UNIVERSITY OF CALIFORNIA, SAN DIEGO

Global Existence and Dispersion of Solutions to Nonlinear Klein-Gordon Equations with Potential

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by

Chad Thornton Wildman

Committee in charge:

Professor Jacob Sterbenz, Chair
Professor Peter Ebenfelt
Professor Kim Griest
Professor Lei Ni
Professor David Tytler

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The dissertation of Chad Thornton Wildman is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2014
DEDICATION

To Dr. John I. Thornton, who taught me the meaning of higher education.
“And I lose your hand through the waves..”
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VITA

2006 B.A. in Mathematics, University of California, San Diego

2007-2012 Graduate Teaching Assistant, University of California, San Diego

2010 M.A. in Mathematics, University of California, San Diego

2014 Ph.D. in Mathematics, University of California, San Diego
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Chad Thornton Wildman

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Professor Jacob Sterbenz, Chair

In this thesis we prove global existence of solutions with small initial data to the perturbed quadratic nonlinear Klein-Gordon equation

\[(\partial_t^2 - \Delta + V + 1)u = u^2\]  \hspace{1cm} (0.1)

in \(n = 3\) space dimensions, subject to assumptions on the potential \(V(x)\). Specifically, we require that \(V\) satisfies the decay estimate \(|\partial^\alpha V(x)| \lesssim C_\alpha \langle x\rangle^{-2-\epsilon}\), that the associated Schrödinger operator \(H = -\Delta + V\) has no eigenvalues, and that 0 is not a resonance of \(H\).

Energy estimates alone are sufficient to establish global existence of solutions to (0.1), but provide only exponential bounds on higher Sobolev norms of a solution. A major part of the paper is thus dedicated to proving dispersion of solutions to (0.1). Dispersive estimates control the global existence problem for cubic nonlinearities, so we employ normal forms methods due to Shatah [2] to prove the full result for (0.1).
Chapter 1

Introduction

The nonlinear Klein-Gordon equation

\[(\partial^2_t - \Delta + 1)u = f(u)\]  \hspace{2cm} (1.1)

has been studied widely in all space dimensions, and in particular global existence of solutions in \(n = 3\) space dimensions can be established for nonlinearities \(f\) vanishing of second order or higher at 0. In this paper, I consider the perturbation of (1.1) obtained by multiplication by a real potential function \(V(x)\):

\[(\partial^2_t - \Delta + V + 1)u = f(u)\]  \hspace{2cm} (1.2)

where \(f\) vanishes to second order. We will take \(f(u) = \lambda u^2\) for \(\lambda \in \mathbb{R}\), but it is straightforward to generalize to more general nonlinearities. In addition we may assume \(\lambda = 1\) since \(\tilde{u} = \lambda u\) solves (1.2) with a nonlinearity of \(\tilde{u}^2\). However, many of the auxiliary results we prove will be stated for general \(f\) and we will point out how generalizations can be made. By the operator \(V\) we mean multiplication by \(V(x)\), a function on which we will impose certain regularity and decay assumptions in addition to technical assumptions related to the spectrum of the operator \(H := -\Delta + V\). We shall also use frequently the notation \(\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}\).
The main theorem of the paper is

**Theorem 1.0.1 (Main Theorem).** Suppose $V : \mathbb{R}^3 \to \mathbb{R}$ is a potential function satisfying the bounds $|\partial^\alpha V(x)| \lesssim C_\alpha \langle x \rangle^{-2-\epsilon}$ and is such that the operator $H := -\Delta + V$ has no eigenvalues and no resonance at 0.

Then given initial data $(u_0, u_1) \in \mathcal{S}(\mathbb{R}^3)$, there exists an $\epsilon > 0$ so that there is a unique global in time solution to the equation

$$(\Box + 1 + V)u = u^2$$

$u(0, x) = \epsilon u_0(x)$

$u_t(0, x) = \epsilon u_1(x)$

satisfying the dispersive estimate

$$\| (u(t, \cdot), u_t(t, \cdot)) \|_{L^\infty} \leq C(u_0, u_1)(1 + t)^{-\frac{3}{2}}$$

Here and in the sequel, the operator $\Box$ is defined by $\Box := \partial_t^2 - \Delta_x$. The general approach is to employ the standard continuity method for existence of solutions; this is discussed in Chapter 3. First a local existence theorem is established, and then the question is whether or not the interval of existence is all of $\mathbb{R}$ or not. To answer this question, we will show that the solution does not “blow up” at any finite time. There are two ways in which we can accomplish this. To obtain the most complete understanding of the solution and hence the existence problem itself, we will prove that solutions to (1.2) satisfy a dispersive estimate that will prevent blowup. The other way exploits the conservation of energy and it is worth noting that energy estimates alone are sufficient to establish global existence for (1.2). Indeed, in the model case $f(u) = u^2$, one considers the quantity

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^3} u_t^2 + |\nabla u|^2 + u^2 + V u^2 - \frac{2}{3} u^3 \, dx \quad (1.3)$$

Differentiating with respect to $t$, integrating by parts, and using (1.2) shows that $\frac{d}{dt} E(t) = 0$, i.e. $E(t)$ is conserved over time. It will be shown in Chapter 3 that this, in conjunction with the local existence theory established there, is sufficient to obtain a global $H^1(\mathbb{R}^3)$ solution to (1.2). However, the argument produces only exponential growth bounds on higher Sobolev norms of the solution in terms of
initial data. This is of course very unsatisfying, and provides motivation even in
the case $f(u) = u^2$ to prove a dispersive estimate giving more complete control
over the solution as $t \to \infty$.

If $f$ vanishes of third order or higher at 0, the a priori energy and dispersive
estimates proved below can be used directly to establish global existence using the
aforementioned continuity method (see, for instance [1]). If $f$ vanishes only to sec-
ond order at 0, the standard approach fails and more sophisticated methods must
be employed. Such methods were developed independently by both Klainerman [3]
and Shatah [2]. They both proved global existence for quadratic nonlinearities, in
two entirely different ways. Klainerman employed his famous vector fields method,
whereas Shatah developed a technique known as a normal forms transformation
based on ODE methods due to Poincaré. The vector field method of Klainerman
has proved extremely versatile in establishing existence for many types of PDE,
but in the case we consider it does not work due to the presence of the potential
$V$. The idea behind his method is to derive estimates using products of the vector
fields $\frac{\partial}{\partial t}, \frac{\partial}{\partial x_j}, t \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial t}$, and $x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}$. If $\Omega^I$ is such a product of these fields,
the key observation is that if $u$ solves (1.1), then we have also $(\Box + 1)\Omega^I u = \Omega^I f(u),
$ i.e. $[\Box, \Omega] = 0$. However, when the term $V(x)u$ is introduced, the commutators
$[\Omega^I, V]$ cause major problems. For instance, if $\Omega = t\partial_{x_j} + x_j \partial_t$, then $[\Omega, V] = t\partial_{x_j} V$
which grows in $t$. It turns out that the normal forms transform of Shatah works
well with a potential $V$, and it is this method that we use in the current work.

Before proceeding, it is worthwhile to answer an obvious question: why
would one consider studying the equation (1.2), as opposed to some perturbation
other than multiplication by a potential? The nonlinear equation (1.1) is known
to arise in the study of particle physics, and the physical applications of (1.2)
have roots in questions of soliton stability. It is known that soliton, or “standing
wave”, solutions to (1.1) exist under certain hypotheses on $f$ (see, for instance, [4])
. Stability of such solitons is a major question in nonlinear equations and was first
addressed in the case of Klein-Gordon in [5] and [6]. As for the case I consider, a
potential term will arise after linearization of an equation around a soliton solution.
We can illustrate this rather easily with a simple example: suppose that a static
function \( u_s(x) \) satisfies the equation
\[
(\Delta + 1) u_s = u_s^2 \tag{1.4}
\]
Now consider the perturbation \( u_s + w \), where \( w(x, t) \) is small. If \( u_s + w \) is to solve the full Klein-Gordon equation
\[
(\partial_t^2 - \Delta + 1)(u_s + w) = (u_s + w)^2 \tag{1.5}
\]
then after expanding \( (u_s + w)^2 \) and using (1.4), we arrive at the identity
\[
(\partial_t^2 - \Delta + 1)w = 2u_sw + w^2
\]
After rearranging terms, we obtain an equation of the prescribed form:
\[
(\partial_t^2 - \Delta + V + 1)w = w^2 \tag{1.6}
\]
where \( V(x) = -2u_s \). This gives some motivation for why we consider the equation (1.2).

A general outline for the flow of the paper is as follows. In Chapter 2 we develop the major machinery from functional analysis that we require to interpret certain operators, as well as introduce some function spaces customized to the problem. We also recall some Sobolev embedding theorems and prove estimates that allow one to move between the new function spaces and the classical ones. Chapter 3 discusses existence techniques for nonlinear PDE in general; in particular it is shown there how the local existence theorem for (1.2) can be used to deduce global existence if additional a priori estimates are known. It also contains the main energy estimates for (1.2) and shows how global existence follows just from energy estimates in the case \( f(u) = u^2 \). This then motivates proving a dispersive estimate, which is sufficiently involved and contained in Chapter 4. Chapter 5 contains the main a priori estimates, which by the remarks in Chapter 3 establish global existence for (1.2). These estimates depend on the normal forms transform in order to handle quadratic nonlinearities; they are stated as a “black box” in Chapter 5 and then in Chapter 6 the normal forms transform is developed and all the remaining estimates are proved. The two appendices contain expositions of standard material concerning the bilinear operators used in the normal forms transform as well as a proof of the boundedness of the transform itself.
Chapter 2

Background Material

In order to keep the main argument as coherent as possible, we state in this chapter some classic theorems in analysis that will be referred to frequently throughout the paper.

2.1 Some Functional Analysis

Given a real potential function $V(x)$, we will define the operator $H := -\Delta + V$ and recast the equation (1.2) as

$$\left(\partial_t^2 + H + 1\right)u = f(u)$$

(2.1)

An understanding of the spectrum of the operator $H$ is essential to proving dispersive bounds for solutions to (2.1), and this in turn is acquired through imposing certain assumptions on the function $V$. First we will require that $V$ and some number of its derivatives have slightly faster than quadratic decay in space. This assumption is also sufficient to ensure the absence of positive eigenvalues of $H$ (in fact Kato showed in [8] that $o(|x|^{-1})$ decay prevents positive embedded eigenvalues), but we will need to assume further that there are no negative eigenvalues and that 0 is not a resonance (defined below). To see why these assumptions are necessary, let us observe what happens when they fail. If $\lambda$ is an eigenvalue of $H$, any chance of a dispersive estimate is lost: suppose $0 \neq \psi(x)$ were such that $H\psi = -\Delta\psi + V\psi = \lambda\psi$. As an ansatz, let $u(t, x) = \alpha(t)\psi(x)$. Then we have
\[(\partial_t^2 + H + 1)u = \psi [\alpha'' + (\lambda + 1)\alpha],\] so if we choose \(\alpha(t) = e^{it\sqrt{\lambda+1}}\) (at least for \(\lambda > -1\)), then \(u\) solves the homogeneous equation \((\partial_t^2 + H + 1)u = 0\), but clearly cannot satisfy any pointwise decay estimate as \(t \to \infty\) since \(\psi\) is not identically 0.

The other requirement has to do with what is referred to as a resonance. Specifically, we define ([9]) a resonance as a distributional solution to \(H\psi = 0\) such that \(\psi \notin L^2(\mathbb{R}^3)\) yet for every \(\sigma < -\frac{1}{2}\), \(\psi \in L^{2,\sigma}(\mathbb{R}^3)\). Here we are referring to the weighted \(L^2\) space with norm
\[
\|u\|_{L^2,\sigma}^2 := \int (1 + |x|^2)^\sigma |u(x)|^2 \, dx
\]
One can similarly define the weighted Sobolev space
\[
\|u\|_{H^{2,\sigma}} := \sum_{|\sigma| \leq 2} \|\partial^\sigma u\|_{L^{2,\sigma}}
\]
As we will see below, dispersion is intimately linked to analysis of the resolvent operator \((H - z)^{-1}\). In [10], for example, it is shown that for \(\Im z > 0\) and \(\Im \frac{1}{2} > 0\), one has an asymptotic expansion
\[
(H - z)^{-1} = -z^{-1} B_{-2} - iz^{-\frac{3}{2}} B_{-1} + B_0 + iz^{\frac{1}{2}} B_1 + \ldots
\]
as \(|z| \to 0\). It is shown there that a 0 resonance corresponds to the presence of the \(z^{-\frac{1}{2}}\) term, so that we will lose control of the resolvent as the energy approaches 0. Note that the rate of blowup for a zero resonance is not quite as bad as for a 0 eigenvalue, which corresponds to the \(z^{-1}\) term. In any event, this behavior of the resolvent for low energies will not allow for a full dispersive estimate (one still might expect \(|t|^{-\frac{1}{2}}\) decay; this is the case for the Schrödinger equation in 3 dimensions as shown by Jensen and Kato in [10]; however this rate is insufficient for our purposes).

To sum up, the two assumptions we will require \(V\) to satisfy are as follows:

**Assumption 2.1.1.** For some \(\epsilon > 0\), we require that \(V\) satisfy the estimate
\[
|\partial^\alpha V(x)| \leq C_\alpha \langle x\rangle^{-2-\epsilon}
\]

**Assumption 2.1.2.** The operator \(H = -\Delta + V\) has no eigenvalues and 0 is not a resonance.
Returning now to the main equation (2.1), note that by the Duhamel principle, solutions with initial data $u_0, u_1$ can be expressed in a formal sense as

$$u(t, \cdot) = \cos(t\langle\sqrt{H}\rangle)u_0 + \sin(t\langle\sqrt{H}\rangle)\langle\sqrt{H}\rangle u_1 + \int_0^t \frac{\sin((t-s)\langle\sqrt{H}\rangle)}{\langle\sqrt{H}\rangle} f(s, \cdot) \, ds \tag{2.2}$$

The tools of functional analysis will allow us to make sense of expressions such as $e^{\pm it\langle\sqrt{H}\rangle}$ as operators, which will in turn allow us to derive dispersive estimates using the expansion (2.2). The two primary results are the spectral theorem and Stone’s formula. They are products of a broad theory that can be found in many functional analysis texts; in this paper we will reference the material in [11] and [12]. In the following sections we will recall the basic definitions and results necessary to explain the expressions that appear in the sequel. The general setting will be that of unbounded operators defined on dense subspaces of Hilbert spaces.

### 2.1.1 The Spectral Theorem and Spectral Measures

The spectral theorem is a tool that will allow for the analysis of expressions such as $f(A)$ for measurable functions $f$ and self-adjoint operators $A$ on a Hilbert space $\mathcal{H}$. As mentioned above, the formal expansion (2.2) suggests that this is a logical place to begin. This theorem has many equivalent formulations. Our path to making sense of the expressions in (2.2) begins with a version of the spectral theorem commonly referred to as the “functional calculus” for self-adjoint unbounded operators, which we recall from Theorem VIII.5 in [11].

**Theorem 2.1.3 (Spectral Theorem – Functional Calculus).** Suppose $A$ is a self-adjoint operator on a Hilbert space $\mathcal{H}$ with (dense) domain $D(A)$, and let $B(\mathbb{R})$ and $\mathcal{L}(\mathcal{H})$ denote the bounded Borel functions on $\mathbb{R}$ and linear operators on $\mathcal{H}$, respectively. Then there exists a unique homomorphism $\phi : B(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ with the following properties:

1. $\phi(f g) = \phi(f)\phi(g)$, \quad $\phi(\lambda f) = \lambda \phi(f)$, \quad $\phi(1) = I$, \quad $\phi(\overline{f}) = \phi(f)^*$

2. $\|\phi(f)\|_{\mathcal{L}(\mathcal{H})} \leq \|f\|_{L^\infty}$

3. If $f_n \in B(\mathbb{R})$ are such that $f_n \to x$ pointwise and $|f_n(x)| \leq |x|$ for all $x, n$, then for any $h \in D(A)$ we have $\phi(f_n)h \to Ah$. 
4. If $f_n \to f$ pointwise and $\{\|f_n\|_{L^\infty}\}_{n=1}^\infty$ is a bounded sequence, then $\phi(f_n) \to \phi(f)$ in the strong operator topology.

From this point on, we shall refer to the image $\phi(f)$ as simply $f(A)$. The first order of business is to use the functional calculus to define the so-called “spectral measures”, which will in turn allow us to begin analysis of the expansion (2.2).

Given a Borel-measurable set $\Omega \subset \mathbb{R}$, we may use the functional calculus (2.1.3) to define the operator $E_\Omega = \chi_\Omega(A)$, where $\chi_\Omega(x)$ is the usual set indicator function. This allows us to define projection-valued measures, which we will require in the next version of the spectral theorem.

**Definition 2.1.4.** A projection-valued measure on $\mathcal{H}$ is a family of operators $\{E_\Omega\}$ satisfying the following properties:

1. Each $E_\Omega$ is an orthogonal projection.
2. $E_\emptyset = 0$ and $E_{(-\infty,\infty)} = I$.
3. If $\{\Omega_n\}$ are pairwise disjoint and $\Omega = \bigcup_{n=1}^\infty \Omega_n$, then $E_\Omega = s\lim_{N \to \infty} \sum_{n=1}^N E_{\Omega_n}$, where the limit is with respect to the strong operator topology.
4. $E_{\Omega_1 \cap \Omega_2} = E_{\Omega_1}E_{\Omega_2}$

Now one can see that for $h \in \mathcal{H}$, the function $\langle h, E_\lambda h \rangle$ induces a Borel measure on $\mathbb{R}$, which we will denote as $d\langle h, E_\lambda h \rangle$. This then produces a complex measure $d\langle h, E_\lambda k \rangle$ through polarization. Given a bounded measurable function $f(\lambda)$, we can then define $f(A)$ via

$$\langle h, f(A)k \rangle = \int_{-\infty}^{\infty} f(\lambda) \, d\langle h, E_\lambda k \rangle$$

(2.3)

For simplicity, we will use the following “spectral representation” notation for the operator $f(A)$:

$$f(A) = \int_{-\infty}^{\infty} f(\lambda) \, dE_\lambda$$

(2.4)
Theorem 2.1.5 (Spectral Theorem – Projection-Valued Measures). There is a 1–1 correspondence between projection-valued measures \( \{ E_\Omega \} \) and self-adjoint operators \( A \) on \( \mathcal{H} \) given by
\[
A = \int_{-\infty}^{\infty} \lambda \, dE_\lambda
\]
If \( f \) is a real-valued Borel function, then (2.3) and (2.4) hold and \( f(A) \) is a self-adjoint operator with dense domain
\[
D_f = \{ h \in \mathcal{H} \mid \int_{-\infty}^{\infty} |f(\lambda)|^2 d\langle h, E_\lambda h \rangle < \infty \}
\]
If \( f \) is bounded, then \( f(A) \) coincides with the image \( \phi(f) \) from Theorem 2.1.3.

2.1.2 The Stone Formula

The final step in being able to analyze the operators in (2.2) then comes from interpreting the measures \( d\langle h, E_\lambda k \rangle \). The main result is due to Stone and links the operators \( E_\Omega = \chi_\Omega(A) \) and the resolvent operators mentioned above. Given a self-adjoint operator \( A \), we define the resolvent operator \( R_A(z) := (A - z)^{-1} \).

Theorem 2.1.6 (Stone’s Formula). Let \( A \) be a self-adjoint operator on a Hilbert space \( \mathcal{H} \). Then
\[
s\lim_{\epsilon \to 0} (2\pi i)^{-1} \int_a^b [R_A(\lambda + i\epsilon) - R_A(\lambda - i\epsilon)] \, d\lambda = \frac{1}{2} \left( \chi_{[a,b]}(A) + \chi_{(a,b)}(A) \right)
\]
where the limit is taken in the strong operator topology.

Proof. This is a simple application of part (4) of the functional calculus (2.1.3). Define
\[
f_\epsilon(x) := (2\pi i)^{-1} \int_a^b \left( \frac{1}{x - \lambda - i\epsilon} - \frac{1}{x - \lambda + i\epsilon} \right) \, d\lambda
\]
Note that the integrand is equal to \( \frac{2i\epsilon}{(\lambda - x)^2 + \epsilon^2} \). Performing the integration shows that
\[
f_\epsilon(x) = \frac{1}{\pi} \left[ \arctan \left( \frac{b - x}{\epsilon} \right) - \arctan \left( \frac{a - x}{\epsilon} \right) \right]
\]
This shows that $\| f_\epsilon \|_{L^\infty}$ is uniformly bounded in $\epsilon$, and in addition we can read off the pointwise limit

$$
\lim_{\epsilon \searrow 0} f_\epsilon(x) = \begin{cases} 
0, & x \not\in [a, b] \\
\frac{1}{2}, & x = a \text{ or } x = b \\
1, & x \in (a, b) 
\end{cases}
$$

This is the same as saying $f_\epsilon \to \frac{1}{2}(\chi_{[a,b]} + \chi_{(a,b)})$ pointwise, so the functional calculus (2.1.3) applies to give (2.5).

\begin{flushright}
$\square$
\end{flushright}

### 2.1.3 Analysis of Resolvent Operators

As mentioned previously, proving dispersive estimates for solutions to (1.2) will rely on a detailed understanding of the resolvent operator $R_V(z) := (H - z)^{-1}$ corresponding to $H = -\Delta + V$. Standard theory (see for instance [19]) implies that with our assumptions on $V$, the operator $H$ can be understood as a self-adjoint operator on $L^2(\mathbb{R}^3)$ with dense domain $D(H) = H^2(\mathbb{R}^3)$. When doing computations with $R_V(z)$, we will often want to connect it to the so-called “free resolvent”, which is defined by

$$
R_0(z) := (-\Delta - z)^{-1}
$$

It is well known that the free resolvent $R_0(z)$ is given explicitly by integration against the kernel

$$
K_0(z)(x, y) = \frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|}
$$

where $\sqrt{z}$ has positive imaginary part.

The key link between the two operators is contained in the resolvent identities:

$$
R_V(z) = R_0(z) - R_0(z)VR_V(z) = R_0(z) - R_V(z)VR_0(z) \\
R_V(z) = (I + R_0(z)V)^{-1}R_0(z) = R_0(z)(I + VR_0(z))^{-1}
$$

At this point the second identity is a formal relationship; it will be made clear in the sequel using results from Agmon [19] that the operator is $I + R_0(z)V$ is actually invertible on the appropriate spaces.
A priori, the operator $R_V(z)$ is only defined for values of $z$ not in the spectrum of $H$. In our case, the spectrum is purely continuous and consists of $[0, \infty)$. However, in view of the Stone Formula (2.5) it is desirable to make some definition of $R_V(\lambda)$ for $\lambda \geq 0$ at least in a limiting sense. The precise results were proved by Agmon in [19], where it is shown that for potentials $V$ satisfying even weaker conditions than those we are imposing, one can define the operators $R_V^\pm(\lambda)$ for $\lambda \geq 0$ as limit operators between the weighted spaces $L^{2,\sigma}(\mathbb{R}^3)$ and $H^{2,-\sigma}(\mathbb{R}^3)$ for $\sigma > \frac{1}{2}$. First it is shown in [19] that the limits

$$ R_0^\pm(\lambda) := \lim_{\epsilon \to 0^+} R_0(\lambda \pm i\epsilon) $$

exist in the uniform operator topology on the space of operators $\mathcal{B}(L^{2,\sigma}, H^{2,-\sigma})$. Note that $R_0^+$ is distinct from $R_0^-$ since we will choose different branches of $\sqrt{z}$ when evaluating each one. Next, the corresponding limits for $R_V$ are proved in [19], i.e. the limits

$$ R_V^\pm(\lambda) := \lim_{\epsilon \to 0^+} R_V(\lambda \pm i\epsilon) $$

exist, also in the uniform operator topology on $\mathcal{B}(L^{2,\sigma}, H^{2,-\sigma})$. Our assumptions on $V$ imply that multiplication by $V$ is a compact operator from $H^{2,-\sigma}(\mathbb{R}^3)$ to $L^{2,\sigma}(\mathbb{R}^3)$ ([19]). Thus, $R_0^\pm(\lambda)V$ is a compact operator from $H^{2,-\sigma}(\mathbb{R}^3)$ to itself.

The Fredholm alternative then implies that $I + R_0^\pm(\lambda)V$ is invertible if $-1$ is not an eigenvalue of $R_0^\pm(\lambda)V$, but such an eigenvalue implies ([19]) that $H$ has $\lambda$ as an eigenvalue, contrary to our assumptions. One now has the analogous resolvent identities

$$ R_V^\pm(\lambda) = R_0^\pm(\lambda) - R_0^\pm(\lambda)VR_V^\pm(\lambda) = R_0^\pm(\lambda) - R_V^\pm(\lambda)VR_0^\pm(\lambda) \tag{2.8} $$

$$ R_V^\pm(\lambda) = (I + R_0^\pm(\lambda)V)^{-1}R_0^\pm(\lambda) = R_0^\pm(\lambda)(I + VR_0^\pm(\lambda))^{-1} $$

We will also define

$$ R_0^\pm(\lambda^2) := \lim_{\epsilon \to 0^+} R_0((\lambda \pm i\epsilon)^2) $$

$$ R_V^\pm(\lambda^2) := \lim_{\epsilon \to 0^+} R_V((\lambda \pm i\epsilon)^2) $$

noting that $R_V^\pm(\lambda^2)$ can also be obtained from the resolvent identity (2.8).
A key property of these operators that we will use later is that the difference
\[ R_+^V(\lambda^2) - R_-^V(\lambda^2) \] is an odd function of \( \lambda \). To see this, consider the difference
\[
\frac{1}{H - (\lambda + i\epsilon)^2} - \frac{1}{H - (\lambda - i\epsilon)^2}
\]
These differences are odd functions of \( \lambda \), and thus the limit will be as well.

The final property of the resolvent operators that we would like to record here for future use has to do with their differentiability properties. We will need to know that the operators \( R_+^\pm V(\lambda^2) \) are differentiable in \( \lambda \). The result is

**Lemma 2.1.7.**

\[
\frac{d}{d\lambda} R_+^\pm(\lambda^2) = (I + R_0^\pm(\lambda^2) V)^{-1} D_0^\pm(\lambda^2) (I + VR_0^\pm(\lambda^2))^{-1}
\]

(2.9)

where \( D_0^\pm(\lambda^2) = \frac{d}{d\lambda} R_0^\pm(\lambda^2) \).

