phi = -\left(\Phi_0 - V\right) \frac{\kappa}{P_o} \left(\frac{1}{\sigma}\right)^{\frac{1}{2\alpha}} = f_{univ}\left(\frac{\pi}{\sigma}, \frac{x}{\sigma}, \left(\frac{x}{2\alpha}\right)^{\frac{1}{2\alpha}}\right) \]  \hspace{1cm} (D-2)

Here, \(f_{univ}\) is a sole function of \(\alpha\) and the stretched distance. It approaches the primary distribution in the edge region, \((x/\sigma)^{1/2\alpha}\), as \(x/\sigma\) approaches infinity.

For the cylinder electrode geometry, \(\alpha = \pi\), and it follows from equation 3.2-2 and 3.2-3 that:

\[ P_o = i_{avg} l_{o,2} P_{avg,l} = i_{avg} r_{o,2} P_{avg,r} \]  \hspace{1cm} (D-3a,b)

For \(\alpha = \pi\), \(f_{univ} = 1.75\) at \(x/\sigma = 0\). Combining equations D-1, D-2 and D-3a,b yields:

\[ \frac{1}{J_{l,avg}^2 P_{avg,l}} \frac{i_n}{i_{n,avg}} = \frac{1}{J_{r,avg}^2 P_{avg,r}} \frac{i_n}{i_{n,avg}} = f_{univ}\left(\frac{l_{o,z}}{\sigma}\right)^{1/2} \]  \hspace{1cm} (3.3-1,2)

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E.1 POTENTIAL FOR SMALL $l_0/\rho_0$

Substituting equations C.1-1a,b,c and C.1-2 into equation 2.2-1 for a constant current density, $i_c$, at the electrode surface, one derives the following expression for the potential on the electrode and insulator:

$$\Phi \frac{\kappa}{i_c \rho_0} = \frac{1}{\pi} \int_{-1}^{1} \left( \frac{2K(m)}{\frac{1}{2} \left( \varepsilon \left( x-x' \right)^2 + 4 \right)} + F \right) dx'. \quad \text{E.1-1}$$

Substituting equations C.1-5, C.1-7, C.1-8 and C.1-9 into equation E.1-1 one obtains the following expression for the potential on the electrode and insulator respectively:

$$\Phi \frac{\kappa}{i_c \rho_0} = \frac{1}{\pi} \left( \ln(64) + 2 - \pi - 2\ln\left(\frac{l_0}{\rho_0}\right) \left( 1 - \frac{z}{l_0} \right) \ln\left( 1 - \frac{z}{l_0} \right) \left( 1 + \frac{z}{l_0} \right) \ln\left( 1 + \frac{z}{l_0} \right) \right) + \mathcal{O}\left(\frac{l_0}{\rho_0}\right), \quad \text{3.4-1}$$

$$\Phi \frac{\kappa}{i_c \rho_0} = \frac{1}{\pi} \left( \ln(64) + 2 - \pi - 2\ln\left(\frac{l_0}{\rho_0}\right) + \left( \frac{z}{l_0} - 1 \right) \ln\left( \frac{z}{l_0} - 1 \right) \left( 1 + \frac{z}{l_0} \right) \ln\left( 1 + \frac{z}{l_0} \right) \right) + \mathcal{O}\left(\frac{l_0}{\rho_0}\right). \quad \text{3.4-2}$$

E.2 POTENTIAL FOR LARGE $l_0/\rho_0$

The potential variation due to a point source at the origin is described by:

$$\Phi = \frac{I}{4\pi \kappa \sqrt{r^2 + z^2}}, \quad \text{E.2-1}$$
where I is the total current emitted from the source. For large $l_0/r_0$, a modified point source equation can be proposed for purposes of integration which gives ring source behavior for distances much greater than $O(r_0)$ accurately but lumps the complicated three dimensional potential variation near the ring source into a functional form suggested by equation 2.1-1. For potentials on the cylinder surface due to a ring source one postulates:

$$\Phi = \frac{I}{4\pi k \sqrt{(z - z')^2 + 4(\beta r_0)^2}}$$  \hspace{1cm} E.2-2

where $\beta$ is assumed to be $O(1)$. Potential behavior near the ring source has been lumped into a “hump” near the ring.

A differential ring source imbedded in an insulating cylinder has total current $2\pi r_0 dz'$. Summing the approximated ring sources over the entire electrode produces:

$$\Phi_0 = \int_{-l_0}^{l_0} \frac{2\pi r_0 j_c}{4\pi k \sqrt{(z - z')^2 + 4(\beta r_0)^2}} \, dz'$$.  \hspace{1cm} 3.4-7

Equation 3.4-7 is formulated to approximate equation 2.2-1 for $l_0 - z = O(l_0)$ or greater. It is expected to break down in the edge region, where the three dimensional nature of the ring source is important in determining potential variation.

Equation 3.4-7 can be written:

$$\frac{\Phi_0 k}{r_0 j_c} = \int_{-\frac{l_0 + z}{2\beta r_0}}^{\frac{l_0 - z}{2\beta r_0}} \frac{1}{\sqrt{y^2 + 1}} \, dy, \quad y = \frac{(z' - z)}{2\beta r_0}$$.  \hspace{1cm} E.2-3

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Equation E.2-3 assumes the following analytic form:

\[
\frac{\Phi_0 \kappa}{i \mathcal{E}_0} = \frac{1}{2} \left( \ln \left( \frac{r_0 - z}{2r_0} + \sqrt{\left( \frac{r_0 - z}{2r_0} \right)^2 + 1} \right) \right) = \frac{1}{2} \left( \ln \left( \frac{r_0 + z}{2r_0} + \sqrt{\left( \frac{r_0 + z}{2r_0} \right)^2 + 1} \right) \right).
\]

