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Author
Sheiman, Jon

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HIGHER ORDER QCD CORRECTIONS TO DOUBLE MOMENT RATIOS IN DEEP INELASTIC SCATTERING

Jon Sheiman, Ian Hinchliffe, and H.E. Haber

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Section I

This paper is concerned with higher order QCD corrections to double moment ratios in deep-inelastic lepto-production. Before entering a discussion of the process in detail we will first make a few comments concerning such higher order QCD corrections* and ambiguities therein. Consider a process calculated through at least two orders in perturbation theory. For convenience write

\[ P = A + B \alpha_s(\mu^2) \]

where \( A \) and \( B \) are the same order in \( \alpha_s(\mu^2) \). \( B \) depends on several factors. If \( A \) contains the coupling constant \( \alpha_s(\mu^2) \), \( B \) will depend on the renormalization scheme used to define \( \alpha_s \) and on the scale \( \mu \) at which it is evaluated.** In cases where the parton model (e.g. the Drell-Yan process [2]) is used and \( P \) is a parton process then \( B \) will depend on the scale at which the parton distributions are evaluated. A criterion for the validity of perturbation theory is \( B \alpha_s << A \). Generally speaking if this criterion is satisfied then QCD is tested by fitting the data to the lowest order term \( A \); few data are sufficiently accurate to be able to detect the presence of a small \( B \) term. If the criterion is not satisfied then the process is useless as a quantitative QCD test.

Unfortunately it is true that by appropriate choice of scheme and scale(s) it is almost always possible to satisfy the criterion. The decision on the status of perturbation theory then rests on

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* For a relatively complete list of processes see Ref.[1].

** Strictly speaking the scheme and the scale \( \mu \) are equivalent, however it is convenient to think of them as independent.
whether those schemes and scales are reasonable (or on a calculation
to yet another order in $g_s$). Ideally one would like to have some
a priori method of choosing schemes and scales in order to remove
ambiguities in $B$. If this choice fails then the perturbation
expansion would be meaningless. Unfortunately no such method exists
rendering it extremely difficult to decide on the merits of
ad hoc a posteriori choices which happen to make $B$ small.

Three schemes are widely used, minimal subtraction (MS), mutilated
minimal subtraction (MS) [3] and momentum space subtraction [4]. The
former is unphysical but there is no reason to prefer either of the other
two, or indeed some other scheme. With regard to the scale of $\mu^2$,
if the process is characterized by only one physical scale $Q^2$, then
it seems natural to choose $\mu^2 = Q^2$. But who is to say that $2Q^2$ or
$Q^2/2$ is unreasonable? In processes with more than one physical
scale such as large $p_T$ hadron scattering [5] the situation is more
complicated and ambiguous.

The outline of this paper is as follows: Section II contains a
detailed discussion of the double moment ratios; Sections III and
IV contain a discussion of the graphs we calculate; finally Section
V contains our results and conclusions. We have elected not to
give detailed formula. A presentation of them would not clarify
this paper and would make it very long. We would be happy to supply
a formula for the final answer to anyone who requests it; either
on a roll of paper or on a stack of (400) fortran cards.

Section II

Consider one particle inclusive neutrino production:

$$W(q) + N(P_1) \rightarrow \pi(P_2) + X$$

where $W(q)$ is a virtual $W$ meson of momentum $q$ ($Q^2 \equiv -q^2 > 0$), $N$ is a
nucleon of momentum $P_1$, and $\pi$ is a pion of momentum $P_2$. We will describe
the process by the scaling variables $x_h$ and $z_h$

\begin{align*}
x_h &= \frac{2P_1 \cdot q}{Q^2} \quad , \quad 0 < x_h < 1 \\
z_h &= \frac{P_2 \cdot P_1}{P_1 \cdot q} \quad , \quad 0 < z_h < 1
\end{align*}

and define the semi-inclusive "cross-section" as follows:

$$\frac{dW_{h+}(x_h, z_h, Q^2)}{dx_h} = \frac{1}{4\pi} \left[ \frac{d^3P_2}{(2\pi)^{3/2}|2P_2|} \right] \int d^4x e^{i\mathbf{q} \cdot \mathbf{x}} \delta \left( z_h - \frac{P_2 \cdot P_1}{P_1 \cdot q} \right)$$

\begin{align*}
\frac{1}{x} \sum_{X < P_1 \cdot |J^\mu(x)||P_2|, X > < P_2 \cdot X J^\nu(0)|P_1 >}
\end{align*}

where $J^\mu$ is the charged weak current.

