Harmonic functions on Walsh's Brownian motion

Permalink
https://escholarship.org/uc/item/4qq7b9fd

Author
Jehring, Kristin Elizabeth

Publication Date
2009

Peer reviewed|Thesis/dissertation
Harmonic Functions on Walsh’s Brownian Motion

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Kristin Elizabeth Jehring

Committee in charge:

Professor Patrick Fitzsimmons, Chair
Professor Tara Javidi
Professor Dimitris Politis
Professor Jason Schweinsberg
Professor Yixiao Sun

2009
The dissertation of Kristin Elizabeth Jehring is approved, and it is acceptable in quality and form for publication on microfilm:

________________________________________

________________________________________

________________________________________

________________________________________

Chair

University of California, San Diego

2009
DEDICATION

To my Busia.
EPIGRAPH

The mountains are calling and I must go.
—John Muir
# TABLE OF CONTENTS

Signature Page .......................................................... iii
Dedication ................................................................. iv
Epigraph ................................................................. v
Table of Contents ......................................................... vi
List of Figures ........................................................... viii
List of Tables ............................................................. ix
Acknowledgements ......................................................... x
Vita ................................................................. xi
Abstract ................................................................. xii

Chapter 1  Introduction ...................................................... 1
  1.1  Notation ............................................................... 4

Chapter 2  Walsh’s Brownian Motion ...................................... 6
  2.1  Existence ............................................................. 6
  2.2  A Construction ....................................................... 8
  2.3  Excursions .......................................................... 11

Chapter 3  Harmonic Functions .......................................... 16
  3.1  Topologies on the State Space .................................. 16
  3.2  Characterization of Harmonic Functions ....................... 19
    3.2.1  Proof of Theorem 3.2 ...................................... 26
    3.2.2  Examples of Harmonic Functions ......................... 32
  3.3  Excessive Functions ............................................. 34
  3.4  Superharmonic Functions ....................................... 35
    3.4.1  Proof of Theorem 3.3 ...................................... 38

Chapter 4  Walsh’s Brownian Motion on a Graph ..................... 41
  4.1  Set-up and General Result ..................................... 41
  4.2  Walsh’s Brownian Motion on a Graph ......................... 44
    4.2.1  Embedded Markov Chains .................................. 45
  4.3  Harmonic Functions .............................................. 47
    4.3.1  Transition Probabilities of Embedded Markov Chains .. 48
    4.3.2  Harmonic Functions for the Embedded Markov Chain .. 49
  4.4  Reversibility ..................................................... 51
    4.4.1  Dirichlet Forms ........................................... 58
  4.5  Passage Times .................................................... 61
| Figure 3.1: | The tree topology is finer than the relative Euclidean metric. | 17 |
| Figure 3.2: | Equivalence of topologies when angular distribution is discrete. | 18 |
| Figure 4.1: | Example of a subgraph. | 42 |
| Figure 4.2: | The triangle graph. | 66 |
| Figure 4.3: | Example 8. | 71 |
List of Tables

Table 4.1: Possible paths between starting and ending vertices. 68
ACKNOWLEDGEMENTS

I would like to first thank my advisor, Professor Fitzsimmons, whose willingness to help a student in need allowed me to persevere through this process. Without his help none of this would have been possible.

I also thank my family and friends for their much needed support and belief in me. I would especially like to thank Larissa, Jason, Nate and Andy for the many hikes and games of cards that we have enjoyed together.

And last, but most definitely not least, I thank Kelly for putting up with me throughout these many years. I truly could not have done it without you by my side.
VITA

2003         B. S. in Mathematics *cum laude*, University of Wisconsin, Madison

2003-2008    Graduate Teaching Assistant, University of California, San Diego

2009         Ph. D. in Mathematics, University of California, San Diego
ABSTRACT OF THE DISSERTATION

Harmonic Functions on Walsh’s Brownian Motion

by

Kristin Elizabeth Jehring

Doctor of Philosophy in Mathematics

University of California San Diego, 2009

Professor Patrick Fitzsimmons, Chair

In this dissertation we examine a variation of two-dimensional Brownian motion introduced in 1978 by Walsh. Walsh’s Brownian motion can be described as a Brownian motion on the spokes of a (rimless) bicycle wheel. We will construct such a process by randomly assigning an angle to the excursions of a reflecting Brownian motion from 0. With this construction we see that Walsh’s Brownian motion in $\mathbb{R}^2$ behaves like one-dimensional Brownian motion away from the origin, but at the origin behaves differently as the process is sent off in another random direction. Taking advantage of this similarity, we provide a characterization of harmonic functions for the process as linear functions on the rays that satisfy a slope-averaging property. We also classify superharmonic functions as concave functions on the rays satisfying some extra conditions. We then generalize the state space to consider a process on any connected, locally finite graph obtained by gluing a number of Walsh’s Brownian motion processes in $\mathbb{R}^2$ together. In this generalized situation, we also classify harmonic functions. We introduce a Markov chain associated to Walsh’s Brownian motion on a graph and explore the relationship between the two processes. We address the reversibility of the process and derive the Dirichlet form of the reversible Walsh’s Brownian motion on a graph.
Chapter 1

Introduction

In 1978 Walsh introduced a variation of two-dimensional Brownian motion. His variation is a generalization of a process constructed by Itô and McKean called skew Brownian motion [14]. To construct skew Brownian motion, let \( \{B_t; t \geq 0\} \) be a one-dimensional Brownian motion and consider the reflecting Brownian motion \( R_t = |B_t| \). Itô and McKean take the excursions of \( R_t \) from 0, and independently assign to each a positive or negative sign at random. The resulting process is a one-dimensional diffusion that behaves like Brownian motion away from 0, but at 0 the process is “skewed” in that it is more likely to head one direction over the other. Walsh’s Brownian motion is constructed in a similar manner, we give the very intuitive description from Walsh himself ([25], p.44):

The main idea is to take each excursion of \( R_t \) and, instead of giving it a random sign, to assign it a random variable \( \theta \) with a given distribution in \([0, 2\pi)\), and to do this independently for each excursion. That is, if the excursion occurs during the interval \((u, v)\), we replace \( R_t \) by the pair \((R_t, \theta)\) for \( u \leq t < v \), \( \theta \) being a random variable with values in \([0, 2\pi)\). This provides a process \( \{(R_t, \theta_t), t \geq 0\} \), where \( \theta_t \) is constant during each excursion from 0, has the same distribution as \( \theta \), and is independent for different excursions. We then consider \( X_t = (R_t, \theta_t) \) as a process in the plane, expressed in polar coordinates. It is a diffusion which, when away from the origin, is a Brownian motion along a ray, but which has what might be called a roundhouse singularity at the origin: when the process enters it, it, like Stephen Leacock’s hero, immediately rides off in all directions at once.

Walsh did not prove that this construction produces a diffusion in \( \mathbb{R}^2 \) when he first introduced it. But since the introduction of Walsh’s Brownian motion many
constructions have been given using resolvents [21], from the infinitesimal generator [2], and using excursion theory [24]. In 1989 Barlow, Pitman, and Yor provided another construction using semigroups [1].

Our interest in this process arose from a paper of Dayanik and Karatzas [12], concerning the optimal stopping problem for one-dimensional diffusions. In this work, the authors provide a simple classification of excessive functions in terms of concave functions. Their results generalize the theorem of Dynkin and Yushkevich for one-dimensional Brownian motion.

**Theorem 1.1 ([5]).** A function is excessive for one-dimensional Brownian motion if and only if it is concave.

In an effort to eventually extend the results of Dayanik and Karatzas to higher dimensional diffusions, we examine the problem for Walsh’s Brownian motion. We begin by classifying the harmonic functions. In this case, we take advantage of the similarities between Walsh’s Brownian motion and one-dimensional Brownian motion away from 0. For one-dimensional Brownian motion we have the following well-known description of harmonic functions in both the analytic and probabilistic senses.

**Theorem 1.2.** Let \( \{B_t; t \geq 0\} \) be a one-dimensional Brownian motion. For any real-valued, locally integrable function \( h \) the following are equivalent:

(i) For every \( x \in \mathbb{R} \) and \( \epsilon > 0 \),

\[
    h(x) = \mathbb{E}^x [h(B_{\tau_x(\epsilon)})],
\]

where \( \tau_x(\epsilon) = \inf\{t \geq 0 : |B_t - x| = \epsilon\} \).

(ii) \( h \) is linear, i.e. \( h(x) = \alpha x + \beta \).

(iii) \( \{h(B_t); t \geq 0\} \) is a continuous martingale.

**Proof.** If \( h \) satisfies (i), then for every \( x \in \mathbb{R} \) and \( \epsilon > 0 \),

\[
    h(x) = \mathbb{E}^x [h(B_{\tau_x(\epsilon)})] = \frac{1}{2} h(x - \epsilon) + \frac{1}{2} h(x + \epsilon). \quad (1.1)
\]

Considering the graph of \( h(x) \), \( x \in \mathbb{R} \), (1.1) says that for any \( \epsilon > 0 \), the points \( h(x - \epsilon) \), \( h(x) \), and \( h(x + \epsilon) \) lie on a single line. Therefore, if \( h \) is continuous, then \( h \) must be linear. We show that \( h \) is in fact a smooth function.
Consider the positive and symmetric mollifier given by

\[
\phi(x) = \begin{cases} 
\exp \left[ \frac{1}{x^2 - 1} \right] & |x| < 1, \\
0 & |x| \geq 1.
\end{cases}
\]

For \( \epsilon > 0 \), define \( \phi_\epsilon(x) = \epsilon^{-1} \phi(\epsilon^{-1} x) \). Since \( h \) is locally integrable, a well-known result in distribution theory says that the convolution \( h_\epsilon = h * \phi_\epsilon \) is a smooth function (see Proposition 9.3 in [6], for example). From (1.1) we have

\[
h_\epsilon(x) = \int h(x - y) \phi_\epsilon(y) dy
\]

\[
= \int_{-1}^{1} h(x - \epsilon z) \phi(z) dz
\]

\[
= \int_{0}^{1} (h(x - \epsilon z) + h(x + \epsilon z)) \phi(z) dz
\]

\[
= 2 \int_{0}^{1} h(x) \phi(z) dz
\]

\[
= h(x).
\]

Thus \( h \) is also a smooth function and we conclude that \( h \) is linear.

If \( h \) is linear, then there are constants \( \alpha, \beta \in \mathbb{R} \) such that \( h(x) = \alpha x + \beta \). Then for any \( x \in \mathbb{R} \) and \( \epsilon > 0 \),

\[
\mathbb{E}^x \left[ h(B_{\tau_{\epsilon}(\epsilon)}) \right] = \frac{1}{2} h(x - \epsilon) + \frac{1}{2} h(x + \epsilon)
\]

\[
= \frac{1}{2} (\alpha (x - \epsilon) + \beta) + \frac{1}{2} (\alpha (x + \epsilon) + \beta)
\]

\[
= h(x).
\]

Thus, \( h \) satisfies (i). Also, since \( h \) is linear, for any \( x \in \mathbb{R} \) and \( t \geq 0 \),

\[
\mathbb{E}^x \left[ h(B_t) \right] = h(x).
\]

Thus, by the Markov property, \( h \) satisfies (iii).

If \( \{h(B_t); t \geq 0\} \) is a martingale, then for every \( x \in \mathbb{R} \) and \( t \geq 0 \),

\[
\mathbb{E}^x \left[ h(B_t) \right] = \mathbb{E}^x \left[ h(B_0) \right] = h(x).
\]

Thus, by the strong Markov property, \( h \) satisfies (i). \( \square \)

The similarity between one-dimensional Brownian motion and Walsh’s Brownian motion away from 0 indicates that the classification of harmonic functions should
also be similar. However, we will see that additional assumptions are required for a function to be harmonic for Walsh’s Brownian motion to account for the behavior at 0.

There is also a corresponding result for superharmonic functions.

**Theorem 1.3.** Let \( \{B_t; t \geq 0\} \) be a one-dimensional Brownian motion. For any real-valued lower semi-continuous function \( h \) the following are equivalent:

(i) For every \( x \in \mathbb{R} \) and \( \epsilon > 0 \),
\[
h(x) \geq E^x \left[ h(B_{\tau_\epsilon(x)}) \right].
\]

(ii) \( h \) is concave.

(iii) \( \{h(B_t); t \geq 0\} \) is a continuous supermartingale.

In Chapter 2 we take a closer look at Walsh’s Brownian motion providing an overview of the construction in [1] as well as the details of another construction using the excursions of reflecting Brownian motion. We also briefly review the basics of excursion theory which we will use in the sequel. In Chapter 3 we define harmonic, excessive, and superharmonic functions for Walsh’s Brownian motion. Chapter 3 gives the main results of this paper as we classify these types of functions for Walsh’s Brownian motion. Furthermore, an important class of examples of harmonic functions for Walsh’s Brownian motion is discussed. This class of examples is the solution of a type of Dirichlet problem. Finally, in Chapter 4 we look at a generalization of Walsh’s Brownian motion by extending the state space to a connected, locally finite graph. We define the embedded Markov chain associated to a Walsh’s Brownian motion on a graph and explore the relationship between the processes addressing harmonic functions, reversibility, and the Dirichlet form. We end with an interesting calculation of the Laplace transform of passage times for Walsh’s Brownian motion on a graph with each edge having unit length.

**1.1 Notation**

Let \( S \) be a measurable space with measure \( \mu \). We denote by \( L^1(S; \mu) \) the set of all measurable functions \( f \) on \( S \) such that
\[
\int_S |f|d\mu < \infty.
\]
Similarly, we denote by $L^2(S; \mu)$ the set of all measurable functions $f$ on $S$ such that

$$\int_S |f|^2 d\mu < \infty.$$  

If $S$ is a topological space, we denote by $C(S)$ the space of all real-valued continuous functions on $S$. For $f \in C(S)$, the support of $f$, denoted $\text{supp}(f)$, is the closure of the set

$$\{s \in S : f(s) \neq 0\}.$$  

We denote by $C_c(S)$ the set of real-valued continuous functions with compact support, i.e.

$$C_c(S) = \{f \in C(S) : \text{supp}(f) \text{ is compact}\}.$$  

We denote by $C^k(S)$ and $C^k_c(S)$ the set of real-valued continuous functions on $S$ and, respectively, the real-valued continuous functions on $S$ with compact support, such that the first $k$ derivatives exist and are also continuous, respectively, have compact support.

For $f \in C(S)$, we say that $f$ vanishes at infinity if for every $\epsilon > 0$ the set

$$\{s \in S : |f(s)| \geq \epsilon\}$$  

is compact. We denote by $C_0(S)$ the set of real-valued continuous functions on $S$ that vanish at infinity, i.e.

$$C_0(S) = \{f \in C(S) : f \text{ vanishes at infinity}\}.$$  

If $S$ is also a metric space, we denote by $\mathcal{B}(S)$ the Borel $\sigma$-algebra generated by the open sets in $S$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $X = \{X_t; t \geq 0\}$ be a stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $S$. For $x \in S$, we denote by $\mathbb{P}^x$ and $\mathbb{E}^x$ the probability and expectation, respectively, conditional on the event that the process starts from $x$, $\{X_0 = x\}$. We denote by $\{\mathcal{F}^X_t\}_{t \geq 0}$ the natural filtration of $X$. That is, $\mathcal{F}^X_t = \sigma\{X_s; 0 \leq s \leq t\}$, the $\sigma$-algebra generated by the process up to time $t$. We denote by $\mathbb{1}$ the indicator function on $\Omega$, i.e. for $A \in \mathcal{F}$, $\mathbb{1}_A$ is the function on $\Omega$ given by $\mathbb{1}_A(\omega) = 1$ if $\omega \in A$, and $\mathbb{1}_A(\omega) = 0$ otherwise.
Chapter 2

Walsh’s Brownian Motion

We uniquely denote points in \( \mathbb{R}^2 \setminus \{0\} \) using polar coordinates \((r, \alpha)\), where \( r > 0 \) and \( \alpha \in [0, 2\pi) \). Let \( \mu \) be a fixed probability measure on \([0, 2\pi)\). Let \( R = \{R_t; t \geq 0\} \), denote a reflecting Brownian motion on \([0, \infty)\). Let \( T_0 = \inf\{t > 0 : R_t = 0\} \) be the hitting time of 0 for \( R \).

A process \( Z = \{Z_t, \mu; t \geq 0\} \) is a Walsh’s Brownian motion in \( \mathbb{R}^2 \) if

(i) \( Z_t = (R_t, A_t) \), for \( t \geq 0 \), where \( A_t \) is a \([0, 2\pi)\)-valued process with \( A_t = 0 \) when \( R_t = 0 \);

(ii) the distribution of the angle \( A_t \) is as follows:

1. If \( Z_0 = 0 \), then for \( t > 0 \), the angle \( A_t \) has distribution given by \( \mu \) independent of \( R_t \).
2. If \( Z_0 = (r, \alpha) \) with \( r > 0 \), then \( A_t \) equals \( \alpha \) on the set \( \{t < T_0\} \), and on the set \( \{t > T_0\} \), \( A_t \) has distribution \( \mu \) independent of \( R_t \).

We denote the state space of \( Z \) by \( E \subseteq \mathbb{R}^2 \).

2.1 Existence

We review the existence of Walsh’s Brownian motion established in [1]. Their approach consists of using a little intuition to write down a possible semigroup for \( Z \) and verifying that the possible semigroup satisfies the necessary conditions. Then using the general theory to produce a canonical process associated to the given semigroup,
they verify that the process possesses the desired characteristics of a Walsh’s Brownian motion. In fact, we will see that a Walsh’s Brownian motion is a Feller diffusion.

Before describing the semigroup, we need the following notation. We equip the state space $E$ of Walsh’s Brownian motion with the relative Euclidean topology inherited from $\mathbb{R}^2$. For $f \in C(E)$, we define

$$f_\alpha(r) = f(r, \alpha),$$

for $r > 0$ and $\alpha \in [0, 2\pi)$. That is, $f_\alpha$ is the restriction of $f$ to the ray at angle $\alpha$. Also define

$$\overline{f}(r) = \int_0^{2\pi} f(r, \alpha) \mu(d\alpha),$$

for $r \geq 0$. The function $\overline{f}$ is the average value of $f$ with respect to the measure $\mu$.

Let $\{P^+_t\}_{t \geq 0}$ be the semigroup of a reflecting Brownian motion on $[0, \infty)$, and let $\{P^0_t\}_{t \geq 0}$ be the semigroup of a Brownian motion on $[0, \infty)$ killed at 0. Then for $t \geq 0$, define $P_t$ to act on $f \in C_0(E)$ as follows:

$$P_tf(0, \alpha) = P^+_t \overline{f}(0),$$
$$P_tf(r, \alpha) = P^+_t \overline{f}(r) + P^0_t (f_\alpha - \overline{f})(r),$$

for $r > 0$ and $\alpha \in [0, 2\pi)$.

**Theorem 2.1** ([1], Theorem 2.1). $\{P_t\}_{t \geq 0}$ is a Feller semigroup on $C_0(E)$.

We can now define a strong Markov process $Z = \{Z_t; t \geq 0\}$ with state space $E \subseteq \mathbb{R}^2$ and semigroup $\{P_t\}_{t \geq 0}$, such that $Z$ is right continuous with finite left limits. We write the process using polar coordinates as $Z_t = (R_t, A_t)$, and set $A_t = 0$ when $R_t = 0$. Then it is shown that the radial process $R = \{R_t; t \geq 0\}$ is a reflecting Brownian motion and that the angular process $\{A_t; t \geq 0\}$ is constant on $[0, T_0)$, where $T_0 = \inf\{t \geq 0 : R_t = 0\}$. Since $Z$ is a strong Markov process, the latter implies that the angular process is also constant on each excursion of $Z$ away from 0. This combined with the continuity of reflecting Brownian motion gives the continuity of $Z$, which establishes the existence of Walsh’s Brownian motion.

**Theorem 2.2** ([1], Corollary 2.5). $Z$ is a Feller diffusion on $E$.

It remains to be seen that the distribution of $A_t$ is given by the probability measure $\mu$, but this follows from excursion theory, which we will say more about in section 2.3.
2.2 A Construction

In this section, we provide a constructive proof of the existence of Walsh’s Brownian motion. We will construct the process $Z$ directly from the sample paths of the reflecting Brownian motion $R$ and a sequence of i.i.d. angles with common distribution $\mu$. Then, after verifying that the constructed process has continuous paths, satisfies the simple Markov property, and has the semigroup $\{P_t\}_{t \geq 0}$ as proposed in [1], we use their results to conclude that the process is a Feller diffusion satisfying the definition of Walsh’s Brownian motion.

We begin by using any enumeration of the positive rational numbers $\mathbb{Q}_+$ to label the excursions of reflecting Brownian motion away from 0. In particular, let $f: \mathbb{Z}_+ \to \mathbb{Q}_+$ be such an enumeration, that is, $f$ is a surjection and the $k$th positive rational number is given by $f(k) = q_k$. Let $e_1$ denote the excursion of $R$ away from 0 that straddles time $q_1 = f(1)$ and let $I_1$ denote the time interval that $e_1$ straddles. In other words, if we let

$$\gamma(t) = \sup\{s \leq t : R_s = 0\} \quad \beta(t) = \inf\{s \geq t : R_s = 0\},$$

then $I_1 = (\gamma(q_1), \beta(q_1))$ and $e_1 = \{R_t; t \in I_1\}$. Now let $\mathbb{Q}^2_+ = \mathbb{Q}_+ \setminus (I_1 \cap \mathbb{Q}_+)$, the set of positive rational numbers remaining after those in $I_1$ are removed. If we define

$$\kappa(2) = \min\{k \in \mathbb{Z}_+ : f(k) \in \mathbb{Q}^2_+\},$$

then $q_{\kappa(2)}$ is the first number in $\mathbb{Q}^2_+$. Then let $e_2$ denote the excursion that straddles $q_{\kappa(2)}$. So $e_2 = \{R_t; t \in I_2\}$, where $I_2 = (\gamma(q_{\kappa(2)}), \beta(q_{\kappa(2)}))$. Now let $\mathbb{Q}^3_+ = \mathbb{Q}^2_+ \setminus (I_2 \cap \mathbb{Q}^2_+)$ and let $q_{\kappa(3)}$ denote the first number in $\mathbb{Q}^3_+$, where

$$\kappa(3) = \min\{k \in \mathbb{Z}_+ : f(k) \in \mathbb{Q}^3_+\}.$$ 

Then let $e_3$ denote the excursion that straddles $q_{\kappa(3)}$, and so on.

In general, the $j$th excursion of $R$ away from 0 $e_j$, $j = 2, 3, \ldots$, is the excursion that straddles $q_{\kappa(j)}$, where

$$\kappa(j) = \min\{k \in \mathbb{Z}_+ : f(k) \in Q^j_+\},$$

with $Q^j_+ = Q^{j-1}_+ \setminus (I_{j-1} \cap Q^{j-1}_+)$ and $I_{j-1}$ is the time interval on which $e_{j-1}$ exists (we set $Q^1_+ = Q_+$).
We have enumerated the excursions of reflecting Brownian motion away from 0 as the sequence \( \{e_j\}_{j=1}^{\infty} \). Let \( \{\alpha_j\}_{j=1}^{\infty} \) be a sequence of i.i.d. \([0,2\pi)\)-valued random variables with common distribution given by \( \mu \), also independent of \( R \). We pair up each excursion \( e_j \) with the angle \( \alpha_j \) to create the process \( Z = \{Z_t, \mu; t \geq 0\} \) with state space \( E \subseteq \mathbb{R}^2 \) and excursions given by

\[
\xi_j = (e_j, \alpha_j).
\]

We define \( Z \) to be 0 whenever the reflecting Brownian motion \( R \) is 0. We can now write \( Z_t = (R_t, A_t) \), where the process \( A_t = \alpha_j \) when \( t \in I_j \), and \( A_t = 0 \) when \( t \in \{s \geq 0 : R_s = 0\} \).

**Theorem 2.3.** The process \( Z \) constructed as above is a Walsh’s Brownian motion.

**Proof.** We begin by checking that \( Z \) has continuous paths. First suppose that \( t \notin \{s : Z_s = 0\} \). Then the continuity of \( Z \) at \( t \) follows from the continuity of the reflecting Brownian motion \( R \), since in a small enough neighborhood of \( t \) the radial part of \( Z \) is given by an excursion of the reflecting Brownian motion \( R \) away from 0 and so the angular part is constant.

For \( t \in \{s : Z_s = 0\} \), note that \( Z_s \) converges to 0 as \( s \to t \) if and only if \( |Z_s| \) converges to 0 as \( s \to t \), where \( |Z_s| \) is the usual Euclidean distance of \( Z_s \) from the origin, which in polar coordinates is equal to the radial part. Thus, the continuity of \( Z \) at \( t \) follows again from the continuity of \( R \), since \( |Z_s| = R_s \) for all \( s \geq 0 \).