E.1.1 Asymptotic Expressions for Equation 3.4-8

For the electrode mid-region; \(z>0, r_0>>r_0, r_0-z>0\) and \(r_0-z=O(r_0)\) implies:

\[
\frac{r_0 + z}{2r_0} \gg 1, \quad \frac{r_0 - z}{2r_0} \gg 1,
\]

and equation 3.4-8 becomes:

\[
\frac{\Phi_0 \kappa}{i \mathcal{E}_0} \equiv \frac{1}{2} \left( \ln \left( \frac{r_0 - z}{r_0} \right) \right).
\]

or:

\[
\frac{\Phi_0 \kappa}{i \mathcal{E}_0} - \ln \left( \frac{r_0}{\beta r_0} \right) \equiv \frac{1}{2} \ln \left( 1 - \frac{z^2}{r_0} \right).
\]

For the electrode-insulator edge region; \(z>0, r_0>>r_0\), and \(r_0-z=O(r_0)\), implying:

\[
\frac{r_0 + z}{2r_0} \gg 1.
\]

Equation 3.4-8 becomes:

\[
\frac{\Phi_0 \kappa}{i \mathcal{E}_0} - \frac{1}{2} \ln \left( \frac{2r_0}{\beta r_0} \right) \equiv \frac{1}{2} \ln \left( \frac{r_0 - z}{2r_0} + \sqrt{\left( \frac{r_0 - z}{2r_0} \right)^2 + 1} \right).
\]
After multiplying the numerator and denominator inside the first logarithm of equation 3.4-8 by:

$$\sqrt{\left(\frac{l_0 z}{2\beta r_o}\right)^2 + 1} - \frac{l_0 z}{2\beta r_o},$$

equation 3.4-8 becomes:

$$\frac{\Phi_0}{i\sigma_0} \cong \frac{1}{2} \ln \frac{\left(\frac{l_0 + z}{2\beta r_o} + \sqrt{\left(\frac{l_0 + z}{2\beta r_o}\right)^2 + 1}\right)}{\left(\frac{z - l_0}{2\beta r_o} + \sqrt{\left(\frac{l_0 - z}{2\beta r_o}\right)^2 + 1}\right)}.$$  

E.2-5

For the insulator region away from the edge region; \(z > l_0, l_0 >> r_0\) and \(z - l_0 = O(l_0)\) or greater; equation E.2-5 becomes:

$$\frac{\Phi_0}{i\sigma_0} = \frac{1}{2} \ln \left(\frac{1 + \frac{z}{l_0}}{\frac{z - l_0}{l_0}}\right).$$  

3.4-13

E.3  POTENTIAL DISTRIBUTION ON THE EDGE REGION FOR CONSTANT \(\ln\)

For large \(l_0/r_0\), and \(l_0 - z << r_0\), the cylinder electrode-insulator edge region appears to lose its curvature. The potential distribution, for constant \(\ln\), is given by the solution to Laplace's equation for an infinite planar electrode-insulator configuration. In cylindrical coordinates we have:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = 0,$$

E.3-1
The solution to this set of equations is:

\[
\frac{\Phi \kappa \pi}{i_c} = - r \ln(r) \cos \theta + r \theta \sin \theta. \tag{E.3.3}
\]

For \(\theta = \pi/2\) we have:

\[
\frac{\kappa \pi}{i_c} \frac{1}{r} \frac{\partial \Phi}{\partial \theta} = 1 + \ln(r). \tag{E.3.4}
\]

Equation E.3.3 and E.3.4 break down at \(l_0 - z = O(r_0)\) for the large \(l_0/r_0\) electrode edge region.
F.1 CURVE FIT FOR F

dq = (z - z')/r_0

function pif(dq)
    implicit double precision(a-h,o-z)
    abvdz=dabs(dq)
    zeta=abvdz/(1.0d0 + abvdz)
    b=0.5d0/(1.0d0 + abvdz)
    c=-1.1083d-3
    d=0.29872d0*zeta
    e= -0.63683d0*zeta**2
    f= 1.5519d0*zeta**3
    g= -2.4429d0*zeta**4
    h= 1.2281d0*zeta**5
    if (abvdz .ge. 20.0d0 .or. abvdz .le. 0.003d0) then
        ef=b
        goto 10
    end if
    ef=b + c + e + f + d + g + h
 10 pi=3.1415926535898d0
    pif=pi*ef
    return
end

F.2 FUNCTION CALCULATING f_{univ}

sqxbar = y^{-1/2} = (J_l(1 - (z'/l_0)^2)^{1/2}

function feebar(sqxbar)
    implicit double precision(a-h,o-z)
    if (sqxbar/2.0d0 .ge. 0.5d0) then
        fractin=1.0d0/dcosh(1.0d0/(sqxbar/2.0d0)**5)
    else
        fractin=0.0d0
    end if
    fractout=1.0d0-fractin
    c
    b0= 1.75302518293d0
    b1= -0.195234399525d0
    b2= -2.05934224121d0
    b3= 2.63679433677d0
    b4= -1.66817831037d0
b5 = 0.629696102982d0
b6 = -0.147232218361d0
b7 = 0.0208781607515d0
b8 = -0.00164370726206d0
b9 = 0.0000550849360993d0

if (sqxbar .le. 6.0d0) then
  feebarin=b0+b1*sqxbar+b2*sqxbar**2+b3*sqxbar**3+b4*sqxbar**4
  1 +b5*sqxbar**5+b6*sqxbar**6+b7*sqxbar**7+b8*sqxbar**8
  2 +b9*sqxbar**9
else
  feebarin=0.0d0
end if

if (sqxbar .ge. 0.1d0) then
  feebarout=1.0d0/sqxbar
else
  feebarout=0.0d0
end if

feebar=fractout*feebarin + fractin*feebarout

return
end

F.3 FUNCTION CALCULATING EQUATION 2.2-20

z = Z'
a = A_r
power = b
rldr0 = l_o / r_o
rjay = J_r
current = r_o / i_c / V_k

function a1(z,a,power,rldr0,rjay,current)
implicit double precision(a-h,o-z)
ybar=0.5d0/rldr0*rjay*(rldr0**2 - z**2)
ybarcent=0.5d0*rldr0*rjay
sqybar=sqrt(abs(ybar))
sqybarcent=sqrt(ybarcent)
a feebar=feebar(sqybar)
c feebar=feebar(sqybarcent)
edge=a*sqrt(rjay)*a feebar
d edgecent=a*sqrt(rjay)*c feebar
ain=current**power + edge**power - edgecent**power
  ai=ain**(1.0d0/power)
return
end
F.4 FUNCTION CALCULATING LEGENDRE POLYNOMIALS

