Twisted Graded Hecke Algebras for Elementary Abelian Groups

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Twisted Graded Hecke Algebras for Elementary Abelian Groups

A Dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Mathematics by Matthew David Highfield

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To my parents for all the support.
Twisted graded Hecke algebras were introduced by S. Witherspoon in [18] as a common generalization of graded Hecke algebras and twisted symplectic reflection algebras. In this thesis, the structure and representation theory of twisted graded Hecke algebras for $(\mathbb{Z}/\ell\mathbb{Z})^n$ are studied. Such an algebra $A_\ell$ is finitely generated as a module over its center. Moreover, for a generic central character $\chi$, there exists a unique simple $A_\ell$-module on which the center acts by $\chi$. 
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Chapter 1

Introduction

In this thesis, we study the twisted graded Hecke algebras associated to the group \((\mathbb{Z}/\ell\mathbb{Z})^{n-1}\). Twisted graded Hecke algebras were introduced by S. Witherspoon in [18] as a common generalization of the graded Hecke algebras of Drinfel’d [6] and Lusztig [11] [12], and the twisted symplectic reflection algebras studied by T. Chmutova [3]. The construction of a twisted graded Hecke algebra for a group \(G\) involves the choice of a two-cocycle for \(G\), and the multiplication in \(G\) is twisted by this two-cocycle. Part of the motivation for the introduction of twisted graded Hecke algebras by Witherspoon was the appearance of discrete torsion of orbifolds studied by physicists, for example in the paper [5], by Douglas and Fiol.

The main results obtained in this thesis are as follows:

- Let \(G = (\mathbb{Z}/\ell\mathbb{Z})^{n-1}\) and \(A_t\) the twisted graded Hecke algebra associated to \(G\). We construct an embedding \(\Theta\) of \(A_t\) into the crossed product algebra \(\mathbb{C}[X_1^{\pm 1} \cdots X_n^{\pm 1}]\# \alpha G\).

This is somewhat analogous to the Dunkl embedding for rational Cherednik algebras, but does not involve differential-difference operators. The embedding \(\Theta\)
gives a new proof that the algebra $A_t$ has a Poincaré-Birkhoff-Witt (PBW) basis. This embedding in the case $n = 3$ is essentially in the paper [5], although they do not define the algebra $A_t$.

- We determine the center $Z_t$ of $A_t$ by showing that it is generated by a set of elements $x_1^\ell, \ldots, x_n^\ell$ and $w_t$, where $w_t$ is an element whose highest degree term is $x_1 \cdots x_n$. In the case $n = 3$ and $\ell = 2$, the element $w_t$ was found in [4], Lemma 7.1. We generalize their formula to any $n$ and $\ell$.

- The algebra $A_t$ is finite as a module over its center $Z_t$, and hence it is a Polynomial Identity (PI) algebra. Moreover, for a generic central character $\chi$, there exists a unique simple $A_t$-module on which the center $Z_t$ acts via $\chi$. In the case $n = 3$, we give an explicit construction of this simple module for generic parameters $t$ and central character $\chi$. More specifically, we show that the algebra has PI-degree $\ell^2$ and we show that, for an idempotent $e$ in $A_t$ and a lifting of $\chi$ to a character $\tilde{\chi}$ of $Z_t[x_3]$, the module $A_t e \otimes_{Z_t[x_3]} \mathbb{C}\tilde{\chi}$ is the unique simple module with central character $\chi$ for generic $t$ and $\chi$.

- We show that, for $n = 3$, the algebra $A_t$ is Morita equivalent to a deformed Sklyanin algebra studied by C. Walton in [17]. (Our results on the center and representation theory of $A_t$ in this case do not appear to be known for the corresponding deformed Sklyanin algebras.)

The thesis is organized as follows. In chapter 2, we introduce the main object of study, the algebra $A_t$, which is a quotient of the twisted crossed product algebra $TV\#_\alpha G$ by certain inhomogeneous relations of degree 2. The algebra is shown to be a PBW-deformation of the twisted crossed product algebra $SV\#_\alpha G$. In chapter 3, the center $Z_t := Z(A_t)$ of $A_t$ is determined. In chapter 4, the representation theory of $A_t$
is considered for $n = 3$ and any $\ell$. Finally, it is shown that the algebra $A_\ell$ is Morita equivalent to a deformed Sklyanin algebra.
Chapter 2

Twisted Graded Hecke Algebras

2.1 Definitions

Let \( \mathbb{N} = \{0, 1, 2, \ldots\} \). We shall work over the field \( \mathbb{C} \) of complex numbers unless otherwise stated; thus, \( \otimes \) means \( \otimes_{\mathbb{C}} \). Let \( V \) be a finite dimensional complex vector space. Denote the tensor algebra of \( V \) as \( TV \), the symmetric algebra as \( SV \). Let \( G \) be a finite subgroup of \( \text{GL}(V) \). Denote the group algebra of \( G \) by \( \mathbb{C}G \). Recall that \( G \) and thus \( \mathbb{C}G \), act on the tensor and symmetric algebras, and these actions are induced by the action of \( G \) on \( V \). Let \( \alpha : G \times G \to \mathbb{C} \) be a function and define the associative algebra \( TV_{\# \alpha}G \) to be \( TV \otimes \mathbb{C}G \) as a vector space with multiplication given by

\[
(r \otimes g)(s \otimes h) = \alpha(g, h)r(g.s) \otimes gh
\]

for all \( r, s \in TV \) and \( g, h \in G \). We may analogously define \( SV_{\# \alpha}G \). In either case, when the product is associative, this is called a twisted crossed product algebra. If \( \alpha(g, h) = 1 \) for all \( g, h \in G \), then we have the ordinary crossed product. Associativity of the product forces the equation
\[ \alpha(g, h) \alpha(gh, k) = \alpha(h, k) \alpha(g, hk) \quad \text{for all } g, h, k \in G. \]

In other words, \( \alpha \) is a two-cocycle. The defining equation of a coboundary \( \beta \) is

\[ \beta(g, h) = \gamma(g) \gamma(h) \gamma(gh)^{-1} \quad \text{for some } \gamma : G \to \mathbb{C}^\times \]

Identifying \( \alpha \) with its cohomology class, the twisted crossed product algebra is well defined up to isomorphism. We assume that \( \alpha \) is a normalized cocycle. This means that we always take a representative with the property

\[ \alpha(1, g) = \alpha(g, 1) = 1 \quad \text{for all } g \in G. \]

We may normalize a representative by multiplication with a coboundary.

The vector space \( \mathbb{C}G \) sits inside the twisted crossed product via the isomorphism \( \mathbb{C}G \to 1 \otimes \mathbb{C}G \hookrightarrow TV_{#\alpha}G \). The multiplication in this vector space is not the ordinary group algebra multiplication, but the twisted group algebra multiplication. In this case, \( \mathbb{C}G \) is denoted by \( \mathbb{C}_\alpha G \). The product of \( g \) and \( h \) in \( \mathbb{C}_\alpha G \) is \( \alpha(g, h) gh \), where the element \( gh \) is obtained from the usual product of \( g \) and \( h \) in the group \( G \).

For each \( g \in G \), choose a skew form \( \omega^g : V \times V \to \mathbb{C} \). Define

\[ A = \frac{TV_{#\alpha}G}{\langle x \otimes y - y \otimes x - \sum_{g \in G} \omega^g(x, y)g \rangle_{x, y \in V}} \]

The denominator here is the two sided ideal generated by all elements of the form given. Denote the canonical projection of \( TV_{#\alpha}G \to A \) by \( \pi \). It is a filtered algebra homomorphism. \( A \) has a filtration given by assigning degree 1 to elements of \( V \) and degree 0 to elements of \( G \). In other words
\[
F_i(A) = \pi \left( \bigoplus_{r \leq i} V^\otimes r \otimes \mathbb{C}G \right)
\]

In the associated graded algebra, \(xy - yx = 0\) for all \(x, y \in V\), so there is a surjective map

\[
SV^\#_a G \twoheadrightarrow \text{gr}(A)
\]

When this map is an isomorphism, we say that \(A\) has the PBW property, or that \(A\) is of PBW type (where PBW stands for Poincaré-Birkhoff-Witt). The PBW property gives us a basis for the crossed product algebra. If \(\{v_1, \ldots, v_n\}\) is a basis for \(V\), then \(\{v_1^{k_1} \ldots v_n^{k_n} \otimes g\}\) is a basis for \(TV^\#_a G\) where \(g \in G\) and \(k_1, \ldots, k_n \in \mathbb{N}\). When \(A\) is of PBW type, we call \(A\) a Twisted Graded Hecke Algebra, abbreviated TGHA in the remainder of this paper.

The PBW property trivially holds when \(\omega_g = 0\) for all \(g \in G\). In other words, \(SV^\#_a G\) is the trivial TGHA for a given \(V\) with a given \(G\)-action. Denote this algebra by \(A_0\). If some set of skew forms containing at least one nonzero \(\omega_g\) yields an algebra of PBW type, then we say that the algebra is a PBW-deformation of \(A_0\).

Let \(n \geq 3, \ell \geq 2, V = \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \cdots \oplus \mathbb{C}x_n \cong \mathbb{C}^n\), and \(\zeta\) a primitive \(\ell\)-th root of unity. Let \(G \cong (\mathbb{Z}/\ell\mathbb{Z})^{n-1}\) be the multiplicative subgroup of \(\text{SL}_n(\mathbb{C})\) generated by

\[
g_1 = \text{diag}(\zeta, \zeta^{-1}, 1, 1, \ldots, 1)
\]
\[
g_2 = \text{diag}(1, \zeta, \zeta^{-1}, 1, \ldots, 1)
\]
\[
g_3 = \text{diag}(1, 1, \zeta, \zeta^{-1}, \ldots, 1)
\]
\[
\vdots
\]
\[
g_{n-1} = \text{diag}(1, 1, \ldots, 1, \zeta, \zeta^{-1})
\]
We write \( g_n = g_1^{-1} \cdots g_{n-1}^{-1} = \text{diag}(\zeta^{-1}, 1, \ldots, 1, \zeta) \)

Equivalently, \( G \) is the subgroup of \( \text{SL}_n(\mathbb{C}) \) consisting of all diagonal matrices \( g \) such that \( g^f = 1 \).