The QCD parton model relates the hadronic cross-section to the
analogously defined partonic cross-section for the process

$$W(q) + \text{Parton} (P_1) \rightarrow \text{Parton} (P_2) + X$$

With the definitions

\begin{align*}
x &= \frac{Q^2}{2P_1 \cdot q} \quad , \quad 0 < x < 1 \\
z &= \frac{P_2 \cdot P_1}{P_1 \cdot q} \quad , \quad 0 < z < 1
\end{align*}

* One could use the variable $\omega_h = \frac{-2P_2 \cdot q}{Q^2}$ instead of $x_h$. The former
choice has the advantage of being insensitive to target mass
effects [6]. We choose $x_h$ because it compresses the fragmentation
region (i.e. $P_2 \cdot P_1 = 0$) down to the single point $x_h = 0$. Thus
moment integrals can extend through all of phase space, simplifying
the calculation.
the relation is
\[
\frac{dW(x_h,z_h,Q^2)}{dx_h} = \int_0^1 \frac{dx}{x} \int_0^1 \frac{dz}{z} \mathcal{F}(x,z) \frac{dW(x,z,Q^2)}{dz} \mathcal{D}(z) .
\] (2.1)

In this equation, \(dW/dz\) is the partonic cross-section, \(\mathcal{F}\) is the partonic distribution function of the nucleon, and \(\mathcal{D}\) is the decay function of the final parton into the observed pion. We have dropped the usual sum over parton types on the assumption that non-singlet differences have been taken for both the nucleon and the pion. The Lorentz indices have been dropped on the assumption that a structure function has been projected out (see later).

Taking moments of Eq. (2.1) with respect to \(x_h\) and \(z_h\) gives
\[
\left\{ \frac{dW}{dx_h} \right\}^{(n,m)} = \mathcal{F}(n) \left( \frac{dW}{dx} \right)^{(n,m)} \mathcal{D}(m) ,
\] (2.2)

where
\[
\left\{ \frac{dW}{dx_h} \right\}^{(n,m)} = \int_0^1 dx_h x_h^{n-1} \int_0^1 dz_h z_h^{m-1} \frac{dW}{dx_h} .
\]

and
\[
\left\{ \mathcal{F} \right\}^{(P)} = \int_0^1 du P^{-1} \left( \frac{\mathcal{F}(u)}{\mathcal{D}(u)} \right) .
\]

The factorization theorem [7] guarantees that the infrared* (IR) singularities of \(dW/dz\) can be factored out and absorbed into a redefinition of \(\mathcal{F}\) and \(\mathcal{D}\). The factorization takes the following form in moment space:
\[
\left\{ \frac{dW}{dx} \right\}^{(n,m)} = \mathcal{F}(n) \left( \frac{dW}{dz} \right)^{(n,m)} \mathcal{D}(m) .
\] (2.3)

where all of the IR singularities of the right hand side reside in \(\mathcal{F}(n)\) and \(\mathcal{D}(m)\).

Performing IR renormalizations of \(\mathcal{F}\) and \(\mathcal{D}\) as follows
\[
\mathcal{F}(n) \equiv \mathcal{F}(n) \mathbf{P}(n) ,
\]
\[
\mathcal{D}(m) \equiv \mathbf{D}(m) \mathbf{P}(m) .
\]

Eq. (2.2) takes the manifestly finite form
\[
\left\{ \frac{dW}{dx_h} \right\}^{(n,m)} = \mathcal{F}(n) \left( \frac{dW}{dz} \right)^{(n,m)} \mathcal{D}(m) .
\] (2.4)

The zeroth order graph for \(dW/dz\) has a single particle final state, so by kinematics
\[
\frac{dW}{dz} = \frac{dW}{dz} + \mathbf{O}(a_s) = \Lambda \delta(l-x) \delta(1-z) + \mathbf{O}(a_s) .
\]

Thus
\[
\left\{ \frac{dW}{dz} \right\}^{(n,m)} = \Lambda \delta(n) + \mathbf{O}(a_s)
\]

and from Eq. (2.4)
\[
\left\{ \frac{dW}{dx_h} \right\}^{(n,m)} = \Lambda \delta(n) \mathbf{D}(m) + \mathbf{O}(a_s) .
\] (2.5)

The zeroth order result thus factorizes into a product of a function of \(n\) and a function of \(m\).

Sakai [8] has invented a double moment ratio whose deviation from unity measures the breaking of this factorization:
\[
R_{\mathcal{S}}(n,m;1,k)(Q^2) = \frac{\left\{ \frac{dW}{dx_h} \right\}^{(n,m)}(1,x) \frac{dW}{dx_h}(1,m) \left\{ \frac{dW}{dz} \right\}^{(n,m)}(1,x) \left\{ \frac{dW}{dz} \right\}^{(n,m)}(1,m)}{\left\{ \frac{dW}{dx_h} \right\}^{(n,m)}(1,x) \left\{ \frac{dW}{dx_h} \right\}^{(n,m)}(1,m) .
\]

* We will consider the term "infrared singularities" to mean both soft and collinear divergences.
The QCD parton model prediction for $R_s$ is, from Eq. (2.4) and Eq. (2.5)

$$ R_s(n,m; l,k) = \frac{\frac{dW}{dz}(n,k)}{\frac{dW}{dz}(n,m)} \frac{\frac{dW}{dz}(1,k)}{\frac{dW}{dz}(1,m)} \tag{2.6} $$

From Eq. (2.3), this can also be written as

$$ R_s(n,m; l,k) = \frac{\frac{dW}{dz}(n,m)}{\frac{dW}{dz}(n,k)} \frac{\frac{dW}{dz}(1,k)}{\frac{dW}{dz}(1,m)} \tag{2.7} $$

Of course, the IR singularities in Eq. (2.7) must cancel, since Eq. (2.6) gives $R_s$ as a manifestly finite quantity.