\( x = z' / l_0 \)

\( n \) is the order of the Legendre Polynomial

\[
\begin{align*}
\text{function } P(n,x) \\
\text{ implicit double precision(a-h,o-z)} \\
p1=1.d0 \\
p2=x \\
\text{ if } (n-1) \geq 1,2,3 \\
P1=p1 \quad \text{return} \\
P2=p2 \quad \text{return} \\
nm1=n-1 \\
do 4 \, nu=1, nm1 \\
P = (x * \text{dfloat}(2*nu+1)) * p2 - \text{dfloat}(nu) * p1 / \text{dfloat}(nu+1) \\
p1=p2 \\
p2=P \quad \text{return} \\
end \\
\end{align*}
\]

F.5 FUNCTION CALCULATING THE ELLIPTICAL INTEGRAL OF THE FIRST KIND

\( w = 1 - m \)

\[
\begin{align*}
\text{function } e1(w) \\
\text{ implicit double precision (a-h,o-z)} \\
\text{ dimension a(5),b(5)} \\
\text{ data a / 1.38629436112d0,.09666344259d0,.03590092383d0 \\
2,.03742563713d0,.01451196212d0/} \\
\text{ data b / .5d0,.12498593597d0,.06880248576d0, \\
3,.03328355346d0,.00441787012d0/} \\
d = a(1) + a(2) * w + a(3) * w**2 + a(4) * w**3 + a(5) * w**4 \\
e1 = d + (b(1) + b(2) * w + b(3) * w**2 + b(4) * w**3 + b(5) * w**4) * dlog(1.0d0/w) \\
\text{return} \\
end \\
\end{align*}
\]

F.6 FUNCTION CALCULATING EQUATION B-8
\[ u = U', \quad a = P_T, \quad \text{power} = b, \quad \text{rldr0} = \frac{l_0}{r_0} \]

function \( f\text{curnt}(u,a,\text{power,rldr0}) \)
       implicit double precision(a-h,o-z)
       param=sqrt(2.0d0*\text{rldr0})
       \text{curcent}=(1.0d0/(1.0d0 + log(\text{rldr0})))**\text{power}
       \text{curcent}=\text{curcent} - (a*sqrt(2.0d0)/sqrt(\text{rldr0})))**\text{power}
       \text{curcent}=(abs(\text{param} - u**2))**\text{power}*(abs(u))**\text{power}\text{curcent}
       \text{curent}=(a*param/sqrt(2.0d0*param-u**2)))**\text{power}
       \text{fcurnt}=(\text{curcent} + \text{curend})**(1.0d0/\text{power})
       return
   end

F.7 FUNCTION CALCULATING EQUATION B-12

\[ u = U', \quad a = P_T, \quad \text{power} = b, \quad \text{rldr0} = \frac{l_0}{r_0} \]
\[ r\text{jay} = J_T \]
\[ \text{current} = \frac{r_0}{\text{current/Vk}} \]

function \( a\text{curnt}(u,a,\text{power,\text{rldr0},r\text{jay,\text{current}}}) \)
       implicit double precision(a-h,o-z)
       param=sqrt(2.0d0*\text{rldr0})
       \text{z}=-\text{rldr0} + 2.0d0*\text{param}\text{u**2} - u**4
       \text{curnt}=ai(\text{z,a,\text{power,\text{rldr0},r\text{jay,\text{current}}})
       \text{acurnt}=\text{curnt}*(\text{param} -u**2)*u
       return
   end

F.8 SUBROUTINE CALCULATING THE INTEGRAND IN EQUATION B-14

\( d_i \) is the value of the integrand in equation B-14
\( \text{rldr0} = \frac{l_0}{r_0} \]
\( \text{zs} = Z* \)
zp = Z' 

nleg is the order of the legendre polynomial

```
subroutine dileg(di,rldr0,zs,zp,nleg)
  implicit double precision (a-h,o-z)
  pi=3.14159265358979d0
  dz=zs-zp
  if (abs(dz) .ge. 1.0d-4) then
      wm=4.0d0/(dz**2 +4.0d0)
      w=1.0d0 - wm
      elipt=e1(w)
      f=-pif(dz)
      g=2.0d0*elipt/(sqrt(dz**2 + 4.0d0)) + f
      zpdl=zp/rldr0
      zsdl=zs/rldr0
      pnzp=p(nleg,zpdl)
      pnzs=p(nleg,zsdl)
      gpnzp=g*pnzp
      aspnzs=0.5d0*pnzs*log((zs-zp)**2)
      di=(gpnzp + aspnzs)/pi
  else
      zsdl=zs/rldr0
      pnzs=p(nleg,zsdl)
      di=1.0d0/pi*(3.0d0*log(2.0d0) - pi/2.0d0)*pnzs
  end if
  return
end
```

di = 1.0d0/pi*(3.0d0*log(2.0d0) - pi/2.0d0)*pnzs

di is the value of the integrand in equation B-4 

F.9 SUBROUTINE CALCULATING THE INTEGRAND IN EQUATION B-4

di is the value of the integrand in equation B-4 

rlr0 = l_o/r_o 

zs = Z* 

zp = Z' 

```
subroutine dinf(di,rldr0,zs,zp)
  implicit double precision (a-h,o-z)
  pi=3.14159265358979d0
  dz=zs-zp
end
```
if (abs(dz) .ge. 1.0d-4) then
  wm=4.0d0/(dz**2 + 4.0d0)
  w=1.0d0 - wm
  elipt=e1(w)
      f=-pif(dz)
  g=2.0d0*elipt/(sqrt(dz**2 + 4.0d0)) + f
  u=sqrt(sqrt(2.0d0*rldr0) - sqrt(rldr0 - zp))
  u1=sqrt(2.0d0*rldr0) - sqrt(rldr0 - zp))
  u2=sqrt(abs(rldr0-zp))
  pctu1=abs((u1 - u2)/u1)
  if (pctu1 .ge. .999999999d0) then
    u=sqrt(u1)
  else
    u=sqrt(u1 - u2)
  end if
  denom = sqrt(2.0d0*sqrt(2.0d0*rldr0) - u**2)
  dnum=g + 0.5d0*log((zs-zp)**2)
      di=4.0d0*rldr0/pi*dnum/denom
else
  u=sqrt(u1 - u2)
end if