It will be useful to have a notation for “distance modulo \( n \)”. Let \( \|i - j\|_n \) denote the smallest non-negative integer \( k \) such that either \( i = j + k \mod n \) or \( i + k = j \mod n \). For example, \( 2 \neq \|2 - 0\|_3 = 1 \). Define the twisting two-cocycle by

\[
\alpha(g_1^{i_1} \cdots g_{n-1}^{i_{n-1}}, g_1^{j_1} \cdots g_{n-1}^{j_{n-1}}) = \zeta^{-\sum_{k=1}^{n-2} ik_j + 1}
\]

A few elementary calculations show that the commutation relations for \( \mathbb{C}_\alpha G \) are as follows:

\[
g_{i+1}g_i = \zeta g_i g_{i+1} \\
g_{ij}g_{j} = g_j g_i \quad \text{for} \quad \|i - j\|_n \neq 1 \tag{2.1}
\]

**Definition 1.** Let \( t = (t_1, \ldots, t_n) \in \mathbb{C}^n \) be an \( n \)-tuple of complex number parameters. Define the algebra \( A_t \) to be the quotient of \( TV \# \alpha G \) by the relations

\[
\begin{align*}
    x_i x_{i+1} - x_{i+1} x_i &= t_i g_i \\
    x_i x_j - x_j x_i &= 0 \quad \text{for} \quad \|i - j\|_n \neq 1 
\end{align*} \tag{2.2}
\]

where the indices are taken modulo \( n \).

In particular, the following equations hold in \( A_t \)

\[
\begin{align*}
    g_i x_i &= \zeta x_i g_i \\
    g_i x_{i+1} &= \zeta^{-1} x_{i+1} g_i \\
    g_i x_j &= x_j g_i \quad \text{when} \quad j \neq i, i + 1 \mod n \tag{2.3}
\end{align*}
\]
For each $i$, let $\omega_{g_i} : V \times V \to \mathbb{C}$ be the skew form defined by

$$\omega_{g_i}(x_i, x_{i+1}) = t_i = -\omega_{g_i}(x_{i+1}, x_i)$$

and

$$\omega_{g_i}(x_j, x_k) = 0$$

for all other values of $j$ and $k$. For all other $g \in G$, define $\omega_g$ to be identically zero. The algebra $A_t$ may be written in the form that defines a TGHA, once the PBW property is shown:

$$A_t = TV\#_\alpha G / \langle x_i \otimes x_j - x_j \otimes x_i - \sum_{g \in G} \omega_g(x_i, x_j)g \rangle_{x_i, x_j \in V}$$

### 2.2 Embedding into $\mathbb{C}[X_1^{\pm 1} \ldots X_n^{\pm 1}]\#_\alpha G$

In order to show that the PBW property does indeed hold, we will use the following theorem. Let $t'_i = \frac{t_i}{\zeta^{-1} - 1}$ for all $i$.

**Theorem 2.** The map

$$x_i \mapsto X_i + \frac{t'_i g_i}{X_{i+1}},$$

$$g_i \mapsto g_i,$$

where $i = 1, \ldots, n$, extends to a unique algebra homomorphism

$$\Theta : A_t \to \mathbb{C}[X_1^{\pm 1} \ldots X_n^{\pm 1}]\#_\alpha G.$$ 

**Proof.** The five equations below must hold

1. $\Theta(x_i x_{i+1} - x_{i+1} x_i) = \Theta(t_i g_i)$

2. $\Theta(x_i x_j - x_j x_i) = 0$ for $\|i - j\|_n \neq 1$
3. $\Theta(g_i x_i) = \Theta(\zeta x_i g_i)$

4. $\Theta(g_i x_{i+1}) = \Theta(\zeta^{-1} x_{i+1} g_i)$

5. $\Theta(g_i x_j) = \Theta(x_j g_i)$ for $j \neq i, i + 1$

To do these calculations, use the following relations in $\mathbb{C}[X_1^\pm \ldots X_n^\pm]#_\alpha G$, which hold for all $1 \leq i \leq n$, where the indices are taken modulo $n$:

\begin{align*}
g_i X_i &= \zeta X_i g_i \\
g_i \frac{1}{X_i} &= \zeta^{-1} \frac{1}{X_i} g_i \\
g_i X_{i+1} &= \zeta^{-1} X_{i+1} g_i \\
g_i \frac{1}{X_{i+1}} &= \zeta \frac{1}{X_{i+1}} g_i
\end{align*}

(2.4)

Also keep in mind that $\frac{t' g_i}{X_{i+1}}$ means that $g_i$ appears to the right: $\left( \frac{t' g_i}{X_{i+1}} \right) g_i$.

1. Proof that $\Theta(x_i x_{i+1} - x_{i+1} x_i) = \Theta(t_i g_i)$. 

\[
\Theta(x_i) \Theta(x_{i+1}) - \Theta(x_{i+1}) \Theta(x_i) \\
= \left( X_i + \frac{t' g_i}{X_{i+1}} \right) \left( X_{i+1} + \frac{t'_{i+1} g_{i+1}}{X_{i+2}} \right) - \left( X_{i+1} + \frac{t'_{i+1} g_{i+1}}{X_{i+2}} \right) \left( X_i + \frac{t' g_i}{X_{i+1}} \right)
\]
\[ X_i X_{i+1} + X_i \left( t'_{i+1} g_{i+1} \frac{X_i}{X_{i+2}} \right) + \left( t'_{i+1} g_i \frac{X_i}{X_{i+1}} \right) X_{i+1} + \left( t'_{i+1} g_i \frac{X_{i+1}}{X_{i+2}} \right) t'_{i+1} g_{i+1} \]

\[ - X_{i+1} X_i - X_{i+1} \left( t'_{i+1} g_i \frac{X_{i+1}}{X_{i+2}} \right) - \left( t'_{i+1} g_{i+1} \frac{X_{i+2}}{X_{i+1}} \right) X_i - \left( t'_{i+1} g_{i+1} \frac{X_{i+2}}{X_{i+1}} \right) t'_{i+1} g_i \]

\[ = t'_{i+1} \left( \frac{1}{X_{i+1}} \right) g_i X_{i+1} + t'_{i+1} \left( \frac{1}{X_{i+2}} \right) g_i g_{i+1} \]

\[ \left( t'_{i+1} \frac{X_{i+1}}{X_i} \right) g_i - \zeta^{-1} t'_{i+1} t'_{i} \left( \frac{1}{X_{i+1}} \right) g_{i+1} \]

\[ = t'_i \left( \frac{1}{X_{i+1}} \right) g_i X_{i+1} - t'_i X_{i+1} \left( \frac{1}{X_{i+1}} \right) g_i \]

\[ = \zeta^{-1} t'_i g_i - t'_i g_i \]

\[ = (\zeta^{-1} - 1) t'_i g_i \]

\[ = t_i g_i \]

\[ = \Theta(t_i g_i) \]

2. Proof that \( \Theta(x_i x_j - x_j x_i) = 0 \) for \( ||i - j||_n \neq 1 \):

\[ \Theta(x_i) \Theta(x_j) - \Theta(x_j) \Theta(x_i) \]

\[ = \left( X_i + \frac{t'_{i} g_i}{X_{i+1}} \right) \left( X_j + \frac{t'_{j} g_j}{X_{j+1}} \right) - \left( X_j + \frac{t'_{j} g_j}{X_{j+1}} \right) \left( X_i + \frac{t'_{i} g_i}{X_{i+1}} \right) \]

\[ = X_i X_j + \frac{t'_{i} g_i}{X_{i+1}} X_j + X_i \frac{t'_{j} g_j}{X_{j+1}} + X_i \frac{t'_{j} g_j}{X_{j+1}} X_i - X_j X_i - \frac{t'_{j} g_j}{X_{j+1}} X_i - X_j \frac{t'_{i} g_i}{X_{i+1}} - X_j \frac{t'_{i} g_i}{X_{i+1}} \]

\[ = 0 \]

3. Proof that \( \Theta(g_i x_i) = \Theta(\zeta x_i g_i) \):
\[ \Theta(g_ix_i) \]
\[ = g_i \left( X_i + \frac{t_i'' g_i}{X_{i+1}} \right) \]
\[ = g_iX_i + g_i \frac{t_i'' g_i}{X_{i+1}} \]
\[ = \zeta X_i g_i + \zeta t_i'' \left( \frac{1}{X_{i+1}} \right) g_i g_i \]
\[ = \zeta \left( X_i + \frac{t_i' g_i}{X_{i+1}} \right) g_i \]
\[ = \Theta(\zeta x_i g_i) \]

4. Proof that \( \Theta(g_ix_{i+1}) = \Theta(\zeta^{-1} x_{i+1} g_i) \):

\[ \Theta(g_ix_{i+1}) \]
\[ = g_i \left( X_{i+1} + \frac{t_{i+1}' g_{i+1}}{X_{i+2}} \right) \]
\[ = g_iX_{i+1} + g_i \frac{t_{i+1}' g_{i+1}}{X_{i+2}} \]
\[ = \zeta^{-1} X_{i+1} g_i + t_{i+1}' g_i \left( \frac{1}{X_{i+2}} \right) g_{i+1} \]
\[ = \zeta^{-1} X_{i+1} g_i + t_{i+1}' \left( \frac{1}{X_{i+2}} \right) g_ig_{i+1} \]
\[ = \zeta^{-1} X_{i+1} g_i + \zeta^{-1} \frac{t_{i+1}'}{X_{i+2}} g_{i+1} g_i \]
\[ = \zeta^{-1} \left( X_{i+1} + \frac{t_{i+1}' g_{i+1}}{X_{i+2}} \right) g_i \]
\[ = \Theta(\zeta^{-1} x_{i+1} g_i) \]

5. Proof that \( \Theta(g_ix_j) = \Theta(x_j g_i) \) for \( j \neq i, i+1 \):
\[ \Theta(g; x_j) \]
\[ = g_i \left( X_j + \frac{t'_j g_j}{X_{j+1}} \right) \]
\[ = g_i X_j + g_i \frac{t'_j g_j}{X_{j+1}} \]
\[ = X_j g_i + g_i \frac{t'_j g_j}{X_{j+1}} \]
\[ = \left( X_j + \frac{t'_j g_j}{X_{j+1}} \right) g_i \]
\[ = \Theta(x_j g_i) \]

This completes the proof that \( \Theta : A_t \to \mathbb{C}[X_1^{\pm 1} \cdots X_n^{\pm 1}]^{\#_G} \) is a homomorphism of algebras. \( \square \)

**Theorem 3.** The homomorphism \( \Theta \) is injective.

**Proof.** Suppose there is a relation

\[ \Theta \left( \sum_{k_1, \ldots, k_n} a_{k_1, \ldots, k_n} x_1^{k_1} \cdots x_n^{k_n} \otimes g \right) = 0. \]

The image of a monomial \( x_1^{k_1} \cdots x_n^{k_n} \otimes g \) under \( \Theta \) has one highest total degree term, which is \( X_1^{k_1} \cdots X_n^{k_n} \otimes g \). Therefore, it suffices to consider only terms in the sum with highest total degree. Since the elements \( X_1^{k_1} \cdots X_n^{k_n} \otimes g \) for \( k_1, \ldots, k_n \geq 0 \) are linearly independent in \( \mathbb{C}[X_1^{\pm 1} \cdots X_n^{\pm 1}]^{\#_G} \), we must have that \( a_{k_1, \ldots, k_n} = 0 \) for all \( n \)-tuples \( (k_1, \ldots, k_n) \) in the sum. This shows that \( \ker(\Theta) = 0. \) \( \square \)

**Corollary 4.** \( A_t \) is a PBW-deformation of \( A_0. \)

**Proof.** The elements \( X_1^{k_1} \cdots X_n^{k_n} \otimes g \) for \( g \in G \) and each \( k_i \in \mathbb{N} \) are linearly independent in \( \mathbb{C}[X_1^{\pm 1} \cdots X_n^{\pm 1}]^{\#_G} \). Since \( \Theta \) is injective, it follows that \( \{ x_1^{k_1} \cdots x_n^{k_n} \otimes g \} \), where \( g \in G \) and each \( k_i \in \mathbb{N} \), is a linearly independent set in \( A_t \). This is the PBW basis for \( A_t. \) \( \square \)
Chapter 3

Center of $A_t$

3.1 Center of $A_0$

Let $Z_t := Z(A_t)$ denote the center of $A_t$ and $Z_0 := Z(A_0)$ the center of $A_0$.