For the Markov property, it suffices to show that

\[
P^x (Z_t \in \Gamma | Z_{t_1}, \ldots, Z_{t_k}) = P^x (Z_t \in \Gamma | Z_{t_k}),
\]

(2.1)

for every finite sequence of times \( 0 \leq t_1 < \cdots < t_k < t \), and \( \Gamma \in \mathcal{B}(E) \). We verify (2.1) for \( \Gamma \) of the special form

\[
\Gamma = F \times G
\]

where \( F \in \mathcal{B}(0, \infty) \) and \( G \in \mathcal{B}(0, 2\pi) \). The \( \pi - \lambda \) Theorem then gives the result for all \( \Gamma \in \mathcal{B}(E) \).

Fix times \( 0 \leq t_1 < \cdots < t_k < t \). Using the Markov property for reflecting
Brownian motion and the independence of the angles \( \{\alpha_j\} \), we have for every \( z \in E \)

\[
P^z \left( Z_t \in F \times G \mid Z_{t_k}, \ldots, Z_{t_1} \right) = P^z \left( R_t \in F, A_t \in G, R_t > 0 \mid Z_{t_k}, \ldots, Z_{t_1} \right)
= \sum_{j=1}^{\infty} P^z \left( R_t \in F, \alpha_j \in G; t \in I_j \mid Z_{t_k}, \ldots, Z_{t_1} \right)
= \sum_{j=1}^{\infty} P^z \left( R_t \in F, \alpha_j \in G; t \in I_j \mid (R_{t_k}, A_{t_k}) \right)
= P^z \left( Z_t \in \Gamma \mid Z_{t_k} \right).
\]

This gives the Markov property for \( Z \).

Finally, we check that the semigroup for the process we constructed is the same as the semigroup in Theorem 2.1. Again using the independence of \( R_t \) and \( A_t \), we have for each \( f \in C_0(E) \),

\[
P_t f(0) = E^0 [f(Z_t)] = E^0 \left[ f(R_t, A_t), R_t > 0 \right]
= \sum_{j=1}^{\infty} E^0 [f(R_t, \alpha_j); t \in I_j]
= \sum_{j=1}^{\infty} E^0 \left[ E^0 \left[ f(R_t, \alpha_j); t \in I_j \mid F_t^R \right] \right]
= \sum_{j=1}^{\infty} E^0 \left [ 1_{\{t \in I_j\}} \cdot \int_0^{2\pi} f(R_t, \alpha) \mu(d\alpha) \right]
= E^0 \left[ \int_0^{2\pi} f(R_t, \alpha) \mu(d\alpha) \right] \sum_{j=0}^{\infty} P^0 (t \in I_j)
= E^0 \left[ \int_0^{2\pi} f(R_t, \alpha) \mu(d\alpha) \right] = P_t^+ \tilde{f}(0).
\]

Now suppose that \( z \in E \) such that \( z = (r, \alpha) \neq 0 \). Note that \( T_0 = \inf\{t \geq 0 : R_t = 0\} = \inf\{t \geq 0 : Z_t = 0\} \). Then

\[
P_t f(z) = E^z [f(Z_t)] = E^z \left[ f(Z_t); t < T_0 \right] + E^z \left[ f(Z_t); t \geq T_0 \right].
\]

Note that

\[
E^z \left[ f(Z_t), t < T_0 \right] = E^r \left[ f(R_t, \alpha); t < T_0 \right] = P_t^0 f_\alpha(r),
\]

since \( A_t = \alpha \) on the set \( \{t < T_0\} \). Also, by the strong Markov property for reflecting
Brownian motion,
\[
\mathbb{E}^z [f(Z_t); t \geq T_0] = \sum_{j=1}^{\infty} \mathbb{E}^z [f(R_t, \alpha_j); t \in I_j, t \geq T_0]
\]
\[
= \sum_{j=1}^{\infty} \mathbb{E}^z [\overline{f}(R_t); t \in I_j, t \geq T_0]
\]
\[
= \mathbb{E}^z [\overline{f}(R_t); t \geq T_0]
\]
\[
= \mathbb{E}^z [\overline{f}(R_t)] - \mathbb{E}^z \left[ \overline{f}(R_t); t < T_0 \right]
\]
\[
= P_t^+ \overline{f}(r) - P_t^0 \overline{f}(r).
\]
Therefore,
\[
P_t f(z) = P_t^0 f_\alpha(r) + P_t^+ \overline{f}(r) - P_t^0 \overline{f}(r)
\]
\[
= P_t^+ \overline{f}(r) + P_t^0 (f_\alpha - \overline{f})(r).
\]
Then since we know that \( \{P_t\}_{t \geq 0} \) is a Feller semigroup by Theorem 2.1, it follows that \( Z \) is a Feller diffusion with the desired characteristics.

\[\square\]

### 2.3 Excursions

An advantage to the direct construction of Walsh’s Brownian motion from the sample paths of a reflecting Brownian motion is the ability to easily write down its excursion space and the associated excursion measure. We review some of the basics of excursion theory and record some of the fundamental results which we will need to prove some of the results in this paper. For details of the proofs, which we will omit, and a more comprehensive treatment of excursion theory see [13], [22], and [23].

We are mainly concerned with excursions away from the origin. As we saw in the previous section, Walsh’s Brownian motion is a continuous process. Therefore, the set \( \{t : Z_t \neq 0\} \) is an open subset of \([0, \infty)\) and so we can write it as a countable union of disjoint open intervals,
\[
\{t : Z_t \neq 0\} = \bigcup_{j=1}^{\infty} (a_j, b_j).
\]
We say that Walsh’s Brownian motion makes an excursion away from 0 over each time interval \((a_j, b_j)\). Each excursion is an element of the excursion space and it follows from the construction given in 2.2 that the excursion space for \( Z \) is given by
\[
\mathcal{U}_Z = \mathcal{U}_R \times [0, 2\pi),
\]
where

$$U_R = \{ \text{continuous } f : [0, \infty) \to [0, \infty) \text{ with } f^{-1}(0, \infty) = (0, \zeta), \text{for some } \zeta > 0 \},$$

is the excursion space for the reflecting Brownian motion $R$. In other words, the excursions of Walsh’s Brownian motion are simply the excursions of the reflecting Brownian motion paired with an angle, which is precisely how we constructed the process. Note that $\zeta$ in the definition of $U_R$ is often referred to as the lifetime of the excursion $f$. It is sometimes written as $\zeta(f)$ to emphasize that it is the lifetime of a specific excursion $f$.

Even though the excursions of Walsh’s Brownian motion are countable in number, there is no “next” excursion, that is, we cannot talk about the first, second, etc. excursions. This is due to the irregularities of the sample paths of reflecting Brownian motion. Specifically, we know that in any interval of time, if a Brownian sample path changes sign at least once, then it does so infinitely many times and, therefore, in any interval of time reflecting Brownian motion makes infinitely many excursions away from 0, it makes any at all. Therefore, in any interval of time, if $Z$ returns to 0 at least once, then it returns infinitely often, which implies that 0 is a regular point.

Although we cannot talk of the next excursion, since 0 is regular, we can “label” excursions of $Z$ using the local time at 0. Intuitively, the local time at 0 is a process that characterizes the amount of time spent at 0 before a fixed time $t$. Formally, the local time at 0 for $Z$ is a continuous and increasing process $\{L_t; t \geq 0\}$ with growth set $\{t : Z_t = 0\}$, which means that $L_t$ only increases when the sample path of the Walsh’s Brownian motion is at 0, otherwise $L_t$ is constant. Since

$$\{t : Z_t = 0\} = \{t : R_t = 0\},$$

the local time at 0 for Walsh’s Brownian motion is equal to the local time at 0 for reflecting Brownian motion. Tanaka’s formula provides a convenient representation of the local time at 0, which says that the process $\{L_t; t \geq 0\}$ satisfies

$$L_t = R_t - \int_0^t \text{sgn}(B_s)dB_s,$$

for $t \geq 0$, where $\{B_t; t \geq 0\}$ is a one-dimensional Brownian motion and

$$\text{sgn}(x) = \begin{cases} +1, & x \geq 0; \\ -1, & x < 0. \end{cases}$$

For more on local time see [15] or [20].
To order the excursions of Walsh’s Brownian motion using the local time at 0, we first label the excursion over an interval \((a, b)\) with the local time \(l = L_a\). We say that the excursion at local time \(l\) is before the excursion at local time \(l'\) if \(l < l'\). This allows us to separate the sample paths of Walsh’s Brownian motion into its excursions away from the origin. We can now represent the path as a point process \(\Pi\) in \([0, \infty) \times U_z\), where the point \((l, \xi)\) is in \(\Pi\) if and only if \(Z\) makes an excursion \(\xi\) at local time \(l\).

The fundamental result in excursion theory, making it extremely useful, was provided by Itô. It explains why we would want to represent the sample paths of a process using the excursion point process, which is seemingly more complicated.

**Theorem 2.4 ([13]).** The excursion point process is a Poisson point process with intensity measure given by the product measure of Lebesgue measure on \([0, \infty)\) with a unique \(\sigma\)-finite measure \(\eta\) on the excursion space.

It is immediate from Theorem 2.4 that the number of points of \(\Pi\) in the set \((0, l) \times U\) is a Poisson random variable with parameter given by

\[
(\text{Leb} \times \eta)((0, l) \times U) = l \cdot \eta(U).
\]

The measure \(\eta\) in Theorem 2.4 is called the excursion measure. From our construction of \(Z\) we see that the excursion measure \(\eta\) on \(U_z\) is given by the product measure \(n \times \mu\), where \(n\) is the excursion measure on \(U_R\). Then for any \(U \subseteq U_R\) and \(A \subseteq [0, 2\pi)\),

\[
\eta(U \times A) = \int_A n(U) \mu(d\alpha).
\]

If \(A = [0, 2\pi)\), we will simply write \(\eta(U)\) for \(\eta(U \times [0, 2\pi))\). That is, for any \(U \subseteq U_R\), we write

\[
\eta(U) = \int_0^{2\pi} n(U) d\mu(\alpha).
\]

We see that calculations involving \(\eta\) depend on \(n\). Thus, we review some basic properties of the excursion measure for reflecting Brownian motion, beginning with the following proposition.

**Proposition 2.1 ([22], Proposition 2).** For every \(\epsilon > 0\), we have

\[
n(\{f \in U_R : \epsilon < \sup_t f(t)\}) = \frac{1}{2\epsilon}.
\]
We now review what is called the Markovian nature of excursions. For \( t > 0 \), define the measure \( n_t \) on \((0, \infty)\) by

\[
n_t(\Gamma) = n(\{f \in \mathcal{U}_R : f(t) \in \Gamma; t < \zeta\}).
\]

We denote by \( P^0_t(r, \Gamma) \) the transition function for one-dimensional Brownian motion killed when it first reaches 0, that is, \( P^0_t(r, \Gamma) \) gives the probability that, starting from \( r \geq 0 \), the process will be in \( \Gamma \) at time \( t \). We have the following.

**Theorem 2.5** ([23], Theorem VI.48.1). For \( 0 < t_1 < \cdots < t_k \), and \( \Gamma_1, \ldots, \Gamma_k \subset (0, \infty) \),

\[
n(\{f : f(t_j) \in \Gamma_j, j = 1, \ldots, k; t_k < \zeta\}) = \int_{\Gamma_1} n_{t_1}(dr_1) \int_{\Gamma_2} P^0_{s_2}(r_1, dr_2) \cdots \int_{\Gamma_k} P^0_{s_k}(r_{k-1}, dr_k),
\]

where \( s_j = t_j - t_{j-1} \).

This theorem says that the behavior of an excursion is initially governed by the measure \( n_t \) and then behaves like a Brownian motion killed at 0. It easily follows from Theorem 2.5 that

\[
\int_0^\infty n_t(dr) P^0_s(r, \Gamma) = n_{t+s}(\Gamma).
\]

This implies that \( \{n_t\}_{t>0} \) is an entrance law for the transition semigroup \( \{P^0_t\}_{t \geq 0} \). So to better understand the excursion measure \( n \) we need to understand the entrance law \( \{n_t\}_{t>0} \). For reflecting Brownian motion, it is known that

\[
n_t(dr) = \frac{2re^{-r^2/2t}dr}{\sqrt{2\pi t^3}}.
\]

(For the derivation of the entrance law see [22].) Using the Laplace transform of the entrance law, it can be shown that

\[
n_t((0, \infty)) = n(\{f \in \mathcal{U}_R : \zeta(f) > t\}) = \frac{2}{\sqrt{2\pi t}}.
\]

We will let \( G \) denote the (countable) set of left endpoints of excursions for \( Z \), which is also the set of left endpoints of excursions for \( R \). We also let \( \xi_t \) denote the excursion starting at \( t \in G \). Note that

\[
\xi_t = (e_t, A_{t+}),
\]

where \( e_t \) is the excursion of the reflecting Brownian motion starting at \( t \in G \) and

\[
A_{t+} = \lim_{s \uparrow t} A_s.
\]

We have the following result.
**Theorem 2.6** ([20], Proposition 2.6). For any non-negative optional process $H$, and any non-negative $\psi : [0, \infty) \times U \rightarrow [0, \infty)$,

$$
E \left[ \sum_{t \in G} H_t \psi(t, \xi_t) \right] = E \left[ \int_0^\infty H_t \eta(t, \cdot) dL_t \right].
$$

As promised, we end this section by showing that the distribution of the angular process $A_t$ of Walsh's Brownian motion is given by the probability measure $\mu$. Note that we can write

$$
\left\{ A_t \in d\alpha \right\} = \bigcup_{s \in G \atop s \leq t} \left\{ A_s+ \in d\alpha; \ T_0 \circ \theta_s > t - s \right\},
$$

where, for $s \geq 0$, $\theta_s : \Omega \rightarrow \Omega$ is the shift operator defined by $Z_t + s(\omega) = Z_s + s(\theta_s \omega)$. Thus, by Theorem 2.6,

$$
P^0 (A_t \in d\alpha) = E^0 \left[ \sum_{s \in G \atop s \leq t} 1_{\left\{ A_s+ \in d\alpha; \ T_0 \circ \theta_s > t - s \right\}} \right]
$$

$$
= E^0 \left[ \int_0^t \eta(\xi_s : A_s+ \in d\alpha; \zeta > t - s) dL_s \right]
$$

$$
= E^0 \left[ \mu(d\alpha) \int_0^t n(\epsilon_s : \zeta > t - s) dL_s \right]
$$

$$
= \mu(d\alpha) E^0 \left[ \int_0^t \frac{2dL_s}{\sqrt{2\pi(t - s)}} \right]
$$

$$
= \mu(d\alpha),
$$

where the last equality follows from the fact that

$$
E^0 \left[ \int_0^t \frac{2dL_s}{\sqrt{2\pi(t - s)}} \right] = 1.
$$

For the details of the calculation of this fact see Proposition A.1 in the Appendix.

Using the distribution of $A_t$ we derive a formula for the exit distribution of Walsh’s Brownian motion from a ball centered at the origin. Fix $\epsilon > 0$. We define the stopping time

$$
\tau(\epsilon) = \inf\left\{ t > 0 : |Z_t| = \epsilon \right\},
$$

which is the exit time from the ball of radius $\epsilon$ centered at 0. Then for any measurable function $h$ on $E$,

$$
E^0 \left[ h(Z_{\tau(\epsilon)}) \right] = \int_0^{2\pi} h(\epsilon, \alpha) P^0 (A_t \in d\alpha) = \int_0^{2\pi} h(\epsilon, \alpha) \mu(\alpha). \quad (2.2)
$$

This formula will be helpful in performing calculations later.
Chapter 3

Harmonic Functions

In this chapter we classify the harmonic functions for Walsh’s Brownian motion in the plane. In general, the (probabilistic) definition of a harmonic function for a given stochastic process includes some type of “averaging” property on the boundary of open subsets of the state space. Therefore, we begin this chapter by examining the different topologies on the state space of Walsh’s Brownian motion, $E$. After establishing the relevant topology under which we work, we give the classification of harmonic functions for Walsh’s Brownian motion. Without too much more effort, we also classify the superharmonic functions.

3.1 Topologies on the State Space

We begin with some terminology concerning the probability measure $\mu$ on $[0, 2\pi)$. We say that $\mu$ is discrete if $\text{supp}(\mu)$ is equal to a finite or countable subset of $[0, 2\pi)$. We say that $\mu$ is continuous if it charges no singletons in $[0, 2\pi)$. If $\mu$ is continuous, then $\text{supp}(\mu)$ is equal to a single closed interval or union of closed intervals in $[0, 2\pi)$. In both cases, the state space of Walsh’s Brownian motion is given by

$$E = \{(r, \alpha) : r > 0, \alpha \in \text{supp}(\mu)\}.$$ 

Since the state space of Walsh’s Brownian motion is a subset of $\mathbb{R}^2$, the obvious first choice for the relevant topology on $E$ is the relative topology induced by the Euclidean metric on $\mathbb{R}^2$. In most cases, this topology is sufficient. However, depending on $\mu$, the relative Euclidean topology on $E$ may not include open intervals along a single ray as the following example shows.
**Example 1.** Consider a sequence of angles \( \{\alpha_k\}_{k=1}^{\infty} \) that converges to some value \( \alpha \) within \([0, 2\pi)\). Assume that the probability measure \( \mu \) gives positive probability to only the angles \( \{\alpha\} \cup \{\alpha_k\}_{k=1}^{\infty} \). Fix \( r_0 > 0 \) and let \( \epsilon < r_0 \). Then the set

\[
\{(r, \alpha) \in E : |r - r_0| < \epsilon\}
\]

is not an open set under the relative Euclidean topology from \( \mathbb{R}^2 \), since any open neighborhood of \((r_0, \alpha)\) under the relative topology will contain points on other rays in \( E \). This is demonstrated by Figure 3.1.

![Figure 3.1: The tree topology is finer than the relative Euclidean metric.](image)

Example 1 gives an example of a discrete \( \mu \) for which open intervals along a single ray in \( E \) are not included in the relative Euclidean topology. The same issue arises for any continuous \( \mu \). In order to include these sets, we need to define a different topology on \( E \).

For \( z_1, z_2 \in E \) with \( z_1 = (r_1, \alpha_1) \) and \( z_2 = (r_2, \alpha_2) \), we define a metric \( \rho \) on \( E \) as

\[
\rho(z_1, z_2) = \begin{cases} 
  r_1 + r_2 & \text{if } \alpha_1 \neq \alpha_2, \\
  |r_1 - r_2| & \text{if } \alpha_1 = \alpha_2.
\end{cases}
\]  

(3.1)

This metric measures distances along rays. We refer to the topology \( \rho \) induces on \( E \) as the tree topology.

As we have seen, depending on the measure \( \mu \), the tree topology may be different from the relative Euclidean topology on \( E \). Example 1 shows that the tree topology is
finer than the relative Euclidean topology, since it always includes sets of the form
\[ \{(r, \alpha) : a \leq r \leq b\}. \]

For this reason, we will work under the tree topology on \( E \).

Remarks. (1) Notice that our decision to work under the tree topology on \( E \) only affects open sets not containing the origin, as open sets containing the origin are equivalent under both topologies: any open set containing the origin under the tree topology is contained in an open set under the relative Euclidean topology, and vice versa (see Figure 3.2(b) below).

(2) If \( \mu \) gives positive probability to only finitely many angles, then the two topologies are the same. This is demonstrated by Figure 3.2.

![Figure 3.2: Equivalence of topologies when angular distribution is discrete.](image)

(3) Finally, notice that in Chapter 2 we proved the continuity of the sample paths of Walsh’s Browian motion under the relative Euclidean topology on \( E \). The same proof also shows that the sample paths are continuous with respect to the tree topology, since the process only travels through the plane along the rays in the state space. The process cannot jump from one ray to another without passing through the origin.

We end this section by defining a third topology on the state space \( E \). We denote by \( \mathcal{B}(E) \) the Borel \( \sigma \)-algebra generated by the tree metric \( \rho \), which is the same as the relative Borel \( \sigma \)-algebra on \( E \) generated by the relative Euclidean topology. We say a subset \( B \) of \( E \) is a nearly Borel set if for every \( z \in B \), there exist sets \( B_1, B_2 \in \mathcal{B}(E) \)
such that $B_1 \subseteq B \subseteq B_2$ and

$$P^z (T_{B_2 \setminus B_1} < \infty) = 0,$$

where $T_{B_2 \setminus B_1} = \inf\{t \geq 0 : Z_t \in B_2 \setminus B_1\}$.

A subset $\Gamma$ of $E$ is **finely open** if for each $z \in \Gamma$, there exists a nearly Borel set $B$ such that $z \in B \subseteq \Gamma$ and

$$P^z (T_B > 0) = 1,$$

where $T_B = \inf\{t \geq 0 : Z_t \in B\}$. In other words, a set $\Gamma$ is finely open if almost every sample path of $Z$ starting in $\Gamma$ initially remains in a nearly Borel subset of $\Gamma$ for a positive interval of time. It easily follows that the collection of finely open sets form a topology on $E$, which we call the **fine topology**. It is also easy to see that open sets in the tree topology are finely open. Thus the fine topology is finer than the tree topology.

A function $f$ on $E$ is **finely continuous** if it is continuous with respect to the fine topology. A function $f$ on $E$ is **nearly Borel measurable** if it is measurable with respect to the nearly Borel sets. The following result shows that for nearly Borel functions, fine continuity is equivalent to the process $\{f(Z_t); t \geq 0\}$ being right-continuous.

**Theorem 3.1** ([4], Theorem 1 in §3.5). *Let $f$ be a nearly Borel measurable function on $E$. Then $f$ is finely continuous if and only if $t \mapsto f(Z_t)$ is a.s. right-continuous on $[0, \infty)$.*

We note that Walsh’s Brownian motion is a reversible process, which roughly means that if we run time backwards the resulting process has the same distribution as the process run forward. Note that one-dimensional Brownian motion has this property. So we can construct a Walsh’s Brownian motion with time run in reverse by pairing excursions of a reversed reflecting Brownian motion with angles according to $\mu$. Thus, the time reversal of a Walsh’s Brownian motion will have the same distribution as the original process. Since $Z_t$ is reversible, it follows that if $t \mapsto f(Z_t)$ is right-continuous for some function $f$ on $E$, then by reversing time it is also left-continuous. Thus, for Walsh’s Brownian motion a function $f$ is finely continuous if and only if the process $\{f(Z_t); t \geq 0\}$ is continuous.

### 3.2 Characterization of Harmonic Functions

We provide a characterization of harmonic functions for Walsh’s Brownian motion from three different perspectives, namely, from a probabilistic, analytic, and stochas-
tic process perspective.

**Theorem 3.2.** For a function $h : E \to \mathbb{R}$ the following are equivalent.

(i) $h$ is locally integrable and for every $z \in E$, $\epsilon > 0$,

$$h(z) = E^z \left[ h(Z_{\tau_z(\epsilon)}) \right],$$

where $\tau_z(\epsilon) = \inf \{ t \geq 0 : \rho(Z_t, z) \geq \epsilon \}$.

(ii) For every $z = (r, \alpha) \in E$,

$$h(z) = m_h(\alpha) \cdot r + h(0),$$

where $m_h : [0, 2\pi) \to \mathbb{R}$ is a measurable function that is integrable with respect to $\mu$ and satisfies

$$0 = \int_0^{2\pi} m_h(\alpha) \mu(d\alpha).$$

(iii) The process $\{h(Z_t); t \geq 0\}$ is a continuous martingale with respect to $\{\mathcal{F}_t^z\}_{t \geq 0}$, for every starting point $z \in E$.

We define a function $h$ to be harmonic (for $Z$) if it satisfies the conditions of Theorem 3.2. We call the function $m_h$ the slope function associated with the function $h$. We refer to (i) in Theorem 3.2 as the “ball-averaging property”, (ii) as the “slope-averaging property” and (iii) as the “martingale property”.

**Remarks.** (1) Integrals of functions on $E$ are taken with respect to the measure on $E$ given by the product measure of Lebesgue measure on $[0, \infty)$ with $\mu$ on $[0, 2\pi)$.

(2) The stopping time $\tau_z(\epsilon)$ defined in (i) of the theorem is the exit time from the set $\{ y \in E : \rho(y, z) < \epsilon \}$, which is the open ball under the tree topology centered at $z$ of radius $\epsilon$. This explains why (i) is called the ball-averaging property.

(3) For the function $m_h$ in (ii) of the theorem to be well-defined we set $m_h(\alpha) = 0$ for any $\alpha \in [0, 2\pi)$ such that $\alpha$ is outside the support of $\mu$. In other words, the support of $m_h$ is the same as the support of $\mu$.

(4) The requirement in (ii) that $m_h$ be measurable guarantees that $h$ is measurable and, therefore, so is the process $\{h(Z_t); t \geq 0\}$. To see this, we define

$$f : E \to \mathbb{R}^2 \quad \text{and} \quad g : \mathbb{R}^2 \to \mathbb{R}$$

$$\begin{align*}
(r, \alpha) &\mapsto (r, m_h(\alpha)) \\
(x, y) &\mapsto h(0) + xy
\end{align*}$$
Note that we set \( f(0) = (0,0) \). Then we can write

\[
h(r, \alpha) = g(f(r, \alpha)) = h(0) + m_h(\alpha) \cdot r.
\]

So \( h \) is measurable if and only if both \( f \) and \( g \) are measurable. It is easy to check that \( g \) is measurable (in fact, it is continuous), and \( f \) is measurable if and only if \( m_h \) is. Thus, \( h \) is measurable if and only if \( m_h \) is measurable.