F.10 SUBROUTINE CALCULATING THE INTEGRAND IN EQUATION B-7

di is the value of the integrand in equation B-7
rldr0 = l0/ro
zs = Z*
zp = Z'
a = Pr
power = b
subroutine fdinf(di, rldr0, zs, zp, a, power)
  implicit double precision (a-h,o-z)
  pi=3.14159265358979d0
  dz=zs-zp
  c
  if (abs(dz) .ge. 1.0d-4) then
    wm=4.0d0/(dz**2 + 4.0d0)
    w=1.0d0 - wm
    elipt=e1(w)
    f=-pif(dz)
    g=2.0d0*elipt/(sqrt(dz**2 + 4.0d0)) + f
    u=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zp))))
    us=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
    dicurp=fcurnt(u,a,power,rldr0)
    dicurs=fcurnt(us,a,power,rldr0)
    di=g*dicurp
    di=di - (3.0d0*log(2.0d0) - pi/2.0d0 - log(abs(dz)))*dicurs
    di=4.0d0/pi*di
  c
  else
    u=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zp))))
    di=0.0d0
  c
  end if
  c
  return
end

F.11 SUBROUTINE CALCULATING THE INTEGRAND IN EQUATION B-11

di is the value of the integrand in equation B-11
rldr0 = l_0/r_0
zs = Z*
zp = Z'
a = P_r
power = b
rjay = J_r
curcent = r_oicenter/V_k

subroutine adinf(di, rldr0, zs, zp, a, power, rjay, curcent)
  implicit double precision (a-h,o-z)
  pi=3.14159265358979d0
  dz=zs-zp
  c
if (abs(dz) .ge. 1.0d-6) then
wm=4.0d0/(dz**2 + 4.0d0)
w=1.0d0 - wm
elipt=e1(w)
f=-pi*dz
Q=2.0d0*elipt/(sqrt(dz**2 + 4.0d0))
U=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zp))))
us=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
dicurp=acurnt(u,a,power,rldr0,rjay,curcent)
dicurs=acurnt(us,a,power,rldr0,rjay,curcent)
di=g*dicurp
di=di - (3.0d0*log(2.0d0) - pi/2.0d0 - log(abs(dz)))*dicurs
di=4.0d0/pi*di
else
u=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zp))))
di=0.0d0
endif
return
end

F.12 SUBROUTINE CALCULATING POTENTIAL FUNCTIONS WITH EQUATION 2.2-13 AS A FIRST CURRENT FUNCTION

zs = Z*
rlrd0 = 10/ro
m is the number of mesh spacings for numerical integration
nleg is the order of the legendre polynomial
theta is the value of the potential function

subroutine trap(zs,rlr0,m,nleg,theta)
implicit double precision (a-h,o-z)
pi=3.1415926535898d0
theta=0.0d0
total=2.0d0*rldr0
part=rlrd0 + zs
fract=part/total
m1=int(fract*dfloat(m)) + 1
m2=int(1.0d0 - fract)*dfloat(m)) + 1
if (nleg .eq. -1) then
if (m1 .ge. 2) then
    delta1=sqrt(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs)))
    eps1=delta1/dfloat(m1-1)
call dinf(di,rldr0,zs,-rldr0)
or1=di
do 1 j=2,m1
    u=eps1*dfloat(j-1)
    zp=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 + u**4
    call dinf(di,rldr0,zs,zp)
or2=di
    theta=theta + eps1*(or2 + or1)/2.0d0
or1=or2
1    continue
celse
c    call dinf(di,rldr0,zs,-rldr0)
beg=di
    call dinf(di,rldr0,zs,zs)
end=di
    uend=sqrt(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0-zs)))
    theta=theta + (end + beg)/2.0d0*abs(uend)
cend if
cif (m2 .ge. 2) then
    delta2=sqrt(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs)))
    eps2=delta2/dfloat(m2-1)
call dinf(di,rldr0,zs,zs)
or1=di
do 3 j=2,m2
    u=sqrt(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs)))
    zp=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 + u**4
    call dinf(di,rldr0,zs,zp)
or2=di
    theta=theta + eps2*(or2 + or1)/2.0d0
or1=or2
3    continue
celse
c    call dinf(di,rldr0,zs,zs)
beg=di
    call dinf(di,rldr0,zs,rldr0)
end=di
    ubeg=sqrt(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs)))
    uend=sqrt(sqrt(2.0d0*rldr0))
\[
\theta = \theta + (\text{end} + \text{beg})/2.0d0*(\text{abs}(\text{end} - \text{beg}))
\]

else

\[
\theta_2 = -0.5d0*rldr0*(2.0d0*\log(rldr0))
1 - 2.0d0*\log(2.0d0))
\]
\[
\theta = \theta + \theta_2
\]

end if

if (m1 \geq 2) then

\[
\delta_1 = rldr0 + z_1
\]
\[
\epsilon_1 = \delta_1/dfloat(2*m1)
\]
\[
call \text{dileg}(d_i, rldr0, z_1, -rldr0, nleg)\]
\[
\theta = \theta + d_i*\epsilon_1/3.0d0
\]
\[
call \text{dileg}(d_i, rldr0, z_1, z_1, nleg)\]
\[
\theta = \theta + d_i*\epsilon_1/3.0d0
\]