Let

$$w_0 = x_1 x_2 \cdots x_n \in A_0.$$ \hspace{1cm} (3.1)

Lemma 5. The center of $A_0$ is

$$Z_0 = \text{SV}^G = \mathbb{C}[x_1^\ell, x_2^\ell, \ldots, x_n^\ell, w_0].$$

Proof. The first step is to show $\text{SV}^G \subset Z_0$. Let $p \otimes g \in A_0$ and $q \in \text{SV}^G$. Then

$$(p \otimes g) \cdot q = \alpha(g, 1) \cdot p(g.q) \otimes g = pq \otimes g = \alpha(1, g) \cdot qp \otimes g = q \cdot (p \otimes g)$$

This shows that $\text{SV}^G \subset Z_0$. To show that $Z_0 \subset \text{SV}^G$, take an element $z \in Z_0$. We can write $z$ in the form $\sum_i p_i \otimes g_i$ where $g_i \neq g_j$ for $i \neq j$. Suppose $g_j \neq 1$ for some $j$. Then there exists $v \in V$ such that $g_j.v \neq v$, and
\[ \sum_i p_i v \otimes g_i = vz = zv = \sum_i p_i (g_i \cdot v) \otimes g_i \]

so \( g_i \cdot v = v \) for all \( i \), thus \( p_j = 0 \). It follows that \( z \in SV \). We have \( g \cdot z = z \) for all \( g \in G \) since \( (g \cdot z) \otimes g = gz = zg = z \otimes g \).

Finally, we show that \( SV^G = \mathbb{C}[x_1^\ell, x_2^\ell, \ldots, x_n^\ell, w_0] \). Note that \( w_0 = x_1 x_2 \cdots x_n \) is \( G \)-invariant, and \( x_i^\ell \) is \( G \)-invariant for all \( i \). If \( p \) is a monomial in \( \mathbb{C}[x_1, x_2, \ldots, x_n]^G \), we can factor out any of these elements until we are left with \( p = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \), where at least one \( k_i = 0 \), and each \( k_i < \ell \). The only such element which is \( G \)-invariant is \( 1 \). For if \( k_i = 0 \), then unless \( k_{i+1} = 0 \), we have that \( g_i \) does not fix \( p \). \( \square \)

### 3.2 Center of \( A_\ell \)

**Lemma 6.** The elements \( \{x_i^\ell\}_{i=1}^n \) belong to \( Z_\ell \).

**Proof.** It is clear that \( g_j x_i^\ell = x_i^\ell g_j \) for all \( j \) and that \( x_j x_i^\ell = x_i^\ell x_j \) for all \( j \) except possibly \( j = i + 1, i - 1 \).

\[
\begin{align*}
  x_{i-1} x_i^\ell \\
  = x_i x_{i-1} x_i^{\ell-1} + [x_{i-1}, x_i] x_i^{\ell-1} \\
  = x_i^2 x_{i-1} x_i^{\ell-2} + x_i [x_{i-1}, x_i] x_i^{\ell-2} + t_{i-1} g_{i-1} x_i^{\ell-1} \\
  = \cdots \\
  = x_i^\ell x_{i-1} + \sum_{k=1}^{\ell} x_i^{k-1} t_{i-1} g_{i-1} x_i^{\ell-k} \\
  = x_i^\ell x_{i-1} + t_{i-1} \sum_{k=1}^{\ell} \zeta^{k-\ell} x_i^{\ell-1} g_{i-1}
\end{align*}
\]
\[
= x_\ell^i x_{i-1} + t_{i-1} x_i^\ell-1 g_{i-1} \left( \sum_{k=1}^{\ell} \zeta^{k-\ell} \right)
\]
\[
= x_\ell^i x_{i-1}
\]
A similar argument holds for \(x_{i+1}\). □

The homogeneous element \(w_0 \in Z_0\) of degree \(n\) is not in \(Z_t\) in general. However, after an adjustment by several lower degree terms involving the parameter \(t\), we can construct an element \(w_t \in Z_t\) which is equal to \(w_0\) when \(t = 0\).

**Example 7.** When \(n = 3\) and \(\ell = 2\), let
\[
w_t = x_1 x_2 x_3 - \frac{1}{2} (t_1 x_3 g_1 + t_2 x_1 g_2 + t_3 x_2 g_3)
\]
It was shown in [4] that \(w_t\) is in the center of \(A_t\).

Informally, for any \(n \geq 3\) and \(\ell \geq 2\), the element \(w_t\) which we will define below is just a linear combination of monomials, each of which can be obtained from \(w_0\) by replacing zero or more occurrences of \(x_i x_{i+1}\) with \([x_i, x_{i+1}] = t_i g_i\). More precisely, we make the following definitions:

\[
I_k = \left\{ (i_1, i_2, \ldots, i_k) \in \mathbb{N}_k \mid \begin{array}{c}
1 \leq i_1 < i_2 < \cdots < i_k \leq n \\
i_{r+1(\text{mod } k)} \neq i_{r+1(\text{mod } n)}
\end{array} \right\}
\]

\[
w(i_1, i_2, \ldots, i_k) = \begin{cases}
\left( \frac{1}{\zeta-1} \right)^k x_1 x_2 \cdots \hat{x}_{i_1} \hat{x}_{i_1+1} \cdots \hat{x}_{i_2} \hat{x}_{i_2+1} \cdots \\
\cdots \hat{x}_{i_k} \hat{x}_{i_k+1} \cdots x_n & \text{if } i_k < n \\
 \zeta \left( \frac{1}{\zeta-1} \right)^k x_2 \cdots \hat{x}_{i_1} \hat{x}_{i_1+1} \cdots \hat{x}_{i_2} \hat{x}_{i_2+1} \cdots \\
\cdots \hat{x}_{i_k-1} \hat{x}_{i_k-1+1} \cdots x_{n-1} & \text{if } i_k = n
\end{cases}
\]
Finally, $w_t$ can be defined.

\[
w_t = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{i \in I_k} t_i w_i g_i
\]  

(3.2)

**Lemma 8.** One has $w_t \in Z_t$

**Proof.** The proof proceeds in two parts. Firstly, show that $w_t$ commutes with $g_j$ for each $j$, and secondly show that $w_t$ commutes with $x_j$ for each $j$.

**Claim 9.** We show that $g_j w_t = w_t g_j$ for all $j \in \{1, \ldots, n\}$.

**Proof.** It suffices to check that $g_j(w_i g_i) = (w_i g_i) g_j$ for each $i \in I_k$ for $k \in \{0, 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\}$.

It is clear that

\[
g_j w_0 = w_0 g_j
\]

\[
g_j(w_i) = \zeta(w_i) g_j \quad \text{if } i \text{ contains } i_{j+1} \text{ and not } i_{j-1}.
\]

In this case, $g_j(g_i) = \zeta^{-1}(g_i) g_j$.

\[
g_j(w_i) = \zeta^{-1}(w_i) g_j \quad \text{if } i \text{ contains } i_{j-1} \text{ and not } i_{j+1}.
\]

In this case, $g_j(g_i) = \zeta(g_i) g_j$.

\[
g_j(w_i) = (w_i) g_j \quad \text{if } i \text{ contains } i_{j-1} \text{ and } i_{j+1}.
\]

In this case, $g_j(g_i) = (g_i) g_j$.

If $i$ contains neither $i_{j-1}$, nor $i_{j+1}$, then it may contain $i_j$ or not. In either case, $g_j(w_i) = (w_i) g_j$ and $g_j(g_i) = (g_i) g_j$.

Thus, in all cases $g_j(w_i g_i) = (w_i g_i) g_j$. 

\[\square\]
Claim 10. We show that \( x_jw_t = w_t x_j \) for all \( j \in \{1, \ldots, n\} \).

