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SOME PROPERTIES OF CAPILLARY SURFACES
ON ELLIPTICAL DOMAINS

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In this report we show there will exist a critical contact angle for an elliptical cross section if the ratio $b/a$ of the minor and major semiaxes is less than 0.6116. (There is no solution of the capillary surface Eqs. (1) and (2) for contact angles less than the critical angle.) We also calculate a lower bound on $\gamma_{\text{crit}}$ for various values of $b/a$. These bounds appear to be close to the values of $\gamma_{\text{crit}}$ found by numerical solution of Eqs. (1) and (2).
I. DEFINITION OF THE PROBLEM

We consider the equilibrium free surface of a liquid partly filling a vertical cylinder for the case in which the surface height $u(x,y)$ is a single-valued smooth function of $x$ and $y$, and in which there is sufficient liquid to cover the base of the cylinder entirely. The gravitational field is taken to be positive when directed vertically downward. The height then satisfies the equation

$$\nabla \cdot \left( \frac{1}{W} \nabla u \right) = \kappa u + 2H \quad (1)$$

where $W = (1 + |\nabla u|^2)^{1/2}$, $\kappa = \rho g / \sigma$ is the capillary constant, $\rho$ is the difference in densities between the liquid and gas phases, $g$ is the acceleration due to gravity, and $\sigma$ is the gas-liquid surface tension. The constant $2H$ is determined by the cross-sectional shape of the cylinder, the volume of liquid, and the boundary condition satisfied by the free surface of the liquid at the cylinder wall.

The boundary condition for a free surface that makes a contact angle $\gamma$ with the cylinder wall is

$$\frac{1}{W} \frac{\partial u}{\partial n} = \cos \gamma \quad \text{at the wall,} \quad (2)$$

where $\partial u / \partial n$ denotes the derivative of $u$ with respect to the outward directed normal at the wall. Only the case of wetting liquids $0^\circ \leq \gamma < 90^\circ$, will be considered here. (The nonwetting case can be obtained from it directly by means of a simple transformation.)
2. CRITERION FOR THE EXISTENCE OF A CRITICAL CONTACT ANGLE FOR ELLIPTICAL CROSS SECTIONS

Let $A$ denote the area and $L$ the perimeter of the cross section of a cylinder. The following theorem holds [1,2].

**Theorem.** Suppose there is a point $p$ on the boundary at which the curvature is greater than $L/A$; then there exists a critical contact angle such that there is no solution of Eqs. (1) and (2) in a neighborhood of $p$ for $0 \leq \gamma < \gamma_{\text{crit}}$.

We shall apply this theorem to an elliptical cross section. For this application we must calculate the area and perimeter of an ellipse and the curvature at any point on it. The area of an ellipse is

$$A = \pi ab$$

where $a$ and $b$ are the semimajor and semiminor axes.

The ellipse is described by the equation

$$y(x) = b \left(1 - \frac{x^2}{a^2}\right)^{1/2}.$$  

(4)

The perimeter of the ellipse is

$$L = 4 \int_0^a \sqrt{1 + (y')^2} \, dx$$

$$= 4 \int_0^a \frac{\left[1 - \left(1 - \frac{b^2}{a^2}\right) \frac{x^2}{a^2}\right]^{1/2}}{\left(1 - \frac{x^2}{a^2}\right)^{1/2}} \, dx.$$  

Let $x/a = \sin \theta$, and

$$m = 1 - \frac{b^2}{a^2}.$$  

Then

$$L = 4aE(m),$$  

(6)
where
\[ E(m) = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{1/2} d\theta \]  \hspace{1cm} (7)

is the complete elliptic integral of the second kind. Values of \( L/A \) for \( a = 1 \) and various values of \( b \) are given in Table I to the indicated number of decimal places.

Table I. Values of \( L/A \) for \( a = 1 \) and various values of \( b \).

<table>
<thead>
<tr>
<th>( b )</th>
<th>( m )</th>
<th>( L/A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.96</td>
<td>6.688</td>
</tr>
<tr>
<td>0.25</td>
<td>0.9375</td>
<td>5.461</td>
</tr>
<tr>
<td>0.33</td>
<td>0.8911</td>
<td>4.290</td>
</tr>
<tr>
<td>0.4</td>
<td>0.84</td>
<td>3.663</td>
</tr>
<tr>
<td>0.5</td>
<td>0.75</td>
<td>3.084</td>
</tr>
<tr>
<td>0.6</td>
<td>0.64</td>
<td>2.708</td>
</tr>
</tbody>
</table>

These will be used in a later section of this report.