Notice that if \( \mu \) is discrete, then a function \( h \) of the form (3.2) is measurable for any choice of \( m_h \). For any \( a, b \in \mathbb{R} \), with \( a < b \),

\[
h^{-1}((a,b)) = \bigcup_k h_k^{-1}((a,b)),
\]

where \( h_k \) is the restriction of \( h \) to the ray at angle \( \alpha_k \), i.e. \( h_k(r) = h(r, \alpha_k) \). Then if \( m_h(\alpha_k) > 0 \),

\[
h_k^{-1}((a,b)) = \{(r, \alpha_k) : r \in (h(0) + m_h(\alpha_k) \cdot a, h(0) + m_h(\alpha_k) \cdot b)\},
\]

and if \( m_h(\alpha_k) < 0 \),

\[
h_k^{-1}((a,b)) = \{(r, \alpha_k) : r \in (h(0) + m_h(\alpha_k) \cdot b, h(0) + m_h(\alpha_k) \cdot a)\}.
\]

So \( h^{-1}((a,b)) \) is a countable union of measurable sets, which is measurable. The countability is lost when \( \mu \) is continuous, which is why we need to assume that \( m_h \) is measurable in general.

Before proving Theorem 3.2, we examine the continuity of harmonic functions under the different topologies on \( E \). Our definition of a harmonic function only guarantees that it will be finely continuous with respect to Walsh’s Brownian motion. The next lemma shows that a necessary and sufficient condition for a function which is radially linear to be finely continuous is that its slope function be integrable on \([0,2\pi)\) with respect to \( \mu \) when \( \mu \) is continuous, which means that the series \( \sum_k m_h(\alpha_k)\mu(\alpha_k) \) converges absolutely if \( \mu \) is discrete. We will need this lemma in the proof of Theorem 3.2.

**Lemma 3.1.** Let \( h : E \to \mathbb{R} \) and suppose that for every \((r, \alpha) \in E\),

\[
h(r, \alpha) = m_h(\alpha) \cdot r + h(0),
\]

for some function \( m_h : [0,2\pi) \to \mathbb{R} \). Then \( h \) is finely continuous if and only if \( m_h \) is integrable with respect to \( \mu \).
Proof. Without loss of generality, we assume that \( h(0) = 0 \). We may do this because constant functions satisfy the conditions of the lemma and sums of radially linear functions are also radially linear.

We assume that \( h \) is integrable. We also assume that \( h \) is not constant, the result being obvious in that case. Note that if \( t \) is not in the zero set of \( Z \), then the continuity of \( h(Z) \) at \( t \) follows from the continuity of the reflecting Brownian motion \( R \).

For \( \epsilon > 0 \), we can choose \( \delta > 0 \) such that if \( |s - t| < \delta \) then \( s \) is less than the first hitting time of 0 for \( Z \) after time \( t \) and such that \( |R_s - R_t| < \epsilon/m \), where \( m \) is the absolute value of the slope function at angle \( A_t = A_s \). Then \( |s - t| < \delta \) implies that

\[
|h(Z_s) - h(Z_t)| = |m_h(A_s)R_s - m_h(A_t)R_t| = m|R_s - R_t| < \epsilon
\]

which gives the a.s. continuity at \( t \notin \{ s : Z_s = 0 \} \).

Now suppose \( t \in \{ s : Z_s = 0 \} \). First we compute the probability that infinitely many excursions of \( h(Z) \) away from 0 reach an absolute value bigger than some \( \epsilon > 0 \) in a local time interval of fixed length. Because the excursion process is a Poisson point process, we know that the number of excursions away from 0 which reach \( \epsilon \) in a local time interval of length \( l \) is a Poisson random variable with mean

\[
l \times \eta(\{ g \in \mathcal{U}_Z : \sup_s |h(g(s))| > \epsilon \}).
\]

Note that

\[
\eta(\{ g \in \mathcal{U}_Z : \sup_s |h(g(s))| > \epsilon \}) = \int_0^{2\pi} n(\{ f \in \mathcal{U}_R : \sup_s |m_h(\alpha)f(s)| > \epsilon \})d\mu(\alpha).
\]

Using Proposition 2.1 gives,

\[
n(\{ f \in \mathcal{U}_R : \sup_s |m_h(\alpha)f(s)| > \epsilon \}) = \frac{|m_h(\alpha)|}{\epsilon}.
\]

Thus, if we let \( F_k \) be the event that \( k \) excursions of \( h(Z) \) reach \( \epsilon \) in a local time interval of length \( l \), then

\[
P(F_k) = \frac{e^{-\lambda} \lambda^k}{k!},
\]

where

\[
\lambda = \frac{1}{\epsilon} \int_0^{2\pi} |m_h(\alpha)|d\mu(\alpha),
\]
which is finite by the integrability of \( m_h \). Then
\[
\sum_{k=0}^\infty P(F_k) = \sum_{k=0}^\infty \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} e^\lambda = 1.
\]
Therefore, by the Borel-Cantelli lemma the probability that infinitely many excursions
of \( h(Z) \) reach \( \epsilon \) on a local time interval of fixed length is 0.

Define
\[
\gamma_l \equiv \inf\{s : L_s > l\}
\]
the (right-continuous) inverse of local time. Choose local time \( l \) such that \( t < \gamma_l \). Then
since almost surely only finitely many excursions from 0 get up to \( \epsilon \) before local time
\( l \) elapses, the same is true on the real time interval \([0, \gamma_l]\). Choose \( \delta > 0 \) such that
\((t - \delta, t + \delta) \subset [0, \gamma_l]\) and no excursion that reaches \( \epsilon \) occurs during \((t - \delta, t + \delta)\). Then
\[
|s - t| < \delta \implies |h(Z_s) - h(Z_t)| = |h(Z_s)| < \epsilon,
\]
and \( h(Z) \) is a.s. continuous at \( t \).

Now assume that \( h \) is finely continuous. If \( m_h \) is not integrable, then
\( P(F_k) = 1 \), for all \( k \). So
\[
\sum_{k=0}^\infty P(F_k) = \infty,
\]
which implies that with probability 1, infinitely many excursions of \( h(Z) \) will reach \( \epsilon \) on
a local time interval of fixed length for every \( \epsilon > 0 \). This contradicts the a.s. continuity
of \( h(Z) \). 

Lemma 3.1 shows that harmonic functions for Walsh’s Brownian motion are
finely continuous. But what about the other topologies on the state space of Walsh’s
Brownian motion? In the next proposition, we see that if we make further assumptions
about the slope function then we obtain continuity under the other topologies on the
state space \( E \) we described.

**Proposition 3.1.** Let \( h : E \to \mathbb{R} \) be of the form
\[
h(r, \alpha) = m_h(\alpha) \cdot r + h(0)
\]
for every \( (r, \alpha) \in E \).

(a) \( h \) is continuous with respect to the tree topology on \( E \) if and only if the slope
function \( m_h \) is bounded on \([0, 2\pi]\).
(b) $h$ is continuous with respect to the Euclidean topology on $E$ if and only if the slope function $m_h$ is continuous on $\text{supp}(\mu)$.

Proof. (a) Suppose that $m_h$ is bounded on $[0, 2\pi)$. So there exists a constant $M$ such that $|m_h(\alpha)| \leq M$ for all $\alpha \in [0, 2\pi)$. Fix $z = (r_z, \alpha_z) \in E$ and let $\epsilon > 0$. If $z \neq 0$, choose $\delta < \min\{r_z, \epsilon/|m_h(\alpha_z)|\}$. Then $\rho(z, y) < \delta$ implies that $\alpha_z = \alpha_y$ and

$$|h(z) - h(y)| = |m_h(\alpha_z)||r_z - r_y| < \epsilon.$$ 

If $z = 0$, choose $\delta < \epsilon/M$. Then $\rho(z, y) < \delta$ implies that $r_y < \epsilon/M$ and

$$|h(0) - h(y)| = r_y|m_h(\alpha_y)| < \epsilon.$$ 

Now suppose that $h$ is continuous with respect to the tree topology on $E$. Fix $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$\rho(0, z) < \delta \quad \Rightarrow \quad |h(0) - h(z)| < \epsilon.$$ 

Fix $r < \delta$. Then for all $\alpha \in [0, 2\pi)$ we have

$$\epsilon > |h(0) - h(r, \alpha)| = r|m_h(\alpha)|.$$ 

Let $M = \epsilon/r$, then $|m_h(\alpha)| \leq M$ for all $\alpha \in [0, 2\pi)$.

(b) Suppose $m_h$ is continuous on $\text{supp}(\mu)$. Fix $z = (r_z, \alpha_z) \in E$ and let $\epsilon > 0$. By the continuity of $m_h$ there exists $\delta_1 > 0$ such that

$$|\alpha - \alpha_z| < \delta_1 \quad \Rightarrow \quad |m_h(\alpha) - m_h(\alpha_z)| < \frac{\epsilon}{2r_z}.$$ 

Also by the continuity of $m_h$ there exists a constant $M$ such that $|m_h(\alpha)| \leq M$ for all $\alpha \in \text{supp}(\mu)$, since $\text{supp}(\mu)$ is a closed subset of $[0, 2\pi)$. Choose $\delta < \min\{\delta_1, \epsilon/2M\}$. Then for all $(r, \alpha) \in E$ such that $|(r, \alpha) - (r_z, \alpha_z)| < \delta$, we have

$$|h(r, \alpha) - h(r_z, \alpha_z)| = |rm_h(\alpha) - r_zm_h(\alpha_z)|$$

$$= |rm_h(\alpha) - r_zm_h(\alpha) + r_zm_h(\alpha) - r_zm_h(\alpha_z)|$$

$$\leq |r - r_z||m_h(\alpha)| + r_z|m_h(\alpha) - m_h(\alpha_z)|$$

$$< \epsilon.$$ 

Now suppose that $h$ is continuous with respect to the Euclidean topology on $E$ and fix $\alpha \in \text{supp}(\mu)$ and $\epsilon > 0$. Then for any fixed $r > 0$ there exists $\delta_1 > 0$ such that

$$|(r', \alpha') - (r, \alpha)| < \delta_1 \quad \Rightarrow \quad |h(r', \alpha') - h(r, \alpha)| < r\epsilon,$$
by the continuity of \( h \). Choose \( \delta > 0 \) such that \( \delta \) is smaller than the angle that spans the open ball centered at \((r, \alpha)\) with radius \( \delta_1 \). Then for all \( \alpha' \in \text{supp}(\mu) \) such that \(|\alpha' - \alpha| < \delta\) we have

\[
|m_h(\alpha') - m_h(\alpha)| = \frac{1}{r}|m_h(\alpha') \cdot r - m_h(\alpha) \cdot r|
= \frac{1}{r}|h(r, \alpha') - h(r, \alpha)|
< \epsilon.
\]

\( \square \)

**Remark.** Notice that if \( \text{supp}(\mu) \) is finite, then any function of the form

\[
h(r, \alpha) = m_h(\alpha) \cdot r + h(0)
\]

is continuous under all three topologies on \( E \).

One may now wonder if there are any harmonic functions that are not continuous under the tree topology, or the relative Euclidean topology, or both. The following examples give harmonic functions in each of the three cases. For the examples, we assume that \( \mu \) is the uniform distribution on \([0, 2\pi)\), i.e. \( \mu(\alpha) = \frac{1}{2\pi} \) for all \( \alpha \in [0, 2\pi) \).

**Example 2.** For an example of a harmonic function that is continuous under the tree topology but not the relative Euclidean topology, we define the slope function

\[
m_h(\alpha) = \begin{cases} 
n & \text{if } \alpha \in [0, \pi) \\
-n & \text{if } \alpha \in [\pi, 2\pi) 
\end{cases}
\]

where \( n \) is any positive constant. Then \( m_h \) is integrable and

\[
\int_0^{2\pi} m_h(\alpha) \mu(\alpha) \, d\alpha = \frac{n}{2} - \frac{n}{2} = 0.
\]

However, \( m_h \) is not continuous on \([0, 2\pi)\).

**Example 3.** For an example of a harmonic function that is continuous under the relative Euclidean topology but not the tree topology, we define the slope function

\[
m_h(\alpha) = (2\pi - \alpha)^{-1/2} - 2(2\pi)^{-1/2}.
\]

Then \( m_h \) is integrable and

\[
\int_0^{2\pi} m_h(\alpha) \mu(\alpha) \, d\alpha = \frac{1}{2\pi} \int_0^{2\pi} m_h(\alpha) \, d\alpha = 0.
\]

However, \( m_h \) is not bounded on \([0, 2\pi)\).
Example 4. For an example of a harmonic function that is not continuous under either the tree topology or the relative Euclidean topology, we define the slope function

\[ m_h(\alpha) = \begin{cases} 
0 & \text{if } \alpha = 0 \\
-\alpha^{-1/2} & \text{if } \alpha \in (0, \pi) \\
(2\pi - \alpha)^{-1/2} & \text{if } \alpha \in [\pi, 2\pi) 
\end{cases} \]

Then \( m_h \) is integrable and

\[ \int_0^{2\pi} m_h(\alpha) \mu(d\alpha) = \frac{1}{2\pi} \int_0^{2\pi} m_h(\alpha) d\alpha = 0. \]

However, \( m_h \) is neither bounded nor continuous on \([0, 2\pi)\).

Notice that in Examples 3 and 4 we can replace every \(-1/2\) exponent with \(-1/p\) for any \(p > 1\), to obtain infinitely many examples of each case.

3.2.1 Proof of Theorem 3.2.

We begin the proof of Theorem 3.2 by showing the equivalence between (i) and (ii). Then we will show that (ii) and (iii) are equivalent. We will prove the theorem in the case that \( \mu \) is continuous. Note that for the discrete case the proof is the same except all integrals over \([0, 2\pi)\) with respect to \( \mu \) are replaced by weighted averages with respect to \( \mu \), that is, infinite sums over \( k \).

(i) \( \Rightarrow \) (ii): If \( h \) is a locally integrable function satisfying the ball-averaging property for every \( z \in E \), then the restriction of \( h \) to a single ray in \( E \) at any angle \( \alpha \in [0, 2\pi) \) is also locally integrable and satisfies the ball-averaging property under the usual Euclidean topology on \((0, \infty)\). Since the process \( Z \) restricted to a single ray is reflecting Brownian motion and since harmonic functions for Brownian motion are linear, \( h \) restricted to a ray at any angle \( \alpha \in [0, 2\pi) \) is linear by Theorem 1.2. So, for \( r > 0 \) and \( \alpha \in [0, 2\pi) \),

\[ h(r, \alpha) = m_h(\alpha) \cdot r + b(\alpha), \]

where \( b(\alpha) \) denotes the intercept and \( m_h(\alpha) \) denotes the slope of \( h \) on the ray at angle \( \alpha \).

For any \( z = (r, \alpha) \in E \) with \( z \neq 0 \), we have

\[ h(z) = \mathbb{E}^z \left[h(Z_{\tau_z(r)})\right] = \frac{1}{2}(h(0) + h(2r, \alpha)), \]
since the process restricted to a single ray is reflecting Brownian motion. Using the fact that $h$ is linear on each ray gives
\[ m_h(\alpha) \cdot r + b(\alpha) = \frac{1}{2} h(0) + m_h(\alpha) \cdot r + \frac{1}{2} b(\alpha), \]
and so
\[ b(\alpha) = h(0). \]
Therefore, the intercepts of $h$ on a ray at any angle $\alpha \in [0, 2\pi)$, are all equal to the value of $h$ at the origin, and we have
\[ h(r, \alpha) = m_h(\alpha) \cdot r + h(0). \]

For any $\epsilon > 0$, let $\tau(\epsilon) = \inf\{ t \geq 0 : |Z_t| = \epsilon \}$. Note that $\tau(\epsilon) = \tau_0(\epsilon)$ is the exit time from the ball centered at the origin of radius $\epsilon$. Then, using the exit distribution for $Z$ given by (2.2),
\[
E^0 \left[ h(Z_{\tau(\epsilon)}) \right] = \int_0^{2\pi} h(\epsilon, \alpha) \mu(d\alpha) = \int_0^{2\pi} (m_h(\alpha) \cdot \epsilon + h(0)) \mu(d\alpha) = h(0) + \epsilon \int_0^{2\pi} m_h(\alpha) \mu(d\alpha),
\]
hence
\[ 0 = \int_0^{2\pi} m_h(\alpha) \mu(d\alpha). \]
Finally, the local integrability of $h$ implies that the slope function $m_h$ is integrable.

(ii) $\Rightarrow$ (i): If $h$ satisfies the slope-averaging property, then $h$ is linear on every ray. Therefore, the restriction of $h$ to a ray at any angle $\alpha \in [0, 2\pi)$, is harmonic for the reflecting Brownian motion. Thus, $h$ satisfies the ball-averaging property on $E \setminus \{0\}$.

Let $\epsilon > 0$. Then, using the averaging property of $m_h$,
\[
E^0 \left[ h(Z_{\tau(\epsilon)}) \right] = \int_0^{2\pi} h(\epsilon, \alpha) \mu(d\alpha) = \int_0^{2\pi} m_h(\alpha) \mu(d\alpha) = h(0),
\]
which gives the ball-averaging property for \( h \) at the origin. The integrability of the slope function gives the local integrability of \( h \) on all of \( E \). Thus we have established the equivalence of the first two definitions of a harmonic function for Walsh’s Brownian motion given by the theorem.

In order to show the equivalence between (ii) and (iii) in Theorem 3.2, we first prove the following lemma. This gives the main result needed to establish the martingale property for harmonic functions of Walsh’s Brownian motion.

**Lemma 3.2.** Let \( h : E \to \mathbb{R} \) be of the form

\[
h(r, \alpha) = m_h(\alpha) \cdot r + h(0)
\]

for \( (r, \alpha) \in E \), where the associated slope function \( m_h \) satisfies

\[
0 = \int_0^{2\pi} m_h(\alpha) \mu(d\alpha).
\]

Then for all \( z \in E \),

\[
h(z) = \mathbb{E}^z[ h(Z_t) ].
\]

**Proof.** Suppose first that \( Z \) starts from 0. For all \( t \geq 0 \), \( h(Z_t) \) can be written in terms of its excursions away from 0 as

\[
h(Z_t) = \sum_{s \in G, s \leq t} h(\xi_s(t-s)) \mathbb{1}_{\{T_0 \circ \theta_s > t-s\}}
\]

\[
= \sum_{s \in G, s \leq t} h(e_s(t-s), A_{s+}) \mathbb{1}_{\{T_0 \circ \theta_s > t-s\}}.
\]

Taking expectations of both sides and using Theorem 2.6, we have

\[
\mathbb{E}^0[h(Z_t)] = \mathbb{E}^0 \left[ \sum_{s \in G, s \leq t} h(e_s(t-s), A_{s+}) \mathbb{1}_{\{T_0 \circ \theta_s > t-s\}} \right]
\]

\[
= \mathbb{E}^0 \left[ \int_0^t \eta(h(e_s(t-s), A_{s+}); \zeta > t-s) dL_s \right],
\]

(3.3)

where \( L_s \) is the local time of \( Z \) at 0 which we recall is the same as the local time for the reflecting Brownian motion. For the expectation on the right-hand side, we have

\[
\eta(h(e_s(t-s), A_{s+}); \zeta > t-s) = \int_0^{2\pi} n(h(e_s(t-s), \alpha); \zeta > t-s) d\mu(\alpha)
\]

\[
= \int_0^{2\pi} n_{t-s}(h(e_s(t-s), \alpha)) d\mu(\alpha),
\]
and for each $\alpha$,
\[
\eta_t(s) = 2 \int_0^\infty \left( h(0) + m_h(\alpha) \cdot r \right) \frac{e^{-r^2/2(t-s)}}{\sqrt{2\pi(t-s)^3}} dr
\]
\[
= \frac{2h(0)}{\sqrt{2\pi(t-s)}} + m_h(\alpha),
\]
where the last equality follows from Proposition A.2 in the Appendix. Thus, since $\mu$ is a probability measure on $[0, 2\pi)$ and $\int_{[0,2\pi]} m_h(\alpha) d\mu(\alpha) = 0$,
\[
\eta(h(\varepsilon(t-s), A_s+); \zeta > t-s) = \int_0^{2\pi} \left( \frac{2h(0)}{\sqrt{2\pi(t-s)}} + m_h(\alpha) \right) d\mu(\alpha)
\]
\[
= \frac{2h(0)}{\sqrt{2\pi(t-s)}},
\]
Therefore, by Fubini’s Theorem and using the fact that
\[
E^0 \left[ \int_0^t \frac{2dL_s}{\sqrt{2\pi(t-s)}} \right] = 1,
\]
which is shown in Proposition A.1 in the Appendix, we have
\[
E^0 \left[ \int_0^t \eta(h(\varepsilon(t-s), A_s+); \zeta > t-s) dL_s \right] = E^0 \left[ \int_0^t \frac{2h(0)}{\sqrt{2\pi(t-s)}} dL_s \right]
\]
\[
= h(0).
\]
Thus, by equation (3.3),
\[
E^0 [h(Z_t)] = h(0).
\]
Now suppose that $Z$ starts from any $z = (r, \alpha) \in E$. In this case, when writing $h(Z_t)$, $t \geq 0$, in terms of its excursions away from 0, an additional term is needed to account for the path of the process before the first excursion away from 0, i.e., an incursion term is needed. This gives
\[
h(Z_t) = h(Z_t) 1_{\{t < T_0\}} + \sum_{\substack{s \in G \\ s \leq t}} h(\varepsilon(s), A_s+) 1_{\{T_0 \circ \theta_s > t-s\}}, \quad t \geq 0,
\]
where the first term on the right-hand side is the incursion term. Using the results from above, we now have
\[
E^z [h(Z_t)] = E^z [h(Z_t); t < T_0] + h(0)E^z \left[ \int_0^t \frac{2dL_s}{\sqrt{2\pi(t-s)}} \right]. \quad (3.4)
\]
In order to compute the expectations in the above sum, notice that on the set \( \{ t < T_0 \} \), we can write \( h(Z_t) = h(0) + m_h(\alpha)R_t \). So,

\[
E^z [h(Z_t); t < T_0] = E^{(r,\alpha)} [h(0) + m_h(\alpha)R_t; t < T_0] \\
= h(0)P^{(r,\alpha)} (t < T_0) + m_h(\alpha)E^r [R_t; t < T_0].
\]

Using the formula for the density of Brownian motion absorbed at 0, we find that

\[
E^r [R_t; t < T_0] = r,
\]

(see Proposition A.3 in the Appendix), which gives

\[
E^z [h(Z_t); t < T_0] = h(0)P^z (t < T_0) + m_h(\alpha) \cdot r.
\]

Again using the excursions of \( Z \) away from 0, we also notice that

\[
P^z (t \geq T_0) = E^z \left[ \sum_{s \in G, s \leq t} 1_{\{T_0 \circ \theta_s > t-s\}} \right] \\
= E^z \left[ \int_0^t \mathbb{1}(t - s \leq t) dL_s \right] \\
= E^z \left[ \int_0^t \frac{2dL_s}{\sqrt{2\pi(t-s)}} \right],
\]

where the first equality follows from writing the event that \( \{ t \geq T_0 \} \) as the union over \( s \in G \) with \( s \leq t \), of events that the lifetime of the excursion starting at time \( s \) is less than \( t-s \). Therefore,

\[
P^z (t < T_0) + E^z \left[ \int_0^t \frac{2dL_s}{\sqrt{2\pi(t-s)}} \right] = 1.
\]

Assembling all this in equation (3.4) gives

\[
E^z [h(Z_t)] = h(0) \left( P^z (t < T_0) + E^z \left[ \int_0^t \frac{2dL_s}{\sqrt{2\pi(t-s)}} \right] \right) + m_h(\alpha) \cdot r \\
= h(0) + m_h(\alpha) \cdot r \\
= h(z).
\]
We can now complete the proof of Theorem 3.2.

(ii) ⇒ (iii): Assume that \( h \) satisfies the slope-averaging property. Then by Lemma 3.1, \( \{h(Z_t); t \geq 0\} \) is continuous. By Lemma 3.2 and the Markov property for \( Z \), we have for any \( z \in E \) and \( s \leq t \),

\[
E^z \left[ h(Z_t) | \mathcal{F}_s \right] = E^{Z_s} \left[ h(Z_{t-s}) \right] = h(Z_s).
\]

Thus \( \{h(Z_t); t \geq 0\} \) is a continuous martingale for every starting point \( z \in E \).

(iii) ⇒ (ii): Now assume that \( \{h(Z_t); t \geq 0\} \) is a continuous martingale for every starting point \( z \in E \). This implies that \( h \) is linear on each ray, since starting from any \( z \neq 0 \), the process \( Z \) restricted to the ray containing \( z \) is equal to \( R \), reflecting Brownian motion. Therefore, on the ray containing \( z \), \( h \) is harmonic for \( R \), and so linear. Thus, \( h \) is of the form

\[
h(r, \alpha) = m_h(\alpha) \cdot r + b(\alpha).
\]

The continuity of \( h(Z) \) forces the intercepts to be equal to \( h(0) \) and the slope function to be integrable by Lemma 3.1. So \( h \) is of the form

\[
h(r, \alpha) = m_h(\alpha) \cdot r + h(0),
\]

where \( m_h \) is \( L^1([0, 2\pi], \mu) \). We let

\[
\overline{m} = \int_0^{2\pi} m_h(\alpha) \mu(d\alpha)
\]

and want to show that \( \overline{m} = 0 \).