**Proof.** \( R_0^\pm(\lambda^2) \) is differentiable as evidenced by its explicit integral kernel (2.6). Thus, we deduce from the identity \( R_+^\pm(\lambda^2) = (I + R_0^\pm(\lambda^2) V)^{-1} R_0^\pm(\lambda^2) \) that \( R_+^\pm(\lambda^2) \) will be differentiable if the factor \( (I + R_0^\pm(\lambda^2) V)^{-1} \) is differentiable. As stated above, \( I + R_0^\pm(\lambda^2) V \) is a compact perturbation of the identity on \( H^{2-\sigma}(\mathbb{R}^3) \), and since \( \lambda^2 \) is not an eigenvalue of \( -\Delta + V \), the Fredholm alternative applies to give that \( (I + R_0^\pm(\lambda^2) V)^{-1} \) is bounded and thus has a derivative in \( \lambda \).

Once \( R_+^\pm(\lambda^2) \) is known to be differentiable, one gets the identity (2.9) by differentiating the formula
\[
R_+^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2) V R_+^\pm(\lambda^2)
\]

This yields the identity
\[
\frac{d}{d\lambda} R_+^\pm(\lambda^2) = D_0^\pm(\lambda^2) - D_0^\pm(\lambda^2) V R_+^\pm(\lambda^2) - R_0^\pm(\lambda^2) V \frac{d}{d\lambda} R_+^\pm(\lambda^2)
\]

which rearranges to
\[
(I + R_0^\pm(\lambda^2) V) \frac{d}{d\lambda} R_+^\pm(\lambda^2) = D_0^\pm(\lambda^2) (I - VR_+^\pm(\lambda^2))
\]

Multiply on the left by \( (I + R_0^\pm(\lambda^2) V)^{-1} \) to obtain
\[
\frac{d}{d\lambda} R_+^\pm(\lambda^2) = (I + R_0^\pm(\lambda^2) V)^{-1} D_0^\pm(\lambda^2) (I - VR_+^\pm(\lambda^2))
\]
The proof is completed by observing that

\[(I - VR^\pm_V(\lambda^2)) = (I + VR_0^\pm(\lambda^2))^{-1}\]

as can be seen by direct multiplication:

\[(I - VR^\pm_V(\lambda^2))(I + VR_0^\pm(\lambda^2)) = I + V[R_0^\pm(\lambda^2) - R_V^\pm(\lambda^2)] - VR^\pm_V(\lambda^2)VR_0^\pm(\lambda^2) = I\]

by the first identity in (2.8). A similar calculation shows that multiplication on the right by \(I - VR^\pm_V(\lambda^2)\) produces \(I\) as well, so we are done.

2.1.4 An Integral Representation Formula

To conclude this section, we show how the results accumulated thus far allow for analysis of the expressions in (2.2). Suppose \(p(x)\) is a bounded Borel function on \(\mathbb{R}\). By imposing the assumptions 2.1.1 and 2.1.2 on \(V\), recall that

\[H = -\Delta + V\]

is understood as a self-adjoint operator on \(L^2(\mathbb{R}^3)\) with domain \(H^2(\mathbb{R}^3)\). Furthermore \(H\) has no eigenvalues and the spectrum \([0, \infty)\) of \(H\) is purely continuous. Thus for \(f, g \in H^2(\mathbb{R}^3)\), one has

\[\langle f, p(H)g \rangle = \int_0^\infty p(\lambda) \langle f, E_\lambda g \rangle d\lambda\]

An application of the Stone Formula 2.5 shows that this can be rewritten as

\[\frac{2\pi i}{-1} \lim_{\epsilon \searrow 0} \int_0^\infty p(\lambda) \langle [R_V(\lambda + i\epsilon) - R_V(\lambda - i\epsilon)]f, g \rangle d\lambda \quad (2.10)\]

Using the definitions and resolvent identities discussed in the previous section, this simplifies to

\[\langle f, p(H)g \rangle = \frac{2\pi i}{-1} \int_0^\infty p(\lambda) \langle [R_V^+(\lambda) - R_V^-(\lambda)]f, g \rangle d\lambda \quad (2.11)\]

It is this final formula (2.11) that will actually be used to derive dispersive estimates for solutions to (2.1).
2.2 Sobolev’s Lemma and $H$-Sobolev Spaces

A frequently used tool in analysis, loosely referred to as “Sobolev Embedding”, is a powerful result that comes in many different versions. Recall that the Sobolev space $W^{k,p}(\mathbb{R}^n)$ is the space of all measurable functions whose (weak) derivatives of order less than or equal to $k$ lie in $L^p(\mathbb{R}^n)$, and when $p = 2$ we write $H^k(\mathbb{R}^n)$ instead of $W^{k,2}(\mathbb{R}^n)$. The norm on $W^{k,p}(\mathbb{R}^n)$ is given by

$$\|f\|_{W^{k,p}(\mathbb{R}^n)} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}$$

For real $s > 0$, we shall also occasionally make use of the Bessel potential spaces $H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) | \int_{\mathbb{R}^n} (1 + |\xi|^2)^{\frac{s}{2}} |\hat{f}(\xi)|^2 \, d\xi < \infty\}$. There is no abuse of notation since for positive integer $s$, the norm induces the same space as above by the Plancherel theory.

For our purposes, a simplified version known as Sobolev’s Lemma will suffice. The following estimate and its proof appear frequently in the literature, but a nice proof can be found in [17]:

**Theorem 2.2.1 (Sobolev’s Lemma).** If $m$ is a positive integer, $n/m < p \leq \infty$, and $\partial^\alpha u \in L^p(\mathbb{R}^n)$ for $|\alpha| \leq m$, then $u \in C(\mathbb{R}^n)$ and there holds the estimate

$$\|u\|_{L^\infty(\mathbb{R}^n)} \lesssim \|u\|_{W^{m,p}(\mathbb{R}^n)} \quad (2.12)$$

I will also make use of the following embedding for the fractional spaces $H^s(\mathbb{R}^n)$ which is proved, for example, in [14]-[16].

**Theorem 2.2.2.** For real $s \in [0, \frac{n}{2})$, one has the inclusion $H^s(\mathbb{R}^n) \subset L^{2n/(n-2s)}(\mathbb{R}^n)$ and with $p^* = 2n/(n-2s)$, the estimate

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \lesssim \|u\|_{H^s(\mathbb{R}^n)}$$

Recall that we have recast the main equation (1.2) in question as

$$(\partial_t^2 + H + 1)u = u^2$$

where $H = -\Delta + V$. Existence arguments for nonlinear problems involve a priori estimates on solutions as we shall see in the sequel. One commonly bootstraps
estimates by differentiating both sides of the equation and applying previously known estimates. Since the operator $H$ includes multiplication by $V(x)$, the simple differential operators $\partial_x^\alpha$ no longer commute with the full operator $\partial_t^2 + H + 1$. Instead of dealing with operator bounds on the commutator $[\partial_x^\alpha, V]$, we will differentiate using the operator $H$ itself, which trivially commutes with $\partial_t^2 + H + 1$. Thus, it will be convenient to define modified “$H$-Sobolev” spaces where the derivatives are in terms of $H$:

**Definition 2.2.3 (H-Sobolev spaces).** For $H = -\Delta + V$ and $k \geq 0$, we define

\[
\| u \|_{W_{H}^{2k,p}} := \sum_{j=0}^{k} \| H^j u \|_{L^p} \\
\| u \|_{W_{H}^{2k+1,p}} := \sum_{j=0}^{k} \| H^j u \|_{W^{1,p}}
\]

Context will determine which norm is being used, according to the parity of the number of derivatives under consideration.

We now prove some basic estimates that allow movement between the classical Sobolev spaces $W^{k,p}$ and the $H$-Sobolev spaces.

**Proposition 2.2.4.** Let $w(\Delta, V)$ denote any word in the operators $\Delta$ and $V$, where the operator $V$ means multiplication by $V(x)$. Then for $1 \leq p \leq \infty$, we have the estimate

\[
\| w(\Delta, V)f \|_{L^p} \lesssim \| f \|_{W^{2a,p}}
\]

where $a$ is the number of occurrences of $\Delta$ in the word $w(\Delta, V)$. The implied constant depends only on $V$ and $p$.

**Proof.** Write $w(\Delta, V) = \Delta^{a_1} V^{b_1} \ldots \Delta^{a_k} V^{b_k}$. Let $a = a_1 + \cdots + a_k$. Our assumptions on $V$ imply that $V$ is bounded as a multiplier on any $W^{s,p}$ space. It follows then that for any nonnegative integers $j, k, m$, we have the bounds $\| V^j g \|_{W^{m,p}} \lesssim \| g \|_{W^{m,p}}$ and $\| \Delta^k f \|_{L^p} \lesssim \| f \|_{W^{2k,p}}$ ($\Delta^k$ is a sum of derivatives of order less than or equal to $2k$). Iterating these bounds yields

\[
\| w(\Delta, V)f \|_{L^p} \lesssim \| f \|_{W^{2(a_1 + \cdots + a_k),p}} = \| f \|_{W^{2a,p}}
\]

\[\square\]
Theorem 2.2.5. We have the bounds
\[
\| f \|_{W^{2k,p}_H} \lesssim \| f \|_{W^{2k,p}} \\
\| f \|_{W^{2k+1,p}_H} \lesssim \| f \|_{W^{2k+1,p}}
\]

Proof. These estimates are established immediately by expanding terms of the form \( H^j = (-\Delta + V)^j \) (keeping in mind that \( \Delta \) and \( V \) do not commute) and applying Proposition 2.2.4. \( \square \)

To go in the other direction is much more delicate. The estimates we require, in addition to a frequently useful product estimate, are collected in the following theorem:

Theorem 2.2.6. For \( 1 \leq p \leq \infty \), we have:
\[
\| u \|_{W^{2k-1,p}} \lesssim \| u \|_{W^{2k,p}_H} \tag{2.13}
\]
\[
\| u \|_{W^{2k,p}} \lesssim \| u \|_{W^{2k+1,p}_H} \tag{2.14}
\]

In the special case \( p = 2 \), we have, for any \( k \), the improved estimate
\[
\| u \|_{W^{k,2}} \lesssim \| u \|_{W^{k,2}_H} \tag{2.15}
\]

Also, one has for \( k \geq 1 \) the product estimate
\[
\| uv \|_{H^{k-1}} \lesssim \| u \|_{H^1} \| v \|_{H^k} + \| u \|_{H^k} \| v \|_{H^1} \tag{2.16}
\]

The proof of (2.16) is a standard computation using the Littlewood-Paley theory. Note that in the cases \( k = 1, 2 \), it follows from Hölder’s inequality and the \( H^1(\mathbb{R}^3) \subseteq L^4(\mathbb{R}^3) \) Sobolev embedding. A proof for general \( k > 2 \) can be found in [13].

Theorem 2.2.6 will be deduced from the following two auxiliary lemmas:

Lemma 2.2.7. For \( j \geq 1 \), we have
\[
\| u \|_{W^{2j-1,p}_H} \lesssim \| \Delta^j u \|_{L^p} + \| u \|_{L^p}
\]
\[
\| u \|_{W^{2j,p}_H} \lesssim \| \Delta^j u \|_{W^{1,p}_H} + \| u \|_{W^{1,p}}
\]
Once we have this, Theorem 2.2.6 will be an immediate consequence of

\[ \text{Lemma 2.2.8. for } j \geq 1, \text{ we have} \]
\[ \| \Delta^j u \|_{L^p} \lesssim \| u \|_{W^{2j,p}_H} \]
\[ \| \Delta^j u \|_{W^{1,p}} \lesssim \| u \|_{W^{2j+1,p}_H} \]

Thus Theorem 2.2.6 has been reduced to proving Lemma 2.2.7 followed by Lemma 2.2.8. Let us also dispose of the special case \( p = 2 \). The estimate (2.15) follows more or less directly from the Parseval identity: for \( |\alpha| \leq k \), one has
\[ \| \partial^\alpha u \|_{L^2}^2 = (2\pi)^{-3} \int |\xi^\alpha| |\hat{u}(\xi)|^2 \, d\xi \lesssim \int |\xi|^{2|\alpha|} |\hat{u}(\xi)|^2 \, d\xi = \| |\xi|^{|\alpha|} \hat{u} \|_{L^2}^2 \]

For \( |\alpha| \) even, we are done since then
\[ \| \partial^\alpha u \|_{L^2} \leq \| |\xi|^{|\alpha|} \hat{u} \|_{L^2} = \| (-\Delta)^{|\alpha|/2} u \|_{L^2} \lesssim \| u \|_{W^{j,2}_{H}} \lesssim \| u \|_{W^{k,2}_{H}} \]

by the first inequality in Lemma 2.2.8. In case \( |\alpha| \) is odd, then for some \( |\beta| = 1 \) we obtain the bound
\[ \| \partial^\alpha u \|_{L^2} \leq \| \partial^\beta (-\Delta)^{|\alpha|-1/2} u \|_{L^2} \leq \| (-\Delta)^{|\alpha|-1/2} u \|_{W^{1,2}} \lesssim \| u \|_{W^{j,2}_{H}} \lesssim \| u \|_{W^{k,2}_{H}} \]

by the second inequality in Lemma 2.2.8. Summing these results over \( |\alpha| \leq k \) concludes the \( p = 2 \) case, so now we turn to the proofs of Lemmas 2.2.7 and 2.2.8.

Proof of Lemma 2.2.7. We will deduce the second inequality from the first at the end of the proof, so let us prove the first inequality. The proof will use the usual Paley-Littlewood decomposition \( u = P_0 u + \sum_{i \geq 1} P_i u \) where \( P_0 \) is projection onto frequencies \( |\xi| \leq 1 \) and \( P_i \) is projection onto frequencies \( 2^{i-1} \leq |\xi| \leq 2^i \). Observe that \( P_0 u = K_0 * u \), where \( \hat{K}_0(\xi) = k_0(\xi) \) and \( k_0 \) is a smooth bump function with support in \( \{ |\xi| \leq 1 \} \). Similarly, \( P_i u = K_i * u \) for \( i \geq 1 \) where the functions \( k_i(\xi) \) are supported in the annuli \( 2^{i-1} \leq |\xi| \leq 2^i \).

First we will estimate \( \| P_0 u \|_{W^{j-1,2}} \). For \( |\alpha| \leq 2j - 1 \), we have
\[ \| \partial^\alpha P_0 u \|_{L^p} = \| (\partial^\alpha K_0) * u \|_{L^p} = \| K_{0,\alpha} * u \|_{L^p} \]
(2.17)
where \( \hat{K}_{0,\alpha}(\xi) = \xi^\alpha k_0(\xi) \) (at least up to some constant involving \( \pi \)). This is a Schwartz function and thus so is its inverse Fourier transform \( K_{0,\alpha} \), which in
particular means that it lies in every $L^p$ space and by Young’s inequality we have the bound

$$\| K_{0,\alpha} * u \|_{L^p} \lesssim \| u \|_{L^p}$$

For high frequencies, we will show that for $i \geq 1$ and $|\alpha| \leq 2j - 1$, we have the bound

$$\| \partial^\alpha P_i u \|_{L^p} \lesssim 2^{-i} \| \Delta^j u \|_{L^p}$$

(2.18)

which will allow us to sum up the high frequency components and establish the lemma. Since $P_i$ is a convolution operator and commutes with derivatives, we can write

$$\partial^\alpha P_i u = \frac{\partial^\alpha}{\Delta} P_i \Delta^j u$$

(2.19)

and finish by proving the estimate $\| \partial^\alpha P_i \|_{L^p \to L^p} \lesssim 2^{-i}$. The operator in question is convolution with the kernel $K_{i,\alpha}$, where $\widehat{K_{i,\alpha}}(\xi) = \frac{\xi^\alpha}{|\xi|^j} p_i(\xi)$. Recall that the $p_i$ are supported away from the origin so that this function is again in the Schwartz class, thus reducing the problem through Young’s inequality to showing that $\| K_{i,\alpha} \|_{L^1} \lesssim 2^{-i}$. Since $p_i(\xi) = p_1(2^{-i}\xi)$, we can write

$$\widehat{K_{i,\alpha}}(\xi) = 2^{-i(2j-|\alpha|)}(\frac{2^{-i}\xi}{2-i\xi})^\alpha \frac{\xi^\alpha}{|\xi|^j} p_1(2^{-i} \xi) = 2^{-i(2j-|\alpha|)} \widehat{K_{1,\alpha}}(2^{-i} \xi)$$

(2.20)

By taking inverse Fourier transforms, we get $K_{i,\alpha}(x) = 2^ni2^{-i(2j-|\alpha|)} K_{1,\alpha}(2^i x)$. Now we compute the $L^1$ norm and find that

$$\| K_{i,\alpha} \|_{L^1} = 2^{-i(2j-|\alpha|)} \| K_{1,\alpha} \|_{L^1} \lesssim 2^{-i}$$

(2.21)

since $2j-|\alpha| \geq 1$ by assumption and $\| K_{1,\alpha} \|_{L^1}$ is independent of $i$. This establishes (2.18) and thus the first inequality in Lemma 2.2.7 by summing.

To recover the second inequality, note that

$$\| u \|_{W^{2j,p}} = \sum_{|\alpha| \leq 2j-1} \| \partial^\alpha u \|_{L^p} + \sum_{|\alpha| = 2j} \| \partial^\alpha u \|_{L^p}$$

By our work so far the first summand is already controlled by $\| \Delta^j u \|_{W^{1,p}} + \| u \|_{W^{1,p}}$. Note then that for $|\alpha| = 2j$, we have $\| \partial^\alpha u \|_{L^p} \lesssim \sum_{|\beta| = 1} \| \partial^\beta u \|_{W^{2j-1,p}}$, and since $\partial^\beta$ commutes with $\Delta^j$ we can apply again the inequality we already proved to get the required bound. □
Using Lemma 2.2.7, we can pause briefly to prove that the $W^{k,p}_H$ spaces chain in the obvious way:

**Proposition 2.2.9.** For $W^{k,p}_H$ defined as above, one has for any $k \geq 0$ the inequalities

\[ \| u \|_{W^{2k,p}_H} \lesssim \| u \|_{W^{2k+1,p}_H} \lesssim \| u \|_{W^{2k+2,p}_H} \]

**Proof.** The first inequality is immediate from the definition. For the second, recall that

\[ \| u \|_{W^{2k+1,p}_H} = \sum_{j=0}^{k} \| H^j u \|_{W^1,p} \]

Consider each summand $\| H^j u \|_{W^1,p} = \sum_{|\alpha| \leq 1} \| \partial^\alpha H^j u \|_{L^p}$. By Lemma 2.2.7, one has for any $|\alpha| \leq 1$ the bound $\| \partial^\alpha H^j u \|_{L^p} \lesssim \| H^j u \|_{L^p} + \| \Delta H^j u \|_{L^p}$. Then since $\Delta = V - H$ and $V$ is bounded as a multiplier on $L^p$, one gets (still for $|\alpha| \leq 1$)

\[ \| \partial^\alpha H^j u \|_{L^p} \lesssim \| H^j u \|_{L^p} + \| H^j u \|_{L^p} + \| H^{j+1} u \|_{L^p} \]

The conclusion is that there is a constant $C$ so that

\[ \sum_{j=0}^{k} \| H^j u \|_{W^1,p} \leq C \sum_{j=0}^{k} \| H^{j+1} u \|_{L^p} \leq C \sum_{j=0}^{k+1} \| H^j u \|_{L^p} = C \| u \|_{W^{2k+2,p}_H} \]

as was to be shown. \(\square\)

Now we can prove Lemma 2.2.8.

**Proof of Lemma 2.2.8.** The proof is by induction. For $j = 1$, we replace $\Delta = V - H$ in both inequalities to obtain

\[ \| \Delta u \|_{L^p} \leq \| V u \|_{L^p} + \| H u \|_{L^p} \lesssim \| u \|_{L^p} + \| H u \|_{L^p} = \| u \|_{W^2,p} \]

and

\[ \| \Delta u \|_{W^1,p} \leq \| V u \|_{W^1,p} + \| H u \|_{W^1,p} \lesssim \| u \|_{W^1,p} + \| H u \|_{W^1,p} = \| u \|_{W^3,p} \]
since $V$ is bounded as a multiplier on any $W^{s,p}$ space. Now suppose these estimates hold up to $j - 1$ and note first that

\[
\| \Delta^j u \|_{L^p} = \| \Delta^{j-1} \Delta u \|_{L^p} \lesssim \| \Delta u \|_{W^{2j-2,p}_H} \\
\lesssim \| V u \|_{W^{2j-2,p}_H} + \| H u \|_{W^{2j-2,p}_H} \\
\lesssim \| V u \|_{W^{2j-2,p}_H} + \| H u \|_{W^{2j-2,p}_H} \\
\lesssim \| u \|_{W^{2j-2,p}_H} + \| H u \|_{W^{2j-2,p}_H} \\
\lesssim \| \Delta^{j-1} u \|_{W^{1,p}_H} + \| u \|_{W^{1,p}_H} + \| H u \|_{W^{2j-2,p}_H} \\
\lesssim \| u \|_{W^{2j-1,p}_H} + \| \Delta u \|_{L^p} + \| u \|_{L^p} + \| H u \|_{W^{2j-2,p}_H} \\
\lesssim \| u \|_{W^{2j,p}_H}
\]

where we have used the induction hypothesis and Lemma 2.2.7 twice in addition to Lemma 2.2.9. A similar calculation shows that $\| \Delta^j u \|_{W^{1,p}_H} \lesssim \| u \|_{W^{2j+1,p}_H}$ so we are done. \qed
Chapter 3

Existence Techniques for Nonlinear Equations

3.1 Picard Iteration

To prove existence of local solutions (i.e. for small time depending on the initial data), we will use a standard Picard iteration argument. The potential term $V$ will be treated as a perturbation, so will write the equation as

\[(\Box + 1)u = -Vu + u^2\]  \hspace{1cm} (3.1)

\[u(0) = u_0\]

\[u_t(0) = u_1\]

With $f(u) = -Vu + u^2$, one can write the solution as

\[u = W(u_0, u_1) + L(f(u))\]  \hspace{1cm} (3.2)

where $w = W(u_0, u_1)$ satisfies the homogeneous equation

\[(\Box + 1)w = 0\quad w(0) = u_0\quad w_t(0) = u_1\]

and $w = L(f(u))$ satisfies the equation

\[(\Box + 1)w = f\quad w(0) = 0\quad w_t(0) = 0\]
We then find a $u$ satisfying (3.2) using an iteration scheme. Define

$$u^{(0)} = W(u_0, u_1)$$ (3.3)

$$u^{(n)} = W(u_0, u_1) + L(f(u^{(n-1)}))$$ (3.4)

The goal is then to find a suitable Banach space in which the sequence $\{u^{(n)}\}_{n \geq 0}$ is Cauchy and thus converges to a solution of (3.2). We will in fact find spaces $\| \cdot \|_{S_T}$ and $\| \cdot \|_{S_0}$, where $[0, T]$ is the time interval on which a solution will be constructed, such that

$$\| u^{(n)} - u^{(n-1)} \|_{S_T} \leq 2^{-n} \| (u_0, u_1) \|_{S_0}$$ (3.5)

This implies that $\{u^{(n)}\}$ is Cauchy with respect to $\| \cdot \|_{S_T}$ by a standard telescoping argument. We will require that there is a uniform constant $C$, depending only on fixed quantities, such that the data norm $S_0$ satisfies

$$\| W(u_0, u_1) \|_{S_T} \leq C \| (u_0, u_1) \|_{S_0}$$ (3.6)

Then to prove (3.5), it will suffice (by linearity of $L$) to prove an estimate of the form

$$\| L(f(\phi)) - L(f(\psi)) \|_{S_T} \leq TC \| \phi - \psi \|_{S_T} (1 + \| \phi \|_{S_T} + \| \psi \|_{S_T})$$ (3.7)

for the same $C$ as above. Instead of proving (3.7) directly, we introduce an auxiliary space $\| \cdot \|_{N_T}$, which we will choose to satisfy the two estimates

$$\| LG \|_{S_T} \leq \| G \|_{N_T}$$ (3.8)

$$\| f(\phi) - f(\psi) \|_{N_T} \leq TC \| \phi - \psi \|_{S_T} (1 + \| \phi \|_{S_T} + \| \psi \|_{S_T})$$ (3.9)

Upon inspection one sees that (3.8) and (3.9) imply (3.7)

Now we are able to calculate precisely how small $T$ needs to be for the iteration to close. It will come from the proof of (3.5), which is by induction. As an intermediate step, we establish the bound

$$\| u^{(n)} \|_{S_T} \leq 2C \| (u_0, u_1) \|_{S_0}$$ (3.10)

which is of independent interest as it allows us to produce a uniform bound on the $S_T$ norm of a solution in terms of the initial data. Indeed, (3.10) is immediate
for \( n = 0 \), so assume that \( \| u^{(n-1)} \|_{S_T} \leq 2C \| (u_0, u_1) \|_{S_0} \). Then (3.7) provides the estimate

\[
\| u^{(n)} \|_{S_T} \leq C \| (u_0, u_1) \|_{S_0} + TC \| u^{(n-1)} \|_{S_T}(1 + \| u^{(n-1)} \|_{S_T}) \\
\leq C \| (u_0, u_1) \|_{S_0} + 2C^2T \| (u_0, u_1) \|_{S_0}(1 + 2C \| (u_0, u_1) \|_{S_0})
\]

Thus, one chooses \( T = \frac{1}{2C}(1 + 2C \| (u_0, u_1) \|_{S_0})^{-1} \) to yield (3.10).

This estimate now allows us to close the iteration scheme by proving (3.5). The base case is trivial, so assume that \( \| u^{(n-1)} - u^{(n-2)} \|_{S_T} \leq 2^{-(n-1)} \| (u_0, u_1) \|_{S_0} \).

Then by (3.7) one has

\[
\| u^{(n)} - u^{(n-1)} \|_{S_T} \leq TC \| u^{(n-2)} \|_{S_T}(1 + \| u^{(n-1)} \|_{S_T} + \| u^{(n-2)} \|_{S_T}) \\
\leq 2^{-(n-1)}TC \| (u_0, u_1) \|_{S_0}(1 + 4C \| (u_0, u_1) \|_{S_0}) \\
\leq 2^{-n} \| (u_0, u_1) \|_{S_0}
\]

if we decrease \( T \) to \( \frac{1}{2C}(1 + 4C \| (u_0, u_1) \|_{S_0})^{-1} \). The conclusion is that we can find a uniform (small) constant \( c \), even for large data \( (u_0, u_1) \), such that the iteration scheme above converges to a solution on \([0, T]\) if \( T \leq T_0 = c(1 + \| (u_0, u_1) \|_{S_0})^{-1} \).

In addition, once we know that the sequence \( u^{(n)} \) converges, we obtain from (3.10) the estimate

\[
\| u \|_{S_T} \leq 2C \| (u_0, u_1) \|_{S_0}
\]

by taking a limit and using continuity of the norm.

