Title
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2012

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UNIVERSITY OF CALIFORNIA, SAN DIEGO

Propagation of Nonlinear Waves in Waveguides and Application to Nondestructive Stress Measurement

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Structural Engineering by

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Professor P. Benson Shing
Professor Chia-Ming Uang

2012
The dissertation of Claudio Nucera is approved and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

2012
DEDICATION

To my family Salvatore, Giuseppina and Michelangelo, and my fiancée Annalisa for their constant love and encouragement.
EPIGRAPH

Live as you were to die tomorrow.
Learn as if you were to live forever.

_Mahatma Gandhi_
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ACKNOWLEDGEMENTS

The research for this dissertation was performed at University of California San Diego (UCSD) under the supervision of Professor Francesco Lanza di Scalea. I would like to express my deepest and most sincere gratitude towards him for his continuous and superb support. He provided me with precious direction during the course of my Ph.D. and his invaluable advices were crucial for my academic and personal growth in the last few years.

Special thanks are given to all the members of my doctoral committee, Professors P. Benson Shing, Chia-Ming Uang, Thomas Bewley and Vlado Lubarda who dedicated their time to help me with the technical issues related to my research. Professor Chia-Ming Uang is acknowledged in particular for his advice on the construction of the Large-Scale Rail NT Test-bed.

I am grateful also to Professor Santi Rizzo from the University of Palermo (Italy) without whom I would not have had the opportunity to perform academic research in the USA and then pursue my Ph.D. Degree. He was for me a rigorous and inspirational teacher during my undergraduate studies in Italy.

Thanks to all the colleagues present and past in the NDE/SHM Laboratory: Dr. Stefano Coccia, Dr. Ankit Srivastava, Prof. Salvatore Salamone, Prof. Ivan Bartoli, Prof. Piervincenzo Rizzo, Arun Manohar, Stefano Mariani, Xuan “Peter” Zhu, Thompson Nguyen and Jeff Tippmann for their technical inputs, humor and friendship. A very special thanks to my colleague Robert Phillips for his continuous and inestimable help in setting up and developing the experimental parts of my research work.
I wish to thank my fellow researchers and co-workers who shared with me highs and lows of the academic life at UCSD: Alexandra Kottari, Andre Barbosa, Juan Murcia-Delso, Marios Mavros, Vasileios Papadopoulos, Ioannis Koutromanos.

I am truly grateful to all my friends in San Diego who shared with me hard work and memorable moments during my Ph.D. In particular, a sincere thank to Giuseppe Lomiento, Noemi Bonessio, Sonya Wilson, Stefano Gentile, Giovanni Castellazzi, Denis Bucher, Giulio Cattarossi, Maurizio Gobbato, Alejandro Amador, Mauro Mileni, Gabriele Guerrini, Flavio Cimadamore, Christian Gazzina, Flavio De Angelis, Houman Ghajari, Giovanni De Francesco, Simone Radice, Raghavendra Poojari.

The most sincere and deepest gratitude to my loving family: Salvatore, Giuseppina and Michelangelo, for their unwavering support and confidence in me. They have always been a solid landmark in my life and taught me how to be continually eager to learn and face everyday challenges with strength and honor.

Last but not least, I am most grateful to my fiancée Annalisa, for her constant love, encouragement and for patiently and immensely supporting me, especially during the last preparation stage of this dissertation. She always brought “colors” to the monotonicity of everyday routine in my life.

The research presented within this thesis was partially funded by the U.S. Federal Railroad Administration under University grant# FR-RRD-0009-10-01-00 with Mahmood Fateh from the FRA Office of Research and Development as the Program Manager. Thanks are extended to Gary Carr, Chief of Rail Research Division of FRA, for the technical advice and discussions. BNSF (especially John Stanford and Scott Staples)
is also acknowledged for the in-kind donation of materials and expertise for the construction of the UCSD Large-scale Rail NT/Buckling Test-bed.

Chapter 3, in part, has been published in the Mathematical Problems in Engineering Journal, Nucera, Claudio; Lanza di Scalea, Francesco; (2012). The title of this paper is *Higher Harmonic Generation Analysis in Complex Waveguides via a Nonlinear Semi-Analytical Finite Element Algorithm*. The dissertation author was the primary investigator and primary author of this paper.

Chapter 3, in part, has also been submitted for publication to the ASCEE Journal of Engineering Mechanics, Nucera, Claudio; Lanza di Scalea, Francesco; (2012). The title of this paper is *Nonlinear Semi-Analytical Finite Element Algorithm for the Analysis of Internal Resonance Conditions in Complex Waveguides*. The dissertation author was the primary investigator and primary author of this paper.

Chapter 4, in part, was presented at the 8th International Workshop on Structural Health Monitoring (IWSHM), Nucera, Claudio; Lanza di Scalea, Francesco; (2012). The title of the article is *Theoretical Considerations and Applications to Thermal Stress Measurement in Continuous Welded Rails*. The dissertation author was the primary investigator and primary author of this article.

Chapter 5, in part, will be submitted for publication to the Journal of the Acoustical Society of America, Nucera, Claudio; Lanza di Scalea, Francesco; (2012). The running title of this paper is *Nonlinear Wave Propagation in Constrained Solids Subjected to Thermal Loads*. The dissertation author will be the primary investigator and primary author of this paper.
Chapter 6, in part, will be submitted for publication to the Structural Health Monitoring Journal, Nucera, Claudio; Lanza di Scalea, Francesco; (2012). The running title of this paper is *Measurement of Neutral Temperature in Continuous Welded Rails: Results from UCSD Large-Scale Rail NT Test-bed*. The dissertation author will be the primary investigator and primary author of this paper.
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ABSTRACT OF THE DISSERTATION

Propagation of Nonlinear Waves in Waveguides and Application to Nondestructive Stress Measurement

by

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Doctor of Philosophy in Structural Engineering

University of California, San Diego, 2012

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Propagation of nonlinear waves in waveguides is a field that has received an ever increasing interest in the last few decades. Nonlinear guided waves are excellent candidates for interrogating long waveguide like structures because they combine high sensitivity to structural conditions, typical of nonlinear parameters, with large inspection ranges, characteristic of wave propagation in bounded media.

The primary topic of this dissertation is the analysis of ultrasonic waves, including ultrasonic guided waves, propagating in their nonlinear regime and their
application to structural health monitoring problems, particularly the measurement of thermal stress in Continuous Welded Rail (CWR).

Following an overview of basic physical principles generating nonlinearities in ultrasonic wave propagation, the case of higher-harmonic generation in multi-mode and dispersive guided waves is examined in more detail. A numerical framework is developed in order to predict favorable higher-order generation conditions (i.e. specific guided modes and frequencies) for waveguides of arbitrary cross-sections. This model is applied to various benchmark cases of complex structures.

The nonlinear wave propagation model is then applied to the case of a constrained railroad track (CWR) subjected to thermal variations. This study is a direct response to the key need within the railroad transportation community to develop a technique able to measure thermal stresses in CWR, or determine the rail temperature corresponding to a null thermal stress (Neutral Temperature – NT). The numerical simulation phase concludes with a numerical study performed using ABAQUS commercial finite element package. These analyses were crucial in predicting the evolution of the nonlinear parameter $\beta$ with thermal stress level acting in the rail.

A novel physical model, based on interatomic potential, was developed to explain the origin of nonlinear wave propagation under constrained thermal expansion. In fact, where the classical physics of nonlinear wave propagation assumes finite strains, the case at hand of constrained thermal expansion is, instead, characterized by infinitesimal (ideally zero) strains.

Hand-in-hand with the theoretical analyses, a comprehensive program of experimental testing has been conducted at UCSD’s Large-Scale Rail NT Test-bed, a
unique 70-ft track with controlled temperature excursions constructed at UCSD’s Powell Laboratories with government and industry funding. A prototype has been constructed for wayside determination of the rail NT based on the measurement of wave nonlinearities. The experimental results obtained with the prototype in the Large-Scale Test-bed are extremely encouraging, showing an accuracy of only a few degrees for the determination of the rail NT. If confirmed in the field, this result could revolutionize the way CWR are maintained to prevent rail buckling with respect to the thermal stress management problem.
Chapter 1

Introduction

1.1 Background

Ultrasonic waves have demonstrated great potential in assessing the state of a variety of engineering structures and have been widely used in the last few decades for Nondestructive Evaluation (NDE) and Structural Health Monitoring (SHM). These are areas of great technical and scientific interest that have been at the center of a continuous growth and innovation for over sixty years. Today ultrasonic signals are applied in a very broad spectrum of applications spanning from the predicting material behavior in structures, to the detection of internal anomalies and delaminations in aircraft components, as well as inspecting human body parts like tumors, bones, and unborn fetus. Besides the practical demand, the incessant progress and development in NDE and SHM has a lot to do with their interdisciplinary nature. In fact, these areas closely link aerospace engineering, civil engineering, electrical engineering, material science, mechanical engineering, nuclear engineering and physics among others. For this reason they benefit dramatically from this massive channeling of scientific efforts coming from different branches of physics and engineering. The attractiveness of NDE/SHM techniques based on ultrasonic waves relies on the fact that, although wave propagation...
phenomena can be quite challenging, the basic concepts behind them are relatively simple.

Ultrasonic waves involve high frequencies and, consequently, short wavelength that lead to a very high sensitivity and efficiency in detecting small structural features such as internal defects, delaminations, cracks, dislocations, etc. However conventional ultrasonic NDE techniques (Birks et al., 1991; Krautkrämer and Krautkrämer, 1990; Ness et al., 1996) are mainly based on point-to-point inspection systems, where the interrogating energy is conveyed in form of shear and longitudinal bulk waves into an area directly below the transducer. In light of this fact it is obvious how they can become extremely time-consuming and inefficient when dealing with large structural systems where the probe must mechanically scan the entire area. Furthermore, their practical applicability is limited by attenuation phenomena that inevitably come into play with stress waves propagating in unbounded media.

When stress waves propagate along an elongated structure (pipes, railroad tracks, beams, plates, etc.) they are constrained between its geometric boundaries and they undergo multiple reflections. A complex mixture of constructive and destructive interferences arises from successive reflections, refractions and mode conversions due to the interaction between waves and boundaries of the waveguide. As a result, so-called “guided waves” are generated. Compared to bulk waves, these waves are able to travel very long distances (in some cases, hundreds of meters) with little loss in energy and a complete coverage of the cross-section of the structure that act as a waveguide with obvious benefits (Figure 1.1). In practical terms this translates to the potential for rapid
screening from a single transducer position and remote inspection of physically inaccessible areas of the structure.

These attracting possibilities justify the considerable research efforts that took place in the 1990’s. Since then guided wave NDE/SHM techniques have grown to be a reality outside the confines of academic research (Cawley and Alleyne, 1996; Ditri and Rose, 1992). Several applications have been explored in the last few years highlighting the big potential of these promising techniques. For example, detection of corrosion and defects in insulated pipes, both critical aspects for the oil and chemical industries, has been addressed using an inspection system based on the reflection properties of guided waves in pipes (Lowe et al., 1998). Railroad tracks are natural waveguides and represent
perfect candidates for ultrasonic guided wave testing, as documented by the very fertile literature production on this topic (Cawley et al., 2003; Coccia et al., 2011; Hayashi et al., 2003; Rose et al., 2002; Wilcox et al., 2003). Ultrasonic guided waves have been successfully used also to interrogate aircraft components such as lap-slice joints, tear straps, landing gears, transmission beams in helicopters and so on (Rose and Soley, 2000; Rose et al., 1998). The ability to concentrate wave energy at the interface between different materials made guided waves a perfect candidate also for adhesive bonding and joining inspections (Hsu and Patton, 1993; Song et al., 2005). The potential of ultrasonic guided waves propagation is exploited also for the inspection of containment structures and concrete (Na et al., 2003). Even imaging applications have been proposed using guided waves (Yan et al., 2010).

However, the theoretical framework governing the propagation of ultrasonic waves in waveguides can be quite challenging even in the linear regime (further complications arise in the nonlinear regime, as discussed in next sections). In fact, at any given frequency an infinite number of different modes (propagating, nonpropagating and evanescent, depending upon the nature of the material) coexist and they are characterized by frequency-dependent velocities (dispersion) and attenuation. In a practical application, a transducer conveys mechanical energy into the structural component in form of ultrasonic waves and generally the frequency content of the interrogating signal is not monochromatic, but involves a packet of frequency because of the practical impossibility of generating a single-frequency signal. Hence the interrogating signal supports many modes traveling at frequency dependent velocities. This leads to the progressive transformations of the shape of the original packet as it travels in the component. This
manifests itself as a spreading of the packet in space and time as it propagates through a structure. It appears as an increase in the duration of the packet in time and a decrease in its amplitude, which is undesirable in long range guided wave testing and consequently reduces the resolution and the sensitivity of the testing system. In order to develop a correct interpretation of the guided wave signals and to fully exploit their potential, various signal-processing techniques have been proposed such as Short Time Fourier Transform (Gabor, 1952), Continuous Wavelet Transform (Daubechies, 1992), and Two-Dimensional Fourier Transform (Alleyne and Cawley, 1991). Despite this work, analysis of such a signal with the aim of predicting structural inhomogeneity based on its features requires a thorough understanding of what to expect from the generated signal in an undamaged component in the first place and accurate prediction of guided wave modal and forced solutions is still indispensable. Dispersion properties as phase velocities and group/energy velocities are important for mode identification. Similarly, the knowledge of the mode attenuation helps maximizing the inspection range by exploiting modes associated to minimum energy attenuation.

In the field of ultrasonic structural monitoring, traditional guided wave techniques rely on measuring “linear” parameter of the interrogating waves (amplitude, speed, phase shift) to infer salient features of the inspected structure. However, it is well documented that “nonlinear” parameters are usually more sensitive to structural conditions than their linear counterparts (Dace et al., 1991). The use of nonlinear guided waves is extremely attractive because they conveniently combine the aforementioned high sensitivity typical of nonlinear parameters with large inspection ranges (Bermes et al., 2008; Cawley and Alleyne, 1996; Rose, 2002). The study of nonlinear acoustics started in the 18th century
(Hamilton and Blackstock, 1988) but very few studies have focused specifically on nonlinear guided wave propagation mostly because of the complexity of the problem. In fact, the governing Navier elastodynamic equations are complicated by nonlinear terms that further sophisticate the already challenging search for a solution, especially for complex waveguides with arbitrary cross-sections. While investigations pertaining to nonlinear effect in solids were reported in the past (de Lima and Hamilton, 2003; Deng, 2003), most of them were limited in their applicability to structures with simple geometries (plates, rods, shells). As discussed in the following sections of the present thesis, the very remarkable potential of nonlinear guided waves is dramatically hindered by the lack of a solid understanding of the involved phenomena. In fact, it is paramount to master the involved phenomena for a successful and profitable transfer of this theoretical knowledge to the field for a practical application.

1.2 Research motivation

With the advent of Continuous-Welded Rail (CWR), the rail industry has experienced an increasing concern due to large longitudinal loads that are caused by restrained thermal expansion and contraction. Excessive tensile loads, occurring in cold weather, can lead to rail breakage. Conversely, excessive compression loads, occurring in hot weather, can lead to rail buckling (Figure 1.2). Both occurrences are causes of train derailments. The US Federal Railroad Administration ranks rail buckling at the top of the causes of rail accidents within the “track” category. The rail industry needs a way to determine the longitudinal forces (or stresses) in the rail as a function of changing rail
temperature. This knowledge can allow the industry to take remedial actions (cutting sections of rail or inserting rail plugs) to avoid rail breakage in cold weather and rail buckling in hot weather.

Figure 1.2 – Examples of buckling failures in CWR tracks.
A crucial rail property is the so-called Neutral Temperature \((T_N)\), defined as the rail temperature at which the thermal longitudinal force (or stress) in the rail is zero. The rail Neutral Temperature is often associated with the rail “laying” or “anchoring” temperature. It should be noted that even at \(T_N\), the rail does has a state of residual stresses caused by manufacturing whose influence needs to be eliminated.

The well-known relation between current longitudinal force, \(P\) (or longitudinal stress \(\sigma\)) and current rail temperature, \(T\), is given by:

\[
P = A\sigma = -EA\alpha \Delta T = -EA\alpha (T - T_N)
\]  

(1.1)

where \(P\) is the applied thermal load, \(\alpha\) is the coefficient of thermal expansion of steel, \(E\) is the Young’s Modulus of steel, \(A\) is the rail cross-sectional area, and \(T\) is the current rail temperature. Hence, when \(T = T_N\) the thermal load is zero, or \(P=0\). Unfortunately, the rail Neutral Temperature changes in service due to several parameters including rail kinematics (creep, breathing, ballast settlement, etc.) and rail maintenance (installation, realignment, distressing, broken rail repairs, etc.). Consequently, even for a known rail “laying” or “anchoring” temperature, the Neutral Temperature for a rail in service is generally unknown. The knowledge of the in-situ rail Neutral Temperature can help preventing rail breakage in cold weather and rail buckling in hot weather.

In light of this scenario, the main object of the present dissertation consists in developing an innovative technique aimed at nondestructively determining the Neutral Temperature (and consequently its state of stress and eventual incipient buckling conditions) of a CWR in-service exploiting the nonlinear behavior of ultrasonic guided waves propagating along the rail running direction. The knowledge of rail thermal stresses is of paramount importance in preventing train derailments due to extreme
temperature fluctuations taking appropriate remedial actions in advance depending on the situation.

The Semi-Analytical Finite Element (SAFE) method is known as one of the best tool to study guided wave propagation phenomena in complex waveguides under linear elastic regime assumption. It describes the wave propagation displacement field by coupling a finite element discretization of the waveguide cross-section with harmonic exponential functions along the wave propagation direction. In practical terms applying SAFE algorithm the original 3D problem is reduced to finding eigensolutions to an eigenvalue problem. These solutions are the wavenumbers and modeshapes of the waveguide at a given frequency.

Compared to standard three-dimensional finite element (FEM) approaches, the SAFE method allows the reduction of one order for the numerical dimension of the problem resulting in a very efficient computational scheme that provides also more physical insight into the problem. Compared to analytical wave based methods, such as Superposition of Partial Bulk Wave (SPBW) methods (Lowe, 1995), the SAFE method presents a wider spectrum of applicability since it can operate on waveguides with arbitrary cross sectional geometries, for which theoretical solutions can be unavailable. In addition, the SAFE method does not suffer from modeling waveguides with a large number of layers, as in the case of composite laminates, for which the determination of the dispersive properties via SPBW methods becomes numerically challenging.

The SAFE method has been traditionally limited to the linear elastic regime. This has prevented the exploitation of its potential to study internal resonance conditions, higher harmonics generation and other forms of wave distortion typical of nonlinear
guided wave propagation. As discussed in the following chapters, the nonlinear analysis involves complex quantities and sets some requirements (including higher-order shape functions for the 2D finite element discretization of the cross-section) that can become quite challenging if implemented in an ad-hoc code. The present dissertation extends the SAFE method to the nonlinear regime and implements it into a highly flexible, yet very powerful and relatively easy to use, commercial Finite Element code to address the aforementioned limitations. Besides all the benefits of the classical Linear SAFE formulation, the proposed scheme is able to efficiently pinpoint optimal combinations of resonant primary and secondary wave modes in waveguides of any complexity. Knowledge of such combinations is critical to the implementation of structural monitoring systems based on nonlinear features of guided waves. Efficiency and versatility of the proposed method have been proved with several case-studies. Results are presented for exemplary cases (exhibiting damping effects, anisotropic multi-layered properties, periodic geometries, heterogeneities) that can all benefit from robust structural monitoring systems and include a viscoelastic plate, a composite quasi-isotropic laminate, and a reinforced concrete slab. In particular the Nonlinear SAFE algorithm (CO.NOSAFE) is applied to study the railroad track. This was a crucial step for the development of the proposed technique for monitoring thermal stresses in CWR rails, since it helped dramatically to understand nonlinear wave propagation phenomena in rails and selecting the specific interrogating modes to be excited.

The traditional mathematical treatment of nonlinear elastic wave propagation relies on the Finite-Strain theory (Murnaghan, 1967). In this scenario a system of nonlinear PDEs is necessary to mathematically describe nonlinear phenomena as
acoustoelasticity (wave speed dependency on state of stress), waves interaction and distortion, higher harmonics generation and so on. In this work this theoretical framework has been extended to analyze nonlinear phenomena appearing in prestressed and axially constrained waveguides. The CWR rail studied in this dissertation falls in this category and benefits from this extension. It is discovered that harmonics generation in constrained waveguides (exempt from experiencing finite strains) can be explained introducing an interatomic potential model (well recognized in molecular dynamics) into the elastic strain energy density form. Experimental tests conducted on a steel specimen corroborated these theoretical predictions.

Figure 1.3 – Large-scale rail test-bed constructed at UCSD’s Powell Structural Laboratories for the development of the Neutral Temperature/Buckling Detection System for CWR.
The knowledge acquired thanks to the proposed numerical algorithm and theoretical model played a crucial role in the development of the Neutral Temperature/Incipient Buckling Detection System for CWR based on specific features of nonlinear guided waves. This technique has been validated and optimized by large-scale testing of a 70 ft-long rail track hosted at UCSD’s Powell Structural Laboratories (Figure 1.3). The final result is a device suitable for a wayside installation on the rail web.

1.3 Outline of the dissertation

The presentation of this research work has been divided into seven chapters, the contents of which are outlined below.