The essential property of $R_s$ is that the distribution and decay functions cancel out of it. Indeed, from Eq. (2.7) we see that the prediction for $R_s$ takes the form of a power series in $\alpha_s(\mu)$ with calculable numerical coefficients. The coefficients depend on $\mu^2$, where $\mu$ is the ultraviolet (UV) renormalization point, and on the UV renormalization scheme.* The only phenomenological input necessary is the value of $\alpha_s$ at some point. Thus, in terms of phenomenology and theoretical ambiguity $R_s$ is analogous to $R$ in e+e- annihilation [9].

$R_s$ is an especially interesting quantity with which to investigate the behavior of perturbation theory. The freedom to define away higher order corrections is limited by the fact that there is only one kinematic scale in the process (Q), and only one theoretical

choice to make (the choice of a coupling constant). Furthermore, one of the usual ways to "improve" a badly converging perturbation series is to pull out $n^2$'s on the assumption that they factor off into exponentials to all orders. Any such terms would cancel out of $R_s$. Thus a large correction to $R_s$ for a reasonable choice of a coupling constant indicates a deeper problem.

The fact that $R_s$ can be expressed directly in terms of partonic cross-sections (see Eq. (1.7)) simplifies the calculation enormously, since we never have to factor the IR singularities. The factorization requires calculation of all of the graphs contributing to $dW/dz$ and careful treatment of the IR singularities as $x$ or $z$ approach unity.

The use of Eq. (2.7) saves us much of this trouble. Consider $dW/dz$ to the 4th order

$$ \frac{dW}{dz} = A \left[ \delta(1-x)\delta(1-z) + \frac{\alpha}{4\pi} d(x,z) + \frac{(\alpha)^2}{4\pi} e(x,z) + \cdots \right] $$

In terms of moments

$$ \frac{dW}{dz} = A \left[ 1 + \frac{\alpha}{4\pi} d(n,m) + \frac{(\alpha)^2}{4\pi} e(n,m) + \cdots \right] $$

* All dependence on the IR renormalization scheme drops out in the ratio.
Thus Eq. (2.7) becomes
\[ R_{s}(n,m;l,k) = 1 + g s \left[ d(n,m) + d(l,k) - d(n,k) - d(l,m) \right] \]
\[ + \frac{g^{2}}{4\pi} \left\{ \frac{1}{2} \left[ \left( d(n,m) \right)^{2} + \left( d(l,k) \right)^{2} \right] - \left( d(n,k) \right)^{2} \left( d(l,m) \right)^{2} \right\} \]
\[ + \frac{g^{2}}{4\pi} \left[ \left( d(n,m) + d(l,k) - d(n,k) - d(l,m) \right)^{2} \right] \]
\[ + O(\alpha^{3}) \] (2.8)

Note that the contribution of the 4th order graphs (i.e. e(x,z)) to \( R_{s} \) is
\[ \frac{g^{2}}{4\pi} \int_{0}^{1} dx \int_{0}^{1} dz \left[ (x^{-1} - x^{-1})(y^{-1} - y^{-1}) \right] e(x,z) + O(\alpha^{3}) \] (2.9)

Thus any singularity in \( e(x,z) \) at \( x \) or \( z \) near 1 is controlled by the moment weighting in square brackets, and therefore requires no special treatment. Furthermore the two loop graphs (e.g. Fig. 2) which are all proportional to \( \delta(1-x)\delta(1-z) \) make no contribution to 4th order.

We conclude this section by listing further simplifications and assumptions made in our calculation.

In order to simplify the Dirac algebra we consider the structure function which is obtained by contracting the Lorentz indices of the \( W \)'s.* Note that this choice eliminates interference between the vector and axial vector weak currents. Thus we did not have to worry about regulating IR divergences in the presence of a \( \gamma_{5} \). We take a strong isospin non-singlet, charge conjugation odd difference for the final state (i.e. \( \pi^{+}-\pi^{-} \)). The relevant linear combination of final state partons is therefore \( u-\bar{d}-u+d \).

For the initial state, we also take a non-singlet difference (e.g. proton-neutron). To fix the relative contributions of quarks and antiquarks in the initial state, we take the Cabibbo angle to be zero. The G parity** of \( J_{\psi} \) and \( J_{R} \) (1 and -1 respectively guarantees that quarks and antiquarks contribute with opposite signs.

Thus we compute the following sum
\[ \frac{d\omega}{dz} = \frac{d\omega_{u+u}}{dz} + \frac{d\omega_{d+d}}{dz} - \frac{d\omega_{u-d}}{dz} - \frac{d\omega_{d-u}}{dz} \]
\[ + \left( \frac{d\omega_{u+u}}{dz} + \frac{d\omega_{d+d}}{dz} - \frac{d\omega_{u-d}}{dz} - \frac{d\omega_{d-u}}{dz} \right) \]

The terms with initial antiquarks simply reproduce the above sum by G parity.