With these calculations in place, we will have an understanding of the local existence problem in any setting where we can define spaces so that (3.6), (3.8), and (3.9) all hold.

### 3.2 Local Existence

Here we will apply the iteration procedure described in the previous section to prove local existence of solutions to the main equation (1.2). The local existence theorem is stated as follows:
Theorem 3.2.1 (Local Existence). Given initial data \((u_0, u_1) \in H^1 \times L^2\), there exists a time \(T_0 > 0\), depending on \(u_0, u_1\), such that the equation

\[
\left( \partial_t^2 - \Delta_x + V + 1 \right) u = u^2
\]

\(u(0) = u_0\)
\(u_t(0) = u_1\)

has a unique solution of class \(H^1\), defined on \([0, T_0] \times \mathbb{R}^3\) satisfying the bound

\[
\sup_{0 \leq t \leq T_0} \|(u, u_t)(t)\|_{H^1 \times L^2} \leq C \| (u_0, u_1) \|_{H^1 \times L^2}
\]

As we will see below, this theorem follows immediately from the work in the previous section after we define the appropriate \(S_T, N_T\) spaces. We can also prove a higher regularity version, which we state as

Theorem 3.2.2 (Higher regularity Local existence). Let \((u_0, u_1) \in H^s \times H^{s-1}\) be initial data with \(s \geq 1\). Then there exists a \(T_0 = T_0(\| (u_0, u_1) \|_{H^1 \times L^2})\) such that there is a unique \(H^s \times H^{s-1}\) solution to \(\left( \partial_t^2 + H + 1 \right) u = u^2\) on \([0, T_0]\). Moreover, there exists a constant \(C > 0\) depending only on \(s\) such that

\[
\sup_{0 \leq t \leq T_0} \|(u, u_t)(t)\|_{H^s \times H^{s-1}} \leq C \|(u_1, u_0)\|_{H^s \times H^{s-1}}
\]

Remark 3.2.3. In the proofs of these theorems we will show in fact that one has the lower bound bound \(T_0 \geq c(1 + \| (u_1, u_0) \|_{H^1 \times L^2})^{-1}\) for some sufficiently small \(c > 0\) that will depend only on \(s\). This is an improvement upon the results in the previous section since even in the higher regularity setting, the time of existence depends only on the initial data in \(H^1 \times L^2\). If \(V\) is such that \(H = -\Delta + V\) has no negative eigenvalues, then one can improve this lower bound to \(c'(1 + \| (u_1, u_0) \|_{H^1 \times L^2})^{-1}\) using the generalized energy estimates presented below. In any event, if one knows a priori that \(\sup_{0 \leq t < T_*} \|(u, u_t)(t)\|_{H^1 \times L^2} < \infty\), then one can continue the solution to \([0, T_* + \epsilon]\) for some (possibly small) \(\epsilon > 0\). As a consequence, if one can show \(\sup_{0 \leq t < T} \|(u, u_t)(t)\|_{H^1 \times L^2} \leq M\) for the solution on every time interval \([0, T]\) on which is exists, then the solution is global in time. In addition, the local higher regularity bound implies the exponential estimate
\[ \| (u, u_t)(t) \|_{H^s \times H^{s-1}} \leq C e^{C t} \| (u_0, u_1) \|_{H^s \times H^{s-1}} \] for any (possibly large) time interval on which a solution exists. Here \( C > 0 \) depends on both \( s \) and \( \| (u_1, u_0) \|_{H^1 \times L^2} \).

This bad growth estimate for higher derivatives motivates proving dispersive estimates even in the case where \( f(u) = u^2 \) and one gets global existence from just the energy estimates (as will be shown later in this chapter).

Since the proof of local existence for higher regularity depends on the results in \( H^1 \times L^2 \), we will handle that case first and then show how to obtain Theorem 3.2.2. The only tools we will need to prove Theorem 3.2.1 are some Sobolev embeddings from (2.2.1) and the basic energy estimates for solutions to (3.1), which we will prove here before proceeding. The method of proof is standard; see for instance [7].

**Lemma 3.2.4.** If \( u \) solves \((\partial_t^2 - \Delta_x + 1)u = f\) on a time interval \([0, T]\) with \((u(0), u_t(0)) = (u_0, u_1)\), then for \( k \geq 1 \) there holds the estimate

\[
\sup_{0 \leq s \leq t} \| (u, u_t)(s) \|_{H^k \times H^{k-1}} \lesssim \| (u_0, u_1) \|_{H^k \times H^{k-1}} + \int_0^t \| f(s, \cdot) \|_{H^{k-1}} \, ds \tag{3.14}
\]

for any \( 0 \leq t \leq T \).

**Proof.** Fix \( t \) in \([0, T]\). Define the energy

\[
E(t) := \frac{1}{2} \left( \| u(t, \cdot) \|_{L^2}^2 + \| u_t(t, \cdot) \|_{L^2}^2 + \| \nabla_x u(t, \cdot) \|_{L^2}^2 \right)
\]

By differentiating and integrating by parts, one gets

\[
E'(t) = \int_{\mathbb{R}^3} u_t(u_{tt} - \Delta_x u + u) \, dx = \int_{\mathbb{R}^3} u_t f \, dx \leq \sqrt{2} E(t)^{\frac{1}{2}} \| f(t, \cdot) \|_{L^2}
\]

where in the last inequality we have used Cauchy-Schwarz. Define the quantity \( M(t) = \sup_{0 \leq s \leq t} E(s)^{\frac{1}{2}} \). Then if \( 0 \leq a \leq t \), we integrate from 0 to \( a \) to obtain

\[
E(a) - E(0) \leq \sqrt{2} \int_0^a E(s)^{\frac{1}{2}} \| f(s, \cdot) \|_{L^2} \, ds \leq \sqrt{2} M(t) \int_0^t \| f(s, \cdot) \|_{L^2} \, ds
\]

Taking the supremum over \( 0 \leq a \leq t \) then gives

\[
M(t)^2 \leq M(0)^2 + \sqrt{2} M(t) \int_0^t \| f(s, \cdot) \|_{L^2} \, ds \tag{3.15}
\]
I claim that this implies the estimate
\[
\sup_{0 \leq s \leq t} E(s)^{\frac{1}{2}} \leq E(0)^{\frac{1}{2}} + \int_0^t \| f(s, \cdot) \|_{L^2} \, ds
\]
To see this, note that if \( M(t) = 0 \) then the left side of the previous inequality is 0 and there is nothing to prove. Otherwise divide by \( M(t) \) in (3.15) to obtain the above formula. This implies in particular the bound
\[
\sup_{0 \leq s \leq t} \| (u, u_t)(s) \|_{H^1 \times L^2} \lesssim \| (u_0, u_1) \|_{H^1 \times L^2} + \int_0^t \| f(s, \cdot) \|_{L^2} \, ds
\]
Differentiating the equation with \( \partial^\alpha \) for \(|\alpha| \leq k - 1\) and reapplying this bound yields 3.14.

**Proof of Theorem 3.2.1.** The Banach spaces used to close the iteration scheme in this case shall be
\[
\| (u, v) \|_{S_0} := \| (u, v) \|_{H^1 \times L^2}
\]
\[
\| u \|_{S_T} := \sup_{0 \leq s \leq T} \| (u, u_t)(s) \|_{H^1 \times L^2}
\]
\[
\| G \|_{N_T} := \int_0^T \| G(s) \|_{L^2} \, ds
\]
First, observe that (3.6) holds with any constant larger than 1. This is an immediate consequence of (3.14) with \( k = 1 \) since \( W(u_0, u_1) \) solves the homogeneous Klein-Gordon equation.

To see that (3.8) holds, we will again use the estimate (3.14). Recall that \( LG \) satisfies the inhomogeneous equation with 0 data and right side \( G \), so the energy estimate in this case is
\[
\sup_{0 \leq s \leq T} \| (u, u_t)(s) \|_{H^1 \times L^2} \lesssim \int_0^T \| G(s) \|_{L^2} \, ds = \| G \|_{N_T}
\]
Thus \( \| u \|_{S_T} \leq \| G \|_{N_T} \) as required.

Finally we will show that (3.9) holds. As previously mentioned, we will take \( f(u) = -Vu + u^2 \) which amounts to treating \( V \) as a perturbation. In this case, we have
\[
\| f(\phi) - f(\psi) \|_{L^2} \leq \| V(\phi - \psi) \|_{L^2} + \| (\phi - \psi)(\phi + \psi) \|_{L^2}
\]
\[
\leq C' \| \phi - \psi \|_{L^2} + \| \phi - \psi \|_{L^4} (\| \phi \|_{L^4} + \| \psi \|_{L^4})
\]
where $C'$ depends only on $V$ (recall that $V$ is bounded as a multiplier on any $L^p$ space). By the Sobolev embedding (2.2.2), we have $H^1 \subseteq L^4$ which implies that there is a constant $C''$ such that

$$\| f(\phi) - f(\psi) \|_{L^2} \leq C'' \| \phi - \psi \|_{L^2} + C'' \| \phi - \psi \|_{H^1}(\| \phi \|_{H^1} + \| \psi \|_{H^1})$$

Integrating from 0 to $T$, taking supremums, and choosing a larger constant $C$ then yields

$$\| f(\phi) - f(\psi) \|_{N_T} \leq C T \| \phi - \psi \|_{S_T}(1 + \| \phi \|_{S_T} + \| \psi \|_{S_T}) \quad (3.16)$$

Now by our work in the previous section, the iteration scheme in Section 3.1 will converge to a solution $(u, u_t) \in H^1 \times L^2$ for any $T \leq T_0 = c(1 + \| (u_0, u_1) \|_{H^1 \times L^2})^{-1}$ for a uniform constant $c$. Additionally, the bound (3.12) is satisfied.

At this point, the only additional estimate we will need to prove Theorem 3.2.2 will be the product estimate (2.16) in Theorem 2.2.6.

**Proof of Theorem 3.2.2.** The higher regularity $S_T$ and $N_T$ spaces are defined as

$$\| u \|_{S_T} := \sup_{0 \leq t \leq T} \| (u, u_t)(t) \|_{H^s \times H^{s-1}}$$

$$\| G \|_{N_T} := \int_0^T \| G(t, \cdot) \|_{H^{s-1}} \, dt$$

Let $S_T^0$ and $N_T^0$ denote the spaces from the lower regularity iteration. Note that as before we have $\| W(u_0, u_1) \|_{S_T} \leq \| (u_0, u_1) \|_{H^s \times H^{s-1}}$ and $\| LG \|_{S_T} \lesssim \| G \|_{N_T}$ as a direct consequence of the energy estimate (3.14). Again we are using $f(\phi) = -V\phi + \phi^2$, so by first using the triangle inequality and then using our assumptions on $V$, we have

$$\| f(\phi) - f(\psi) \|_{H^{s-1}} \leq C'' \| \phi - \psi \|_{H^{s-1}} + \| (\phi - \psi)(\phi + \psi) \|_{H^{s-1}} \quad (3.17)$$

We will prove a slightly different form of (3.9), and from this we will be able to deduce Theorem 3.2.2 and the improvements mentioned in the remark after its statement.

First, the product estimate (2.16) applies to obtain a constant $C''$ so that

$$\| (\phi - \psi)(\phi + \psi) \|_{H^{s-1}} \leq C'' \| \phi - \psi \|_{H^1} \| \phi + \psi \|_{H^s} + C'' \| \phi - \psi \|_{H^s} \| \phi + \psi \|_{H^1}$$
Inserting this last bound into the right side of (3.17), integrating both sides of the resulting inequality from 0 to $T$, and taking supremums gives, with a new constant $C$, the estimate

$$
\| f(\phi) - f(\psi) \|_{N_T} \leq TC \| \phi - \psi \|_{S_T} + TC \| \phi - \psi \|_{S_T^0} (\| \phi \|_{S_T} + \| \psi \|_{S_T}) +
+ TC \| \phi - \psi \|_{S_T} (\| \phi \|_{S_T^0} + \| \psi \|_{S_T^0})
$$

Thus to prove the bound $\| u^{(n+1)} - u^{(n)} \|_{S_T} \leq 2^{-(n+1)} \| (u_0, u_1) \|_{H^s \times H^{s-1}}$ and close the iteration, we must control the two terms

$$
TC \| u^{(n)} - u^{(n-1)} \|_{S_T} (1 + \| u^{(n)} \|_{S_T^0} + \| u^{(n-1)} \|_{S_T^0}) \tag{3.18}
$$

$$
TC \| u^{(n)} - u^{(n-1)} \|_{S_T^0} (\| u^{(n)} \|_{S_T} + \| u^{(n-1)} \|_{S_T}) \tag{3.19}
$$

So suppose that $\| u^{(j)} - u^{(j-1)} \|_{S_T} \leq 2^{-j} \| (u_0, u_1) \|_{H^s \times H^{s-1}}$ for $j = 0, \ldots, n$. By the low regularity iteration, we have $\| u^{(n)} \|_{S_T^0} \leq 2C \| (u_0, u_1) \|_{H^s \times L^2}$ for any $n \geq 0$ and a constant $C$. Thus (3.18) is bounded by $2^{-(n+1)} \| (u_0, u_1) \|_{H^s \times H^{s-1}}$ as long as $T \leq \frac{1}{2C} (1 + 4C \| (u_0, u_1) \|_{H^s \times L^2})^{-1}$.

For (3.19), note that by telescoping we obtain

$$
\| u^{(n)} \|_{S_T} + \| u^{(n-1)} \|_{S_T} \leq \| u^{(n)} - u^{(n-1)} \|_{S_T} + 2 \| u^{(n-1)} \|_{S_T}
$$

$$
\leq \| u^{(n)} - u^{(n-1)} \|_{S_T} + 2 \| u^{(n-1)} - u^{(n-2)} \|_{S_T} + 2 \| u^{(n-2)} \|_{S_T}
$$

$$
\leq \cdots
$$

Using the inductive bound $\| u^{(j)} - u^{(j-1)} \|_{S_T} \leq 2^{-j} \| (u_0, u_1) \|_{H^s \times H^{s-1}}$ in each summand and summing provides the estimate

$$
\| u^{(n)} \|_{S_T} + \| u^{(n-1)} \|_{S_T} \leq 2(1 - 2^{-(n+1)}) \| (u_0, u_1) \|_{H^s \times H^{s-1}} \leq 2 \| (u_0, u_1) \|_{H^s \times H^{s-1}}
$$

Now from the low regularity iteration we have a constant $c$ such that if $T \leq c(1 + \| (u_0, u_1) \|_{H^s \times L^2})^{-1}$, then $\| u^{(n)} - u^{(n-1)} \|_{S_T^0} \leq 2^{-n} \| (u_0, u_1) \|_{H^s \times L^2}$. Shrinking $c$ (if necessary) only to ensure that $c \leq (4C)^{-1}$ and collecting what we have so far bounds (3.19) by $2^{-(n+1)} \| (u_0, u_1) \|_{H^s \times H^{s-1}}$ if $T \leq c(1 + \| (u_0, u_1) \|_{H^s \times L^2})^{-1}$, which completes the proof of Theorem 3.2.2.

Note that if a solution exists on a time interval $[0, T]$, one can divide up the interval into subintervals and iterate the local bound (3.13) to obtain for any $t \in
This gives us cause to prove dispersive estimates even though we will get global existence in $H^1$ just from the energy in the next section.

### 3.3 Global Existence from Energy Estimates

As mentioned in the previous section, all that is necessary to establish global existence of small amplitude $H^1$ solutions to (3.11) is to prove that the quantity
$$\sup_{0 \leq t \leq T} \left\| (u, u_t)(t) \right\|_{H^1 \times L^2}$$
remains uniformly bounded on any time interval $[0, T]$ where the solution exists. The following a priori estimate implies that this is true.

**Theorem 3.3.1.** Suppose $u$ solves
$$(\partial_t^2 - \Delta_x + V + 1)u = u^2$$
on a time interval $[0, T]$ with initial data $u_0, u_1$. Then one has the estimate
$$\sup_{0 \leq t \leq T} \left\| (u, u_t)(t) \right\|_{H^1 \times L^2}^2 \leq C \left( \left\| (u_0, u_1) \right\|_{H^1 \times L^2}^2 + \left\| u_0 \right\|_{L^3}^3 + \sup_{0 \leq t \leq T} \left\| (u, u_t)(t) \right\|_{H^1 \times L^2}^3 \right)$$

**Remark 3.3.2.** This bound implies the global existence. To see this, suppose the data $u_0, u_1$ are of size $\epsilon$. It will suffice to find an $M > 0$, independent of $T$ and $\epsilon$, such that if
$$\sup_{0 \leq t \leq T} \left\| (u, u_t)(t) \right\|_{H^1 \times L^2} \leq M\epsilon$$
then the same bound holds with $M$ replaced by $M/2$ if $\epsilon$ is sufficiently small. Theorem 3.3.1 implies the bound
$$\left\| (u, u_t) \right\|_{H^1 \times L^2}^2 \leq C \left( \epsilon^2 + \epsilon^3 + M^3 \epsilon^3 \right)$$
If $M > 12C$ and $\epsilon$ is sufficiently small, the previous expression is bounded by $\frac{M^2 \epsilon^2}{4}$, so by taking square roots we are done.

Now we return to the proof of the a priori estimate.

**Proof of Theorem 3.3.1.** Define
$$E(t) = \int_{\mathbb{R}^3} u_t^2 + |\nabla_x u|^2 + u^2 + Vu^2 - \frac{2}{3}u^3 \, dx$$
Differentiating and integrating by parts in the usual fashion shows that \( E'(t) = 0 \), i.e. \( E(t) \equiv E(0) \) is constant. In Section 11 of Lieb & Loss [23], it is proved that in the absence of eigenvalues of \(-\Delta + V\) one has the two bounds

\[
0 \leq \int |\nabla_x u|^2 + Vu^2 \, dx
\]  

(3.22)

and

\[
\int |\nabla_x u|^2 \, dx \leq C \int |\nabla_x u|^2 + Vu^2 + u^2 \, dx
\]  

(3.23)

Using (3.22) one finds that \( \int u_t^2 + u^2 \, dx \leq E(t) + \frac{2}{3} \int u^3 \, dx \leq E(t) + \frac{2}{3} \| u \|_{L^3}^3 \). Then (3.23) implies that \( \int |\nabla_x u|^2 \, dx \leq C \left( E(t) + \frac{2}{3} \| u \|_{L^3}^3 \right) \). By combining the above and choosing a larger constant \( C \) one has (recalling that \( E(t) = E(0) \) is constant)

\[
\| (u, u_t)(t) \|_{H^1 \times L^2}^2 \leq C(E(0) + \| u(t) \|_{L^3}^3)
\]

Using Sobolev embedding first and then taking supremums, we obtain (with another new constant)

\[
\sup_{0 \leq t < T} \| (u, u_t)(t) \|_{H^1 \times L^2} \leq C(E(0) + \sup_{0 \leq t < T} \| (u, u_t)(t) \|_{H^1}^3)
\]

Now the theorem follows from the definition of \( E(0) \) and the fact that \( V \) is bounded as a multiplier on \( L^1 \).

\[
\Box
\]

### 3.4 General Energy Estimates

Previously we have treated the potential \( V \) as a perturbation. Alternatively, we can consider it as part of the differential operator \( \partial_t^2 + H + 1 \). It turns out that very similar energy estimates hold for this operator, but there are two slight hurdles. The first is that the derivatives \( \partial^\alpha \) no longer commutes with the equation. However, the operators \( H^j \) (where again \( H = -\Delta + V \)) do commute with the equation, and we can prove some preliminary estimates in the \( W^{k,p}_H \) spaces. Then the bounds in section 2.2 will allow us to recover the corresponding estimates in the classical spaces.

The second obstruction is that the quantity

\[
E_V(t) := \int u_t^2 + |\nabla_x u|^2 + u^2 + Vu^2 \, dx
\]
is a priori not known to be positive. This is overcome by our assumptions on the potential \( V \), since we recall from [23] that when \(-\Delta + V\) has no eigenvalues, the quantity \( \mathcal{E}(u) := \int |\nabla_x u|^2 + Vu^2 \, dx \) is nonnegative. Then for \( 0 < \epsilon < 1 \), a simple calculation shows that

\[
E_V(t) = \mathcal{E}(u) + \|(u, u_t)\|_{L^2_x L^2}^2 \\
\geq \epsilon \|(u, u_t)\|_{H^1_x L^2}^2 + (1 - \epsilon) \mathcal{E}(u) + \int (1 - \epsilon + \epsilon V)u^2 \, dx
\]

Now if we choose \( \epsilon \ll 1 \), the last term above is positive due to the decay assumptions on \( V \). We conclude that there is a small constant \( c \), depending only on \( V \), such that

\[
E_V(t) \geq c \|(u, u_t)\|_{H^1_x L^2}^2
\]

(3.24)

Now we may define the energy

\[
E_V(t) = \|(u, u_t)\|_{H^1_x L^2}^2 + \int Vu^2 \, dx
\]

(3.25)

as above and proceed with proving the corresponding energy estimate.

**Lemma 3.4.1.** Suppose \( V \) is such that \( H = -\Delta + V \) has no eigenvalues. If \( u \) solves (3.11) for \( 0 \leq t \leq T \) and \( k \geq 1 \) is odd, then

\[
\|(u, u_t)(t)\|_{H^k_x H^{k-1}} \lesssim \|(u_0, u_1)\|_{H^k_x H^{k-1}} + \int_0^t \| f(s, \cdot) \|_{H^{k-1}} ds
\]

(3.26)

for \( 0 \leq t \leq T \).

**Proof.** We have as before that \( E_V(t)^{\frac{1}{2}} \leq E_V(0)^{\frac{1}{2}} + \int_0^t \| f(s, \cdot) \|_{L^2} ds \). Observe also that as a consequence of the boundedness of \( V \) as an \( L^p \) multiplier, the following two bounds hold:

\[
\|(u, u_t)(t)\|_{H^1_x L^2} \lesssim E_V(t)^{\frac{1}{2}}
\]

\[
E_V(0)^{\frac{1}{2}} \lesssim \|(u_0, u_1)\|_{H^1_x L^2}
\]

These combine with our work above to give the same estimate from the previous section:

\[
\|(u, u_t)(t)\|_{H^1_x L^2} \lesssim \|(u_0, u_1)\|_{H^1_x L^2} + \int_0^t \| f(s, \cdot) \|_{L^2} ds
\]
Now as an intermediate step to obtaining (3.26), we may differentiate the equation with the operator $H^j$ for each $j \geq 1$ with $2j \leq k$ and apply the previous bound to immediately obtain the estimate

$$\| (u, u_t)(t) \|_{W^{2j+1,2}_H \times W^{2j,2}_H} \lesssim \| (u_0, u_1) \|_{W^{2j+1,2}_H \times W^{2j,2}_H} + \int_0^t \| f(s, \cdot) \|_{W^{2j,2}_H} ds$$

Then we can simply appeal to Theorems 2.2.5 and 2.2.6 to obtain (3.26). □
Chapter 4

The Dispersive Estimate

In this chapter we prove the primary dispersive estimate for solutions to the full equation \((\partial_t^2 + H + 1)u = f\), where \(H = -\Delta + V\) and \(V\) is subject to the assumptions 2.1.1 and 2.1.2. Assumption 2.1.1 in particular implies that \(V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)\) for some choice of \(p < \frac{3}{2} < q\), a fact which we will use later to apply a theorem of Goldberg. The estimate will be proved through a series of reductions, but we state the main theorem first.

4.1 Main Dispersive Estimate

**Theorem 4.1.1.** Suppose that \(H, V, f\) are as above and \(u\) solves the equation

\[
(\partial_t^2 + H + 1)u = f \tag{4.1}
\]

\[
u(0) = u_0
\]

\[
u_t(0) = u_1
\]

for \(u_0, u_1 \in \mathcal{S}(\mathbb{R}^3)\). Then for \(t > 0\) and \(k\) odd we have the dispersive estimate

\[
\| (u(t, \cdot), u_t(t, \cdot)) \|_{W^{k, \infty}} \lesssim (1 + t)^{-\frac{3}{2}} \sum_{j=0}^{1} \| u_j \|_{W^{k+6-j, 1}} +
\]

\[
+ \int_0^t (1 + t - s)^{-\frac{3}{2}} \| f(s, \cdot) \|_{W^{k+5, 1}} ds
\]
4.2 Preliminary Reductions

Theorem 4.1.1 will be proved in a long series of reductions. The first and most simple reduction is to prove the corresponding $L^\infty$ estimate, which is given by

$$
\| (u(t, \cdot), u_t(t, \cdot)) \|_{L^\infty} \lesssim (1 + t)^{-\frac{3}{2}} \sum_{j=0}^{1} \| u_j \|_{W^{5-j,1}} +
$$

$$
+ \int_0^t (1 + t - s)^{-\frac{3}{2}} \| f(s, \cdot) \|_{W^{5,1}} ds
$$

(4.3)

To recover (4.2) from (4.3), we just differentiate the equation with powers of $H$ and apply Theorems 2.2.5 and 2.2.6 to obtain for $k$ odd

$$
\| (u, u_t) \|_{W^{k,\infty}} \lesssim \| (u, u_t) \|_{W^{k+1,\infty}} \lesssim (1 + t)^{-\frac{3}{2}} \sum_{j=0}^{1} \| u_j \|_{W^{5+k+1-j,1}} +
$$

$$
+ \int_0^t (1 + t - s)^{-\frac{3}{2}} \| f(s, \cdot) \|_{W^{k+5,1}}
$$

The Duhamel formulation of solutions to (4.1) is given by

$$
u(t, \cdot) = \cos(t<\sqrt{H}>)u_0 + \sin(t<\sqrt{H}>)u_1 + \int_0^t \cos(t<s><\sqrt{H}>)f(s, \cdot) ds
$$

(4.4)

Note that the corresponding formula for $u_t$ is

$$
u_t(t, \cdot) = \cos(t<\sqrt{H}>)u_1 - <\sqrt{H}> \sin(t<\sqrt{H}>)u_0 + \int_0^t \cos(t<s><\sqrt{H}>)f(s, \cdot) ds
$$

(4.5)

This reduces the proof of (4.2) to proving estimates for the homogeneous equations, since the generic term $e^{\pm it<\sqrt{H}>}<\sqrt{H}>^\pm u_j$ satisfies a homogeneous Klein-Gordon equation. Thus Theorem 4.2 will be deduced from the following theorem for the homogeneous equation:

**Theorem 4.2.1.** Suppose that $H$ and $V$ are as above and $u$ solves $(\partial_t^2 + H + 1)u = 0$ with initial data $u_0, u_1$. Then we have

$$
\| (u(t, \cdot), u_t(t, \cdot)) \|_{L^\infty_x} \lesssim \| u_j \|_{W^{5-j,1}}, \quad \text{if } |t| \leq 1
$$

(4.6)

$$
\| (u(t, \cdot), u_t(t, \cdot)) \|_{L^\infty_x} \lesssim |t|^{-\frac{3}{2}} \sum_{j=0}^{1} \| u_j \|_{W^{5-j,1}}, \quad \text{if } |t| > 1
$$

(4.7)
Corollary 4.2.2. For the solution of \((\partial^2_t + H + 1)u = 0\) with initial data \(u_0, u_1\), one has the estimate

\[
\| (u(t, \cdot), u_t(t, \cdot)) \|_{L^\infty_x} \lesssim (1 + t)^{-\frac{3}{2}} \sum_{j=0}^{1} \| u_j \|_{W^{5-j,1}}
\]

Proof. This is obtained by simply adding the two estimates in Theorem 4.2.1 and noticing that for \(0 < t < 1\), one has \(1 \lesssim (1 + t)^{-\frac{3}{2}}\), and for \(t \geq 1\), one has \(t^{-\frac{3}{2}} \lesssim (1 + t)^{-\frac{3}{2}}\) (the implied constants are universal).