Proof. Writing \( x_jw_t \) and \( w_t x_j \) in terms of the PBW basis will show that \( x_jw_t - w_t x_j \) is zero. In order to do this, it is helpful to partition the set of terms of \( w_t \) in such a way that the sum over each set commutes with \( x_j \). Consider the case where \( j \in \{3, 4, \ldots, n - 3, n - 2\} \). To define the partition, first choose two terms of \( w_t \).

\[
\begin{aligned}
s &= t_a w_a g_a \quad \text{for } a = (a_1, \ldots, a_{k_1}) \in I_{k_1} \\
q &= t_b w_b g_b \quad \text{for } b = (b_1, \ldots, b_{k_2}) \in I_{k_2}
\end{aligned}
\]

Note that \( w_a \) and \( w_b \) are scalar multiples of \( x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n} \) and \( x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n} \) respectively, where each \( \mu_i \) and each \( \nu_i \) is either zero or one. By using associativity of multiplication, \( x^\mu \) and \( x^\nu \) may be written as

\[
\begin{aligned}
x^\mu &= s_1 \left( x_{j-1}^{\mu_{j-1}} x_j^{\mu_j} x_{j+1}^{\mu_{j+1}} \right) s_2 \\
x^\nu &= q_1 \left( x_{j-1}^{\nu_{j-1}} x_j^{\nu_j} x_{j+1}^{\nu_{j+1}} \right) q_2
\end{aligned}
\]

Place \( s \) and \( q \) in the same equivalence class if \( s_1 = q_1 \) and \( s_2 = q_2 \). It is clear that this is an equivalence relation. The number of equivalence classes depends on \( n \). If \( s \) is an element of an equivalence class \( E \), then \( s \) is completely determined by \( \left( x_{j-1}^{\mu_{j-1}} x_j^{\mu_j} x_{j+1}^{\mu_{j+1}} \right) \), and by whether \( g_{j-1} \) or \( g_j \) is a factor of \( s \) in the case that \( \mu_{j-1} = \mu_j = \mu_{j+1} = 0 \). For each of the seven possible elements \( s \in E \), We rewrite \( x_j s \) and \( -s x_j \) in terms of the PBW basis, omitting all factors determined by \( E \):

\[
\begin{align*}
1 \left\{ 
\begin{array}{l}
 x_j(x_{j-1} x_j x_{j+1}) = x_{j-1} x_j x_{j+1} - \zeta^{-1} t_{j-1} x_j x_{j+1} g_{j-1} \\
 - (x_{j-1} x_j x_{j+1}) x_j = - x_{j-1} x_j x_{j+1} + t_j x_j x_{j+1} g_j 
\end{array}
\right.
\end{align*}
\]
\[
\begin{align*}
\text{2} & \quad x_j \left( \frac{1}{\zeta - 1} \right) (t_{j-2}x_jx_{j+1}g_{j-2}) = \left( \frac{1}{\zeta - 1} \right) t_{j-2}x_j^2x_{j+1}g_{j-2} \\
& \quad + \left( \frac{1}{\zeta - 1} \right) t_{j-2}tx_jg_{j-2}g_j \\
\text{3} & \quad x_j \left( \frac{1}{\zeta - 1} \right) (t_{j-1}x_{j+1}g_{j-1}) = \left( \frac{1}{\zeta - 1} \right) t_{j-1}x_jx_{j+1}g_{j-1} \\
& \quad - \zeta^{-1} \left( \frac{1}{\zeta - 1} \right) t_{j-1}tx_jx_{j+1}g_{j-1} \\
\text{4} & \quad x_j \left( \frac{1}{\zeta - 1} \right) (t_{j-1}x_{j-1}g_j) = \left( \frac{1}{\zeta - 1} \right) t_{j-1}x_jx_{j-1}g_j \\
& \quad - \zeta \left( \frac{1}{\zeta - 1} \right) t_{j-1}tx_jx_{j-1}g_j \\
\text{5} & \quad x_j \left( \frac{1}{\zeta - 1} \right) (t_{j+1}x_{j-1}x_jg_{j+1}) = \left( \frac{1}{\zeta - 1} \right) t_{j+1}x_jx_{j-1}^2g_{j+1} \\
& \quad - \zeta^{-1} \left( \frac{1}{\zeta - 1} \right) t_{j+1}tx_jx_{j-1}g_{j+1} \\
\text{6} & \quad x_j \left( \frac{1}{\zeta - 1} \right)^2 (t_{j-2}t_jg_jg_{j-2}) = \left( \frac{1}{\zeta - 1} \right)^2 t_{j-2}t_jx_jg_jg_{j-2} \\
& \quad - \zeta \left( \frac{1}{\zeta - 1} \right)^2 t_{j-2}tx_jx_jg_jg_{j-2} \\
\text{7} & \quad x_j \left( \frac{1}{\zeta - 1} \right)^2 (t_{j-1}t_{j+1}g_jg_{j-1}) = \left( \frac{1}{\zeta - 1} \right)^2 t_{j-1}t_{j+1}x_jg_jg_{j-1} \\
& \quad - \zeta^{-1} \left( \frac{1}{\zeta - 1} \right)^2 t_{j-1}t_{j+1}tx_jg_jg_{j-1} \\
\end{align*}
\]

Note that the terms in (2) and (6) only occur in those equivalence classes having \( \mu_{j-2} = 0 \). No terms except “d” and “e” terms occur in (2) and (6), and all such terms are
included. Similarly, the terms in (5) and (7) only occur if \( \mu_{j+2} = 0 \). Moreover, no terms except “\(g\)” and “\(h\)” terms occur in (5) and (7), and all such terms are included. Therefore, if the terms of the RHS sum to zero, then they sum to zero independently of whether \( x_{j-2} \) and \( x_{j+2} \) occur as factors of \( s \). It is easy to check that the terms associated to each label, “\(a\)”, “\(b\)”, etc., sum to zero. Therefore, for \( j \in \{3, 4, \ldots, n - 3, n - 2\} \), we have \( w_t x_j = x_j w_t \).

The other cases follow from a certain change of basis. Since the defining relations for \( A_t \) depend only on the indices modulo \( n \), there is an automorphism \( \phi \) of \( A_t \) sending \( t_i \) to \( t_{i+1} \mod n \), \( x_i \) to \( x_{i+1} \mod n \), and \( g_i \) to \( g_{i+1} \mod n \). Let \( w'_t \) be the image of \( w_t \) under this automorphism.

**Claim 11.** We show that \( w_t = w'_t \).

**Proof.** Let \( S \) denote the set of PBW basis elements with nonzero coefficient in \( w_t \). For \( k \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor \} \), let \( c_k = \frac{1}{(\zeta - 1)^k} \). Then, an arbitrary term of \( w_t \) is of the form

\[
t_i w_i g_i = \begin{cases} 
    c_k s & \text{if } i_k < n \\
    \zeta c_k s & \text{if } i_k = n 
\end{cases}
\]

for some \( s \in S \), \( k \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor \} \), and \( i = (i_1, \ldots, i_k) \in I_k \). Make the following definition for such indices \( i \):

\[
i + 1 = \begin{cases} 
    (i_1 + 1, i_2 + 1, \ldots, i_k + 1) & \text{if } i_k < n \\
    (1, i_1 + 1, i_2 + 1, \ldots, i_{k-1} + 1) & \text{if } i_k = n
\end{cases}
\]

Then the automorphism can be described as

\[
\phi(t_i w_i g_i) = \begin{cases} 
    t_{i+1} w_{i+1} g_{i+1} & \text{if } i_k < n \\
    t_{i+1} w_{i+1} g_{i+1} + ( \text{ lower degree terms } ) & \text{if } i_k = n
\end{cases}
\]

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Note that $\phi(t_i) = t_{i+1}$ and $\phi(g_i) = g_{i+1}$ even if $i_k = n$, since $g_i g_j = g_j g_i$ for $\|i - j\|_n \neq 1$.

If $i$ is an entry in $i$, then we say that $x_i$ and $x_{i+1}$ are “paired away” in $t_i u_1 g_i$. We must use caution with this terminology, because if $i$ contains $i - 1$ and $i + 1$, then $w_1$ does not have a factor of $x_i x_{i+1}$, however, $x_i$ and $x_{i+1}$ are not paired away.

Let $S_i$ be the subset of $S$ consisting of all monomials whose corresponding term in $w_t$ has $x_i$ and $x_{i+1}$ paired away. Write $S_{(i, \ldots, j)}$ for $S_i \cup \cdots \cup S_j$. Then the set of all monomials with a factor of $x_i x_{i+1}$ is exactly the set $S_{i-1,i,i+1}^c$. For each $i \in \{1, \ldots, n\}$, let $\psi_i : S_{i-1,i,i+1}^c \rightarrow S_i$ be the bijective map of sets that removes the factors $x_i$ and $x_{i+1}$, and inserts the factors $t_i$ and $g_i$ in the PBW order. This map corresponds to pairing away $x_i$ and $x_{i+1}$ for terms of $w_t$, but it ignores coefficients. Partition $S$ into the following subsets:

\begin{align*}
A_1 &= S_{n-2}^c \cap S_{n-1,n}^c \cap S_1^c \\
A_2 &= S_{n-2}^c \cap S_{n-1,n}^c \cap S_1 \\
A_3 &= S_{n-2} \cap S_{n-1,n}^c \cap S_1^c \\
A_4 &= S_{n-2} \cap S_{n-1,n} \cap S_1 \\
A_5 &= S_{n-1} \cap S_1^c \\
A_6 &= S_{n-1} \cap S_1 \\
A_7 &= S_{n-2}^c \cap S_n \\
A_8 &= S_{n-2} \cap S_n
\end{align*}

It is easy to check that these are pairwise disjoint sets whose union is $S$. The first four sets are those including $x_n$ as a factor, so after applying $\phi$, the commutation relations of $A_t$ apply to give several lower degree terms. Therefore, for a monomial $s$ in
any of the first four subsets of $S$, $\phi(s) \in A_t$ is not even in $S$. Write $\varphi$ for the function that adds 1 modulo $n$ to the indices, but rewrites the variables in the PBW order (as if all commutation relations were zero). Notice that $\phi$ and $\varphi$ are equal functions on $S_{n-1} \cup S_n = \bigcup_{i=5}^{8} A_t$, because there is no $x_1$ at the end of the resulting monomial. Notice also that if $s \in S$, then the $A_i$ that contains $s$ is completely determined by which variables are present, so $\varphi$ applied to this partition remains a partition. Finally, a monomial $s \in S$ has coefficient of the form $\zeta c_k = \frac{\zeta - 1}{\zeta} c_k$ in $w_t$ if and only if $s \in A_7 \cup A_8 = S_n = \phi(A_5 \cup A_6)$. If $s \in \bigcup_{i=1}^{6} A_i$, its coefficient has the form $c_k = \frac{1}{\zeta - 1} c_k$.