Write

\[
h(Z_t) = h(R_t, A_t) = h(0) + m_h(A_t)R_t
\]

\[
= h(0) + (m_h(A_t) - \overline{m})R_t + (\overline{m}R_t - \overline{m}L_t) + \overline{m}L_t.
\]

Since \( \overline{m}R_t - \overline{m}L_t \) is a martingale, this implies that the remaining sum of terms

\[
h(0) + (m_h(A_t) - \overline{m})R_t + \overline{m}L_t
\]

is a martingale. Note that the function \( g(r, \alpha) = h(0) + (m_h(\alpha) - \overline{m})r \) satisfies (ii) of Theorem 3.2. Thus from the previous direction, we know that

\[
g(Z_t) = h(0) + (m_h(A_t) - \overline{m})R_t
\]

is a martingale. Therefore, we must also have that the only remaining term in equation (3.5), \( \overline{m}L_t \), is a martingale. Since \( L_t \) is a continuous and increasing process, this implies that \( \overline{m}L_t \) is constant and so \( \overline{m} \) must equal zero.
3.2.2 Examples of Harmonic Functions

We end this section by giving examples of harmonic functions for Walsh’s Brownian motion. The first class of examples is given by solving a Dirichlet problem, from which we derive a formula for hitting probabilities. Then we show that the only non-negative harmonic functions for Walsh’s Brownian motion are constant.

**Example 5.** Let $D$ be a bounded open subset of $E$ (under the tree topology) and suppose a function $f : \partial D \to \mathbb{R}$ is given. We want to find a function $h : \overline{D} \to \mathbb{R}$ that is harmonic in $D$ with boundary values given by $f$. Such a function is often referred to as a solution to the Dirichlet problem on $D$ for the boundary function $f$. This is a classical problem in analysis for which we present a probabilistic solution in this context, using the results of the previous section.

We define the first exit time from $D$ as

$$\tau_D = \inf\{t \geq 0 : Z_t \in D^c\}.$$  

Then a solution to the Dirichlet problem on $D$ for $f$ is given by

$$h(z) = \mathbb{E}^z[f(Z_{\tau_D})],$$

for $z \in D$, provided that

$$\mathbb{E}^z[|f(Z_{\tau_D})|] < \infty$$

for all $z \in D$. By definition of $\tau_D$, $h(z) = f(z)$ for all $z \in \partial D$. For $z \in D$, choose any $\epsilon > 0$ such that

$$B_\rho(z, \epsilon) = \{z' \in E : \rho(z, z') < \epsilon\} \subseteq D.$$  

Then by the strong Markov property for Walsh’s Brownian motion,

$$h(z) = \mathbb{E}^z[f(Z_{\tau_D})]$$

$$= \mathbb{E}^z\left[\mathbb{E}^z[f(Z_{\tau_D})|\mathcal{F}_{\tau_D(z)}]\right]$$

$$= \mathbb{E}^z[h(Z_{\tau_D(z)})]$$

Thus, for $z \in D$, $h$ satisfies the ball-averaging property and, therefore, by Theorem 3.2 $h$ is harmonic in $D$.

As an application, we derive a formula for the hitting probabilities of Walsh’s Brownian motion. Suppose $D$ is any open neighborhood of the origin and fix an angle $\alpha \in \text{supp}(\mu)$. We want the probability that the Walsh’s Brownian motion starting inside
$D$ will first hit the boundary of $D$ on the ray at angle $\alpha$, i.e. the function defined on $D$ by

$$h(z) = \mathbb{P}^z (Z_{r_D} = z_\alpha),$$

where $z_\alpha$ is the point in $\partial D$ on the ray at angle $\alpha$. If we define the function

$$f(z) = \begin{cases} 1 \text{ if } z = z_\alpha \\ 0 \text{ otherwise} \end{cases}$$

for $z \in \partial D$, then $h$ is a solution to the Dirichlet problem on $D$ for $f$, since

$$h(z) = \mathbb{E}^z [f(Z_{r_D})].$$

Hence, $h$ is a harmonic function on $D$. We derive an explicit formula for $h$ when $\mu$ is discrete.

Suppose that $\mathrm{supp}(\mu) = \{\alpha_j\}_{j=1}^{\infty}$. Let

$$D_j = \{(r, \alpha_j) : 0 \leq r < r_j\},$$

where $r_j > 0$, for $j = 1, 2, \ldots$. Then

$$D = \bigcup_{j=1}^{\infty} D_j$$

is an open neighborhood of 0. For each $j = 1, 2, \ldots$, the function

$$h_j(z) = \mathbb{P}^z (Z_{r_D} = (r_j, \alpha_j))$$

is a harmonic function on $D$. By Theorem 3.2, we know that $h_j$ is linear on each ray and satisfies the slope-averaging property. This implies that

$$h_j(r, \alpha_i) = \begin{cases} h_j(0) \left(1 - \frac{r}{r_i}\right) & \text{if } i \neq j \\ \left(\frac{1 - h_j(0)}{r_j}\right) r + h_j(0) & \text{if } i = j \end{cases}$$

where

$$h_j(0) = \left(\frac{r_j}{\mu(\alpha_j)} \sum_{i=1}^{\infty} \frac{\mu(\alpha_i)}{r_i}\right)^{-1}.$$

Notice that if $D = B_\mu(0, \epsilon)$ for some $\epsilon > 0$, then

$$h_j(0) = \frac{1}{\mu(\alpha_j)},$$

for each $j = 1, 2, \ldots$. This says that, starting from the origin, the probability of Walsh’s Brownian motion exiting the ball $B_\mu(0, \epsilon)$ along the $j^{th}$ ray is given by the inverse of the probability assigned to that ray by $\mu$. 


Example 6. As a trivial example, we note that the only non-negative functions on $E$ which are harmonic for Walsh’s Brownian motion are constant. If we let $h : E \to \mathbb{R}$ be a harmonic function such that $h(z) \geq 0$, for all $z \in E$, then by the linearity of harmonic functions, we know that on each ray $h$ must be either increasing or constant in order to remain non-negative. Now if $h$ is increasing on a given ray, then the value of the slope function $m_h$ is positive for that angle. However, since $h$ also satisfies the slope-averaging property, $m_h$ must be negative for at least one $\alpha \in \text{supp}(\mu)$ in order to cancel this positive slope, thereby insuring that the average value of $m_h$ is 0. But we know this is not possible if $h$ is non-negative on $E$. Thus, $h$ must be constant on each ray and, in fact, the constant on each ray is the same since the values must agree at the origin.

3.3 Excessive Functions

A measurable function $f : E \to [0, \infty]$ is excessive for $Z$ if

(a) $P_t f \leq f$, for every $t > 0$;

(b) $\lim_{t \to 0} P_t f = f$.

Since property (a) implies that $\lim_{t \to 0} P_t f \leq f$, we can replace property (b) with the equivalent property

(b’) $\lim_{t \to 0} P_t f \geq f$.

Proposition 3.2. If $f$ is excessive, then $\{f(Z_t); t \geq 0\}$ is a supermartingale with respect to $\{\mathcal{F}_t^Z\}_{t \geq 0}$, for every starting point $z \in E$.

Proof. By the Markov property for $Z$ and property (a) in the definition of excessive, for all $s \leq t$,

$$
\mathbb{E}^z [h(Z_t) | \mathcal{F}_s^Z] = \mathbb{E}^{Z_s} [h(Z_{t-s})]
$$

$$
= P_{t-s} f(Z_s)
$$

$$
\leq f(Z_s).
$$

Notice that Proposition 3.2 does not say anything about the continuity of the process $\{f(Z_t); t \geq 0\}$. In fact, property (b) in the definition guarantees that excessive functions are finely continuous. For a proof of this see [3].
3.4 Superharmonic Functions

We now provide a classification of superharmonic functions for Walsh’s Brownian motion.

**Theorem 3.3.** Let \( h : E \rightarrow (-\infty, \infty] \) and assume that \( h \) is lower semi-continuous. Then the following are equivalent.

(i) For every \( z \in E \) and every \( \epsilon > 0 \),

\[
h(z) \geq E^z \left[ h(Z_{\tau_\varepsilon(z)}) \right].
\]

(ii) The restriction \( h_\alpha(r) = h(r, \alpha) \) is concave for every \( \alpha \in [0, 2\pi) \), and the derivative function \( h'_\alpha(0+) \) satisfies

\[
0 \geq \int_0^{2\pi} h'_\alpha(0+) \mu(d\alpha).
\]

(iii) The process \( \{h(Z_t); t \geq 0\} \) is a continuous supermartingale with respect to \( \{\mathcal{F}_t^Z\}_{t \geq 0} \), for every starting point \( z \in E \).

**Remark.** The derivative function in (ii) is defined for each \( \alpha \) by

\[
h'_\alpha(0+) = \lim_{r \downarrow 0} \frac{h(r, \alpha) - h(0)}{r}.
\]

We define a function \( h \) on \( E \) to be superharmonic (for \( Z \)) if it satisfies the conditions of Theorem 3.3. We say that a function is radially concave if its restriction to each ray in \( E \) is a concave function.

Let \( f \) be a concave function on \([0, \infty)\). We recall the following properties of concave functions in one-dimension, which we will use below.

(i) Fix \( a \geq 0 \). For every \( r \geq a \),

\[
\frac{f(r) - f(a)}{r - a} \leq f'(a),
\]

where if \( a = 0 \), \( f'(a) \) is replaced with \( f'(0+) \). Furthermore, as \( r \downarrow a \), the left-hand side of (3.6) increases.

(ii) Equation (3.6) implies that for any point \( a \geq 0 \),

\[
f(r) \leq f(a) + f'(a)(r - a),
\]

for every \( r \geq a \).
(iii) For any \( r \geq a \), \( f'(r) \leq f'(a) \).

Before we prove Theorem 3.3 we show that positive superharmonic functions are excessive. For this, we need the following lemma. The proof is similar to Lemma 3.2, the analogous result for harmonic functions.

**Lemma 3.3.** Let \( h : E \rightarrow (-\infty, \infty] \) be radially concave. If

\[
0 \geq \int_0^{2\pi} h'_\alpha(0+) \mu(d\alpha),
\]

then for all \( z \in E \),

\[
h(z) \geq \mathbb{E}^z [h(Z_t)].
\]

**Proof.** First we assume that \( Z \) starts from 0. Then we write

\[
h(Z_t) = \sum_{s \in \mathcal{G}, s \leq t} h(e_s(t-s), A_{s+}) \mathbb{1}_{\{T_0 \circ \theta_s > t-s\}}.
\]

Taking expectations we get

\[
\mathbb{E}^0 [h(Z_t)] = \mathbb{E}^0 \left[ \int_0^t \eta(h(e_s(t-s), A_{s+}); \zeta > t-s) dL_s \right].
\]  

(3.7)

For the right-hand side we have

\[
\eta(h(e_s(t-s), A_{s+}); \zeta > t-s) = \int_0^{2\pi} n(h_\alpha(e_s(t-s)); \zeta > t-s) d\mu(\alpha)
\]

\[
= \int_0^{2\pi} n_{t-s}(h_\alpha(e_s(t-s))) d\mu(\alpha)
\]

\[
\leq \int_0^{2\pi} n_{t-s}(h(0) + h'_\alpha(0+) \cdot e_s(t-s)) d\mu(\alpha)
\]

\[
\leq \frac{2h(0)}{\sqrt{2\pi(t-s)}},
\]

where the first inequality follows from the radial concavity of \( h \) and the second inequality follows from the sub-averaging property of \( h'_\alpha(0+) \). Then

\[
\mathbb{E}^0 \left[ \int_0^t \eta(h(e_s(t-s), A_{s+}); \zeta > t-s) dL_s \right] \leq \mathbb{E}^0 \left[ \int_0^t \frac{2h(0)}{\sqrt{2\pi(t-s)}} dL_s \right] = h(0).
\]

Therefore, by equation (3.7),

\[
\mathbb{E}^0 [h(Z_t)] \leq h(0).
\]
Now, if $Z$ starts from any $z = (r, \alpha) \in E$ with $z \neq 0$, we write
\[
h(Z_t) = h(Z_t) \mathbb{1}_{\{t < T_0\}} + \sum_{s \notin \mathcal{G}} h(e_s(t - s), A_{s+}) \mathbb{1}_{\{T_0 \cap \theta_s > t-s\}}, \quad t \geq 0,
\]
Note that without loss of generality we may assume that $h(0) = 0$. Taking expectations of the incursion-excursion expansion gives
\[
\mathbb{E}^z [h(Z_t)] \leq \mathbb{E}^z [h(Z_t); t < T_0] + h(0) \mathbb{E}^z \left[ \int_0^t \frac{2dL_s}{\sqrt{2\pi(t-s)}} \right]
\]
\[
= \mathbb{E}^z [h(Z_t); t < T_0]
\]
On the set \{\{t < T_0\}, h(Z_t) = h_\alpha(R_t)\}. Thus, since the restriction $h_\alpha$ is concave on $[0, \infty)$, we have
\[
P^0_t h_\alpha \leq h_\alpha,
\]
where $P^0_t$ is the semigroup for Brownian motion on $[0, \infty)$ killed at 0. So,
\[
\mathbb{E}^z [h(Z_t); t < T_0] = P^0_t h_\alpha(r)
\]
\[
\leq h_\alpha(r).
\]
Therefore,
\[
\mathbb{E}^z [h(Z_t)] \leq h(z).
\]
\[\square\]

**Theorem 3.4.** *If $h$ is a positive superharmonic function, then $h$ is excessive.*

**Proof.** By Lemma 3.3, $h$ satisfies condition (a) in the definition of excessive functions. Restricting $Z$ to a single ray in $E$ at a time gives that $h$ satisfies (b) on $E \setminus \{0\}$, by the result for one-dimensional Brownian motion. At the origin we show that $h$ satisfies the equivalent condition (b'). This follows from Fatou’s lemma, since $h$ is positive and lower semi-continuous at 0:
\[
\liminf_{t \to 0} \mathbb{E}^0 [h(Z_t)] \geq \mathbb{E}^0 \left[ \liminf_{t \to 0} h(Z_t) \right] \geq h(0).
\]
\[\square\]
The next lemma shows that superharmonic functions are finely continuous. In the proof of this lemma, we use the fact that lower semi-continuity at 0 implies local lower boundedness. Theorem 3.4 explains why we require superharmonic functions to be lower semi-continuous at the origin, rather than the weaker condition of local lower boundedness.

**Lemma 3.4.** If $h$ is superharmonic for $Z$, then the process \{h(Z_t); t \geq 0\} is a.s. continuous.

**Proof.** If $t \notin \{s : Z_s = 0\}$, then the continuity of $h(Z_t)$ follows from the continuity of superharmonic functions applied to Brownian motion. If $t \in \{s : Z_s = 0\}$, then $h(Z_t) = h(0)$. Since $h$ is lower semi-continuous at 0, for every $\epsilon > 0$ there exists a constant $M \in \mathbb{R}$ such that

$$h(r, \alpha) \geq M,$$

for all $r < \epsilon, \alpha \in [0, 2\pi)$. Then the function $g : [0, \epsilon) \times [0, 2\pi) \rightarrow [0, \infty]$ given by

$$g(r, \alpha) = h(r, \alpha) - M,$$

is a positive superharmonic function. Thus, by Theorem 3.4, $g$ is excessive. This gives the continuity of $h(Z_t)$ since excessive functions are finely continuous.

\[ \Box \]

**3.4.1 Proof of Theorem 3.3**

(i) $\Rightarrow$ (ii): By restricting $h$ to a single ray in $E$ at a time, we get that $h$ is radially concave from the result on superharmonic functions for Brownian motion.

Fix $\epsilon > 0$. Again using the lower semi-continuity of $h$ at 0, there exists $M \in \mathbb{R}$ such that $h(r, \alpha) \geq M$ for all $r \leq \epsilon$ and $\alpha \in [0, 2\pi)$. This implies that there also exists $K \in \mathbb{R}$ such that

$$\frac{h(r, \alpha) - h(0)}{\epsilon} \geq K,$$

for $r \leq \epsilon$ and $\alpha \in [0, 2\pi)$. Thus, the functions

$$H(\epsilon, \alpha) = \frac{h(r, \alpha) - h(0)}{\epsilon} - K$$

form an increasing sequence of non-negative functions that converge to $h'_{\alpha}(0+) - K$ as
$\epsilon \to 0$. By the sub-averaging property at 0

$$h(0) \geq \mathbb{E}^0 [h(Z_{\tau(\epsilon)})] = \int_{0}^{2\pi} h(\epsilon, \alpha) d\mu(\alpha).$$

From this we obtain

$$0 \geq \int_{0}^{2\pi} \frac{h(\epsilon, \alpha) - h(0)}{\epsilon} \mu(d\alpha),$$

and so

$$-K \geq \int_{0}^{2\pi} H(\epsilon, \alpha) \mu(d\alpha).$$

Taking the limit as $\epsilon \to 0$, it follows from the Monotone Convergence Theorem that

$$-K \geq \int_{0}^{2\pi} (h'(\alpha)(0+) - K) \mu(d\alpha),$$

hence

$$0 \geq \int_{0}^{2\pi} h'(\alpha)(0+) \mu(d\alpha).$$

(ii) $\Rightarrow$ (i): By the radial concavity, we know that $h$ restricted to any ray is superharmonic for Brownian motion. Thus, $h$ satisfies (i) on $E\setminus\{0\}$.

Fix $\epsilon > 0$. Then by the radial concavity of $h$ and the sub-averaging property of the derivative function $h'(\alpha)(0+)$,

$$\mathbb{E}^0 [h(Z_{\tau(\epsilon)})] = \int_{0}^{2\pi} h(\alpha) d\mu(\alpha)$$

$$\leq h(0) + \epsilon \int_{0}^{2\pi} h'(\alpha)(0+) d\mu(\alpha)$$

$$\leq h(0).$$

(ii) $\Rightarrow$ (iii): By Lemma 3.4, $\{h(Z_t); t \geq 0\}$ is a.s. continuous. By Lemma 3.3 and the Markov property for $Z$, we have for any $z \in E$ and $s \leq t$,

$$\mathbb{E}^z [h(Z_t)|\mathcal{F}_s] = \mathbb{E}^{Z_s} [h(Z_{t-s})]$$

$$\leq h(Z_s).$$

Therefore, $\{h(Z_t); t \geq 0\}$ is a continuous supermartingale for every starting point $z \in E$.

(iii) $\Rightarrow$ (ii): Finally, we assume that $\{h(Z_t); t \geq 0\}$ is a continuous supermartingale for every starting point $z \in E$. Since the process $Z$ restricted to any one of the rays in $E$ is reflecting Brownian motion, we obtain the radial concavity of $h$. 

Fix $\epsilon > 0$. Then there exists $M \in \mathbb{R}$ such that $h(z) \geq M$, for all $z \in E$ with $|z| \leq \epsilon$. Define the process

$$Y_t = h(Z_{t \wedge \tau(\epsilon)}) - M,$$

for $t \geq 0$. Then $\{Y_t; t \geq 0\}$ is a supermartingale and $Y_t \geq 0$. Thus, letting $t \to \infty$, the Martingale Convergence Theorem gives

$$\mathbb{E}^0 [Y_0] \geq \mathbb{E}^0 [h(Z_{\tau(\epsilon)}) - M]$$

and hence

$$h(0) \geq \mathbb{E}^0 [h(Z_{\tau(\epsilon)})].$$

Now, proceeding as in the proof that (i) $\Rightarrow$ (ii), we obtain

$$0 \geq \int_0^{2\pi} h'_\alpha(0+) \mu(d\alpha),$$

which completes the proof.
Chapter 4

Walsh’s Brownian Motion on a Graph

We now consider Walsh’s Brownian motion with the more general state space of a graph. Diffusion processes on graphs have been studied in [10] and [17]. These processes occur when modeling certain physical processes such as the spread of pollutants in a stream, electrical networks, circulatory systems, and nerve impulse propagation ([7], [8], [18]). They also arise in limiting theorems for many classical processes.

We begin by defining diffusions on graphs in general using the infinitesimal generator, and see that Walsh’s Brownian motion in $\mathbb{R}^2$ is a special case. We then define what we mean by a Walsh’s Brownian motion on a graph and introduce an embedded Markov chain associated to the process. We will see that analogous results to those in Chapter 3 can be obtained for harmonic functions in this more general case. We then examine conditions under which the process is reversible and derive the Dirichlet form for the reversible process. We end with a derivation of the Laplace transform of passage times for Walsh’s Brownian motion on a graph with unit edges.

4.1 Set-up and General Result

Let $G$ denote any undirected, connected graph with vertex set $V$. If there is an edge in $G$ between vertices $u, v \in V$ we denote it by $\{u, v\}$. If the edge $\{u, v\}$ is in $G$, we say that $u$ and $v$ are neighboring vertices and write $u \sim v$. We consider graphs $G$ that are locally finite in the sense that each vertex $u \in V$ has only finitely many neighboring
vertices, i.e. the set 
\[ V_u = \{ v \in V : u \sim v \} \]
is finite. The total number of vertices in \( G \) may still be infinite though.

Each edge in \( G \) is thought of as being homeomorphic to an interval of the real line. We assign to each edge \( \{u, v\} \) in \( G \) a length, denoted by \( l_{uv} \). If we orient the edge \( \{u, v\} \) to be directed from \( u \) to \( v \), then \( \{u, v\} \) can be viewed as the interval \([0, l_{uv}]\), where 0 is identified with \( u \) and \( l_{uv} \) is identified with \( v \). So we can think of \( G \) as a union of intervals of the form \((0, l_{uv})\) and vertices \( V \). A non-vertex point in \( G \) is represented as an ordered pair of the form \((x, \{u, v\})\), where \( x \in (0, l_{uv}) \). For two points \( x, y \) on the same edge \( \{u, v\} \), we use the Euclidean metric, i.e. \( d(x, y) = |x - y| \). For points not on the same edge we use the tree metric defined in (3.1). We assume that \( \inf \{l_{uv} : \{u, v\} \} > 0 \).

Note that we allow infinite edges in the graph. An infinite edge is homeomorphic to \([0, \infty)\). To include these edges we define an infinite vertex, denoted by \( \delta \), which is simply a point at \( \infty \). Then we require that every infinite edge in \( G \) must have \( \delta \) as one endpoint and any \( u \in V \) as the other. Therefore, infinite edges can only be connected to a single finite vertex. We will not consider infinite vertices as neighbors to any other vertex, and do not include them in the set of neighboring vertices for any finite vertex.

We introduce notation which will be very useful later on. For each vertex \( u \in V \), we define the subgraph \( G_u \) to be the graph with vertices \( V_u \cup \{u\} \) and edges \( \{u, v\} \), for \( v \in V_u \). In other words, \( G_u \) is the graph containing the vertex \( u \), its neighbors, and the edges connecting \( u \) with its neighbors. See Figure 4.1 for an example.

![Diagram: Original graph G and subgraph G_{u3}](Figure 4.1: Example of a subgraph.)

For any function \( f \) defined on \( G \) we denote by \( f_{u,v} \) the restriction of \( f \) to the
edge \{u, v\}. A real-valued function \( f \) defined on \( G \) is continuous if for each edge \{u, v\}, \( f_{u,v} \) is continuous on the interval \([0, l_{uv}]\), and, if \{u, v\} is directed from u to v, then there exist constants \( c_{f,u} \) and \( c_{f,v} \) such that

\[
\lim_{x \to 0} f_{u,v}(x) = c_{f,u} \quad \text{and} \quad \lim_{x \to l_{uv}} f_{u,v}(x) = c_{f,v}.
\]

Note that for every vertex \( u \in V \), the constant \( c_{f,u} \) is given by \( f(u) \), the value of the function at \( u \). We define the derivative \( f' \) of a real-valued continuous function \( f \) along each edge \{u, v\} as follows: if we direct \{u, v\} from \( u \) to \( v \), then

\[
f'(x, \{u, v\}) = f'_{u,v}(x), \quad \text{for } x \in (0, l_{uv});
\]

\[
f'_{u,v}(u) = \lim_{x \to 0} f'_{u,v}(x) = f'_{u,v}(0+);
\]

\[
f'_{u,v}(v) = \lim_{x \to l_{uv}} f'_{u,v}(x) = f'_{u,v}(l_{uv}-);
\]

where \( f'_{u,v}(x) \) is the usual derivative of \( f_{u,v} \) at \( x \) in \((0, l_{uv})\). At each vertex, the derivatives along each incident edge will be measured in the outward direction. We denote the derivative of \( f \) at \( v \) along the edge \{u, v\} by \( f'_{v,u}(v) \) to emphasize that it is measured in the outward direction from \( v \). So if along the edge \{u, v\} vertex \( v \) is identified with \( l_{uv} \), then the derivative of \( f \) at \( v \) along \{u, v\} is given by \( f'_{v,u}(v) = -f'_{u,v}(l_{uv}-) \), note the negative sign so that we are measuring outward from \( v \). Notice that this definition can be applied many times to define higher order derivatives of continuous functions on \( G \).

