Elasticity in drift-wave–zonal-flow turbulence

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We present a theory of turbulent elasticity, a property of drift-wave–zonal-flow (DW-ZF) turbulence, which follows from the time delay in the response of DWs to ZF shears. An emergent dimensionless parameter $|\langle \nabla' \rangle|/\Delta \omega_k$ is found to be a measure of the degree of Fickian flux-gradient relation breaking, where $|\langle \nabla' \rangle|$ is the ZF shearing rate and $\Delta \omega_k$ is the turbulence decorrelation rate. For $|\langle \nabla' \rangle|/\Delta \omega_k > 1$, we show that the ZF evolution equation is converted from a diffusion equation, usually assumed, to a telegraph equation, i.e., the turbulent momentum transport changes from a diffusive process to wavelike propagation. This scenario corresponds to a state very close to the marginal instability of the DW-ZF system, e.g., the Dimits shift regime. The frequency of the ZF wave is $\Omega_{ZF} = \pm \gamma_d^{1/2} \gamma_{\text{mode}}^{1/2}$, where $\gamma_d$ is the ZF friction coefficient and $\gamma_{\text{mode}}$ is the net ZF growth rate for the case of the Fickian flux-gradient relation. This insight provides a natural framework for understanding temporally periodic ZF structures in the Dimits shift regime and in the transition from low confined mode to high confined mode in confined plasmas.

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delay time is crucial to understanding the physical essence of turbulent elasticity, we also give a heuristic discussion on the scaling of the delay Time.

Zonal-flow wave induced by turbulent elasticity. The nonzero response time is a clue that the Fickian flux-gradient relation (which assumes an instant response) for the ZF is incomplete, so we must deal with evolution equations for momentum and momentum flux simultaneously. The constitutive equations of the DW-ZF system are the momentum balance equation for ZF

$$\frac{\partial}{\partial t} \langle v \rangle = -\frac{\partial}{\partial x} \Pi - \gamma_d \langle v \rangle$$  \hspace{1cm} (1)$$

and the wave kinetic equation [4]

$$\frac{\partial}{\partial t} N_k + v_{g,k} \frac{\partial}{\partial x} N_k - k_y \langle v \rangle \frac{\partial}{\partial k_x} N_k = -\gamma_{N_k}(N_k - N_{0,k})$$  \hspace{1cm} (2)$$

where $\Pi = \langle \hat{v}_x \hat{v}_y \rangle$ is the poloidally averaged turbulent momentum flux, $\hat{v}_x$ and $\hat{v}_y$ are velocity fluctuations in the radial and poloidal directions, and $\gamma_d$ is the friction coefficient for ZF. Here $v_{g,k} = \theta \omega_k / k_x$ is the linear group velocity of the DW, $N_k = E_k / \omega_k = (1 + k_x^2 \rho_e^2)^2 |\phi_k|^2 / 2 \omega_k$ is the wave action density, $E_k$ is the wave energy density, $\omega_k = \omega_s / (1 + k_x^2 \rho_e^2)$ is the linear DW frequency, $\phi_k$ is electrostatic potential, $\omega_s$ is the ion-sound Larmor radius. Equation (2) includes a relaxation modeled by a Krook operator, i.e., $-\gamma_{N_k}(N_k - N_{0,k})$. Here $\gamma_{N_k}$ is the relaxation time of the wave action density and $N_{0,k}$ is the wave action density at the equilibrium state. The wave action density is equivalent to the potential enstrophy density, so this collision term also accounts for the forward cascade of the potential enstrophy. Since the forward potential enstrophy cascade is a consequence of potential vorticity mixing [13], $\gamma_{N_k}$ can also be interpreted as the rate of local potential vorticity mixing.