Remark 4.2.3. We can dispense with the case \(|t| \leq 1\) in Theorem 4.2.1 using Sobolev embedding. Indeed, by using the energy estimates from Chapter 3 and the Sobolev estimates from Chapter 2, one has

\[
\| (u, u_t) \|_{L^\infty \times L^\infty} \lesssim \| (u, u_t) \|_{H^2 \times H^2} \lesssim \| (u, u_t) \|_{H^3 \times H^2} \lesssim \| (u_0, u_1) \|_{H^3 \times H^2}
\]

The Sobolev embedding \(W^{k,1} \subseteq W^{k-l,2}\), valid for \(l \geq 2\) and \(k \geq l\), then bounds the right side by \(\| u_0 \|_{W^{5,1}} + \| u_1 \|_{W^{4,1}}\). Thus, all that remains is to prove the estimate (4.7). This is the heart of the dispersive estimate and the rest of the chapter is dedicated to its proof.

In light of the Duhamel formulations (4.4) and (4.5), it will be sufficient to prove instead the following estimate for the half wave operators \(e^{\pm it\langle \sqrt{H} \rangle}\):

Proposition 4.2.4. Suppose \(H\) and \(V\) are as above. Then for \(|t| > 1\) we have the estimate

\[
\| e^{\pm it\langle \sqrt{H} \rangle} \langle \sqrt{H} \rangle^m f \|_{L^\infty} \lesssim |t|^{-\frac{3}{2}} \| f \|_{W^{4,1}} \quad \text{for } m = 0, \pm 1 \quad (4.8)
\]

This estimate then controls all terms in the Duhamel formulation of solutions to \((\partial^2_t + H + 1)u = 0\) as well as the first time derivative by decomposition of sine and cosine into complex exponentials. The proof of Proposition 4.2.4 follows a standard procedure of introducing a cutoff function \(\chi\) and proving estimates on dyadic shells. The final reduction is the following proposition, which allows us to deduce Proposition 4.2.4 and hence the dispersive estimate (4.7).
Proposition 4.2.5. Let $\chi_k(\lambda)$ be a smooth cutoff function compactly supported where $\lambda \sim 2^k$ and satisfying the derivative bounds

$$\| \partial_s^s \chi_k \|_{L^\infty} \lesssim 2^{ks}$$

Then for $H$ and $V$ defined as above, we have the estimate

$$\| e^{\pm it \langle \sqrt{H} \rangle} \chi_k(\sqrt{H}) f \|_{L^\infty} \lesssim |t|^{-\frac{3}{2}} (2^k)^{\gamma} \| f \|_{L^1(\mathbb{R}^3)}$$

(4.9)

for $\gamma = \frac{5}{2}$. This estimate also holds with $k = 0$ and $\chi_0$ a smooth cutoff function supported near $[-1, 1]$.

Let us first observe how this implies Proposition 4.2.4. A key result is due to Jensen and Nakamura and is proved in [20]. In that paper the potential $V$ is required to belong to $L^2_{\text{loc}}(\mathbb{R}^3)$ and satisfy the “Kato” condition:

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^3} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|} dy = 0$$

Our assumptions imply that both of these requirements hold. The result from [20] is stated as follows.

Theorem 4.2.6. Assume that $V \in L^2_{\text{loc}}(\mathbb{R}^3)$ satisfies the Kato condition. Then for $H = -\Delta + V$ and $g \in C_0^\infty(\mathbb{R})$, one has for $1 \leq p \leq \infty$,

$$\| g(\theta H) \|_{L^p \to L^p} \lesssim 1$$

uniformly in $0 < \theta \leq 1$.

We shall sum up the estimates from each dyadic shell, so let us assume as we may that the identity $I = \chi_0 + \sum_{k \geq 1} \chi_k$ holds. This then allows us to write

$$\| e^{\pm it \langle \sqrt{\Pi} \rangle} \langle \sqrt{H} \rangle^m f \|_{L^\infty} \leq \| e^{\pm it \langle \sqrt{\Pi} \rangle} \chi_0(\sqrt{H}) \langle \sqrt{H} \rangle^m f \|_{L^\infty} + \sum_{k \geq 1} \| e^{\pm it \langle \sqrt{\Pi} \rangle} \chi_k(\sqrt{H}) \langle \sqrt{H} \rangle^m f \|_{L^\infty}$$

For each $k \geq 0$, let us choose $\tilde{\chi}_k$ such that $\chi_k \tilde{\chi}_k = \chi_k$. By Proposition 4.2.5 and by Theorem 4.2.6, the first term above is controlled by

$$|t|^{-\frac{3}{2}} \| \tilde{\chi}_0(\sqrt{H}) \langle \sqrt{H} \rangle^m f \|_{L^1} \lesssim |t|^{-\frac{3}{2}} \| f \|_{L^1}$$
On the other hand, by Proposition 4.2.5 we have for $k \geq 1$ the estimate
\[
\| e^{\pm it\sqrt{\lambda}} \chi_k(\sqrt{\lambda}) \langle \sqrt{\lambda} \rangle^m f \|_{L^1} \leq |t|^{-\frac{3}{2}} (2^k)^{\frac{3}{2}} \| \tilde{\chi}_k(\sqrt{\lambda}) \langle \sqrt{\lambda} \rangle^m f \|_{L^1}
\]
Define $\psi_{k,m}(\lambda) = \tilde{\chi}_k(\lambda) \langle \lambda \rangle^m$. Then for any $\epsilon > 0$, we can write
\[
\sum_{k \geq 1} \| e^{\pm it\sqrt{\lambda}} \chi_k(\sqrt{\lambda}) \langle \sqrt{\lambda} \rangle^m f \|_{L^1} \leq |t|^{-\frac{3}{2}} \sum_{k \geq 1} (2^k)^{\frac{3}{2}} \| \psi_{k,m}(\sqrt{\lambda}) f \|_{L^1}
\]
\[
= |t|^{-\frac{3}{2}} \sum_{k \geq 1} (2^{-\epsilon^2} (2^k)^{\frac{3}{2} + \epsilon} \| \psi_{k,m}(\sqrt{\lambda}) f \|_{L^1}
\]
\[
\leq C_\epsilon |t|^{-\frac{3}{2}} \sup_{k \geq 1} (2^k)^{\frac{3}{2} + \epsilon} \| \psi_{k,m}(\sqrt{\lambda}) f \|_{L^1}
\]
Now we have reduced the problem to showing that for $m = 0, \pm 1$ we have the estimate
\[
\| (2^k)^{\frac{3}{2} + \epsilon} \psi_{k,m}(\sqrt{\lambda}) f \|_{L^1} \lesssim \| f \|_{W^{4,1}}
\]
where the implied constant is independent of $k$.

If $m = 0$, let $g(t) = \tilde{\chi}(\sqrt{t}) t^{-2}$ and $\theta = 2^{-2k}$, where $\tilde{\chi}_k(\lambda) = \tilde{\chi}(2^{-k} \lambda)$. Then $g(\theta H) = (2^k)^4 \tilde{\chi}(2^{-k} \sqrt{H}) H^{-2} = (2^k)^4 \psi_{k,0}(\sqrt{H}) H^{-2}$. Thus, take $\epsilon = \frac{3}{2}$ in (4.10) and apply Theorem 4.2.6 to obtain the bound
\[
\| (2^k)^4 \psi_{k,0}(\sqrt{H}) f \|_{L^1} \lesssim \| H^2 f \|_{L^1} \lesssim \| f \|_{W^{4,1}}
\]
by Theorem 2.2.5.

If $m = 1$, choose instead $g(t) = \tilde{\chi}(\sqrt{t}) t^{-2}$; then one has
\[
g(\theta H) = (2^k)^3 \psi_{k,1}(\sqrt{H}) \frac{(2^{-k} \sqrt{H})}{(2^{-k} \sqrt{H})} H^{-2}
\]
Thus we set $\epsilon = \frac{1}{2}$ in (4.10) and use Theorem 4.2.6 again to get
\[
\| (2^k)^3 \psi_{k,1}(\sqrt{H}) f \|_{L^1} \lesssim \| H^2 f \|_{L^1} \lesssim \| f \|_{W^{4,1}}
\]
Finally, if $m = -1$, take $g(t) = \tilde{\chi}(\sqrt{t}) t^{-1}$. Then
\[
g(\theta H) = (2^k)^3 \psi_{k,-1}(\sqrt{H}) \frac{2^{-k} (\sqrt{H})}{(2^{-k} \sqrt{H})} H^{-1}
\]
so if we take $\epsilon = \frac{1}{2}$ again, Theorem 4.2.6 yields the estimate
\[
\| (2^k)^3 \psi_{k,-1}(\sqrt{H}) f \|_{L^1} \lesssim \| H f \|_{L^1} \lesssim \| f \|_{W^{2,1}}
\]
which establishes the bound (4.10) for $m = 0, \pm 1$ and hence Proposition 4.2.4.

Thus, everything has been reduced to the proof of (4.9) and the rest of the section will accomplish this using the machinery developed in Chapter 2.
4.3 Proof of the Reduced Estimate

Now we present a proof of Proposition 4.2.5. To estimate the left side of (4.9), we use duality and consider

\[ \sup_{\| g \|_{L^1} = 1} \left| \langle e^{\pm it\sqrt{\lambda}} \chi_k(\sqrt{\lambda}) f, g \rangle \right| \]  

(4.11)

Note that although \( L^1(\mathbb{R}^3) \) is not the full dual space of \( L^\infty(\mathbb{R}^3) \), one can still compute the \( L^\infty \) norm in this way. We will begin by using the main integral representation (2.11) derived in Chapter 2, Section 2.1, which allows us to consider instead the explicit integral

\[ \left| \int_0^\infty e^{it\sqrt{\lambda}} \chi_k(\sqrt{\lambda}) (\| f, g \| \]  

(4.12)

Recall that with the assumptions we have made on \( V \) above, the spectrum of \( H \) is purely absolutely continuous on \([0, \infty)\). We further reduce by changing variables \( \sqrt{\lambda} \mapsto \lambda \), producing

\[ \left| \int_0^\infty e^{it\lambda} \lambda \chi_k(\lambda) (\| f, g \| \]  

(4.13)

Using the resolvent identities from Chapter 2 will allow us to express the difference \( R^+_V(\lambda^2) - R^-_V(\lambda^2) \) in terms of the free resolvent \( R^\pm_0(\lambda^2) \) and the operators \( T(\lambda) \) := \((I + V R^+_0(\lambda^2))^{-1} \) and \( T^-(\lambda) \) := \((I + V R^-_0(\lambda^2))^{-1} \). This notation matches that of Goldberg [18], where it is shown that the operators \( T^\pm(\lambda) \) are bounded on \( L^1(\mathbb{R}^3) \), uniformly in \( \lambda \).

Recall that \( \chi_0 = \psi \) is a smooth even cutoff function with support around \([-1, 1]\). Define \( T_L(\lambda) = T(\lambda)\psi(\frac{x}{L}) \) for \( L \geq 1 \). Then it follows that \( \int_{\mathbb{R}} \| T_L(\lambda) \|_{1 \rightarrow 1} d\lambda \) is finite and thus the operators \( T_L \) and \( T^-_L \) have Fourier transforms in \( \lambda \). A key part of our analysis will be a result from [18], in the form of the following estimate on the Fourier transform of \( T_L \):

**Theorem 4.3.1** (Goldberg). If \( V \) is as in Theorem 1.1 above, then one has

\[ \sup_{L \geq 1} \int_{\mathbb{R}^3} \left| \widehat{T_L}(\rho) f(x) \right| d\rho dx \lesssim \| f \|_{L^1} \]

for any \( f \in L^1(\mathbb{R}^3) \). The same estimate holds for \( T^-_L(\lambda) \).
Return now to the quantity (4.13). By our remarks in Chapter 2, the operators \( R_\pm^\dagger (\lambda^2) \) are differentiable in \( \lambda \), and thus an integration by parts is valid and yields

\[
\frac{1}{it} \int_0^\infty e^{it\lambda} \left[ \langle \lambda \rangle \chi_k(\lambda) \left\langle \left[ \frac{d}{d\lambda} R_\pm^\dagger (\lambda^2) - \frac{d}{d\lambda} R_\mp (\lambda^2) \right] f, g \right\rangle + \right.
\]
\[
+ \langle \lambda \rangle \chi'_k(\lambda) \left\langle \left[ R_\mp^\dagger (\lambda^2) - R_\mp (\lambda^2) \right] f, g \right\rangle + \right.
\]
\[
+ \lambda(\lambda)^{-1} \chi_k(\lambda) \left\langle \left[ R_\mp^\dagger (\lambda^2) - R_\mp (\lambda^2) \right] f, g \right\rangle d\lambda
\] (4.14)

Let us introduce some notation in order to write the above terms more succinctly. If \( D_0^\pm (\lambda^2) = \frac{d}{d\lambda} R_0^\pm (\lambda^2) \) and \( D^\pm (\lambda^2) = \frac{d}{d\lambda} R_\pm^\dagger (\lambda^2) \), then the terms can be written in the form

\[
\frac{1}{it} \int_0^\infty e^{it\lambda} \eta(\lambda) A(\lambda) d\lambda
\] (4.15)

where

\[
\eta(\lambda) = \langle \lambda \rangle \chi_k(\lambda) \quad A(\lambda) = \left\langle \left[ D^+ (\lambda^2) - D^- (\lambda^2) \right] f, g \right\rangle
\]

(4.16a)

\[
\eta(\lambda) = \langle \lambda \rangle^2 \chi'_k(\lambda) \quad A(\lambda) = \langle \lambda \rangle^{-1} \chi_k(\lambda) \left\langle \left[ R_\mp^\dagger (\lambda^2) - R_\mp (\lambda^2) \right] f, g \right\rangle
\]

(4.16b)

\[
\eta(\lambda) = \lambda \chi_k(\lambda) \quad A(\lambda) = \langle \lambda \rangle^{-1} \chi_k(\lambda) \left\langle \left[ R_\mp^\dagger (\lambda^2) - R_\mp (\lambda^2) \right] f, g \right\rangle
\]

(4.16c)

Here we have chosen \( \tilde{\chi}_k(\lambda) \) to be a slightly larger cutoff that includes the support of \( \chi_k(\lambda) \).

The general plan to estimate the quantity (4.15) is to extend the integral to all of \( \mathbb{R} \) and apply the Parseval identity to the factors \( F(\lambda) := e^{it\lambda} \eta(\lambda) \) and an appropriately truncated version of \( A(\lambda) \). The Fourier transform of \( F(\lambda) \) will be controlled by a standard stationary phase calculation which is contained in its own section, whereas \( \hat{A} \) will be estimated using Theorem 4.3.1. In order to proceed, we must extend the integral in (4.15) to all of \( \mathbb{R} \). This is done differently in the low and high energy regimes: for \( \chi_k \) with \( k \geq 1 \), one can immediately extend the integral in (4.15) to be over all of \( \mathbb{R} \) since \( \chi_k \) is supported near \( \lambda \sim 2^k \) (away from 0). However, for \( k = 0 \) and \( \chi_0 = \psi \) a smooth cutoff function supported around \([-1, 1] \), this is not as clear. To see why it works, first note that it is no loss to assume also that \( \psi \) is even. In Chapter 2 it was shown that that \( R_\dagger^\mp (\lambda^2) - R_-^\mp (\lambda^2) \) is an odd function of \( \lambda \). It follows then that \( D^+ (\lambda^2) - D^- (\lambda^2) \) is even. Since \( \psi' \)
is odd, it then follows by inspection that every term in (4.14) is even so that we may extend the integral to all of $\mathbb{R}$ by introducing a factor of 2.

We now seek to control

$$\frac{1}{it} \int_{-\infty}^{\infty} e^{it\lambda} \eta(\lambda) A(\lambda) \, d\lambda$$

(4.17)

for any of the choices in (4.16).

In order to interface with Goldberg’s results in [18], it will be convenient to choose a value of $L \geq 3 \cdot 2^k \geq 1$. This implies then that $\eta(\lambda) = \eta(\lambda) \psi_j(\frac{\lambda}{L})$ for any power $j \geq 1$. This allows us to rewrite the integral in (4.17) as

$$\frac{1}{it} \int_{-\infty}^{\infty} e^{it\lambda} \eta(\lambda) \psi_j(\frac{\lambda}{L}) A(\lambda) \, d\lambda = \frac{1}{it} \int_{-\infty}^{\infty} e^{it\lambda} \eta(\lambda) B(\lambda) \, d\lambda$$

(4.18)

where $B(\lambda) := \psi_j(\frac{\lambda}{L}) A(\lambda)$. With $\hat{F}(\rho)$ denoting the Fourier transform of $F(\lambda) = e^{it\lambda} \eta(\lambda)$, an application of the Parseval theorem would then require us to control

$$\frac{1}{2\pi it} \int_{-\infty}^{\infty} \hat{F}(\rho) \overline{\hat{B}(\rho)} \, d\rho$$

(4.19)

The next section provides an estimate of $\| \hat{F}(\rho) \|_{L^\infty}$, so it is now just a matter of estimating $\| \hat{B} \|_{L^1_\rho}$ for each of the choices of $A(\lambda)$ in (4.16). It is sufficient just to consider $\| \hat{B} \|_{L^1_\rho}$, and in fact we will find that in all cases we have the bound $\| \hat{B} \|_{L^1_\rho} \lesssim \| f \|_{L^1} \| g \|_{L^1}$, leaving us only to show in the following section that $\| \hat{F}(\rho) \|_{L^\infty}$ contributes the rest of the estimate.

Begin with (4.16a). By Lemma 2.1.7 we have

$$D^+(\lambda^2) = (I + R_0^+(\lambda^2)V)^{-1}D_0^+(\lambda^2)(I + VR_0^+(\lambda^2))^{-1} = (T^-(\lambda))^* D_0^+(\lambda^2) T(\lambda)$$

since $R_0^-(\lambda)$ is the formal adjoint of $R_0^+(\lambda)$. To estimate this term we do not need to exploit any cancellation in the expression $D^+(\lambda^2) - D^-(\lambda^2)$, so to bound (4.16a) it will suffice by the triangle inequality to replace $A(\lambda)$ by $\langle D^+(\lambda^2) f, g \rangle$. So with the above definitions for $T_L$ and $T_L^-$, choose $j = 3$ in (4.18) to obtain

$$B(\lambda) = \langle \psi(\frac{\lambda}{L}) D_0^+(\lambda^2) T_L(\lambda) f, T_L^-(\lambda) g \rangle = \int_{\mathbb{R}^3} P(\lambda, x) \overline{Q(\lambda, x)} \, dx$$
where we have defined $P(\lambda, x) = \psi_L(\lambda)D_0^+(\lambda^2)T_L(\lambda)f(x)$ and $Q(\lambda, x) = T_L^-(\lambda)g(x)$ to help us unravel the expressions carefully. By $\psi_L(\lambda)$ we still mean $\psi(\lambda/L)$, but it will help simplify later calculations. Now since the Fourier transform sends products to convolutions, we get

$$\hat{B}(\rho) = \frac{1}{2\pi} \int_\mathbb{R} \int_\mathbb{R} \hat{P}(\tau, x)\overline{Q}(\rho - \tau, x) \, d\tau \, dx = \frac{1}{2\pi} \int_\mathbb{R} \int_\mathbb{R} \hat{P}(\tau, x)\overline{Q}(\tau - \rho, x) \, d\tau \, dx$$

(4.20)

Note that we obtain the integration kernel for $D_0^+(\lambda^2)$ by differentiating the kernel for the free resolvent. The result is $\frac{1}{4\pi}e^{i\lambda|x-y|}$, so we can compute $\hat{P}$ via

$$P(\lambda, x) = \frac{1}{4\pi} \int_\mathbb{R} \int_\mathbb{R} e^{i\lambda|x-y|}\psi_L(\lambda)(T_L(\lambda)f)(y) \, dy$$

$$\hat{P}(\tau, x) = \frac{1}{4\pi} \int_\mathbb{R} \int_\mathbb{R} e^{-i\lambda(\tau - |x-y|)}\psi_L(\lambda)(T_L(\lambda)f)(y) \, d\lambda \, dy$$

$$= \frac{1}{8\pi^2} \int_\mathbb{R} \int_\mathbb{R} \hat{\psi}_L(\tau - |x-y| - \sigma)(\hat{T}_L(\sigma)f)(y) \, d\sigma \, dy$$

$$= \frac{L}{8\pi^2} \int_\mathbb{R} \int_\mathbb{R} \hat{\psi}(L(\tau - |x-y| - \sigma))(\hat{T}_L(\sigma)f)(y) \, d\sigma \, dy$$

Inserting this and the expansion of $\overline{Q(\tau - \rho, x)}$ gives

$$\hat{B}(\rho) = \frac{L}{16\pi^3} \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} \hat{\psi}(L(\tau - \sigma - |x-y|))(\hat{T}_L(\sigma)f)(y)(\hat{T}_L^-(\tau - \rho)g)(x) \, d\tau \, d\sigma \, dy \, dx$$

$$= \frac{L}{16\pi^3} \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} \hat{\psi}(L(\rho + \tau - \sigma - |x-y|))(\hat{T}_L(\sigma)f)(y)(\hat{T}_L^-(\tau)g)(x) \, d\tau \, d\sigma \, dy \, dx$$

Now it is clear how to proceed. Integrate in $\rho$ and use the triangle inequality and Fubini-Tonelli to interchange the order of integration so as to process the $\rho$ integral first. For any choice of $s \in \mathbb{R}$, we have $L \int_\mathbb{R} |\hat{\psi}(L(\rho - s))| \, d\rho = ||\hat{\psi}||_{L^1}$, so we have

$$\|\hat{B}\|_{L^p} \lesssim ||\hat{\psi}||_{L^1} \int_\mathbb{R} \int_\mathbb{R} ||(\hat{T}_L(\sigma)f)(y)|| \, d\sigma \, dy \int_\mathbb{R} \int_\mathbb{R} ||(\hat{T}_L^-(\tau)g)(x)|| \, d\tau \, dx$$

$$\lesssim \|f\|_{L^1} \|g\|_{L^1}$$

We need only one computation to handle (4.16b) and (4.16c) since $A(\lambda)$ is the same for both. Recalling our earlier notation, we compute

$$R_V^+(\lambda^2) - R_V^-(\lambda^2) = (I + R_0^-(\lambda^2)V)^{-1}(R_0^+(\lambda^2) - R_0^-(\lambda^2))(I + VR_0^+(\lambda^2))^{-1}^{-1}$$

$$= (T(\lambda))^*(R_0^+(\lambda^2) - R_0^-(\lambda^2))T(\lambda)$$
This time we will only choose \( j = 2 \) in (4.18), so that \( B \) looks like

\[
B(\lambda) = \langle \lambda \rangle^{-1} \tilde{\chi}_k(\lambda) \left\langle \left[ R_0^+ (\lambda^2) - R_0^- (\lambda^2) \right] T_L(\lambda)f, T_L(\lambda)g \right\rangle
\]

From the definition, we see that the operator \( R_0^+ (\lambda^2) - R_0^- (\lambda^2) \) has the integration kernel \( \frac{\sin(\lambda|x-y|)}{4\pi|x-y|} = \frac{\sin(\lambda|x-y|)}{4\pi|x-y|} - e^{-i\lambda|x-y|} \). Here we will make the definitions

\[
P(\lambda, x) = a(\lambda)[R_0^+ (\lambda^2) - R_0^- (\lambda^2)](T_L(\lambda))f(x)
\]

and \( Q(\lambda, x) = (T_L(\lambda)g)(x) \), where we have abbreviated \( \langle \lambda \rangle^{-1} \tilde{\chi}_k(\lambda) \) by \( a(\lambda) \). As before, we begin with

\[
\hat{B}(\rho) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{P}(\tau, x) \overline{Q(\tau - \rho, x)} \, dx \, d\tau
\]

This time we have

\[
P(\lambda, x) = \frac{1}{4\pi} a(\lambda) e^{i\lambda|x-y|} \frac{e^{-i\lambda|\tau-x-y|}}{|x-y|} (T_L(\lambda)f)(y) \, dy
\]

\[
\hat{P}(\tau, x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}} e^{-i\lambda(\tau-|x-y|)} - e^{-i\lambda(\tau+|x-y|)} a(\lambda)(T_L(\lambda)f)(y) \, d\tau \, dy
\]

\[
= \frac{1}{8\pi^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \tau e^{-i\lambda|\tau-x-y|} - \tau e^{-i\lambda|\tau+|x-y|} a(\lambda)(T_L(\lambda)f)(y) \, d\tau \, dy
\]

Now we return to computation of \( \hat{B}(\rho) \):

\[
\hat{B}(\rho) = \frac{1}{16\pi^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \tau e^{-i\lambda|\tau-x-y|} - \tau e^{-i\lambda|\tau+|x-y|} a(\lambda)(T_L(\lambda)f)(y) (\overline{T_L(\sigma)g})(x) \, d\tau \, d\sigma \, dy \, dx
\]

Proceeding as in [18], we integrate in \( \rho \) and use Fubini-Tonelli again, first evaluating

\[
\int_{\mathbb{R}} \int_{\rho + |x-y| - \sigma} \frac{\tau e^{-i\lambda|\tau-x-y|}}{|x-y|} \, d\tau \, d\sigma \leq \int_{\mathbb{R}} \int_{\rho - |x-y|} |\tau e^{-i\lambda|\tau-x-y|}| \, d\tau \, d\rho
\]

\[
= \int_{\mathbb{R}} \int_{|x-y|} |\tau e^{-i\lambda|\tau-x-y|}| \, d\tau \, d\rho
\]

\[
= 2 \int_{\mathbb{R}} |\tau e^{-i\lambda|\tau-x-y|}| \, d\tau
\]
This produces the bound
\[
\| \hat{B} \|_{L^1_\rho} \leq \frac{1}{8\pi^3} \| (\hat{a})' \|_{L^1} \int_{\mathbb{R}^3} |(\widehat{T_L}(\sigma)f)(y)| d\sigma dy \int_{\mathbb{R}} |(\widehat{T_L}(\tau)g)(x)| d\tau dx \quad (4.21)
\]
To finish the proof, we have to prove that \( \| (\hat{a})' \|_{L^1} \) is bounded independent of \( k \). Recall that \( a(\lambda) = \langle \lambda \rangle^{-1} \tilde{\chi}_k(\lambda) \). Then \( (\hat{a})' = -i\hat{\lambda}a \). Thus
\[
(\hat{a})'(\xi) = -i \int_{-\infty}^{\infty} e^{-i\lambda \xi} \frac{\lambda}{\sqrt{\lambda^2 + 1}} \tilde{\chi}(2^{-k} \lambda) d\lambda
\]
\[
= -i2^k \int_{-\infty}^{\infty} e^{-is(2^k \xi)} \frac{s}{\sqrt{s^2 + 2^{-2k}}} \tilde{\chi}(s) ds
\]
where \( \phi_\epsilon(s) = \frac{s\tilde{\chi}(s)}{\sqrt{s^2 + \epsilon}} \) and \( \epsilon = 2^{-2k} \). Thus \( \| (\hat{a})' \|_{L^1} \leq 2^k \| \hat{\phi}_\epsilon(2^k \cdot) \|_{L^1} = \| \hat{\phi}_\epsilon \|_{L^1} \).