Suppose $t_1 w_1 g_1 = c_k s_1$, where $s_1 \in A_1$. then $s_1$ includes the factors $x_1, x_{n-1}$ and $x_n$, and thus $\phi(s_1)$ includes the factors $x_2, x_n$ and $x_1$, in that order. Therefore, when applying $\phi$ to $w_t$, we have

$$
\phi(c_k s_1) = \phi(t_1 w_1 g_1) = c_k t_{i+1} x_2 \cdots x_n x_1 g_{i+1}
$$

$$
= c_k t_{i+1} x_2 \cdots x_n x_1 g_{i+1} + c_k t_{i+1} x_2 \cdots x_1 \hat{x}_n t_n g_n g_{i+1}
$$

$$
= c_k t_{i+1} x_2 \cdots x_n g_{i+1} + c_k t_{i+1} x_2 \cdots x_1 \hat{x}_n (t_n g_n) g_{i+1} - c_k t_{i+1} (t_1 g_1) \hat{x}_1 \hat{x}_n \cdots x_n g_{i+1}
$$

$$
= t_{i+1} w_1 g_{i+1} + c_k \psi_n(t_{i+1} x_2 \cdots x_n g_{i+1}) - c_k \psi_1(t_{i+1} x_1 x_2 \cdots x_n g_{i+1})
$$

$$
= c_k \varphi(s_1) + c_k \psi_n(\varphi(s_1)) - c_k \psi_1(\varphi(s_1))
$$

The first term here, $c_k \varphi(s_1)$, is just another term of $w_t$. The next two terms come from commuting $x_1$ past $x_n$ and $x_2$, respectively. If we remove the factor $x_2$, or $x_n$, or if we remove both from each monomial in $\varphi(A_1)$, we obtain the sets $\varphi(A_2), \varphi(A_3)$, and $\varphi(A_4)$, respectively. Therefore, taking $s_2, s_3,$ and $s_4$ in the sets $A_2, A_3, A_4$, respectively, we obtain
\[ \phi(s_1) = \varphi(s_1) + \psi_n(\varphi(s_1)) - \psi_1(\varphi(s_1)) \]

\[ \phi(s_2) = \varphi(s_2) + \psi_n(\varphi(s_2)) \]

\[ \phi(s_3) = \varphi(s_3) - \psi_1(\varphi(s_3)) \]

\[ \phi(s_4) = \varphi(s_4) \]

For the first equation, note that \( \psi_{n-1} : A_1 \to A_5 \) and thus \( \psi_n : \phi(A_1) \to \phi(A_5) \) are bijective maps which lower degree by 2. Suppose \( s_1 \), and thus \( \varphi(s_1) \) each have coefficient \( c_k \) in \( w_t \). If \( \psi_{n-1}(s_1) = s_5 \in A_5 \), then \( s_5 \) has coefficient \( c_{k+1} \) in \( w_t \) and \( \phi(s_5) = \varphi(s_5) = \psi_n(\varphi(s_1)) \) has coefficient \( \zeta c_{k+1} \) in \( w_t \). Therefore, when applying \( \phi \) to \( w_t \), one portion reads

\[ \phi(c_k s_1 + c_{k+1} s_5) = c_k \varphi(s_1) + c_{k+1} \varphi(s_5) + c_k \psi_n(\varphi(s_1)) - c_k \psi_1(\varphi(s_1)) \]

\[ = c_k \varphi(s_1) + (c_{k+1} + c_k) \varphi(s_5) - c_k \psi_1(\varphi(s_1)) \]

\[ = c_k \varphi(s_1) + (\zeta c_{k+1}) \varphi(s_5) - c_k \psi_1(\varphi(s_1)) \]

The second equation is similar, except that the useful correspondence is \( \psi_n : \phi(A_1) \to \phi(A_6) \). For \( \psi_n(\varphi(s_2)) = \phi(s_6) = \varphi(s_6) \), the analogous calculation is:

\[ \phi(c_k s_2 + c_{k+1} s_6) = c_k \varphi(s_2) + c_{k+1} \varphi(s_6) + c_k \psi_n(\varphi(s_2)) \]

\[ = c_k \varphi(s_2) + (c_{k+1} + c_k) \varphi(s_6) \]

\[ = c_k \varphi(s_2) + (\zeta c_{k+1}) \varphi(s_6) \]

So far, every monomial of the form \( \varphi(A_1 \cup A_2 \cup A_5 \cup A_6) \) is accounted for, and has the correct coefficient by the equation \( c_{k+1} + c_k = \zeta c_{k+1} \). In the same way, use the bijections \( \psi_n : A_1 \to A_7 \) and \( \psi_n : A_3 \to A_8 \) to account for other terms, using the same equation in this form: \( \zeta c_{k+1} - c_k = c_{k+1} \). The set \( A_4 \) stands alone permuting terms.
of $w_t$ under the automorphism $\phi$. Therefore, $\phi(w_t) = w'_t$ is equal to $w_t$, proving claim (11).

In turn, the proof of this claim proves claim (10), because it shows that $w_t$ commutes with all $x_j$ for $j \in \{1, \ldots, n\}$.

The proof that $w_t \in Z_t$ is now therefore complete.

We are now able to state and prove the main theorem of chapter 3:

**Theorem 12.** $Z_t = \mathbb{C}[x_1^\ell, \ldots, x_n^\ell, w_t]$

It remains to show that $Z_t \subset \mathbb{C}[x_1^\ell, \ldots, x_n^\ell, w_t]$. In order to do this we use an associated graded argument. Note that the filtration on $A_t$ induces a filtration on $Z_t$. Denote both filtrations by $F_*$. We say $p$ has filtration degree $k$ if $p \in F_k A_t$, but $p \notin F_{k-1} A_t$. Denote the symbol map for $A_t$ by $\sigma : A_t \to \text{gr}(A_t)$, and write $\sigma$ for its restriction to $Z_t$ also. The map $\sigma$ takes an element $q$ of filtration degree $m$ to $\sigma_m(q)$, which is defined to be the image of $q$ under the natural projection map $F_m \to F_m A_t / F_{m-1} A_t \to \text{gr}(A_t)$. The symbol map is multiplicative, but not additive. Let $q \in A_t$, and let $p \in Z_t$ have filtration degree $k$.

$$\sigma(p) \cdot \sigma(q) = \sigma(p \cdot q) = \sigma(q \cdot p) = \sigma(q) \cdot \sigma(p) \quad (3.3)$$

Thus $\sigma(p) = \sigma_k(p) \in F_k A_t / F_{k-1} A_t$ is in the center of $\text{gr}(A_t)$, which is just $A_0$ by the PBW property.

Let $M_k$ be the set of all $n$-tuples of non-negative integers $\mu = (\mu_1, \ldots, \mu_{n+1})$ such that $n \mu_{n+1} + \sum_{i=1}^n \ell \mu_i = k$. Since $Z_0 = \mathbb{C}[x_1^\ell, \ldots, x_n^\ell, w_0]$, we can choose a representative $\psi$ of $\sigma_k(p)$ in $F_k A_t$ as follows:
\[ \psi = \sum_{\mu \in M_k} \nu_\mu x_1^{\ell_\mu_1} \cdots x_n^{\ell_\mu_n} w_{n+1}^{\mu_n+1} \]

where \( \{ \nu_\mu \}_{\mu \in M_k} \) is a set of complex number coefficients. Let

\[ \tilde{\psi} = \sum_{\mu \in M_k} \nu_\mu x_1^{\ell_\mu_1} \cdots x_n^{\ell_\mu_n} w_{l+1}^{\mu_n} \]

Then \( \tilde{\psi} \in Z_l \) and \( \sigma(\tilde{\psi}) = \psi \). So \( p - \tilde{\psi} \in Z_l \) and it has filtration degree strictly less than \( k \). By induction on the filtration degree, \( p \) is a polynomial in \( x_1^{\ell} \cdots x_n^{\ell} \), and \( w_l \).
Chapter 4

Representation theory of $A_t$ for $n = 3$

4.1 PI Algebras

For the convenience of the reader, we collect here some of the definitions and results on PI algebras which we shall use; more details may be found in [2]. Let $\mathbb{Z}(x_1, \ldots, x_n) = \mathbb{Z}(\mathbf{x})$ denote the free algebra generated by $x_1, \ldots, x_n$. A ring $R$ satisfies $f \in \mathbb{Z}(\mathbf{x})$ if $f(r_1, \ldots, r_n) = 0$ for all $r_i \in R$. In this case we call $f$ a polynomial identity of $R$. The ring $R$ is a Polynomial Identity ring or PI ring if it satisfies $f$ for some monic $f$. (The function $f$ is monic if at least one of its highest degree monomials has coefficient 1.)

**Definition:** The **minimal degree** of a PI ring $R$ is the least possible degree of a polynomial identity $f$. If $R$ is commutative, $f(r_1, r_2) = r_1 r_2 - r_2 r_1 = 0$ for all $r_1, r_2 \in R$, so commutative rings are examples of $PI$ rings with minimal degree 2.

The following theorem, due to Posner, applies to rings that are prime. A ring
R is prime if \( a, b \in R \) and \( arb = 0 \) for all \( r \in R \) implies either \( a = 0 \) or \( b = 0 \). For example, any integral domain is prime, and any simple ring is prime.

**Theorem 13.** (Posner) If \( R \) is a prime PI ring with center \( Z \) and minimal degree \( d \), let \( S = Z \setminus \{0\} \), \( Q = RS^{-1} \), and \( F = ZS^{-1} \). Then \( d \) is even and \( \dim_F(Q) = \left(\frac{d}{2}\right)^2 \).

In this case, we define the **PI-degree** of \( R \) to be \( \sqrt{\dim_F(Q)} = d/2 \) and write PI-deg(\( R \)) to denote this number. Suppose PI-deg(\( R \)) = \( n \). If \( R \) can be generated as a \( Z \)-module by \( t \) elements, then \( \dim_F(Q) \leq t \). In other words, \( n^2 \leq t \).

**Proposition 14.** \( A_t \) is a prime PI algebra.

**Proof.** Since \( A_t \) is a finitely generated module over its center, it is a PI algebra. Since the \( G \)-action on \( SV \) is faithful, the algebra \( A_0 \) is prime by ([15], Cor. 12.6). Since \( \text{gr}(A_t) = A_0 \), it follows that \( A_t \) is prime. \( \square \)

For the sake of generality, The results (15) through (18) below apply to any algebraically closed field \( k \), though in the rest of this paper \( \mathbb{C} \) will be used. Assume for these results that \( R \) is finitely generated as a \( k \)-algebra and that \( R \) is finitely generated as a module over its center, \( Z \) (so \( R \) is a PI algebra).

**Lemma 15.** (Artin, Tate) \( Z \) is a finitely generated \( k \)-algebra, hence Noetherian.

**Proof.** A proof can be found in ([16], Thm. 4.2.1). \( \square \)

**Proposition 16.** Any simple \( R \)-module \( M \) is finite dimensional.

**Proof.** A proof can be found in ([2], Prop. III.1.1, part 4). \( \square \)

**Proposition 17.** Let \( R \) be a prime PI algebra with PI-deg(\( R \)) = \( n \), and let \( M \) be a simple \( R \)-module. Then
1. $ann_R(M) \cap Z$ is a maximal ideal $\mathfrak{m}$ of $Z$, and every maximal ideal of $Z$ arises in this way.

2. $\dim(M) \leq n$

3. If $\dim(M) = n$, then $M$ is the unique simple $R$-module annihilated by $\mathfrak{m}$.

Proof. The statement (1) is proved in ([2], Prop. III.1.1). Part (2) is proved in ([1], Prop. 3.1a). Lastly, part (3) follows from ([1], Prop. 3.1b), together with ([2], Prop. III.1.6).