Multiplying by $k_x v_{g,k}$ on both sides of Eq. (2) yields the evolution equation for the wave momentum flux

$$\frac{\partial}{\partial t} \Pi + \frac{\partial}{\partial x} \Pi + \alpha \frac{\partial}{\partial x} \langle v \rangle = -\gamma_N(\Pi - \Pi_0)$$  \hspace{1cm} (3)$$

where $\Pi = \sum_k \Pi_k = \sum_k k_x v_{g,k} N_k = \langle \hat{v}_x \hat{v}_y \rangle$ is the total wave momentum flux. Since the momentum of the electrostatic field is zero, $\Pi$ is also the turbulent flow (i.e., nonresonant particles in kinetic picture) momentum flux in Eq. (1) [14,15]. This follows since for fluidlike dynamics, the total momentum equals the nonresonant particle momentum, which equals the wave momentum. Here $\gamma_N \equiv \sum_k \gamma_{N_k} N_k$ accounts for the characteristic response time of DW turbulence. Then $\Pi = \gamma_{N_k} N_k$ is the transport of wave momentum flux. Without the first two terms on the left-hand side, Eq. (3) reduces to the familiar Fickian relation $\Pi = \Pi_0 = -\alpha / \gamma_N \hat{\nabla} \Pi$. Here

$$\alpha / \gamma_N = \sum_k \frac{2 \omega_k k_x^2 \rho_e^2}{(1 + k_x^2 \rho_e^2)^2} \hat{\nabla} N_k / \gamma_N < 0$$

is the negative viscosity, which describes local growth of the ZF through modulational instability.

Here $\Gamma_{\Pi}$ may be thought of as a flux of momentum flux of the DW gas. The nonzero divergence of $\Gamma_{\Pi}$ is critical in exciting zonal flow, as a second sound wave. A consistent derivation of the momentum flux requires the proper closure of Eq. (3), i.e., we need to find the relation between $\Gamma_{\Pi}$ and $\Pi$. In general, the flux-gradient relation can be expressed as

$$\Gamma_{\Pi} = -\int d\chi K(x,x') \frac{\partial}{\partial x} \Pi(x')$$  \hspace{1cm} (4)$$

where $K(x,x')$ is a kernel function, representing the generalized diffusivity. The form of $\Gamma_{\Pi}$ is sensitive to whether the transport of $\Pi$ is local or nonlocal. If the mean free path $l_{\text{MFP}}$ of the DW is much shorter than the characteristic scale length $L_{\Pi}$, then the transport is driven via strong local DW-DW scattering. This scenario corresponds to a diffusive flux and $K(x,x')$ can be taken to be a delta function, i.e., $K(x,x') = D \delta(x - x')$ or $D$ is the Fickian diffusion coefficient. From Eq. (4) the usual flux-gradient relation $\Gamma_{\Pi} = -D \delta_{\Pi}$ is recovered. In the weak local scattering scenario, DW packets can propagate a longer distance, so $l_{\text{MFP}}$ is comparable to $L_{\Pi}$ and the Fickian diffusion ansatz fails. In other words, we need a nonlocal integral kernel in the flux equation. In this Rapid Communication we focus on in the flux-gradient relation when $l_{\text{MFP}} \gg L_{\Pi}$, which is equivalent to the nonlocal interaction exceeding the local interaction. Given the continuity of the content, we delay the discussion of the physical meaning of this limit to the next section. In this scenario, each scatterer position makes an equal contribution to $K(x,x')$ and hence $K(x,x')$ can be expressed as a step function, i.e., $K(x,x') = \nu_{\Pi}(x - x')$, with $\Theta = 1$ for $x \geq x'$ and $\Theta = 0$ for $x < x'$. Here $\nu_{\Pi}$ is the characteristic transport speed. Putting the step function into Eq. (4), $\Gamma_{\Pi}$ is readily derived as

$$\Gamma_{\Pi} = -\nu_{\Pi} \Pi$$  \hspace{1cm} (5)$$

which gives a convective flux-gradient relation. A more general method of calculating $\Gamma_{\Pi}$ is via flux-limited diffusion theory [16]. Similar to heat transfer in radiation hydrodynamics [17], the limited-flux–gradient relation can be cast in the phenomenological form [18]