do 2 j=1, m1

\[
z_p = -rldr0 + \epsilon_1*dfloat(2*j - 1)
\]
\[
call \text{dileg}(d_i, rldr0, z_1, z_p, nleg)\]
\[
\theta = \theta + 4.0d0/3.0d0*\epsilon_1*d_i
\]
2 continue

do 21 j=1, m1-1

\[
z_p = -rldr0 + \epsilon_1*dfloat(2*j)
\]
\[
call \text{dileg}(d_i, rldr0, z_1, z_p, nleg)\]
\[
\theta = \theta + 2.0d0/3.0d0*\epsilon_1*d_i
\]
21 continue

else

\[
call \text{dileg}(d_i, rldr0, z_1, -rldr0, nleg)\]
\[
beg = d_i
\]
\[
zpmid = (-rldr0 + z_1)/2.0d0
\]
\[
call \text{dileg}(d_i, rldr0, z_1, zpmid, nleg)\]
\[
\amid = d_i
\]
\[
call \text{dileg}(d_i, rldr0, z_1, z_1, nleg)\]
\[
\text{end} = d_i
\]
\[
\epsilon_1 = \text{abs}(z_1 + rldr0)/2.0d0
\]
\[
\theta = \theta + \epsilon_1*(\text{beg} + 4.0d0*\amid + \text{end})/3.0d0
\]
end if

if (m2 \geq 2) then

\[
\delta_2 = 2.0d0*rldr0 - (rldr0 + z_1)
\]
\[
\epsilon_2 = \delta_2/dfloat(2*m2)
\]
\[
call \text{dileg}(d_i, rldr0, z_1, z_1, nleg)\]
\[
\theta = \theta + \epsilon_2*d_i/3.0d0
\]
\[
call \text{dileg}(d_i, rldr0, z_1, rldr0, nleg)\]
\[
\theta = \theta + \epsilon_2*d_i/3.0d0
\]
do 4 j=1,m2
zp=zs + eps2*dfloat(2*j -1)
call dileg(di,rlr0,zs,zp,nleg)
theta=theta + 4.0d0/3.0d0*eps2*di
4 continue

do 41 j=1, m2-1
zp=zs + eps2*dfloat(2*j)
call dileg(di,rlr0,zs,zp,nleg)
theta=theta + 2.0d0/3.0d0*eps2*di
41 continue

c else

c call dileg(di,rlr0,zs,zs,nleg)
beg=di
zpmid=(zs + rlr0)/2.0d0
call dileg(di,rlr0,zs,zpmd,nleg)
amid=di
call dileg(di,rlr0,zs,rlr0,nleg)
end=di
eps2=(rlr0 - zs)/2.0d0
theta=theta + eps2*(beg + 4.0d0*mid + end)/3.0d0

c end if

c zsd=zs/rlr0
pnzs=p(nleg,zsd)
a=rlr0-zs
b=rlr0+zs
if (abs(a) .le. 1.0d-11) then
theta2=-1.0d0/pi*pnzs*(b*log(b) - 2.0d0*rlr0)
else if (abs(b) .le. 1.0d-11) then
theta2=-1.0d0/pi*pnzs*(a*log(a) - 2.0d0*rlr0)
else
theta2=-1.0d0/pi*pnzs*(a*log(a)+b*log(b)-2.0d0*rlr0)
end if
theta=theta + theta2
end if
return
end

F.13 SUBROUTINE CALCULATING POTENTIAL FUNCTIONS WITH EQUATION 2.2-14 AS A FIRST CURRENT FUNCTION

zs = Z*
rlr0 = l0/ro
m is the number of mesh spacings for numerical integration
nleg is the order of the legendre polynomial
theta is the value of the potential function

\[ a = P_r \]
\[ \text{power} = b \]

```fortran
subroutine fsim(zs,rldr0,m,nleg,theta,a,power)
implicit double precision (a-h,o-z)
pi=3.1415926535898d0
theta=0.0d0
if (nleg .eq. -2) then
    total=sqrt(sqrt(2.0d0*rldr0))
    part=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
    fract=part/total
    m1=int(fract*dfloat(m)) + 1
    m2=int((1.0d0 - fract)*dfloat(m)) + 1
else
    total=2.0d0*rldr0
    part=rldr0 + zs
    fract=part/total
    m1=int(fract*dfloat(m)) + 1
    m2=int((1.0d0 - fract)*dfloat(m)) + 1
end if
if (nleg .eq. -2) then
    if (m1 .ge. 2) then
        delta1=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
        eps1=delta1/dfloat(2*m1)
        call fdinf(di,rldr0,zs,-rldr0,a,power)
        up=di
        call fdinf(di,rldr0,zs,zs,a,power)
        down=di
        theta=theta + (up + down)*eps1/3.0d0
    end if
    do 1 j=1,m1
        u=eps1*dfloat(2**j-1)
        zp=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4
        call fdinf(di,rldr0,zs,zp,a,power)
        theta=theta + di*eps1*4.0d0/3.0d0
    1 continue
    do 12 j=1,m1-1
        u=eps1*dfloat(2**j)
        zp=-rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4
        call fdinf(di,rldr0,zs,zp,a,power)
        theta=theta + di*eps1*2.0d0/3.0d0
    12 continue
```