Next we shall calculate the curvature. The differential of arc length is
\[ ds = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} \, dx. \]

The unit vector \( \hat{t} \) tangent to the curve \( y(x) \) in two dimensions is
\[ \hat{t} = \left( \frac{dx}{ds}, \frac{dy}{ds} \right) = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{-1/2} \left( 1, \frac{dy}{dx} \right). \]  \hspace{1cm} (8)
The curvature \( C \) and the unit vector \( \hat{n} \) normal to the curve \( y(x) \) are defined by

\[
C \hat{n} = \frac{df}{ds} \\
\frac{df}{ds} = \left[1 + (y')^2\right]^{-1/2} \frac{df}{dx} \\
= y'' \left[1 + (y')^2\right]^{-2} (-y',1). \tag{9}
\]

Therefore

\[
\hat{n} = \text{sgn}(y'') \left[1 + (y')^2\right]^{-1/2} (-y',1) \tag{10}
\]

and

\[
C = |y''| \left[1 + (y')^2\right]^{-3/2}. \tag{11}
\]

Note that \( C \) also can be written

\[
C = \left| \frac{d}{dx} \left\{ y' \left[1 + (y')^2\right]^{1/2} \right\} \right|. \tag{12}
\]

Let \( y(x) \) be an ellipse. Then

\[
y' = -b^2x/a^2y \]
\[
y'' = -b^4/a^2y^3
\]

Thus

\[
C = \frac{4b^4/(a^4y^2 + b^4x^2)^{3/2}}{a^4b^4}. \tag{13}
\]

The curvature is maximum at the point \( x = a, y = 0 \) where the major axis intersects the ellipse. Thus

\[
C_{\text{max}} = a/b^2. \tag{13}
\]

According to the theorem, there will exist a critical contact angle for ellipses with \( C_{\text{max}} \) greater than \( L/A \), that is, for

\[
b/a < \pi/4E(m). \tag{14}
\]

This inequality can be solved numerically. The result is that a critical angle will exist for ellipses with \( b/a < 0.6116 \).
3. A LOWER BOUND ON $\gamma_{\text{crit}}$

Consider the cross section of the cylinder shown in Fig. 1. Let $A$ be the area of the entire cross section and $L$ be its perimeter. The cross section is cut by a curve $\Gamma$, which intersects the perimeter at points $p_1$ and $p_2$, and whose length we also denote by $\Gamma$. Let $A^*$ denote the part of the domain cut off by $\Gamma$ and denote also the area of this part. Let $L^*$ denote the part of the perimeter cut off by $\Gamma$ and denote also the length of this part.

If a solution to Eqs. (1) and (2) exists for $\kappa = 0$ and contact angle $\gamma$, then integration of Eq. (1) over $A^*$ gives

$$2HA^* = \int_{A^*} \nabla \cdot \left( \frac{1}{W} \nabla u \right) \, dA = \int_{\Gamma+L^*} \left( \frac{1}{W} \nabla u \right) \cdot \hat{n} \, dL$$

$$= L^* \cos \gamma + \int_{\Gamma} \left( \frac{1}{W} \nabla u \right) \cdot \hat{n} \, dL .$$

Now $|\hat{n} \cdot \nabla u| \leq |\nabla u| \leq W$; therefore

$$-\Gamma \leq \int_{\Gamma} \left( \frac{1}{W} \nabla u \right) \cdot \hat{n} \, dL \leq \Gamma .$$

Thus if a solution exists for contact angle $\gamma$, then for any curve $\Gamma$

$$-\Gamma \leq 2HA^* - L^* \cos \gamma \leq \Gamma . \quad (15)$$

This can be written

$$\frac{2HA^* - \Gamma}{L^*} \leq \cos \gamma \leq \frac{2HA^* + \Gamma}{L^*} . \quad (16)$$

Integration of Eq. (1) over the entire cross section gives

$$2HA = L \cos \gamma .$$
Eliminating $H$ from Eq. (15) gives

$$-\Gamma \leq (L^* - A^*L/A) \cos \gamma \leq \Gamma.$$  

For $0 \leq \gamma \leq 90^\circ$ and $L^* - A^*L/A$ nonzero, this can be written

$$\cos \gamma \leq V,$$  

where

$$V = \frac{\Gamma}{|L^* - A^*L/A|}.$$  

Equations (16) and (17) are Lemma 6 and Corollary 3.2 of Ref. 2 for the special case of a two-dimensional domain.