Chapter 1 is an introduction to the topic of guided wave propagation and their fruitful application to Nondestructive Evaluation and Structural Health Monitoring areas. It provides a brief overview of the attractive benefits in using nonlinear features of guided wave propagation when compared to traditional approaches based on linear parameters. It outlines the main challenges posed by the ever-increasing use of Continuous Welded Rails (CWR). It also defines the research motivation and stresses the need of a new methodology to assess nondestructively the state of stress acting in this type of rails to avoid the dangerous and, unfortunately, sometimes catastrophic consequences associated with rail buckling occurrence.

Chapter 2 offers a concise overview of the theoretical framework governing guided wave propagation in solids in their linear and nonlinear regime. Linear elastodynamic equations for ultrasonic bulk waves in unbounded media and guided
waves in plate-like structures are briefly revised. A survey of the analytical and numerical techniques typically adopted to describe guided wave propagation phenomena is provided. The attention is then focused on the nonlinear regime. Energy representation in nonlinear media, modal decomposition of forced solution in waveguides, and method of perturbation are described in detail.

Chapter 3 presents an extension of the classical linear Semi-Analytical Finite Element formulation (SAFE) to the nonlinear regime and its convenient and efficient implementation into a highly flexible yet powerful commercial finite element package, namely COMSOL. After a discussion on the mathematical fundamentals of the method, the proposed algorithm (CO.NO.SAFE) is benchmarked in three waveguides that do not lend themselves to alternative analyses such as closed-form solutions because of different levels of complexity in terms of material properties and geometrical features. More specifically, a viscoelastic plate, a multilayered composite panel and a heterogeneous reinforced concrete slab are considered. The successful identification of favorable combinations of resonant primary and secondary waveguide modes in these three case-studies is discussed, highlighting the promising potential of the proposed algorithm. Emphasis is placed on the correct identification of resonant modes and on the crucial role they play for the actual implementation of nondestructive condition assessment systems based on the measurement of nonlinear ultrasonic guided waves.

Chapter 4 starts unveiling in more details the need for the present research and provides statistics and historical background on continuous welded rails and annexed problems. It stresses the importance of evaluating in-situ the rail Neutral Temperature (NT) which represents the temperature at which the rail stress level is zero. Furthermore,
a brief survey of the proposed techniques up to date aimed at evaluating the rail NT is
provided, pinpointing both their potential benefits and drawbacks. The second part of the
chapter summarizes the computational efforts that guided and optimized the development
of the proposed nondestructive inspection system for neutral temperature and incipient
buckling detection in continuous welded rails. CO.NO.SAFE algorithm was employed
first to analyze dispersion characteristics and internal resonance conditions, and identify
optimal combinations of resonant modes for an AREMA 136 RE railroad track. ABAQUS commercial finite element code was used next to predict the evolution of the
nonlinearity (quantified via the nonlinear parameter $\beta$) with the thermal stress level.

Chapter 5 presents a new constitutive model aimed at describing nonlinear effects
characterizing ultrasonic wave propagation in thermo-elastic axially constrained
waveguides, as CWR rails. Starting from a classical interatomic potential formulation, a
relationship between thermal stress (arising in constrained waveguides subjected to
thermal variations) and nonlinear effects is established. Numerical interpolation is
employed during the development and a new nonlinear parameter is defined. The
proposed model is validated via a series of experimental tests performed on a steel block.
The influence of the axial constraints is analyzed repeating the test in presence and in
absence of axial constraints. An alternative formulation leading to a closed-form
expression for the equation of motion in constrained waveguides under thermal stresses is
also discussed.

Chapter 6 is dedicated to the description of the RAIL-NT system development.
Two possible wayside installations and a potential in-motion implementation are detailed.
The chapter summarizes also the extensive proof-of-principle experimental investigations
performed at UCSD Powell Structural Laboratories. The prototype proposed for a wayside installation on the rail web is also described. This prototype will be field tested in the Summer 2012 at TTCI (Transportation Technology Center, Inc.), Pueblo, CO.

Chapter 7 summarizes the research work performed, emphasizes the important original contributions and findings of this dissertation, and discusses future research directions and recommendations.
Chapter 2

Ultrasonic guided wave propagation – Theoretical fundamentals

2.1 Introduction

A body elongated in one direction and having a cross-section of finite dimension represents a waveguide. Typical examples in structural mechanics include plates, rods, shells, strands, rails, etc. These structures trap the energy of the propagating wave between their boundaries leading to a particular type of complex waves which are standing in the finite cross-sectional area and traveling in the extended direction. These waves are called guided waves, and they exhibit multimodal dispersive behavior. Infinite modes coexist at any given frequency and they are characterized by frequency-dependent velocity and attenuation. The description of the three dimensional motion of an elastic waveguide is already challenging in the linear elastic regime, especially when dealing with complex geometrical and/or material properties. However, a detailed derivation would show that the general system of equations in the material description is strongly nonlinear and linear theory is nothing more than an approximate tool to obtain relatively adequate results for wave propagation problems but this is not always the case. The original linear problem is further complicated by the need to introduce higher-order descriptions in order to correctly explore nonlinear phenomena.
In this chapter the basic equations for guided wave propagation in both linear and nonlinear elastic regimes are summarized for the purpose of reference.

### 2.2 Guided waves in linear elastic regime

When stress waves generated by a generic transducer interact with the boundaries of a waveguide, multiple reflections and mode conversions take place following a complex mixture of constructive and destructive interference. This mechanism lasts until the superposition of reflected, refracted and converted longitudinal and shear waves, at a certain distance from the source, form coherent wave packets i.e. ultrasonic guided waves (Figure 2.1).

![Guided Wave Diagram](image)

**Figure 2.1 – Guided waves generation in an isotropic homogeneous plate.**

Ultrasonic guided waves represent a very efficient tool to assess nondestructively the integrity of solids with waveguide geometry. Principal advantages in using Guided
Ultrasonic Waves (GUW) in nondestructive evaluation and structural health monitoring are:

1. Long inspection range;
2. Complete coverage of the waveguide cross-section;
3. Increased sensitivity to small features.

The main drawback is the dispersive behavior with coexistence of multiple modes and frequency-dependent velocities and attenuation. This scenario is significantly complicated when nonlinear regime is introduced.

The theoretical description of guided wave propagation in linear elastic regime has its origins in Berlin at the end of the 19th century with the work of Pochhammer on the vibration phenomena occurring in semi-infinite elastic cylindrical waveguides (Pochhammer, 1876). A few years later, Chree treated the same problem independently (Chree, 1889). For both cases, due to the complexity of the governing problem, detailed calculation of the roots did not appear until the middle of the 20th century. Bancroft was the first to study the lowest branch of the roots of the Pochhammer-Chree frequency equation and to evaluate the relation between phase velocity and wavenumbers (Bancroft, 1941). Other milestones in the development of theoretical analysis of guided waves in cylindrical waveguides are represented by the works of Gazis on the propagation phenomena for hollow, single layer, elastic circular cylinder in vacuum (Gazis, 1959) and Zemanek who was one of the first authors to present a complete analytical and experimental study (Zemanek, 1972).

Flat layered waveguides were firstly studied by Lord Rayleigh in response to the lack of understanding of elastic wave phenomena during the first recorded earthquake
seismograms in the early 1880s. He derived the equation for waves traveling along the free surface of a semi-infinite elastic half space (Figure 2.2) and showed also that the effect of this particular type of waves decrease rapidly with depth and their velocity of propagation is smaller than that of bulk waves (Rayleigh, 1887). Love demonstrated in his work that transverse modes were also possible in a half-space covered by a layer of finite thickness and different elastic properties. The modes he discovered involve transverse (shear) motion in the plane of the layer (Figure 2.3). Stoneley carried out a generalization of the single interface problem studied by Rayleigh performing a study of interface waves propagating without leakage at the boundary between two solid half spaces (Stoneley, 1924).

Horace Lamb was another pioneer in the development of the wave propagation theory in flat waveguides. In particular he was the first to study in detail (Lamb, 1917) the propagation of waves in infinite domains bounded by two surfaces (free plates) which were named after him. His derivation consists of two distinct expressions (Rayleigh-
Lamb equations) whose roots represent symmetric and antisymmetric plate modes (Figure 2.4). A plot of these roots in the frequency domain gives the well-known Lamb wave dispersion curves. However, it was only after the experimental work of Worlton that the possibility of using Lamb waves for nondestructive testing was demonstrated (Worlton, 1961). Classical treatises on guided wave propagation in isotropic media can be found in literature (Achenbach, 1973; Auld, 1990; Graff, 1991; Rose, 1999; Viktorov, 1967).

Figure 2.3 – Displacement field for Love Wave.

Figure 2.4 – Displacement field for Lamb Wave. (a) Symmetric Mode. (b) Antisymmetric Mode.
2.2.1 Linear elastodynamics of unbounded media: bulk waves

We consider a body $B$ occupying a regular region $V$ in space, which may be bounded or unbounded, with interior $V$, closure $\overline{V}$ and boundary $S$. The system of equations governing the motion of a homogeneous, isotropic, linearly elastic body consists of the stress equations of motion, Hooke's law and the strain-displacement relations. Using indicial notation these equations read:

\[
\sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i \quad (2.1)
\]

\[
\sigma_{ij} = C_{ijkl} \epsilon_{kl} \quad (2.2)
\]

\[
\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (2.3)
\]

where $\sigma_{ij}$ is the symmetric stress tensor at a point, $\epsilon_{ij}$ is the strain tensor at a point, $C_{ijkl}$ is the fourth order stiffness tensor, $u_i$ is the displacement vector of a material point, $f_i$ is the body force per unit volume and $\rho$ is the density. The Einstein summation convention for repeated subscripts is assumed here and in what follows.

The stiffness tensor possesses a number of symmetries and allows to write the constitutive laws in the simplified notation named after Woldemar Voigt:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\sigma_{yz} \\
\sigma_{xz} \\
\sigma_{xy}
\end{bmatrix}
= 
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\
c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\
c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\
c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\
c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\
c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\epsilon_z \\
\epsilon_{yz} \\
\epsilon_{xz} \\
\epsilon_{xy}
\end{bmatrix} \quad (2.4)
\]

An elastically isotropic material has no preferred directions and the elastic constants are independent of the orientation of the Cartesian coordinates. As a result, for isotropic
elastic bodies, the 21 independent stiffness constants $c_{ij}$ of Eq. (2.4) reduce to just two material constants, for example the Young’s modulus ($E$) and the Poisson’s ratio ($v$) or, equivalently, the two Lamé elastic constants $\lambda$ and $\mu$. Eqs. (2.2) through (2.4) then simplify to:

$$\sigma_{ij} = \lambda \varepsilon_{i,k} \delta_{ij} + 2\mu \varepsilon_{ij}$$  \hspace{1cm} (2.5)

$$\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\sigma_{yz} \\
\sigma_{xz} \\
\sigma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 \\
\lambda & \lambda + 2\mu & 0 & 0 \\
\lambda & 0 & \lambda + 2\mu & 0 \\
0 & 0 & 0 & 2\mu \\
0 & 0 & 0 & 2\mu \\
0 & 0 & 0 & 2\mu 
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\varepsilon_{yz} \\
\varepsilon_{xz} \\
\varepsilon_{xy}
\end{bmatrix}$$  \hspace{1cm} (2.6)

where $\delta_{ij}$ is the Kroencker Delta defined as:

$$\delta_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}$$  \hspace{1cm} (2.7)

Substituting the strain-displacement relations (2.3) into the constitutive law (2.5) and the expressions for the stresses subsequently in the stress equations of motion (2.1), the displacement equations of motion (Navier’s elastodynamic equations) are obtained as:

$$\left(\lambda + \mu\right)u_{i,j,i} + \mu u_{i,j,j} + \rho f_i = \rho \ddot{u}_i$$  \hspace{1cm} (2.8)

The vector equivalent of this expression is:

$$\left(\lambda + \mu\right)\nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} + \rho \mathbf{f} = \rho \ddot{\mathbf{u}}$$  \hspace{1cm} (2.9)

where $\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$ is the vector operator “nabla”.

In terms of rectangular scalar notation, Eqs. (2.8)-(2.9) represent the system of three equations:
\[(\lambda + \mu) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \right) + \mu \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2} \right) + \rho f_x = \rho \frac{\partial^2 u}{\partial t^2} \]

\[(\lambda + \mu) \left( \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y \partial z} \right) + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 v}{\partial z^2} \right) + \rho f_y = \rho \frac{\partial^2 v}{\partial t^2} \]

\[(\lambda + \mu) \left( \frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 v}{\partial z \partial y} + \frac{\partial^2 w}{\partial z \partial z} \right) + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \right) + \rho f_z = \rho \frac{\partial^2 w}{\partial t^2} \]

where \(u, v, w\) are the particle displacements in the \(x, y, z\) directions.

The above equations must be satisfied at every interior point of the undeformed body \(B\), i.e., in the domain \(V\). On the surface \(S\) of the undeformed body, boundary conditions must be prescribed. The common boundary conditions are

1. Displacement boundary conditions (Dirichlet BC): the three displacement components \(u_i\) are prescribed on the boundary.

2. Traction boundary conditions (Neumann BC): the three traction components, \(t_i\) are prescribed on the boundary with unit normal \(\mathbf{n}\). Through Cauchy's formula:

\[ t_i = \sigma_{ij} n_j \]  \hspace{1cm} (2.11)

this case corresponds to conditions on the three components of the stress tensor.

3. Displacement boundary conditions on part \(S_1\) of the boundary and traction boundary conditions on the remaining part \(S - S_1\).

To complete the problem statement, initial conditions are defined; in \(V\) at time \(t = 0\), we have:

\[ u_i (x, 0) = u_i^0 (x) \]  \hspace{1cm} (2.12)

\[ \dot{u}_i (x, 0) = \ddot{u}_i^0 (x) \]  \hspace{1cm} (2.13)

In the absence of body forces, Eqs. (2.8)-(2.9) become:

\[(\lambda + \mu) u_{j,ii} + \mu u_{i,jj} = \rho \ddot{u}_i \]  \hspace{1cm} (2.14)
\((\lambda + \mu) \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} = \rho \mathbf{u} \cdot \mathbf{a} \) \quad (2.15)

The main disadvantage of the system of equations above consists in the fact that it couples the three displacement components. A convenient approach to address this issue is to use Helmotz decomposition (Morse and Feshbach, 1999) to split the displacement field \(u\) into a rotational component \(\nabla \times \mathbf{\psi}\) and an irrotational component \(\nabla \phi\):

\[ u = \nabla \phi + \nabla \times \mathbf{\psi} \quad (2.16) \]

where \(\phi\) is a compressional scalar potential and \(\mathbf{\psi}\) is an equivoluminal vector potential.

Using Eq. (2.16) in Eq. (2.15) leads to:

\[ \mu \nabla^2 \left[ \nabla \phi + \nabla \times \mathbf{\psi} \right] + (\lambda + \mu) \nabla \cdot \left[ \nabla \phi + \nabla \times \mathbf{\psi} \right] = \rho \frac{\partial}{\partial t} \left[ \nabla \phi + \nabla \times \mathbf{\psi} \right] \quad (2.17) \]

Since \(\nabla \cdot \nabla \phi = \nabla^2 \phi\) and \(\nabla \cdot \nabla \times \mathbf{\psi} = 0\), Eq. (2.17) results in:

\[ \nabla \left[ (\lambda + 2\mu) \nabla^2 \phi - \rho \ddot{\phi} \right] + \nabla \times \left[ \mu \nabla^2 \mathbf{\psi} - \rho \ddot{\mathbf{\psi}} \right] = 0 \quad (2.18) \]

The identity derived is satisfied if either of the two terms on the l.h.s. in Eq. (2.18) vanishes providing the following uncoupled wave equations:

\[ \nabla^2 \phi = \frac{1}{c_L^2} \ddot{\phi} \quad (2.19) \]

\[ \nabla^2 \mathbf{\psi} = \frac{1}{c_T^2} \ddot{\mathbf{\psi}} \quad (2.20) \]

where

\[ c_L^2 = \frac{\lambda + 2\mu}{\rho} \quad (2.21) \]

\[ c_T^2 = \frac{\mu}{\rho} \quad (2.22) \]

It can be shown that harmonic potential functions of the form:
\[ \phi = \Phi e^{i(k_L \cdot x - \omega t)} \]
\[ \psi = \Psi e^{i(k_T \cdot x - \omega t)} \]  

(2.23)
satisfy the decoupled Eqs. (2.19)-(2.20). The exponential terms, which are wholly imaginary, describe the harmonic propagation of the waves in space and time. The wavenumber vector \( k \) describes the spatial distribution of the wave. The homogeneous waves propagate in the direction of the wavevector \( k \) with a spatial frequency of wavelength \( \lambda = 2\pi / |k| \) and a temporal circular frequency \( \omega = 2\pi f \). The substitution of Eqs. (2.21)-(2.22) into Eqs. (2.19)-(2.20) leads to:

\[ |k_L|^2 = \frac{\omega^2}{c_L^2} \quad |k_T|^2 = \frac{\omega^2}{c_T^2} \]  

(2.24)

It can be noted that two types of homogeneous plane waves may travel through the medium in any direction (Figure 2.5):

- dilatational (pressure) waves propagating with longitudinal speed \( c_L \)
- transverse (shear) waves propagating with speed \( c_T \).

![Figure 2.5 - Deformations caused by bulk plane waves. (a) Longitudinal waves. (b) Transverse waves.](image-url)
These bulk waves represent the eigensolutions for the equation of motion in an infinite elastic isotropic medium.

### 2.2.2 Linear elastodynamics of bounded media: Lamb waves

The attention now moves to the case of harmonic waves propagating in a bounded isotropic elastic plate with thickness $2h$ (Figure 2.6). The coordinate $y=0$ is taken to be at the mid-plane of the plate and $x$ is assumed as direction of propagation. In this particular system harmonic waves propagate in the same way as bulk waves in the unbounded media except the fact that a continuous succession of reflections of the internal shear and longitudinal waves between the two planar boundary surfaces takes place (Figure 2.1). Once the steady state condition is reached the systems of incident and reflected waves experience constructive interference and a standing wave across the thickness of the plate is created. This guided wave propagates in the direction of the layer ($x$).

![Figure 2.6 – Schematic representation of reflections and refractions taking place at the boundaries of a plate-like structure.](image)

The principle of constructive interference can be used to analyze the time harmonic motion in plane strain condition for an elastic layer (Tolstoy and Usdin, 1953).
A simpler approach consists in introducing expressions for the field variables representing a standing wave in the thickness $y$-direction of the layer and traveling waves in the $x$-direction.