Along each edge \{u, v\} in \( G \) we associate two real-valued functions \( \sigma_{u,v} \) and \( b_{u,v} \) which are in \( C^\infty((0, l_{uv})) \), and such that \( \sigma_{u,v} \) is strictly positive. We define the differential operator \( L_{u,v} \) on each edge \{u, v\} by

\[
L_{u,v} f_{u,v}(x) = \frac{1}{2} \sigma_{u,v}^2(x) f''_{u,v}(x) + b_{u,v}(x) f'_{u,v}(x).
\]

For each vertex \( u \in V \), we associate a probability \( \mu(u, v) \) with the edge \{u, v\}, for all \( v \in V_u \), where

\[
0 \leq \mu(u, v) \leq 1 \quad \text{and} \quad \sum_{v \in V_u} \mu(u, v) = 1.
\]

We define a linear operator \( \mathcal{A} \) on the space \( C(G) \) by

\[
\mathcal{A} f(x, \{u, v\}) = L_{u,v} f_{u,v}(x),
\]
where the domain of definition $\mathcal{D}(A)$ contains all $f \in C(G)$ such that $f_{u,v} \in C^2((0, l_{uv}))$ for each edge $\{u, v\}$, and for each vertex $u$ satisfy

$$\sum_{v \in V_u} \mu(u, v) f'_{u,v}(u) = 0. \quad (4.1)$$

We refer to (4.1) as the *gluing condition*. The following theorem gives the existence and uniqueness of a Markov process on $G$ with infinitesimal generator $A$.

**Theorem 4.1** ([10], Theorem 3.1). The operator $A$ on $C(G)$ generates a Feller Markov process on $G$ with continuous sample paths. On the edge $\{u, v\}$, the Markov process generated by $A$ coincides with the diffusion process generated by $L_{u,v}$. The operator $L_{u,v}$ and the gluing condition at the vertices define such a process in a unique way (in the sense of distribution).

The proof of Theorem 4.1 is similar to the proof of analogous results for diffusions on an interval.

**Example 7.** Let $G$ be a graph with a single finite vertex and $k$ infinite edges. This graph could be represented in $\mathbb{R}^2$ by a set of rays from 0 at angles $\alpha_1, \ldots, \alpha_k \in [0, 2\pi)$ made with the positive horizontal axis. We identify each edge in $G$ with an angle. Then $G$ can be thought of as the state space of a Walsh’s Brownian motion associated to a measure $\mu$ on $[0, 2\pi)$ whose support is given by the set $\{\alpha_1, \ldots, \alpha_k\}$. For any function $f$ on $G$ denote by $f_{\alpha_j}$, $j = 1, \ldots, k$, the restriction to the ray at angle $\alpha_j$. Then define the operator

$$Af(x, \alpha_j) = \frac{1}{2} f''_{\alpha_j}(x),$$

with domain all $f \in C(G)$ such that $f_{\alpha_j} \in C^2((0, \infty))$ on each ray and at the origin satisfy

$$\sum_{j=1}^{k} \mu(\alpha_j) f'_{\alpha_j}(0) = 0.$$ 

The process generated by $A$ is Walsh’s Brownian motion in $\mathbb{R}^2$ with angular distribution $\mu$.

### 4.2 Walsh’s Brownian Motion on a Graph

Let $G$ be a graph as in the previous section. That is, $G$ is an undirected, connected, and locally finite graph with edges satisfying

$$\inf_{\{u,v\} \in G} l_{uv} > 0.$$
Also let \( \{ \mu(u, v), u \in V, v \in V_u \} \), be a set of probabilities associated to \( G \). We define Walsh’s Brownian motion on \( G \) in two ways. The first definition will be in terms of the infinitesimal generator, and the second in terms of excursion processes.

For each edge \( \{u, v\} \in G \), define the operator

\[
L_{u,v}g(x) = \frac{1}{2} g''(x)
\]

for \( g \in C((0, l_{uv})) \). Then define the following linear operator \( A \) on \( C(G) \):

\[
Af(\{u, v\}, x) = L_{u,v}f_{u,v}(x),
\]

for \( x \in (0, l_{uv}) \), where the domain \( \mathcal{D}(A) \) contains all \( f \in C(G) \) such that \( f_{u,v} \in C^2((0, l_{uv})) \) for each edge \( \{u, v\} \), and for each vertex \( u \) satisfy (4.1), the gluing condition. We define the continuous Feller Markov process generated by \( A \) to be Walsh’s Brownian motion on a graph and denote it by \( Z = \{Z_t; t \geq 0\} \). Theorem 4.1 guarantees the existence of \( Z \) and implies that the process inside any edge \( \{u, v\} \) is reflecting Brownian motion.

Notice that Walsh’s Brownian motion on a graph can be thought of as a collection of Walsh’s Brownian motions in \( \mathbb{R}^2 \) (with discrete \( \mu \)) glued together. If we restrict \( Z \) to the subgraph \( G_u \), for any vertex \( u \in V \), then we obtain a Walsh’s Brownian motion in \( \mathbb{R}^2 \) killed at the neighboring vertices of \( u \). Notice that for Walsh’s Brownian motion in \( \mathbb{R}^2 \) all the edges are infinite, which is not necessarily the case for all edges in \( G \). Because of this, there may be finite vertices with only one neighboring vertex and therefore only one incident edge. We call such a vertex a boundary vertex in \( G \). The definition of Walsh’s Brownian motion on \( G \) implicitly specifies the behavior of the process at these boundary vertices. Specifically, in the definition of Walsh’s Brownian motion on \( G \) given above, the behavior of the process at boundary vertices is reflecting. This is because \( f'(v) = 0 \) for all boundary vertices \( v \) and \( f \in \mathcal{D}(A) \), which follows from the gluing condition. The behavior at boundary vertices can be changed by modifying the domain of the infinitesimal generator \( A \) and changing the boundary conditions. For example, if we instead want the process to be absorbing at boundary vertices, then we require that all \( f \in \mathcal{D}(A) \) also satisfy \( f(v) = 0 \), for all boundary vertices \( v \).

### 4.2.1 Embedded Markov Chains

As in the original setting with Walsh’s Brownian motion in \( \mathbb{R}^2 \), we can also describe Walsh’s Brownian motion on \( G \) by using the excursions of reflecting Brownian
motion. Since Walsh’s Brownian motion on a graph can be viewed as a collection of Walsh’s Brownian motions in $\mathbb{R}^2$, we use a collection of reflecting Brownian motions. We associate with each vertex $u \in V$ a reflecting Brownian motion $R^{(u)}$. Then at each vertex $u$, we enumerate the excursions of the reflecting Brownian motion $R^{(u)}$ away from $0$, which we identify with $u$, and obtain the sequence $\{e_j^{(u)}\}_{j=1}^\infty$. We also associate with each vertex $u$ a sequence $\{v_j^{(u)}\}_{j=1}^\infty$ of i.i.d. random variables such that $v_j^{(u)} = \{u, v\}$ with probability $\mu(u, v)$ for $v \in V_u$.

The construction is similar to the construction of Walsh’s Brownian motion in $\mathbb{R}^2$, inasmuch as the idea is to pair up excursions $e_j^{(u)}$ with edges $v_j^{(u)}$ at each vertex $u$. However, it is complicated by the fact that there are two different types of excursions at each vertex, the ones that return to $u$ before leaving an incident edge and the ones that do not. For the ones that do not return before leaving the incident edge, we kill that excursion once it reaches the other vertex. Then the behavior of the Walsh’s Brownian motion is governed by the reflecting Brownian motion associated to the new vertex. So to construct Walsh’s Brownian motion on a graph, we must also keep track of which excursion of the reflecting Brownian motion at each vertex to use, according to the enumeration picked. One way to do this is to place a counter at each vertex that increases by one each time the process returns to the vertex. We omit the details of this construction as it is not very insightful and the notation is extremely cumbersome.

With this construction, however, we do see that an important piece of information related to Walsh’s Brownian motion on a graph is the sequence of distinct vertices visited by the process. We define the embedded Markov chain $\{X_n; n \in \mathbb{N}\}$ associated to Walsh’s Brownian motion on a graph to be a Markov chain with state space $V$ and transition probabilities given by the probability that the Walsh’s Brownian motion starting at vertex $u$ will hit neighboring vertex $v \in V_u$ first. If we let $T_u = \inf\{t \geq 0 : Z_t \in V_u\}$ denote the first hitting time of the set of neighboring vertices of $u$ for Walsh’s Brownian motion on a graph, then we can write the transition probabilities of the embedded Markov chain $X_n$ as

$$P(u, v) = \begin{cases} \mathbb{P}^{u}(Z_{T_u} = v) & \text{if } v \in V_u, \\ 0 & \text{if } v \notin V_u. \end{cases}$$

Note that the embedded Markov chain associated to Walsh’s Brownian motion on $G$ is irreducible since the graph is assumed to be connected. Also note that the embedded
Markov chain will never visit infinite vertices in $G$, if any exist. So the state space of $X$ is actually given by the set containing only finite vertices in $G$.

The remainder of this chapter will be devoted to exploring the relationship between the Markov process $\{Z_t; t \geq 0\}$ and the Markov chain $\{X_n; n \in \mathbb{N}\}$.

### 4.3 Harmonic Functions

As we saw in the previous section, Walsh’s Brownian motion on a graph is locally the same as Walsh’s Brownian motion in $\mathbb{R}^2$. Since harmonic is a local property of a function, the characterization of harmonic functions for Walsh’s Brownian motion on a graph is similar to the case in $\mathbb{R}^2$.

**Theorem 4.2.** Let $G$ be a locally finite, connected graph and let $Z$ be a Walsh’s Brownian motion on $G$. For a function $h \in C(G)$ the following are equivalent.

(i) For every $(x, \{u, v\}) \in G$ and every $\epsilon > 0$,

$$h(x, \{u, v\}) = \mathbb{E}^{(x, \{u, v\})}[h(Z_{\tau_x(\epsilon)})].$$

(ii) $h$ is linear on each edge in $G$ and satisfies the slope-averaging property at each vertex.

**Proof.** Fix a point $(x, \{u, v\}) \in G$. The result follows by restricting the process to the subgraph $G_u$ consisting of $\{u, w\}$, for $w \in V_u$, since the restriction is a Walsh’s Brownian motion in $\mathbb{R}^2$.

A function $h$ on $G$ is **harmonic for $Z$** if it satisfies the conditions of Theorem 4.2

**Remarks.** (i) Note that harmonic functions are defined to be continuous functions on $G$. Since we are only considering locally finite graphs, this is forced: functions satisfying any of the conditions in the theorem will necessarily be continuous (under the tree topology on $G$).

(ii) For the slope-averaging property at each vertex, slopes are measured outward along each incident edge to the vertex. This follows from the corresponding result for Walsh’s Brownian motion in $\mathbb{R}^2$, since in that case edges were always directed away from the origin.
The infinitesimal generator of $Z$ provides another characterization of harmonic functions. We notice first that if $h$ is harmonic for $Z$, then $h \in D(A)$. For if $h$ is harmonic then $h \in C(G)$ and $h_{u,v} \in C^2(0,l_{uv})$, since $h_{u,v}$ is a linear function for every edge $\{u,v\}$. Furthermore, the slope-averaging property of $h$ implies that $h$ satisfies the gluing condition and $h \in D(A)$. It easily follows from Theorem 4.2 that $h$ is harmonic for $Z$ if and only if $Ah = 0$. We use this equivalent characterization to obtain a martingale property for harmonic functions of Walsh’s Brownian motion on a graph.

**Theorem 4.3.** A function $h$ is harmonic for $Z$ if and only if $\{h(Z_t); t \geq 0\}$ is a continuous martingale with respect to $\{\mathcal{F}_t^Z\}$ for every starting point $(x, \{u,v\}) \in G$.

**Proof.** If $h$ is harmonic, then $h \in D(A)$. Then the process

$$h(Z_t) - \int_0^t Ah(Z_s) ds$$

is a $\{\mathcal{F}_t^Z\}$-martingale. But $Ah \equiv 0$, so $\{h(Z_t); t \geq 0\}$ must be a martingale. Since $h \in C(G)$, it is a continuous martingale.

For the other direction, restrict the process $\{h(Z_t); t \geq 0\}$ to the subgraphs $G_u$, $u \in V$, and use the results for Walsh’s Brownian motion in $\mathbb{R}^2$ to obtain the linearity of $h$ on each edge and the slope-averaging property at every vertex. $\square$

### 4.3.1 Transition Probabilities of Embedded Markov Chains

We now derive an explicit formula for the transition probabilities $P(u,v)$ of the embedded Markov chain associated to a Walsh’s Brownian motion $Z$ on $G$. Fix a vertex $u$ and one of its neighbors $v \in V_u$. We want to compute the probability that starting from $u$, the process will hit $v$ before any other $w \in V_u$. Notice that the probability we want is the value at $u$ of a harmonic function for a Walsh’s Brownian motion in $\mathbb{R}^2$ defined on the subgraph $G_u$, as we proved in Example 5. Specifically, the harmonic function $h$ on $G_u$ we want satisfies

$$h(v) = 1,$$

$$h(w) = 0, \text{ for all } w \in V_u, w \neq v.$$
We know by Theorem 4.2 that $h$ must be linear on each edge and the slopes satisfy the averaging property. This gives

\[
1 = h(v) = h(u) + m_v l_{uv},
\]

\[
0 = h(w) = h(u) + m_w l_{uw}, \quad \text{for } w \in V_u, w \neq v,
\]

\[
0 = \sum_{w \in V_u} \mu(u, v) m_w,
\]

where $m_w$ denotes the slope of $h$ along $\{u, w\}$. Note that we orient each edge to direct away from $u$ so that the slopes are measured outward from $u$. By solving for the slopes in terms of $h(u)$ and substituting into the averaging condition, we get

\[
0 = \mu(u, v) \frac{(1 - h(u))}{l_{uv}} + \sum_{w \in V_u, w \neq v} \mu(u, w) \frac{(-h(u))}{l_{uw}}.
\]

Solving for $h(u)$ gives

\[
h(u) = \frac{\mu(u, v) l_v^{(v)}}{\sum_{w \in V_u} \mu(u, w) l_w^{(w)}},
\]

where

\[
l_u^{(v)} = \prod_{w \in V_u, w \neq v} l_{uw}
\]

is the product of the lengths of all incident edges to $u$ except $\{u, v\}$. Thus, associated to every Walsh’s Brownian motion $Z$ on $G$ is the embedded Markov chain $X_n$ on $V$ with transition probabilities

\[
P(u, v) = \begin{cases} \frac{\mu(u, v) l_v^{(v)}}{\sum_{w \in V_u} \mu(u, w) l_w^{(w)}}, & \text{if } v \in V_u \\ \sum_{w \in V_u} \mu(u, w) l_w^{(w)}, & \text{otherwise}. \end{cases}
\]

Notice that if all the lengths are equal to 1, then $P(u, v) = \mu(u, v)$.

### 4.3.2 Harmonic Functions for the Embedded Markov Chain

A function $h$ defined on $V$ is harmonic for the Markov chain $\{X_n; n \in \mathbb{N}\}$ with transition probabilities $P(u, v)$ if for every $u \in V$,

\[
h(u) = \sum_{v \in V} P(u, v) h(v).
\]
There is a one to one correspondence between harmonic functions for Walsh’s Brownian motion on $G$, which are constant on any infinite edges in $G$, and harmonic functions for the associated embedded Markov chain on $V$.

**Theorem 4.4.** Let $Z = \{Z_t; t \geq 0\}$ be a Walsh’s Brownian motion on $G$ with embedded Markov chain $X = \{X_n; n \in \mathbb{N}\}$.

(a) If $h$ is a harmonic function for $Z$ such that $h$ is constant on any infinite edges in $G$, then the restriction of $h$ to $V$ is harmonic for $X$.

(b) If $h$ is harmonic for $X$, then there is a unique extension of $h$ to $G$ that is harmonic for $Z$ and constant on infinite edges.

**Proof.** (a): Fix a vertex $u \in V$. If $h$ is harmonic for $Z$, then by Theorem 4.2, $h$ satisfies the slope-averaging property at $u$. Assuming that $h$ is constant on any infinite edges in $G$ implies that along any infinite edge the slope of $h$ is equal to 0. Therefore, we may ignore infinite edges in the slope-averaging property.

Measuring slopes outward from $u$, we have

$$0 = \sum_{v \in V_u} \mu(u, v) \frac{h(v) - h(u)}{l_{uv}}.$$ 

Multiplying by the product of the lengths of all incident edges to $u$ gives

$$0 = \sum_{v \in V_u} \mu(u, v) l_u^{(v)} (h(v) - h(u)),$$

and, therefore,

$$h(u) \sum_{v \in V_u} \mu(u, v) l_u^{(v)} = \sum_{v \in V_u} \mu(u, v) l_u^{(v)} h(v).$$

Solving for $h(u)$ and using (4.2),

$$h(u) = \sum_{v \in V_u} P(u, v) h(v).$$

So $h$ is harmonic for $X$.

(b): If $h$ is harmonic for the embedded Markov chain, we extend $h$ to all of $G$ using linear interpolation on the finite edges. That is, for $(x, \{u, v\}) \in G$, $x \in (0, l_{uv})$, if $\{u, v\}$ is directed from $u$ to $v$, we define an extension of $h$ to $G$ as

$$h(x, \{u, v\}) = \left( \frac{h(v) - h(u)}{l_{uv}} \right) \cdot x + h(u).$$
On an edge in $G$ with infinite length, the extension of $h$ is defined to equal the constant value of $h$ at the finite vertex the edge is connected to. So for $(x, \{u, \delta\}) \in G, x > 0$,

$$h(x, \{u, \delta\}) = h(u).$$

This gives a unique extension of $h$ to $G$ for a fixed orientation of the edges. Since $h$ is linear on the edges, it suffices to show that $h$ satisfies the slope-averaging property at each vertex. Since $h$ is constant on infinite edges, we need not consider them when computing the average of the slopes. Rewriting the harmonic property for $h$ on $V$,

$$0 = \sum_{v \in V_u} P(u, v)(h(v) - h(u))$$

$$= \sum_{v \in V_u} \mu(u, v) l_{uv}^v (h(v) - h(u))$$

$$= \sum_{v \in V_u} \mu(u, v) \frac{h(v) - h(u)}{l_{uv}}$$

where the last equality follows from dividing each side by the product of the lengths of all incident edges to $u$. This gives the result.

4.4 Reversibility

We saw in the previous chapter that Walsh’s Brownian motion in $\mathbb{R}^2$ is a reversible process. For Walsh’s Brownian motion on a graph, we will see that reversibility is tied to the reversibility of the embedded Markov chain. First we define what we mean for a process to be reversible in this case, for which we need the following. For any measure $m$ on $G$ we define the following inner product on $L^2(G; m)$, the space of real-valued, square integrable, $m$-measurable functions on $G$:

$$(f, g)_m = \int_G f \cdot g \, dm.$$

Let $Z = \{Z_t; t \geq 0\}$ be a Walsh’s Brownian motion on $G$. Let $\{P_t\}_{t \geq 0}$ denote the semigroup associated to $Z$. Then $P_t$ is a linear operator on $L^2(G; m)$ and we can define the infinitesimal generator $A$ of $Z$ as an operator on $L^2(G; m)$, in which case the domain of $A$ is given by

$$\mathcal{D}(A) = \{f \in L^2(G; m) : f'' \in L^2(G; m), \text{for every } \{u, v\} f_{u,v} \text{ is differentiable and}$$

$$f'_{u,v} \text{ is absolutely continuous, } f \text{ satisfies the gluing condition for every } u\}.$$
We say that $Z$ is reversible if there exists a $\sigma$-finite measure $m$ on $G$ such that

$$(P_t f, g)_m = (f, P_t g)_m,$$

for $f, g \in L^2(G; m)$ and $t \geq 0$. We refer such a measure $m$, if it exists, as a reversibility measure for $Z$.

**Theorem 4.5.** Walsh’s Brownian motion on a locally finite, connected graph is reversible if and only if the associated embedded Markov chain is.

**Proof.** First, we assume that the embedded Markov chain is reversible. In this case, there exists a stationary measure $\pi$ on the set of vertices $V$ such that

$$\pi(u)P(u, v) = \pi(v)P(v, u)$$

for all $u, v \in V$. Using this condition we define a measure $m$ on $G$ by placing weights on the edges and then crossing that set with the Lebesgue measure on $(0, \infty)$. We assign the weight $c_{u,v}$ to the edge $\{u, v\}$ by

$$c_{u,v} = \pi(u)P(u, v).$$

Note that by the reversibility of the Markov chain, $c_{u,v} = c_{v,u}$. Also, since $P(u, v)$ is proportional to $\mu(u, v)$ by (4.2), we also have that the weights $c_{u,v}$ are proportional to the probabilities $\mu(u, v)$. We claim that this measure gives a reversibility measure for the Walsh’s Brownian motion.

In order to verify this claim, we show that the infinitesimal generator is self-adjoint with respect to the measure $m$. That is, we want to show that

$$(f, Ag)_m = (Af, g)_m$$

for all $f, g \in \mathcal{D}(A)$. Since $L^2(G; m)$ is a reflexive Banach space, we know that the adjoint semigroup of $P_t$ has infinitesimal generator given by the adjoint of $A$ (see Corollary 10.6 in [19]). Therefore, if $A$ is self-adjoint, $P_t$ is and the Walsh’s Brownian motion is reversible.
Note that by using integration by parts twice we get

\[(f, Ag)_m = \frac{1}{2} \int_G f \cdot g'' \, dm \]

\[= \frac{1}{2} \sum_{\{u,v\} \in G} c_{u,v} \left[ f_{u,v}(x)g''_{u,v}(x) \right]_{0}^{l_{u,v}} - f'_{u,v}(x)g_{u,v}(x) \right|_{0}^{l_{u,v}} \]

\[+ \int_{0}^{l_{u,v}} f''_{u,v}(x)g_{u,v}(x) \, dx \]

\[= (Af, g)_m + \frac{1}{2} \sum_{\{u,v\} \in G} c_{u,v} \left[ f_{u,v}(x)g'_{u,v}(x) \right]_{0}^{l_{u,v}} - f'_{u,v}(x)g_{u,v}(x) \right|_{0}^{l_{u,v}} . \]

So we need to show that

\[\sum_{\{u,v\} \in G} c_{u,v} \left[ f_{u,v}(x)g'_{u,v}(x) \right]_{0}^{l_{u,v}} - f'_{u,v}(x)g_{u,v}(x) \right|_{0}^{l_{u,v}} = 0. \quad (4.3)\]

Fix an edge \(\{u, v\}\) in \(G\). We orient \(\{u, v\}\) to be directed from \(u\) to \(v\). That is, vertex \(u\) is identified with 0 and vertex \(v\) is identified with \(l_{uv}\). Then

\[c_{u,v} \left[ f_{u,v}(x)g'_{u,v}(x) \right]_{0}^{l_{u,v}} - f'_{u,v}(x)g_{u,v}(x) \right|_{0}^{l_{u,v}} \]

\[= c_{u,v} \left[ f_{u,v}(l_{uv})g'_{u,v}(l_{uv}) - f_{u,v}(0)g'_{u,v}(0) - f'_{u,v}(l_{uv})g_{u,v}(l_{uv}) + f'_{u,v}(0)g_{u,v}(0) \right] \]

\[= c_{u,v} \left[ -f(v)g'_{v,u}(l_{uv}) + f(u)g'_{u,v}(0) - f'_{v,u}(l_{uv})g(v) + f'_{u,v}(0)g(u) \right], \quad (4.4)\]

where \(f'_{u,v}(0)\) denotes the derivative of \(f\) at \(u\) measured along the edge \(\{u, v\}\) directed from \(u\) to \(v\), and \(f'_{v,u}(l_{uv})\) is the derivative at \(v\) measured along \(\{u, v\}\) directed instead from \(v\) to \(u\) instead (similarly for \(g\)). The sign changes on the first and third terms in the sum of (4.4) come from switching the order in which the derivatives at vertex \(v\) are taken. Thus, all derivatives at a given vertex are measured in the outward direction from that vertex.

We see that the integral along each edge \(\{u, v\}\) contributes a term of the form

\[c_{u,v} \left[ -f(u)g'_{u,v}(u) + g(u)f'_{u,v}(u) \right] \]

at vertex \(u\), and

\[c_{u,v} \left[ -f(v)g'_{v,u}(v) + g(v)f'_{v,u}(v) \right] \]
at vertex $v$. Notice that at any vertex, regardless of the orientation, the derivative of $g$ at the vertex will always be multiplied by the opposite of the value of $f$ at the vertex and the derivative of $f$ at the vertex will always be multiplied by the value of $g$ at the vertex. Therefore,

$$
\sum_{\{u,v\} \in G} c_{u,v} \left[ f_{u,v}(x) g'_{u,v}(x) \right] = \sum_{u \in V} \sum_{v \in V} c_{u,v} \left[ -f(u)g'_{u,v}(u) + g(u)f'_{u,v}(u) \right].
$$

Since $f, g \in \mathcal{D}(A)$, we know that

$$
\sum_{v \in V} \mu(u,v) f'_{u,v}(u) = \sum_{v \in V} \mu(u,v) g'_{u,v}(u) = 0,
$$

for all vertices $u \in V$. Since the weights $c_{u,v}$ and probabilities $\mu(u,v)$ are proportional, this implies that

$$
\sum_{v \in V_u} c_{u,v} f'_{u,v}(u) = \sum_{v \in V_u} c_{u,v} g'_{u,v}(u) = 0.
$$

Thus, (4.3) holds and $A$ is self-adjoint.