$$\Gamma_{\Pi} = -\frac{D \nabla \Pi}{\sqrt{1 + (l_{\text{MFP}} \nabla \ln L)^2}}$$  \hspace{1cm} (6)$$

where $1 / \sqrt{1 + (l_{\text{MFP}} \nabla \ln L)^2}$ is a flux-limiting factor. For small values of mean free path, Eq. (6) reduces to the diffusion form. In the strong nonlocality scenario, the factor is approximated as

$$1 / \sqrt{1 + (l_{\text{MFP}} \nabla \ln L)^2} \approx \text{sgn}(\nabla \ln L) 1 / (l_{\text{MFP}} \nabla \ln L).$$

Putting it into Eq. (6), one obtains $\Gamma_{\Pi} = -\text{sgn}(\nabla \ln L) D / l_{\text{MFP}} \frac{\Pi}{\nabla L}$ or $D = \text{sgn}(\nabla \ln L) \nu_{\Pi} l_{\text{MFP}}$ is used. This result is the same as Eq. (5) and hence our choice of the step function $\Theta$ as the integral kernel in Eq. (4) appears proper.

From Eqs. (3) and (5) the evolution equation for the perturbed momentum flux is

$$\frac{\partial}{\partial t} \Pi = \nu_{\Pi} \frac{\partial}{\partial x} \Pi + \frac{\partial}{\partial x} \langle v \rangle = -\gamma_N \Pi$$  \hspace{1cm} (7)$$
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Thus, combining Eqs. (1) and (7) yields at last a telegraph evolution equation for the ZF

\[
\frac{\partial^2}{\partial t^2} \langle v \rangle + \left( \gamma_N + \gamma_d + \nu_c \frac{\partial}{\partial x} \right) \frac{\partial}{\partial t} \langle v \rangle + \left( -\gamma_d \nu_c \frac{\partial}{\partial x} - \alpha \frac{\partial^2}{\partial x^2} + \gamma_N \gamma_d \right) \langle v \rangle = 0, \quad (8)
\]

which immediately suggests wavelike solutions. In the limit of short delay time, when the term with the second time derivative in Eq. (8) is negligible, that equation reduces to the familiar parabolic momentum equation. We linearize Eq. (8) when the deviation of \( N_k \) from its equilibrium \( N_{0,k} \) is small. To obtain the dispersion relation of the ZF wave, using transformations \( \partial_x \to -i \Omega_{ZF} + \gamma_{ZF} \) and \( \partial_x \to iq \) (\( q \) is the radial wave number of the ZF), the real and imaginary parts of Eq. (8) are readily obtained as

\[
-\Omega_{ZF}^2 + \gamma_{ZF}^2 + \gamma_{ZF} (\gamma_N + \gamma_d) - q \Omega_{ZF} \nu_c + \alpha_0 q^2 + \gamma_N \gamma_d = 0, \quad (9a)
\]

\[
(\gamma_N + \gamma_d) \Omega_{ZF} + q \gamma_d \nu_c + 2 \Omega_{ZF} \gamma_{ZF} + q \nu_c \gamma_N = 0. \quad (9b)
\]

Here

\[
\alpha_0 = \sum_k \frac{2 \omega_k k_x k_y^2 \rho_y^2}{\left(1 + k_x^2 \rho_x^2\right)^2} \frac{\delta N_{0,k}}{\delta k_x}.
\]

Equation (9b) shows that \( v_c \neq 0 \) implies the existence of a steady ZF wave solution. It should be emphasized that Eqs. (9a) and (9b) are obtained in the laboratory frame, therefore their wave solutions are also in the laboratory frame; the wave is not a propagating localized pulse. Since we seek a stationary solution, by setting \( \gamma_{ZF} = 0 \) the dispersion relation follows as

\[
\Omega_{ZF} \simeq \pm \left( \frac{\gamma_d |\alpha_0| q^2}{\gamma_N} - \gamma_d^2 \right)^{1/2}. \quad (10)
\]