Recalling Eq. (2.16), the displacement field components in Cartesian coordinates are:

\[
\begin{align*}
\frac{\partial \phi}{\partial x} + \frac{\partial \psi_z}{\partial y} - \frac{\partial \psi_x}{\partial z} & = u_x \\
\frac{\partial \phi}{\partial y} - \frac{\partial \psi_z}{\partial x} + \frac{\partial \psi_x}{\partial z} & = u_y \\
\frac{\partial \phi}{\partial z} + \frac{\partial \psi_z}{\partial x} - \frac{\partial \psi_y}{\partial y} & = u_z
\end{align*}
\]

(2.25)

The existence of plain strain conditions is assumed such that the displacement component $u_z = 0$ and $\frac{\partial}{\partial z}(\ ) = 0$. Therefore Eqs. (2.25) can be rewritten as:

\[
\begin{align*}
\frac{\partial \phi}{\partial x} + \frac{\partial \psi_z}{\partial y} & = u_x \\
\frac{\partial \phi}{\partial y} - \frac{\partial \psi_z}{\partial x} & = u_y
\end{align*}
\]

(2.26)

The substitution of the expressions for $u_x$ and $u_y$ into Navier’s elastodynamic equations produces the following system of two partial differential wave equations:

\[
\begin{align*}
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} & = \frac{1}{c_L^2} \frac{\partial^2 \phi}{\partial t^2} \\
\frac{\partial^2 \psi_z}{\partial x^2} + \frac{\partial^2 \psi_z}{\partial y^2} & = \frac{1}{c_T^2} \frac{\partial^2 \psi_z}{\partial t^2}
\end{align*}
\]

(2.27)

In order to investigate the harmonic wave motion in the elastic plate, the following solutions can be considered:
\[
\phi = \Phi(y)e^{i[kx-\omega t]} \\
\psi_z = \Psi(y)e^{i[kx-\omega t]} 
\]  
(2.28)

where \( \Phi(y) \) and \( \Psi(y) \) are functions of the position along the thickness direction and represent standing waves while the exponential term \( e^{i[kx-\omega t]} \) describes an harmonic wave propagating in the x direction with wave speed equal to \( c=\omega/k \). The terms \( \omega \) and \( k \) are well-known in acoustics as angular temporal frequency and wavenumber of the wave, respectively. There exist a direct proportionality between the angular temporal frequency \( \omega \) and the linear temporal frequency \( f (\omega=2\pi f) \) and an inverse proportionality between the wavenumber \( k \) and the wavelength \( \lambda \ (k=2\pi/\lambda) \). Inputting Eqs. (2.28) into the two partial differential equations (2.27) allows the reduction to two ordinary differential equations whose solutions can be obtained as:

\[
\Phi(y) = A_1 \sin(py) + A_2 \cos(py) \\
\Psi(y) = B_1 \sin(qy) + B_2 \cos(qy) 
\]  
(2.29)

where \( A_1, A_2, B_1 \) and \( B_2 \) are the wave amplitudes determined from the boundary conditions. The terms \( p \) and \( q \) are defined as:

\[
p = \sqrt{\frac{\omega^2}{c_L^2} - k^2} \\
q = \sqrt{\frac{\omega^2}{c_T^2} - k^2} 
\]  
(2.30)

Next step is consists in introducing the harmonic solutions expressed in Eqs. (2.28) in the displacement component functions in Eqs. (2.26) and in the stress components in Eqs. (2.5). Doing so the resultant expressions for displacement components and stresses are:
\[ u_x = [ik\Phi + \frac{d\Psi}{dy}] e^{i(kx-\omega t)} \]
\[ u_y = [\frac{d\Phi}{dy} - ik\Psi] e^{i(kx-\omega t)} \]
\[ \sigma_{xy} = [\mu(2ik \frac{d\Phi}{dy} + k^2 \Psi + \frac{d^2\Psi}{dy^2})] e^{i(kx-\omega t)} \]
\[ \sigma_{yy} = [\lambda(-k^2 \Phi + \frac{d^2\Phi}{dy^2} + 2\mu(\frac{d^2\Phi}{dy^2} - ik \frac{d\Psi}{dy})] e^{i(kx-\omega t)} \]

(2.31)

Using the potential functions of Eqs. (2.29) in Eqs. (2.31), the modes of wave propagation in the elastic layer may be split up into two systems of symmetric and antisymmetric modes, schematically represented in 2D views in Figure 2.7:

- SYMMETRIC MODES

\[ \Phi(y) = A_2 \cos(py) \]
\[ \Psi(y) = B_1 \sin(qy) \]
\[ u_x = [ikA_2 \cos(py) + qB_1 \cos(qy)] e^{i(kx-\omega t)} \]
\[ u_y = [-pA_2 \sin(py) - ikB_1 \sin(qy)] e^{i(kx-\omega t)} \]
\[ \sigma_{xy} = \mu[-2ikpA_2 \sin(py) + (k^2 - q^2)B_1 \sin(qy)] e^{i(kx-\omega t)} \]
\[ \sigma_{yy} = \{-\lambda \left(k^2 + p^2\right)A_2 \cos(py) - 2\mu \left[p^2A_2 \cos(py) + ikqB_1 \cos(qy)\right]\} e^{i(kx-\omega t)} \]

(2.32)

- ANTISYMMETRIC MODES

\[ \Phi(y) = A_1 \sin(py) \]
\[ \Psi(y) = B_2 \cos(qy) \]
\[ u_x = [ikA_1 \sin(py) - qB_2 \sin(qy)] e^{i(kx-\omega t)} \]
\[ u_y = [pA_1 \cos(py) - ikB_2 \cos(qy)] e^{i(kx-\omega t)} \]
\[ \sigma_{xy} = \mu[2ikpA_1 \cos(py) + (k^2 - q^2)B_2 \cos(qy)] e^{i(kx-\omega t)} \]
\[ \sigma_{yy} = \{-\lambda \left(k^2 + p^2\right)A_1 \sin(py) - 2\mu \left[p^2A_1 \sin(py) - ikqB_2 \sin(qy)\right]\} e^{i(kx-\omega t)} \]

(2.33)

The integration constants \(A_1, A_2, B_1\) and \(B_2\) can be calculated imposing the stress free boundary conditions at the upper and lower surfaces of the plate \((y = \pm h)\):
\[ \sigma_{xy} = \sigma_{yy} = 0 \]  \hspace{1cm} (2.34)

\[ \sigma_{xy} = \sigma_{yy} = 0 \]  \hspace{1cm} (2.34)

**Figure 2.7 – Displacement field for Lamb wave modes. (a) Symmetric. (b) Antisymmetric.**

For the symmetric modes, application of these boundary conditions yields a system of two homogeneous equations for the constants \( A_2 \) and \( B_1 \). Similarly, for the antisymmetric modes we obtain two homogeneous equations for the constants \( A_1 \) and \( B_2 \). Since the systems are homogeneous, the determinant of the coefficients must vanish, yielding to the frequency equations:

\[
\frac{(k^2 - p^2) \sin(qh)}{2ikp \sin(ph)} = -\frac{2\mu kq \cos(qh)}{(\lambda k^2 + \lambda p^2 + 2\mu p^2) \cos(ph)} \tag{2.35}
\]

\[
\frac{(k^2 - q^2) \cos(qh)}{2ikp \sin(ph)} = \frac{2\mu kq \sin(qh)}{(\lambda k^2 + \lambda p^2 + 2\mu p^2) \sin(ph)} \tag{2.36}
\]

Eqs. (2.35)-(2.36) in a more compact form known as Rayleigh-Lamb equation:

\[
\frac{\tan(ph)}{\tan(qh)} = \left[ \frac{4k^2 pq}{(k^2 - q^2)^2} \right]^{\pm 1} \tag{2.37}
\]

where the exponent +1 applies for symmetric modes (S) while exponent -1 applies to antisymmetric modes (A). An infinite number of eigensolutions exist for Eq. (2.37), thereby an infinite number of guided wave modes exist. Each eigenvalue corresponds to a particular angular frequency and mode of propagation, namely symmetric or
antisymmetric. At low frequencies, only two propagating modes exist corresponding to the fundamental symmetric mode, $S_0$ and antisymmetric mode, $A_0$. For each eigenvalue, a corresponding set of eigencoefficients also exist: $(A_2, B_1)$ and $(A_1, B_2)$ for the symmetric and antisymmetric case respectively. These coefficients can be used in Eqs. (2.31) to evaluate the Lamb mode shapes across the plate depth.

The speed of propagation of each individual wave crest is called phase velocity ($c_p = \omega/k$) and is a function of the frequency. This speed is generally different from the speed of the wave packet as a whole. The speed at which the wave packet (or wave envelope) propagates is the one measured through experiments and is referred to as group velocity ($c_g = d\omega/dk$). It also depends on the frequency and provides information about the celerity at which a given mode transports energy. The latter speed is the one of interest when it is necessary to isolate particular modes, observing different arrival times in the signals obtained through experimental measures.

Guided wave dispersion solutions for a particular system are generally represented via wavenumber and/or phase velocity and/or group velocity vs. frequency plots, commonly referred as dispersion curves. Examples of such plots are shown in Figure 2.8 and Figure 2.9 where Lamb mode wavenumbers and phase velocity dispersion curves are depicted for a typical engineering structure consisting in an aluminum plate mechanically characterized by $\rho = 2770$ kg/m$^3$, $\lambda = 6.049\times10^9$ Pa and $\mu = 2.593\times10^9$ Pa. Using the half-thickness frequency products as abscissa values normalizes these curves so that they can be used for other aluminum plates with different thickness values simply scaling the frequency. The dispersive behavior is well evident in both the curves.
Highlighted in the phase velocity dispersion curve are the first two symmetric modes, commonly referred as $S_0$ and $S_1$, and the first two antisymmetric modes, commonly referred as $A_0$ and $A_1$ at 1.5 MHz mm.

Figure 2.8 – Wavenumber dispersion curves for an aluminum plate ($h = \text{half-thickness, } \rho = 2770 \text{ kg/m}^3, \lambda = 6.049E10 \text{ Pa and } \mu = 2.593E10 \text{ Pa}$).

Figure 2.9 – Phase velocity dispersion curves for an aluminum plate ($h = \text{half-thickness, } \rho = 2770 \text{ kg/m}^3, \lambda = 6.049E10 \text{ Pa, } \mu = 2.593E10 \text{ Pa and } c_T = 3059.58 \text{ m/sec}$).
Modes $S_0$ and $A_0$ are the fundamental modes and, with the $S_{H0}$ mode, are the only ones propagating in the entire frequency range. Modes with index larger or equal to 1 are characterized by the so-called “cut-off frequency” which represents the specific frequency limit beyond which these modes start to propagate and can be estimated on the horizontal axis of the wavenumber dispersion curve. The cut-off frequency, in fact, corresponds to a zero value of the wavenumber $k_i=2\pi/\lambda_i$. As a consequence, every mode at its cut-off frequency has infinite wavelength. For the same reason, the fundamental modes have infinite wavelength at zero frequency that can be considered their cut-off frequency.

Figure 2.10 and Figure 2.11 illustrate the displacement field for the first two antisymmetric modes $A_0$ and $A_1$, while Figure 2.12 and Figure 2.13 depict the same field for the first two symmetric modes $S_0$ and $S_1$. 
Figure 2.10 – Displacement field associated with the first antisymmetric mode $A_0$ at 1.5 MHz mm (the displacement is antisymmetric with respect to the $x$-axis).

Figure 2.11 - Displacement field associated with the second antisymmetric mode $A_1$ at 1.5 MHz mm (the displacement is antisymmetric with respect to the $x$-axis).
Figure 2.12 - Displacement field associated with the first symmetric mode $S_0$ at 1.5 MHz mm (the displacement is symmetric with respect to the $x$-axis).

Figure 2.13 - Displacement field associated with the second symmetric mode $S_1$ at 1.5 MHz mm (the displacement is symmetric with respect to the $x$-axis).
2.3 Guided waves in nonlinear elastic regime

Elastic wave propagation in solids has been classically studied within the linear elastic regime where a simple linear theoretical framework can be applied and superposition principle holds. This approach relies on assuming small-signal regime and consequent infinitesimal deformations (coincidence between deformed and initial configurations) with wave amplitudes sufficiently weak that linear equations may be adequate to mathematically describe the problem. However, the actual existence of ideal linear elastic waves is extremely doubtful despite the wide use of such terminology in literature.

Nonlinear effects in elastic wave propagation may arise from several different causes. First, the amplitude of the elastic wave may be sufficiently large so that finite deformations arise. Second, a material which behaves in a linear way when undeformed may respond nonlinearly when infinitesimal ultrasonic waves are propagated, provided that a sufficient amount of external static stress is superimposed. Furthermore, the material itself may exhibit various energy absorbing mechanisms or particular forms of energy potentials that, under specific boundary conditions and external excitations, lead to a nonlinear response. The latter is analyzed in detail in a following chapter via the introduction of a new thermo-elastic material model for nonlinear guided wave propagation in constrained waveguides.

The aforementioned mechanisms produce increasingly noticeable nonlinear effects that must be introduced in the analytical framework to obtain a correct description of the response (linear theory cannot explain such effects). Among the manifestations of
the nonlinear behavior, higher harmonic generation is considered in detail in the present dissertation. In this scenario, an initially sinusoidal stress wave of a given frequency distorts as it propagates, and energy is transferred from the fundamental to the higher harmonics that appear. In practical terms, supposing to excite an ultrasonic wave into the waveguide structure at a fixed frequency, \( \omega \) (fundamental frequency), the nonlinearity manifests itself in the generation of multiple harmonics of \( \omega \), e.g. \( 2\omega \) (Second Harmonic), \( 3\omega \) (Third Harmonic) and so on. When certain resonance requirements (discussed in the following) are met, the nonlinear response is cumulative and grows with distance.

Traditional NDE techniques are based on linear theory and rely on measuring some particular parameter (wave speed, attenuation, transmission and reflection coefficients, phase shifts) of the interrogating signal to infer salient features of the inspected structure. In this case the frequency of the input and output signals are the same and no wave distortion effects take place. These conventional techniques are sensitive to gross defects, open cracks, delaminations in advanced state and other similar macroscopic features where the specific condition we want to avoid nondestructively is already well developed. Thus they are effective just where there is an effective barrier that actively influences the linear wave propagation. Nonlinear techniques efficiently overcome these limitations. They are much more sensitive to structural condition than the aforementioned conventional methods (Dace et al., 1991; Jhang, 2009) and able to pinpoint the presence of a defect or other particular structural states (early stage damages, microstructural changes) that are very difficult or even impossible to be detected by linear NDE/SHM techniques (Herrmann et al., 2006; Kim et al., 2001; Nagy, 1998; Zheng et al., 2000). The key difference between linear and nonlinear SHM/NDE
techniques is that in the latter the existence and characteristics of defects and/or other structural features are related to an acoustic signal whose frequency differs from that of the input signal because of nonlinear distortion effects (Donskoy and Sutin, 1998; Ekimov et al., 1999; Van den Abeele et al., 2000a; Van den Abeele et al., 2000b). The attractive potential in using nonlinear guided waves relies in the optimal combination of high sensitivity of nonlinear parameters and large inspection ranges characterizing guided waves that they offer (Ahmad et al., 2009; Bermes et al., 2008; Bray and Stanley, 1996; Chimenti, 1997; Kwun and Bartels, 1996; Lowe et al., 1998; Rose, 1999, 2002). Therefore their application to nondestructive evaluation and structural health monitoring has drawn considerable research interest (Ekimov et al., 1999; Rudenko, 1993; Zaitsev et al., 1995).

### 2.3.1 Historical background

A detailed understanding of linear acoustic phenomena has developed from experiments and theories dating back to several centuries ago. The study of linear wave propagation has been extensive and linear properties such as refraction, absorption, dispersion, reflection and transmission through interfaces, to mention some, are today very well explored and established as means of interrogating structures nondestructively.

The study of nonlinear acoustics has increased dramatically only in the last fifty years and, in comparison to the linear wave propagation, the matured understanding is exceedingly limited especially when dealing with nonlinear guided wave propagation. However the general field of nonlinear wave propagation is very old. Pioneering works of brilliant mathematicians like Euler, Riemann, Earnshow on finite-amplitude sound waves
in fluids and gases date back to the 18th century (Hamilton and Blackstock, 2008). It was Euler, in particular, who triggered the study of nonlinear acoustics, establishing linear acoustic as one of its subsets realizing that all wave phenomena in nature are strongly nonlinear. For the first 200 years the progress was very slow mainly because the need to understanding nonlinear phenomena in wave propagation was not such a big concern for the era and because the nonlinear mathematics necessary to describe finite-amplitude sound has been very challenging to face. In the specific area of nonlinear elastic wave propagation in solids, most of the research has focused on bulk (Gedroits and Krasilnikov, 1963; Thompson and Breazeale, 1963) and surface waves (Hamilton et al., 1999; Mayer, 2008). Despite the focus on plates, rods, rails, shells and so on in NDE/SHM applications, very few studies have been developed considering the propagation of nonlinear elastic waves in structural waveguides due to the mathematical complexity of the problem. The already challenging nonlinear Navier’s equations (valid for bulk and surface waves) are further complicated by geometrical constraints essential to the generation and sustenance of guided waves.

The modern theory of nonlinear elastic wave propagation started in Russia in the 1950s thanks to the pioneering work of Landau and Lifshitz published in a short section of their book (Landau and Lifshitz, 1959) entitled “Anharmonic Vibrations”. They introduced the concept of internal resonance and discussed, for the first time, the concepts of wave distortion, higher harmonic generation and nonlinear interaction between plane longitudinal and transverse waves in isotropic solids. In the development of their theory, they assumed the elastic medium to be homogeneous, isotropic and hyperelastic, expanding the strain energy density up to third order terms in the particle
displacement and introducing a set of third-order elastic constants that today bears their names. The framework put forth by Landau and Lifshitz, because of its focus on wave propagation, triggered the subsequent prolific publication of a series of papers concerning the theory of nonlinear wave propagation in solids. In the last thirty years, several successful applications of nonlinear guided waves have been discussed, spanning from assessing the fatigue damage of metals (Cantrell, 2006; Cantrell and Yost, 2001; Yost and Cantrell, 1992) and concrete (Shah and Ribakov, 2009), to the efficient location of internal cracks and dislocations (Arias and Achenbach, 2004; Bermes et al., 2008; Kim et al., 2010; Kuchler et al., 2009). The author of the present work recently exploited the features of nonlinear guided wave propagation in seven-wire steel strands and proposed a methodology to measure the stress level acting on these structural elements based on the theory of Contact Acoustic Nonlinearity (Nucera and Lanza Di Scalea, 2011).

Recent investigations (de Lima and Hamilton, 2003; Deng, 2003) analyzed the problem of nonlinear guided waves in isotropic plates by using normal mode decomposition and forced response (Auld, 1990). The generation of double harmonic and the cumulative growth of a phase-matched higher harmonic resonant mode were explained. Double harmonic generation was also investigated for rods and shells (de Lima and Hamilton, 2005). More recently, the theoretical treatment of guided waves in plates and rods has been extended to include all higher harmonics (Srivastava and di Scalea, 2009, 2010). In these studies it is found that the nonlinear behavior of antisymmetric Lamb mode in plates and the first order flexural mode in rods is analogous to the transverse plane wave in unconstrained isotropic media, in line with classical results (Goldberg, 1960).
While several investigations pertaining to nonlinear effect in solids and higher harmonic generation were reported in the past, most of them were limited in their applicability to structures with simple geometries (plates, rods, shells) where analytical solutions for the primary (linear) wave field are available in literature. In the present work, the propagation of waves in nonlinear solid waveguides with complex geometrical and material properties is investigated theoretically and numerically. An innovative numerical algorithm based on an extension of the semi-analytical finite element formulation to the nonlinear regime is proposed and implemented into a commercial multipurpose FEM package. Its ability to efficiently predict and explore the nonlinear wave propagation phenomena in several types of structural waveguides is discussed and validated in case-studies.

2.3.2 Nonlinear hyperelastic strain energy expression

Assuming nonlinear elastic regime, the generalized theory of nonlinear elasticity needs to be applied. Even a brief description of the fundamentals of nonlinear elasticity would have been a very challenging task going well beyond the scope of the present work. Concentrating on nonlinear guided wave problems, in the present work some key definitions are repeated referring to classical references for any further detail (Engelbrecht, 1983; Eringen, 1962; Lurie, 1990).

Different types of nonlinearities can be introduced in the analysis of wave propagation phenomena in nonlinear regime (Engelbrecht, 1997). However, for this particular case, geometrical and intrinsic physical nonlinearities are the ones exerting the strongest influence on wave propagation and are considered in the following.
According to finite strain theory (Murnaghan, 1967), geometric nonlinearity is described by the exact expression of the strain tensor (Green-Lagrange strain tensor) defined as:

\[ E_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} \right) \]  

(2.38)

where \( u_i(x_k,t) \) is the displacement vector in the Lagrange variables and the comma between index denotes derivation with respect to the corresponding coordinate (i.e. \( u_{i,j} = \frac{\partial u_i}{\partial x_j} \)). Through Eq. (2.38) potential large but finite geometrical variations of initial configuration are introduced. By definition, the Green-Lagrange strain tensor is a symmetric second-order tensor which reduces to the linear Cauchy strain tensor when infinitesimal deformations are assumed (so that the quadratic terms in Eq. (2.38) can be disregarded). As a result of this nonlinear deformation process there appear stresses that should be connected with the strains. This represents the source of physical nonlinearity discussed in the following sections.

The concept of an elastic material as a simple material whose behavior does not depend on strain history, in the nonlinear framework now augmented by the requirement of the existence of a stress potential of either the deformation gradient or other possible and alternative strain measures. The notion of potential was introduced for the first time almost two centuries ago (Green, 1839). Its existence is associated with the property of the elastic medium to store the work done by external forces on loading and to return the stored energy on unloading. In ideally elastic materials the deformation components at an arbitrary point of the material are uniquely determined by the corresponding stress components at the same point and no electrical, chemical or thermal phenomena occur.
due to the application of external load. The body possesses a distinguished state defined the natural state (undeformed and unstressed) to which it returns when all external loads are removed.

In the present treatment the physical nonlinearity is introduced considering the body hyperelastic. An elastic solid is said to be hyperelastic if it possesses a strain energy density $U$ that is an analytic function of the strain tensor $E_{ij}$ such that the second Piola-Kirchoff stress tensor $S_{ij}$ can be expressed as:

$$S_{ij} = \frac{\partial U}{\partial E_{ij}} = \frac{\partial U}{\partial \left( \frac{\partial u_i}{\partial x_j} \right)}$$

(2.39)

where $\rho_0$ is the initial density of the body. It is apparent that physical nonlinearity depends on the constitutive law governing the mechanical behavior of a specific nonlinear material and, in particular, it is related to the structure of the internal strain energy density. Eq. (2.39) was used first by Green for infinitesimal strain (Green, 1839), and later Cosserat extended the method to finite strain (Cosserat, 1896). For the general case of anisotropic and heterogeneous materials, the strain energy density depends on the material coordinates $X_i$, material descriptors $G_i$ and the invariants of any one of the material or spatial strain measures, as $I_1, I_2$ and $I_3$ (Eringen, 1962):

$$U = U \left( X_i, G_i, I_1, I_2, I_3 \right)$$

(2.40)

where the invariants of the strain tensor are defined as:

$$I_1 = E_{ii}$$
$$I_2 = E_{ij} E_{ji}$$
$$I_3 = E_{ij} E_{jk} E_{ki}$$

(2.41)
If the material is homogeneous, the strain energy density benefits of invariance with respect to a group of transformation determined by material symmetry and, consequently, the dependence on material coordinates $X_i$ drops out:

$$U = U\left(G_i, I_1, I_2, I_3\right)$$ (2.42)

If the material is isotropic, then the directional independence in the natural state requires the independence of $U$ from the material descriptors $G_i$:

$$U = U\left(X_i, I_1, I_2, I_3\right)$$ (2.43)

As final result, an isotropic homogeneous hyperelastic solid may be mechanically described by a strain energy density function $U$ which represents a single-valued function of invariants of any one of the material or spatial strain measures:

$$U = U\left(I_1, I_2, I_3\right)$$ (2.44)

Francis Murnaghan was the first to propose a development of the strain energy density for homogeneous isotropic hyperelastic solids as a power series in the three invariants of the strain tensor using constant coefficients that were determined experimentally:

$$U = \frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2 + \frac{l + 2m}{3} I_1^3 - 2ml I_2 I_3 + n I_3 + O\left(E_0^4\right)$$ (2.45)

This approximation has proved to be the most useful for compressible nonlinearly elastic materials under small deformations. The first two terms of Eq. (2.45) account for linear elasticity assuming infinitesimal deformations, hence the second order elastic moduli $\lambda$ and $\mu$ (Lamé’s Coefficients) characterize linear elastic properties of the material. The remaining terms account for first order material nonlinearity (up to
displacements strain invariants of the third order) through the use of the third order elastic constants \((l,m,n)\).