Section III

We calculate the graphs which contribute to the cross-section with two body final states (virtual graphs) in this section. The graphs are computed in the Euclidean region where \( x > 1 \) and there are no discontinuities. Analytic continuation then yields the

* In the notation of Ellis [10] this corresponds to the combination
\[ -M_{1}^{2} + \frac{V^{2}}{M_{2}^{2}} \]

** The G parity we need here consists of a weak isospin rotation by \( \pi \) followed by charge conjugation.
correct result in the physical region $0 < x < 1$. All the calculations are performed in Feynman gauge. Dimensional regularization is used to control both the UV and IR divergences [11]. Integrals are performed in $n$ dimensions and the singularities associated with UV and IR divergences appear as poles in $\varepsilon(n = 4 - 2\varepsilon)$. If UV subtraction is necessary then it is important to distinguish between the UV and IR poles so that the former can be subtracted. This can always be done by first evaluating the graph with the external legs off shell, extracting the UV poles, and then taking the on shell limit when the IR poles appear. However with the use of Feynman parameters the UV and IR singularities are always distinct. The former appear in momentum integrals and the latter in Feynman parameter integrals.

The order $\alpha_s$ contribution to $\frac{dW}{d\varepsilon}$, \{d(x,z)\}, is obtained from Figs. 3(a), 3(b), and 3(c). Figure 3(a) gives a contribution proportional to $\delta(1-x) \delta(1-z)$ which cancels from $R_s$ to this order. Figs. 3(b) and 3(c) give

$$d(x,z) = \frac{8}{3} \frac{2(1-x-z) + (x+z)^2}{(1-x)(1-z)} + O(\varepsilon) .$$

Before considering the order $\alpha^2$ virtual graphs proper let us first dispose of the order $\alpha^2$ contributions to $R_s$ coming from the order $\alpha$ contributions. $(d^{mn} + d^{kl} - d^{nk} - d^{ml})^2$ is IR finite since it is simply the square of the order $\alpha$ piece. However the term $(d_{mn})^2 + (d_{lk})^2 - (d_{nk})^2 - (d_{ml})^2$ contains IR singularities which cancel against those from the order $\alpha^2$ graphs (e.g. etc.). In order to obtain the cancellation before integration over $x$ and $z$ it is sufficient to notice that

$$(d_{mn})^2 = \int x^{-1} z^{-1} (d\varepsilon) dxdz$$

and $d(x,z)$ comes from the order $\alpha$ graphs shown in Fig. 3. This convolution is easily performed. Note that the virtual graph Fig. 3(a) which did not contribute to the order $\alpha$ piece of $R_s$ must now be included in the convolution. It is important that $d(x,z)$ be retained in $n$ dimensions until after the convolution has been performed when a Laurent expansion about $\varepsilon = 0$ reveals double and single IR poles. The order $\alpha^2$ piece of $R_s$ can now be written as

$$\left(\frac{\alpha_s}{4\pi}\right)^2 \int dxdz (x^{n-1} - x^{-1}) (z^{m-1} - z^{-1}) \left[ d(x,z) - \frac{1}{2} d\varepsilon \right]$$

We now turn to the virtual diagrams contributing to $e(x,z)$. The first set of such graphs consists of the one loop corrections to the vertices and propagators in Figs. 3(b) and 3(c). The corrections to the external quark lines and the $W$-quark vertex can be considered together. A Ward identity ensures that the UV divergences arising from these graphs will cancel so that it is not necessary to perform a UV subtraction on them. This is especially convenient since the unsubtracted self energy of an on shell massless fermion is zero in dimensional regularization.

The corrections to the internal (off shell) fermion lines, external gluon lines and gluon vertices need UV subtractions. We must specify the subtraction scheme and hence the coupling constant. We use the $\overline{\text{MS}}$ scheme [3] which entails subtracting the UV poles
as well as attendant Riemann numbers and log (4π). Momentum space subtraction will be briefly mentioned later. After performing the subtractions all the remaining integrals are straightforward; no function worse than a log^2 appears.

The most difficult virtual diagrams are the box graphs shown in Fig. 4. These graphs have no UV singularities but contain single and double IR poles. The integrals are complicated by the need to retain these terms as well as ε^3 terms. Some comments on our technique may prove useful elsewhere. It is convenient to delay integrals over loop momenta until after forming the cross-section by multiplying by the lower order graphs and performing the Dirac algebra. Feynman parameters are introduced and the loop momentum integral performed. This leaves Feynman parameter integrals of the following type

$$I_p = \int_0^1 \int_0^{1-\alpha_1} \int_0^{1-\alpha_2} \int_0^{1-\alpha_3} f(a_1, a_2, a_3) \, da_1 \, da_2 \, da_3 \, \left[ a_1 a_2 a_3 + A a_1 (a_2 + a_3) + B a_2 a_3 + C a_1 a_2 \right]^p,$$

where A, B, and C are scalar products of various combinations of external momenta and f is a polynomial. There are 17 such integrals with p = 2 + ε and 4 with p = 1 + ε. These can be simplified and reduced in number as follows:

change variables

$$w = a_2,$$
$$u(l - v) = a_3,$$
$$x(l - u) = a_1.$$

then

$$I_p = \int_0^1 \int_0^{1-P} \int_0^{1-P} \int_0^{1-P} \int_0^{1-P} f(x, u, v) \, dx \, du \, dv \, dw \, dz \, \left[ A a_1 a_2 a_3 + B (a_2 + a_3) + C a_1 a_2 \right]^p.$$
\[ F_1(1, 2-x, 3-x, x) = \frac{(2-x)}{x^2} \left[ -\ln(1-x) - x + e^2 \left( \text{Li}_2(x) - x \right) \right. \]
\[ \left. + e^2 \left( \text{Li}_2(x) - x \right) \right] \]

\[ F_1(1, 3-x, 4-x, x) = \frac{(3-x)}{x^2} \left[ -\ln(1-x) - x - \frac{x^2}{2} \right. \]
\[ \left. + e^2 \left( \text{Li}_2(x) - \frac{x^2}{4} - x \right) \right] \]

and \[ \ln(x) = \sum_{n=0}^{\infty} \frac{x^n}{n} \] is the usual polylogarithm [12]. Although

dilogslogarithms appear in these expansions they are absent from the
final result for the box graphs which does however contain some 17
dilogslogarithms.

Section IV

In this section, we discuss the graphs with three body final
states.

The graphs contributing to a 2 gluon, 1 quark final state are
shown in Fig. 5. The diagrams are drawn as squared amplitudes. The
lines crossing the cut are the unobserved partons. One must take
care not to sum over the (unphysical) longitudinal polarization of the
final gluons. This presents no problem when there is only one
external gluon, but when there are two, the full spin sum operator

\[ \sum_{\text{physical polarization}} \epsilon^\alpha(P, \lambda) \epsilon^\beta(P, \lambda) = -\left\{ g^2 \beta - \frac{P^\alpha \beta + P^\beta \alpha}{P \cdot \hat{P}} \right\} \]

\[ \hat{P} = (E, \hat{P}) , \quad \hat{P} = (E, -\hat{P}) \]

must be used on one of the gluons. The usual sum over physical and
non-physical polarizations may then be used on the other one, by the

Ward identities. Note that since either of the gluons crossing the
cut in Fig. 5 may be the "special" one, we must average over both
possibilities. It is convenient to label the cut momenta so that
only one non-covariant vector \( \hat{P} \) appears. The Dirac algebra can be
reduced by using the interchange \( P_1 \leftrightarrow -P_2 \) to generate graphs from
each other \( \text{e.g.} \) Figs. 5(a) and 5(f) are related by this interchange.

When the results were added up, the vector \( \hat{P} \) dropped out.

We can use arguments involving flavor to discard many of the
graphs contributing to a 3 fermion final state. For instance, Fig. 6(a)
and Fig. 6(b) each are zero because the charged weak current changes
flavor.* Figure 6(c) vanishes due to the non-singlet sum on the
final observed quark.

The remaining graphs are divided by their flavor-topological
structure into Figs. (7 + 13).

Table I lists the processes to which each class contributes
(assuming an incoming \( W^+ \) boson), and the associated weight from
Eq. (2.10) (of course this weight is the same for a \( W^- \) by G parity).
Also listed is the factor (if any) which arises from the sum over
the flavors of the unobserved fermions. \( N_f \) is the number of quark
flavors.

A careful application of Wick's theorem shows that Figs. (9 + 13)
all receive Fermi minus signs.

We have indicated the usefulness of interchanging Fermion
momenta to generate graphs from each other. Numerous such
transformation exist involving the graphs with 3 final state fermions.
The Dirac algebra for Figs. 7 and 8 can be generated from

\[ \text{These graphs do contribute to } R_s \text{ in electroproduction.} \]
Figs. 7(a) and 7(b). Similarly Figs. 9-13 can be generated from Figs. 9(a)-9(c).

We now describe the phase space integrals for the real emission graphs. Let $P_3$ and $P_4$ be the momenta of the unobserved partons. For fixed $P_1$ and $P_2$, $P_3 + P_4$ is fixed by momentum conservation. The remaining integral over the unobserved momenta is performed in their center of mass frame (which is the usual Gottfried-Jackson (GJ) frame), i.e.

$$\vec{P}_3 + \vec{P}_4 = \vec{P}_1 + \vec{q} - \vec{P}_2 = 0$$  \hspace{1cm} (4.1)

We further define

$$y = \frac{\left[ z + \frac{2\vec{P}_2 \cdot \vec{q}}{\vec{q}^2} \right]}{(1-x)(1-z)} \quad 0 \leq y \leq 1$$