A necessary but not sufficient condition for the existence of a ZF wave is \( |\alpha_0| q^2 / \gamma_N > \gamma_d \), which states that the growth rate \( |\alpha_0| q^2 / \gamma_N \) of the modulational instability must overcome the frictional damping of the ZF. Equation (10) also gives a critical ZF wave number \( q_c = (\gamma_d \gamma_N / |\alpha_0|)^{1/2} \) and \( |q| \geq q_c \) is a necessary condition for the existence of a ZF wave. In the large-wave-number regime \( |q| \gg q_c \), one has \( \Omega_{ZF} \simeq \pm (\gamma_d |\alpha_0| / \gamma_N)^{1/2} q \), which is just the second sound dispersion relation with \( (\gamma_d |\alpha_0| / \gamma_N)^{1/2} \) being the phase velocity. We can rewrite \( \Omega_{ZF} \) as the geometric mean of \( \gamma_{modu} \) and \( \gamma_d \), i.e., \( \Omega_{ZF} = \sqrt{\gamma_{modu} \gamma_d} \), where \( \gamma_{modu} = |\alpha_0| q^2 / \gamma_N - \gamma_d \) is the net ZF growth rate for the case of the Fickian flux-gradient relation.

Here we give a physical picture of the propagation mechanism for the ZF wave (Fig. 1). For the initial ZF pattern, the response of the ZF friction force is transient, but the divergence of turbulent momentum flux (i.e., Reynolds force) at this moment is zero because of the delayed response of the DW turbulence to the ZF pattern. Thus, the amplitude of the ZF pattern tends toward the value zero. In this sense, we can view the friction force as a kind of restoring force. As it approaches the value zero, the restoring force becomes weaker and weaker, but the Reynolds force gradually increases, which drives the ZF pattern away from zero. Thus, the Reynolds force acts as kind of repulsive force. Once the ZF reaches zero, the restoring force disappears, but the repulsive force is maximal. In other words, the momentum carried by the initial ZF pattern is totally converted into pseudomomentum carried by the DW packets. Like classical molecules, these wave packets can scatter into the no-flow region by mutual collisions, i.e., nonlinear interactions that mix potential vorticity. Because of this spatial mixing of the momentum, a new ZF pattern can be created in the no-flow region. When the new ZF pattern attains its peak value, the restoring force will again assert itself. By repeating this process, a sequence of spatiotemporal structures will occur. The characteristic time scales of the restoring and repulsive forces are \( \gamma_d^{-1} \) and \( \gamma_{modu}^{-1} \), respectively, so one might expect that the period of the ZF wave will scale as \( \sim \gamma_d^{-1} / \gamma_{modu} \), with \( \beta + \delta = 1 \). This heuristic argument is consistent with our analytical result (10), where \( \beta = \delta = 1/2 \). The propagation direction of the ZF wave is affected by the boundary conditions and mean flow profiles. For example, at the edge of tokamaks, there exist strong mean shear flows that can reflect an outward propagating ZF wave back inward. Hence one can expect that most ZF waves propagate inward from the edge. Indeed, two-way pulse propagation, reflection from the edge, and the ultimate predominance of the inward propagating population have been observed in a recent experiment [10].

**Structure of the delay time.** From the above discussion, we know that the delay time \( \tau_N (\tau_N \equiv \gamma_N^{-1}) \) is a fundamental quantity and measures the elastic strength of DW turbulence. Also, a more complete understanding of the ZF wave requires knowing the structure of \( \tau_N \). Thus, we here seek a deeper understanding and characterization of \( \tau_N \). This is an important element of the theory of turbulent elasticity. Physically, \( \tau_N \) originates from the finite collision time during the turbulent relaxation of the DW-ZF system. There are two types of collision processes: direct DW-DW collisions, which are local interactions, and indirect DW-DW collisions, i.e., DW-DW scattering mediated by ZF, which are nonlocal interactions.