Now, conversely, assume that the Walsh’s Brownian motion is reversible. Therefore, there exists a measure $m$ on $G$ under which the transition semigroup is self-adjoint. This implies that the infinitesimal generator is also self-adjoint. In this case, we have weights $c_{u,v}$ on the edges $\{u,v\}$ of $G$ such that for any $f, g \in \mathcal{D}(A)$,

$$
0 = \sum_{\{u,v\} \in G} c_{u,v} \left[ f \cdot g'_{\{u,v\}} - g \cdot f'_{\{u,v\}} \right]
= \sum_{u \in V} \sum_{v \in V} c_{u,v} \left[ -f(u)g'_{u,v}(u) + g(u)f'_{u,v}(u) \right],
$$

which is obtained using the same argument as above, i.e. using the self-adjoint property of $A$ and applying the integration by parts formula twice. We can single out one vertex $u \in V$ by considering a function $f \in \mathcal{D}(A)$ that vanishes at every other vertex, and by letting $g \equiv 1$. Then equation (4.5) becomes

$$
0 = \sum_{v \in V_u} c_{u,v} f'_{u,v}(u).
$$

Thus, for any fixed function $f \in \mathcal{D}(A)$ that vanishes at every vertex except $u$, we have the following implication:

$$
\sum_{v \in V_u} \mu(u,v) f'_{u,v}(u) = 0 \implies \sum_{v \in V_u} c_{u,v} f'_{u,v}(u) = 0.
$$
Using a standard argument from linear algebra (namely, by looking at the kernels of appropriate linear mappings and counting dimensions of vector spaces), it follows from this implication that for each vertex \( u \in V \) there is a constant \( k_u \) such that

\[
\mu(u, v) = k_u c_{u,v},
\]

for all \( v \in V_u \). Notice that

\[
1 = \sum_{v \in V_u} \mu(u, v) = \sum_{v \in V_u} k_u c_{u,v} = k_u \sum_{v \in V_u} c_{u,v}.
\]

If we let \( c_u = \sum_{v \in V_u} c_{u,v} \), then

\[
k_u = \frac{1}{c_u}.
\]

Therefore, \( \mu(u, v) = \frac{c_{u,v}}{c_u} \). We can now rewrite the transition probabilities of the embedded Markov chain in terms of the weights \( c_{u,v} \): for \( u \sim v \),

\[
P(u, v) = \frac{\mu(u, v) l^{(v)}_u}{\sum_{w \in V_u} \mu(u, w) l^{(w)}_u} = \frac{(c_{u,v}/c_u) l^{(v)}_u}{\sum_{w \in V_u} (c_{u,w}/c_u) l^{(w)}_u} = \frac{c_{u,v} l^{(v)}_u}{\sum_{w \in V_u} c_{u,w} l^{(w)}_u}.
\]

Let \( \pi \) denote the following measure on \( V \): for \( u \in V \),

\[
\pi(u) = \frac{\left( \sum_{v \in V_u} l^{(v)}_u c_{u,v} \right) l^{(u)}_u}{\sum_{w \in V} \left( \sum_{v \in V_w} l^{(v)}_w c_{v,w} \right) l^{(w)}_u},
\]

where

\[
l^{(v)}_u = \prod_{w \in V_u, w \neq v} l_{uw},
\]

for \( v \in V_u \), and

\[
l^{(u)} = \prod_{\{v,w\} \in G \text{ in } G} l_{vw}.
\]
In other words, \( l_u^{(v)} \) is the product of edge lengths for all the incident edges of \( u \) except the one connected to \( v \), and \( l^{(u)} \) is the product of all edge lengths in \( G \) except for those edges containing \( u \). We claim that \( \pi \) is a reversible measure for the Markov chain.

To see this, note that for any \( u_1, u_2 \in V \) such that \( u_1 \sim u_2 \),

\[
\pi(u_1)P(u_1, u_2) = \frac{\left( \sum_{v \in V_{u_1}} l_{u_1}^{(v)} c_{u_1,v} \right) l^{(u_1)}}{\sum_{w \in V} \left( \sum_{v \in V_w} l_{v,w}^{(v)} c_{v,w} \right) l^{(w)}} \cdot \frac{l_{u_1}^{(u_2)} c_{u_1,u_2}}{\left( \sum_{v \in V_{u_2}} l_{u_2}^{(v)} c_{u_2,v} \right) l^{(u_2)}}.
\]

(4.6)

and

\[
\pi(u_2)P(u_2, u_1) = \frac{\left( \sum_{v \in V_{u_2}} l_{u_2}^{(v)} c_{u_2,v} \right) l^{(u_2)}}{\sum_{w \in V} \left( \sum_{v \in V_w} l_{v,w}^{(v)} c_{v,w} \right) l^{(w)}} \cdot \frac{l_{u_2}^{(u_1)} c_{u_1,u_2}}{\left( \sum_{v \in V_{u_1}} l_{u_1}^{(v)} c_{u_1,v} \right) l^{(u_1)}}.
\]

(4.7)

Notice that the numerators in both equations (4.6) and (4.7) are equal to the product of the weight \( c_{u_1,u_2} \) and all the edge lengths of \( G \) except for \( \{u_1, u_2\} \), the edge connecting \( u_1 \) and \( u_2 \). The denominators are the same normalizing factor that makes \( \pi \) a probability measure. Therefore,

\[
\pi(u_1)P(u_1, u_2) = \pi(u_2)P(u_2, u_1).
\]

Notice that the lengths of the edges in \( G \) do not directly influence the reversibility of Walsh’s Brownian motion. As we saw in (4.2), however, the transition probabilities of the embedded Markov chain do depend on the edge lengths. What happens to the reversibility if the lengths are changed? We find that if the edge lengths are changed in such a way that the embedded Markov chain remains the same, i.e. the transition probabilities are the same, then the new edge lengths must be proportional to the original...
lengths. If the edge lengths are changed in any way, even if the embedded Markov chain remains the same, the Walsh’s Brownian motion will be different. But the new process will remain reversible if the original one is, by the theorem.

**Proposition 4.1.** Let $Z$ be a Walsh’s Brownian motion on $G$ with the set of probabilities $\{\mu(u, v), u \in V, v \in V_u\}$ and edge lengths $\{l_{uv}, u \in V, v \in V_u\}$. Let $\tilde{Z}$ be another Walsh’s Brownian motion on $G$ with the same set of probabilities but different edge lengths $\{\tilde{l}_{uv}, u \in V, v \in V_u\}$. Then $Z$ and $\tilde{Z}$ have the same embedded Markov chain if and only if there exists a constant $k > 0$ such that $l_{uv} = kl_{uv}$ for $u \in V, v \in V_u$.

**Proof.** First assume that there exists a constant $k$ such that $l_{uv} = kl_{uv}$. We denote by $P(u, v)$ and $\tilde{P}(u, v)$ the transition probabilities of the embedded Markov chains for $Z$ and $\tilde{Z}$, respectively. Then by (4.2),

$$P(u, v) = \frac{\mu(u, v)l_{uv}^{(v)}}{\sum_{w \in V_u} \mu(u, w)l_{uw}^{(w)}} = \frac{\mu(u, v)kl_{uv}^{(v)}}{\sum_{w \in V_u} \mu(u, w)kl_{uw}^{(w)}} = \frac{\mu(u, v)\tilde{l}_{uv}^{(v)}}{\sum_{w \in V_u} \mu(u, w)\tilde{l}_{uw}^{(w)}} = \tilde{P}(u, v).$$

Thus the embedded Markov chains for $Z$ and $\tilde{Z}$ have the same transition probabilities and so are the same process.

Now we assume that $Z$ and $\tilde{Z}$ are associated to the same embedded Markov chain. Fix vertices $u_1, u_2 \in V$ such that $u_1 \sim u_2$. For $v_1 \in V_{u_1}$, with $v_1 \neq u_2$,

$$P(u_1, v_1) = \frac{\mu(u_1, v_1)l_{u_1v_1}^{(v_1)}}{\sum_{w \in V_{u_1}} \mu(u_1, w)l_{u_1w}^{(w)}} = \frac{\mu(u_1, v_1)\tilde{l}_{u_1v_1}^{(v_1)}}{\sum_{w \in V_{u_1}} \mu(u_1, w)\tilde{l}_{u_1w}^{(w)}}. \quad (4.8)$$

Then by separating the edge lengths $l_{u_1u_2}$ and $\tilde{l}_{u_1u_2}$ from the products $l_{u_1v_1}^{(v_1)}$ and $\tilde{l}_{u_1v_1}^{(v_1)}$, respectively, we obtain from (4.8)

$$l_{u_1u_2} = k_{u_1}\tilde{l}_{u_1u_2}, \quad (4.9)$$
where \( k_{u_1} \), a constant depending on \( u_1 \), is equal to the remaining terms in (4.8). Similarly, for \( v_2 \in V_{u_2} \), with \( v_2 \neq u_1 \), by setting the formulas for \( P(u_2, v_2) \) equal, we obtain a constant \( k_{u_2} \) such that

\[
l_{u_1u_2} = k_{u_2} \tilde{t}_{u_1u_2}.
\]

(4.10)

Setting equations (4.9) and (4.10) equal gives \( k_{u_1} = k_{u_2} \). Thus, there is a constant \( k \), which does not depend on any vertex, such that \( l_{uv} = k \tilde{l}_{uv} \) for every \( u \in V, v \in V_u \).

Proposition 4.1 says that if we only have the embedded Markov chain associated to a Walsh’s Brownian motion on a graph, then we can recover the Walsh’s Brownian motion up to a constant. We associate the probabilities to the graph; however, the edge lengths are viewed as parameters associated to the Walsh’s Brownian motion.

### 4.4.1 Dirichlet Forms

Let \( \{Z_t; t \geq 0\} \) be a reversible Walsh’s Brownian motion on \( G \). Then, as we saw above, a reversibility measure for \( Z \) is the measure \( m \) on \( G \) given by the Lebesgue measure on \((0, \infty)\) crossed with a set of weights \( c_{u,v} \) assigned to the edges, where the weights are proportional to the probabilities associated to the graph.

The Dirichlet form for \( Z \) is the closed symmetric form \( E \) on \( L^2(G;m) \) given by

\[
E(f, g) = \lim_{t \to 0} \frac{1}{t} (f - P_t f, g)_m,
\]

(4.11)

where \( f \) is in the domain of \( E \), denoted by \( D[E] \), if the limit in (4.11) exists. Note that the symmetry of \( E \) follows from the symmetry of \( P_t \):

\[
E(f, g) = \lim_{t \to 0} \frac{1}{t} (f - P_t f, g)_m = \lim_{t \to 0} \frac{1}{t} ((f, g)_m - (P_t f, g)_m)
\]

\[
= \lim_{t \to 0} \frac{1}{t} ((f, g)_m - (f, P_t g)_m)
\]

\[
= \lim_{t \to 0} \frac{1}{t} (f, g - P_t g)_m
\]

\[
= E(g, f).
\]

Note that \( D(A) \subset D[E] \). There is a strong relationship between the Dirichlet form for a reversible process and the infinitesimal generator. In fact, we have the following equivalence between Dirichlet forms and self-adjoint operators on \( L^2(G;m) \).
Theorem 4.6 ([11], Theorem 1.3.1). There is a one to one correspondence between the family of closed symmetric forms $E$ on any Hilbert space $H$ and the family of self-adjoint operators $A$ on $H$. The correspondence can be characterized by

$$
\begin{cases}
    \mathcal{D}(A) \subset \mathcal{D}[E] \\
    E(f,g) = (-Af,g), \text{ for } f \in \mathcal{D}(A), g \in \mathcal{D}[E].
\end{cases}
$$

We claim that the Dirichlet form corresponding to the generator of a reversible Walsh’s Brownian motion on $G$ is given by

$$
\begin{cases}
    E(f,g) = \frac{1}{2} \int_G f' \cdot g' \, dm \\
    \mathcal{D}[E] = \{ f \in L^2(G;m) : f \text{ is absolutely continuous and } f' \in L^2(G;m) \}
\end{cases}
$$

(4.12)

In order to verify this claim, we show that the self-adjoint operator on $L^2(G;m)$ corresponding to $E$, call it $A_E$, coincides with the infinitesimal generator of Walsh’s Brownian motion on $G$.

First, suppose $f \in \mathcal{D}(A_E)$. Then by the correspondence given in Theorem 4.6, $f \in \mathcal{D}[E]$ and

$$
E(f,g) = (-A_E f,g)_m,
$$

(4.13)

for all $g \in \mathcal{D}[E]$. Since $C_\infty^\infty(G) \subset \mathcal{D}[E]$, (4.13) implies that

$$
\frac{1}{2} \int_G f' \cdot \phi' \, dm = - \int_G A_E f \cdot \phi \, dm,
$$

(4.14)

for all $\phi \in C_\infty^\infty(G)$.

Fix an edge $\{u,v\}$ in $G$ and use the orientation that identifies $u$ with 0 and $v$ with $l_{uv}$. For any $\phi \in C_\infty^\infty(G)$ with support contained in , but not equal to, the edge $\{u,v\}$, (4.14) becomes

$$
\frac{1}{2} \int_0^{l_{uv}} f'_{u,v}(x) \phi'(x) \, dx = - \int_0^{l_{uv}} A_E f(x) \phi(x) \, dx.
$$

(4.15)

If we denote by $\partial g$ the derivative in the sense of distributions for any function $g \in L^2((0,l_{uv}))$, then (4.15) implies that

$$
\frac{1}{2} \partial f'_{u,v} = A_E f_{u,v},
$$

from which it follows that $f'_{u,v}$ is absolutely continuous and its distributional derivative is equal to the ordinary derivative. Hence,

$$
A_E f_{u,v} = \frac{1}{2} f''_{u,v},
$$
for every edge \{u,v\}, and \(f'' \in L^2(G;m)\) since \(A_E f \in L^2(G;m)\).

Fix a vertex \(u\) in \(G\). We orient each edge in the subgraph \(G_u\) away from \(u\). For any \(\phi \in C^\infty_c(G)\) with support strictly contained in \(G_u\), (4.14) becomes

\[
\frac{1}{2} \sum_{v \in V_u} c_{u,v} \int_{l_{uv}}^{l_{uv}} f'_{u,v}(x) \phi'(x) dx = \frac{1}{2} \sum_{v \in V_u} c_{u,v} \int_{l_{uv}}^{l_{uv}} f''_{u,v}(x) \phi(x) dx. \tag{4.16}
\]

Using the absolute continuity of \(f'_{u,v}\), for each \(v \in V_u\), we have

\[
\int_{l_{uv}} f'_{u,v}(x) \phi'(x) dx = \int_{l_{uv}} \left( f'_{u,v}(0) + \int_{0}^{x} f''_{u,v}(t) dt \right) \phi'(x) dx
\]

\[
= f'_{u,v}(0) \int_{0}^{l_{uv}} \phi'(x) dx + \int_{0}^{l_{uv}} \int_{0}^{x} f''_{u,v}(t) \phi'(x) dt dx
\]

\[
= f'_{u,v}(0) \phi(0) + \int_{0}^{l_{uv}} \int_{0}^{t} f''_{u,v}(t) \phi(0) dx dt
\]

\[
= f'_{u,v}(0) \phi(0) - \int_{0}^{l_{uv}} f''_{u,v}(t) \phi(t) dt.
\]

Therefore, (4.16) becomes

\[
\sum_{v \in V_u} c_{u,v} f'_{u,v}(0) = 0,
\]

and since \(c_{u,v} = \mu(u,v)/k\) for some constant \(k\) and all \(v \in V_u\), we obtain the gluing condition at \(u\):

\[
\sum_{v \in V_u} \mu(u,v) f'_{u,v}(0) = 0.
\]

Thus, \(\mathcal{D}(A_E) \subset \mathcal{D}(A)\).

Now, conversely, suppose that \(f \in \mathcal{D}(A)\). Then \(f \in \mathcal{D}[E]\) and an integration by parts gives

\[
\mathcal{E}(f, g) = \frac{1}{2} \int_{G} f' \cdot g' \ dm
\]

\[
= \frac{1}{2} \sum_{\{u,v\}} c_{u,v} \int_{l_{uv}} f'_{u,v}(x) g''_{u,v}(x) dx
\]

\[
= \frac{1}{2} \sum_{\{u,v\}} c_{u,v} \left[ f'_{u,v}(x) g''_{u,v}(x) \right]_{0}^{l_{uv}} - \int_{0}^{l_{uv}} f''_{u,v}(x) g_{u,v}(x) dx
\]

\[
= -\frac{1}{2} \sum_{\{u,v\}} c_{u,v} \int_{0}^{l_{uv}} f''_{u,v}(x) g_{u,v}(x) dx
\]

\[
= -\frac{1}{2} \int_{G} f'' \cdot g \ dm
\]

\[
= (-A_E f, g)_m,
\]
where the second to last equality follows from the fact that
\[ \sum_{\{u,v\}} c_{u,v} - \frac{1}{2} f'_{u,v}(x) g_{u,v}(x) |_{\{u,v\}} = 0, \]
for \( f \in \mathcal{D}(A) \), as was shown in the proof of Theorem 4.5. Thus \( (A, \mathcal{D}(A)) = (A_\varepsilon, \mathcal{D}(A_\varepsilon)) \) and we have shown the Dirichlet form for Walsh’s Brownian motion on \( G \) is given by (4.12).

### 4.5 Passage Times

Let \( Z \) be a Walsh’s Brownian motion on a locally finite, connected graph \( G \). For any vertex \( u \in V \) we denote the first hitting time of \( u \) by the process as
\[ \tau(u) = \inf \{ t \geq 0 : Z_t = u \}. \]
We denote by the function \( \varphi_u \) on \( G \) the Laplace transform of the passage time for the process to hit \( u \), that is
\[ \varphi_u(z) = \mathbb{E}^z [ e^{-\beta \tau(u)} ], \]
for \( z \in G, \beta > 0 \). From the general theory for Markov processes, we know that \( \varphi_u \) satisfies

(i) \( A \varphi_u = \beta \varphi_u \) on \( G \setminus \{u\} \),

(ii) \( 0 \leq \varphi_u(z) \leq 1 \) for \( z \in G \),

(iii) \( \varphi_u(u) = 1 \).

The goal of this section is to derive a formula for \( \varphi_u \).

Before deriving \( \varphi_u \) for a general graph \( G \), we consider two special cases of graphs. The first type of graph we consider is the state space of a Walsh’s Brownian motion in \( \mathbb{R}^2 \) with discrete probability measure \( \mu \) on \([0, 2\pi)\), killed when the radial part hits 1. That is, let \( G \) be a graph consisting of a single interior vertex \( u \), \( n \) boundary vertices \( u_i \), and \( n \) edges \( \{u, u_i\} \) each with unit length, for \( i = 1, \ldots, n \). We write
\[ p_i = \mu(u, u_i), \]
for \( i = 1, \ldots, n \). We fix the orientation that directs each edge \( \{u, u_i\} \) from \( u \) to \( u_i \), \( i = 1, \ldots, n \). So \( u \) is always identified with 0 and \( u_i \) is identified with 1 for each \( i \).
Let \( f \) be a function defined on \( G \). We denote the restriction of \( f \) to the edge \( \{u, u_i\} \) by \( f_i \), \( i = 1, \ldots, n \). Then \( Z = \{Z_t; t \geq 0\} \) is a Walsh’s Brownian motion on \( G \) with infinitesimal generator

\[
\mathcal{A}f(x, \{u, u_i\}) = \frac{1}{2} f_i''(x),
\]

for \( x \in (0, 1) \), \( i = 1, \ldots, n \), where \( f \in \mathcal{D}(\mathcal{A}) \) if \( f \) is continuous on \( G \), \( f_i \in C^2((0, 1)) \), \( f_i(1) = 0 \) for each each \( i = 1, \ldots, n \), and \( f \) satisfies the gluing condition at \( u \):

\[
0 = \sum_{i=1}^{n} p_i f_i'(0),
\]

where \( f_i'(0) = \lim_{x \downarrow 0} f_i'(x) \). Notice the additional condition that functions in the domain of the generator equal 0 at the boundary vertices, this gives the absorption of \( Z \) upon reaching one unit away from the origin, radially. We now derive a formula for \( \varphi_{u_i} \), \( i = 1, \ldots, n \).

**Proposition 4.2.** Let \( G \) be the state space of \( Z \), a Walsh’s Brownian motion in \( \mathbb{R}^2 \) with discrete \( \mu \) and killed when the radial part reaches one unit away from the origin, as described above. Let \( x \in (0, 1) \), then

\[
\varphi_{u_i}(z) = \frac{\sinh(x \sqrt{2\beta})}{\sinh(\sqrt{2\beta})} + \frac{\sinh((1-x) \sqrt{2\beta})}{\sinh(\sqrt{2\beta})} \frac{p_i}{\cosh(\sqrt{2\beta})},
\]

for \( z = (x, \{u, u_i\}) \),

\[
\varphi_{u_i}(z) = \frac{\sinh((1-x) \sqrt{2\beta})}{\sinh(\sqrt{2\beta})} \frac{p_i}{\cosh(\sqrt{2\beta})},
\]

for \( z = (x, \{u, u_j\}), j = 1, \ldots, n, j \neq i; \) and

\[
\varphi_{u_i}(u) = \frac{p_i}{\cosh(\sqrt{2\beta})}.
\]

**Proof.** We prove the result for \( i = 1 \), the other cases being handled similarly. We know that \( \varphi_{u_i} \) satisfies the following:

(i) \( \mathcal{A}\varphi_{u_1} = \beta \varphi_{u_1} \) on \( G \setminus \{u_1\} \),

(ii) \( 0 \leq \varphi_{u_1}(z) \leq 1 \) for \( z \in G \),

(iii) \( \varphi_{u_1}(u_1) = 1 \).
Condition (i) implies that on each edge in $G$, the function $\varphi_{u_1}$ must be of the form

$$e^+ e^{x \sqrt{2\beta}} + e^- e^{-x \sqrt{2\beta}},$$

for $x \in (0, 1)$ and constants $c^+, c^-$. However, the constants $c^+, c^-$ need not be the same for all edges $\{u, u_i\}, i = 1, \ldots, n$. So we define functions $\varphi_i$, $i = 1, \ldots, n$, on the interval (0, 1), such that

$$\varphi_{u_1}(z) = \varphi_i(x) = c^+_i e^{x \sqrt{2\beta}} + c^-_i e^{-x \sqrt{2\beta}},$$

for $z = (x, \{u, u_i\})$ and constants $c^+_i, c^-_i$. It follows from (iii) that

$$\varphi_1(1) = c^+_1 e^{\sqrt{2\beta}} + c^-_1 e^{-\sqrt{2\beta}} = 1,$$
$$c^+_1 = e^{-\sqrt{2\beta}} - c^-_1 e^{-2\sqrt{2\beta}}.$$

Therefore,

$$\varphi_1(x) = (e^{-\sqrt{2\beta}} - c^-_1 e^{-2\sqrt{2\beta}}) e^{x \sqrt{2\beta}} + c^-_1 e^{-x \sqrt{2\beta}},$$

for some constant $c_1$. By (i) we must have for $i = 2, \ldots, n$,

$$\varphi_{u_1}(u_i) = \varphi_i(1) = c^+_i e^{\sqrt{2\beta}} + c^-_i e^{-\sqrt{2\beta}} = 0,$$
$$c^+_i = -c^-_i e^{-2\sqrt{2\beta}},$$

so that $\varphi_{u_1}$ is in the domain of $\mathcal{A}$. Therefore,

$$\varphi_1(x) = -c_i e^{-2\sqrt{2\beta}} e^{x \sqrt{2\beta}} + c_i e^{-x \sqrt{2\beta}},$$

for constants $c_i, i = 2, \ldots, n$. Also from (i), we know that $\varphi_{u_1}$ must be continuous on $G \setminus \{u_1\}$. The continuity at $u$ implies that we must have

$$\varphi_1(0) = \varphi_2(0) = \cdots = \varphi_n(0).$$

So for $i, j = 2, \ldots, n$ and $i \neq j$,

$$\varphi_i(0) = \varphi_j(0)$$
$$c_i \left(1 - e^{-2\sqrt{2\beta}}\right) = c_j \left(1 - e^{-2\sqrt{2\beta}}\right)$$
$$c_i = c_j.$$

Thus, for $i = 2, \ldots, n$,

$$\varphi_i(x) = -c e^{-2\sqrt{2\beta}} e^{x \sqrt{2\beta}} + c e^{-x \sqrt{2\beta}},$$
for some constant $c$. Also by the continuity at $u$, we must have that

$$\varphi_1(0) = \varphi_i(0),$$

for any $i = 2, \ldots, n$, which gives

$$e^{-\sqrt{2\beta}} + c_1 \left( 1 - e^{-2\sqrt{2\beta}} \right) = c \left( 1 - e^{-2\sqrt{2\beta}} \right).$$

We rewrite this equation as

$$c_1 - c = \frac{-e^{-\sqrt{2\beta}}}{1 - e^{-2\sqrt{2\beta}}}.$$  \hfill (4.17)

By condition (i), we also know that $\varphi_{u_1}$ must satisfy the gluing condition at $u$.