It is important to emphasize at this point that the quadratic displacement gradients terms in Eq. (2.38) account for the geometrical nonlinearity, whereas terms in Eq. (2.45) proportional to the higher-order moduli \(A, B, C\) account for the physical or material nonlinearity. This distinction, however, is mainly a colloquial one because of the intrinsic overlap of the two terms due to the values of the invariants \(I_k\). It is therefore necessary to take into consideration both geometrical and physical nonlinearity simultaneously in the development of the theoretical framework for nonlinear wave propagation.

The energy expression (2.45) may be written in an alternative form using another set of invariants (Lurie, 1990):

\[
\begin{align*}
J_1 &= I_1 \\
J_2 &= I_1^2 - 2I_2 \\
J_3 &= I_1^3 - 3I_1I_2 + 3I_3
\end{align*}
\]  

resulting in:

\[
U = \frac{\lambda}{2} J_1^2 + \mu J_2 + \frac{\nu_1}{6} J_1^2 + \nu_2 J_1J_2 + \nu_3 J_3 + O\left( E_{ij}^4 \right)
\]  

where \(\nu_1=2l-2m+n, \nu_2=m-n/2\) and \(\nu_3=n/4\) are the third order Lamè constants.

An alternative series expansion of the strain energy for cartesian geometry was proposed few years later by Landau and Lifshitz (Landau and Lifshitz, 1959),

\[
U = \frac{1}{2} \lambda I_1^2 + \mu I_2 + \frac{1}{3} CI_3^3 + BI_1I_2 + \frac{1}{3} AI_3 + O\left( E_{ij}^4 \right)
\]  

where \(A, B, C\) are the new third order moduli. This form is probably the most widely used to study nonlinear wave propagation phenomena in isotropic hyperelastic systems.
and it is considered in detail in the present work. Table 2.1 presents the relations between different third-order elastic constants proposed by various authors for isotropic solids.

<table>
<thead>
<tr>
<th>Table 2.1 – Third-order elastic constants for isotropic solids.</th>
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<tbody>
<tr>
<td>( l, m, n )</td>
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Using the definitions in Eqs. (2.38) and (2.41), Eq. (2.48) can be expressed in terms of displacement differentials as:

\[
U = \frac{1}{2} \lambda (u_{m,m})^2 + \frac{1}{4} \mu (u_{i,k} + u_{k,i})^2 + \left( \mu + \frac{1}{4} \right) u_{i,k} u_{m,h} u_{m,i} + \frac{1}{2} (\lambda + B) u_{m,m} (u_{i,k})^2 + \\
+ \frac{1}{12} A u_{i,k} u_{k,m} u_{m,i} + \frac{1}{12} B u_{i,k} u_{k,m} u_{m,m} + \frac{1}{3} C (u_{m,m})^3 + \frac{1}{4} A (u_{n,m})^4 + \frac{1}{4} \mu (u_{n,l} u_{n,k})^2 + \\
+ \frac{1}{8} A [(u_{i,k} + u_{k,i}) (u_{i,m} + u_{m,i}) + (u_{i,k} + u_{k,i}) (u_{i,m} + u_{m,i}) + (u_{i,m} + u_{m,i}) (u_{i,k} + u_{k,i})] + \frac{1}{2} B [(u_{i,k} + u_{k,i}) (u_{n,l} u_{n,k}) u_{m,m} + \\
+ \frac{1}{2} (u_{i,m} + u_{m,i}) u_{m,m} + \frac{3}{2} C (u_{m,m})^2 (u_{m,m})^2 + \frac{1}{2} A [(u_{i,k} + u_{k,i}) (u_{s,l} u_{s,m})] + \\
+ (u_{i,m} + u_{m,i}) (u_{s,l} u_{s,m}) + (u_{s,m} + u_{m,i}) (u_{i,k} u_{i,m}) + \\
+ (u_{i,k} + u_{k,i}) (u_{n,s} u_{n,k}) (u_{s,m})^2 + \frac{1}{12} C u_{m,m} (u_{m,m})^4 + \frac{1}{24} A (u_{n,l} u_{n,k}) (u_{s,l} u_{s,m}) + \\
+ \frac{1}{8} B (u_{n,l} u_{n,k})^2 (u_{s,m})^2 + \frac{1}{24} C (u_{m,m})^6
\]

\[ (2.49) \]
Eq. (2.49) represents the full representation of the first order (quadratic) nonlinearity. It contains displacement gradients whose order spans from quadratic to sextic. However the contribution of quartic, quantic and sextic terms in the general wave propagation mechanism is significantly smaller than that of quadratic and cubic terms. For this reason in the past the majority of studies on nonlinear wave propagation have assumed a first order nonlinearity approximated up to cubic displacement gradients. With this assumption Eq. (2.49) simplifies to:

\[
U = \frac{1}{2} \lambda (u_{i,j})^2 + \frac{1}{4} \mu (u_{i,k} + u_{j,i})^2 + \left( \mu + \frac{1}{4} \right) (u_{i,j} u_{i,j} u_{i,k}) + \frac{1}{2} \left( \lambda + B \right) \left[ u_{i,j} \left( u_{i,k} \right)^2 \right] + \\
+ \frac{1}{12} A (u_{i,j} u_{k,j} u_{i,j}) + \frac{1}{12} B (u_{i,k} u_{i,k} u_{i,j}) + \frac{1}{3} C (u_{i,j})^3 + ...
\] (2.50)

Eq. (2.50) is the form originally used in the pioneering works on nonlinear wave propagation in solids (Goldberg, 1960; Jones and Kobett, 1963). Over the years several alternative expressions of the strain energy density for hyperelastic solids have been proposed, extending the validity of the approach to micromorphic solids (Eringen, 1972), Cosserat continua and pseudocontinua (Cattani and Rushchitskii, 2003), two-phase elastic mixtures (Erofeyev, 2003) and so on.

It is worth noticing that Eq. (2.48) describes the first order nonlinearity which, in addition, is considered to be “weak” because the components of the strain tensor are sufficiently small that convergence of the proposed series expansions is guaranteed. If higher-order nonlinearities need to be modeled, the series expansion discussed before must proceed beyond strain invariants of cubic order. Forth (and eventually higher) order moduli need to be introduced. According to the framework set by Landau and Lifshitz,
the strain energy density for a second order (cubic) nonlinear hyperelastic solid, also defined as nine-constant theory of elasticity, is (Konyukhov and Shalashov, 1974):

\[
U = \frac{1}{2} \lambda I_1^2 + \mu I_2 + \frac{1}{3} A I_3 + BI_4 + \frac{1}{3} CI_5^3 + DI_6^4 + GI_7 + HI_8^2 + JI_9^2 + O\left( E_{ij}^5 \right)
\] (2.51)

where \( D, G, H \) and \( J \) are forth order Landau-Lifshitz moduli. Further order nonlinearities can be introduced following the same path.

### 2.3.3 Nonlinear elastodynamic equations for waveguides

The general momentum equation for hyperelastic bodies is:

\[
\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial S_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ \frac{\partial U}{\partial u_i} \right]
\] (2.52)

Assuming first order weak nonlinearity approximated up to cubic displacement gradients, the nonlinear hyperelastic constitutive equation can be obtained substituting Eq. (2.48) into Eq. (2.39) and keeping up to second-order terms:

\[
S_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij} + \delta_{ij} \left( CE_{kk} E_{ii} + BE_{ki} E_{ik} \right) + 2BE_{kk} E_{ij} + AE_{jk} E_{ki}
\] (2.53)

Using Eq. (2.53) in the general momentum equation, Eq. (2.52), the equations of motion governing the nonlinear wave propagation phenomena in homogeneous, hyperelastic, isotropic solids can be formulated as (Goldberg, 1960):
Before characterizing the treatment to the nonlinear wave propagation in waveguides it is convenient to rearrange the stress tensor \( S_{ij} \). Substituting Eq. (2.38) into Eq. (2.53), the tensor \( S_{ij} \) can be written as:

\[
S_{ij} = S_{ij}^L + S_{ij}^{NL}
\]  

(2.55)

where \( S_{ij}^L \) and \( S_{ij}^{NL} \) are the linear and nonlinear parts of the stress tensor, respectively. The linear part, \( S_{ij}^L \), is given by:

\[
S_{ij}^L = \lambda u_{i,k} \delta_{ij} + \mu \left( u_{i,j} + u_{j,i} \right)
\]  

(2.56)

The substitution of Eq. (2.55) and Eq. (2.56) into the momentum equation, Eq. (2.52), leads to the nonlinear Navier equation of motion

\[
(\lambda + \mu) u_{j,i,j} + \mu u_{i,j,j} + f_i = \rho_0 \ddot{u}_i
\]  

(2.57)

where the vector \( f_i \) includes all the nonlinear terms and acts as a vector of body force. A stress free boundary condition is necessary for the creation and the sustenance of guided waves. The latter condition reads:

\[
S_{ij}^L n_j = -\bar{S}_y n_j \quad \text{on} \quad \Gamma
\]  

(2.58)

where \( n_j \) is the unit vector normal to the surface of the waveguide \( \Gamma \). The expressions for \( f_i \) and \( \bar{S}_y \) are:
The resultant nonlinear boundary value problem represented by Eqs. (2.57)-(2.58) is solved by perturbation writing the displacement vector \( \mathbf{u} \) as the sum of a primary and a secondary solution:

\[
\mathbf{u} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)}
\]

(2.61)

where \( |\mathbf{u}^{(2)}| \ll |\mathbf{u}^{(1)}| \) is assumed (perturbation condition). The original nonlinear boundary value problem is split into two linear boundary value problems, namely the first-order and second-order approximations of the nonlinear boundary value problem. In the first approximation the boundary value problem is:

\[
\left( \lambda + \mu \right) \mathbf{u}^{(1)} + \mu \mathbf{u}^{(1)} = \rho_0 \mathbf{u}^{(1)}
\]

\[
S^{(1)}_{ij} n_j = 0 \quad \text{on} \quad \Gamma
\]

(2.62)

where \( S^{(1)}_{ij} = S^{(1)}_{ij} (\mathbf{u}^{(1)}) \) is the first order approximation of the second Piola-Kirchoff stress tensor. The problem stated in Eq. (2.62) represents a relatively simple system of linear Navier equations for a waveguide that can be solved analytically for simple geometries and numerically (using methods such as traditional linear SAFE algorithm) in case of complex waveguides.
As long as the second-order approximation is concerned, the governing equations become a system of forced linear partial differential equations:

\[
(\lambda + \mu)u_{j,ji}^{(2)} + \mu u_{i,ij}^{(2)} + f_i^{(1)} = \rho_0 \ddot{u}_i^{(2)}
\]

\[
S_{ij}^{(2)} n_j = -\ddot{S}_{ij}^{(1)} n_j \quad \text{on} \quad \Gamma
\]

where \( u_i^{(2)} \) represents the secondary solution, \( S_{ij}^{(2)} = S_{ij}^L (u_i^{(2)}) \) is the second order approximation of the second Piola-Kirchoff stress tensor, and \( f_i^{(1)} \) and \( \ddot{S}_{ij}^{(1)} \) are obtained by replacing \( u_i = u_i^{(1)} \) in Eqs. (2.59) and (2.60), respectively.

The original system of unforced partial differential equations governing the harmonic generation phenomena due to nonlinear distortion of a guided wave propagating in a quadratically nonlinear waveguide has been presented. By successive approximations, it has been reduced to two sets of linear inhomogeneous partial differential equations. The first is a homogeneous system of linear PDEs whose solutions are the dispersion solutions of the considered waveguide. This represents the primary solution \( u_i^{(1)} \). The second is an inhomogeneous system of linear PDEs where the forcing terms emanate from the primary solution. Its solution represents the secondary solution \( u_i^{(2)} \). Hence, the original problem reduces to seeking the solution for an elastic wave generated by two external forces, the surface force \( \ddot{S}_{ij}^{(1)} n_j \) and the body force \( f_i^{(1)} \).

At this stage it is possible to explain qualitatively the second harmonic generation mechanism. In the second approximation, the forcing function, Eq. (2.59), contains terms which are products of displacement gradients. If the primary wave field (primary solution) exhibits simple time dependence with frequency \( \omega \), this fact will give rise to terms in the forcing function which will exhibit \( 2\omega \) dependence. In light of this fact, the
particular solution to Eq. (2.63) will also show $2\omega$ dependence, i.e. the second harmonic is generated by a single monochromatic input signal in first-order nonlinear waveguides (Figure 2.14). The mechanism qualitatively explained here was characterized to quadratic nonlinearity. It can easily be extended to higher-order nonlinearities in order to explain the generation of higher-order harmonics.

![Figure 2.14 – Schematic of second harmonic generation phenomenon in an exemplary nonlinear waveguide.](image)

### 2.3.4 Solution for linear forced waveguides

In this section an efficient technique introduced by Auld for the analysis of linear forced waveguides (Auld, 1990) is briefly reproduced for the sake of completeness. It is based on normal mode expansion at the double harmonic to represent the secondary approximation of the solution.
2.3.4.1 Waveguide mode orthogonality

Considering two elastodynamic states \((v_1, T_1, F_1)\) and \((v_2, T_2, F_2)\) where \(v\) is the velocity vector, \(T\) is the surface traction and \(F\) is the body force, the complex reciprocity relation can be formulated as (Auld, 1990):

\[
\nabla \cdot \left[ -v_2^* \cdot T_1 - v_1 \cdot T_2^* \right] = \left[ v_2^* \cdot F_1 - v_1 \cdot F_2^* \right]
\]

(2.64)

The derivation of the waveguide mode orthogonality condition is derived here for guided waves in plates (Lamb modes) but the essential form of the final solution holds for arbitrary waveguides. To derive the orthogonality relations, body forces \((F_1, F_2)\) are neglected. In what follows solutions 1 and 2 are considered to be free modes with propagation factors \(k_m\) and \(k_n\), respectively. Furthermore, \(z\) is assumed as wave propagation direction, while \(y\) as thickness direction. For plane strain conditions the velocity fields for the two mentioned solutions can be expressed as:

\[
v_1 = e^{ik_mz}v_m(y) \quad \quad \quad v_2 = e^{ik_nz}v_n(y)
\]

(2.65)

In light of these assumptions, the complex reciprocity relation simplifies to:

\[
\nabla \cdot \left\{ \right\} = \frac{\partial}{\partial z} \left\{ \right\} \cdot \hat{z} + \frac{\partial}{\partial y} \left\{ \right\} \cdot \hat{y}
\]

(2.66)

where

\[
\left\{ \right\} = -v_2^* \cdot T_1 - v_1 \cdot T_2^* \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
After the above equation is integrated with respect to \( y \) across the waveguide, the right hand side of it reduces to the value of \( \{ \} \) at the plate edges. Considering a stress free or rigid boundary condition at these edges, i.e.:

\[
\mathbf{T}_i \cdot \hat{\mathbf{y}} = 0 \quad \text{or} \quad \mathbf{v} = 0 \quad \text{at} \quad y = 0, b
\]  

(2.69)

the right hand side of Eq. (2.68) is zero. Consequently, Eq. (2.68) simplifies to:

\[
i(k_m - k_n^*) P_{mn} = 0
\]  

(2.70)

Hence the orthogonality condition for the elastic waveguide modes reads:

\[
P_{mn} = 0 \quad \text{if} \quad k_m \neq k_n
\]  

(2.71)

It is worth noticing that for propagating modes the term \( P_{mn} \) represents the average power flow in the \( z \) direction (in other words it is the \( z \) component of the Poynting Vector). In this case it is defined as:

\[
P_{mn} = \Re \left\{ \frac{1}{2} \int \{ -v_n^* \cdot \mathbf{T}_m \} \cdot \hat{z} dy \right\}
\]  

(2.72)

**2.3.4.2 Complex reciprocity relation**

The necessary complex reciprocity relation for the second order problem is (Auld, 1990):

\[
-\frac{\partial}{\partial z} \left[ \left( -\mathbf{v}_n^* \cdot \mathbf{S}^2 + \mathbf{v}^2 \cdot \sigma_n^* \right) \cdot \mathbf{n}_z e^{i k_n z} \right] - \nabla_\perp \cdot \left( \mathbf{v}_n^* \cdot \mathbf{S}^2 + \mathbf{v} \cdot \sigma_n^* \right) e^{i k_n z} = \mathbf{v}_n^* \cdot \mathbf{f}^{i k_n z}
\]  

(2.73)

where \( \mathbf{v}_n(r) \) is the \( n^{th} \) modal velocity for a stress-free waveguide, \( \sigma_n(r) \) is the \( n^{th} \) modal stress obtained from \( \mathbf{v}_n(r) \), \( k_n \) is the wavenumber for the \( n^{th} \) mode, \( \nabla^2 = \partial^2 / \partial t^2 \) is the solution for the particle velocity, \( \mathbf{S}^2(r) \) is the stress obtained from \( \mathbf{v}^2(r) \), \( \mathbf{n}_z \) is the unit vector in the direction of propagation, and the differential operator \( \nabla_\perp \) is defined as:
\[
\n\nabla_\perp = n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y}
\]

(2.74)

The expressions for \( v^2 \) and \( S^2 \cdot n_z \) can be expanded in terms of waveguide modes as:

\[
v^2(r, z, t) = \frac{1}{2} \sum_{m=1}^{\infty} A_m(z) v_m(r) e^{-\alpha_m z} + \text{c.c.} 
\]

(2.75)

\[
S^2(r, z, t) \cdot n_z = \frac{1}{2} \sum_{m=1}^{\infty} A_m(z) \sigma_m(r) \cdot n_z e^{-\alpha_m z} + \text{c.c.}
\]

(2.76)

where \( A_m(z) \) is the modal amplitude and c.c. stands for complex conjugate. Using Eqs. (2.75)-(2.76) in Eq. (2.73), integrating the result over the waveguide cross-sectional area \( \Omega \), and applying the divergence theorem to the second term on the left-hand side produces:

\[
-P_{mn} \frac{\partial}{\partial z} \left[ e^{ikz} \sum_m 4A_m(z) \right] - e^{ikz} \left[ \int_{\Gamma} \left( v^2 \cdot \sigma_n^* + v^* \cdot S^2 \right) \cdot n d\Gamma \right] = e^{ikz} \int_{\Omega} f^1 \cdot v^* d\Omega
\]

(2.77)

where \( \Gamma \) is the curve enclosing the volume \( \Omega \), \( n \) is the unit vector normal to \( \Gamma \) and:

\[
P_{mn} = -\frac{1}{4} \int_{\Omega} \left( v^* \cdot \sigma_m + v_m \cdot \sigma_n^* \right) \cdot n_z d\Omega
\]

(2.78)

In the secondary problem, only traction is prescribed and the modes correspond to a stress-free waveguide \( (\sigma_n \cdot n = 0) \). Therefore, Eq. (2.77) can be reformulated as:

\[
-P_{mn} \frac{\partial}{\partial z} \left[ e^{ikz} \sum_m 4A_m(z) \right] - e^{ikz} \left[ \int_{\Gamma} v^* \cdot \tilde{S}^1 \cdot n d\Gamma \right] = e^{ikz} \int_{\Omega} f^1 \cdot v^* d\Omega
\]

(2.79)

The orthogonality relation of modes, according to Auld, reads:

\[
P_{mn} = 0 \quad \text{if} \quad k_m \neq k_n^*
\]

(2.80)

Using the above result in Eq. (2.79) leads to:
\[ 4P_{mn} \left( \frac{d}{dz} + i k_n \right) A_m(z) = \left[ f_n^{\text{surf}}(z) + f_n^{\text{vol}}(z) \right] e^{i(k_n z_k)z} \] (2.81)

where the \( n^{th} \) mode is the only mode not orthogonal to the \( m^{th} \) mode and:

\[ f_n^{\text{surf}}(z) = \int_{S} \left( \mathbf{v}_n^* \cdot \mathbf{S} \cdot \mathbf{n} \right) d\Gamma \] (2.82)

\[ f_n^{\text{vol}}(z) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_n^* d\Omega \] (2.83)

The terms \( f_n^{\text{surf}}(z) \) and \( f_n^{\text{vol}}(z) \) are identified as the complex external power due to the surface stress \( \mathbf{S} \cdot \mathbf{n} \) and volume force \( \mathbf{f} \). If the source condition is assumed to be:

\[ u^2 = 0 \quad \text{at} \quad z = 0 \] (2.84)

the solution to Eq. (2.81) is:

\[ A_m(z) = \overline{A}_m(z) e^{i(k_n z_k)z} - \overline{A}_m(0) e^{-iu_n z} \] (2.85)

The amplitude \( A_m(z) \) quantifies how strong the excitation of the \( m^{th} \) secondary mode in the modal expansion is. In Eq. (2.85), the amplitude of the secondary modes is expressed in two different forms depending on the existence of the phase-matching condition (synchronism). The latter occurs between two modes having the same phase velocity. The expressions are:

\[ \overline{A}_m(z) = \frac{i(f_n^{\text{surf}} + f_n^{\text{vol}})}{4P_{mn} \left( k_n^* - (k_a \pm k_b) \right)} \quad \text{if} \quad k_n^* \neq (k_a \pm k_b) \quad \text{(ASYNCHRONISM)} \] (2.86)

\[ \overline{A}_m(z) = \frac{(f_n^{\text{surf}} + f_n^{\text{vol}})z}{4P_{mn}} \quad \text{if} \quad k_n^* = (k_a \pm k_b) \quad \text{(SYNCHRONISM)} \] (2.87)

The analytical treatment has been maintained so far to a general level where two propagating waveguide modes with real wavenumbers \( k_a \) and \( k_b \) are conveyed into the
nonlinear waveguide. In this case frequency-mixing phenomena in addition to higher-harmonic generation occur because of nonlinear wave distortions. Second-harmonic generation represents just a special case in which only a single mode is excited. In this case the nonlinearity of the waveguide transforms a monochromatic (single frequency) wave input into a distorted output where primary wave and secondary harmonic coexist. Characterizing the treatment to second-harmonic generation, the expressions for first order nonlinear solution and higher-order modal amplitude read:

$$v(r, z, t) = \frac{1}{2} \sum_{m=1}^{\infty} A_m(z) v_m(r) e^{-i 2 \omega t} + c.c. \quad (2.88)$$

$$A_m(z) = A_m(z) e^{i(2\omega z)} - A_m(0) e^{i \omega z} \quad (2.89)$$

where

$$\overline{A}_m(z) = i \left( \frac{f_n^{surf} + f_n^{vol}}{4P_{mn}} \right) \frac{k_n^* - 2k}{k_n^* - 2k} \quad \text{if} \quad k_n^* \neq 2k \quad \text{(ASYNCHRONISM)} \quad (2.90)$$

$$\overline{A}_m(z) = \left( \frac{f_n^{surf} + f_n^{vol}}{4P_{mn}} \right) z \quad \text{if} \quad k_n^* = 2k \quad \text{(SYNCHRONISM)} \quad (2.91)$$

It is possible to notice that the modal amplitude of the generic $m^{th}$ secondary mode oscillates in value if the solution is asynchronous, while it increases with propagation distance $z$ if the solution synchronous. The latter is the known cumulative behavior occurring for nonlinear resonant modes in presence of the so-called internal resonance. This mechanism relies on the simultaneous occurrence of two conditions, namely:

1. Phase Matching: $k_n^* = 2k$
2. Non-zero power transfer from primary to secondary wave field: $f_n^{surf} + f_n^{vol} \neq 0$
Recent investigations performed by Deng et al. have analyzed the influence of an additional requirement for the occurrence of internal resonance, namely the group velocity matching (Deng et al., 2011). In this study the authors proved both analytically and experimentally that, as long as the two aforementioned conditions (phase-matching and non-zero power transfer) are satisfied, the cumulative effect of the secondary resonant mode takes place even when the group velocity matching condition is not satisfied. They concluded that group velocity matching does not represent a necessary requirement for cumulative second-harmonic generation. For this reason phase-matching and power transfer only are considered in detail in the present work.