Then the 3 body phase space integral is

$$\frac{1}{4\pi} \left[ \left( \frac{d^{n-1}P_2}{(2\pi)^{n/2}} \right) \delta \left[ z - \frac{P_2 \cdot P_2}{P_1 \cdot P_1} \right] \left( \frac{d^{n-1}P_3}{(2\pi)^{n/2}} \right) \left( \frac{d^{n-1}P_4}{(2\pi)^{n/2}} \right) \right] \times (2\pi)^n \delta^{(n)}(P_1 + q - P_3 - P_4) |M|^2$$

$$= \frac{1}{2\Gamma(1-\varepsilon)} \left[ \frac{1}{6\pi^3} \frac{(1-x)^2(1-z)^2}{x^2} \right]^{\varepsilon}$$

$$\times \int_0^1 dy \ y^{\varepsilon} (1-y)^{-\varepsilon} \int d\Omega_{GJ} |M|^2$$  \hspace{1cm} (4.2)

Equation (4.1) implies that $\hat{P}_1$, $\hat{q}$ and $\hat{P}_2$ span a 2 dimensional plane; call it the $xz$ plane. Introduce angles $\theta$ and $\phi$ such that

$$\left( \vec{P}_3 \right)_z = P_3 \cos\theta, \quad 0 \leq \theta \leq \pi$$
$$\left( \vec{P}_3 \right)_x = P_3 \sin\theta \cos\phi, \quad 0 \leq \phi \leq \pi$$

Then

$$d\Omega_{GJ} = d(\cos\theta)(1-\cos^2\theta)^{-\varepsilon} d\phi (\sin\theta)^{-2\varepsilon} (4\pi)^{-\varepsilon} \frac{(1-\varepsilon)}{(1-2\varepsilon)}$$  \hspace{1cm} (4.3)

The angular integrals are performed by partial-fractioning the integrand. The resulting integrals are performed analytically as Laurent expansions in $\varepsilon$, keeping only terms which become order $\varepsilon^0$ or lower after the $y$ integration (as we shall see, the $y$ integration can introduce a factor of $\frac{1}{\varepsilon}$). There were 4 basic types of integrals. Integrals of the type

$$I_1 = \int d\Omega_{GJ} \frac{1}{1 - \hat{P}_3 \cdot \hat{v}} \quad \hat{v} = \hat{P}_1 + \hat{P}_2$$

have singularities when $\hat{P}_3 = \hat{v}$. These poles are regulated by the Jacobian in Eq. (4.3), giving rise to a simple pole in $\varepsilon$.

Integrals of the form

$$I_2 = \int d\Omega_{GJ} \frac{1}{1 - \hat{P}_3 \cdot \hat{P}_3}$$

arise from propagators of the form $1/(P_1 - P_2 - P_3)^2$. $L = 1$ can occur only when $P_1 \cdot P_2 = 0$, i.e. $z = 0$. Thus the $L = 1$ singularity requires no special treatment since it is removed by the moment weighting in Eq. (2.9). $I_1$ and $I_2$ are straightforward, and the results are

$\ast$ The symbol $^\ast$ denotes a unit vector.
I_1 = \left( -\frac{1}{\epsilon} \right) (2\pi)^{1-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}

I_2 = (2\pi)^{1-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{1}{L} \left[ \log \frac{1+L}{1-L} + A \right] + O(\epsilon^2)

where

A = -\frac{\epsilon}{2} \log^2 \left( \frac{1+L}{1-L} + 2\epsilon \text{Li}_2 \left( \frac{-2L}{1-L} \right) \right).

Integrals of the form

I_3 = \int \frac{d\Omega_{ij}}{((1-\hat{p}_3 \cdot \hat{V}) (1-\hat{p}_3 \cdot \hat{W}))}

can be performed by combining the denominators using Feynman parameters.

The result is

I_3 = \left( -\frac{1}{\epsilon} \right) (4\pi)^{1-\epsilon} \frac{\Gamma(1-\epsilon)}{2\Gamma(1-2\epsilon)} \left[ 1 + \epsilon \ln \left( \frac{1-L^2}{(1-L \cdot V)^2} \right) \right] \frac{\epsilon A \cdot L \cdot \hat{V}}{1-L \cdot \hat{V}}.

Finally, we had integrals of the form

I_4 = \int \frac{d\Omega_{ij}}{((1-\hat{p}_3 \cdot \hat{V}) (1-\hat{p}_3 \cdot \hat{W}))}

\hat{V} = \hat{P}_1, \quad \hat{W} = \hat{P}_2.

The integrand has poles at \( \hat{p}_3 = \hat{V} \) and at \( \hat{p}_3 = \hat{W} \). As \( \hat{V} + \hat{W} \), the 2 singularities collide. This occurs when one of the following is satisfied:

- (case A) \( \hat{P}_1 = \hat{P}_2 \)
- (case B) \( \hat{P}_1 = -\hat{P}_2 \).

* The A type singularities occur in Fig. 13 and Fig. 5(h)-5(k).

The B type occur in Figs. 5(b), 5(c), 5(d), 5(e), 5(o), and 5(p).

Kinematics dictates that case A occurs at \( y = 1 \), case B at \( y = 0 \).