Equation (17) holds for each contact angle for which a solution to Eqs. (1) and (2) exists. In particular it holds for $\gamma_{\text{crit}}$. Let $V_0$ be the minimum of $V$ with respect to all possible curves $\Gamma$. $V_0$ is an upper bound for $\cos \gamma_{\text{crit}}$.

$$\cos \gamma_{\text{crit}} \leq V_0.$$  

This gives a lower bound on $\gamma_{\text{crit}}$.

4. MINIMIZATION OF $V$ FOR FIXED $p_1$ AND $p_2$

The minimization of $V$ can be done in two steps. First, find $\Gamma$ that minimizes $V$ for a fixed pair of intersection points $p_1$ and $p_2$; then vary $p_1$ and $p_2$. It can be shown that the minimizing $\Gamma$ is the arc of a circle [2].

Let $R$ be the radius of the circular arc $\Gamma$, and let $O$ be the center of the circle as shown in Fig. 2. Let $2y$ be the length of the chord from $p_1$ to $p_2$. Let $A$ be the area of the region bounded by this chord and by $\Gamma$. Let $2\theta$ be the angle $p_1O p_2$. Then the following relations hold.
\[ \theta = \arcsin \left( \frac{y}{R} \right) \]  
(20)

\[ \Gamma = 2R\theta \]  
(21)

\[ A_1 = R^2\theta - yR \cos \theta \]  
(22)

Let us hold \( p_1 \) and \( p_2 \) fixed and vary \( R \). Then \( y \) is fixed, but \( \theta \), \( \Gamma \), and \( A_1 \) vary. Also \( \delta A_1 = -\delta A_1 \) because \( A_* + A_1 \) is fixed.

Varying \( R \) in Eqs. (20)-(22) and eliminating \( \delta \theta \) gives

\[ \delta \Gamma = 2(\theta - \tan \theta) \delta R \]

\[ \delta A_* = -2R(\theta - \tan \theta) \delta R \]

thus

\[ \delta A_* = -R \delta \Gamma . \]  
(23)

Let \((L_* - A_*L/A)\) be positive; then

\[ (L_* - A_*L/A) \delta V = \delta \Gamma + (VL/A) \delta A_* . \]  
(24)

Let \( V_p \) be the minimum value of \( V \) for fixed points \( p_1 \) and \( p_2 \), and let \( R_p \) be the minimizing radius. Setting \( \delta V = 0 \) in Eq. (24) and using Eq. (23) gives

\[ R_p = A/(VL_p) . \]  
(25)

5. MINIMIZATION OF \( V \) WITH \( p_1 \) AND \( p_2 \)

Next, let us hold \( R \) fixed and vary \( p_1 \) and \( p_2 \). Let \( \phi_1 \) denote the acute angle of intersection of \( \Gamma \) with the cross section at \( p_1 \), and let \( \phi_2 \) denote the corresponding angle at \( p_2 \). Let \( \delta L_1 \) be the variation in \( L_* \) at \( p_1 \), and \( \delta L_2 \) be the variation at \( p_2 \). Then

\[ \delta \Gamma = \cos \phi_1 \delta L_1 + \cos \phi_2 \delta L_2 \]  
(26)

\[ \delta A_* = (\Gamma/2) \sin \phi_1 \delta L_1 + (\Gamma/2) \sin \phi_2 \delta L_2 . \]  
(27)
Let \((L^* - A^* L/A)\) be positive; then

\[
(L^* - A^* L/A) \delta V = \delta \Gamma + (VL/A) \delta A^* - V \delta L^* .
\]  

(28)

Setting \(\delta V = 0\) in Eq. (28) and using Eqs. (26) and (27) gives two equations for the coefficients of the independent variations \(\delta L_1^*\) and \(\delta L_2^*\).