2.3.5 Analogy with a SDOF system

To capture immediately the effect of nonlinearities in wave propagation phenomena it is instructive to qualitatively analyze the response of a very simple dynamical system. It is single-degree-of-freedom (SDOF) oscillator (Figure 2.15) exhibiting a cubic stiffness (Duffing system), viscoelastic damping and subjected to a harmonic excitation.

In this exemplary model material nonlinearity only is assumed through a nonlinear spring restoring force:

\[ F(x) = k_1 x + k_3 x^3 \]  \hspace{1cm} (2.92)

The stiffness of this spring:

\[ k(x) = \frac{dF(x)}{dx} = k_1 + 3k_3 x^2 \]  \hspace{1cm} (2.93)
is a quadratic function of the stretch $x$. It either monotonically decreases (*softening*) or increases (*hardening*) with the stretch, depending on the sign of the ratio $k_3/k_1$ (Figure 2.16).

![Damped forced Duffing oscillator](image1)

**Figure 2.15** – Damped forced Duffing oscillator.

![Nonlinear spring behaviors for Duffing oscillator](image2)

**Figure 2.16** – Nonlinear spring behaviors for Duffing oscillator, depending on the ratio $k_3/k_1$.

The equation governing the motion of this oscillator is:

$$m\ddot{x} + c\dot{x} + k_1x + k_3x^3 = F_0 \cos(\Omega t)$$  \hspace{1cm} (2.94)

where $c$ is the linear damping coefficient, $m$ is the mass of the oscillator, $F_0$ is the amplitude of the applied harmonic excitation and $\Omega$ is its frequency. Under the excitation
of the simple harmonic forcing function $F(t) = F_0 \cos(\Omega t)$, the oscillator long-term responses (after any transients have died out), because of its intrinsic material nonlinearity, will be markedly different from those of the approximate linear system, that is, the considered system for which $k_3 = 0$. This fact is depicted in Figure 2.17.

![Figure 2.17 – Long-term responses for linear and nonlinear SDOF systems subjected to harmonic excitation.](image)

For linear systems the response is unique and harmonic input translates to a harmonic response with the very same frequency as the input and no wave distortion phenomenon takes place. This fact is a strict consequences of the principle of superposition (Worden and Tomlinson, 2001), which holds for linear systems. The motion of nonlinear systems, on the other side, can be periodic or aperiodic and, in addition, multiple responses may coexist since solution uniqueness does not hold anymore. Possible periodic responses include the primary resonant response (a modification of the sole resonant response of the linear system) and secondary resonant responses, which include subharmonics and higher harmonics generation. Possible
aperiodic responses are quasiperiodic responses (motions with periodically modulated amplitude and/or phase) and chaotic responses.

2.3.6 Nonlinear parameter $\beta$

It is widely recognized that the nonlinear signature in the response of a nonlinear waveguide is strongly related to an intrinsic material property and can be quantified defining the nonlinear parameter $\beta$. This parameter represents the key feature to be monitored in nondestructive assessments of structures based on nonlinear wave propagation.

In the following sections the propagation of a longitudinal plane wave in a nonlinear unbounded medium is considered in detail to consolidate the understanding of nonlinear phenomena and, in particular, second harmonic generation. In this case the acoustic nonlinearity parameter $\beta$ can be defined analytically. In contrast to bulk waves, no analytical expression exists for the calculation of this parameter in presence of guided waves propagation and proportional relative nonlinearity parameter $\beta'$ is generally used to overcome this limitation.

For a longitudinal plane wave propagating in the $z$ direction, Eq. (2.57) simplifies to:

\[
(\lambda + 2\mu) \frac{\partial^2 u_z}{\partial z^2} - \rho_0 \frac{\partial^2 u_z}{\partial t^2} - \left[3(\lambda + 2\mu) + 2A + 6B + 2C \right] \frac{\partial u_z}{\partial z} \frac{\partial^2 u_z}{\partial z^2} = 0
\] (2.95)

Introducing the aforementioned acoustic nonlinearity parameter $\beta$:

\[
\beta = \left(\frac{3}{2} + \frac{A + 3B + C}{(\lambda + 2\mu)}\right)
\] (2.96)

Eq. (2.95) can be further simplified as:
\[
\frac{\partial^2 u_z}{\partial t^2} = c_L^2 \left[ 1 - \beta \left( \frac{\partial u_z}{\partial z} \right) \right] \frac{\partial^2 u_z}{\partial z^2}
\]  
(2.97)

where \(c_L = \left[ \left( \lambda + 2\mu / \rho_0 \right) \right]^{1/2} \) is the longitudinal wave velocity.

Alternative definitions of the acoustic nonlinearity parameter can be obtained by using different third order elastic moduli (Kundu, 2004). From Eq. (2.96), it can be noted that the nonlinear parameter is dimensionless. Furthermore, when this parameter is equal to zero, the nonlinear wave equation (2.97) reduces to its linear counterpart.

Assuming to launch a simple monochromatic sinusoidal wave of the form \(P = P_0 \cos(\omega t)\) into the medium at \(z = 0\), the solution, as detailed in previous sections, can be obtained employing a perturbation expansion of the displacement field in the form

\[u_z = u_z^{(1)} + u_z^{(2)}\]  
(2.98)

The contribution \(u_z^{(1)}\) is the solution to the linear wave equation

\[\frac{\partial^2 u_z^{(1)}}{\partial t^2} - c_L^2 \frac{\partial^2 u_z^{(1)}}{\partial z^2} = 0\]  
(2.99)

The solution, taking into account the above boundary condition, is represented by the plane wave:

\[u_z^{(1)} = A \cos(\omega t)\]  
(2.100)

where \(k\) is the wavenumber of the primary mode while \(\omega\) is its angular frequency.

A first-order perturbation equation for the contribution \(u_z^{(2)}\) is obtained using the solution given by Eq. (2.100) into the nonlinear term of Eq. (2.97). The resulting expression is:

\[\frac{\partial^2 u_z^{(2)}}{\partial t^2} = c_L^2 \frac{\partial^2 u_z^{(2)}}{\partial z^2} - \frac{1}{2} c_L^2 \beta k^3 (A_i)^2 \sin 2(\omega t)\]  
(2.101)
It can be noticed from Eq. (2.101) how the nonlinearity manifests itself as a forcing function with a temporal frequency dependency of the type $2\omega$. This gives rise to the second harmonic in the nonlinear response. In order to solve Eq. (2.101) a general d’Alembert solution is assumed:

$$u_z^{(2)} = f(z)\sin 2(kz - \omega t) + g(z)\cos 2(kz - \omega t)$$  \hspace{1cm} (2.102)

The two functions $f$ and $g$ are assumed to be $z$-dependent only and null at $z=0$ because the boundary condition used for the solution enforces the existence of the fundamental wave only at $z = 0$. Substitution of Eq. (2.102) into Eq. (2.101) yields the expression:

$$\left(\frac{-4\omega^2 f}{c_L^2}\right)\sin 2(kz - \omega t) + \left(\frac{-4\omega^2 g}{c_L^2}\right)\cos 2(kz - \omega t) =$$

$$= \left(\frac{\partial^2 f}{\partial z^2} - 4k^2 f - 4k \frac{\partial g}{\partial z} - \frac{1}{2} \beta k^3 A^2\right)\sin 2(kz - \omega t) +$$

$$+ \left(4k \frac{\partial f}{\partial z} + \frac{\partial^2 g}{\partial z^2} - 4k^2 g\right)\cos 2(kz - \omega t)$$  \hspace{1cm} (2.103)

Using the definition of longitudinal wave velocity as $c_L = \omega/k$ and equating coefficients of the sinusoidal and cosinusoidal terms between right-hand side and left-hand side in Eq. (2.103), the following system of two partial differential equations is obtained:

$$\frac{\partial^3 f}{\partial z^3} - 4k \frac{\partial g}{\partial z} - \frac{1}{2} \beta k^3 A^2 = 0$$

$$\frac{\partial^3 g}{\partial z^3} + 4k \frac{\partial f}{\partial z} = 0$$  \hspace{1cm} (2.104)
Enforcing the functions \( f \) and \( g \) and their derivatives to be zero at \( z=0 \) (in accordance to the mentioned boundary condition), it is possible to solve the system in Eq. (2.104). The solution is:

\[
f = 0 \\
g = -\frac{1}{8} \beta k^2 (A_1)^2 z
\]  

(2.105)

In light of this result the final solution for Eq. (2.97) is found as:

\[
u_z = u_z^{(1)} + u_z^{(2)} = A_1 \cos(kz - \omega t) - \frac{1}{8} \beta k^2 A_1^2 z \sin(2(kz - \omega t))
\]  

(2.106)

Eq. (2.106) shows that, in addition to the fundamental wave of amplitude \( A_1 \) and angular frequency \( \omega \), a second harmonic signal is generated of amplitude \( A_2 = (1/8) \beta k^2 (A_1)^2 z \). It is evident how the second harmonic wave depends on the nonlinearity parameter \( \beta \) and grows linearly with propagation distance \( z \). The latter result is obvious if we consider that bulk waves are nondispersive and that phase-matching and power transfer are always guaranteed.

Eq. (2.106) suggests that \( \beta \) may be quantified experimentally by measuring the absolute amplitudes of the fundamental and second harmonic signals as:

\[
\beta = 8 \frac{A_2}{k^2 A_1^2 z}
\]  

(2.107)

The above definition has been derived for the specific case of waves propagating in a nonlinear unbounded medium. It is not trivial to extend this definition to the guided wave propagation regime. Studies in the past have proposed alternative definitions for the nonlinear parameter for guided waves using scaling factors (Herrmann et al., 2006) but a consolidated approach to obtain an analytical expression, especially for waves
propagating in complex waveguides, is still nonexistent. A relative nonlinear parameter $\beta'$ will be used in the present work in accordance with the majority of studies published in literature. This parameter is defined as:

$$\beta' = \frac{A_z}{A_i^2} = \frac{1}{8} \beta k^2 z$$  \hspace{1cm} (2.108)

It is noted that the relative nonlinearity parameter is linearly proportional to the absolute nonlinearity parameter and to the propagation distance from the source.
Chapter 3

Nonlinear Semi-Analytical Finite Element algorithm

(CO.NO.SAFE) – Internal resonance analysis of nonlinear structural waveguides

3.1 Introduction

It is well recognized that the great potential of guided waves in NDE and SHM applications strongly relies on a solid understanding of the complex propagation phenomena involved. These complexities include the existence of multiple modes, the frequency-dependent velocities (dispersion), and the frequency-dependent attenuation. This scenario is further complicated when transitioning from linear to nonlinear regime, where, aside from dispersion characteristics, internal resonance conditions and wave distortion manifestations must be unveiled and managed. The knowledge of dispersion curves and mode shapes is of paramount importance for the development of any application based on the use of linear guided waves. Internal resonance analysis, instead, is crucial in identifying optimal combinations of resonant primary and secondary modes to be used for successful application of nonlinear guided waves in NDE/SHM.

Focusing on the linear regime, analytical wave propagation methods generally based on the superposition of bulk waves (Lowe, 1995; Soldatos and Ye, 1994) are well-established algorithms for guided wave future extraction in simple problems, such as
plates or cylinders made of homogeneous or multilayered isotropic materials. In these methods, the dispersive equations of motion are formulated via constructive interference of bulk waves with respect to the waveguide boundary conditions. Despite these methods were conceived for multilayered structures of viscoelastic, anisotropic materials, the root searching routines in the complex plane of wavenumbers are not straightforward, especially for waveguides with a large number of layers, and may miss some solutions. This shortcoming and the necessity to investigate a large number of layers such as composite laminates and that of modeling waveguides of arbitrary cross-section (for which exact solutions do not generally exist) triggered notable research efforts in developing both numerical and hybrid numerical-analytical techniques to model guided waves propagation. Several different approaches based on Finite Element modeling to predict dispersion curves for guided waves have emerged. The most intuitively obvious, but also the most computationally expensive to reach this goal is the time-domain modeling (Moser et al., 1999). The theoretical framework behind this approach consists in considering a finite element model of a length of waveguide, applying a particular time dependent excitation force at one location and analyzing the subsequent wave propagation. Quantitative dispersion data are then extracted from the model response employing specialized techniques such as two-dimensional Fourier transform (Alleyne and Cawley, 1991) and wavelet transform (Benz et al., 2003; Presser et al., 1999). The main disadvantages of this method are the length of waveguide that needs to be modeled in order to allow guided modes to develop and to allow separation from end reflections (hence high computational demand) and the post-processing phase, not always trivial, required to extract dispersion data. An alternative FE method consists in modeling a
relatively short length of waveguide with the nodes at the ends constrained to move only in the plane perpendicular to the length of the waveguide (Sanderson and Smith, 2002; Thompson, 1997). The resonant frequencies and mode shapes of such a model are then calculated using an eigensolver. These correspond to frequencies where standing waves are set up over the length of waveguide that has been modeled. The number of periods along the length of the waveguide in the mode shape associated with a particular resonant frequency enables the wavelength and thus the phase velocity of a guided wave mode to be calculated. Similar models with different lengths of waveguide are then used to obtain more points in phase velocity–frequency space.

Semi-Analytical Finite Element (SAFE) formulation certainly represents the most powerful, sophisticated and well-suited numerical alternative to overcome the limitations of the analytical methods and explore wave propagation phenomena in prismatic waveguides in a very efficient computational manner. In fact, this approach requires a finite element discretization of the 2D cross-section only when compared to the FE methods discussed above, thus reducing the dimension of the model by one. The displacements along the wave propagation direction are conveniently described in an analytical fashion as harmonic exponential functions. A SAFE method for waveguides of arbitrary cross-section was proposed for the first time four decades ago (Aalami, 1973; Lagasse, 1973), even though the authors of these works limited their investigation to propagative modes only (i.e. real wavenumbers only). Ten years later Huang and Dong used a similar approach to calculate propagative, nonpropagative and evanescent modes (i.e. complex and imaginary wavenumbers) for anisotropic cylinders (Huang and Dong,
The three different types of waveguide modes obtained as solution of the SAFE eigenproblem in the most general case are depicted graphically in next Figure 3.1.

![Figure 3.1 – Schematic illustration of possible waveguide modes. (a) Propagative mode (real wavenumber). (b) Propagating evanescent mode (complex wavenumber). (c) Nonpropagative mode (imaginary wavenumber).](image)

More recently other researchers applied the SAFE algorithm to several different structural systems, including thin-walled waveguides (Gavrić, 1994), railroad tracks (Gavrić, 1995), rib stiffened plates (Orrenius and Finnveden, 1996), rods (Mazuch, 1996), wedges (HladkyHennion, 1996) and fluid filled pipes (Finnveden, 1997). Waveguides with periodic geometric along the direction of wave propagation and separating into several branches have also been addressed (Mencik and Ichchou, 2005). Other applications of the general SAFE method include non-homogeneous anisotropic beams (Volovoi et al., 1998) and twisted waveguides (Onipede and Dong, 1996). Reflection phenomena from the end of a waveguide were studied Taweel et al. (Taweel et al., 2000). The propagative modes in built-up thin-walled structures, including a channel beam and a plate in a wind, were investigated recently (Finnveden, 2004). In this study an innovative and advantageous derivation of the group velocity from the individual solutions of the SAFE eigenproblem was proposed. Laminated composite waveguides were studied using SAFE algorithm for the first time by Dong and Huang (Dong and Huang, 1985) and, subsequently, by Mukdadi et al. (Mukdadi et al., 2002). Recently the
SAFE method has been extended to dissipative waveguides using an efficient viscoelastic model (Bartoli et al., 2006) and to prestressed waveguides (Loveday, 2009).

From the brief literature review presented above, it is clear that SAFE formulation, in its linear fashion, has been the object of extensive research efforts during the last decades. It is also evident that, while several investigations pertaining to nonlinear effects in solids and harmonic generation were reported in the past, almost all of them were limited in their applicability to structures with simple geometries (plates, rods, shells) and simple material configuration (isotropic, homogeneous) where analytical solution for the primary (linear) wave field are available in literature. In the present work the propagation of waves in nonlinear solid waveguides with complex geometrical and material properties is investigated theoretically and numerically. For the solution of the nonlinear boundary value problem, perturbation theory and modal expansion discussed in previous sections are used. An innovative numerical algorithm, able to efficiently predict and explore the nonlinear wave propagation phenomena in several types of structural waveguides, is proposed. It is based on the implementation of a Nonlinear SAFE formulation into a highly flexible and powerful commercial Finite Element package (COMSOL®). The resulting CO.NO.SAFE algorithm does not require any new element to be developed (which is the case for ad-hoc written SAFE codes) and it combines the full power of existing libraries and routines of the commercial code with its ease of use and extremely capable post-processing functions and multi-core processing support. Hence internal resonance conditions of structural waveguides with different level of complexity can be conveniently analyzed via user-friendly interfaces. Ready-to-use high-order shape functions can be easily utilized in the model. This aspect is crucial for the
development of the present theory since the nonlinear post-processing analysis involves gradients of the displacement field up to the third order. In addition, immediate and extensive post-processing for all the required quantities can be developed through friendly GUIs. Figure 3.2 summarizes graphically the most salient features and benefits offered by the proposed numerical algorithm.

The applicability of the proposed analysis is quite wide, since it can efficiently handle general prismatic structures, viscoelastic waveguides with damping effects, multilayered composite laminate panels and heterogeneous systems, all cases where theoretical wave solutions are either nonexistent or extremely difficult to determine. Furthermore, the proposed approach requires simple modifications to the original
commercial FEM code so that the nonlinear semi-analytical formulation can be taken into account and translated to match the required formalism.

After a brief discussion on the background of the present work and the proposed algorithm, several case-studies have been analyzed in detail to emphasize the potential of the method. Appropriate combinations of primary and secondary modes (nonlinear resonance conditions) were identified for relatively complex waveguides that include: a viscoelastic plate, a composite quasi-isotropic laminate, and a reinforced concrete slab. A railroad track is considered in detail in the next chapter.

It is important to emphasize how the knowledge of these nonlinear resonance conditions is of primary importance for the actual implementation of conditions assessment systems for these structures that are based on the measurement of nonlinear ultrasonic guided waves.

### 3.2 CO.NO.SAFE algorithm - Mathematical framework

The Nonlinear SAFE mathematical model presented in this dissertation has been implemented into COMSOL commercial finite element code in two different stages starting from the very general three-dimensional elasticity approach to avoid any simplification in the treatment. In a first phase the linear elastic regime is assumed. The quantitative analysis of dispersion characteristics of a given waveguide is developed in line with the classical linear SAFE formulation. This linear solution of the governing eigenproblem constitutes the starting point for the following nonlinear analysis where the modeshapes are used to calculate the velocity vectors in Eq. (2.88) and the associated
eigenvalues are the wavenumbers in expressions (2.89), (2.90) and (2.91). The nonlinear part of the algorithm has been originally implemented in MATLAB and seamlessly integrated with COMSOL environment in real time by establishing an associative connection between the two platforms via a specialized livelink application. This step was essential for the calculation of all the complex quantities involved in the nonlinear analysis, exploiting the extensive programming capabilities of MATA LB and the full power of COMSOL commercial code.