Thus if we can show that \( \| \hat{\phi}_\epsilon \|_{L^1} \) is bounded independent of \( \epsilon \in (0, 1] \), we return to (4.21) and use Theorem 4.3.1 to get the required bound \( \| \hat{B} \|_{L^1_\rho} \lesssim \| f \|_{L^1} \| g \|_{L^1} \).

To see that \( \| \hat{\phi}_\epsilon \|_{L^1} \) can be bounded independent of \( \epsilon \), note that \( \phi_\epsilon \) is \( C^\infty_0 \), so in particular \( \hat{\phi}_\epsilon \in S(\mathbb{R}) \). Furthermore, \( \phi_\epsilon \) and all its derivatives are uniformly bounded in \( L^1 \), independent of \( \epsilon \in (0, 1] \). Since \( \xi \hat{\phi}_\epsilon = i\hat{\phi}_\epsilon' \) and \( \xi^2 \hat{\phi}_\epsilon = -\hat{\phi}_\epsilon'' \), this implies that any power of \( \xi \) multiplied by \( \hat{\phi}_\epsilon \) is uniformly bounded in \( L^\infty \), again independent of \( \epsilon \), so we can estimate
\[
\int_{-\infty}^{\infty} |\hat{\phi}_\epsilon(\xi)| d\xi \leq \sup_{\xi \in \mathbb{R}} |(1 + |\xi|)^2\hat{\phi}_\epsilon(\xi)| \int_{-\infty}^{\infty} (1 + |\xi|)^{-2} d\xi
\]

By our remarks above we have a uniform bound, independent of \( \epsilon \), on the final expression. All that remains now is to compute the contribution of \( \| \hat{F} \|_{L^2_\rho} \) in (4.19).

### 4.4 The Fourier transform of \( e^{it\langle \lambda \rangle} \eta(\lambda) \)

In this section, we complete the proof of Proposition 4.2.4 by analyzing the oscillatory integral
\[
\hat{F}(\rho) = \int_{-\infty}^{\infty} e^{-i\lambda \rho} e^{it\langle \lambda \rangle} \eta(\lambda) d\lambda \quad (4.22)
\]
where \( \eta(\lambda) \) is as in (4.16a), (4.16b), or (4.16c). Our main estimate is
Theorem 4.4.1. For $\hat{F}(\rho)$ as above, we have for $k \geq 1$ the bound

$$\|\hat{F}\|_{L^\infty} \lesssim (2^k)^{\frac{3}{2}} |t|^{-\frac{1}{2}}$$

(4.23)

This estimate also holds with $k = 0$ and $\chi_0 = \psi$ in the definition of $\hat{F}$.

We will need to examine separately the high energy ($k \geq 1$) and low energy ($k = 0$) cases. In both cases, however, the general idea is to carefully use a combination of standard stationary phase estimates and integration by parts. Thus, the first step is to rewrite the integral in (4.22) as

$$I(t) := \int_{-\infty}^{\infty} e^{itp(\lambda)} \eta(\lambda)\, d\lambda$$

(4.24)

where the phase function $p(\lambda)$ is given by

$$p(\lambda) = \langle \lambda \rangle - \frac{\lambda \rho}{t}$$

We now recall the standard stationary phase estimate; this version and its proof can be found in [21].

Proposition 4.4.2 (Stationary Phase). let $p \in C^\infty(X)$ have the non-degenerate critical point $\lambda_0 \in X$ and suppose $p'(\lambda) \neq 0$ for $\lambda \neq \lambda_0$. Then there are differential operators $A_{2\nu}(D)$ of order $\leq 2\nu$ such that for every compact set $K \subset X$ and $N \in \mathbb{N}$, there is a constant $C = C_{K,N}$ such that for every $u \in C^\infty_0(K)$ one has for $t \geq 1$ the estimate

$$\left| \int e^{itp(\lambda)}u(\lambda)\, d\lambda - \left( \sum_{\nu=0}^{N-1} (A_{2\nu}(D\lambda)u)(\lambda_0)t^{-\nu-\frac{n}{2}} \right) e^{itp(\lambda_0)} \right| \leq C t^{-N-\frac{n}{2}} \|u\|_{W^{2N+n+1,\infty}}$$

Furthermore $A_0 = \frac{(2\pi)^{\frac{n}{2}} e^{\frac{\pi}{4} \text{sgn} p''(\lambda_0)}}{|\det \phi''(\lambda_0)|^\frac{1}{2}}$

This does not quite apply to (4.24) since the support of $\eta$ can depend on $k$. However, if we have a fixed $u \in C^\infty_0$ whose support is independent of $k$, then taking $n = N = 1$ in Proposition 4.4.2 and using the triangle inequality yields the bound
$$\left| \int e^{ip(\lambda)}u(\lambda)\,d\lambda \right| \leq t^{-\frac{1}{2}}\frac{\sqrt{2\pi}}{(p''(\lambda_0))^\frac{1}{2}} + C t^{-\frac{1}{2}} \| u \|_{W^{4,\infty}}$$

(4.25)

where the constant $C$ is uniform if the support of $u$ is fixed.

It should be noted that there are hidden constants in the estimates above that depend on derivatives of the phase function. Thus, if our phase function depends on a large parameter such as $2^k$, we will have to also ensure that the phase and its derivatives are bounded uniformly in $k$ in a neighborhood of any critical point $s_0$ that lies in the support of $u_0$.

### 4.4.1 High Energy

Here we will assume $k \geq 1$ so that $\eta$ is supported where $\lambda \sim 2^k$. Note that each of the choices of $\eta$ in (4.16a), (4.16b), and (4.16c) obey the same derivative bounds, so we will assume here that $\eta(\lambda) = \lambda \chi_k(\lambda)$. In order to use the stationary phase estimate (4.25), let us first change variables $s = 2^{-k} \lambda$ in (4.24). This produces

$$I_k(t) := 2^{2k} \int e^{i p(2^k s)} s \chi_k(2^k s)\,ds$$

Note that $\chi_k(2^k s)$ is smooth and supported near $[1, 2]$, so let $u_0(s) = s \chi_k(2^k s)$. In addition we will define $p_k(s) = 2^k p(2^k s)$, so that the above integral reduces to

$$2^{2k} \int e^{i (2^{-k} t)p_k(s)} u_0(s)\,ds$$

By our comments above, it will be convenient to extract the portion of the phase that grows in $k$; indeed by rationalizing the numerator, one has

$$p_k(s) = 2^k \langle 2^k s \rangle - 2^k \frac{\rho s}{t} = \frac{1}{s + \sqrt{2 - 2^k s^2}} + 2^k s (1 - \frac{\rho}{t})$$

A critical point $s_0$ of $p_k'$ exists if

$$\frac{2^k s_0}{\langle 2^k s_0 \rangle} - \frac{\rho}{t} = 0$$

which can only occur if $|\rho| < t$. Since we are only interested if $s_0 \in [1, 2]$, we can see by manipulating this equation that we have

$$1 - \frac{\rho}{t} = \frac{2^{-2k}}{2^{-2k} + s_0^2 + s_0 \sqrt{2^{-2k} + s_0^2}} \leq 2^{-2k}$$
Thus there is a constant $c_k$, with $|c_k| \lesssim 1$ uniformly in $k$, such that

$$p_k(s) = \frac{1}{s + \sqrt{2^{-2k}} + s^2} + c_k s$$

From this expression we see that $p_k$ and all its derivatives are uniformly bounded in $k$ which will allow us to safely apply the estimate 4.25 in the case where $2^{-k}t \geq 1$.

First we compute the second derivative $p_k''(s_0) = 2^{3k}(2^k s_0)^{-3}$; as claimed above it is uniformly bounded above and below in $k$. Therefore if $2^{-k}t \geq 1$, we apply (4.25) and obtain

$$|I_k(t)| \lesssim (2^{-k}t)^{-\frac{1}{2}} 2^{2k} + t^{-\frac{3}{2}} \|u_0\|_{W^{4,\infty}} \lesssim (2^k)^{\frac{5}{2}} t^{-\frac{1}{2}}$$

If, on the other hand, we have $2^{-k}t \leq 1$, we use the trivial bound of the entire integral $|I_k(t)| \lesssim 2^{2k}$ and simply observe that $2^{2k} = (2^k)^{\frac{3}{2}} (2^k)^{-\frac{1}{2}} \leq (2^k)^{\frac{3}{2}} t^{-\frac{1}{2}}$ when $2^{-k}t \leq 1$. Thus in either case we have the required bound for Theorem 4.4.1.

In the other case $|\rho| \geq t$, there is no critical point so we can integrate by parts once:

$$I(t) = \frac{1}{it} \int \left( \frac{d}{d\lambda} e^{itp(\lambda)} \frac{\eta(\lambda)}{p'(\lambda)} \right) d\lambda$$

Thus the original integral (4.24) is bounded by

$$|\hat{F}(\rho)| \leq |t|^{-1} \left( \int_{-\infty}^{\infty} \left| \eta'(\lambda) \right| \frac{\eta(\lambda)}{|p'(\lambda)|^2} + \frac{\left| \eta(\lambda) \right| |p''(\lambda)|}{|p'(\lambda)|^2} d\lambda \right)$$

Since $|\rho| \geq t$, one has

$$|p'(\lambda)| \geq \frac{|\rho|}{t} - \frac{\lambda}{\sqrt{\lambda^2 + 1}} \geq 1 - \frac{\lambda}{\sqrt{\lambda^2 + 1}} \gtrsim \frac{1}{\lambda^2}$$

so using the fact that $|p''(\lambda)| \lesssim \lambda^{-3}$ we estimate

$$|\hat{F}(\rho)| \lesssim |t|^{-1} (2^k)^2 \|\eta\|_{L^1} + |t|^{-1} 2^k \|\eta\|_{L^1} \lesssim (2^k)^3 |t|^{-1}$$

This is not quite what we want, but we can just interpolate it with the trivial bound $|\hat{F}(\rho)| \leq \|\eta\|_{L^1} \lesssim (2^k)^2$ to get precisely the required result of $\|\hat{F}\|_{L^p} \lesssim (2^k)^{\frac{3}{2}} |t|^{-\frac{1}{2}}$. 
4.4.2 Low Energy

Here we will prove the estimate for $k = 0$, i.e. with $\eta(\lambda) = \langle \lambda \rangle \chi_0(\lambda)$, where $\chi_0$ is even and compactly supported in $[-1, 1]$. In particular we will return to the original integral

$$I(t) = \int_{-\infty}^{\infty} e^{itp(\lambda)} \eta(\lambda) \, d\lambda$$

with $p(\lambda) = \langle \lambda \rangle - \frac{\rho}{t}$. In the case $|t| \leq 1$, one can bound the entire integral by $|I(t)| \lesssim \| \eta \|_{L^1} \lesssim 1 \leq |t|^{-\frac{1}{2}}$ as required.

If $|t| \geq 1$, there is a single critical point $\lambda_0 = \frac{\rho}{\sqrt{t^2 - \rho^2}}$ only when $|\rho| < t$. Since $|\lambda| \leq 1$ in the support of $\eta$, one has a uniform lower bound on $p''(\lambda_0) = \langle \lambda_0 \rangle^{-3}$, so the stationary phase estimate (4.25) applies to get

$$|I(t)| \lesssim t^{-\frac{1}{2}} + Ct^{-\frac{3}{2}} \lesssim t^{-\frac{1}{2}}$$

as required. When $|\rho| \geq t$, there are no critical points so we can recover the decay by integrating by parts:

$$I(t) = \frac{1}{it} \int \left( \frac{d}{d\lambda} e^{itp(\lambda)} \right) \frac{\eta(\lambda)}{p'(\lambda)} \, d\lambda$$

$$= \frac{1}{it} \int e^{itp(\lambda)} \left( \frac{\eta'(\lambda)}{p'(\lambda)} - \frac{\eta(\lambda)p''(\lambda)}{p'(\lambda)^2} \right) \, d\lambda$$

One can see directly that the entire integrand is bounded uniformly for $\lambda \in [-1, 1]$, since $p''(\lambda) = \langle \lambda \rangle^{-3}$ and

$$|p'(\lambda)| \geq \frac{|\lambda|}{\sqrt{\lambda^2 + 1}} - \frac{|\rho|}{t} = \frac{|\rho|}{t} - \frac{|\lambda|}{\sqrt{\lambda^2 + 1}} \geq 1 - \frac{|\lambda|}{\sqrt{\lambda^2 + 1}} \geq \frac{2 - \sqrt{2}}{2}$$

Thus $|I(t)| \lesssim t^{-1}$ which is more than enough to establish Theorem 4.4.1 in this case.
Chapter 5

A Priori Estimates

By our remarks after the local existence theorem in Chapter 3, all that is needed to establish global existence of solutions to the main equation (1.2) are certain a priori estimates on solutions. In this chapter we prove these estimates. Some of these estimates depend on the normal forms transform discussed in Chapter 6 and will be proved there, but we state them here for completeness.

5.1 Main Estimates

Theorem 5.1.1. Suppose $u$ is a classical solution to:

$$(\partial_t^2 + H + 1)u = u^2, \quad (u(0), u_t(0)) = (u_0, u_1),$$

on a time interval $[0, T]$. Then exists a uniform constant $C > 0$ such that for any integer $m \geq 5$, we have both:

$$\sup_{0 \leq t \leq T} \| (u, u_t)(t) \|_{H^{4m+1} \times H^{4m}} \lesssim \| (u_0, u_1) \|_{H^{4m+1} \times H^{4m}} +$$

$$+ \sup_{0 \leq t \leq T} \| u(t) \|_{H^{4m}} \cdot \sup_{0 \leq t \leq T} (1 + t)^{\frac{3}{2}} \| u(t) \|_{W^{2m+1, \infty}}, \quad (5.1)$$
and:

\[ \sup_{0 \leq t \leq T} (1 + t)^{3/2} \| (u, u_t)(t) \|_{W^{2m+1, \infty}} \lesssim \| (u_0, u_1) \|_{W^{2m+7.1 \times W^{2m+6.1}}} + \]

\[ + \| (u_0, u_1) \|_{H^{4m+1 \times H^{4m}}} + \| (u_0, u_1) \|_{H^{4m+1 \times H^{4m}}} + \]

\[ + \sup_{0 \leq t \leq T} (1 + t) \| (u, u_t)(t) \|_{H^{4m+1 \times H^{4m}}} \cdot \sup_{0 \leq t \leq T} (1 + t)^{3/2} \| (u, u_t)(t) \|_{W^{2m+1, \infty}} \]

These estimates imply global existence for small data \( \epsilon u_0, \epsilon u_1 \) as per the discussion in Chapter 3. To prove this, let \( m \geq 5 \) be fixed. We must find an \( M > 0 \), independent of \( T \) and \( \epsilon \), such that if

\[ \sup_{0 \leq t \leq T} \| (u, u_t)(t) \|_{H^{4m+1 \times H^{4m}}} \leq M \epsilon , \]  

(5.3)

\[ \sup_{0 \leq t \leq T} (1 + t)^{3/2} \| (u, u_t)(t) \|_{W^{2m+1, \infty}} \leq M \epsilon . \]  

(5.4)

then we can improve the bounds by replacing \( M \) with \( M/2 \) for sufficiently small \( \epsilon \). Indeed, suppose that (5.3) and (5.4) hold. Applying first (5.1) and then (5.2) shows that the left side of (5.3) is bounded by \( C(\epsilon + M^2 \epsilon^2) \). This is in turn bounded by \( \frac{M \epsilon}{2} \) if \( M > 2C \) and \( \epsilon \) is small enough. Similarly, by applying first (5.2) and then (5.1) to the quantity on the left side of (5.4), we find that it is controlled by \( C(\epsilon + \epsilon^2 + \epsilon^3 + (M \epsilon + M^3 \epsilon^3)M \epsilon) \), which is again bounded by \( \frac{M \epsilon}{2} \) for suitable \( M \) (depending only on \( C \)) and small \( \epsilon \).

Now we will collect the pieces that constitute the proof of Theorem 5.1.1. First, recall our main energy and dispersive estimates from Chapters 3 and 4: for odd values of \( k \) we have

\[ \| (u(t, \cdot), u_t(t, \cdot)) \|_{W^{k, \infty}} \lesssim (1 + t)^{-\frac{3}{2}} \sum_{i=0}^{1} \| u_i \|_{W^{k+6-1,1}} + \]

\[ + \int_0^t (1 + t - s)^{-\frac{3}{2}} \| f(s, \cdot) \|_{W^{k+5,1}} \, ds \]

and

\[ \| (u, u_t)(t) \|_{H^k \times H^{k-1}} \lesssim \| (u_0, u_1) \|_{H^k \times H^{k-1}} + \int_0^t \| f(s, \cdot) \|_{H^{k-1}} \, ds \]

(5.6)

We need to add to these two estimates a renormalization lemma (which will be proved in the next chapter):
Theorem 5.1.2. For given \( u \) on \([0, T]\) there exists a \( U \) defined on \([0, T]\) such that:

\[
(\partial_t^2 + H + 1) U = u^2 + R,
\]

and for \( m \geq 5 \) one has the estimates:

\[
\| R(t) \|_{W^{2m+6,1}} \lesssim \left( \| (u, u_t)(t) \|_{H^{4m+1} \times H^{4m}} + \right)
+ \| (u, u_t)(t) \|_{H^{4m+1} \times H^{4m}}^3 \cdot \| (u, u_t)(t) \|_{W^{2m+1,\infty}}.
\]

\[
\| (U, U_t)(t) \|_{W^{2m+1,\infty}} \lesssim \left( \| (u, u_t)(t) \|_{H^{4m+1} \times H^{4m}} + \right)
+ \| (u, u_t)(t) \|_{H^{4m+1} \times H^{4m}}^2 \cdot \| (u, u_t)(t) \|_{W^{2m+1,\infty}}.
\]

and

\[
\| U(0), U_t(0) \|_{W^{2m+7,1} \times W^{2m+6,1}} \lesssim \left( \| (u_0, u_1) \|_{H^{4m+1} \times H^{4m}}^2 + \| (u_0, u_1) \|_{H^{4m+1} \times H^{4m}}^3 \right)
\]

Proof. The proof of these estimates is relegated to the next chapter which develops the normal forms transform that produces the \( U \) mentioned above.

Proof of Theorem 5.1.1. Let us begin with (5.1). Suppose \( m \geq 5 \). By (5.6) we have to control the term \( \int_0^t \| u^2(s, \cdot) \|_{H^{4m}} ds \) for any \( t \in [0, T] \). Expansion of the integrand gives

\[
\| u^2(s, \cdot) \|_{H^{4m}} \leq \sum_{|\alpha| \leq 4m} c_{\alpha, \alpha''} \| (\partial^{\alpha'} u)(\partial^{\alpha''} u) \|_{L^2}
\]

At most one of \( \alpha' \), \( \alpha'' \) can account for more than \( 2m \) derivatives in each summand, so by symmetry we will assume without loss of generality that \( |\alpha'| \leq 2m \). Then

\[
\| (\partial^{\alpha'} u)(\partial^{\alpha''} u) \|_{L^2} \lesssim \| u \|_{W^{2m,\infty}} \| u \|_{H^{4m}},
\]

so by choosing a constant that depends only on \( m \) we obtain

\[
\int_0^t \| u^2(s, \cdot) \|_{H^{4m}} \lesssim \sup_{0 \leq t \leq T} \| u \|_{H^{4m}} \int_0^t (1 + s)^{-\frac{3}{2}} (1 + s)^{\frac{3}{2}} \| u(s, \cdot) \|_{W^{2m,\infty}} ds
\]

\[
\lesssim \sup_{0 \leq t \leq T} \| u(t) \|_{H^{4m}} \cdot \sup_{0 \leq t \leq T} (1 + t)^{\frac{3}{2}} \| u(t) \|_{W^{2m+1,\infty}}
\]
Combining this estimate with (5.6) and taking supremums completes the proof of (5.1).

Now we give the proof of (5.2). By Theorem 5.1.2, we may write $u = U + W$, where $W = u - U$. Fix $t \in [0, T]$. For convenience we define

$$\mathcal{E}_T := \sup_{0 \leq t \leq T} \| (u, u_t)(t) \|_{H^{4m+1} \times H^{4m}} + \sup_{0 \leq t \leq T} \| (u, u_t)(t) \|_{H^{4m+1} \times H^{4m}}^3$$

Then we have

$$(1 + t)^{\frac{3}{2}} \|(u, u_t)(t)\|_{W^{2m+1, \infty}} \leq (1 + t)^{\frac{3}{2}} \|(U, U_t)(t)\|_{W^{2m+1, \infty}} + (1 + t)^{\frac{3}{2}} \|(W, W_t)(t)\|_{W^{2m+1, \infty}}$$

(5.10)

The bound for $(U, U_t)$ in Theorem 5.1.2 gives

$$(1 + t)^{\frac{3}{2}} \|(U, U_t)(t)\|_{W^{2m+1, \infty}} \lesssim \mathcal{E}_T \sup_{0 \leq t \leq T} (1 + t)^{\frac{3}{2}} \|(u, u_t)(t)\|_{W^{2m+1, \infty}}$$

which is precisely the right side of (5.2) (except for the data). To bound the second term in (5.10), note that $(\partial_t^2 + H + 1)W = -R$. Suppose $(W(0), W_t(0)) = (W_0, W_1)$. We apply (5.5) to obtain

$$(1 + t)^{\frac{3}{2}} \|(W, W_t)(t)\|_{W^{2m+1, \infty}} \lesssim \|(W_0, W_1)\|_{W^{2m+7, 1} \times W^{2m+6, 1}} + (1 + t)^{\frac{3}{2}} \int_0^t (1 + t - s)^{-\frac{3}{2}} \| R(s) \|_{W^{2m+6, 1}} \, ds$$

Using the bound from Theorem 5.1.2 shows that the integral in the previous inequality is bounded above by

$$\mathcal{E}_T (1 + t)^{\frac{3}{2}} \int_0^t (1 + t - s)^{-\frac{3}{2}} \| u(s, \cdot) \|_{W^{2m+1, \infty}} \, ds$$

Multiplying by $(1 + s)^{\frac{3}{2}}$ inside the integral and taking a supremum leaves us with

$$\mathcal{E}_T \sup_{0 \leq s \leq T} (1 + s)^{\frac{3}{2}} \|(u, u_t)(s)\|_{W^{2m+1, \infty}} (1 + t)^{\frac{3}{2}} \int_0^t (1 + t - s)^{-\frac{3}{2}} (1 + s)^{-\frac{3}{2}} \, ds$$

Note that the final integral is on the order of $(1 + t)^{-\frac{3}{2}}$:

$$\int_0^t (1 + t - s)^{-\frac{3}{2}} (1 + s)^{-\frac{3}{2}} \, ds = 2 \int_0^{t/2} (1 + t - s)^{-\frac{3}{2}} (1 + s)^{-\frac{3}{2}} \, ds$$

$$\lesssim (1 + t)^{-\frac{3}{2}} \int_0^\infty (1 + s)^{-\frac{3}{2}} \, ds$$

$$\lesssim (1 + t)^{-\frac{3}{2}}$$
which makes the contribution \( \mathcal{E}_T \cdot \sup_{0 \leq t \leq T}(1 + t)^{\frac{3}{2}} \|(u, u_t)\|_{W^{2m+1, \infty}}, \) as required.

Finally, note that

\[
\|(W_0, W_1)\|_{W^{2m+7,1} \times W^{2m+6,1}} \leq \|(u_0, u_1)\|_{W^{2m+7,1} \times W^{2m+6,1}} + \|(U(0), U_t(0))\|_{W^{2m+7,1} \times W^{2m+6,1}}
\]

so the proof is completed by the last bound in Theorem 5.1.2.

5.2 Conclusion

These estimates combined with the remarks in Chapter 3 have now established global existence of solutions to (1.2); however we have postponed the proof of Theorem 5.1.2. This proof is sufficiently involved and will be proved in the next Chapter using the normal forms transform of Shatah.
Chapter 6

The Normal Forms Transform

6.1 Motivation

The key technique in handling the quadratic nonlinearity in (1.2) is the normal forms transformation of Shatah. To provide some motivation for the basic ideas involved and for the terms involved in the transform, it is useful to consider first the simple nonlinear ODE

\[ \ddot{u} + u = u^2 \]  

(6.1)

If we assume that we can solve and control solutions to (6.1) with cubic nonlinearity instead of quadratic, the general idea is to seek a decomposition \( u = U + W \), where \( U \) is given explicitly in terms of \( u \) and its derivatives, and \( W \) satisfies an equation similar to (6.1) but with only cubic nonlinear terms.