77
else
  call fdinf(di, rldr0, zs, -rldr0, a, power)
  beg = di
  umid = sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))/2.0d0
  zpmid = -rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*umid**2 - umid**4
  call fdinf(di, rldr0, zs, zpmid, a, power)
  rmid = di
  call fdinf(di, rldr0, zs, zs, a, power)
  rend = di
  uend = sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
  eps1 = uend/2.0d0
  theta = theta + (beg + 4.0d0*rmid + rend)*eps1/3.0d0
end if

if (m2 .ge. 2) then
  delta2 = sqrt(sqrt(2.0d0*rldr0))
  1 + sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
  eps2 = delta2/dfloat(2*m2)
  call fdinf(di, rldr0, zs, zs, a, power)
  up = di
  call fdinf(di, rldr0, zs, rldr0, a, power)
  down = di
  theta = theta + (up + down)*eps2/3.0d0
  do 3 j = 1, m2
    u = sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
    1 + eps2*dfloat(2**j-1)
    zp = rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4
    call fdinf(di, rldr0, zs, zp, a, power)
    theta = theta + di*eps2*4.0d0/3.0d0
  3 continue
  do 31 j = 1, m2-1
    u = sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
    1 + eps2*dfloat(2**j)
    zp = rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4
    call fdinf(di, rldr0, zs, zp, a, power)
    theta = theta + di*eps2*2.0d0/3.0d0
  31 continue
  else
    call fdinf(di, rldr0, zs, zs, a, power)
    beg = di
    umid = (sqrt(sqrt(2.0d0*rldr0)) +
    1 sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs)))))/2.0d0
    zpmid = -rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*umid**2 - umid**4
    call fdinf(di, rldr0, zs, zpmid, a, power)
    rmid = di
    call fdinf(di, rldr0, zs, rldr0, a, power)
    rend = di

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ubeg=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
wend=sqrt(sqrt(2.0d0*rldr0))
eps2=(wend - ubeg)/2.0d0
theta=theta + (beg + 4.0d0*rmid + rend)*eps2/3.0d0
end if

const=sqrt(sqrt(2.0d0*rldr0))
us=sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0 - zs))))
if (abs(const - us) .ge. 1.0d-12) then
a1=const - us
a2=const + us
a3=sqrt(2.0d0*const**2 - us**2) - const
a4=sqrt(2.0d0*const**2 - us**2) + const
theta2=a1*log(a1) + a2*log(a2) - a3*log(a3)
1 + a4*log(a4) - 4.0d0*const
else
theta2=4.0d0*const*log(2.0d0*const) - 4.0d0*const
end if
theta2=(3.0d0*log(2.0d0) - pi/2.0d0)*const - theta2
curnts=fcurnt(us,a,power,rldr0)
theta2=4.0d0/pi*curnts*theta2
theta=theta + theta2
else
if (m1 .ge. 2) then
delta1=rldr0 + zs
eps1=delta1/dfloat(2*m1)
call dileg(di,rldr0,zs,-rldr0,nleg)
theta=theta + di*eps1/3.0d0
call dileg(di,rldr0,zs,zs,nleg)
theta=theta + di*eps1/3.0d0
do 2 j=1, m1
zp=-rldr0 + eps1*dfloat(2**j - 1)
call dileg(di,rldr0,zs,zp,nleg)
theta=theta + 4.0d0/3.0d0*eps1*di
2 continue
do 21 j=1, m1-1
zp=-rldr0 + eps1*dfloat(2**j)
call dileg(di,rldr0,zs,zp,nleg)
theta=theta + 2.0d0/3.0d0*eps1*di
21 continue
else

call dileg(di,rldr0,zs,-rldr0,nleg)
beg=di
zpmid=(-rldrO + zs)/2.0d0
call dileg(di,rldrO,zs,zpmid,nleg)
amid=di
   call dileg(di,rldrO,zs,zs,nleg)
end=di
eps1=abs(zs + rldrO)/2.0d0
theta=theta + eps1*(beg + 4.0d0*amid + end)/3.0d0
end if

if (m2 .ge. 2) then
   delta2=2.0d0*rldrO - (rldrO + zs)
   eps2=delta2/dfloat(2**m2)
   call dileg(di,rldrO,zs,zs,nleg)
      theta=theta + eps2*di/3.0d0
   call dileg(di,rldrO,zs,rldrO,nleg)
      theta=theta + eps2*di/3.0d0
   do 4 j=1,m2
      zp=zs + eps2*dfloat(2*j -1)
      call dileg(di,rldrO,zs,zp,nleg)
      theta=theta + 4.0d0/3.0d0*eps2*di
   4   continue
   do 41 j=1, m2-1
      zp=zs + eps2*dfloat(2*j)
      call dileg(di,rldrO,zs,zp,nleg)
      theta=theta + 2.0d0/3.0d0*eps2*di
   41  continue
else
   call dileg(di,rldrO,zs,zs,nleg)
   beg=di
   zpmid=(zs + rldrO)/2.0d0
   call dileg(di,rldrO,zs,zpmid,nleg)
   amid=di
      call dileg(di,rldrO,zs,rldrO,nleg)
   end=di
   eps2=(rldrO - zs)/2.0d0
   theta=theta + eps2*(beg + 4.0d0*amid + end)/3.0d0
end if

zsdl=zs/rldr0
pnzs=p(nleg,zsdl)
aa=rldrO-zs
b=rldrO+zs
if (abs(aa) .le. 1.0d-11) then
   theta2=-1.0d0/pi*pnzs*(b*log(b) - 2.0d0*rldrO)
else if (abs(b) .le. 1.0d-11) then
   theta2=-1.0d0/pi*pnzs*(aa*log(aa) - 2.0d0*rldrO)
else
    theta2 = -1.0d0/pi*pnzs*(aa*log(aa)+b*log(b)-2.0d0*rlr0)
end if
theta = theta + theta2
end if
return
end