The equations of motion in tensor notation can be written as:

$$\sigma_{i,j} = \rho \ddot{u}_i$$  \hspace{1cm} (3.1)

where \( \sigma_{i,j} \) is the stress tensor associated with the propagating wave, \( x_j \) are the Cartesian coordinates, \( u_i \) are the displacement components along each Cartesian direction, the indices \( i \) and \( j \) run from 1 to 3, comma denotes derivation and the Einstein summation convention is adopted over repeated indices. For the general case of a linear elastic anisotropic solid, the constitutive equation is:

$$\sigma_{i,j} = C_{ijkl} \varepsilon_{ij}$$  \hspace{1cm} (3.2)

where \( C_{ijkl} \) is the fourth-order elasticity tensor and \( \varepsilon_{kl} \) is the Cauchy strain tensor (infinitesimal deformations) defined as:

$$\varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right)$$  \hspace{1cm} (3.3)

Substitution of Eq. (3.3) into Eq. (3.2) leads to the following relation between stress and displacement:

$$\sigma_{i,j} = \frac{1}{2} C_{ijkl} u_{k,l} + \frac{1}{2} C_{ijkl} u_{l,k}$$  \hspace{1cm} (3.4)
Exploiting the symmetry of stress and strain tensors:

\[ C_{ijkl} = C_{ijlk} \]  \hspace{1cm} (3.5)

Eq. (3.4) can be rewritten as:

\[ \sigma_{ij} = \frac{1}{2} C_{ijkl} u_{k,j} + \frac{1}{2} C_{ijlk} u_{l,k} \]  \hspace{1cm} (3.6)

Furthermore, being \( k \) and \( l \) dummy indices which are summed out, the constitutive equation can be reformulated as:

\[ \sigma_{ij} = C_{ijkl} u_{k,l} \]  \hspace{1cm} (3.7)

Using the above result in the original equations of motion (3.1) yields to the final form:

\[ C_{ijkl} u_{k,l} = u_t \hat{u}_i \text{ on } \Omega \]  \hspace{1cm} (3.8)

with summation implied over dummy indices \( j, k \) and \( l \) from 1 to 3. The associated Neumann and Dirichlet boundary conditions can be defined, respectively, as:

\[ \sigma_{ij} n_j = \hat{t}_i \text{ on } \Gamma_\sigma \]  \hspace{1cm} (3.9)

\[ u_i = \hat{u}_i \text{ on } \Gamma_u \]  \hspace{1cm} (3.10)

In Eqs. (3.9) and (3.10), \( n_j \) represents the unit normal vector pointing outward from the surface of the waveguide, \( \Omega \) is the volume of the waveguide, \( \Gamma_\sigma \) is the portion of the exterior surface \( \Gamma \) where surface tractions are prescribed and \( \Gamma_u \) is the remaining part of the surface where boundary displacement are prescribed.

For a Cartesian reference system, the waveguide cross-section is set in the \( x-y \) plane while the axis \( z \) is along the wave propagation direction (Figure 3.3). The classical SAFE key approximation is applied, enforcing the displacement field to be harmonic along the wave propagation direction, \( z \), and using spatial shape functions to describe its amplitude in the cross-sectional plane \( x-y \). This condition mathematically translates to
\[ u_i(x, y, z, t) = N_i(x, y)e^{i(kz-\omega t)} \]  

(3.11)

where \( k \) is the wavenumber, \( \omega \) is the angular frequency, \( i \) is the imaginary unit and \( N_i(x,y) \) are the shape functions.

Figure 3.3 – Schematic illustration of waveguide reference system.

Subdividing the cross-section via finite elements (Figure 3.4), for the generic \( e^{th} \) element the above displacement field reads:

\[ u_i^e(x, y, z, t) = N_i^e(x, y)e^{i(kz-\omega t)} = N_q^e(x, y)q_i^e e^{i(kz-\omega t)} \]  

(3.12)

In Eq. (3.12) \( N_q(x,y) \) represents the shape function matrix whose order is \((3x3n)\), being \( n \) the number of nodes per element. It is defined as:
In the same Eq. (3.12) the term \( q_i^e \) denotes the nodal displacement vector for the \( e^{th} \) element defined as:

\[
q^e = \begin{bmatrix} U_{x1} & U_{y1} & U_{z1} & U_{x2} & U_{y2} & U_{z2} & \ldots & \ldots & U_{xn} & U_{yn} & U_{zn} \end{bmatrix}^T
\]  

(3.14)

In light of the discussed SAFE assumption, the gradients of the displacement field in Eq. (3.11) reduce to:

\[
\begin{align*}
\dot{u}_{i,x} &= N_{i,x} e^{i(kz-\omega t)} \\
\dot{u}_{i,y} &= N_{i,y} e^{i(kz-\omega t)} \\
\dot{u}_{i,z} &= ik N_{i,z} e^{i(kz-\omega t)} \\
\ddot{u}_i &= \frac{\partial^2 u_i}{\partial t^2} = -\omega^2 N_i e^{i(kz-\omega t)}
\end{align*}
\]  

(3.15)

Making use of the above definitions, the system of partial differential equations of motion (3.8) and the associated boundary conditions (3.9) and (3.10) can be expressed as:
\[ C_{jkl}N_{jkl} + \rho \omega^2 N_i = 0 \quad \text{in} \quad \Omega \quad (3.16) \]

\[ C_{jkl}N_{jkl} n_k = i \hat{t}_i \quad \text{on} \quad \Gamma_\sigma \quad (3.17) \]

\[ u_i = \hat{u}_i \quad \text{on} \quad \Gamma_u \quad (3.18) \]

with, \( i=1,2,3 \), and summation implied over the indices \( j, k \) and \( l \). After some intermediate transformations, Eqs. (3.16)-(3.17) can be reformulated as (Predoi et al., 2007):

\[ C_{jkl}N_{jkl} + i \left( C_{13jk} + C_{iklj} \right) \left( kN_j \right)_k - kC_{13jk} \left( kN_j \right)_k + \rho \omega^2 \delta_{ij} N_j = 0 \quad \text{in} \quad \Omega \quad (3.19) \]

\[ C_{jkl}N_{jkl} n_k + iC_{iklj} \left( kN_j \right)_k n_k = i \hat{t}_i \quad \text{on} \quad \Gamma_\sigma \quad (3.20) \]

where \( j=1,2,3 \) and \( k,l=1,2 \).

COMSOL commercial code offers a number of powerful physics interfaces for equation-based modeling which support several PDE formulations as well as general ways of adding ODEs, algebraic equations, and other global (space-independent) equations. The so-called *Coefficient Form PDE* interface covers many well-known PDEs and it is very well suited for solving linear and almost linear PDEs via finite element method. This form has been applied in the present work. In coefficient form, COMSOL input formalism to model the most general PDE problem reads (COMSOL, 2011):

\[ e_a \left( \frac{\partial^2 u}{\partial t^2} + d_a \left( \frac{\partial u}{\partial t} + \nabla \cdot (-c \nabla u - \alpha u + \gamma) + \beta \cdot \nabla u + au \right) + f \right) \quad \text{in} \quad \Omega \quad (3.21) \]

\[ n \cdot (c \nabla u + \alpha u - \gamma) + qu - g = h^T \mu \quad \text{on} \quad \Gamma_\sigma \quad (3.22) \]

\[ hu = r \quad \text{on} \quad \Gamma_u \quad (3.23) \]

where \( \Omega \) is the computational domain (union of all subdomains) corresponding to the meshed 2D cross-section of the waveguide, \( \Gamma_\sigma \) is the portion of the domain boundary \( \Gamma \) where surface tractions are prescribed, \( \Gamma_u \) is the remaining part of the domain boundary
where displacement are prescribed, $n$ is the outward unit normal vector on $\Gamma$, $e_a$ is the mass coefficient, $d_a$ is the damping/mass coefficient, $c$ is the diffusion coefficient, $\alpha$ is the conservative flux convection coefficient, $\beta$ is the convection coefficient, $a$ is the absorption coefficient, $\gamma$ is the conservative flux source term, $f$ is the source term and $u$ represent the set of variables to be determined. The above coefficients must be established via an identification procedure. Their identification depends on the physical problem under investigation. Figure 3.5 provides graphically a physical insight into the various term of the governing equation in the PDE coefficient form solver engine.

![Figure 3.5 – Physical interpretation of the terms in the COMSOL PDE coefficient form interface.](image)

The symbol $\nabla$ is the vector differential operator defined as:

$$\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right)$$

(3.24)

where $n$ is the number of space dimensions.

Eq. (3.21) is the PDE, which must be satisfied in $\Omega$. Eqs. (3.22)-(3.23) represent the natural (Generalized Neumann BC) and essential (Dirichlet BC) boundary conditions, respectively, which must hold in $\Gamma$. If $u$ is a single variable, all the coefficients in the
above system of equations are scalars except \( \alpha, \beta \) and \( \gamma \), which are vectors with \( n \) components and \( c \), which may be a \( n \)-by-\( n \) matrix. They all admit complex values, which is essential for viscoelastic waveguides, as highlighted in one the analyzed case-studies, discussed in the following work.

As discussed in previous sections, SAFE modeling of the guided wave propagation problem in the linear regime can be mathematically represented by a boundary value eigenproblem comprising Navier equations of motion and an associated stress-free boundary condition on \( \Gamma = \Gamma_\sigma \). Considering Eqs. (3.21)-(3.22), the original PDE problem be reformulated as a scalar eigenvalue problem via the correspondence \( \partial / \partial t \leftrightarrow t \), linking the time derivative to the eigenvalue \( \lambda \). The result of this manipulation, dismissing unnecessary forcing terms, reads:

\[
-\lambda^2 e_a + \lambda d_u u + \nabla \cdot \left( c \nabla u + \alpha u - \gamma \right) - \beta \nabla u - \alpha u = 0 \quad \text{in} \quad \Omega \tag{3.25}
\]

\[
n \cdot \left( c \nabla u + \alpha u - \gamma \right) + qu = 0 \quad \text{on} \quad \Gamma_\sigma \tag{3.26}
\]

If \( \gamma = e_a = 0 \), Eqs. (3.25)-(3.26) can be rewritten as:

\[
C_{ijkl} u_{j,k} + \left( \alpha_{ijk} - \beta_{ijk} \right) u_{j,k} = \alpha_{ij}u_j + \lambda d_{ij} u_j = 0 \quad \text{in} \quad \Omega \tag{3.27}
\]

\[
C_{ijkl} u_{j,k} + \alpha_{ijkl} n_k u_j + q_{ijkl} u_j = 0 \quad \text{on} \quad \Gamma \tag{3.28}
\]

It is evident how the eigenproblem formulated by Eqs. (3.27)-(3.28) effectively represent the original eigenproblem in Eqs. (3.19)-(3.20) once all the coefficients have been correctly defined. Nontrivial solutions can be found by solving this twin-parameter generalized eigenproblem in \( k \) and \( \omega \). The frequency \( \omega \) is a real positive quantity. The wavenumber \( k \) can be real, complex or imaginary and can have both positive and negative signs, associated with so-called right-propagating and left-propagating
waveguide modes, respectively. The full dispersion curve spectrum can be simply obtained by using the efficient parametric sweep analysis (supported by COMSOL) over the desired range of frequencies, with \( \omega \) as parameter of the sweep. The resulting eigenvalues, complex in the most general case, are used to describe the velocity of the traveling waves through their real part, \( k_{Re} \), and their amplitude decay through the imaginary part, \( k_{Im} \). However, for each frequency \( \omega \) a relatively complex second-order polynomial eigenvalue problem needs to be solved. This scenario can be computationally optimized using a classic technique consisting in recasting the original eigenproblem to a first-order eigensystem by introducing a new vector variable \( \mathbf{v} \) defined as:

\[
M \cdot \mathbf{v} = k M \cdot \mathbf{u}
\]  

(3.29)

where \( M \) is an arbitrary diagonal matrix. In order to correctly formulate the original problem described by Eqs. (3.19)-(3.20) in the form of Eqs. (3.27)-(3.28) the following set of variables is introduced:

\[
\mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 & v_1 & v_2 & v_3 \end{bmatrix}^T
\]  

(3.30)

With this new set of variables the coefficients appearing in the FEM formalism discussed above must be (Predoi et al., 2007):

\[
\begin{align*}
\mathbf{d}_e & = \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} ; & \alpha & = \begin{bmatrix} 0 & iA \\ 0 & 0 \end{bmatrix} ; & \beta & = \begin{bmatrix} 0 & -iB \\ 0 & 0 \end{bmatrix} ; \\
\mathbf{c} & = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} ; & \mathbf{a} & = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} .
\end{align*}
\]  

(3.31)

In Eq. (3.31), \( i \) represents the complex unit, the term “0” represents a zero matrix of appropriate dimensions, and the submatrices are defined as:
\[
M = \begin{bmatrix}
-\rho \omega^2 & 0 & 0 \\
0 & -\rho \omega^2 & 0 \\
0 & 0 & -\rho \omega^2
\end{bmatrix}; \quad D = \begin{bmatrix}
-C_{55} & -C_{54} & -C_{53} \\
-C_{45} & -C_{44} & -C_{43} \\
-C_{35} & -C_{34} & -C_{33}
\end{bmatrix}.
\]

\[
A = \begin{bmatrix}
C_{15} & C_{14} & C_{13} \\
C_{65} & C_{64} & C_{63} \\
C_{25} & C_{24} & C_{23} \\
C_{55} & C_{54} & C_{53} \\
C_{45} & C_{44} & C_{43}
\end{bmatrix}; \quad B = \begin{bmatrix}
C_{51} & C_{56} & C_{55} \\
C_{56} & C_{52} & C_{54} \\
C_{41} & C_{46} & C_{45} \\
C_{31} & C_{36} & C_{35} \\
C_{36} & C_{32} & C_{34}
\end{bmatrix};
\]

\[
C = \begin{bmatrix}
C_{11} & C_{16} & C_{12} \\
C_{61} & C_{66} & C_{62} \\
C_{21} & C_{26} & C_{22} \\
C_{51} & C_{56} & C_{52} \\
C_{41} & C_{46} & C_{42}
\end{bmatrix}; \quad (3.32)
\]

where \( \rho \) is the density, \( \omega \) is the frequency and \( C_{ij} \) \((i,j = 1,...,6)\) are the stiffness coefficients (generally complex) expressed in Voigt notation.

The discussed manipulation doubles the algebraic size of the original eigensystem. This size depends on the finite element mesh used to discretize the cross-section of the waveguide and, consequently, on the number of degrees of freedom of the finite element model. As a result, being \( 2M \) the size of the linearized eigensystem, at each frequency \( \omega \), \( 2M \) eigenvalues \( k_m \) and \( 2M \) associated eigenvectors are obtained. The eigenvectors are the \( M \) forward and the corresponding \( M \) backward waveguide modes. The eigenvalues occur as pairs of real numbers \((\pm k \Re)\), representing propagative waves in the \( \pm x \)-directions, as pairs of complex conjugate numbers \((\pm k \Re \pm ik \Im)\), representing propagative evanescent waves decaying in the \( \pm x \)-directions, or as pairs of purely imaginary numbers \((\pm ik \Im)\), representing the nonoscillating evanescent waves in the \( \pm x \)-
directions. The phase velocity can be then evaluated by $c_{ph} = \omega/k_{Re}$ and the attenuation, in Nepers per meter, by $k_{lm}$.

### 3.2.1 Periodic Boundary Conditions

At a definition level, a periodic structure consists fundamentally of a number of identical structural components which are joined together end-to-end and/or side-by-side to form the whole structure. In mathematical models, periodic boundary conditions (PBCs) are widely used in order to simulate a large system exhibiting material/geometrical periodicity along a particular principal direction by modeling a small part that is far from its edges (periodic cell). In the present work an extension of the classical SAFE formulation is employed as suggested in (Predoi et al., 2007). This analytical expedient allows an efficient study of guided wave propagation in structures exhibiting material/geometrical periodicity along their width (which is normal to the direction of propagation and to the thickness and considered infinite). With this powerful tool, a generally complex periodic structure (grooved panel, reinforced concrete elements, just to mention a couple) can be modeled simply by considering a very small cell and applying PBCs on its sides. Mathematically, they represent a particular case of Neumann boundary conditions: the variables and their derivatives up to the element order are forced to take identical values on the pair of boundaries of the structure where the PBCs are applied.

Considering a simple rectangular domain of base $B$ and height $H$, the above conditions must represent continuity of displacements and stresses between the two edges and can be implemented as:
\[ u \big|_{-B/2} = u \big|_{B/2} \]
\[ \nabla u \big|_{-B/2} = \nabla u \big|_{B/2} \]  

(3.33)

The fact that continuity only is imposed at the PBCs constitutes a significant benefit for plate systems since the shear horizontal modes (SH) are always included in the eigensolution. Furthermore, the value for the width B in the algorithm with PBCs is not important for the solution given that the resulting structure is an infinitely wide plate made of identical, adjacent blocks with continuity of both displacement and stresses at their junction.

This tool is very attractive because it opens up new possibilities to study the guided wave propagation (linear and nonlinear) for a general class of periodic structures by developing the analysis just on a small portion of the whole waveguide.

### 3.2.2 Axial load influence in prestressed waveguides

The proposed algorithm was further extended to account for the effect of axial load on guided wave propagation following a procedure recently proposed for a different implementation of the semi-analytical finite element algorithm (Loveday, 2009). This extension is essential in order to analyze dispersion characteristics (starting point of the internal resonance analysis) for prestressed waveguides, likewise continuous welded rails. After a brief overview of the theoretical fundamentals, the proposed extension is validated on an aluminum rod subjected to axial prestress, extensively analyzed in literature (Chen and Wilcox, 2007; Loveday, 2009).

Loveday pointed out that the mass matrix of the model is derived from the kinetic energy and the stiffness matrix (dependent on the wavenumber) is derived from the strain
energy. The application of an initial prestress leads to new terms in the strain energy which therefore produce additions to the stiffness matrix. After a straightforward mathematical development, Loveday, in the same paper, also showed that the additional strain energy to be considered to take the axial load into account has the form:

\[ K^{(0)} = \frac{1}{2} \sigma_{zz}^{(0)} \cdot k^2 \begin{bmatrix} -u & -v & -w \end{bmatrix} \begin{bmatrix} -u \\ -v \\ -w \end{bmatrix} \] (3.34)

where \( k \) is the wavenumber, \( \sigma_{zz}^{(0)} \) is the axial load applied to the waveguide and \( u, v, \) and \( w \) represent the amplitude of the displacement field along Cartesian axes \( x, y, z \) (Figure 3.3). Realizing that the form of this term identical to that of the kinetic energy, the new term to be implemented is actually a stiffness term proportional to the mass matrix and defined as:

\[ K^{(0)} = \frac{\sigma_{zz}^{(0)}}{\rho} M \] (3.35)

where \( \rho \) is the density of the waveguide and \( M \) is the mass matrix.

In light of the result above and in agreement with the theoretical framework discussed in Section 3.2, the only modification required in CO.NOSAFE algorithm to account for axial load is trivial and involves just the absorption coefficient matrix \( a \) and the damping/mass coefficient matrix \( d_a \) (Figure 3.5), defined in Eqs. (3.31)-(3.32). After introducing the new term quantifying the effect of the axial load, these two matrices read:
The influence of an axial load on dispersion curves was analyzed on an aluminum rod in order to validate the proposed extension. This particular problem has been considered in the past by other authors (Chen and Wilcox, 2007; Loveday, 2009) and serves here as a benchmark.

The influence of a tensile axial load, $T$, on the phase velocity, $c_{ph}$, at the frequency, $\omega$, for a beam with Young’s modulus, $E$, second moment of area, $I$, and mass per unit length, $m$, is provided in Eq. (Chen and Wilcox, 2007).

$$c_{ph} = \omega \sqrt{\frac{2EI}{\sqrt{T^2 + 4mEI\omega^2} - T}}$$ (3.37)

A 1 mm diameter rod was modeled in CO.NO.SAFE using 250 linear Lagrangian triangular elements. Figure 3.6 depicts the employed finite element mesh with a contour plot of the mesh quality index.
The dispersion analysis was performed inside the (0-100) kHz range in two stages, first with no axial load and then with a tensile load applied corresponding to 0.1% axial strain. Phase velocity dispersion curves were calculated numerically using CO.NO.SAFE code in both cases (with and without axial load) and are presented in Figure 3.7. It is clear from this figure that only three propagating waveguide modes coexist in the considered frequency range, namely the flexural mode, the torsional mode and the axial mode. Furthermore, the curves reveal that only the flexural mode is sensitive to the presence of an axial load and this influence is more pronounced at lower frequencies (yellow inset in Figure 3.7).
Figure 3.7 – Phase velocity dispersion curves for a 1 mm diameter aluminum rod with and without axial load. Three propagating modes present in the considered frequency range are highlighted (contour plot for the out-of-plane displacement field and vector plot for the in-plane displacement field).

Figure 3.8 – Comparison between numerical results (CO.NO.SAFE) and closed-form solution (Euler-Bernoulli) for the flexural mode in both loaded and unloaded cases.
Figure 3.8 shows a comparison between dispersion curves calculated numerically (CO.NO.SAFE) and in closed-form (Euler-Bernoulli) for the flexural mode. A logarithmic scale is employed to accentuate the deviation between loaded and unloaded cases. It is clear that the numerical results employing CO.NO.SAFE extension are practically identical to the Euler-Bernoulli beam model and in accordance to other results recently presented in literature (Chen and Wilcox, 2007; Loveday, 2009).