Let $U$ be the set of $\mathfrak{m} \in \text{MaxSpec } Z$ such that there is a simple $R$-module $M$ with $\dim(M) = n$ and $\mathfrak{m} \subset \text{ann}_R(M)$.

Theorem 18. $U$ is an open dense subset of $\text{MaxSpec}(Z)$.

The set $U$ in Theorem (18) is known as the Azumaya locus of $R$ over $Z$. The theorem states that for a generic central character $\chi$ of $Z$, there exists a unique simple $R$-module $M$ with $\dim(M) = n$ and the center of $R$ acts on $M$ via $\chi$. For a proof, consult ([2], III.1.7).

4.2 Generic simple modules for $n = 3$ and $t = 0$

From now on, we assume that $n = 3$. Moreover, we shall consider $t_1, t_2, t_3$ as variables. In other words, we now consider $A_t$ as an algebra over $\mathbb{C}[t_1, t_2, t_3]$. Let us first examine the simple modules for $A_0 = SV_{\#_a}G$ in the case $n = 3$, so that $G \cong (\mathbb{Z}/\ell\mathbb{Z})^2$.

The center of this algebra is $(SV)^G = \mathbb{C}[x_1^\ell, x_2^\ell, x_3^\ell, x_1x_2x_3]$. For a given central character $\chi : (SV)^G \to \mathbb{C}$, if

$$a = \chi(x_1^\ell), \ b = \chi(x_2^\ell), \ c = \chi(x_3^\ell), \ d = \chi(x_1x_2x_3),$$
then the equation \( abc = d^\ell \) must hold. Conversely, for all \( a, b, c, d \in \mathbb{C} \) satisfying this equation, there is some representation with central character \( \chi \). For the remainder of the section, fix a generic central character \( \chi \), and let \( a, b, c, d \in \mathbb{C} \) be as above. Consider the following \( G \)-orbit in \( V \):

\[
\mathcal{O}_\chi = \{ (x, y, z) \in V \mid x^\ell = a, y^\ell = b, z^\ell = c, xyz = d \}
\]

It is clear that \( \mathcal{O}_\chi \) is a non-empty \( G \)-stable subset of \( V \) on which the action of \( G \) is both transitive and free. Thus we may choose an identification of \( \mathcal{O}_\chi \) with \( G \). By writing each element of \( \mathcal{O}_\chi \) as its corresponding element \( g \in G \), the action of \( G \) on \( \mathcal{O}_\chi \) becomes simply the action of \( G \) on itself by left translation. Let \( \text{Fun}(\mathcal{O}_\chi) \) denote the vector space of \( \mathbb{C} \)-valued functions on \( \mathcal{O}_\chi \), and denote the characteristic function of \( g \in \mathcal{O}_\chi \) by \( e_g \in \text{Fun}(\mathcal{O}_\chi) \). The set \( \{e_g\}_{g \in \mathcal{O}_\chi} \) forms a basis for the vector space, which therefore has dimension \( \ell^2 = |G| \). The \( G \)-action on \( \mathcal{O}_\chi \) induces an action on \( \text{Fun}(\mathcal{O}_\chi) \) by \((g.f)(h) = f(g^{-1}h)\) for all \( g \in G, f \in \text{Fun}(\mathcal{O}_\chi) \), and \( h \in \mathcal{O}_\chi \). In terms of basis elements \( e_h \), the action is given by the formula \( g.e_h = e_{gh} \) for all \( g \in G \).

We define multiplication on the vector space \( \text{Fun}(\mathcal{O}_\chi) \) by pointwise multiplication of functions. Then it is easy to show that the map

\[
\text{res}_\chi : SV \to \text{Fun}(\mathcal{O}_\chi) : f \mapsto (f|_{\mathcal{O}_\chi} : (x, y, z) \mapsto f(x, y, z)),
\]

is a \( \mathbb{C}G \)-algebra homomorphism.

**Claim 19.** The map \( \text{res}_\chi \) is surjective, and factors through \( (\ker \chi) \), the ideal in \( SV \) generated by \( \ker \chi \). The dimension of \( \frac{SV}{(\ker \chi)} \) is therefore greater than or equal to \( \ell^2 = |G| \).

**Proof.** Let \( f \in \ker \chi \). Since \( f \in (SV)^G \), it is a polynomial in \( x_1^\ell, x_2^\ell, x_3^\ell \), and \( x_1x_2x_3 \). Therefore, for any \( (x, y, z) \in \mathcal{O}_\chi \), \( f(x, y, z) \) is a polynomial in \( a, b, c, \) and \( d \) which is equal
to $\chi(f)$. Since $\chi(f) = 0$, the function $\text{res}_\chi(f) : O_\chi \to \mathbb{C}$ is the zero function. Moreover, any function on a finite subset of a vector space can be extended to a polynomial function on that vector space. Thus, we have a surjective map

$$\text{res}_\chi : \frac{SV}{(\ker \chi)} \to \text{Fun}(O_\chi),$$

which shows that $\dim \frac{SV}{(\ker \chi)} \geq \ell^2$. 

\textbf{Claim 20.} The domain $\frac{SV}{(\ker \chi)}$ is spanned by less than or equal to $\ell^2$ elements.

\textbf{Proof.} Clearly, the set $\{x_1^{p_1}x_2^{p_2}x_3^{p_3} | 0 \leq p_1, p_2, p_3 < \ell\}$ of $\ell^3$ monomials spans $\frac{SV}{(\ker \chi)}$. Any element $x_1^{p_1}x_2^{p_2}x_3^{p_3}$ from this spanning set may be written as

$$\frac{1}{\chi(x_1x_2x_3)^{\ell-p_3}(x_1x_2x_3)^{p_1}} \chi(x_1^\ell) = \chi(x_3^\ell) \frac{x_1^{p_1+\ell-p_3}x_2^{p_2}x_3^{\ell-p_3}}{x_1^{p_1}x_2^{p_2}} \chi(x_1x_2x_3)^{-p_3} \chi(x_1x_2x_3)^{\ell-p_3}$$

If $p_1$ or $p_2$ are greater than or equal to $p_3$, then factor out $x_1^{\ell}$ or $x_2^{\ell}$, respectively, to obtain the additional scalars $\chi(x_1^{\ell})$ or $\chi(x_2^{\ell})$. \qedsymbol

So in fact, $\frac{SV}{(\ker \chi)}$ is exactly of dimension $\ell^2$ and $\text{res}_\chi$ gives a $\mathbb{C}G$-algebra isomorphism from $\frac{SV}{(\ker \chi)}$ to $\text{Fun}(O_\chi)$. The simple modules of $SV\#_\alpha G$ for a generic central character $\chi$ are precisely the simple modules of $SV/(\ker \chi)\#_\alpha G \cong \text{Fun}(O_\chi)\#_\alpha G$.

\textbf{Claim 21.} There is a $\mathbb{C}$-algebra isomorphism $\text{Fun}(O_\chi)\#_\alpha G \cong \text{Mat}_{|G| \times |G|}(\mathbb{C})$.

\textbf{Proof.} We will need the converse part of the statement in Theorem 3.2.1 of [10], originally a statement of Wedderburn.

\textbf{Theorem 22.} (Wedderburn). If $\mathcal{A}$ is any finite-dimensional simple algebra over $\mathbb{C}$ with unit, then there is a finite-dimensional complex vector space $V$ such that $\mathcal{A} \cong \text{End}(V)$.  

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The unit in $\text{Fun}(O_\chi)^\#_a G$ is $1^\#_a 1$. As vector spaces, one has

$$\text{Fun}(O_\chi)^\#_a G = \text{Fun}(O_\chi) \otimes_C CG,$$

so both have dimension $|G|^2 = \ell^4$. By Wedderburn’s theorem above, it suffices to show that $\text{Fun}(O_\chi)^\#_a G$ is simple. Suppose $I$ is a nonzero two-sided ideal of $\text{Fun}(O_\chi)^\#_a G$. Let $m = \sum_{k,h} a_{k,h} e_k \otimes h$ be a nonzero element in the ideal. Say $a_{i,g} \neq 0$. By replacing $m$ with $mg^{-1}$, we can assume $a_{i,1} \neq 0$. Then

$$e_i m = \sum_h a_{i,h} e_i \otimes h$$

is a nonzero element in the ideal. We have,

$$e_i me_i = \sum_h a_{i,h} e_i e_{hi} \otimes h = a_{i,1} e_i \otimes 1 \neq 0$$

is in the ideal. So $e_i \otimes 1 \in I$. But then $e_{h_1} \otimes h_2 \in I$ for any $h_1, h_2 \in G$, since $h_1i^{-1}(e_i \otimes 1)h_1^{-1}i h_2 = e_{h_1} \otimes h_2$. This proves the claim.

\[ \square \]

The algebra $\text{Mat}_{|G| \times |G|}(C)$ has a unique simple module, $C^{[G]}$. So $\frac{SV}{(565, X)}^\#_a G$ has a unique simple module of dimension $|G|$. Therefore, by proposition (17), the PI degree of $A_t$ has to be greater than or equal to $|G| = \ell^2$.

### 4.3 Generic simple modules for $n = 3$

Let $\tilde{Z}_t$ be the subalgebra of $A_t$ generated by $Z_t$ and $x_3$. Since $x_3^t \in Z_t$, $\tilde{Z}_t$ is a finitely generated $Z_t$-module. Therefore, the Going Up theorem from commutative algebra implies that any homomorphism $\chi : Z_t \rightarrow \mathbb{C}$ can be extended to a homomorphism $\tilde{\chi} : \tilde{Z}_t \rightarrow \mathbb{C}$. 

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Let $M(t, \tilde{\chi}) = A_t e \otimes_{\tilde{Z}_t} C_{\tilde{\chi}}$, where $C_{\tilde{\chi}}$ is the one-dimensional $\tilde{Z}_t$-module with action given by $\tilde{\chi}$, and $e$ is the idempotent defined by equation (4.1).

$$e = \frac{1}{\ell} \sum_{i=0}^{\ell-1} g_i^j$$

(4.1)

The centralizer of $\tilde{Z}_t$ in $A_t$ contains $e$, so $A_t e$ is a right $\tilde{Z}_t$-module. If $\alpha_t \in A_t$, then write $\alpha_t e$ for the element $\alpha_t e \otimes 1 \in A_t e \otimes_{\tilde{Z}_t} C_{\tilde{\chi}}$. Since $g_1 e = e$ and $x_3 e = ex_3$, $M(t, \tilde{\chi})$ is spanned by the monomials $x_1^i x_2^j g_2^p e$ for $0 \leq i, j, p < \ell$.