The functions $\varphi_i$, $i = 1, \ldots, n$, have derivatives

$$\varphi'_1(x) = \sqrt{2\beta} \left[ \left( e^{-\sqrt{2\beta}} - c_1 e^{-2\sqrt{2\beta}} \right) e^{x\sqrt{2\beta}} - c_1 e^{-x\sqrt{2\beta}} \right],$$

$$\varphi'_i(x) = \sqrt{2\beta} \left[ -c e^{-2\sqrt{2\beta}} e^{x\sqrt{2\beta}} - c e^{-x\sqrt{2\beta}} \right], \quad i = 2, \ldots, n.$$

Thus,

$$\varphi'_1(0) = \sqrt{2\beta} \left[ e^{-\sqrt{2\beta}} - c_1 \left( 1 + e^{-2\sqrt{2\beta}} \right) \right]$$

$$\varphi'_i(0) = \sqrt{2\beta} \left[ -c \left( 1 + e^{-2\sqrt{2\beta}} \right) \right], \quad i = 2, \ldots, n,$$

and the gluing condition for $\varphi_{u_1}$ at vertex $u$ is the equation

$$0 = \sum_{i=1}^{n} p_i \varphi'_i(0)$$

$$= p_1 \left[ e^{-\sqrt{2\beta}} - c_1 \left( 1 + e^{-2\sqrt{2\beta}} \right) \right] + \sum_{i=2}^{n} p_i \left[ -c \left( 1 + e^{-2\sqrt{2\beta}} \right) \right],$$

which we rewrite as

$$c_1 p_1 + c \sum_{i=2}^{n} p_i = \frac{p_1 e^{-\sqrt{2\beta}}}{1 - e^{-2\sqrt{2\beta}}}. \hfill (4.18)$$

We have two unknowns $c_1$ and $c$. Equations (4.17) and (4.18) give the following system of equations involving $c_1$ and $c$:

$$\begin{cases}
  c_1 - c = \frac{-e^{-\sqrt{2\beta}}}{1 - e^{-2\sqrt{2\beta}}}
  \\
  c_1 p_1 + c \sum_{i=2}^{n} p_i = \frac{p_1 e^{-\sqrt{2\beta}}}{1 + e^{-2\sqrt{2\beta}}}
\end{cases}$$

Solving this system and substituting the values of $c$ and $c_1$ into the expressions for $\varphi_i$ gives the result (see A.1 in the Appendix for the details).
Remark. There is an intuitive interpretation of the formulas for $\varphi_{u_i}$ given in Proposition 4.2. Recall that $Z$ restricted to any edge $\{u, u_i\}$ is one-dimensional Brownian motion on the interval $[0, 1]$, absorbed at the endpoints. It is a well-known result (see [15], for example) that for one-dimensional Brownian motion the Laplace transform of the exit times from the interval $[0, 1]$ are given by

$$
E^x \left[ e^{-\beta T_0} \mathbb{1}_{\{T_0<T_1\}} \right] = \frac{\sinh((1-x)\sqrt{2\beta})}{\sinh(\sqrt{2\beta})},
$$

$$
E^x \left[ e^{-\beta T_1} \mathbb{1}_{\{T_1<T_0\}} \right] = \frac{\sinh(x\sqrt{2\beta})}{\sinh(\sqrt{2\beta})},
$$

for $x \in [0, 1]$, where $T_0$ and $T_1$ are the hitting times of 0 and 1, respectively, for one-dimensional Brownian motion. Thus, if $Z$ starts from $z = (x, \{u, u_i\})$, $x \in [0, 1]$, then the Laplace transforms of the time it takes the process to hit $u$ or $u_i$ are given by

$$
E^z \left[ e^{-\beta \tau(u)} \mathbb{1}_{\{\tau(u)<\tau(u_i)\}} \right] = \frac{\sinh((1-x)\sqrt{2\beta})}{\sinh(\sqrt{2\beta})},
$$

$$
E^z \left[ e^{-\beta \tau(u_i)} \mathbb{1}_{\{\tau(u_i)<\tau(u)\}} \right] = \frac{\sinh(x\sqrt{2\beta})}{\sinh(\sqrt{2\beta})}.
$$

Consider the formulas for $\varphi_{u_1}$. If $Z$ starts from a point $z$ on $\{u, u_i\}$, $i = 2, \ldots, n$, then the process must first hit $u$ or $u_i$ before $u_1$. If $u_i$ is hit first, then the process is killed. If $u$ is hit first, then $\varphi_{u_1}(z)$ is equal to the product of $\varphi_{u_1}(u)$ and the Laplace transform of the time it takes the process to first hit $u$ starting from $z$, which agrees with the formula in Proposition 4.2.

If $Z$ starts from a point $z$ on $\{u, u_1\}$, then the process may hit $u_1$ before $u$. In which case, the Laplace transform of the passage time to $u_1$ from $z$ is given by the time it takes for one-dimensional Brownian motion on $[0, 1]$ to hit 1 first. If $Z$ hits $u$ before $u_1$, then the Laplace transform of the passage time to $u_1$ form $z$ is the product of $\varphi_{u_1}(u)$ and the Laplace transform of the time it takes the process to first hit $u$ starting from $z$. Thus, $\varphi_{u_1}(z)$ is given by the sum of the formulas provided by these two cases, which agrees with the formula derived in Proposition 4.2.

The second special type of graph we consider is a triangle. That is, $G$ consists of three vertices $u_1, u_2, u_3$, and three edges $\{u_1, u_2\}, \{u_2, u_3\}, \{u_1, u_3\}$. See Figure 4.2. For the probabilities at the vertices we write

$$p_1 = \mu(u_1, u_2),$$

$$p_2 = \mu(u_2, u_3),$$

$$p_3 = \mu(u_3, u_1).$$
Therefore, \( \mu(u_1, u_3) = 1 - p_1, \mu(u_2, u_1) = 1 - p_2, \) and \( \mu(u_3, u_2) = 1 - p_3. \) Notice that in this special case, the transition probabilities for the embedded Markov chain are exactly equal to the probabilities at the vertices. This follows from (4.2) and the assumption that each edge has unit length.

![Figure 4.2: The triangle graph.](image)

We fix the orientation of \( G \) that directs each edge away from the vertex with the smaller index. For example, under this orientation \( \{u_1, u_2\} \) is homeomorphic to the interval \([0, 1]\) with \( u_1 \) identified with 0 and \( u_2 \) identified with 1. We derive a formula for

\[
\varphi_{u_1}(z) = \mathbb{E}^z \left[ e^{-\beta \tau(u_1)} \right],
\]

\( z \in G, \) the Laplace transform of \( \tau(u_1). \)

**Proposition 4.3.** Let \( \{Z_t; t \geq 0\} \) be a Walsh’s Brownian motion on the triangle graph \( G \) described above. Let \( x \in [0, 1], \) then

\[
\varphi_{u_1}(z) = \frac{\sinh(x \sqrt{2\beta})}{\sinh(\sqrt{2\beta})} \sum_{k=0}^{\infty} \left( \frac{p_2^k (1 - p_3)^k (1 - p_2)}{\cosh^{2k+1}(2\sqrt{\beta})} + \frac{p_2^{k+1} (1 - p_3)^k p_3}{\cosh^{2k+2}(\sqrt{2\beta})} \right)
\]

\[
+ \frac{\sinh((1-x) \sqrt{2\beta})}{\sinh(\sqrt{2\beta})},
\]

for \( z = (x, \{u_1, u_2\}); \)

\[
\varphi_{u_1}(z) = \frac{\sinh((1-x) \sqrt{2\beta})}{\sinh(\sqrt{2\beta})} \sum_{k=0}^{\infty} \left( \frac{(1 - p_3)^k p_2^k p_3}{\cosh^{2k+1}(2\sqrt{\beta})} + \frac{(1 - p_3)^k (1 - p_2)}{\cosh^{2k+2}(\sqrt{2\beta})} \right)
\]

\[
+ \frac{\sinh(x \sqrt{2\beta})}{\sinh(\sqrt{2\beta})} \sum_{k=0}^{\infty} \left( \frac{(1 - p_3)^k p_2^k p_3}{\cosh^{2k+1}(2\sqrt{\beta})} + \frac{(1 - p_3)^k (1 - p_2)}{\cosh^{2k+2}(\sqrt{2\beta})} \right),
\]
for \( z = (x, \{u_2, u_3\}) \):

\[
\varphi_{u_1}(z) = \frac{\sinh(x \sqrt{2 \beta})}{\sinh(\sqrt{2 \beta})} \sum_{k=0}^{\infty} \left( \frac{(1 - p_3)^k p_2^k p_3}{\cosh^{2k+1}(2 \sqrt{\beta})} + \frac{(1 - p_3)^{k+1} p_2^k (1 - p_2)}{\cosh^{2k+2}(2 \sqrt{\beta})} \right) + \frac{\sinh((1 - x) \sqrt{2 \beta})}{\sinh(\sqrt{2 \beta})},
\]

for \( z = (x, \{u_1, u_3\}) \).

**Proof.** The derivation of the formulas for \( \varphi_{u_1} \) are obtained by observing the sequence of distinct vertices visited by \( Z \) before hitting \( u_1 \) as given by the embedded Markov chain. Then \( \tau(u_1) \) is the sum of the times the process spends between consecutive vertices in the sequence, and so the Laplace transform of \( \tau(u_1) \) is the product of the Laplace transforms for these times. We begin by deriving formulas for the Laplace transform of the passage times between vertices.

If the process starts from vertex \( u_2 \), then the next (distinct) vertex visited by \( Z \) will be either \( u_1 \) or \( u_3 \). To compute the Laplace transform of the passage time from \( u_2 \) to \( u_1 \) or \( u_3 \), we apply Proposition 4.2 to the process restricted to the subgraph \( G_{u_2} \), killed at \( u_1 \) and \( u_3 \). Then

\[
E_{u_2} \left[ e^{-\beta \tau(u_1)} \mathbb{1}_{\{\tau(u_1) < \tau(u_3)\}} \right] = \frac{1 - p_2}{\cosh(\sqrt{2 \beta})},
\]

\[
E_{u_2} \left[ e^{-\beta \tau(u_3)} \mathbb{1}_{\{\tau(u_3) < \tau(u_1)\}} \right] = \frac{p_2}{\cosh(\sqrt{2 \beta})}.
\]

Similarly, if the process starts from \( u_3 \), then

\[
E_{u_3} \left[ e^{-\beta \tau(u_1)} \mathbb{1}_{\{\tau(u_1) < \tau(u_2)\}} \right] = \frac{p_3}{\cosh(\sqrt{2 \beta})},
\]

\[
E_{u_3} \left[ e^{-\beta \tau(u_2)} \mathbb{1}_{\{\tau(u_2) < \tau(u_1)\}} \right] = \frac{1 - p_3}{\cosh(\sqrt{2 \beta})}.
\]

We now return to the task of finding a formula for \( \varphi_{u_1} \). If \( Z \) starts from \( u_2 \) or \( u_3 \), then using the sequence of vertices visited before \( u_1 \) provided by the embedded Markov chain, \( \varphi_{u_1} \) is the product of Laplace transforms of the passage times between consecutive vertices visited, which we just derived. By summing over all possible sequences, we obtain the formula for \( \varphi_{u_1} \) at \( u_2 \) or \( u_3 \). We work this out in the case that \( Z \) starts from vertex \( u_2 \). Notice that there is one possible sequence of vertices for a given number of vertices visited before \( u_1 \). Table 4.1 lists the sequences with the number of vertices visited before \( u_1 \) for \( n = 1, \ldots, 6 \). The last column of the table gives the product of the Laplace transforms of passage times between consecutive vertices visited for the given sequence.
Table 4.1: Possible paths between starting and ending vertices.

<table>
<thead>
<tr>
<th>sequence of vertices</th>
<th># of vertices visited before (u_1)</th>
<th>numerator</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u_2u_1)</td>
<td>1</td>
<td>(\frac{1 - p_2}{\cosh(\sqrt{2\beta})})</td>
</tr>
<tr>
<td>(u_2u_3u_1)</td>
<td>2</td>
<td>(p_2p_3)</td>
</tr>
<tr>
<td>(u_2u_3u_2u_1)</td>
<td>3</td>
<td>(p_2(1 - p_3)(1 - p_2))</td>
</tr>
<tr>
<td>(u_2u_3u_2u_3u_1)</td>
<td>4</td>
<td>(p_2^3(1 - p_3)p_3)</td>
</tr>
<tr>
<td>(u_2u_3u_2u_3u_2u_1)</td>
<td>5</td>
<td>(p_2^3(1 - p_3)^2(1 - p_2))</td>
</tr>
<tr>
<td>(u_2u_3u_2u_3u_2u_3u_1)</td>
<td>6</td>
<td>(p_2^5(1 - p_3)^2p_3)</td>
</tr>
</tbody>
</table>

Notice the pattern in the table. For a given sequence of vertices, the numerator of the products in the last column is equal to the probability that the embedded Markov chain follows that sequence, while the denominator is simply the function \(\cosh(\sqrt{2\beta})\) raised to the \(n\)th power, where \(n\) is the number of vertices visited before \(u_1\).

If \(n\) is odd, then the last vertex the process visits before hitting \(u_1\) is necessarily \(u_2\), since the process starts there, and the first \(n - 1\) vertices visited must be split evenly between \(u_2\) and \(u_3\). Therefore, starting from \(u_2\), the probability of the Markov chain visiting an odd number of vertices before hitting \(u_1\), i.e. \(n = 2k + 1\), is given by

\[
p_2^k(1 - p_3)^k(1 - p_2),
\]

where the \(k\) powers of \(p_2\) correspond to the \(k\) transitions from \(u_2\) to \(u_3\), the \(k\) powers of \(1 - p_3\) correspond to the \(k\) transitions from \(u_3\) back to \(u_2\), and \(1 - p_2\) is the probability of the chain going from \(u_2\) to \(u_1\) on the last step.

If \(n\) is even, then the last vertex visited before \(u_1\) must be \(u_3\), and of the first \(n - 1\) vertices visited, \(u_2\) is visited once than \(u_3\), since the process starts there. Therefore, starting from \(u_2\), the probability of the Markov chain visiting an even number of vertices before hitting \(u_2\), i.e. \(n = 2k + 2\), is given by

\[
p_2^{k+1}(1 - p_3)^kp_3,
\]

where the \(k + 1\) powers of \(p_2\) correspond to the \(k + 1\) transitions from \(u_2\) to \(u_3\), the \(k\) powers of \(1 - p_3\) correspond to the \(k\) transitions from \(u_3\) to \(u_2\), and \(p_3\) gives the probability
of the last transition from $u_3$ to $u_1$.

Using the pattern, we find that the product of the Laplace transforms of passage times between consecutive vertices visited for a sequence that visits $n$ vertices before hitting $u_1$, is given by

$$
\begin{cases}
  p^k_2 (1 - p_3)^k (1 - p_2) \cosh^{2k+1}(\sqrt{2\beta}), & \text{if } n = 2k + 1, \\
  p^{k+1}_2 (1 - p_3)^k p_3 \cosh^{2k+2}(\sqrt{2\beta}), & \text{if } n = 2k + 2.
\end{cases}
$$

Thus, summing over all possible sequences of vertices gives

$$
\varphi_{u_1}(u_2) = \sum_{k=0}^{\infty} \left( p^k_2 (1 - p_3)^k (1 - p_2) \cosh^{2k+1}(\sqrt{2\beta}) + p^{k+1}_2 (1 - p_3)^k p_3 \cosh^{2k+2}(\sqrt{2\beta}) \right).
$$

A similar argument at $u_3$ gives

$$
\varphi_{u_1}(u_3) = \sum_{k=0}^{\infty} \left( (1 - p_3)^k p^k_2 p_3 \cosh^{2k+1}(\sqrt{2\beta}) + (1 - p_3)^{k+1} p^k_2 (1 - p_2) \cosh^{2k+2}(\sqrt{2\beta}) \right).
$$

Now suppose that $Z$ starts from a point on edge $\{u_1, u_2\}$, i.e. $z = (x, \{u_1, u_2\})$ for some $x \in (0, 1)$. To compute $\varphi_{u_1}(z)$ we need to first find the Laplace transforms for the passage times from $z$ to $u_1$ and $u_2$. As we saw in the remark following Proposition 4.2, these are given by the Laplace transform of the exit time from the interval $[0,1]$ for one-dimensional Brownian motion:

$$
E^z \left[ e^{-\beta \tau(u_1)} 1_{\{\tau(u_1) < \tau(u_2)\}} \right] = \frac{\sinh((1 - x)\sqrt{2\beta})}{\sinh(\sqrt{2\beta})},  \quad (4.19)
$$

$$
E^z \left[ e^{-\beta \tau(u_2)} 1_{\{\tau(u_2) < \tau(u_1)\}} \right] = \frac{\sinh(x\sqrt{2\beta})}{\sinh(\sqrt{2\beta})}  \quad (4.20)
$$

Starting from $z$, if the process hits $u_1$ first, then the Laplace transform of the passage time to $u_1$ is given by (4.19); if the process hits $u_2$ first, then Laplace transform of the passage time to $u_1$ is given by the product of (4.20) with $\varphi_{u_1}(u_2)$, the Laplace transform of the passage time to $u_1$ for the process starting at $u_2$. Thus, we have

$$
\varphi_{u_1}(z) = \frac{\sinh(x\sqrt{2\beta})}{\sinh(\sqrt{2\beta})} \sum_{k=0}^{\infty} \left( p^k_2 (1 - p_3)^k (1 - p_2) \cosh^{2k+1}(\sqrt{2\beta}) + p^{k+1}_2 (1 - p_3)^k p_3 \cosh^{2k+2}(\sqrt{2\beta}) \right)
+ \frac{\sinh((1 - x)\sqrt{2\beta})}{\sinh(\sqrt{2\beta})}.
$$
The same argument applies in the case that $Z$ starts from $z = (x, \{u_1, u_3\})$, for some $x \in (0, 1)$, giving

$$
\varphi_{u_1}(z) = \frac{\sinh(x \sqrt{2\beta})}{\sinh(\sqrt{2\beta})} \sum_{k=0}^{\infty} \left( \frac{(1 - p_3)^k p_2^k p_3}{\cosh^{2k+1}(\sqrt{2\beta})} + \frac{(1 - p_3)^{k+1} p_2^k (1 - p_2)}{\cosh^{2k+2}(\sqrt{2\beta})} \right) \\
+ \frac{\sinh((1 - x) \sqrt{2\beta})}{\sinh(\sqrt{2\beta})}.
$$

Finally, suppose that $Z$ starts from $z = (x, \{u_2, u_3\})$, for some $x \in (0, 1)$. Again since $Z$ restricted to $\{u_2, u_3\}$ is one-dimensional Brownian motion on $[0, 1]$, we have a formula for the Laplace transform of the passage times to either $u_2$ or $u_3$ from $z$. Once the process hits one of these vertices, we use the formula for the Laplace transform of the passage time to $u_1$ for the process starting from either $u_2$ or $u_3$. Thus,

$$
\varphi_{u_1}(z) = \frac{\sinh((1 - x) \sqrt{2\beta})}{\sinh(\sqrt{2\beta})} \sum_{k=0}^{\infty} \left( \frac{p_2^k (1 - p_3)^k (1 - p_2)}{\cosh^{2k+1}(\sqrt{2\beta})} + \frac{p_2^{k+1} (1 - p_3)^k p_3}{\cosh^{2k+2}(\sqrt{2\beta})} \right) \\
+ \frac{\sinh(x \sqrt{2\beta})}{\sinh(\sqrt{2\beta})} \sum_{k=0}^{\infty} \left( \frac{(1 - p_3)^k p_2^k p_3}{\cosh^{2k+1}(\sqrt{2\beta})} + \frac{(1 - p_3)^{k+1} p_2^k (1 - p_2)}{\cosh^{2k+2}(\sqrt{2\beta})} \right).
$$

For a general graph $G$, with all edges unit length, a formula for $\varphi_u$ can be derived in an analogous way to the special case of a triangle. First, we need to introduce some notation.

For any vertices $u, v \in V$, we denote by $\Gamma(u, v)$ the set of all sequences of neighboring vertices beginning with $u$ and ending with $v$ such that the vertex $v$ appears exactly once. In other words, $\Gamma(u, v)$ is the set of all possible paths from $u$ to $v$. We denote an element of $\Gamma(u, v)$ by $\gamma$. We denote by $P(\gamma)$, for $\gamma \in \Gamma(u, v)$, the product of probabilities between consecutive vertices in the sequence given by $\gamma$. In the case that all the edge lengths are 1, $P(\gamma)$ gives the probability that the embedded Markov chain associated to a Walsh’s Brownian motion on $G$ starting from $u$, follows the sequence of vertices given by $\gamma$. This follows from the fact that the transition probabilities of the embedded Markov chain are given by the probabilities at the vertices in a graph with unit edges. We also denote by $n(\gamma)$ the length of the sequence $\gamma$, which is equal to the number of vertices in the sequence. To clarify the notation we consider the following example.
Example 8. Suppose that \( G \) is given by Figure 4.3. Two elements of the set \( \Gamma(u, v) \) are
\[
\gamma_1 = u, u_1, u_3, v \\
\gamma_2 = u, u_1, u_3, u_2, u, u_2, u_3, v
\]
Then \( n(\gamma_1) = 4 \) and \( n(\gamma_2) = 8 \). The probabilities of the paths are given by
\[
P(\gamma_1) = \mu(u, u_1)\mu(u_1, u_3)\mu(u_3, v), \\
P(\gamma_2) = \mu(u, u_1)\mu(u_1, u_3)\mu(u_3, u_2)\mu(u_2, u)\mu(u, u_2)\mu(u_2, u_3)\mu(u_3, v).
\]

![Figure 4.3: Example 8.](image)

The results from Proposition 4.2 combined with the methods in the proof of Proposition 4.3 give the following result for \( \varphi_u \) on a general graph \( G \), but still assuming all edges have unit length.

Proposition 4.4. Let \( Z \) be a Walsh’s Brownian motion on a locally finite, connected graph \( G \) with all edges having unit length. Fix a vertex \( u \). Then for \( z = (x, \{u_1, u_2\}) \), with \( x \in [0, 1] \) and \( \{u_1, u_2\} \) directed from \( u_1 \) to \( u_2 \),
\[
\varphi_u(z) = \frac{\sinh((1 - x)\sqrt{2}\beta)}{\sinh(\sqrt{2}\beta)} \sum_{\gamma \in \Gamma(u_1, u)} \frac{P(\gamma)}{\cosh^{n(\gamma)-1}(\sqrt{2}\beta)} + \frac{\sinh(x\sqrt{2}\beta)}{\sinh(\sqrt{2}\beta)} \sum_{\gamma \in \Gamma(u_2, u)} \frac{P(\gamma)}{\cosh^{n(\gamma)-1}(\sqrt{2}\beta)}.
\]

Now we make no assumptions on the edge lengths in \( G \). In order to derive a formula for \( \varphi_u \) in this case, we begin by revisiting the first type of graph that we considered. Only now, we do not assume that all the edge lengths are 1.

Let \( G \) be a graph consisting of a single interior vertex \( u \), \( n \) boundary vertices \( u_i \), and \( n \) edges \( \{u, u_i\} \) each with finite length \( l_i \), for \( i = 1, \ldots, n \). As before, we write
\[ p_i = \mu(u, u_i), \] and fix the orientation that directs each edge \( \{u, u_i\} \) from \( u \) to \( u_i \), for \( i = 1, \ldots, n \). Then \( G \) is the state space of a Walsh’s Brownian motion in \( \mathbb{R}^2 \) killed along each edge \( \{u, u_i\} \) when the radial part reaches a distance \( l_i \) away from the origin, i.e. at the boundary vertices of \( G \). The following proposition gives the Laplace transform of the passage time to a boundary vertex. In particular, we give the formula for vertex \( u_1 \).