In the following CO.NO.SAFE algorithm is benchmarked in three exemplary case-studies involving waveguide of different level of complexity in terms of geometrical features and material properties.

### 3.3 Benchmark case-studies

This section illustrates predictions of nonlinear second-harmonic generation in complex waveguides. The proposed analysis can easily take into account damping effects, anisotropic multi-layered properties, periodic geometries and other complex waveguide properties in a computational efficient and accurate manner. In terms of flowchart, the code firstly reveals the guided wave propagation properties in the linear regime (dispersion curves and waveguide mode shapes). In a second step the nonlinear part of the algorithm uses the above eigensolutions for the mode expansion in order to obtain the nonlinear solution with a perturbative approach. Favorable combinations of resonant primary (fundamental harmonic at $\omega$) and secondary (double harmonic at $2\omega$) waveguide modes are identified for three exemplary cases including a viscoelastic plate, a composite quasi-isotropic laminate, and a reinforced concrete slab.
3.3.1 Viscoelastic isotropic plate

As first case-study, a viscoelastic isotropic high performance polyethylene (HPPE) plate was investigated to benchmark the applicability of the proposed algorithm in dissipative waveguides. This structural system has been studied in the past in the linear elastic regime only to obtain dispersion curves and associated waveguide modes (Bartoli et al., 2006; Bernard et al., 1999; Bernard et al., 2001). In the present work these results are confirmed and extended to the nonlinear regime; an efficient combination of resonant primary and secondary modes is identified and discussed in detail.

Material and geometrical properties for the plate are illustrated in Table 3.1 (Bernard et al., 1999; Bernard et al., 2001), where $\rho$ is the density, $h$ is the thickness, $c_L$ is the longitudinal bulk wave velocity, $c_T$ is the shear bulk wave velocity, $k_L$ is the longitudinal bulk wave attenuation and $k_T$ is the shear bulk wave attenuation.

<table>
<thead>
<tr>
<th>$\rho$ [kg/m$^3$]</th>
<th>$h$ [mm]</th>
<th>$c_L$ [m/s]</th>
<th>$c_T$ [m/s]</th>
<th>$k_L$ [Nepers/wavelength]</th>
<th>$k_T$ [Nepers/wavelength]</th>
</tr>
</thead>
<tbody>
<tr>
<td>953</td>
<td>12.7</td>
<td>2344</td>
<td>953</td>
<td>0.055</td>
<td>0.286</td>
</tr>
</tbody>
</table>

The dissipative behavior of the plate was implemented via the Hysteretic formulation (Bartoli et al., 2006). Hence the resultant stiffness matrix is frequency-independent and was calculated just once at the beginning of the analysis once the complex Lame’s constants were evaluated. The results for the present case are:

$$\tilde{\lambda} = \rho c_T^2 \frac{3\tilde{c}_L^2 - 4\tilde{c}_T^2}{\tilde{c}_L^2 - \tilde{c}_T^2} \tilde{\nu} = 3.51 + 0.06i \text{ GPa} \quad (3.38)$$
In Eqs. (3.38)-(3.39) the complex bulk wave velocities (longitudinal and transverse) are calculated as:

\[ \tilde{c}_{L,T} = \tilde{c}_{L,T} \left( 1 + i \frac{k_{L,T}}{2\pi} \right)^{-1} \]  

(3.40)

The resultant viscoelastic stiffness matrix, with terms expressed in GPa, is given by:

\[
\tilde{C} = \begin{bmatrix}
\tilde{\lambda} + 2\tilde{\mu} & \tilde{\lambda} & \tilde{\lambda} & 0 & 0 & 0 \\
\tilde{\lambda} & \tilde{\lambda} + 2\tilde{\mu} & \tilde{\lambda} & 0 & 0 & 0 \\
\tilde{\lambda} & \tilde{\lambda} & \tilde{\lambda} + 2\tilde{\mu} & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{\mu} & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{\mu} & 0 \\
0 & 0 & 0 & 0 & 0 & \tilde{\mu}
\end{bmatrix} =
\begin{bmatrix}
5.23 - 0.09i & 3.51 + 0.06i & 3.51 + 0.06i & 0 & 0 & 0 \\
3.51 + 0.06i & 5.23 - 0.09i & 3.51 + 0.06i & 0 & 0 & 0 \\
3.51 + 0.06i & 3.51 + 0.06i & 5.23 - 0.09i & 0 & 0 & 0 \\
0 & 0 & 0 & 0.86 - 0.08i & 0 & 0 \\
0 & 0 & 0 & 0 & 0.86 - 0.08i \\
0 & 0 & 0 & 0 & 0 & 0.86 - 0.08i
\end{bmatrix}
\]

(3.41)

First, the plate system was solved in the linear regime to calculate dispersion curves and propagative modes, necessary for the nonlinear analysis. PBCs were employed in relation to that. According to this approach, the present plate system was modeled using a mesh of just 60 quadrilateral cubic Lagrangian elements mapped and deployed in a (3.17 × 12.7) mm periodic cell (Figure 3.9). The resulting Lamb wave
solutions are displayed in Figure 3.10 and Figure 3.11 in the (0-500) kHz frequency range. They are found to be in perfect agreement with well-known results previously published in literature. Primary and secondary modes for the nonlinear analysis are highlighted with green circles in the same figures.

Due to the lack of studies in literature concerning specifically the HPPE material, the third order Landau-Lifshitz elastic constants characterizing a very similar plastic polymer (Polystyrene) were adopted for the nonlinear analysis (Cattani and Rushchitskii, 2007). The assumed values are $A = -10.8$ GPa, $B = -7.85$ GPa and $C = -9.81$ GPa.

Figure 3.9 – Geometry and associated mesh for a 2D periodic cell representative of the 12.7 mm thick HPPE plate (dimensions in mm).
Figure 3.10 – Phase velocity dispersion curves in the (0-500) kHz frequency range with primary and secondary modes selected for nonlinear analysis highlighted (green circles).

Figure 3.11 – Attenuation curves (expressed in dB/m) in the (0-500) kHz frequency range with primary and secondary waveguide modes selected for nonlinear analysis highlighted (green circles).
The nonlinear analysis was developed between 250 kHz (primary mode) and 500 kHz (secondary mode). Being the waveguide dissipative, all the eigenvalues and eigenvectors are complex. Propagative modes were separated from evanescent and non-propagative solutions by using a threshold of 10% between imaginary and real parts of each eigenvalue. After a preliminary analysis on different potential combinations among the propagative modes, one particular mode was selected as input (primary mode) for the nonlinear post-processing. It is associated with a complex eigenvalue $k = 669.62 + 87.56i$ and a corresponding phase velocity $c_{ph} = 2345.80$ m/s at 250 kHz.

The application of CO.NO.SAFE algorithm in this case is simplified because of the assumption of 2D strain regime (the plate is considered infinite in the width direction). For this reason all the terms used in the nonlinear post-processing discussed in Chapter 2 are evaluated on a line segment running through the thickness. This approach is sometimes referred as 1D SAFE (Predoi et al., 2007), and appeared in literature for the first time almost four decades ago (Dong and Nelson, 1972; Nelson and Dong, 1973).

The results of the analysis pinpointed the presence of a resonant secondary mode. As mentioned before, while the contribution of all other modes is oscillatory and bounded (Eq. (2.90)), this secondary mode shows a cumulative behavior and represents the dominant term in the expansion Eq. (2.88), participating with a contribution that linearly increases with distance. In fact, after all the secondary modal amplitudes were calculated from Eq. (2.91)) for the synchronous case, the identified resonant secondary mode exhibits a value which is orders of magnitude larger than those associated to the asynchronous modes (Figure 3.12).
The selected primary mode is detailed in Figure 3.13. In particular, Figure 3.13a illustrates the out-of-plane (wave propagation direction) displacement field associated with this mode with a contour plot with heights and color gradients proportional to the displacement amplitude. Figure 3.13b shows the in-plane displacement field via a contour plot with superimposed a vectorial plot where the length of the arrows results proportional to the in-plane displacement amplitude. Using a rendered 3D view, Figure 3.13c depicts the global modeshape considering a length of 1 cm for the waveguide.

The amplitudes of the displacement fields are not normalized and, consequently, they supply exact information about the mode shapes. At the same time, the values are therefore not comparable from one mode to another.

The selected primary mode is a complex axial symmetric mode. The mode at the double harmonic exhibits also features typical of axial modes. This resonant secondary mode at 500 kHz looks very promising in a possible structural monitoring system.
because it keeps the majority of the energy in the central area of the cross-section and minimizes wave leakage into the surrounding medium. Furthermore, Figure 3.11 shows that both primary and secondary modes have very small attenuation values (especially the secondary mode at 500 kHz); this fact makes the studied combination even more attractive because of the large inspection range that can potentially be achieved.
Figure 3.13 – Selected primary waveguide mode propagating in the HPPE plate at 250 kHz. (a) Contour plot of out-of-plane displacement field. (b) Contour plot with superimposed vectorial plot of in-plane displacement field. (c) 3D view of global displacement field.
Figure 3.14 - Resonant secondary waveguide mode propagating in the HPPE plate at 500 kHz. (a) Contour plot of out-of-plane displacement field. (b) Contour plot with superimposed vectorial plot of in-plane displacement field. (c) 3D view of global displacement field.
3.3.2 Anisotropic elastic composite laminate

A multi-layered composite laminate with unidirectional laminae in a quasi-isotropic layup was examined next. More specifically, the selected system consists of eight unidirectional T800/924 graphite-epoxy plies with a stacking sequence of \([\pm 45/0/90]_s\). The same laminate was investigated in the linear regime by Pavlakovic and Lowe using the software DISPERSE developed at Imperial College, London, UK (Pavlakovic and Lowe, 2003). Each layer has a thickness of 0.125 mm resulting in a total laminate thickness of 1 mm. The material properties for each single lamina in the principal directions of material symmetry are: \(\rho = 1500 \text{ kg/m}^3\), \(E_{11} = 161 \text{ GPa}\), \(E_{22} = 9.25 \text{ GPa}\), \(G_{12} = 6.0 \text{ GPa}\), \(\nu_{12} = 0.34\) and \(\nu_{23} = 0.41\) (Percival and Birt, 1997). The corresponding stiffness matrix, expressed in GPa, is given by:

\[
C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}

= \begin{bmatrix}
168.4 & 5.45 & 5.45 & 0 & 0 & 0 \\
5.45 & 11.3 & 4.74 & 0 & 0 & 0 \\
5.45 & 4.74 & 11.3 & 0 & 0 & 0 \\
0 & 0 & 0 & 3.28 & 0 & 0 \\
0 & 0 & 0 & 0 & 6.0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6.0
\end{bmatrix}
\]

(3.42)

The stiffness matrix for each lamina needs to be opportuneely rotated according to the angle between the fiber direction and the wave propagation direction (Bartoli et al., 2006). In the following, a wave propagation direction forming a 0° angle with respect to the fiber direction 1 was assumed (the extension to cases where this angle assumes different values is trivial). After all the matrix rotations were developed, the governing eigenvalue problem was solved as in the previous case-study, using the rotated stiffness matrices in the constitutive relations.
A mapped mesh made of 48 quadrilateral cubic Lagrangian elements was adopted. It was used to model a (0.3 × 1) mm rectangular cell with PBCs on both lateral sides. Geometric characteristics, composite layout and finite element model for the laminate periodic cell are shown in Figure 3.15. Resultant Lamb wave solutions between 50 kHz and 5 MHz, are illustrated in Figure 3.16. The dispersion curves match extremely well with the results already published in literature (Bartoli et al., 2006; Pavlakovic and Lowe, 2003). The primary and secondary modes adopted for the nonlinear analysis, along with two particular propagative modes at 3 MHz (labeled as Mode M1 and Mode M2), are highlighted in Figure 3.16 using different symbols. Modes M1 and M2 were analyzed in further detail to emphasize how the abrupt changes in material properties between adjacent plies lead to complex modeshapes; they are significantly different than the ones characterizing an equivalent isotropic homogeneous system. In fact, abrupt changes in slope in the displacement fields can be observed at the interfaces between adjacent layers, as depicted in Figure 3.17 and Figure 3.18. More specifically, Figure 3.17a and Figure 3.18a depict the out-of-plane displacement field (along the direction of propagation) as a 3D contour plot (height and color gradients proportional to the out-of-plane displacement amplitudes) for the two selected modes, respectively.

Figure 3.17b and Figure 3.18b show, for the same two modes, the in-plane displacement field (cross-section components) via a vector plot (arrows represent the resultant of the two displacement components as direction and amplitude) superimposed to a contour plot (color gradients proportional to in-plane displacement amplitudes).
Figure 3.15 - (a) Geometrical details for a 2D periodic cell representative of a 1 mm thick elastic composite 8-layer quasi-isotropic laminate (dimensions in mm). (b) Finite element mesh with periodic boundary conditions highlighted.

Figure 3.16 – Phase velocity dispersion curves in the (0.05-5) MHz range with exemplary propagative modes at 3 MHz, along with selected primary and secondary modes for nonlinear analysis.
Figure 3.17 - Selected mode M1 propagating at 3 MHz in the composite laminate. (a) Contour plot of out-of-plane displacement field. (b) Vector plot of in-plane displacement field.

Figure 3.18 - Selected mode M2 propagating at 3 MHz in the composite laminate. (a) Contour plot of out-of-plane displacement field. (b) Vector plot of in-plane displacement field.
The third-order elastic constants assumed for each lamina are: \( A = 15 \text{ GPa}, \ B = -33 \text{ GPa} \) and \( C = -14 \text{ GPa} \) (Prosser, 1987). The nonlinear analysis was developed between 2.5 MHz and 5.0 MHz. A complex primary mode combining attributes typical of axial and flexural horizontal modes was selected as the input. One of the propagative modes at the double harmonic (5 MHz) was found to be in internal resonance with a very dominant secondary modal amplitude and, consequently, very attractive for an actual application.

The results, in terms of modal amplitude plots, are shown in Figure 3.19.

![Figure 3.19 – Modal amplitude plot of secondary propagative modes for the anisotropic elastic composite laminate.](image)

Figure 3.19 emphasizes how drastic the predominance of the only resonant mode in terms of modal amplitude is, when compared with all the other propagative secondary modes existing at 5 MHz. Out-of-plane and in-plane displacement fields along with global modeshape for the investigated combination of resonant modes are represented in Figure 3.20 and Figure 3.21 in the same fashion as in the previous case.
Figure 3.20 - Selected primary waveguide mode propagating in the anisotropic elastic composite laminate at 2.5 MHz. (a) Contour plot of out-of-plane displacement field. (b) Contour plot with superimposed vectorial plot of in-plane displacement field. (c) 3D view of global displacement field.
Figure 3.21 - Resonant secondary waveguide mode propagating in the anisotropic elastic composite laminate at 5 MHz. (a) Contour plot of out-of-plane displacement field. (b) Contour plot with superimposed vectorial plot of in-plane displacement field. (c) 3D view of global displacement field.
Both primary and secondary modes concentrate the axial wave energy near the center of the waveguide; consequently, this combination appears appealing for the inspection of the laminate because of the expected reduced wave leakage into surrounding areas.

### 3.3.3 Reinforced concrete slab

This section discusses the suitability of the proposed algorithm to analyze guided wave propagation phenomena and internal resonance conditions in heterogeneous and geometrically periodic structures. A reinforced concrete slab is considered for this purpose. The complexity here arises from the coexistence of two domains with very different material properties (concrete and steel). Previous studies have shown the influence of the reinforcement on the concrete slab guided wave dispersion curves (Predoi et al., 2007). In the present work, for the first time it is attempted to analyze the nonlinear features of the guided wave propagation in this complex system, with the final goal of identifying appropriate combinations of resonant waveguide modes.

Periodic Boundary Conditions are applied also in this case on the lateral sides to deal with the geometrical periodicity of the slab. The 2D periodic cell considered is 6 cm wide and 8 cm tall. The steel bars are assumed to be 1.6 cm in diameter. A finite element mesh consisting of 528 triangular cubic Lagrangian elements was created using Delaunay’s algorithm (Knuth, 1992). Material properties assumed for the concrete domain were: \(\rho = 2133 \text{ kg/m}^3\), \(C_{11} = 33.2 \text{ GPa}\), \(C_{66} = 11.8 \text{ GPa}\) (Bouhadjera, 2004). For the steel bars, the following values were used: \(\rho = 7900 \text{ kg/m}^3\), \(C_{11} = 280 \text{ GPa}\), \(C_{66} = 80\)
GPa (Predoi et al., 2007). Geometric properties, boundary conditions and mesh detail of the periodic cell are presented in Figure 3.22.

Figure 3.22 – Geometrical details and finite element mesh for a periodic cell representative of a 8 cm thick reinforced concrete slab (dimensions in cm).

Figure 3.23 – Quality index distribution characterizing the assumed reinforced concrete slab finite element model.
The finite element mesh has been developed using the “advancing front” algorithm (COMSOL, 2011). The mesh quality is quantified using an index spanning from 0 (degenerated element) to 1 (completely symmetric element). The result in terms of quality index (average value equal to 0.9198) is illustrated in Figure 3.23.

Guided wave solutions for the reinforced concrete slab were obtained in the (0-100) kHz frequency range and are presented in Figure 3.24. Due to the conspicuous difference in material properties between concrete and steel, quite complex modes, with abrupt variations at the steel-concrete interface, can be seen.
Two of the propagative modes at 40 kHz (both highlighted in Figure 3.24 and labeled as Mode M3 and Mode M4) are represented in detail in the following to highlight the complexity of wave propagation phenomena in such a complex waveguide. The influence of the steel bars on the wave propagation characteristics is evident.

Figure 3.25 – Selected mode M3 propagating at 40 kHz in the reinforced concrete slab. (a) Contour plot of out-of-plane displacement field. (b) Vector plot of in-plane displacement field.

Figure 3.26 - Selected mode M4 propagating at 40 kHz in the reinforced concrete slab. (a) Contour plot of out-of-plane displacement field. (b) Vector plot of in-plane displacement field.
CO.NO.SAFE algorithm was used with 40 kHz as the primary frequency. The primary mode selected as input exhibits essentially a flexural horizontal displacement field. Its wavenumber is $k = 65.59 \text{ rad/m}$ while its phase velocity is $c_{ph} = 3831.79 \text{ m/s}$. In order to optimize the computational efficiency without any loss in accuracy, a smaller cell (half in width) was adopted for the nonlinear analysis. This smaller periodic cell was defined by appropriately applying the Periodic Boundary Conditions on its lateral sides, likewise in previous cases.

Results of the nonlinear analysis are presented in Figure 3.27 in terms of modal amplitude plot. They reveal the presence of few asynchronous modes characterized by a relatively large power transfer (modal amplitude values inside the circle) and only a single resonant secondary mode able to verify also the phase-matching condition. This resonant mode is characterized by $k = 124.04 \text { rad/m}$ and, being synchronous, it has the...
same phase velocity as the primary mode (taking opportune into account inevitable numerical errors).

Figure 3.28 – Complex power transfer distribution through the volume (top) and through the surface (bottom) between primary and resonant secondary modes propagating in the reinforced concrete slab in the (40-80) kHz frequency range.

The nature of the identified favorable combination of modes is illustrated in detail in Figure 3.29 and Figure 3.30.
Figure 3.29 – Selected primary waveguide mode propagating in the reinforced concrete slab at 80 kHz. (a) Contour plot of out-of-plane displacement field. (b) Contour plot with superimposed vectorial plot of in-plane displacement field. (c) 3D view of global displacement field (concrete domain in gray, reinforcement domain in red).
Figure 3.30 - Resonant secondary waveguide mode propagating in the reinforced concrete slab at 80 kHz. (a) Contour plot of out-of-plane displacement field. (b) Contour plot with superimposed vectorial plot of in-plane displacement field. (c) 3D view of global displacement field (concrete domain in gray, reinforcement domain in red).
3.4 Conclusions

The use of nonlinear guided waves is gaining increasing attention in the nondestructive evaluation and structural health monitoring communities. Proper application of nonlinear measurements requires a solid understanding of the higher-harmonic generation that can be expected for the test waveguide. The present section has demonstrated the potential of an innovative numerical algorithm for internal resonance analysis of complex waveguides. It extends the classical SAFE algorithm to the nonlinear regime and is implemented in a powerful multipurpose commercial code (COMSOL). The result is a new tool that opens up new possibilities for the analysis of dispersion characteristics and, most importantly, nonlinear internal resonance conditions, for a variety of complex structural waveguides that do not lend themselves to alternative analyses such as purely analytical solutions. The specific “complex” cases that were examined include: viscoelastic waveguides with damping effects (HPPE plate), multilayered composite panels (8-ply quasi-isotropic laminate), and heterogeneous periodic systems (reinforced concrete slab). In all these cases, the proposed algorithm successfully identified appropriate combinations of resonant primary and secondary modes that exhibit the desired conditions of synchronism and large cross-energy transfer. These properties can be exploited in an actual system aimed at monitoring the structural condition of the waveguide by nonlinear waves (detect defects, measure quasi-static loads or instability conditions, etc...).

The next chapter discusses, among other topics, the use of proposed algorithm to guide and optimize the design of an innovative nonlinear technique for thermal stress
monitoring in Continuous Welded Rails (CWR). The complexity, here, is mainly related to the particular geometrical features of the rail cross-section.

3.5 Acknowledgements

This chapter, in part, has been published in the Mathematical Problems in Engineering Journal, Nucera, Claudio; Lanza di Scalea Francesco; (2012). The title of this paper is *Higher Harmonic Generation Analysis in Complex Waveguides via a Nonlinear Semi-Analytical Finite Element Algorithm*. The dissertation author was the primary investigator and primary author of this paper.