The simplest approach in this case turns out to work: let \( U = au^2 + bu^2 \), with \( a \) and \( b \) constants to be determined. Then simply let \( W = u - U \). Let us denote by \( CT(u) \) any sum of cubic or higher terms in \( u \) and its derivatives (e.g. a term like \( u^2 \dot{u} + 2 \dot{u}^3 \)). We calculate

\[ \dot{U} = 2au\dot{u} + 2b\ddot{u} \]

\[ \ddot{U} = 2a(\dot{u}^2 + u\ddot{u}) + 2b(\ddot{u}^2 + \dot{u}^2) \]  

(6.2)

From the equation (6.1) and the equation for \( \dot{u} \) obtained by differentiating both
sides of (6.1), we get the identities
\[ \ddot{u} = u^2 - u \]
\[ \dddot{u} = 2u\dot{u} - \dot{u} \]

Substituting these into (6.2) and adding, we obtain
\[ \ddot{U} + U = 2au^2 - 2au^2 - 2b\dot{u}^2 + 2bu^2 + au^2 + bu^2 + CT(u) \]

Thus it follows that \( W \) satisfies
\[ \ddot{W} + W = \ddot{u} + u - \ddot{U} - U \]
\[ = u^2 - 2a\dot{u}^2 + 2a\dot{u}^2 + 2b\dot{u}^2 - 2bu^2 - au^2 - b\dot{u}^2 + CT(u) \]
\[ = (1 + a - 2b)u^2 + (b - 2a)\dot{u}^2 + CT(u) \]

Thus we simply choose \( a \) and \( b \) so as to make the first two terms vanish (in this case \( a = \frac{1}{3}, b = \frac{2}{3} \)). Then \( W \) will satisfy an equation with only cubic nonlinearities and is thus controlled by our previous assumption. Given the explicit form of \( U \), we have thus gained control over \( u = U + W \).

When dealing with the nonlinear PDE (1.2), we will replace the constants \( a \) and \( b \) by distributions acting on \( u \) and its derivatives, and sufficient work will be necessary to establish control over what corresponds to the \( U \) term.

### 6.2 The Transform and Associated Error Terms

Now we return again to the main equation
\[ (\partial_t^2 - \Delta + V + 1)u = u^2 \]

We will follow our motivation from the previous section; that is we seek a decomposition \( u = U + W \), where \( U \) can be controlled more or less directly and \( W \) satisfies a Klein-Gordon equation with cubic nonlinearity.

We first introduce some notation.
Definition 6.2.1. Let $D' = -i\partial'$ and $D'' = -i\partial''$, and for $u, v \in S(\mathbb{R}^3)$, define $B(D', D'') [u][v]$ to be the function whose Fourier transform evaluated at $\xi$ is

$$(2\pi)^{-3} \int B(\xi - \eta, \eta) \hat{u}(\xi - \eta) \hat{v}(\eta) d\eta$$

For a complete introduction to these operators and their properties, the reader is directed to Appendix A. To handle a quadratic nonlinearity, we will define

$$U := \mathcal{K}_0(D', D'') [u][u] + \mathcal{K}_1(D', D'') [\partial_t u][\partial_t u]$$

where the $\mathcal{K}_i$ are to be chosen later. Recall our earlier notation $CT(u)$ for terms that are of cubic order or higher in $u$ and its derivatives. We now introduce the notation $CT_V(u)$. In addition to terms such as $B(D', D'') [u^2][u]$, $CT_V(u)$ will also include terms with the potential $V$ such as $B(D', D'') [Vu][u]$. The reason for this grouping is that our assumptions on $V$ allow us to handle them similarly to cubic terms. Anagolously to the simple case we examined in the previous section, the goal is to choose $\mathcal{K}_0, \mathcal{K}_1$ such that $(\Box + 1 + V) U = u^2 + VU + CT_V(u) = u^2 + R$, where $R$ is the term referenced in the renormalization theorem 5.1.2. With $W = u - U$, this forces $(\Box + 1 + V) W = -R$.

The idea is to simply compute $(\Box + 1) U$, derive a system of equations that the $\mathcal{K}_i$ must satisfy to meet our requirements, and then solve the system. We split the computation of $(\Box + 1) U = (\partial_t^2 - \Delta + 1) U$ into two steps, dealing with the time derivatives first. Note that the derivative $\partial_t$ commutes with the Fourier transform in the spatial variables, so by Fourier inversion and the usual Leibniz rule, we can compute

$$\partial_t^2 U = \mathcal{K}_0(D', D'') ([\partial_t^2 u][u] + 2[\partial_t u][\partial_t u] + [u][\partial_t^2 u]) + \mathcal{K}_1(D', D'') ([\partial_t^3 u][\partial_t u] + 2[\partial_t^2 u][\partial_t^2 u] + [\partial_t u][\partial_t^3 u])$$

In (6.3), we replace the terms that have time derivatives of second order or higher using the original equation:

$$\partial_t^2 u = \Delta u - u - Vu + u^2 = (\Delta - 1) u - Vu + u^2$$

$$\partial_t^3 u = \partial_t(\Delta u - u - Vu + u^2) = (\Delta - 1) \partial_t u - V \partial_t u + \partial_t(u^2)$$
Using Definition 6.2.1 in reverse, we collect all terms in (6.3):

$$[\partial^2_t u][u] = -(|D'|^2 + 1)[u][u] + CT^1_V(u) \quad (6.4)$$
$$[u][\partial^2_t u] = -(|D''|^2 + 1)[u][u] + CT^2_V(u) \quad (6.5)$$
$$[\partial^3_t u][\partial_t u] = -(|D'|^2 + 1)[\partial_t u][\partial_t u] + CT^3_V(u) \quad (6.6)$$
$$[\partial_t u][\partial^3_t u] = -(|D''|^2 + 1)[\partial_t u][\partial_t u] + CT^4_V(u) \quad (6.7)$$
$$[\partial^2_t u][\partial^2_t u] = (|D'|^2 + 1)(|D''|^2 + 1)[u][u] + CT^5_V(u) \quad (6.8)$$

Now define $CT_V(u) = \sum_{j=1}^5 CT^j_V(u)$. Then we have

$$CT_V(u) = K_0(D', D'')\left(u^2[u] - [Vu][u] - [u][Vu] + [u][u^2]\right) + K_1(D', D'')\left(2[u\partial_t u][\partial_t u] - [V\partial_t u][\partial_t u] - [\partial_t u][V\partial_t u] + 2[\partial_t u][u\partial_t u] + 2[Vu][Vu] + 2[u^2][u^2] - 2[Vu][u^2] - 2[u^2][Vu] + 2(|D'|^2 + 1)[u][Vu] + 2(|D''|^2 + 1)[Vu][u] - 2(|D'|^2 + 1)[u][u^2] - 2(|D''|^2 + 1)[u^2][u]\right) \quad (6.9)$$

Now we return to the computation of $(\Box + 1 + V)U$. The second half of the computation concerns $-\Delta U$. The results from Appendix A guide the calculations. By collecting and cancelling, we obtain

$$(-\Delta)U = (|D'|^2 + |D''|^2 + 2\langle D', D'' \rangle)K_0(D', D'')[u][u] + (|D'|^2 + |D''|^2 + 2\langle D', D'' \rangle)K_1(D', D'')[\partial_t u][\partial_t u] \quad (6.10)$$

$$+ (|D'|^2 + |D''|^2 + 2\langle D', D'' \rangle)K_1(D', D'')[\partial_t u][\partial_t u] \quad (6.11)$$

It follows then that

$$(\Box + 1 + V)U = T_1(D', D'')[u][u] + T_2(D', D'')[\partial_t u][\partial_t u] + VU + CT_V(u)$$

where

$$T_1(D', D'') = (2\langle D', D'' \rangle - 1)K_0 + 2(|D'|^2 + 1)(|D''|^2 + 1)K_1$$

$$T_2(D', D'') = 2K_0 + (2\langle D', D'' \rangle - 1)K_1$$

Thus to ensure that $(\Box + 1 + V)U = u^2 + VU + CT_V(u)$, we are led to solve the system

$$(2\langle \xi, \eta \rangle - 1)K_0(\xi, \eta) + 2(|\xi|^2 + 1)(|\eta|^2 + 1)K_1(\xi, \eta) = 1$$

$$(2\langle \xi, \eta \rangle - 1)K_1(\xi, \eta) + 2K_0(\xi, \eta) = 0 \quad (6.12)$$
With \( a = a(\xi, \eta) := 2\langle \xi, \eta \rangle - 1 \) and \( b = b(\xi, \eta) := 2(|\xi|^2 + 1)(|\eta|^2 + 1) \), we simply have to row reduce the matrix:

\[
\begin{bmatrix}
a & b & 1 \\
2 & a & 0
\end{bmatrix}
\]

A short calculation shows that the row reduced version is:

\[
\begin{bmatrix}
1 & 0 & a \\
0 & 1 & \frac{2}{2b-a^2}
\end{bmatrix}
\]

With the above choices of \( a(\xi, \eta), b(\xi, \eta) \), we have

\[2b - a^2 = 4 \left( |\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2 + |\xi|^2 + |\eta|^2 + \langle \xi, \eta \rangle \right) + 3\]

Let us then define

\[\mathcal{K}(\xi, \eta) = \left( 4(|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2 + |\xi|^2 + |\eta|^2 + \langle \xi, \eta \rangle) + 3 \right)^{-1}\]

so that we may express the solution compactly as

\[
\begin{align*}
\mathcal{K}_0(\xi, \eta) &= (1 - 2\langle \xi, \eta \rangle)\mathcal{K}(\xi, \eta) \\
\mathcal{K}_1(\xi, \eta) &= 2\mathcal{K}(\xi, \eta)
\end{align*}
\]

### 6.3 Proof of Renormalization Lemma

Now we are prepared to begin the proof of Theorem 5.1.2. We have already established the existence of the term \( U \); recall that it is given by

\[U = \mathcal{K}_0(D', D''')[u][u] + \mathcal{K}_1(D', D''')[\partial_t u][\partial_t u]\]

(6.15)

Then we can just take \( W = u - U \) to give the decomposition \( u = U + W \). Recall again the designations

\[
\begin{align*}
\mathcal{K}_0(\xi, \eta) &= (1 - 2\langle \xi, \eta \rangle)\mathcal{K}(\xi, \eta) \\
\mathcal{K}_1(\xi, \eta) &= 2\mathcal{K}(\xi, \eta)
\end{align*}
\]

where

\[\mathcal{K}(\xi, \eta) = \left( 4(|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2 + |\xi|^2 + |\eta|^2 + \langle \xi, \eta \rangle) + 3 \right)^{-1}\]

(6.17)

Theorem 5.1.2 will be deduced as a consequence of the following two results:
**Theorem 6.3.1.** If \( p, q \in \{1, 2, \infty\} \) are such that \( \frac{1}{p} + \frac{1}{q} \leq 1 \), and \( r \) is chosen so that \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \), then for \( \mathcal{K} \) as defined above and \( u, v \in \mathcal{S}(\mathbb{R}^3) \), one has

\[
\| \mathcal{K}(D', D'') [u][v] \|_{L^r} \lesssim \| u \|_{W^{1,p}} \| v \|_{W^{1,q}}
\]

(6.18)

**Corollary 6.3.2.** For \( \mathcal{K}_0, \mathcal{K}_1 \) as defined in (6.16) and \( p, q, r \) as in Theorem 6.3.1, then the estimate

\[
\| \mathcal{K}_i(D', D'')[u][v] \|_{L^r} \lesssim \| u \|_{W^{2-i,p}} \| v \|_{W^{2-i,q}}
\]

holds for \( u, v \in \mathcal{S}(\mathbb{R}^3) \).

**Proof.** Since \( \mathcal{K}_1(\xi, \eta) = 2\mathcal{K} \) and \( \mathcal{K}_0(\xi, \eta) = (1 - 2\langle \xi, \eta \rangle)\mathcal{K}(\xi, \eta) \), the estimate for \( \mathcal{K}_0 \) follows immediately from Theorem 6.3.1. Applying the theorem to \( \mathcal{K}_0 \) results in one extra differentiation of each factor, corresponding to the presence of the \( \langle \xi, \eta \rangle \) term.

Now we can restate Theorem 5.1.2 and give a proof based on these estimates. The proof of Theorem 6.3.1 follows an outline that can be found in [1] and is contained in Appendix A.

**Theorem (5.1.2).** For given \( u \) on \([0, T]\) there exists a \( U \) defined on \([0, T]\) such that:

\[
(\partial_t^2 + H + 1)U = u^2 + R ,
\]

(6.19)

and one has for any \( m \geq 5 \) the estimates:

\[
\| R(t) \|_{W^{2m+6,1}} \lesssim (\| (u, u_t)(t) \|_{H^{4m+1} \times H^{4m}} + \| (u, u_t)(t) \|_{H^{4m+1} \times H^{4m}}^3) \cdot \| (u, u_t)(t) \|_{W^{2m+1,\infty}}
\]

(6.20)

\[
\| (U, U_t)(t) \|_{W^{2m+1,\infty}} \lesssim (\| (u, u_t)(t) \|_{H^{4m+1} \times H^{4m}} + \| (u, u_t)(t) \|_{H^{4m+1} \times H^{4m}}^2) \cdot \| (u, u_t)(t) \|_{W^{2m+1,\infty}}
\]

(6.21)

and

\[
\| U(0), U_t(0) \|_{W^{2m+7,1 \times W^{2m+6,1}}} \lesssim (\| (u_0, u_1) \|_{H^{4m+1} \times H^{4m}}^2 + \| (u_0, u_1) \|_{H^{4m+1} \times H^{4m}}^3)
\]

(6.22)
Proof. First we will prove the bound for $\mathcal{R}$. By our work in this section, we have that $\mathcal{R} = VU + CT_V(u)$, where $CT_V(u)$ is given above. We first handle the estimate for $VU$.

One has

$$
\| VU \|_{W^{2m+6,1}} = \sum_{|\alpha| \leq 2m+6} \| \partial^\alpha (VU) \|_{L^1} \leq \sum_{|\alpha| \leq 2m+6} c_{\alpha,\alpha''} \| (\partial^{\alpha'} V)(\partial^{\alpha''} U) \|_{L^1}
$$

Using the usual Cauchy-Schwarz inequality, we get

$$
\| (\partial^{\alpha'} V)(\partial^{\alpha''} U) \|_{L^1} \leq \| V \|_{H^{[\alpha',1]}} \| U \|_{H^{2m+6}}
$$

By our assumptions on $V$ the first factor is bounded by a constant depending only on $V$ and $m$. For the second factor, recall that $U$ is defined as in (6.15). By the Leibniz rule, one finds that

$$
\| U \|_{H^{2m+6}} = \sum_{|\gamma| \leq 2m+6} \| \partial^\gamma U \|_{L^2} \leq \sum_{|\gamma| \leq 2m+6} c_{\gamma,\gamma''} \| K_i(D',D'')[\partial_x^{\gamma'} \partial_t^{\gamma''} u][\partial_x^{\gamma'} \partial_t^{\gamma''} u] \|_{L^2}
$$

To each summand, we shall apply Corollary 6.3.2 with either $r = 2$, $p = \infty$, $q = 2$ or $r = 2$, $p = 2$, $q = \infty$. The choice of indices is determined by requiring that the smaller number of derivatives go in $L^\infty$. By symmetry, we will assume without loss that $|\gamma'| \leq |\gamma''|$ so that

$$
\| K_i(D',D'')[\partial_x^{\gamma'} \partial_t^{\gamma''} u][\partial_x^{\gamma'} \partial_t^{\gamma''} u] \|_{L^2} \lesssim \| \partial_x^{\gamma'} \partial_t^{\gamma''} u \|_{W^{2m+1,\infty}} \| \partial_x^{\gamma'} \partial_t^{\gamma''} u \|_{W^{2m+1,2}}
$$

Now $|\gamma'| + 2 - i \leq m + 5 \leq 2m + 1$ when $m \geq 5$, so the first factor is controlled by $\| (u, u_t)(t) \|_{W^{2m+1,\infty}}$. For the second factor, we have $|\gamma''| + 2 - i \leq 2m + 8 - i$, and since $m \geq 5$ we get a bound of $\| \partial_t^i u \|_{H^{4m+1-i}}$. Returning to the sum (6.23), we have shown that

$$
\| U \|_{H^{2m+6}} \lesssim \| (u, u_t) \|_{W^{2m+1,\infty}} \sum_{i=0,1} \| \partial_t^i u \|_{H^{4m+1-i}} \leq \| (u, u_t) \|_{W^{2m+1,\infty}} \| (u, u_t) \|_{H^{4m+1} \times H^{4m}}
$$
where the implied constant depends only on $m$. The conclusion is that with a constant depending only on $m$ and $V$, one has

$$\| VU \|_{W^{2m+6,1}} \lesssim \| (u, u_t) \|_{H^{4m+1} \times H^{4m}} \| (u, u_t) \|_{W^{2m+1, \infty}}$$

Now we proceed to estimating the error terms appearing in $CT_V(u)$. There are many symmetric terms appearing in the list (6.9) and it will be helpful to list here precisely those terms that we need to estimate. These are given by

$$\mathcal{K}_1(D', D'') \left( 2[u \partial_t u][\partial_t u] - [V \partial_t u][\partial_t u] + [V u][V u] - [V u][u^2] + [u^2][u^2] - [\Delta u][V u] + [u][V u] + [\Delta u][u^2] - [u][u^2] \right) + \mathcal{K}_0(D', D'')(u^2)[u] - [V u][u]$$

Thus we can collect all the possible estimates we need to control $CT_V(u)$ in the following proposition.

**Proposition 6.3.3.** For $m \geq 5$, $i = 0, 1$, and $|\eta| \leq 2$ one has the following bounds on error terms in $CT_V(u)$:

$$\| \mathcal{K}_1(D', D'')[V \partial_t^i u][\partial_t^j u] \|_{W^{2m+6,1}} \lesssim \| \partial_t^i u \|_{H^{4m+1-i}} \| (u, u_t) \|_{W^{2m+1, \infty}}$$  \hspace{1cm} (6.24)

$$\| \mathcal{K}_1(D', D'')[\partial_t^2 u][V u] \|_{W^{2m+6,1}} \lesssim \| u \|_{H^{4m+1}} \| (u, u_t) \|_{W^{2m+1, \infty}}$$  \hspace{1cm} (6.25)

$$\| \mathcal{K}_1(D', D'')[\partial_t^2 u][u^2] \|_{W^{2m+6,1}} \lesssim \| u \|_{H^{4m+1}}^2 \| (u, u_t) \|_{W^{2m+1, \infty}}$$  \hspace{1cm} (6.26)

$$\| \mathcal{K}_1(D', D'')[V u][V u] \|_{W^{2m+6,1}} \lesssim \| u \|_{H^{4m+1}} \| (u, u_t) \|_{W^{2m+1, \infty}}$$  \hspace{1cm} (6.27)

$$\| \mathcal{K}_1(D', D'')[u^2][u^2] \|_{W^{2m+6,1}} \lesssim \| u \|_{H^{4m+1}}^3 \| (u, u_t) \|_{W^{2m+1, \infty}}$$  \hspace{1cm} (6.28)

$$\| \mathcal{K}_1(D', D'')[V u][u^2] \|_{W^{2m+6,1}} \lesssim \| u \|_{H^{4m+1}}^2 \| (u, u_t) \|_{W^{2m+1, \infty}}$$  \hspace{1cm} (6.29)

$$\| \mathcal{K}_1(D', D'')[u \partial_t u][\partial_t u] \|_{W^{2m+6,1}} \lesssim \| (u, u_t) \|_{H^{4m+1} \times H^{4m}} \| (u, u_t) \|_{W^{2m+1, \infty}}$$  \hspace{1cm} (6.30)

Furthermore, the implied constants depend only on $m$ and $V$.

**Remark 6.3.4.** Note that these bounds, in conjunction with the estimate already proved for $VU$, establish the required estimate (6.20) for $\mathcal{R}$.

**Proof.** The common first step in bounding each of the above quantities will be an expansion of the norm in question and then an application of the Leibniz rule as
follows:
\[
\| \mathcal{K}_i(D', D'')[f][g] \|_{W^{2m+6,1}} = \sum_{|\alpha| \leq 2m+6} \| \partial_\alpha^2 \mathcal{K}_i(D', D'')[f][g] \|_{L^1} = \]
\[
\leq \sum_{|\alpha| \leq 2m+6, \alpha'+\alpha''=\alpha} c_{\alpha,\alpha''} \| \mathcal{K}_i(D', D'')[\partial_\alpha' f][\partial_\alpha'' g] \|_{L^1} \tag{6.31}
\]
where \(c_{\alpha,\alpha''} = \left( \frac{\alpha}{\alpha''} \right)\). Note that \(c_{\alpha,\alpha''}\) is bounded by a fixed constant once \(m\) is fixed, so we have reduced the problem to proving estimates for
\[
\| \mathcal{K}_i(D', D'')[\partial_\alpha' f][\partial_\alpha'' g] \|_{L^1}
\]
where \(f, g\) will be replaced with the appropriate functions appearing in the above list and \(\alpha', \alpha''\) are any multi-indices with \(|\alpha'| + |\alpha''| \leq 2m + 6\).

We shall proceed in order. By our remarks in the previous paragraph, for the first entry (6.24) in Proposition 6.3.3, we will estimate
\[
\| \mathcal{K}_i(D', D'')[\partial_\alpha' (V\partial_t^i u)][\partial_\alpha'' \partial_t^i u] \|_{L^1} \tag{6.32}
\]
If \(|\alpha'| \leq |\alpha''|\), then we apply Corollary 6.3.2 with \(r = 1, p = 2, q = 2\) to get an upper bound of
\[
\| \partial_\alpha' (V\partial_t^i u) \|_{H^{2-i}} \| \partial_\alpha'' \partial_t^i u \|_{H^{2-i}}
\]
Expanding further shows that this is bounded by
\[
\sum_{\alpha'_1 + \alpha'_2 = \alpha'} c_{\alpha',\alpha''} \| (\partial_\alpha' V)(\partial_\alpha'' \partial_t^i u) \|_{H^{2-i}} \| \partial_\alpha'' \partial_t^i u \|_{H^{2-i}}
\]
In this case, we have \(|\alpha'| \leq m + 3\), which implies that \(|\alpha'_2| + 2 - i \leq |\alpha'| + 2 - i \leq m + 5 \leq 2m + 1\) since \(m \geq 5\). On the other hand we have \(|\alpha''| + 2 - i \leq 2m + 8 - i \leq 4m + 1 - i\), so we can choose a constant \(C\) depending only on \(m\) such that the above is controlled by
\[
C \| \partial_\alpha' V \|_{H^2} \| (u,u_t) \|_{W^{2m+1,\infty}} \| \partial_t^i u \|_{H^{4m+1-i}}
\]
By our assumptions on \(V\), we can absorb that factor into a larger constant, concluding that the total contribution of (6.32) in this case is
\[
C' \| (u,u_t) \|_{W^{2m+1,\infty}} \| \partial_t^i u \|_{H^{4m+1-i}}
\]
If instead we have $|\alpha'| > |\alpha''|$, we must apply Corollary 6.3.2 with $r = 1$, $p = 1$, $q = \infty$ to see that (6.32) is bounded by

$$\| \partial_x^\alpha (V \partial_t^i u) \|_{W^{2+1,1}} \| \partial_x^\alpha \partial_t^i u \|_{W^{2+1,\infty}}$$

Since this time $|\alpha''| \leq m + 3$, $|\alpha''| + 2 - i \leq 2m + 1 - i$ so the second factor is controlled by $\| (u, u_t) \|_{W^{2m+1, \infty}}$. The first factor is equal to

$$\sum_{|\beta| \leq 2-i} \| \partial_x^\alpha \partial_t^i u \|_{L^1}$$

We expand the derivative like before, but then apply Cauchy-Schwarz to obtain

$$\sum_{\alpha_1', \alpha_2' = \alpha', \beta \atop |\beta| \leq 2-i} c_{\alpha', \alpha_1', \beta} \| (\partial_x^\alpha V)(\partial_x^\alpha \partial_t^i u) \|_{L^1} \leq C \| V \|_{H^{2m+8}} \| \partial_t^i u \|_{H^{2m+8-i}}$$

where $C$ depends only on $m$. Again we will absorb the factor with $V$ into a larger constant, and since $m \geq 5$ one has $2m + 8 - i \leq 4m + 1 - i$, which establishes the bound (6.24).

Next is (6.25). By (6.31), we must estimate

$$\| \mathcal{K}_1(D', D'')[\partial_x^\alpha u][\partial_x^\alpha (V u)] \|_{L^1}$$

(6.33)

If $|\alpha'| \leq |\alpha''|$, then apply Corollary 6.3.2 with $r = 1$, $p = \infty$, $q = 1$ to bound (6.33) by

$$\| \partial_x^\alpha u \|_{W^{1,\infty}} \| \partial_x^\alpha (V u) \|_{W^{1,1}} \leq \| u \|_{W^{2m+1, \infty}} \| \partial_x^\alpha (V u) \|_{W^{1,1}}$$

since here $|\alpha'| \leq m + 3$ so that $|\alpha'| + |\eta| + 1 \leq m + 6 \leq 2m + 1$ when $m \geq 5$. We expand the second factor to obtain

$$\sum_{|\beta| \leq 1} \| \partial_x^\alpha + \beta (V u) \|_{L^1} \leq \sum_{\alpha_1'' + \alpha_2'' = \alpha'' \atop |\beta| \leq 1} c_{\alpha'' \alpha_1'', \beta} \| (\partial_x^\alpha V)(\partial_x^\alpha u) \|_{L^1}$$

By Cauchy-Schwarz each summand is bounded by $\| V \|_{H^{4m+1}} \| u \|_{H^{4m+1}}$, which is sufficient to establish (6.25) in this case. If instead $|\alpha'| > |\alpha''|$, then apply Corollary 6.3.2 with $r = 1$, $p = 2$, $q = 2$ to bound (6.33) by

$$\| \partial_x^\alpha u \|_{H^{1}} \| \partial_x^\alpha (V u) \|_{H^{1}} \leq \| u \|_{H^{4m+1}} \| \partial_x^\alpha (V u) \|_{H^{1}}$$
since in this case $|\alpha'| + |\eta| + 1 \leq 2m + 9 \leq 4m + 1$. Expand the second factor to obtain

$$
\sum_{|\beta| \leq 1} \| \partial_x^{\alpha' + \beta} (Vu) \|_{L^2} \leq \sum_{\alpha'' + \alpha_2'' = \alpha' + \beta, |\beta| \leq 1} c_{\alpha'', \alpha_2'', \beta} \| (\partial_x^{\alpha''} V)(\partial_x^{\alpha_2''} u) \|_{L^2}
$$

In this case one has $|\alpha''| \leq |\alpha''| + 1 \leq m + 4 \leq 2m + 1$, so we get a bound of $\| V \|_{H^{4m + 1}} \|(u, u_t)\|_{W^{2m + 1, \infty}}$, which proves (6.25) in this case due to our assumptions on $V$.