In this work, we mainly focus on wave momentum transport in a quasistationary, near marginal state, where both the growth rate and the amplitude of the DW are small. In this respect, the collision via the direct DW-DW interaction [Fig. 2(a)] is weaker than that via the indirect DW-DW interaction.
nearly quenched by the ZFs and \( \Delta_1\omega_k \ll \tau_N \), turbulence and hence the long excursion condition \( l \) wave. With Eq. (11), the scaling of turbulence intensity spreading associated with the ZF is determined mainly by the ZF shearing time, i.e.,

\[
\tau_N \simeq \langle |v'| \rangle^{-1}.
\]

With this scaling, one has \( l_{\text{MFP}} |q| = |v_{\text{f}}/\langle v' \rangle|\tau_N \langle v' \rangle \simeq |v_{\text{f}}/\langle v' \rangle|\), with \( l_{\text{MFP}} = v_{\text{f}}\tau_N \), \( |q| = L^{-1}_N = \langle |v'|/\langle v' \rangle \rangle \), and \( v_{\text{f}} = \sum_k v_{\text{f},k} N_k / \sum_k N_k \) the characteristic group velocity of the DW packet. In general, \( v_{\text{f}} \) is much larger than \( \langle |v'| \rangle \) in DW-ZF turbulence and hence the long excursion condition \( l_{\text{MFP}} \gg |q| \) is satisfied, i.e., a DW packet can retain its identity while propagating through many ZF bands [Fig. 2(b)] and the Fickian flux-gradient fails. A physical setting of this scenario is the Dimits shift regime [12], where the DW turbulence is nearly quenched by the ZFs and \( \Delta_0 k \ll \langle |v'| \rangle \) tends to occur.

With Eqs. (9b) and (11), \( v_{\text{f}} \) scales as \( v_{\text{f}} \sim \langle |v'| \rangle^{1/2} y_d^{-1/2} \gamma_d^{1/2} \), which can be interpreted as the scaling of turbulence intensity spreading associated with the ZF wave. With Eq. (11), the scaling of \( \Omega_{ZF} \) is derived as

\[
\Omega_{ZF} \sim \langle |v'| \rangle^{-1/2} y_d^{1/2} \gamma_d^{1/2} q.\]

Since turbulent relaxation occurs by spatial scattering of the DW packets, \( \tau_N \) is necessarily set by the relative dispersion of the DW packets. Thus, the structure of \( \tau_N \) can also be analyzed in a more rigorous way, i.e., studying the relative dispersion of two DW packets by using the two-point correlation function of the wave action density [19]. It is straightforward to show that the more rigorous way gives the same scaling with Eq. (11); details are beyond the scope of the present paper.

Turbulent elasticity, as proposed in this Rapid Communication, may be tested in the following three ways. (i) A ZF wave is detected. Wavelike propagation of a ZF pattern provides a dissipationless means for momentum transport. In fact, there is numerical and experimental evidence for the inward radial propagation of ZF during LH transition experiments [9,10]. (ii) Wavelike turbulence spreading is observed. This type of turbulence spreading can be very important near the marginal state, as the characteristic scale (i.e., \( l_{\text{MFP}} \)) of elastic spreading is much larger than that of viscous spreading (i.e., \( l_{\text{shd}} \)) [20]. (iii) The turbulent elasticity can alter the dynamical structure of the usual predator-prey model [4], so access to an alternative limit cycle solution become possible. Simulation tests of this theory seem most viable at least initially. Such simulation tests could focus on (a) identifying and quantifying fast turbulence spreading, where “spreading” refers to the expansion of an ensemble of coupled turbulent eddies and zonal flows and “fast” refers to a superdiffusive process, and (b) identifying zonal waves in the laboratory frame and demonstrating consistency with the predicted dispersion characteristics.

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