F.14 SUBROUTINE CALCULATING POTENTIAL FUNCTIONS WITH EQUATION 2.2-20 AS A FIRST CURRENT FUNCTION

zs = Z*
rlr0 = l0/ro
m is the number of mesh spacings for numerical integration
nleg is the order of the legendre polynomial
theta is the value of the potential function
a = Pr
power = b
rjay = Jr
curcent = roicenter/Vk

subroutine asim(zs,rlr0,m,nleg,theta,a,power,rjay,curcent)
implicit double precision (a-h,o-z)
p = 3.1415926535898d0
theta = 0.0d0

if (nleg .eq. -2) then
    total = sqrt(sqrt(2.0d0*rlr0))
    part = sqrt(abs(sqrt(2.0d0*rlr0) - sqrt(abs(rlr0 - zs))))
    fract = part/total
    m1 = int(fract*dfloat(m)) + 1
    m2 = int((1.0d0 - fract)*dfloat(m)) + 1
else
    total = 2.0d0*rlr0
    part = rlr0 + zs
    fract = part/total
    m1 = int(fract*dfloat(m)) + 1
    m2 = int((1.0d0 - fract)*dfloat(m)) + 1
end if

if (nleg .eq. -2) then
    if (m1 .ge. 2) then
        delta1 = sqrt(abs(sqrt(2.0d0*rlr0) - sqrt(abs(rlr0 - zs))))
    else
        total = sqrt(sqrt(2.0d0*rlr0))
        part = sqrt(abs(sqrt(2.0d0*rlr0) - sqrt(abs(rlr0 - zs))))
        fract = part/total
        m1 = int(fract*dfloat(m)) + 1
        m2 = int((1.0d0 - fract)*dfloat(m)) + 1
    end if
else
    total = 2.0d0*rlr0
    part = rlr0 + zs
    fract = part/total
    m1 = int(fract*dfloat(m)) + 1
    m2 = int((1.0d0 - fract)*dfloat(m)) + 1
end if

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\[ \text{eps1 = delta1/dfloat(2*m1)} \]
\[ \text{call adinf(di,rldr0,zs,-rldr0,a,power,rajy,curcent)} \]
\[ \text{up = di} \]
\[ \text{call adinf(di,rldr0,zs,zs,a,power,rajy,curcent)} \]
\[ \text{down = di} \]
\[ \theta = \theta + (\text{up} + \text{down}) \times \text{eps1}/3.0d0 \]
\[ \text{do 1 } j = 1, m1 \]
\[ \text{u = eps1*dfloat(2*j-1)} \]
\[ \text{zp = rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4} \]
\[ \text{call adinf(di,rldr0,zs,zp,a,power,rajy,curcent)} \]
\[ \theta = \theta + di \times \text{eps1}/4.0d0/3.0d0 \]
\[ \text{continue} \]
\[ \text{do 12 } j = 1, m1-1 \]
\[ \text{u = eps1*dfloat(2*j)} \]
\[ \text{zp = rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4} \]
\[ \text{call adinf(di,rldr0,zs,zp,a,power,rajy,curcent)} \]
\[ \theta = \theta + di \times \text{eps1}/2.0d0/3.0d0 \]
\[ \text{continue} \]
\[ \text{else} \]
\[ \text{call adinf(di,rldr0,zs,-rldr0,a,power,rajy,curcent)} \]
\[ \text{beg = di} \]
\[ \text{umid = sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0-zs))))/2.0d0} \]
\[ \text{zpmid = rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*umid**2 - umid**4} \]
\[ \text{call adinf(di,rldr0,zs,zpmid,a,power,rajy,curcent)} \]
\[ \text{rmid = di} \]
\[ \text{call adinf(di,rldr0,zs,zs,a,power,rajy,curcent)} \]
\[ \text{rend = di} \]
\[ \text{uend = sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0-zs))))} \]
\[ \text{eps1 = uend/2.0d0} \]
\[ \theta = \theta + (\text{beg} + 4.0d0*\text{rmid} + \text{rend}) \times \text{eps1}/3.0d0 \]
\[ \text{end if} \]
\[ \text{else if (m2 .ge. 2) then} \]
\[ \text{delta2 = sqrt(sqrt(2.0d0*rldr0)} \]
\[ \text{1 - sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0-zs))))} \]
\[ \text{eps2 = delta2/dfloat(2*m2)} \]
\[ \text{call adinf(di,rldr0,zs,zs,a,power,rajy,curcent)} \]
\[ \text{up = di} \]
\[ \text{call adinf(di,rldr0,zs,rldr0,a,power,rajy,curcent)} \]
\[ \text{down = di} \]
\[ \theta = \theta + (\text{up} + \text{down}) \times \text{eps2}/3.0d0 \]
\[ \text{do 3 } j = 1, m2 \]
\[ \text{u = sqrt(abs(sqrt(2.0d0*rldr0) - sqrt(abs(rldr0-zs))))} \]
\[ \text{1 + eps2*dfloat(2*j-1)} \]
\[ \text{zp = rldr0 + 2.0d0*sqrt(2.0d0*rldr0)*u**2 - u**4} \]
\[ \text{call adinf(di,rldr0,zs,zp,a,power,rajy,curcent)} \]
\[ \theta = \theta + di \times \text{eps2}/4.0d0/3.0d0 \]

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continue
    do 31 j=1, m2-1
        u = sqrt(abs(sqrt(2.0d0*rlr0) - sqrt(abs(rlr0 - zs))))
        1 + eps2 * dfloat(2^j)
        zp = rlr0 + 2.0d0 * sqrt(2.0d0 * rlr0) * u**2 - u**4
        call adinf(di, rlr0, zs, zp, a, power, rjay, curcent)
        theta = theta + di * eps2 * 2.0d0 / 3.0d0
    31 continue

else
    call adinf(di, rlr0, zs, zs, a, power, rjay, curcent)
    beg = di
    umid = (sqrt(sqrt(2.0d0*rlr0)) + 1 sqrt(abs(sqrt(2.0d0*rlr0) - sqrt(abs(rlr0 - zs)))))/2.0d0
    zpmid = rlr0 + 2.0d0 * sqrt(2.0d0 * rlr0) * umid**2 - umid**4
    call adinf(di, rlr0, zs, zpmid, a, power, rjay, curcent)
    rmid = di
    call adinf(di, rlr0, zs, rlr0, a, power, rjay, curcent)
    rend = di
    ubeg = sqrt(abs(sqrt(2.0d0*rlr0) - sqrt(abs(rlr0 - zs))))
    uend = sqrt(2.0d0*rlr0)
    eps2 = (uend - ubeg) / 2.0d0
    theta = theta + (beg + 4.0d0 * rmid + rend) * eps2 / 3.0d0
end if