Next we consider (6.26). By (6.31), we must estimate

$$
\| \mathcal{K}_i(D', D'')[\partial_x^{\alpha' + \eta} u][\partial_x^{\alpha''}(u^2)] \|_{L^2} \leq \sum_{|\beta| \leq 2-i} \| \partial_x^{\alpha'' + \beta}(u^2) \|_{L^2} \leq \sum_{\alpha'' + \alpha_2'' = \alpha' + \beta, |\beta| \leq 2-i} c_{\alpha'', \alpha_2'', \beta} \| (\partial_x^{\alpha''} u)(\partial_x^{\alpha_2''} u) \|_{L^2}
$$

As usual, both factors cannot have more than half the total possible number of derivatives, so by symmetry we may assume that $|\alpha''| \leq m + 4 \leq 2m + 1$. Then

$$
\| (\partial_x^{\alpha''} u)(\partial_x^{\alpha_2''} u) \|_{L^2} \leq \|(u, u_t)\|_{W^{2m+1, \infty}} \| \partial_x^{\alpha''} u \|_{L^2} \leq \|(u, u_t)\|_{W^{2m+1, \infty}} \| u \|_{H^{4m+1-i}}.
$$

So by choosing a constant depending only on $m$ and combining our estimates so far, we get a contribution of

$$
C \| u \|_{H^{4m+1-i}}^2 \|(u, u_t)\|_{W^{2m+1, \infty}}
$$

which establishes (6.26).

For (6.27), we consider the quantity

$$
\| \mathcal{K}_1(D', D'')[\partial_x^{\alpha'} (Vu)][\partial_x^{\alpha''}(Vu)] \|_{L^2} \leq \sum_{|\beta| \leq 2-i} \| \partial_x^{\alpha'' + \beta}(Vu) \|_{L^2} \leq \sum_{\alpha'' + \alpha_2'' = \alpha' + \beta, |\beta| \leq 2-i} c_{\alpha'', \alpha_2'', \beta} \| (\partial_x^{\alpha''} Vu)(\partial_x^{\alpha_2''} Vu) \|_{L^2}
$$

Again, symmetry will allow us to consider only the case where $|\alpha'| \leq |\alpha''|$ which gives $|\alpha'| \leq m + 3$. Then Corollary 6.3.2 with $r = 1, p = \infty, q = 1$ bounds (6.35)
by

\[ \| \partial_x^\alpha (Vu) \|_{W^{1,\infty}} \| \partial_x^\beta (Vu) \|_{W^{1,1}} \]

The first factor is bounded by

\[
\sum_{|\beta| \leq 1} \| \partial_x^{\alpha + \beta} (Vu) \|_{L^\infty} \leq \sum_{\alpha_1', \alpha_2' = \alpha' + \beta, |\beta| \leq 1} c_{\alpha', \alpha_1', \alpha_2', \beta} \| (\partial_x^{\alpha_1'} V)(\partial_x^{\alpha_2'} u) \|_{L^\infty}
\]

\[
\leq C \sum_{\alpha_1' + \alpha_2' = \alpha' + \beta, |\beta| \leq 1} \| \partial_x^{\alpha_1'} u \|_{L^\infty}
\]

where \( C \) depends on \( m \) and \( V \). Since \(|\alpha' + |\beta| \leq m + 4 \leq 2m + 1\), by choosing a larger constant we bound the last line by \( C \| (u, u_t) \|_{W^{2m+1, \infty}} \). To complete the bound (6.27), we estimate

\[
\| \partial_x^{\alpha''} (Vu) \|_{W^{1,1}} \leq \sum_{\alpha_1'' + \alpha_2'' = \alpha'' + \beta, |\beta| \leq 1} c_{\alpha'', \alpha_1'', \alpha_2'', \beta} \| (\partial_x^{\alpha_1''} V)(\partial_x^{\alpha_2''} u) \|_{L^1}
\]

\[
\leq C \sum_{\alpha_1'' + \alpha_2'' = \alpha'' + \beta, |\beta| \leq 1} \| \partial_x^{\alpha_1''} V \|_{L^2} \| \partial_x^{\alpha_2''} u \|_{L^2}
\]

\[
\leq C \| u \|_{H^{4m+1}}
\]

using Cauchy-Schwarz followed by our assumptions on \( V \) and the fact that \(|\alpha''| + |\beta| \leq 4m + 1\). This completes the estimate (6.27).

Next is (6.28). We will estimate

\[
\| K_1(D', D'')(\partial_x^{\alpha'} (u^2))[(\partial_x^{\alpha''} (u^2))] \|_{L^1}
\]

(6.36)

Once again there is symmetry in the factors so we may assume \(|\alpha'| \leq m + 3\). Corollary 6.3.2 with \( r = 1, p = \infty, q = 1 \) then yields a bound of

\[
\| \partial_x^{\alpha'} (u^2) \|_{W^{1,\infty}} \| \partial_x^{\alpha''} (u^2) \|_{W^{1,1}}
\]
Expansion of the first factor results in

\[
\| \partial_x^{\alpha'}(u^2) \|_{W^{1,\infty}} \leq \sum_{\alpha'_1 + \alpha'_2 = \alpha' + \beta} c_{\alpha', \alpha'_2, \beta} \| (\partial_x^{\alpha'_1} u)(\partial_x^{\alpha'_2} u) \|_{L^\infty}
\]

\[
\leq C \sum_{\alpha'_1 + \alpha'_2 = \alpha' + \beta} \| \partial_x^{\alpha'_1 + \gamma} u \|_{L^2} \| \partial_x^{\alpha'_2} u \|_{L^\infty}
\]

\[
\leq C' \| u \|_{H^{4m+1}} \| (u, u_t) \|_{W^{2m+1, \infty}}
\]

Note that we have used Sobolev’s Lemma (Theorem 2.2.1) as well as the fact that when \( m \geq 5 \), one has both \(|\alpha'| + |\beta| + |\gamma| \leq m + 6 \leq 2m + 1 \) and \(|\alpha'| + |\beta| \leq m + 4 \leq 4m + 1 \). The other factor \( \| \partial_x''(u^2) \|_{W^{1,1}} \) is bounded using the same argument as for \( \| \partial_x''(Vu) \|_{W^{1,1}} \) in the previous estimate of (6.27), producing a bound of \( C' \| u \|_{H^{4m+1}}^2 \). Here \( C' \) depends only on \( m \), and this establishes (6.28).

The penultimate bound is (6.29). Here the quantity in question is

\[
\| K_1(D', D'')[\partial_x''(V u)][\partial_x''(u^2)] \|_{L^1}
\]

(6.37)

Since the terms are no longer symmetric, we have to consider separately the cases \(|\alpha'| \leq |\alpha''| \) and \(|\alpha'| > |\alpha''| \). In the first case, we have \(|\alpha'| \leq m + 3 \) like usual and we apply Corollary 6.3.2 with \( r = 1, p = \infty, q = 1 \) to bound (6.37) by

\[
\| \partial_x^{\alpha'}(V u) \|_{W^{1,\infty}} \| \partial_x''(u^2) \|_{W^{1,1}}
\]

Note that from our work on (6.27) we have the bound \( \| (u, u_t) \|_{H^{2m+1, \infty}} \) for the first factor, and the bound of \( \| u \|_{H^{4m+1}}^2 \) for the second factor was proved in the analysis of (6.28), with implied constants depending on \( m \) and \( V \). This is sufficient to establish (6.29) if \(|\alpha'| \leq |\alpha''| \). If instead \(|\alpha'| > |\alpha''| \), then we want to apply the corollary with \( r = 1, p = 1, q = \infty \) which shows that (6.37) is bounded by

\[
\| \partial_x^{\alpha'}(V u) \|_{W^{1,1}} \| \partial_x''(u^2) \|_{W^{1,\infty}}
\]

Both of these factors have been controlled previously as well; the first during our work on (6.27) and the second in (6.28). The bound is \( \| u \|_{H^{4m+1}}^2 \| (u, u_t) \|_{W^{2m+1, \infty}} \) with an implied constant depending on \( V \) and \( m \), as required.
Finally we consider (6.30). We must estimate
\[ \| \mathcal{K}_1(D', D'')[\partial_x^\alpha (u \partial_t u)][\partial_x^{\alpha''} (\partial_t u)] \|_{L^1} \]  
(6.38)

An application of Corollary 6.3.2 with \( r = 1, \ p = q = 2 \) shows that (6.38) is controlled by
\[ \| \partial_x^\alpha (u \partial_t u) \|_{H^1} \| \partial_x^{\alpha''} \partial_t u \|_{H^1} \]

Since \( |\alpha''| + 1 \leq 2m + 7 \leq 4m \) for \( m \geq 5 \), the second factor is bounded by \( C \| u_t \|_{H^{4m}} \) for a constant depending only on \( m \). We expand and estimate the first factor as follows:
\[ \| \partial_x^\alpha (u \partial_t u) \|_{H^1} \leq \sum_{\alpha', \alpha''} c_{\alpha', \alpha'', \beta} \| (\partial_x^{\alpha'} u)(\partial_x^{\alpha''} \partial_t u) \|_{L^2} \]

For those summands with \( |\alpha'_1| \leq |\alpha'_2| \), we have \( |\alpha'_1| \leq m + 4 \) since \( |\alpha'| + |\beta| \leq 2m + 7 \) is the maximum possible number of derivatives. Since \( m + 4 \leq 2m + 1 \) we could then choose a constant \( C \) depending only on \( m \) such that the above sum is bounded by \( C \| u_t \|_{H^{4m}} \| u \|_{W^{2m+1, \infty}} \), since no matter what we have \( |\alpha'_2| \leq |\alpha'| + |\beta| \leq 2m + 7 \leq 4m \). On the other hand, if \( |\alpha'_2| \leq m + 4 \), we get a bound of \( C \| u \|_{H^{4m+1}} \| u_t \|_{W^{2m+1, \infty}} \) by the same reasoning. This proves (6.30) and the proof of Proposition 6.3.3 is complete.

Now we will prove the bound (6.21) for \((U, U_t)\). First consider \( U \), recalling that \( U = \sum_{i=0,1} \mathcal{K}_i(D', D'')[\partial_i^\alpha u][\partial_i^\alpha u] \). Thus for \( |\alpha| \leq 2m + 1 \), we have by the Leibniz rule
\[ \| U \|_{W^{2m+1, \infty}} = \sum_{|\alpha| \leq 2m + 1} \| \partial^\alpha u \|_{L^\infty} \leq \sum_{\alpha', \alpha''} c_{\alpha, \alpha''} \| \mathcal{K}_i(D', D'')[\partial_x^{\alpha'} \partial_i^\alpha u][\partial_x^{\alpha''} \partial_i^\alpha u] \|_{L^\infty} \]

(6.39)
The coefficients in the above sum are again bounded when \( m \) is fixed. Thus, it suffices to bound each summand using Corollary 6.3.2 with \( p = q = r = \infty \), which yields
\[ \| \mathcal{K}_i(D', D'')[\partial_x^{\alpha'} \partial_i^\alpha u][\partial_x^{\alpha''} \partial_i^\alpha u] \|_{L^\infty} \lesssim \| \partial_x^{\alpha'} \partial_i^\alpha u \|_{W^{2-1, \infty}} \| \partial_x^{\alpha''} \partial_i^\alpha u \|_{W^{2-1, \infty}} \]
Since the factors are symmetric we may assume that $|\alpha'| \leq |\alpha''|$, which then implies that $|\alpha'| \leq m + 1$. But then the first factor is controlled by $\| (u, u_t) \|_{W^{2m+1, \infty}}$ since $|\alpha'| + 2 - i \leq m + 3 \leq 2m + 1$ when $m \geq 5$. The second factor will be handled using Sobolev embedding:

$$\| \partial_x^{\alpha''} \partial_t^i u \|_{W^{2-i, \infty}} = \sum_{|\beta| \leq 2-i} \| \partial_x^{\alpha''+\beta} \partial_t^i u \|_{L^\infty} \lesssim \sum_{|\beta| \leq 2-i, |\eta| \leq 2} \| \partial_t^{\alpha''+\beta+\eta} \partial_t^i u \|_{L^2}$$

We have $|\alpha''| + |\beta| + |\eta| \leq 2m + 5 - i \leq 4m + 1 - i$, so by enlarging the constant from the Sobolev embedding we obtain

$$\| \partial_x^{\alpha''} \partial_t^i u \|_{W^{2-i, \infty}} \leq C \| \partial_t^i u \|_{H^{4m+1-i}}$$

Combining this with the estimate of the other factor and returning to the original sum (6.39) gives the bound

$$\| U \|_{W^{2m+1, \infty}} \leq C \| (u, u_t) \|_{H^{4m+1, \infty}} \| (u, u_t) \|_{W^{2m+1, \infty}}$$

which establishes (6.21) for $U$.

To get an expression for $U_t$ we apply the Leibniz rule to $K_i(D', D'')[\partial_t^i u][\partial_t^i u]$ and find that

$$U_t = K_0(D', D'')[u][u_t] + K_0(D', D'')[u_t][u] + K_1(D', D'')[u, u_t] + K_1(D', D'')[u_t][u]$$

(6.40)

We can estimate the first two terms in $W^{2m+1, \infty}$ as follows. Beginning with the expansion

$$\| K_0(D', D'')[u][u_t] \|_{W^{2m+1, \infty}} \leq \sum_{\alpha'+\alpha''=\alpha, |\alpha| \leq 2m+1} c_{\alpha, \alpha''} \| K_0(D', D'')[\partial_x^{\alpha'} u][\partial_x^{\alpha''} \partial_t u] \|_{L^\infty}$$

we apply Corollary 6.3.2 with $r = \infty$ and $p = q = \infty$ to get

$$\| K_0(D', D'')[\partial_x^{\alpha'} u][\partial_x^{\alpha''} \partial_t u] \|_{L^\infty} \lesssim \| \partial_x^{\alpha'} u \|_{W^{2, \infty}} \| \partial_x^{\alpha''} \partial_t u \|_{W^{2, \infty}}$$

At this point, we proceed exactly as in the estimate for $U$, obtaining the bound

$$\| (u, u_t) \|_{H^{4m+1, \infty}} \| (u, u_t) \|_{W^{2m+1, \infty}}$$

for the first two terms in (6.40).
In the second set of terms in (6.40), we must make the substitution $u_{tt} = \Delta u - u - Vu + u^2$. The model term is then

$$K_1(D', D'')[u_{tt}][u_t] = K_1(D', D'')[\Delta u][u_t] - K_1(D', D'')[u][u_t] - K_1(D', D'')[V u][u_t] + K_1(D', D'')[u^2][u_t]$$

We can handle the first two summands at once by proving that for $|\eta| \leq 2$ one has

$$\|K_1(D', D'')[\partial_x^\eta u][\partial_t u]\|_{W^{2m+1, \infty}} \lesssim \|(u, u_t)\|_{H^{4m+1} \times H^{4m}} \|(u, u_t)\|_{W^{2m+1, \infty}} \quad (6.41)$$

Using the same reductions we have used thus far leads us to estimate the quantity

$$\|K_1(D', D'')[\partial_x^{\alpha'} \partial_t u][\partial_x^{\alpha''} \partial_t u]\|_{L^\infty}$$

where $|\alpha'| + |\alpha''| \leq 2m + 1$. Using Corollary 6.3.2 with $r = p = q = \infty$ bounds this by

$$\|\partial_x^{\alpha'} \partial_t u\|_{W^{1, \infty}} \|\partial_x^{\alpha''} \partial_t u\|_{W^{1, \infty}}$$

As before, we will leave the factor with fewer derivatives in $L^\infty$ and use Sobolev embedding in the other (adding 2 derivatives to get into $L^2$). This works since if $|\alpha'| \leq m + 1$, then $|\alpha'| + |\eta| + 1 \leq 2m + 1$ and also $|\alpha''| + 2 \leq 2m + 3 \leq 4m$, giving a bound of $\|u_t\|_{H^{4m}} \|(u, u_t)\|_{W^{2m+1, \infty}}$. On the other hand, if $|\alpha''| \leq m + 1$ then $|\alpha'| + |\eta| + 2 \leq 4m + 1$ and the bound is $\|u\|_{H^{4m+1}} \|(u, u_t)\|_{W^{2m+1, \infty}}$. These combine to give (6.41).

All that remains now is prove the two estimates

$$\|K_1(D', D'')[V u][\partial_t u]\|_{W^{2m+1, \infty}} \lesssim \|(u, u_t)\|_{H^{4m+1} \times H^{4m}} \|(u, u_t)\|_{W^{2m+1, \infty}} \quad (6.42)$$

$$\|K_1(D', D'')[u^2][\partial_t u]\|_{W^{2m+1, \infty}} \lesssim \|(u, u_t)\|_{H^{4m+1} \times H^{4m}}^2 \|(u, u_t)\|_{W^{2m+1, \infty}}^2 \quad (6.43)$$

To prove (6.42), we use the usual setup and apply Corollary 6.3.2 with $r = p = q = \infty$ to obtain

$$\|K_1(D', D'')[\partial_x^{\alpha'} (V u)][\partial_x^{\alpha''} \partial_t u]\|_{L^\infty} \lesssim \|\partial_x^{\alpha'} (V u)\|_{W^{1, \infty}} \|\partial_x^{\alpha''} \partial_t u\|_{W^{1, \infty}}$$

The strategy is again to use Sobolev embedding in the factor with more derivatives and leave the other in $L^\infty$, which by the same reasoning as before produces the bound in (6.42).
For (6.43), we arrive in the usual way at the reduced quantity

$$\| \mathcal{K}_1(D', D'')[\partial^\alpha_x (u^2)][\partial^\alpha_x \partial_t u]\|_{L^\infty} \lesssim \| \partial^\alpha_x (u^2) \|_{W^{1, \infty}} \| \partial^\alpha_x \partial_t u \|_{W^{1, \infty}}$$

$$\leq \sum_{\alpha'_1 + \alpha'_2 = \alpha + \beta, |\beta| \leq 1} c_{\alpha'_1, \alpha'_2, \beta} \| (\partial^\alpha_x u)(\partial^\alpha_x u) \|_{L^\infty} \| \partial^\alpha_x \partial_t u \|_{W^{1, \infty}}$$

This time, if $|\alpha'| \leq |\alpha''|$, we will leave $\partial^\alpha_x u$ in $L^\infty$ and use Sobolev embedding on both $\partial^\alpha_x u$ and in the factor $\| \partial^\alpha_x \partial_t u \|_{W^{1, \infty}}$ to produce the bound (6.43). If instead $|\alpha''| \leq |\alpha'|$, one can leave the factor $\| \partial^\alpha_x \partial_t u \|_{W^{1, \infty}}$ in $L^\infty$ and use Sobolev embedding on both $\partial^\alpha_x u$ and $\partial^\alpha_x u$ to get (6.43).

Finally, we must prove (6.22). Using our previous expansion for $U_1$, we find that

$$U(0) = \mathcal{K}_0(D', D'')[u_0][u_0] + \mathcal{K}_1(D', D'')[u_1][u_1]$$

and

$$U_1(0) = \mathcal{K}_0(D', D'')[u_0][u_1] + \mathcal{K}_0(D', D'')[u_1][u_0] +$$

$$+ \mathcal{K}_1(D', D'')[u_1][\Delta u_0 - u_0 - V u_0 + u_0^2] +$$

$$+ \mathcal{K}_1(D', D'')[\Delta u_0 - u_0 - V u_0 + u_0^2][u_1]$$

For $U(0)$, the analysis is the same as that of $U$ earlier except that we are now applying Corollary 6.3.2 with $r = 1, p = q = 2$ to obtain

$$\| U(0) \|_{W^{2m+7,1}} \lesssim \| u_0 \|_{H^{4m+1}}^2 + \| u_1 \|_{H^{4m}}^2 \lesssim \| (u_0, u_1) \|_{H^{4m+1} \times H^{4m}}^2$$

Note that in this case, the maximum number of derivatives that can fall on any term is $2m + 9$, which is less than $4m$ since $m \geq 5$.

The analysis for $U_1(0)$ is also similar to the previous analysis of $U_1$: the contribution of $\| \mathcal{K}_1(D', D'')[u_1][\Delta u_0 - u_0 - V u_0] \|_{W^{2m+7,1}}$ is no more than $\| u_1 \|_{H^{4m+1}} \| u_0 \|_{H^{4m+1}}$ (using Corollary 6.3.2 with $r = 1, p = q = 2$ again), and the contribution of $\| \mathcal{K}_1(D', D'')[u_1][u_0^2] \|_{W^{2m+7,1}}$ is handled in two cases. If the most derivatives fall on $u_1$, we use $r = 2, p = 2, q = 2$ in Corollary 6.3.2 followed by Sobolev embedding to get a bound of

$$\| u_1 \|_{H^{4m}} \| u_0 \|_{H^{4m+1}}^2 \lesssim \| (u_0, u_1) \|_{H^{4m+1} \times H^{4m}}^3$$
If more derivatives fall on $u_0^2$, then we use Corollary 6.3.2 with $r = 1$, $p = \infty$, and $q = 1$ followed by Sobolev embedding and Cauchy-Schwarz to get the same result. This completes the proof of Theorem 5.1.2.
Appendix A

The Fourier Transform and Bilinear Operators

This appendix records some basic results concerning the Fourier transform as well as the bilinear operators that we used in the material concerning normal forms.

A.1 The Fourier Transform

Recall that the Fourier transform of a function is defined by

**Definition A.1.1** (Fourier Transform). If \( f \in L^1(\mathbb{R}^n) \), then the Fourier transform of \( f \) is the function \( \hat{f}(\xi) \) is given by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx
\]

The transform is defined for any \( L^1 \) function, but the resulting function will not necessarily remain in \( L^1 \). However, standard theory shows that the transform is an isomorphism when considered either as a map on \( \mathcal{S}(\mathbb{R}^n) \) or \( L^2(\mathbb{R}^n) \).

A.2 Bilinear Operators

Now we can define the key bilinear differential operators from the normal forms transforms in Chapter 6. We also prove the key properties of these operators
that were used.

**Definition A.2.1.** Let $D' = -i\partial'$ and $D'' = -i\partial''$, and for $u, v \in S(\mathbb{R}^3)$, define $\mathcal{B}(D', D'')[u][v]$ to be the function whose Fourier transform evaluated at $\xi$ is

$$(2\pi)^{-3} \int B(\xi - \eta, \eta) \hat{u}(\xi - \eta) \hat{v}(\eta) d\eta$$

To see what is going on here, notice that $D'$ acts by differentiating the first factor in $[u][v]$, whereas $D''$ differentiates the second. We will illustrate the definition with an example. For instance, let $B(\xi, \eta) = -|\xi|^2 + \langle \xi, \eta \rangle$. This expression will appear in later chapters so it is useful to compute it here. I claim that $\mathcal{B}(D', D'')[u][v] = v\Delta u - \nabla u \cdot \nabla v$. By our definition, $(-|D'|^2)[u][v]$ is the function whose Fourier transform at $\xi$ is equal to

$$-(2\pi)^{-3} \int |\xi - \eta|^2 \hat{u}(\xi - \eta) \hat{v}(\eta) d\eta$$

Then we simply compute the Fourier transform of $v\Delta u$:

$$\mathcal{F}[v\Delta u](\xi) = (2\pi)^{-3}(\hat{v} \ast \Delta \hat{u})(\xi) = -(2\pi)^{-3} \int \hat{v}(\xi - \eta) |\eta|^2 \hat{u}(\eta) d\eta$$

$$= -(2\pi)^{-3} \int |\xi - \eta|^2 \hat{u}(\xi - \eta) \hat{v}(\eta) d\eta$$

where the last line follows by the change of variables $\eta \mapsto \xi - \eta$. Since the Fourier transform is an isomorphism of the Schwartz class onto itself, we conclude that $(-|D'|^2)[u][v] = v\Delta u$. Compare this with $(-|D''|^2)[u][v] = u\Delta v$. Similarly, the Fourier transform of $(\langle D', D'' \rangle)[u][v]$ is

$$(2\pi)^{-3} \int \eta \cdot (\xi - \eta) \hat{u}(\xi - \eta) \hat{v}(\eta) d\eta$$

We will use the notation $u_j := \partial_{x_j} u$. Then the Fourier transform of $\nabla u \cdot \nabla v$ is

$$\sum_j \mathcal{F}[u_j v_j] = (2\pi)^{-3} \sum_j \int \hat{u}_j(\xi - \eta) \hat{v}_j(\eta) d\eta$$

$$= -(2\pi)^{-3} \sum_j \eta_j(\xi_j - \eta_j) \hat{u}(\xi - \eta) \hat{v}(\eta) d\eta$$

$$= -(2\pi)^{-3} \int \eta \cdot (\xi - \eta) \hat{u}(\xi - \eta) \hat{v}(\eta) d\eta$$

By invoking Fourier inversion again, we conclude $\mathcal{B}(D', D'')[u][v] = v\Delta u - \nabla u \cdot \nabla v$.

These operators satisfy a Leibniz rule; indeed we can immediately establish
Proposition A.2.2.

\[ \partial_{x_j} \mathcal{B}(D', D'')[u][v] = \mathcal{B}(D', D'')[\partial_{x_j} u][v] + \mathcal{B}(D', D'')[u][\partial_{x_j} v] \]

Proof. This identity follows immediately from taking the Fourier transform of both sides, rearranging expressions, and then invoking Fourier inversion. \qed

This implies the general Leibniz rule:

Corollary A.2.3 (Leibniz Rule). One has the identity

\[ \partial^\alpha \mathcal{B}(D', D'')[u][v] = \sum_{\alpha' + \alpha'' = \alpha} c_{\alpha, \alpha'} \mathcal{B}(D', D'')[\partial^{\alpha'} u][\partial^{\alpha''} v] \]

where \( c_{\alpha, \alpha''} = \binom{\alpha}{\alpha''} \).

It will be nice to have an analogy to the usual formula \( \Delta(fg) = f\Delta g + g\Delta f + 2\nabla f \cdot \nabla g \). It turns out we can apply Proposition A.2.2 twice and sum to obtain

Corollary A.2.4. For the operator \( \mathcal{B}(D', D'')[u][v] \), there holds the identity

\[ \Delta \mathcal{B}(D', D'')[u][v] = (-|D'|^2 - |D''|^2 - 2\langle D', D'' \rangle) \mathcal{B}(D', D'')[u][v] \]

A.2.1 Integral Representation

Although our initial definition of \( \mathcal{B}(D', D'')[u][v] \) allowed us to derive basic properties, it turns out not to be the most useful one for proving estimates. We require certain product-type estimates on these bilinear operators, and to prove them it will be indispensable to have a tractable integral representation.