**Proposition 23.** For generic $(t, \chi)$, $M(t, \tilde{\chi})$ is spanned as a vector space by the $\ell^2$ elements $x_1^i x_2^j g_2^p e$, for $0 \leq j, p < \ell$.

**Proof.** It must be shown that $x_1^i x_2^j g_2^p e$ can be written as a linear combination of elements of the form $x_1^{i'} x_2^{j'} g_2^{p'} e$. Say that a term is “o.k.” if it is a scalar multiple of $x_1^{i'} x_2^{j'} g_2^{p'} e$ for some $0 \leq j', p' < \ell$. The proof is by induction on $i$, the exponent of $x_1$. If $i = 0$, there is nothing to prove. Suppose $i \geq 1$, and assume by induction hypothesis that $x_1^{i'} x_2^j g_2^p e$ can be written as a sum of o.k. terms when $i' < i$. 

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\[ x_1^j x_2^p e \]
\[ = \tilde{x}(x_3)^{-1} \tilde{x}(x_3) x_1^j x_2^p e \]
\[ = \tilde{x}(x_3)^{-1} x_1^j x_2^p x_3 e \]
\[ = \zeta^{-p} \tilde{x}(x_3)^{-1} x_1^j x_2^p x_3 e \]
\[ = \zeta^{-p} \tilde{x}(x_3)^{-1} x_1^{j-1} x_2^{j-1} (x_1 x_2 x_3) g_2^p e + (\text{o.k. terms}) \]
\[ = \zeta^{-p} \tilde{x}(x_3)^{-1} x_1^{j-1} x_2^{j-1} (w_1 - \frac{t_1}{\zeta - 1} x_3 g_1 - \frac{t_2}{\zeta - 1} x_1 g_2 - \frac{t_3}{\zeta - 1} x_2 g_3) g_2^p e + (\text{o.k. terms}) \]
\[ = \zeta^{-p} \tilde{x}(x_3)^{-1} \left( x_1^{j-1} x_2^{j-1} w_1 g_2^p e \right) \]
\[ \quad - \frac{t_2}{\zeta - 1} x_1^{j-1} x_2^{j-1} x_1 g_2 g_1 e \]
\[ \quad - \frac{t_3}{\zeta - 1} x_1^{j-1} x_2^{j-1} x_2 g_3 g_2 e \right) + (\text{o.k. terms}) \]
\[ = \left( -\frac{t_2}{\zeta - 1} \right) \zeta^{-p} \tilde{x}(x_3)^{-1} x_1^{j-1} x_2^{j-1} g_2^{p+1} e + (\text{o.k. terms}) \]

So we have, for all \(0 \leq j, p < \ell\),
\[ x_1^j x_2^p e \]
\[ + \left( \frac{t_2}{\zeta - 1} \right) \zeta^{-p} \tilde{x}(x_3)^{-1} x_1^{j-1} x_2^{j-1} g_2^{p+1} e = (\text{o.k. terms}) \]

Or equivalently
\[ \zeta^p \tilde{x}(x_3) x_1^j x_2^p e + \left( \frac{t_2}{\zeta - 1} \right) x_1^j x_2^{j-1} g_2^{p+1} e = (\text{o.k. terms}) \quad (4.2) \]

Setting \(j + p = \ell - 1\), we get a system of \(\ell\) equations in \(\ell\) unknowns. In matrix form:
\[
\begin{bmatrix}
  x_1^0 x_2^\ell e \\
  x_1^1 x_2^{\ell-1} e \\
  \vdots \\
  x_1^\ell x_2^0 e
\end{bmatrix} =
\begin{bmatrix}
  (\text{o.k. terms}) \\
  (\text{o.k. terms}) \\
  \vdots \\
  (\text{o.k. terms})
\end{bmatrix}
\]

\[ A =
\]

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where $A$ can be easily calculated as

$$\begin{bmatrix}
    \zeta^{-1}\tilde{\chi}(x_3x_2^\ell) & 0 & 0 & \cdots & \frac{t_2}{\zeta-1} \\
    \frac{t_2}{\zeta-1} & \zeta^{-2}\tilde{\chi}(x_3) & 0 & \cdots & 0 \\
    0 & \frac{t_2}{\zeta-1} & \zeta^{-3}\tilde{\chi}(x_3) & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & \frac{t_2}{\zeta-1} & \tilde{\chi}(x_3)
\end{bmatrix}$$

If $A$ is invertible, we may solve this system:

$$\begin{bmatrix}
    x_1^i x_2^0 g_2^{\ell-1} e \\
    x_1^i x_2^1 g_2^{\ell-2} e \\
    \vdots \\
    x_1^i x_2^{\ell-1} g_2^0 e
\end{bmatrix} = A^{-1} \begin{bmatrix}
    \text{(o.k. terms)} \\
    \text{(o.k. terms)} \\
    \vdots \\
    \text{(o.k. terms)}
\end{bmatrix} \quad (4.3)$$

For an arbitrary monomial $x_1^i x_2^j g_2^p e$ (where $j + p$ is not necessarily equal to $\ell - 1$), choose the equation coming from row $j + 1$ of system (4.3), and multiply both sides by $g_2^{p-(\ell-1-j)}$ on the left. This may have a rescaling effect, but still allows us to write an arbitrary monomial with a factor of $x_1^i$ as a linear combination of monomials with lower powers of $x_1$.

To see that $A$ is invertible, calculate the determinant. Expanding along the first row gives:
\[ \text{det}(A) = \zeta^{-1} \tilde{\chi}(x_3 x_2^\ell) \left( \prod_{k=2}^{\ell} \zeta^{-k} \tilde{\chi}(x_3) \right) + (-1)^{\ell-1} \frac{t_2}{\zeta - 1} \left( \frac{t_2}{\zeta - 1} \right)^{\ell-1} \]

\[ = \chi(x_2^\ell x_3^\tau) \left( \prod_{k=1}^{\ell} \zeta^{-k} \right) + (-1)^{\ell-1} \frac{t_2}{\zeta - 1} \left( \frac{t_2}{\zeta - 1} \right)^{\ell} \]

\[ = \zeta^{-\ell(\ell-1)/2} \chi(x_2^\ell x_3^\tau) + (-1)^{\ell-1} \left( \frac{t_2}{\zeta - 1} \right)^{\ell} \]

\[ = (-1)^{\ell-1} \left[ \chi(x_2^\ell x_3^\tau) + \left( \frac{t_2}{\zeta - 1} \right)^{\ell} \right] \]

Since the determinant of \( A \) is nonzero for generic \((t,\chi)\), the proof is complete. \( \square \)

**Lemma 24.** The module \( M(t, \tilde{\chi}) \) is nonzero.

**Proof.** Let \( z_1, \ldots, z_r \) be a maximal set of \( \tilde{Z}_t \)-linearly independent elements in \( A_t e \), and \( \alpha_1, \ldots, \alpha_s \) a set of generators of \( A_t e \) as a \( \tilde{Z}_t \)-module. For each \( i \) such that \( \alpha_i \not\in \{z_1, \ldots, z_r\} \), there are coefficients \( c_{ij}, d_i \in \tilde{Z}_t, d_i \neq 0 \) with

\[ d_i \alpha_i + c_{i1} z_1 + \cdots + c_{ir} z_r = 0 \quad (4.4) \]

For each \( i \) such that \( \alpha_i \in \{z_1, \ldots, z_r\} \), set \( d_i = 1 \). Let \( d = d_1 \cdots d_s \in \tilde{Z}_t \). Since \( \tilde{Z}_t \) is a domain (its associated graded algebra is a domain), \( d \neq 0 \). Consider the localization.

\[ \bigoplus_{i=1}^r \tilde{Z}_t \left[ d^{-1} \right] \to A_t e \otimes \tilde{Z}_t \left[ d^{-1} \right] \]

\[ (k_1, \ldots, k_r) \mapsto \sum_{i=1}^r z_i \otimes k_i \quad (4.5) \]

**Claim 25.** The map in (4.5) is an isomorphism

**Proof.** To show that this map is injective, choose an element \( \sum_{i=1}^r z_i \otimes k_i \) from the image and suppose \( \sum_{i=1}^r z_i \otimes k_i = 0 \). Then, for all \( 1 \leq i \leq r \) and for \( N \in \mathbb{Z} \) sufficiently large, \( d^N k_i \in \tilde{Z}_t \) and

\[ 0 = \sum_{i=1}^r z_i \otimes k_i d^N = \sum_{i=1}^r z_i (k_i d^N) \otimes 1, \]

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so \( \sum_{i=1}^r (k_i d^N) z_i = 0 \). By linear independence, \( k_i d^N = 0 \) for all \( i \), and so \( k_i = 0 \) for all \( i \).

To show the map is surjective, choose \( i \in \{1, \ldots, s\} \) and note that it suffices to show \( \alpha_i \otimes 1 \) is in the image. Equation (4.4) says \( d_i \alpha_i = \sum_{j=1}^r (-c_{ij}) z_j \). Write \( \lambda_j \) for \( \frac{d}{dt}(-c_{ij}) \in \tilde{Z}_t \), so that \( d \alpha_i = \sum_{j=1}^r \lambda_j z_j \). Consider the element

\[
\left( \frac{\lambda_1}{d}, \frac{\lambda_2}{d}, \ldots, \frac{\lambda_r}{d} \right) \in \bigoplus_{i=1}^r \tilde{Z}_t \left[ d^{-1} \right].
\]

The image of this element is

\[
\sum_{j=1}^r \frac{z_j}{d} \otimes \frac{\lambda_j}{d} = \sum_{j=1}^r \lambda_j z_j \otimes \frac{1}{d} = d \alpha_i \otimes \frac{1}{d} = \alpha_i \otimes 1.
\]

This completes the proof that (4.5) is an isomorphism. \( \square \)

The remaining portion of the proof that \( M(t, \tilde{\chi}) \neq 0 \) will use the fact that, for generic \( \chi \), one has \( \tilde{\chi}(d) \neq 0 \). This follows from:

**Claim 26.** If \( I \) is a prime ideal in the polynomial ring \( \mathbb{C}[X_1, \ldots, X_n] \) and \( d \) is not contained in \( I \), then \( V(I) \) is not contained in \( V(d) \).

(Here, \( V(I) \) is the zero-locus of \( I \), and \( V(d) \) is the zero-locus of \( d \). The claim says that we can find a point on which \( I \) vanishes but \( d \) does not vanish.)