\[
\cos \phi_1 + U \sin \phi_1 = V \]  

(29)

\[
\cos \phi_2 + U \sin \phi_2 = V ,
\]  

(30)

where

\[
U = VL\Gamma/2A .
\]

For a given value of \(V, L, \Gamma,\) and \(A,\) Eq. (29) has the solutions

\[
\sin \phi_1 = \left\{ U \pm \sqrt{U^2 + (1 + U^2)(1 - V^2)} \right\}^{1/2} / (1 + U^2) .
\]

For \(V < 1,\) Eq. (29) has one positive solution for \(\sin \phi_1.\) For \(V > 1,\) it has two positive solutions; however, \(V > 1\) gives no useful bound on \(\cos \gamma_{\text{crit}},\) so we shall ignore this case. Thus if \(V\) has a relative minimum that is less than one, then \(\phi_1\) will equal \(\phi_2\) for the minimizing \(\Gamma.\)

Consider a cross section that is symmetric about some straight line, as is the ellipse of Fig. 3 about the x axis. Let the curve \(\Gamma\) have intersection points \(p_1\) and \(p_2\) on opposite sides of that line. If \(V\) has a relative minimum that is less than one, then the minimizing \(p_1\) and \(p_2\) will be symmetric about that line.
6. LOWER BOUND ON $\gamma_{\text{crit}}$ FOR ELLIPTICAL CROSS SECTIONS

We apply the preceding results to the elliptical cross section shown in Fig. 3. The ellipse is

$$y(x) = b(1 - x^2/a^2)^{1/2}.$$  \hspace{1cm} (31)

Let $p_1$ be the point $[x_1, y(x_1)]$; it is sufficient to take $p_2$ to be $[x_1, -y(x_1)]$. Let $A_2$ be the area $A_1 + A^*$; then

$$A_2 = 2 \int_{x_1}^{a} y(x) \, dx$$  \hspace{1cm} (32)

$$A^* = A_2 - A_1$$  \hspace{1cm} (33)

$$L^* = 2 \int_{x_1}^{a} \sqrt{1 + (y')^2} \, dx.$$  \hspace{1cm} (34)

Equations (18), (20)-(22), (25), and (31)-(34) can be solved for $V_p$ for each value of $x_1$. By calculating $V_p$ for a sequence of values of $x_1$, we find the minimum value $V_0$.

The circular arc $\Gamma$ must lie inside the ellipse. This is assured if $R$ is greater than $R_1$. $R_1$ is the length of the line $0_1 p_1$ that is perpendicular to the ellipse at $p_1$ as shown in Fig. 3.

$$R_{1}^2 = [y(x_1)]^2 + \frac{b^4 x_1^2}{a^4}$$  \hspace{1cm} (35)

$$R = \max (R_1, R_p)$$  \hspace{1cm} (36)

The preceding equations give the following results for the lower bound on $\gamma_{\text{crit}}$ for $a = 1$ and various values of $b$. 

<table>
<thead>
<tr>
<th>b</th>
<th>lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>35.12°</td>
</tr>
<tr>
<td>0.25</td>
<td>26.95°</td>
</tr>
<tr>
<td>0.33</td>
<td>16.88°</td>
</tr>
<tr>
<td>0.40</td>
<td>10.32°</td>
</tr>
<tr>
<td>0.50</td>
<td>3.71°</td>
</tr>
<tr>
<td>0.60</td>
<td>0.12°</td>
</tr>
</tbody>
</table>

These bounds appear to be close to the values of $\gamma_{\text{crit}}$ found by numerical solution of Eqs. (1) and (2) [3]. The lower bound as a function of $b$ is shown in Fig. 4.

ACKNOWLEDGMENTS

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REFERENCES


Fig. 1. Cross section of a cylinder. (XBL 776-949)

Fig. 2. Cross section of a cylinder with variables defined. (XBL 776-949)

Fig. 3. Cross section of an ellipse. (XBL 776-949)
Fig. 4. Lower bound on $Y_{\text{crit}}$ as a function of the ratio $b$ of the semiminor and semimajor axes. (XBL 776-1117)
This report was done with support from the United States Energy Research and Development Administration. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the United States Energy Research and Development Administration.