To motivate the representation we will derive, take an arbitrary function \( P(\xi, \eta) \) on \( \mathbb{R}^{3+3} \) that is bounded and continuous. Suppose the inverse Fourier transform \( P(x, y) \) is in \( L^1(\mathbb{R}^{3+3}) \). Then \( P \) is a tempered distribution with action on \( w \in S(\mathbb{R}^6) \) given by \( \int \int P(y, z)w(y, z)dydz \). Now given \( u, v \in S(\mathbb{R}^3) \), fix \( x \in \mathbb{R}^3 \). Then \( w_x(y, z) := u(x - y)v(x - z) \) is an element of \( S(\mathbb{R}^6) \) and one can consider the function

\[ x \mapsto \int \int P(y, z)w_x(y, z)dydz = \int \int P(y, z)u(x - y)v(x - z)dydz \]

In order to connect this to Definition A.2.1, we prove
Proposition A.2.5. If $P \in L^1(\mathbb{R}^{3+3})$, then for $u, v \in S(\mathbb{R}^3)$ one has

$$
P(D', D'')[u][v](x) = \int\int P(y, z)u(x - y)v(x - z) \, dy \, dz \quad (A.1)
$$

This identity is stated in Hörmander, but many of the details are left out of the proof so we include them here.

**Proof.** Since $P \in L^1(\mathbb{R}^{3+3})$, it holds that $\hat{P} = P$ is continuous and bounded. First we will assume further that $\hat{P} \in S(\mathbb{R}^{3+3})$. Then a limiting argument will be used to recover the full result.

We begin the computation with the Fourier transform of the left side, which by definition is equal to

$$
(2\pi)^{-3} \int \mathcal{P}(\xi - \eta, \eta) \hat{u}(\xi - \eta) \hat{v}(\eta) \, d\eta \quad (A.2)
$$

Since $\mathcal{P}$ is a Schwartz function, the Fubini theorem implies

$$
\mathcal{P}(\xi - \eta, \eta) = \int \int e^{-iy \cdot (\xi - \eta)}e^{-iz \cdot \eta}P(y, z) \, dy \, dz
$$

Substitution of this and the definition of $\hat{u}(\xi - \eta)$ into (A.2), along with more applications of Fubini, gives

$$
(2\pi)^{-3} \int \int \int \int e^{-iy \cdot \xi}e^{-is \cdot \xi}e^{i(y - z + s) \cdot \eta} \hat{v}(\eta)P(y, z)u(s) \, d\eta \, ds \, dz \, dy
$$

Integrating in $\eta$ and using Fourier inversion yields

$$
\int \int P(y, z) \int e^{-iy \cdot \xi}e^{-is \cdot \xi}u(s)v(y - z + s) \, ds \, dz \, dy
$$

Changing variables $s \mapsto s - y$ in the inner integral and then changing order of integration again gives the equivalent expression

$$
\int e^{-is \cdot \xi} \int \int P(y, z)u(s - y)v(s - z) \, dy \, dz \, ds
$$

which is the Fourier transform of the right hand side of (A.1).

In the general case, choose $\phi_n \in S(\mathbb{R}^{3+3})$ with $\phi_n \to P$ in $L^1(\mathbb{R}^{3+3})$. To make the notation easier to read, I will first introduce $F_n(x) = \hat{\phi}_n(D', D'')[u][v](x)$.
and $F(x) = \mathcal{P}(D', D'')[u][v](x)$. Beginning with the right side of (A.1), we get
\[
\int\int P(y, z)u(x - y)v(x - z) \, dy \, dz = \lim_n \int\int \phi_n(y, z)u(x - y)v(x - z) \, dy \, dz
= \lim_n \hat{\phi}_n(D', D'')[u][v](x)
= \lim_n F_n(x)
\]

Thus the problem is reduced to showing that $F_n(x) \to F(x)$. However, since $\phi_n \to P$ in $L^1$, it follows that $\hat{\phi}_n \to \mathcal{P}$ uniformly, so we estimate
\[
\int |\hat{F}_n(\xi) - \hat{F}(\xi)| \, d\xi \leq (2\pi)^{-3} \int\int |\hat{\phi}_n(\xi - \eta) - \mathcal{P}(\xi - \eta)||\hat{u}(\xi - \eta)||\hat{v}(\eta)| \, d\eta \, d\xi
\]
The last expression tends to 0 since $u, v \in \mathcal{S}$ and $\hat{\phi}_n \to \mathcal{P}$ uniformly. Finally, we can estimate
\[
|F_n(x) - F(x)| = \left|(2\pi)^{-3} \int e^{ix\cdot\xi}[\hat{F}_n(\xi) - \hat{F}(\xi)] \, d\xi\right| \lesssim \|\hat{F}_n - \hat{F}\|_{L^1} \to 0
\]
so the proof is complete.

This representation of the operators $\mathcal{B}(D', D'')[u][v]$ will be sufficient to prove the required product estimates.
Appendix B

Boundedness of the Normal Forms Transform

This appendix contains the proof of Theorem 6.3.1. The result is built up through a series of smaller parts. The general outline for this proof can be found in Hörmander [1], but for the sake of completeness and due to the fact that many of the details are left out in [1], I record them here.

First, a product estimate of the required type is proved for functions satisfying certain hypotheses, and the results following it are dedicated to showing that $\mathcal{K}$ satisfies these hypotheses.

B.1 Preliminary Results

The first lemma will use the integral representation developed in Section A for the operators $\mathcal{B}(D', D'')[u][v]$ to prove an initial product estimate.

Lemma B.1.1. Suppose $P \in L^1(\mathbb{R}^{3+3})$ (which implies $\mathcal{P} = \hat{P}$ is continuous and bounded). If $p, q \in \{1, 2, \infty\}$ are such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and $r$ satisfies $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then

$$\| \mathcal{P}(D', D'')[u][v] \|_{L^r} \leq \| P \|_{L^1} \| u \|_{L^p} \| v \|_{L^q}$$  \hspace{1cm} (B.1)

for any $u, v \in \mathcal{S}(\mathbb{R}^3)$. 
Proof. First, dispense with the case \((p,q) = (\infty, \infty)\). In this case \(r = \infty\) also. Then by Proposition A.1, one can estimate
\[
\left| \int \int P(y, z) u(x - y) v(x - z) \, dy \, dz \right| \leq \| u \|_{L^\infty} \| v \|_{L^\infty} \int \int |P(y, z)| \, dy \, dz
\]
which certainly implies \(\| P(D', D'') [u] [v] \|_{L^1} \leq \| P \|_{L^1} \| u \|_{L^\infty} \| v \|_{L^\infty} \).

Now if \((p, q) = (1, \infty)\) or \((p, q) = (\infty, 1)\), then \(r = 1\). It suffices to prove just the \((p, q) = (1, \infty)\) case since \(u\) and \(v\) can be swapped to obtain the other. So if \(p = 1\), \(q = \infty\), one can see that since \(u,v \in S(\mathbb{R}^3)\) and \(P \in L^1(\mathbb{R}^{3+3})\), it holds that \(P(\cdot, \cdot)u(x - \cdot)v(x - \cdot) \in L^1(\mathbb{R}^{3+3})\). This implies that
\[
\left| \int \int P(y, z) u(x - y) v(x - z) \, dy \, dz \right| \leq \int \int |P(y, z)| |u(x - y)||v(x - z)| \, dy \, dz \tag{B.2}
\]
Integrating over \(x \in \mathbb{R}^3\) and then using this bound followed by Tonelli’s theorem produces the estimate
\[
\| P(D', D'') [u] [v] \|_{L^1} \leq \| P \|_{L^1} \| u \|_{L^\infty} \| v \|_{L^\infty},
\]
The right side of the last inequality is equal to \(\|P\|_{L^1} \| u \|_{L^1} \| v \|_{L^\infty}\), which is what was to be shown.

A slight modification of the argument in the previous paragraph also handles the \((p, q) = (2, 2), r = 1\) case: after integrating the right side of (B.2) and applying Tonelli’s theorem, one has
\[
\int \int |P(y, z)| \int |u(x - y)||v(x - z)| \, dx \, dy \, dz
\]
which, according to the usual Hölder estimate, is controlled by \(\| P \|_{L^1} \| u \|_2 \| v \|_{L^2}\).

Thus, the only cases that remain are \((p, q) = (2, \infty), (\infty, 2)\). Again it is only necessary to prove one of them. In fact either one can be obtained by interpolating between the results I already have. To be precise about it, suppose \(p = \infty\) and \(q = 2\) so that \(r = 2\). I will interpolate between the two estimates
\[
\| P(D', D'') [u] [v] \|_{L^1} \leq \| P \|_{L^1} \| u \|_{L^\infty} \| v \|_{L^1}, \tag{B.3}
\]

To do this, choose \(u,v \in S(\mathbb{R}^3)\). Define the map \(T^u : L^1 + L^\infty \to L^1 + L^\infty\) via \(T^u(v) = P(D', D'')[u][v]\). Then by the two estimates in (B.3), we
have \( \| T^u(v) \|_{L^1} \leq M_1 \| v \|_{L^1} \) and \( \| T^u(v) \|_{L^\infty} \leq M_\infty \| v \|_{L^\infty} \), where \( M_1 = M_\infty = \| P \|_{L^1} \| u \|_{L^\infty} \). Then apply the Riesz-Thorin interpolation theorem (the version in [22] is sufficient) to obtain

\[
\| \mathcal{P}(D', D'') [u][v] \|_{L^2} \leq M_1^{1 - \frac{1}{2}} M_\infty^{\frac{1}{2}} \| v \|_{L^2} = \| P \|_{L^1} \| u \|_{L^\infty} \| v \|_{L^2}
\]

\( \square \)

Remark B.1.2. The Minkowski integral inequality can be used to directly estimate the integral in the last case instead of using interpolation. However, it is instructive to point out the way the interpolation works since it can be used to prove this estimate for the full range of exponents \( p, q, r \) satisfying \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \). This lemma covers only the more common sets of exponents one would expect to encounter in basic nonlinear problems; in particular every combination we have used in this paper is covered here.

The next lemma provides a tractable sufficient condition for the inverse Fourier transform \( P(x, y) \) of a given \( \mathcal{P}(\xi, \eta) \) to lie in \( L^1(\mathbb{R}^{3+3}) \).

Lemma B.1.3. Suppose \( \hat{P} = \mathcal{P} \). Then if \( \partial_\xi^\alpha \partial_\eta^\beta \mathcal{P}(\xi, \eta) \in L^2(\mathbb{R}^{3+3}) \) for all \( \alpha, \beta \) with \( |\alpha|, |\beta| \leq 2 \), one has \( P \in L^1(\mathbb{R}^{3+3}) \).

Proof. Let \( \hat{P} = \mathcal{P} \). By Parseval we have \( \mathcal{P} \in L^2(\mathbb{R}^{3+3}) \) and \( x^\alpha y^\beta \mathcal{P} \in L^2(\mathbb{R}^{3+3}) \) for any \( |\alpha|, |\beta| \leq 2 \). By multiplying and dividing by \( (1 + |x|^2)(1 + |y|^2) \) and applying the Cauchy-Schwarz inequality, we can estimate

\[
\int\int |P(x, y)| \, dx \, dy \leq C \int\int (1 + |x|^2)^2 (1 + |y|^2)^2 |P(x, y)|^2 \, dx \, dy
\]

\[
\leq C' \sum_{|\alpha|, |\beta| \leq 2} \int\int |x^\alpha y^\beta P(x, y)|^2 \, dx \, dy < \infty
\]

where \( C = \int\int (1 + |x|^2)^{-2} (1 + |y|^2)^{-2} \, dx \, dy < \infty \). \( \square \)

The previous two lemmas suggest the investigation of whether \( \mathcal{K} \) (as defined in 6.17) and its derivatives are in \( L^2(\mathbb{R}^{3+3}) \). Given the loss of differentiability in (6.18), one might expect to have to weight \( \mathcal{K} \) to get it into \( L^2 \). Lemma B.1.6 confirms this, but before we proceed, we will quickly prove two technical results which will serve to greatly simplify the integration involved.
Proposition B.1.4. With $K$ defined as in 6.17, one has the upper bounds

\[
K(\xi, \eta) \leq \left[ |\xi|^2 |\eta|^2 (1 - (\xi, \eta)^2) + |\xi|^2 + |\eta|^2 + 1 \right]^{-1} \tag{B.4}
\]

\[
K(\xi, \eta) \leq (4 \sqrt{|\xi|^2 + |\eta|^2})^{-1} \tag{B.5}
\]

for any $\xi, \eta \in \mathbb{R}^3$.

Proof. I will prove instead the corresponding inequalities for the reciprocals. To simplify notation, let $A = |\xi||\eta|$, $B = (\xi, \eta)$, and $C = |\xi|^2 + |\eta|^2$. Then we have $-A \leq B \leq A$ by the Cauchy-Schwarz inequality, and by Young’s inequality we also have $2A \leq C$.

Starting with the bound (B.4), note that it is equivalent to

\[
4(A^2 - B^2 + C + B) + 3 \geq A^2(1 - B^2) + C + 1
\]

so by rearranging terms it will suffice to show that

\[
3A^2 + 4B + A^2B^2 - 4B^2 + 3C \geq 0 \tag{B.6}
\]

Two cases seem to present themselves: when $A \geq 1$, it is natural to group the left side of (B.6) as

\[
3(A^2 - B^2) + 4(A + B) + B^2(A^2 - 1) + 3C
\]

which is nonnegative by the inequalities mentioned above. Otherwise $0 \leq A < 1$, in which case $|B| < 1$ is forced as well. In particular, $B^2 \leq |B|$ is true. If $B < 0$, one can see that

\[
3A^2 + 4B + A^2B^2 - 4B^2 + 3C \geq 3(A^2 - B^2) + 5B + 3C
\]

But $3C \geq 6A$ and $5B + 6A \geq 0$ so this case is complete. On the other hand if $B \geq 0$ then $-4B^2 \geq -4B$ which immediately makes the left side of the above inequality nonnegative and completes the proof of (B.4).

For the bound (B.5), first note that $K(\xi, \eta) \leq \left[(2|\xi|^2 + 2|\eta|^2) + 3\right]^{-1}$. This is equivalent to the inequality

\[
4A^2 - 4B^2 + 4B + 4C + 3 \geq 2C + 3
\]
which rearranges to
\[ 4(A^2 - B^2) + 4B + 2C \geq 0 \]
This last inequality is true since \( 2C \geq 4A \) and \( 4A + 4B \geq 0 \). To deduce (B.5), all that remains is then to show that \( 2(|\xi|^2 + |\eta|^2) + 3 \geq 4\sqrt{|\xi|^2 + |\eta|^2} \). This, however, is equivalent to just \( 2C + 3 \geq 4\sqrt{C} \) which is clear (just consider the parabola \( f(x) = 2x^2 - 4x + 3 \)).

The preceding bound will make it much easier to estimate the \( L^2 \) norm of \( K \). Additionally, we can make use of the following useful integration formula from Hörmander [1].

**Proposition B.1.5.** Suppose \( h : \mathbb{R}^3 \to \mathbb{R} \) is nonnegative and continuous. Then
\[
\int \int \int_{\mathbb{R}^6} h(|\xi|^2, \langle \xi, \eta \rangle, |\eta|^2) \, d\xi \, d\eta = C \int_0^\infty \int_0^\infty \int_{-1}^1 h(x^2, xyz, y^2) x^2 y^2 \, dz \, dy \, dx \tag{B.7}
\]
Proof. We will begin with the left side. Fix \( \eta \) and consider the inner integral with respect to \( \xi \). Note that since the integrand is a function of only \(|\xi|, \langle \xi, \eta \rangle, |\eta|\) (and is thus unchanged under the transformation \( (\xi, \eta) \mapsto (-\xi, -\eta) \)), this integral depends only on \(|\eta|\). It is no loss of generality to assume that \( \eta = (|\eta|, 0, 0) \), since otherwise we could apply a rotational change of variables (with Jacobian determinant equal to 1). So first let \( y = |\eta| \). Then \( \langle \xi, \eta \rangle = \xi_1 y \) and we can think of \(|\xi|^2 \) as \( \xi_1^2 + (\xi_2^2 + \xi_3^2) \). Then using the formula for integration in polar coordinates (see, for example, Corollary 2.51 in [22]) the left side of (B.7) is equal to
\[
C' \int_0^\infty \int \int_{\mathbb{R}^2} h(|\xi|^2, \xi_1 y, y^2) y^2 \, d\xi_1 \, dy \, dx = C' \int_0^\infty \int_{\mathbb{R}^2} h(|\xi|^2, \xi_1 y, y^2) y^2 \, d\xi_1 \, dy \, dx \tag{B.7}
\]
In the inner integral, use polar coordinates in \( \mathbb{R}^2 \) given by \( r = |\xi| \) to obtain
\[
2\pi C' \int_0^\infty \int_0^\infty \int_{\mathbb{R}} h(\xi_1^2 + r^2, \xi_1 y, y^2) \, r y^2 \, d\xi_1 \, dr \, dy \tag{B.8}
\]
The clear choice of \( x \) is \( x = \sqrt{\xi_1^2 + r^2} \). Then to ensure that we have \( \xi_1 y = xyz \), let \( z = \xi_1 / \sqrt{\xi_1^2 + r^2} \). Note that then \( z \) ranges between \(-1\) and \( 1 \). The Jacobian of this transformation is given by the reciprocal of
\[
\begin{vmatrix}
\xi_1 x^{-1} & r x^{-1} \\
r^2 x^{-3} & -\xi_1 r x^{-3}
\end{vmatrix} = | -r \xi_1^2 x^{-4} - r^3 x^{-4} | = r x^{-2}
\]
Inserting this into (B.8) yields
\[ 2\pi C' \int_0^\infty \int_0^\infty \int_{-1}^1 h(x^2, xyz, y^2) ry^2 r^{-1} x^2 \, dz \, dx \, dy \]
which is precisely (B.7) with \( C = 2\pi C' \).

The strategy to estimate \( \mathcal{K} \) in \( L^2 \) is then to use Propositions B.1.4 and B.1.5 to instead study a very concrete triple integral that can be estimated directly.

**Lemma B.1.6.** Define
\[ \mathcal{P}(\xi, \eta) = \langle \xi \rangle^{-1} \langle \eta \rangle^{-1} \mathcal{K}(\xi, \eta) \]
Then the functions \( \partial_\xi^\alpha \partial_\eta^\beta \mathcal{P}(\xi, \eta) \) are in \( L^2(\mathbb{R}^{3+3}) \) for \( |\alpha|, |\beta| \leq 2 \).

**Proof.** By Proposition B.1.4, we have the bound
\[ \mathcal{P}(\xi, \eta) \leq h(|\xi|^2, \langle \xi \rangle, |\eta|^2) := \langle \xi \rangle^{-1} \langle \eta \rangle^{-1} (|\xi|^2 |\eta|^2 (1 - \langle \xi \rangle, \eta)^2) + |\xi|^2 + |\eta|^2 + 1)^{-1} \]
Then \( h^2 \geq \mathcal{P}^2 \), so it is enough to estimate
\[
\int_0^\infty \int_0^\infty \int_{-1}^1 \frac{x^2 y^2}{(x^2 (y^2(1-z^2) + x^2 + y^2 + 1))^2} \, dz \, dy \, dx \tag{B.9}
\]
We will split up the integrals in \( x \) and \( y \) as follows: in the region where \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \), there is nothing to prove because the integrand is bounded there. The remaining region will be divided up into the two subregions \( R_1 = \{(x, y) \mid x \leq y, y \geq 1\} \) and \( R_2 = \{(x, y) \mid y \leq x, x \geq 1\} \). By symmetry of the integrand, it will suffice to show that the integral over region \( R_1 \) is finite. In the entire region \( R_1 \), one can estimate the integrand as follows:
\[
\frac{x^2 y^2}{(x^2 (y^2(1-z^2) + x^2 + y^2 + 1))^2} \leq \frac{x^2 y^2}{y^2(1+x^2)(x^2 (1-z^2) + y^2)^2} = \frac{x^2 y^2}{y^2(1+x^2)(x^2 (1-z^2) + 1)^2} = \frac{x^2}{y^2(1+x^2)(x^2 (1-z^2) + 1)^2}
\]
Denoting the final expression by \( f(x, y, z) \), we thus have to control the two integrals
\[
\int_0^1 \int_1^\infty \int_{-1}^1 f(x, y, z) \, dz \, dy \, dx + \int_1^\infty \int_x^\infty \int_{-1}^1 f(x, y, z) \, dz \, dy \, dx \tag{B.10}
\]
The first integral in (B.10) converges since for $0 \leq x \leq 1$, we have $f(x, y, z) \leq y^{-4}$.

For the second integral, first perform the integration in $y$ to obtain
\[
\frac{1}{3} \int_{-1}^{1} \int_{1}^{\infty} \frac{1}{x(1 + x^2)(x^2(1 - z^2) + 1)^2} \, dx \, dz \lesssim \int_{-1}^{1} \int_{1}^{\infty} \frac{1}{x^3(x^2(1 - z^2) + 1)^2} \, dx \, dz
\]

When $|z| < \frac{1}{2}$, there is no trouble since then the integrand is controlled by a constant times $x^{-7}$ which is integrable on $(1, \infty)$. When $1 \geq |z| \geq \frac{1}{2}$, change variables to $s = 1 - z^2$. Then since $|z|$ is bounded above and below in this region, the final integral can be controlled by
\[
\int_{0}^{1} \int_{1}^{\infty} \frac{1}{x^3(sx^2 + 1)^2} \, dx \, ds
\]

Integrating first in $s$ yields
\[
\int_{1}^{\infty} \frac{1}{x^3(x^2 + 1)} \, dx \leq \int_{1}^{\infty} x^{-5} \, dx < \infty
\]

The conclusion is that $\mathcal{P}$ is in $L^2(\mathbb{R}^{3+3})$.

It now remains to show that the same is true for the $\xi, \eta$ derivatives of order less than 2. This is accomplished by showing that the quantity $\partial^2_\xi \partial^2_\eta \mathcal{P}(\xi, \eta)$ is controlled by $\mathcal{P}(\xi, \eta)$ itself, for $|\alpha|, |\beta| \leq 2$. To see this, let $\mathcal{J}(\xi, \eta)\mathcal{K}(\xi, \eta) = 1$, i.e.
\[
\mathcal{J}(\xi, \eta) = 4(|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2 + |\xi|^2 + |\eta|^2 + \langle \xi, \eta \rangle) + 3
\]

To get a handle on the derivatives of $\mathcal{J}$, first observe that
\[
\partial_\xi (|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2) = 2\xi_j|\eta|^2 - 2\eta_j \langle \xi, \eta \rangle
\]
\[
\partial_\eta (|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2) = 2\eta_j|\xi|^2 - 2\xi_j \langle \xi, \eta \rangle
\]

Combining these two identities and summing gives
\[
|\nabla_{\xi, \eta}(|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2)|^2 = 4(|\xi|^2 + |\eta|^2)(|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2)
\]

The bound (B.5) from Proposition B.1.4 is equivalent to $\mathcal{J}(\xi, \eta)^2 \geq 16(|\xi|^2 + |\eta|^2)$, so using this and (B.11), one can deduce that for first order derivatives, the bound $|\nabla_{\xi, \eta}\mathcal{J}(\xi, \eta)| \lesssim \mathcal{J}(\xi, \eta)$ holds. Similar calculations hold for derivatives of second order and we obtain
\[
|\partial^2_\xi \partial^2_\eta \mathcal{J}(\xi, \eta)| \lesssim \mathcal{J}(\xi, \eta) \quad |\alpha|, |\beta| \leq 2
\]
Using (B.12), one can then induct to show that the same bound holds for $\mathcal{K}$: since $J(\xi, \eta)\mathcal{K}(\xi, \eta) = 1$, one has, for instance,

$$0 = \nabla_\xi (JK) = \nabla_\xi JK + J\nabla_\xi K$$

Then $|J(\xi, \eta)\nabla_\xi K(\xi, \eta)| = |\nabla_\xi J(\xi, \eta)K(\xi, \eta)| \lesssim J(\xi, \eta)K(\xi, \eta)$ from which it can be concluded that $|\nabla_\xi K(\xi, \eta)| \lesssim K(\xi, \eta)$. Proceeding by induction gives the full estimate $|\partial^{\alpha}_\xi \partial^{\beta}_\eta K(\xi, \eta)| \lesssim K(\xi, \eta)$ which in turn proves that (still for $|\alpha|, |\beta| \leq 2$)

$$|\partial^{\alpha}_\xi \partial^{\beta}_\eta P(\xi, \eta)| \lesssim P(\xi, \eta) \quad (\text{B.13})$$

This estimate implies then that $\partial^{\alpha}_\xi \partial^{\beta}_\eta P(\xi, \eta) \in L^2(\mathbb{R}^{3+3})$ for the given range of $\alpha, \beta$.

\[ \square \]

### B.2 Proof of Theorem 6.3.1

We now assemble these results to prove Theorem 6.3.1.

**Proof of Theorem 6.3.1.** First write $\mathcal{K}(\xi, \eta) = \langle \xi \rangle^2 \langle \eta \rangle^2 \langle \xi \rangle^{-2} \langle \eta \rangle^{-2} \mathcal{K}(\xi, \eta)$. Using the notation we defined earlier, this is equivalent to

$$\mathcal{K}(\xi, \eta) = (1 + |\xi|^2)(1 + |\eta|^2)\langle \xi \rangle^{-1} \mathcal{P}(\xi, \eta)$$

$$= (1 + \xi_1^2 + \cdots + \xi_3^2)(1 + \eta_1^2 + \cdots + \eta_3^2)\langle \xi \rangle^{-1} \mathcal{P}(\xi, \eta)$$

$$= \sum_{|\alpha|, |\beta| \leq 1} c_{\alpha, \beta} \xi^\alpha \eta^\beta \mathcal{K}_{\alpha, \beta}(\xi, \eta)$$

where $\mathcal{K}_{\alpha, \beta}(\xi, \eta) = \xi^\alpha \langle \xi \rangle^{-1} \eta^\beta \langle \eta \rangle^{-1} \mathcal{P}(\xi, \eta)$. It was established in Lemma B.1.6 that $\mathcal{P}$ and its derivatives up to second order are in $L^2(\mathbb{R}^{3+3})$, and since $|\alpha|, |\beta| \leq 1$, the weights $\xi^\alpha \langle \xi \rangle^{-1} \eta^\beta \langle \eta \rangle^{-1}$ present no problem (they remain bounded after successive differentiation). Thus the conclusion of Lemma B.1.6 holds for $K_{\alpha, \beta}$. Finally, observe that

$$\mathcal{K}(D', D'')[u][v] = \sum_{|\alpha|, |\beta| \leq 1} c_{\alpha, \beta} \xi^\alpha \eta^\beta \mathcal{K}_{\alpha, \beta}(D', D'')[u][v]$$

$$= \sum_{|\alpha|, |\beta| \leq 1} c_{\alpha, \beta} \mathcal{K}_{\alpha, \beta}(D', D'') [D^\alpha u][D'^\beta v]$$
To complete the proof, simply apply Lemma B.1.1 to obtain

$$\|K(D', D'')[u][v]\|_{L^r} \lesssim \|u\|_{W^{1,p}} \|v\|_{W^{1,q}}$$
Bibliography