const = sqrt(2.0d0*rlr0)
us = sqrt(abs(sqrt(2.0d0*rlr0) - sqrt(abs(rlr0 - zs))))
if (abs(const - us) .ge. 1.0d-12) then
    a1 = const - us
    a2 = const + us
    a3 = sqrt(2.0d0*const**2 - us**2) - const
    a4 = sqrt(2.0d0*const**2 - us**2) + const
    theta2 = a1*log(a1) + a2*log(a2) + a3*log(a3)
    1 + a4*log(a4) - 4.0d0*const
else
    theta2 = 4.0d0*const*log(2.0d0*const) - 4.0d0*const
end if

theta2 = (3.0d0*log(2.0d0) - pi/2.0d0) * const - theta2
curnts = acurnt(us, a, power, rlr0, rjay, curcent)
theta2 = 4.0d0*pi*curnts*theta2
theta = theta + theta2
else
    if (m1 .ge. 2) then
        delta1 = rlr0 + zs
        eps1 = delta1 / dfloat(2^m1)
call dileg(di,rlr0,zs,-rlr0,nleg)
theta=theta + di*eps1/3.0d0
call dileg(di,rlr0,zs,zs,nleg)
theta=theta + di*eps1/3.0d0

    do 2 j=1, m1
       zp=-rlr0 + eps1*dfloat(2*j - 1)
       call dileg(di,rlr0,zs,zp,nleg)
       theta=theta + 4.0d0/3.0d0*eps1*di
       continue
    do 21 j=1, m1-1
       zp=-rlr0 + eps1*dfloat(2*j)
       call dileg(di,rlr0,zs,zp,nleg)
       theta=theta + 2.0d0/3.0d0*eps1*di
    21 continue
    else

    call dileg(di,rlr0,zs,-rlr0,nleg)
    beg=di
    zpmd=(-rlr0 + zs)/2.0d0
    call dileg(di,rlr0,zs,zpmd,nleg)
    amid=di
    call dileg(di,rlr0,zs,zs,nleg)
    end=di
    eps1=abs(zs + rlr0)/2.0d0
    theta=theta + eps1*(beg + 4.0d0*amid + end)/3.0d0
    end if

    if (m2 .ge. 2) then
       delta2=2.0d0*rlr0 - (rlr0 + zs)
       eps2=delta2/dfloat(2*m2)
       call dileg(di,rlr0,zs,zs,nleg)
       theta=theta + eps2*di/3.0d0
       call dileg(di,rlr0,zs,rlr0,nleg)
       theta=theta + eps2*di/3.0d0
    else

    do 4 j=1,m2
       zp=zs + eps2*dfloat(2*j - 1)
       call dileg(di,rlr0,zs,zp,nleg)
       theta=theta + 4.0d0/3.0d0*eps2*di
       continue
    do 41 j=1, m2-1
       zp=zs + eps2*dfloat(2*j)
       call dileg(di,rlr0,zs,zp,nleg)
       theta=theta + 2.0d0/3.0d0*eps2*di
    41 continue
    else

end if

end
call dileg(di,rldr0,zs,zs,nleg)
beg=di
zpmid=(zs + rldr0)/2.0d0
call dileg(di,rldr0,zs,zpmid,nleg)
amid=di
call dileg(di,rldr0,zs,rldr0,nleg)
end=di
eps2=(rldr0 - zs)/2.0d0
theta=theta + eps2*(beg + 4.0d0*amid + end)/3.0d0
end if
c
zsdl=zs/rldr0
pnzs=p(nleg,zsdl)
aa=rldr0-zs
b=rldr0+zs
if (abs(aa) .le. 1.0d-11) then
theta2=-1.0d0/pi*pnzs*(b*log(b) - 2.0d0*rldr0)
else if (abs(b) .le. 1.0d-11) then
theta2=-1.0d0/pi*pnzs*(aa*log(aa) - 2.0d0*rldr0)
else
theta2=-1.0d0/pi*pnzs*(aa*log(aa)+b*log(b)-2.0d0*rldr0)
end if
theta=theta + theta2
end if
return
end
APPENDIX G: TABLES OF NUMERICAL RESULTS

G.1 $P_{\text{avg}}$

<table>
<thead>
<tr>
<th>$\log Q_0$</th>
<th>$P_{\text{avg}, l}$</th>
<th>$P_{\text{avg}, r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.450</td>
<td>0.0142</td>
</tr>
<tr>
<td>0.01</td>
<td>0.449</td>
<td>0.0449</td>
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<td>0.138</td>
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<td>0.383</td>
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<tr>
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<td>0.761</td>
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<td>0.111</td>
<td>1.11</td>
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<tr>
<td>300.0</td>
<td>0.0721</td>
<td>1.25</td>
</tr>
<tr>
<td>1000.0</td>
<td>0.0421</td>
<td>1.35</td>
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G.2 RESISTANCE

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<tr>
<th>$\log Q_0$</th>
<th>$R_{Kl0}$</th>
<th>$R_{Kr0}$</th>
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<tbody>
<tr>
<td>0.001</td>
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<td>0.411</td>
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<tr>
<td>0.01</td>
<td>0.00295</td>
<td>0.295</td>
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<tr>
<td>0.1</td>
<td>0.0181</td>
<td>0.181</td>
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<td>0.0809</td>
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<tr>
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<td>0.220</td>
<td>0.0220</td>
</tr>
<tr>
<td>30.0</td>
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<td>0.0102</td>
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<tr>
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<td>0.00403</td>
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<tr>
<td>300.0</td>
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<td>0.589</td>
<td>0.000589</td>
</tr>
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G.3 POTENTIAL DIFFERENCE BETWEEN ELECTRODE CENTER AND EDGE FOR UNIFORM $i$

<table>
<thead>
<tr>
<th>$\log Q_0$</th>
<th>$\Delta \Phi_{Q_{\text{loic}}}$</th>
<th>$\Delta \Phi_{Q_{\text{roic}}}$</th>
</tr>
</thead>
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<tr>
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<td>0.439</td>
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<td>0.322</td>
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