Any Laurent expansion performed at this stage (i.e. before the \( y \) integration), must be uniform in \( y \). Thus our expansion for \( I_4 \) must be uniform in \( (\hat{V} \cdot \hat{W}) \). This integral is performed in Ref. (5). The result is

\[ I_4 = \left( -\frac{1}{\epsilon} \right) (4\pi)^{1-\epsilon} \frac{\Gamma(1-\epsilon)}{2\Gamma(1-2\epsilon)} \left( 1-x \right)^{-1-\epsilon} \left( 1+\epsilon^2 \text{Li}_2(x) \right) \]

where \( x = \frac{(1-\hat{V} \cdot \hat{W})^2}{4} \). Since

\[ \hat{P}_1 \cdot \hat{P}_2 = 1 + O(1-y) \quad \text{as} \quad y \to 1 \]
\[ \hat{P}_1 \cdot \hat{P}_2 = -1 + O(y) \quad \text{as} \quad y \to 0 \],

case A singularities result in singularities of the form \((1-y)^{-1-\epsilon}\) as \( y \to 1 \) similarly case B gives \( y^{-1-\epsilon} \) as \( y \to 0 \).

The result of the angular integral as a function of \( y \) was complicated enough to prevent us from doing the \( y \) integral analytically. However, we still need to have the result as a Laurent expansion, even if the terms in it are integrals over \( y \). We are prevented from simply expanding the integrand because of singularities of the form \((1-y)^{-1-\epsilon}\) as \( y \to 1 \) and \( y^{-1-\epsilon} \) as \( y \to 0 \). We handle these by the usual "plussing" technique, e.g.

\[ \frac{1}{(\hat{P}_3 \cdot \hat{P}_4)^2} = \frac{x}{(1-x) (1-y) (1-z)} \]
\[(1-y)^{-1-\varepsilon} = -\frac{1}{\varepsilon} \delta(1-y) + \sum \left( \frac{\log(1-y)}{1-y} \right) + O(\varepsilon^2)\]

and similarly for \(y^{-1-\varepsilon}\). This equation is valid when multiplying a function of \(y\) which is continuous at \(y = 1\).

We have shown the great care with which the endpoint singularities in \(y\) must be handled. No such subtleties occur in the moment integration of Eq. (2.9) because the moment weighting cancels out all remaining endpoint singularities.

Section V

The IR singularities (i.e. poles and double poles in Eq. (2.8)) can now be assembled in the formula for \(R_{\mathcal{S}}\). Their contribution is given by the pole part of Eq. (3.1). The contribution of the real emission graphs to \(e(x,z)\), discussed in the previous section is in the form of an integral over \(y\). The integral is sufficiently simple that it can be performed analytically. The factorization theorem guarantees that \(R_{\mathcal{S}}\) is IR finite for all moment indices. This can be true only if the integrand \(e(x,z) - 2 \delta\) is IR finite (up to delta functions at \(x = 1\) or \(z = 1\), which we have discarded). We verified explicitly that this was the case.

It now remains to evaluate the remaining integrals numerically in order to obtain values of \(R_{\mathcal{S}}\). The integral consists of two parts; a double integral over \(x\) and \(z\) coming from the IR finite parts of the graphs discussed in Sec. III and the parts from Sec. IV which result from the plussing operation; and a triple integral over the remaining terms from Sec. IV. These integrals are of course all finite, however they contain integrable singularities of the form \(\log^2(1-x)\), \(\log^2(1-z)\) etc. It is important to change variables to render these singularities more amenable to numerical integration. The following variable transformation is very effective

\[
\int_0^1 f(x) dx = \int_0^{1-1/y} \frac{e^{-1/y}}{y^2} f(e^{1-1/y}) dy.
\]

If \(f(x) - \log^2(x)\) as \(x \to 0\), the left hand side needs of order 500 steps to achieve a convergence on the integral, 10 steps are sufficient for the right hand side. The convergence of regular function integrals is unaffected by this variable change. Numerical integration of the (IR) finite parts of Eq. (2.8) is now simple and the results are displayed in Table II. Writing

\[
R_{\mathcal{S}}(n,m;1,k) = 1 + A(n,m;1,k) \left[ \frac{\alpha_s(\mu^2)}{\pi} \right]
\]

\[
+ B(n,m;1,k) \left[ \frac{\alpha_s(\mu^2)}{\pi} \right]^2
\]

the table shows values for \(A\) and \(B\). It is clear that the corrections are large.

The numbers shown in the table are for the \(\overline{\text{MS}}\) scheme with \(\mu^2 = Q^2\). The order \(\alpha^2\) terms are of the order of 60\% of the order \(\alpha\) terms for \(\alpha = 0.2\). \(B\) is of course scheme dependent; in the momentum space scheme (Landau gauge) with \(\mu^2 = Q^2\), \(B\) is given from the \(\overline{\text{MS}}\) in the table by [4]

\[
B_{\text{mom}} = B_{\overline{\text{MS}}} - 3.07 A \left[ \frac{\alpha_s(\mu^2)}{\pi} \right].
\]

Using a momentum space subtracted \(\alpha\) therefore reduces the corrections to about 40\%. The corrections get larger as the difference between \(n\) and 1 and \(k\) and \(m\) increases. This presumably reflects pieces of the formula for \(R_{\mathcal{S}}\) which go like \(\log^2(n)\) etc. Notice that the lowest
order term $A$ is symmetric under the interchanges $m \leftrightarrow n$ and $k \leftrightarrow l$.