**Proposition 4.5.** Let \( G \) be the state space of \( Z \), a Walsh’s Brownian motion in \( \mathbb{R}^2 \) with discrete \( \mu \) and killed when the radial part reaches a distance \( l_i \) away from the origin along the edge \( \{u, u_i\} \), \( i = 1, \ldots, n \), as described above. Then

\[
\varphi_{u_1}(x, \{u, u_1\}) = \frac{\sinh((l_1 - x)\sqrt{2}\beta)}{\sinh(l_1\sqrt{2}\beta)} \cdot \frac{p_1 \prod_{j=2}^{n} \sinh(l_j\sqrt{2}\beta)}{\sum_{j=1}^{n} p_j \cosh(l_j\sqrt{2}\beta) \prod_{k=1, k \neq j}^{n} \sinh(l_k\sqrt{2}\beta)} \cdot \\
\frac{\sinh(x\sqrt{2}\beta)}{\sinh(l_1\sqrt{2}\beta)},
\]

for \( x \in (0, l_1) \);

\[
\varphi_{u_1}(x, \{u, u_i\}) = \frac{\sinh((l_i - x)\sqrt{2}\beta)}{\sinh(l_i\sqrt{2}\beta)} \cdot \frac{p_1 \prod_{j=2}^{n} \sinh(l_j\sqrt{2}\beta)}{\sum_{j=1}^{n} p_j \cosh(l_j\sqrt{2}\beta) \prod_{k=1, k \neq j}^{n} \sinh(l_k\sqrt{2}\beta)} ,
\]

for \( x \in (0, l_i) \), \( i = 2, \ldots, n \); and

\[
\varphi_{u_1}(u) = \frac{p_1 \prod_{j=2}^{n} \sinh(l_j\sqrt{2}\beta)}{\sum_{j=1}^{n} p_j \cosh(l_j\sqrt{2}\beta) \prod_{k=1, k \neq j}^{n} \sinh(l_k\sqrt{2}\beta)}. \quad (4.21)
\]

**Proof.** As before, we know that \( \varphi_{u_1} \) satisfies the following conditions:

(i) \( \varphi_{u_1}(x, \{u, u_i\}) = \varphi_i(x) = c_i^+ e^{x\sqrt{2}\beta} + c_i^- e^{-x\sqrt{2}\beta} \), for \( x \in (0, l_i) \), \( i = 1, \ldots, n \);

(ii) \( \varphi_{u_1}(u_1) = \varphi_1(l_1) = 1 \);

(iii) \( \varphi_{u_1}(u_i) = \varphi_i(l_i) = 0 \), for \( i = 2, \ldots, n \);
(iv) \( \varphi_i(0) = \varphi_j(0) \), for \( i, j = 1, \ldots, n \);

(v) \( \sum_{i=1}^{n} p_i \varphi_i'(0) = 0 \).

The second condition implies that

\[ c_1^+ = e^{-l_1 \sqrt{2\beta}} - c_1^- e^{-2l_1 \sqrt{2\beta}}, \]

and we write

\[ \varphi_1(x) = \left( e^{-l_1 \sqrt{2\beta}} - c_1 e^{-2l_1 \sqrt{2\beta}} \right) e^{x \sqrt{2\beta}} + c_1 e^{-x \sqrt{2\beta}}, \]

for \( x \in (0, l_1) \) and some constant \( c_1 \). Similarly, the third condition implies that for each \( i = 2, \ldots, n \),

\[ c_i^+ = -c_i^- e^{-2l_i \sqrt{2\beta}}, \]

and we write

\[ \varphi_i(x) = -c_i e^{-2l_i \sqrt{2\beta}} e^{x \sqrt{2\beta}} + c_i e^{-x \sqrt{2\beta}}, \]

for \( x \in (0, l_i) \) and some constant \( c_i \).

The functions \( \varphi_i, i = 1, \ldots, n \), have derivatives

\[ \varphi'_1(x) = \sqrt{2\beta} \left[ e^{-l_1 \sqrt{2\beta}} - c_1 \left( 1 + e^{-2l_1 \sqrt{2\beta}} \right) \right] e^{x \sqrt{2\beta}} - c_1 e^{-x \sqrt{2\beta}} \],

\[ \varphi'_i(x) = \sqrt{2\beta} \left[ -c_i \left( 1 + e^{-2l_i \sqrt{2\beta}} \right) \right], \quad i = 2, \ldots, n. \]

Therefore,

\[ \varphi'_1(0) = \sqrt{2\beta} \left[ e^{-l_1 \sqrt{2\beta}} - c_1 \left( 1 + e^{-2l_1 \sqrt{2\beta}} \right) \right], \]

\[ \varphi'_i(0) = \sqrt{2\beta} \left[ -c_i \left( 1 + e^{-2l_i \sqrt{2\beta}} \right) \right], \quad i = 2, \ldots, n. \]

So the last condition that \( \varphi_{u_1} \) satisfies, which is the gluing condition at vertex \( u \), is the equation

\[ 0 = p_1 \left[ e^{-l_1 \sqrt{2\beta}} - c_1 \left( 1 + e^{-2l_1 \sqrt{2\beta}} \right) \right] + \sum_{i=2}^{n} p_i \left[ -c_i \left( 1 + e^{-2l_i \sqrt{2\beta}} \right) \right], \]

which becomes

\[ \sum_{i=1}^{n} c_i p_i \left( 1 + e^{-2l_i \sqrt{2\beta}} \right) = p_1 e^{-l_1 \sqrt{2\beta}}. \] (4.22)

The fourth condition that \( \varphi_{u_1} \) satisfies provides a relationship between the constants \( c_i \), allowing us to eliminate all but two constants from the gluing condition. Namely, for \( i = 3, \ldots, n \), \( c_i \) is related to \( c_2 \) by the following:

\[ c_i = c_2 \frac{1 - e^{-2l_i \sqrt{2\beta}}}{1 - e^{-2l_i \sqrt{2\beta}}}. \]
Then (4.22) becomes

\[ p_1 e^{-l_1 \sqrt{2\beta}} = \]

\[ c_1 p_1 \left( 1 + e^{-2l_1 \sqrt{2\beta}} \right) + c_2 \left[ p_2 \left( 1 + e^{-2l_2 \sqrt{2\beta}} \right) + \sum_{i=3}^{n} \frac{\left( 1 - e^{-2l_i \sqrt{2\beta}} \right) \left( 1 + e^{-2l_i \sqrt{2\beta}} \right)}{\left( 1 - e^{-2l_i \sqrt{2\beta}} \right)} \right] \]

which we rewrite as

\[ c_1 p_1 \left( 1 + e^{-2l_1 \sqrt{2\beta}} \right) + c_2 \left[ \sum_{i=2}^{n} p_i \left( 1 + e^{-2l_i \sqrt{2\beta}} \right) \prod_{j=2, j \neq i}^{n} \left( 1 - e^{-2l_j \sqrt{2\beta}} \right) \right] = p_1 e^{-l_1 \sqrt{2\beta}}. \]

(4.23)

Using condition (iv) again, we know that \( \varphi_1(0) = \varphi_2(0) \), providing another equation involving \( c_1 \) and \( c_2 \):

\[ c_1 \left( 1 - e^{-2l_1 \sqrt{2\beta}} \right) - c_2 \left( 1 - e^{-2l_2 \sqrt{2\beta}} \right) = -e^{-l_1 \sqrt{2\beta}}. \]

(4.24)

Solving equations (4.23) and (4.24), we find that

\[ c_2 = \frac{2p_1 e^{-l_1 \sqrt{2\beta}} \prod_{i=3}^{n} \left( 1 - e^{-2l_i \sqrt{2\beta}} \right)}{\sum_{i=1}^{n} p_i \left( 1 + e^{-2l_i \sqrt{2\beta}} \right) \prod_{j=1, j \neq i}^{n} \left( 1 - e^{-2l_j \sqrt{2\beta}} \right)}. \]

Substituting this into the formula for \( \varphi_2 \) gives

\[ \varphi_{u_1}(u) = \varphi_2(0) = c_2 \left( 1 - e^{-2l_2 \sqrt{2\beta}} \right) \]

\[ = \frac{2p_1 e^{-l_1 \sqrt{2\beta}} \prod_{i=2}^{n} \left( 1 - e^{-2l_i \sqrt{2\beta}} \right)}{\sum_{i=1}^{n} p_i \left( 1 + e^{-2l_i \sqrt{2\beta}} \right) \prod_{j=1, j \neq i}^{n} \left( 1 - e^{-2l_j \sqrt{2\beta}} \right)} \]

\[ \begin{aligned} &= \frac{p_1 \prod_{i=2}^{n} \sinh(l_i \sqrt{2\beta})}{\sum_{i=1}^{n} p_i \cosh(l_i \sqrt{2\beta}) \prod_{j=1, j \neq i}^{n} \sinh(l_j \sqrt{2\beta})}. \end{aligned} \]
The formulas for \( \varphi_{u_i}(x, \{u, u_i\}) \), with \( x \in (0, l_i) \), \( i = 1, \ldots, n \), follow as before from the fact that for a one-dimensional Brownian motion the Laplace transform of the exit times from the interval \([0, l_i]\) are given by

\[
\mathbb{E}^x \left[ e^{-\beta T_0} \mathbb{1}_{\{T_0 < T_{l_i}\}} \right] = \frac{\sinh((l_i - x) \sqrt{2\beta})}{\sinh(l_i \sqrt{2\beta})},
\]

\[
\mathbb{E}^x \left[ e^{-\beta T_{l_i}} \mathbb{1}_{\{T_{l_i} < T_0\}} \right] = \frac{\sinh(x \sqrt{2\beta})}{\sinh(l_i \sqrt{2\beta})}.
\]

We can now give a formula for \( \varphi_u \) on a completely general graph \( G \). But first, a bit more notation is needed. For \( \gamma \in \Gamma(u, v) \) we denote by \( \Phi(\gamma) \) the product of Laplace transforms of passage times between consecutive vertices in the sequence given by \( \gamma \). In other words, if \( \gamma = u_1, u_2, \ldots, u_{n(\gamma)} \in \Gamma(u, v) \), where \( u_1 = u \) and \( u_{n(\gamma)} = v \), then

\[
\Phi(\gamma) = \prod_{i=1}^{n(\gamma)-1} \varphi_{u_{i+1}}(u_i),
\]

where \( \varphi_{u_{i+1}}(u_i) \) is given by (4.21) in Proposition 4.5.

**Theorem 4.7.** Let \( Z \) be a Walsh’s Brownian motion on a locally finite, connected graph \( G \). Fix vertex \( u \in V \). Then for \( z = (x, \{u_1, u_2\}) \), with \( x \in (0, l_{u_1u_2}) \) and \( \{u_1, u_2\} \) directed from \( u_1 \) to \( u_2 \),

\[
\varphi_u(z) = \frac{\sinh((l_{u_1u_2} - x) \sqrt{2\beta})}{\sinh(l_{u_1u_2} \sqrt{2\beta})} \sum_{\gamma \in \Gamma(u_1, u)} \Phi(\gamma)
\]

+ \[
\frac{\sinh(x \sqrt{2\beta})}{\sinh(l_{u_1u_2} \sqrt{2\beta})} \sum_{\gamma \in \Gamma(u_2, u)} \Phi(\gamma).
\]
Appendix A

Calculations

We now provide the details of certain facts used above. The first fact is used to show that the distribution of the angular part of Walsh’s Brownian motion is given by the measure $\mu$. It is also used in the proof of Lemma 3.2.

**Proposition A.1.** If $\{L_t; t \geq 0\}$ is the local time at 0 for one-dimensional Brownian motion, then for $t \geq 0$,

$$E^0 \left[ \int_0^t \frac{2dL_t}{\sqrt{2\pi(t-s)}} \right] = 1.$$ 

**Proof.** We begin by showing that

$$E^0 [dL_t] = \frac{dt}{\sqrt{2\pi t}}.$$ 

Let $\{B_t; t \geq 0\}$, be a standard one-dimensional Brownian motion. Then for $t \geq 0$,

$$E^0 [||B_t||] = \int_{-\infty}^{\infty} \frac{|x|e^{-x^2/2t}}{\sqrt{2\pi t}} dx$$

$$= 2 \int_0^{\infty} \frac{x e^{-x^2/2t}}{\sqrt{2\pi t}} dx$$

$$= 2 \sqrt{t} \int_0^{\infty} \frac{e^{-u}}{\sqrt{2\pi}} du$$

$$= 2 \sqrt{t} \frac{1}{\sqrt{2\pi}}$$

Tanaka’s formula gives

$$|B_t| = \int_0^t \text{sgn}(B_s) dB_s + L_t,$$
for \( t \geq 0 \). Therefore, by taking expectations of both sides,

\[
\frac{2\sqrt{t}}{\sqrt{2\pi}} = E^0[|B_t|] = E^0[L_t],
\]

and differentiating,

\[
E^0[dL_t] = \frac{dt}{\sqrt{2\pi t}}.
\]

Next we show

\[
\int_0^t \frac{ds}{\sqrt{s(t-s)}} = \pi.
\]

Completing the square and making the trigonometric substitution \( s - \frac{t}{2} = \frac{t}{2} \sin \theta \), for \(-\pi/2 \leq \theta \leq \pi/2\), gives

\[
\int_0^t \frac{ds}{\sqrt{s(t-s)}} = \int_0^{\frac{t}{2}} \frac{ds}{\sqrt{s(t-s)}} = \int_{s=0}^{s=\frac{t}{2}} \frac{\frac{t}{2} \cos \theta d\theta}{\frac{t}{2} \cos \theta} = \int_{s=0}^{s=\frac{t}{2}} d\theta = \theta \bigg|_{s=0}^{s=\frac{t}{2}} = \sin^{-1}\left(\frac{2s-t}{t}\right) \bigg|_0^t = \pi
\]

Fubini’s Theorem and equations (A.1) and (A.2) give the desired result:

\[
E^0\left[\int_0^t \frac{2dL_s}{\sqrt{2\pi(t-s)}}\right] = \int_0^t \frac{2ds}{\sqrt{2\pi(t-s)\sqrt{2\pi s}}} = \frac{1}{\pi} \int_0^t \frac{ds}{\sqrt{s(t-s)}} = 1.
\]

The following integral arises in the proof of Lemma 3.2.

**Proposition A.2.** For any constants \( a, b \in \mathbb{R} \) and \( t > 0 \),

\[
\int_0^\infty (a + bx) \frac{xe^{-x^2/2t}}{\sqrt{2\pi t^3}} dx = \frac{a}{\sqrt{2\pi t}} + b.
\]

**Proof.** We have

\[
\int_0^\infty (a + bx) \frac{xe^{-x^2/2t}}{\sqrt{2\pi t^3}} dx = a \int_0^\infty \frac{xe^{-x^2/2t}}{\sqrt{2\pi t^3}} dx + b \int_0^\infty \frac{x^2 e^{-x^2/2t}}{\sqrt{2\pi t^3}} dx.
\]
To compute the first integral in the sum (A.3), we make the substitution $y = x^2/2t$:

$$a\int_0^\infty \frac{x e^{-x^2/2t}}{\sqrt{2\pi t^3}} \, dx = a \int_0^\infty \frac{e^{-y}}{\sqrt{2\pi t}} \, dy = \frac{a}{\sqrt{2\pi t}} \quad (A.4)$$

For the second integral in the sum (A.3), we use the Integration by Parts formula with $u = x$ and $dv = x e^{-x^2/2t} \, dx$, which gives

$$b\int_0^\infty x^2 e^{-x^2/2t} \, dx = b \int_0^\infty te^{-x^2/2t} \, dx. \quad (A.5)$$

Note that since $(2\pi t)^{-1/2} e^{-x^2/2t}$ is the density function of a Normal distribution with mean 0, we have

$$\int_0^\infty e^{-x^2/2t} \, dx = \frac{1}{2}. \quad (A.6)$$

So the integral in (A.5) is equal to

$$b \int_0^\infty t e^{-x^2/2t} \, dx = \frac{b}{2}t \sqrt{2\pi t} = \frac{b}{2} \sqrt{2\pi t^3}. \quad (A.6)$$

Combining (A.4) and (A.6) gives

$$\int_0^\infty (a + bx) \frac{x e^{-x^2/2t}}{\sqrt{2\pi t^3}} \, dx = a \int_0^\infty \frac{x e^{-x^2/2t}}{\sqrt{2\pi t^3}} \, dx + b \int_0^\infty \frac{x^2 e^{-x^2/2t}}{\sqrt{2\pi t^3}} \, dx = \frac{a}{\sqrt{2\pi t}} + b. \quad \square$$

In the context of the proof of Lemma 3.2, we have $a = h(0)$ and $b = m_h(\alpha)$, for $\alpha$ fixed, where $h$ is a harmonic function. Also, $t$ is replaced with $t - s$. The following is a calculation of the expected value of a Brownian motion killed at the origin also needed for the proof of Lemma 3.2.

**Proposition A.3.** For $x \geq 0$, $E^x [R_t; t < T_0] = x$.

**Proof.** Recall from classical theory the following formula, obtained by the reflection principle, for the density of Brownian motion absorbed at 0,

$$P^x (B_t \in dy; t < T_0) = \frac{1}{\sqrt{2\pi t}} \left( e^{-(y-x)^2/2t} - e^{-(y+x)^2/2t} \right) \, dy,$$

for $t > 0$, $x > 0$, and $y > 0$. Thus,

$$E^x [R_t; t < T_0] = \int_0^\infty \frac{y}{\sqrt{2\pi t}} \left( e^{-(y-x)^2/2t} - e^{-(y+x)^2/2t} \right) \, dy$$

$$= \int_0^\infty \frac{y}{\sqrt{2\pi t}} e^{-(y-x)^2/2t} \, dy - \int_0^\infty \frac{y}{\sqrt{2\pi t}} e^{-(y+x)^2/2t} \, dy$$
For the first integral, we make the change of variables \( z = y - x \) to obtain

\[
\int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi t}} e^{-y^2/2t} dy = \int_{-x}^{\infty} \frac{z}{\sqrt{2\pi t}} e^{-z^2/2t} dz + \int_{-x}^{\infty} \frac{x}{\sqrt{2\pi t}} e^{-x^2/2t} dz
\]

\[
= \frac{t}{\sqrt{2\pi t}} e^{-x^2/2t} + x\Phi(x/\sqrt{t}),
\]

(A.7)

where \( \Phi \) denotes the standard normal cumulative distribution function. Similarly, for the second integral, we make the change of variables \( z = y + x \) to obtain

\[
\int_{0}^{\infty} \frac{y}{\sqrt{2\pi t}} e^{-(y+x)^2/2t} dy = \int_{x}^{\infty} \frac{z}{\sqrt{2\pi t}} e^{-z^2/2t} dz - \int_{x}^{\infty} \frac{x}{\sqrt{2\pi t}} e^{-x^2/2t} dz
\]

\[
= \frac{t}{\sqrt{2\pi t}} e^{-x^2/2t} - x[1 - \Phi(x/\sqrt{t})].
\]

(A.8)

Subtracting (A.8) from (A.7) gives the desired result.

\[ \square \]

### A.1 Proof of Proposition 4.2

In the proof of Proposition 4.2, we derive a formula for \( \varphi_{u_1} \), the Laplace transform of the passage time of Walsh’s Brownian motion to vertex \( u_1 \). We found that for \( z = (x, \{u, u_1\}) \), \( x \in (0, 1) \),

\[
\varphi_1(x) = \left( e^{-\sqrt{2\beta}} - c_1 e^{-2\sqrt{2\beta}} \right) e^{x\sqrt{2\beta}} + c_1 e^{-x\sqrt{2\beta}},
\]

for some constant \( c_1 \). We also found that for \( z = (x, \{u, u_i\}) \), \( x \in (0, 1) \), \( i = 2, \ldots, n \),

\[
\varphi_i(x) = -c_i e^{-2\sqrt{2\beta}} e^{x\sqrt{2\beta}} + c_1 e^{-x\sqrt{2\beta}},
\]

for some constant \( c \). The continuity and gluing conditions on \( \varphi_{u_1} \) gave the following system of equations:

\[
\begin{align*}
    c_1 - c & = \frac{-e^{-\sqrt{2\beta}}}{1 - e^{-2\sqrt{2\beta}}} \\
    c_1 p_1 + c \sum_{i=2}^{n} p_i & = \frac{p_1 e^{-\sqrt{2\beta}}}{1 + e^{-2\sqrt{2\beta}}}
\end{align*}
\]

(A.9)

Note that

\[
\sum_{i=2}^{n} p_i = 1 - p_1.
\]
So we can rewrite (A.9) as

\[
\begin{aligned}
  c_1 - c &= \frac{-1}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \\
  c_1p_1 + c(1 - p_1) &= \frac{p_1}{e^{\sqrt{2\beta}} + e^{-\sqrt{2\beta}}}
\end{aligned}
\]

Solving this system for \( c \) and \( c_1 \) gives

\[
\begin{aligned}
  c &= \frac{p_1}{e^{\sqrt{2\beta}} + e^{-\sqrt{2\beta}}} + \frac{p_1}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} = \frac{2p_1 e^{\sqrt{2\beta}}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \\
  c_1 &= \frac{p_1}{e^{\sqrt{2\beta}} + e^{-\sqrt{2\beta}}} - \frac{(1 - p_1)}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} = \frac{2p_1 e^{\sqrt{2\beta}} - (e^{\sqrt{2\beta}} + e^{-\sqrt{2\beta}})}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}}
\end{aligned}
\]

Substituting \( c \) into the formula for \( \varphi_i, i = 2, \ldots, n \), we find

\[
\begin{aligned}
  \varphi_i(x) &= \frac{2p_1 e^{\sqrt{2\beta}}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \left( e^{-x\sqrt{2\beta}} - e^{(x-2)\sqrt{2\beta}} \right) \\
  &= \frac{2p_1 e^{(1-x)\sqrt{2\beta}} - 2p_1 e^{-(1-x)\sqrt{2\beta}}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \\
  &= \frac{p_1 \sinh((1 - x)\sqrt{2\beta})}{2 \sinh(2\sqrt{2\beta})} \\
  &= \frac{p_1 \sinh((1 - x)\sqrt{2\beta})}{\cosh(\sqrt{2\beta}) \sinh(\sqrt{2\beta})}
\end{aligned}
\]

for \( x \in (0, 1) \). Now substituting \( c_1 \) into the formula for \( \varphi_1 \), we have

\[
\begin{aligned}
  \varphi_1(x) &= e^{-(1-x)\sqrt{2\beta}} + \frac{2p_1 e^{\sqrt{2\beta}}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \left( e^{-x\sqrt{2\beta}} - e^{(x-2)\sqrt{2\beta}} \right) \\
  &= \frac{2p_1 e^{\sqrt{2\beta}}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \left( e^{-x\sqrt{2\beta}} - e^{(x-2)\sqrt{2\beta}} \right) \\
  &\quad + e^{-(1-x)\sqrt{2\beta}} - \frac{\left( e^{\sqrt{2\beta}} + e^{-\sqrt{2\beta}} \right) \left( e^{-x\sqrt{2\beta}} - e^{(x-2)\sqrt{2\beta}} \right)}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \\
  &= e^{-(1-x)\sqrt{2\beta}} - \frac{\left( e^{\sqrt{2\beta}} + e^{-\sqrt{2\beta}} \right) \left( e^{-x\sqrt{2\beta}} - e^{(x-2)\sqrt{2\beta}} \right)}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}}
\end{aligned}
\]

The first term in (A.10) is identical to \( \varphi_i(x), i = 2, \ldots, n \). For the remaining terms in (A.10) we have

\[
\begin{aligned}
  e^{-(1-x)\sqrt{2\beta}} - \frac{\left( e^{\sqrt{2\beta}} + e^{-\sqrt{2\beta}} \right) \left( e^{-x\sqrt{2\beta}} - e^{(x-2)\sqrt{2\beta}} \right)}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}}
  &= \frac{e^{(1+x)\sqrt{2\beta}} - e^{-(1+x)\sqrt{2\beta}} - e^{(1-x)\sqrt{2\beta}} + e^{-(1-x)\sqrt{2\beta}}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \\
  &= \frac{\sinh((1 + x)\sqrt{2\beta}) - \sinh((1 - x)\sqrt{2\beta})}{\sinh(2\sqrt{2\beta})} \\
  &= \frac{\sinh(x\sqrt{2\beta})}{\sinh(\sqrt{2\beta})}.
\end{aligned}
\]
Thus, for $x \in (0, 1)$,

$$\varphi_1(x) = \frac{p_1 \sinh((1 - x)\sqrt{2\beta})}{\cosh(\sqrt{2\beta}) \sinh(\sqrt{2\beta})} + \frac{\sinh(x\sqrt{2\beta})}{\sinh(\sqrt{2\beta})}. $$

We have verified the formulas for $\varphi_{u_1}(z)$ for $z = (x, \{u, u_i\})$, $x \in (0, 1)$, $i = 1, \ldots, n$. At the vertex $u$, we use the fact that $\varphi_{u_1}$ is a continuous function on the graph $G$. Therefore, we must have

$$\varphi_{u_1}(u) = \lim_{x \to 0} \varphi_i(x),$$

for every $i = 1, \ldots, k$. Note that for $i = 2, \ldots, n$,

$$\lim_{x \to 0} \varphi_i(x) = \lim_{x \to 0} \frac{p_1 \sinh((1 - x)\sqrt{2\beta})}{\cosh(\sqrt{2\beta}) \sinh(\sqrt{2\beta})} = \frac{p_1}{\cosh(\sqrt{2\beta})},$$

and since $\sinh(0) = 0$,

$$\lim_{x \to 0} \varphi_1(x) = \lim_{x \to 0} \left( \frac{p_1 \sinh((1 - x)\sqrt{2\beta})}{\cosh(\sqrt{2\beta}) \sinh(\sqrt{2\beta})} + \frac{\sinh(x\sqrt{2\beta})}{\sinh(\sqrt{2\beta})} \right) = \frac{p_1}{\cosh(\sqrt{2\beta})}.$$ 

Therefore,

$$\varphi_{u_1}(u) = \frac{p_1}{\cosh(\sqrt{2\beta})}$$

which completes the proof of Proposition 4.2.
Bibliography