**Proof.** By the Hilbert Nullstellensatz, the ideal of all \( f \) in \( \mathbb{C}[X_1, \ldots, X_n] \) such that \( V(f) \) contains \( V(I) \) is equal to the radical \( \sqrt{I} \) of the ideal \( I \), where \( \sqrt{I} = \{ f \text{ such that sufficiently large powers of } f \text{ are in } I \} \). Thus, if \( V(I) \) is contained in \( V(d) \), then some power of \( d \) has to be in \( I \). But \( I \) is a prime ideal, so \( d \) is in \( I \), a contradiction. \( \square \)

Since \( \tilde{\chi}(d) \neq 0 \), we can extend \( \tilde{\chi} \) from \( \tilde{Z}_t \) to \( \tilde{Z}_t[d^{-1}] \). In this way, \( \mathbb{C}_{\tilde{\chi}} \) becomes a left \( \tilde{Z}_t[d^{-1}] \)-module, and when considered as such, it will be denoted \( \mathbb{C}_{\tilde{\chi}} \).
\[ M(t, \tilde{\chi}) = A_t e \otimes_{\mathbb{Z}_t} C_{\tilde{\chi}} \]
\[ = A_t e \otimes_{\mathbb{Z}_t} \hat{\mathbb{Z}}_{[d^{-1}]} \otimes_{\mathbb{Z}_t} [d^{-1}] C_{\tilde{\chi}} \]
\[ \cong \bigoplus_{k=1}^{r} \hat{\mathbb{Z}}_{[d^{-1}]} \otimes_{\mathbb{Z}_t} [d^{-1}] C_{\tilde{\chi}} \]
\[ = \bigoplus_{k=1}^{r} C_{\tilde{\chi}} \]
\[ \neq 0 \]

The following theorem is one of the main results of this thesis.

**Theorem 27.** The PI-degree of \( A_t \) is \( \ell^2 \). For generic \((t, \chi)\), \( M(t, \tilde{\chi}) \) is the unique simple \( A_t \)-module with central character \( \chi \).

**Proof.** Since we have a nonzero module \( M(t, \tilde{\chi}) \) with dimension at most \( \ell^2 \) for generic \( t \) and \( \chi \), it follows from Theorem (18) that the PI-degree of \( A_t \) is at most \( \ell^2 \). On the other hand, we have shown in Section 4.2 that the PI-degree of \( A_t \) is at least \( \ell^2 \). Therefore, the PI-degree of \( A_t \) is equal to \( \ell^2 \), and \( M(t, \tilde{\chi}) \) must be the unique simple \( A_t \)-module with central character \( \chi \) for generic \((t, \chi)\). \( \square \)

### 4.4 Morita equivalence with deformed Sklyanin algebras

For a fixed ring \( R \), one can form the category whose objects are all left \( R \)-modules, and whose morphisms are all of the \( R \)-module homomorphisms. Denote this category by \( \text{R-mod} \). If there is another ring \( S \), such that \( \text{R-mod} \) and \( \text{S-mod} \) are equivalent as categories, then we say that \( R \) and \( S \) are **Morita equivalent**. It is a well-known fact that if \( R \) is an associative algebra with an idempotent \( e \) satisfying \( ReR = R \), then \( R \) and \( eRe \) are Morita equivalent. For a proof, see ([9], Cor 2.3.4)
Definition 28. For \( i = 1, 2, 3 \), let \( e_i, d_i \in \mathbb{C} \). Let \( W = \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z \cong \mathbb{C}^3 \) be a 3-dimensional complex vector space. The deformed Sklyanin algebra \( S_{def} \) is the quotient of the tensor algebra \( TW \) by the relations

\[
\begin{align*}
ayz + bzy + cx^2 + d_1x + e_1 &= 0 \\
azx + bxz + cy^2 + d_2y + e_2 &= 0 \\
axy + byx + cz^2 + d_3z + e_3 &= 0
\end{align*}
\] (4.6)

where \((a, b, c) \in \mathbb{C}^3\) is generic.

Here, “generic” means that not all of \(a, b,\) and \(c\) are cube roots of unity, and \((a, b, c)\) does not lie on any coordinate axis, as in ([17], Def. IV.2).

Theorem 29. Let \( n = 3 \) and \( A_t \) the twisted graded Hecke algebra with parameters \( t_1, t_2, t_3 \in \mathbb{C} \). Then \( A_t \) is Morita equivalent to \( S_{def} \), where the parameters for \( S_{def} \) are

\[
\begin{align*}
a &= 1 \\
b &= \zeta \\
c &= 0 \\
d_1 &= 0 \\
d_2 &= 0 \\
d_3 &= 0 \\
e_1 &= -\zeta t_1 \\
e_2 &= -\zeta t_2 \\
e_3 &= -\zeta t_3
\end{align*}
\] (4.7)

More specifically, for the idempotent element \( e \) defined in equation (4.1), we have \( A_t e A_t = A_t \) and \( e A_t e \cong S_{def} \).

Proof. Let us first check that \( A_t e A_t = A_t \). It suffices to show that 1 is a linear combination of elements of the form \( g_2^j g_2^k \) for \( 0 \leq j, k \leq \ell - 1 \). This will also show that \( \mathbb{C}_\alpha Ge \mathbb{C}_\alpha G = \mathbb{C}_\alpha G \). Since \( g_2 g_1 = \zeta g_1 g_2 \), we have

\[
g_2^j g_2^k = g_2^j \left( \frac{1}{\ell} \sum_{i=0}^{\ell-1} g_1^i \right) g_2^k = \frac{1}{\ell} \sum_{i=0}^{\ell-1} \zeta^{ij} g_1^i g_2^{i+k} \] (4.8)
Define $e_{j,k}$ to be equal to $g_1^j g_2^k$. Then equation (4.8) implies $e_{j,-j} = \frac{1}{\ell} \sum_i \zeta^{ij} g_1^i$. Consider the following matrix equation:

$$\frac{1}{\ell} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \cdots & \zeta^{\ell-1} \\ 1 & \zeta^2 & \zeta^4 & \cdots & \zeta^{2(\ell-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{\ell-1} & \zeta^{(\ell-1)2} & \cdots & \zeta^{(\ell-1)(\ell-1)} \end{bmatrix} \begin{bmatrix} 1 \\ g_1 \\ g_1^2 \\ \vdots \\ g_1^{\ell-1} \end{bmatrix} = \begin{bmatrix} e_{0,0} \\ e_{1,-1} \\ e_{2,-2} \\ \vdots \\ e_{\ell-1,1-\ell} \end{bmatrix} (4.9)$$

By the Vandermonde formula, the determinant of this matrix is

$$\prod_{0 \leq i < j \leq \ell-1} (\zeta^j - \zeta^i) \neq 0$$

Therefore, by solving the corresponding system of $\ell - 1$ simultaneous equations and using the first equation, 1 can be written as a linear combination of the elements $e_{j,-j} \in \mathbb{C}_\alpha Ge \mathbb{C}_\alpha G$. This shows that $\mathbb{C}_\alpha Ge \mathbb{C}_\alpha G = \mathbb{C}_\alpha G$ and that $A_t e A_t = A_t$.

As the next step, we show that $e \mathbb{C}_\alpha Ge$ is one-dimensional. Any element of $e \mathbb{C}_\alpha Ge$ can be written as a linear combination of the elements $eg_1^j g_2^k e$ for $0 \leq j, k \leq \ell - 1$. But $g_1 e = e = eg_1$. So we need only consider elements of the form $eg_2^k e$. However, we have that

$$eg_2^k e = \frac{e}{\ell} \left( \sum_{i=0}^{\ell-1} \zeta^{ik} g_1^i \right) g_2^k = \frac{e}{\ell} \left( \sum_{i=0}^{\ell-1} \zeta^{ik} \right) g_2^k = \begin{cases} e & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

Therefore $e \mathbb{C}_\alpha Ge = e \mathbb{C} \cong \mathbb{C}$

Let

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\[ y_1 = x_1g_2 \]
\[ y_2 = x_2g_3 \]  \hspace{1cm} (4.10)
\[ y_3 = x_3g_1 \]

Let \( S \) be the subalgebra of \( A_t \) generated by \( y_1, y_2, y_3 \). Observe that \( g_1, g_2, \text{ and } g_3 \) each commute with each of \( y_1, y_2, \text{ and } y_3 \). For we have

\[ g_1y_1 = g_1x_1g_2 = \zeta x_1g_1g_2 = x_1g_2g_1 \]
\[ g_1y_2 = g_1x_2g_3 = \zeta^{-1} x_2g_1g_3 = x_2g_3g_1 \]  \hspace{1cm} (4.11)
\[ g_1y_3 = g_1x_3g_1 = x_3g_1g_1 \]

The calculations for \( g_2 \) and \( g_3 \) are essentially the same.

**Definition 30.** Let \( q \) be any complex number. For elements \( a_1 \) and \( a_2 \) of an algebra \( A \), define the \( q \)-commutator of \( a_1 \) with \( a_2 \) to be \([a_1,a_2]_q = a_1a_2 - qa_2a_1\).

Let us compute the \( \zeta \)-commutation relations for the \( y_i \), keeping in mind the ordinary commutation relations (2.1), (2.2), and (2.3) for \( A_t \).

\[ [y_1, y_2]_\zeta \]
\[ = x_1g_2x_2g_3 - \zeta x_2g_3x_1g_2 \]
\[ = \zeta x_1x_2g_2g_3 - x_2x_1g_3g_2 \]
\[ = x_1x_2g_3g_2 - x_2x_1g_3g_2 \]
\[ = (x_1x_2 - x_2x_1)g_3g_2 \]
\[ = [x_1, x_2]_\alpha (g_3, g_2)g_1^{-1} \]
\[ = \zeta^{-1}t_1g_1g_1^{-1} \]
\[ = \zeta t_1 \]  \hspace{1cm} (4.12)
We can compute the other two $\zeta$–commutators similarly. The three resulting equations are:

\begin{align*}
[y_1, y_2]_{\zeta} &= \zeta t_1 \\
[y_2, y_3]_{\zeta} &= \zeta t_2 \\
[y_3, y_1]_{\zeta} &= \zeta t_3
\end{align*}

(4.13)

These are exactly equations (4.6) with parameters set as in (4.7), if $x$, $y$, and $z$ are set to $y_3$, $y_1$ and $y_2$, respectively. Therefore $S$ is the deformed Sklyanin algebra $S_{def}$.

As shown in (4.11), $g_1$ and $g_2$ commute with the generators of $S$. As algebras, therefore, $A_t \cong S \otimes C_\alpha G$. So $eA_te = S \otimes eC_\alpha Ge \cong S$, since $eC_\alpha Ge \cong C_\alpha$. This shows that $eA_te$ is isomorphic to $S_{def}$. 

$\square$
Bibliography