This is a reflection of the fact that $d(x,z)$ is symmetric with respect to interchange of $x$ and $z$. This symmetry is maintained in the $B$ terms within the errors on our numerical integration. The fact that the ratio $B/A$ is almost independent of the indices indicates that the corrections can be made uniformly small by using some fraction of $Q^2$ as the scale in $\alpha_s$. Unfortunately we need $Q^2/8$ in the momentum space scheme to make the $\alpha^2$ term 10% of the $\alpha$ term. It seems difficult to see why this should be the correct scale although it is worth pointing out that within the momentum space scheme such a scale leads to reasonable (negative) corrections of the order of 20% to the usual formulae for the $Q^2$ evolution of moments in inclusive lepto-production. It can be argued [13] that a natural scale for $\mu^2$ is $Q^2(1-z)(1-x)$; this scale being a typical off shell-ness of parton in the process. Now using this scale the lowest order contribution to $R_s$ changes. Choosing the coupling constant such that $\alpha_s(100 \text{ GeV}^2) = 0.2$ Table III now shows the order $\alpha$ and order $\alpha^2$ contributions to $R$. It is immediately clear that the corrections in order $\alpha^2$ are now small, and the perturbation expansion appears to be reliable. Unfortunately to see that this is really the case needs a proof that such a choice of $\mu^2$ will work to all orders in reducing the corrections. Such a proof is lacking, making it difficult to decide whether the smallness of the order $\alpha^2$ terms is a mere coincidence.

In conclusion we have computed the order $\alpha^2_s$ terms in the double moment ratio in semi-inclusive deep inelastic scattering. The corrections are large enough that one should worry about the status of the perturbation expansion. However they are not as large as those found in some other processes [14], and the fact that they depend only slightly on the moment indices encourages one to think that something can be salvaged.

ACKNOWLEDGMENTS

This research was supported by the High Energy Physics Division of the U. S. Department of Energy under contract no. W-7405-ENG-48. All the algebraic manipulations in this paper were performed with the aid of MACSYMA [15]. We are grateful for the support provided by the MATHLAB group at MIT.
<table>
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<th>Graphs</th>
<th>Processes</th>
<th>Weight from Eq. (2.10)</th>
<th>Sum Over Unobserved Flavors</th>
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<tr>
<td>Fig. (7)</td>
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<td>$N_f$</td>
</tr>
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<td>$\left[ \frac{N_f}{2} \right]$</td>
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<td>1</td>
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<td>Fig. (10)</td>
<td>$d + \bar{d} + x$ or u + u + x</td>
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<tr>
<td>Fig. (11)</td>
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<td>Fig. (12)</td>
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<td>Fig. (13)</td>
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**TABLE I**

List of flavor weights associated with Figs. 7-13.

<table>
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<tr>
<th>n</th>
<th>m</th>
<th>l</th>
<th>k</th>
<th>A</th>
<th>B</th>
<th>% Correction ($\alpha = 0.2$)</th>
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**TABLE II**

Values of A and B in the formula.

$$R_{\bar{s}}(n,m;1,k) = 1 + A(n,m;1,k) \left[ \frac{a_{\bar{s}}(m)}{\pi} \right] + B(n,m;1,k) \left[ \frac{a_{\bar{s}}(m)}{\pi} \right]^2$$

in the $\overline{\text{MS}}$ scheme with $\mu^2 = q^2$. 
Values of LO and NLO, and the percentage correction, in the formula $R_s = 1 + \text{LO} + \text{NLO}$, where LO is the lowest order contribution with $\mu^2 = Q^2(1-x)(1-z)$ in the $\overline{\text{MS}}$ scheme (with $\alpha_{\overline{\text{MS}}} (Q^2) = 0.2$), and NLO is the next correction.

<table>
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<tr>
<th>$n$</th>
<th>$m$</th>
<th>$l$</th>
<th>$k$</th>
<th>LO</th>
<th>NLO</th>
<th>% Correction</th>
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<td>0.0037</td>
<td>$3.8 \times 10^{-4}$</td>
<td>10</td>
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</table>

TABLE III

REFERENCES


13 S. Brodsky and P. Lepage, SLAC Publication 2446.

14 R. K. Ellis, D. A. Ross and A. E. Terrano, CALT 68 785, see also Refs. [2] and [5].


FIGURE CAPTIONS

Figure 1: The lowest order graph in semi-inclusive deep inelastic scattering.

Figure 2: A two loop graph which does not contribute to $R_s$.

Figure 3: Graph contributing to $\frac{dW}{dz}$ in order $a_s$.

Figure 4: Virtual diagrams contributing to $\frac{dW}{dz}$ in order $a_s^2$.

Figure 5: Graphs contributing to $\frac{dW}{dz}$ with two gluons in the final state.

Figure 6: Graphs which do not contribute to $\frac{dW}{dz}$ by virtue of their flavor structure.

Figures 8-13: Graphs contributing to $\frac{dW}{dz}$ with one gluon and two fermions in the final state.