This chapter, in part, has been recently submitted to the ASCEE Journal of Engineering Mechanics, Nucera, Claudio; Lanza di Scalea Francesco; (2012). The title of this paper is *Nonlinear Semi-Analytical Finite Element Algorithm for the Analysis of Internal Resonance Conditions in Complex Waveguides*. The dissertation author was the primary investigator and primary author of this paper.
Chapter 4

Application to nondestructive thermal stress measurement in Continuous Welded Rails (CWR)

4.1 Need for the study

Railroad tracks have appeared more than four centuries ago. In fact, the underlying technology developed over a long period, starting with primitive timber rails in mines in the 17th century. For a long period of time railroad tracks were connected end-to-end to produce a continuous surface on which trains may run (jointed rails). This was traditionally accomplished bolting the two adjacent rail portions using metal fishplates (Figure 4.1). In this way each rail section could expand and accommodate temperature-related physical expansion and contraction arising from seasonal thermal variations.

The diffusion of portable flash-butt welding machines along with the possibility of improving ride comfort, expand rail life and obtain higher traveling speed, reduced track maintenance and costs and a better geometry of the track triggered the use of Continuous Welded Rails (CWRs). Their attractiveness is evident when modern trends towards heavier axle loads and higher train velocities are considered. In this form of track, two successive rail sections are welded together to form continuous structural lengths of a few hundreds of meters and, in some instances, even a few kilometers
(Figure 4.2). The major problem with CWRs consists in the almost total absence of expansion joints, which can create severe issues in terms of safety.

Figure 4.1 – Fishplate bolted to join two successive rail sections in a jointed railroad track.

Figure 4.2 – Continuous welded rail section.
Due to the impossibility to expand or contract lengthwise, the welded rail, during cold weather, develops substantial tension along the direction of travel. This tension, if sufficiently large, will initiate a cracking mechanism at the weakest point in the rail that progressively evolves until the whole rail cross-section fractures completely, leading to rail breakage or pulling-apart (Figure 4.3). These pull-aparts generally occur at the welded joints or at the ends of the string and result in a gap in the rails. These gaps can result in a derailment and traffic must be safely slowed down or halted until appropriate corrective action is taken. However this dangerous event is not very common and is also often detectable through the loss of electrical signals carried in the track.

Figure 4.3 – CWR breakage due to tensile stresses (cold weather).

A more dangerous and frequent issue for the safety of rail transportation can happen in presence of hot weather. In Chapter 1 it has already been discussed that any difference of actual rail temperature from rail neutral temperature $T_N$ (defined in section 1.2) generates a longitudinal thermal stress. In fact, when the ambient temperature is
higher than $T_N$, a CWR enters into a state of compression, according to Eq. (1.1). When this state of compressive state reaches sufficiently high levels, the rail can exhibit buckling collapse mechanisms characterized by sudden and rapid lateral or vertical movement over a relatively short length (Figure 4.1) (Kerr, 1975, 1978b; Kish and Clark, 2004; Kish and Samavedam, 2005). This form of collapse is referred as rail thermal buckling or sun kink. It constitutes an extremely dangerous situation because if the buckling occurs under a train, a derailment is likely; if it occurs between trains, traffic must either be stopped or slowed down until the buckled track condition is fixed.

The risk of thermal buckling occurrence is emphasized considering that, for an exemplary location with typical seasonal variations in temperature (like Europe, North
America), the difference between rail temperatures and ambient air temperature can be as high as 20 °C on hot summer days (this difference is around 5 °C during winter) (Szelążek, 1992). To minimize this risk, CWRs are generally built connecting portions of tracks that are conveniently prestressed prior to welding. The applied tensile load is set according to the temperature at the time.

The key point to avoid such catastrophes consists in studying and exploiting particular mechanisms to determine longitudinal forces (or stresses) in the rail as a function of changing rail temperature. By knowing the existence and/or location of excessive tensile loads and/or excessive compressive loads remedial actions can be taken, such as for example cutting sections of rail and/or inserting rail plugs to avoid rail breakage in cold weather or rail buckling in hot weather.

According to Federal Railroad Administration (FRA) Safety Statistics Data, rail buckling was responsible for 48 derailments and nearly $30M in costs during 2006 alone in the U.S. The analysis of this scenario at a broader extent, considering the whole FRA database between January 1975 and February 2012 (Figure 4.5), suggests how significant is the collection on the in-situ thermal stress level to prevent rail buckling. In this period range more than two thousands derailments and more than $300M in costs associated with rail buckling were recorded. The severity of these accidents is, unfortunately, variable. There have been cases with only economic losses (Figure 4.6), other cases where several passengers were injured (Figure 4.7) and more serious cases where fatalities were recorded (Figure 4.8).

It is worth noticing that in addition to purely thermal stress, other mechanisms actively contribute to build up rail longitudinal stress (and potentially lead to rail
buckling) such as residual stress patterns, welding stresses and train-induced forces
dynamically generated during acceleration and braking. However, thermal stress
constitutes by far the main source of longitudinal stress in CWRs and is considered in
detail in the present dissertation.

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<td>92</td>
<td>0.2</td>
<td>-</td>
<td>92</td>
</tr>
<tr>
<td>T215- Joint bar broken (noninsulated)</td>
<td>649</td>
<td>1.1</td>
<td>1</td>
<td>648</td>
</tr>
<tr>
<td>T216- Joint bolts, broken, or missing</td>
<td>233</td>
<td>0.4</td>
<td>1</td>
<td>232</td>
</tr>
<tr>
<td>T217- Mismatched rail-head contour</td>
<td>324</td>
<td>0.6</td>
<td>2</td>
<td>319</td>
</tr>
<tr>
<td>T218- Piped rail</td>
<td>54</td>
<td>0.1</td>
<td>1</td>
<td>53</td>
</tr>
<tr>
<td>T219- Rail defect with joint bar repair</td>
<td>59</td>
<td>0.1</td>
<td>-</td>
<td>58</td>
</tr>
</tbody>
</table>

Figure 4.5 – Federal Railroad Administration Statistics on rail accidents due to track conditions in the period January 1975 - February 2012 (http://safetydata.fra.dot.gov/OfficeofSafety/default.aspx).
Despite many years of experience with CWR, the measurement of the applied stress (or $NT$) still represents a long-standing challenge for railway owners and operators. As briefly introduced in Chapter 1, the rail neutral temperature is defined as the temperature at which the thermal longitudinal force (or stress) in the rail is zero. Rail $T_N$ is often associated with the rail “laying” or “anchoring” temperature. The main difficulty in tracking this value arises from the fact that $T_N$ of the rail while in service is relatively
dynamic in the sense that it may change due to numerous factors (creep, breathing, ballast settlement, rail installation, distressing, realignment, broken rail repairs, etc...). Even for a rail with a known laying temperature or anchoring temperature, the neutral temperature for a rail in service may not be known.

In light of the discussion above it is clear how much the railroad industry would benefit in terms of both safety and economy of operation from a non-invasive technique able to develop in-situ measurement of thermal stress in rails and detect incipient buckling conditions, with sensitivities large enough to overcome the effects of tie-to-tie variations, changing temperature, and changing steel microstructure. This dissertation proposes an innovative system aimed at nondestructively assess the $T_N$ and potential incipient buckling conditions of a CWR in-service by measuring the thermal stress in the rail, $\sigma$, at a rail given temperature, $T$.

In this work the evaluation of the thermal stress is accomplished by measuring the nonlinear behavior of ultrasonic guided waves propagating along the rail running direction. Specific guided wave modes and specific guided wave frequencies need to be selected to gain sufficient sensitivity to thermal stress. In presence of nonlinearity the harmonic motion of a structural component at a given excitation frequency $f$ is distorted by the existence of higher-harmonics that are multiples of the fundamental harmonic. The presence/magnitude of these higher-harmonics is the particular nonlinear phenomenon exploited to track the rail thermal stress via the relative nonlinear parameter $\beta'$ defined in Eq. (2.108). The novelty here comes from the direct linking between this nonlinear response and the state of thermal stress in the rail geometry.
CO.NOSAFE algorithm was used to pinpoint a favorable combination of resonant primary and secondary modes. Nonlinear 3D finite element simulations using ABAQUS commercial code were used later to predict numerically the behavior of the nonlinear parameter with respect to temperature variations. In order to strengthen the proposed system, several proof-of-principle investigations were conducted at the UCSD Large-Scale Rail NT/Buckling Test-bed constructed at the Powell Structural Laboratories under FRA funding. Based on several experimental validations (detailed in the following), a prototype has been designed for a wayside stationary installation on the rail web.

After a brief illustration of the state of the art and the proposed techniques up to date to identify $T_N$ in CWRs, the present chapter discusses the computational efforts that guided and optimized the development of the proposed system.

### 4.2 State of the art

Nondestructive experimental stress measurement in structures still represents an open issue in research. Several methods have been proposed in the past. However, the vast majority of them is destructive and is generally based on measuring the strain relaxation when material is removed and deducing the original stress level consequently (Withers and Bhadeshia, 2001). The need for a non-invasive technique able to determine the state of stress in structural components is of great importance in structural engineering. There is an urgent need of a nondestructive technique that is easy to
implement and that does not interfere with the operation of the railway in order to avoid breakage in cold weather and buckling/instability in hot weather.

Some of the methodologies proposed so far to measure $T_N$ in CWRs are currently in the evaluation phases and none of them, however, has gained widespread use yet for more or less severe limitations in applicability and precision in results. Their potential in measuring the $T_N$ is often shadowed by comparable sensitivity to other structural and non-structural features (such as microstructure, tie-to-tie variations, temperature alone). The methods under consideration today are briefly described below, highlighting benefits and drawbacks.

- **Measurement of static rail stiffness (VERSE Method):** This technique was developed jointly by VORTOK International and AEA Technology. Although effective, this approach is quite cumbersome and not very practical since it is time-demanding and interferes with railway operation since it requires unfastening of ~100 ft of rail. Furthermore it cannot measure the neutral temperature when the rail is compression, as highlighted in test studies conducted at TTCI (Transportation Technology Center, Inc.) in 2000 (Tunna, 2000) and in Prague in 2001 (Railway-Research-Institute, 2001).

![Figure 4.9 – VERSE Equipment for neutral temperature measurement in CWRs. (a) Field deployment. (b) Schematic layout.](image)
- **Measurement of dynamic resonance of torsional mode of vibration (D’STRESEN Method):** The D’stresen technique is based on the measurement of dynamic resonance (acceleration amplitude) below 90 Hz for the torsional mode of vibration of the rail. The approach does not require unfastening, hence the potential attractiveness. Unfortunately, as confirmed in recent investigations at TTCI (Read and Shust, 2007), the method is highly sensitive to rail fastening/support conditions. Consequently normal tie-to-tie variations can make the stress or \( T_N \) measurement challenging and unreliable with this method. Furthermore, D’Stresen method works for rails which are in tension or compression but, unfortunately, it is necessary to know what the actual state of stress before to testing is, in order to collect correct results. Because of these problems, the industry has considered this method more as an “estimator”, rather than a direct measurement, of rail neutral temperature.

- **Measurement of ultrasonic velocity of bulk waves (Acoustoelastic Method):** The acoustoelastic method for stress measurement in rails has been known for more than thirty years (Egle and Bray, 1976). Acoustoelastic stress measurement is based on the theory of finite deformations (Murnaghan, 1967) which produce a change in ultrasonic velocity with applied stress. It typically uses longitudinal, shear or surface (Rayleigh) waves in the ~MHz frequency range. The biggest challenge of the technique is that the acoustoelastic variation of wave velocity with stress is extremely small (~0.1% velocity change per GPa of stress for rail steel). This low sensitivity often masks the stress indications by other parameters affecting wave velocity (namely temperature variations and steel microstructure.
variations). For this reason, the acoustoelastic method for stress measurement in rails has been challenging to implement in practical field conditions.

- **Measurement of magnetic permeability of steel (MAPS-SFT Method):** Introduced in the early 1990’s, this technique (Figure 4.10) exploits the relationship between stress and the magnetic properties of ferromagnetic materials. The magnetic properties are sensitive to the rail total stresses (residual stress plus thermal stress due to restrained expansion). Only the thermal stress due to restrained expansion – with the current rail temperature - is needed to determine the rail $T_N$ from Eq. (1.1). Consequently, the technique needs to eliminate the effect of residual stresses by using calibration curves obtained from different rail manufacturers/rail types where magnetic measurements are taken at zero stress to isolate the residual stress component. The magnetic probes are attached to the rail web to attempt to determine the thermal stresses at the rail neutral axis. The thermal stress determination from this technique currently requires 8 scans, or at least 30 minutes, to produce a single value of rail Neutral Temperature. A statistical analysis is also performed to fit the data to a statistical trend. Consequently, the technique can currently only be used at wayside and not in-motion. Finally, to the author’s knowledge, this technique cannot distinguish tension vs. compression stresses.

- **Measurement of phase or group velocity of ultrasonic guided waves:** A few studies have attempted to overcome the low sensitivity of ultrasonic velocity to stress for bulk waves by exploiting the dispersive behavior of ultrasonic guided
waves that depend on the waveguide geometry of the rail structure (Chen and Wilcox, 2007; Loveday and Wilcox, 2010). However, these techniques are affected by the following two problems: (1) the phase velocity technique was proposed at low frequencies, hence the sensitivity to rail support conditions (tie-to-tie variation); (2) phase and group velocities alone are highly sensitive to changes in the material constants due to temperature alone. Consequently, it has been extremely challenging to eliminate the temperature-dependent elastic constant effects from the effect of the actual thermal stress from restrained expansion. As discussed above for MAPS-SFT, the latter effect is the only parameter directly related to the rail Neutral Temperature.

Figure 4.10 – MAPS-SFT equipment installed on a typical rail section (http://www.maps-technology.com).
- **Measurement of Ultrasonic Backscattering**: Research studies are on-going at the University of Nebraska aimed at developing a nondestructive system to assess the thermally-induced longitudinal stress in CWRs using ultrasonic backscatter. This phenomenon results from the multitude of reflections that occur at grain boundaries and is potentially influenced by the material’s stress state. However, research progress and practical implementation are at an early stage.

- **Measurement of Rayleigh Wave Polarization**: Texas A&M University is studying a technique to measure rail longitudinal stress due to thermal variations based on the polarization of Rayleigh waves. This is defined as the ratio between the in-plane and out-of-plane displacements. As in the previous case, the research and system implementation are at an early stage of development.

### 4.3 CO.NOSAFE application to Continuous Welded Rails

The identification of a favorable combination of primary and secondary modes able to meet the requirements for internal resonance (as discussed in Chapter 2) is of paramount importance for the actual implementation of the proposed nondestructive system aimed at measuring the Neutral Temperature and detect incipient buckling conditions in CWRs. As discussed in Chapter 6, the knowledge of these particular mode combinations will help designing optimum transducer configurations for a practical implementation of the proposed technique. Hence, the first step of the system development process consisted in applying the CO.NOSAFE algorithm to analyze internal resonance conditions in a CWR and pinpoint convenient combinations of
primary and secondary waveguide modes. The AREMA 136RE typology has been adopted (Figure 4.11). This type is widely employed and is considered in detail because it is the one used in the large-scale experimental investigations, discussed in next sections. Due to the complex geometry of the cross-section, solutions for the dispersion curves and, consequently, for the higher harmonic generation analysis cannot be calculated analytically.

Figure 4.11 – AREMA 136 RE railroad track geometrical details.
Two exemplary cases in the low frequency range (primary mode at 80 kHz and secondary mode at 160 kHz) were selected first as representative to benchmark the potential of the proposed algorithm in analyzing higher harmonic generation in geometrically complex waveguides. Real resonant combinations and “false positive”, where either one of the two requirements for internal resonance is not satisfied, were correctly distinguished. In the first case, phase-matching between primary and secondary modes is verified. However, due to the characteristic energy distribution over the rail cross-section, no power transfer is present between the modes and, consequently, internal resonance does not occur; hence, the secondary modal amplitude is bounded in value and oscillates with distance along the direction of wave propagation (Eq. (2.90)). In the second case, instead, both required conditions are verified and internal resonance takes place, leading to a resonant secondary wave field growing linearly with wave propagation distance.

Later the analysis is extended to higher frequencies, considering a primary mode in input at 200 kHz. Moving towards higher frequency is particularly beneficial in view of a wayside installation for the proposed system because, even though more propagative modes appear as part of the eigensolution, the majority of them focus the wave energy in confined portions of the rail section. The interrogating combination of modes selected inside this frequency range can potentially exhibit smaller sensitivity to tie-to-tie variations and other external influences that can corrupt the applicability of the proposed technique. The material properties considered are given in Table 4.1Table 4.1 – Material properties assumed for railroad nonlinear track analysis.. Landau-Lifshitz third-order elastic constants are detailed in (Sekoyan and Eremeev, 1966).
Table 4.1 – Material properties assumed for railroad nonlinear track analysis.

<table>
<thead>
<tr>
<th>$\rho$ [kg/m$^3$]</th>
<th>$\lambda$ [GPa]</th>
<th>$\mu$ [GPa]</th>
<th>$A$ [GPa]</th>
<th>$B$ [GPa]</th>
<th>$C$ [GPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>7932</td>
<td>116.25</td>
<td>82.754</td>
<td>-340</td>
<td>-646.667</td>
<td>-16.667</td>
</tr>
</tbody>
</table>

The finite element mesh used for the analysis is shown in Figure 4.12. It has been developed using the “advancing front” algorithm (COMSOL, 2011). The mesh quality is (average quality index equal to 0.9319) is illustrated in Figure 4.13. In order to correctly explore the displacement field and all the derived quantities (essential for the calculation of all the terms during the nonlinear post-processing), 351 cubic Lagrangian 10-node triangular isoparametric elements (Figure 4.14) were employed (Onate, 2009).

![Finite element mesh](image-url)
Figure 4.13 - Quality index distribution for the AREMA 136 RE finite element model.

Figure 4.14 – Cubic Lagrangian 10-node triangular elements analytical description (Onate, 2009).
In Figure 4.14, $L_1$, $L_2$, and $L_3$ represent the area coordinates which can be interpreted as the ratio between the distance from a generic point $P$ inside the triangle to the opposite side divided by the distance from the node to that side (Figure 4.15). These coordinates have proved to be very useful to derive the shape functions of triangular finite elements.

![Figure 4.15 – Area coordinates for an exemplary triangular element (Onate, 2009).](image)

Using the above coordinate system, the shape functions for the 10-node cubic Lagrangian triangular element can be expressed as (Onate, 2009):

$$
\begin{align*}
N_1 &= \frac{1}{2} L_1(3L_1-1)(3L_1-2) \\
N_2 &= \frac{1}{2} L_2(3L_2-1)(3L_2-2) \\
N_3 &= \frac{1}{2} L_3(3L_3-1)(3L_3-2) \\
N_4 &= \frac{9}{2} (3L_1-1)L_1L_2 \\
N_5 &= \frac{9}{2} (3L_2-1)L_2L_3 \\
N_6 &= \frac{9}{2} (3L_3-1)L_3L_1 \\
N_7 &= \frac{9}{2} (3L_1-1)L_1L_3 \\
N_8 &= \frac{9}{2} (3L_2-1)L_2L_1 \\
N_9 &= \frac{9}{2} (3L_3-1)L_3L_2 \\
N_{10} &= 27L_1L_2L_3
\end{align*}
$$

(4.1)

In Figure 4.16, the resultant wavenumber and phase velocity dispersion curves in the (0-600) kHz frequency range are represented.
Figure 4.16 – AREMA 136 RE railroad track dispersion properties in the (0-600) kHz frequency range. (a) Wavenumber curve. (b) Phase velocity dispersion curve.
The complexity of the guided wave propagation for this particular waveguide is clear considering the abundance of propagative modes present and their dispersion characteristics (especially at higher frequencies). By analyzing the two zoomed insets in Figure 4.17, it can be noticed that relatively few propagative modes exist for this particular waveguide until around 100 kHz, while at higher frequencies the picture is way more complex. Several modes coexist with intricate dispersion characteristics.

![Figure 4.17 – Zoomed views on rail phase velocity dispersion curve around 80 kHz and 330 kHz, respectively.](image)

Figure 4.18 shows some propagative modes found inside the range (80-160) kHz. It can be noted how differently the energy is concentrated within the waveguide. As stated before, two exemplary cases are used as benchmark and discussed in detail in the
following sections. Figure 4.19 illustrates these combinations of modes in the phase-velocity dispersion curve.

![Figure 4.18 - Propagative modes in the (80-160) kHz frequency range. (a) Flexural vertical mode (energy mainly concentrated in the rail head). (b) Flexural horizontal mode (energy exclusively confined in the rail web). (c) Axial mode. (d) Complex mode involving a mixture of axial, torsional and flexural displacements.](image1)

![Figure 4.19 - Selected combinations of synchronous primary and secondary rail waveguide modes in the (80-160) kHz frequency range.](image2)
4.3.1 Non-resonant combination

A flexural horizontal primary mode (Figure 4.20) was selected as primary excitation (input for the CO.NO.SAFE algorithm). The nonlinear analysis revealed the presence of a synchronous secondary mode at $2\omega$ (Figure 4.21) exhibiting a similar flexural horizontal displacement distribution. However, the power transfer through the volume and the surface of the waveguide (Figure 4.22) is such that the other necessary requirement for internal resonance is not met for this particular combination, leading to an oscillating secondary modal amplitude value and absence of internal resonance. At the same time, a conspicuous power transfer occurs between the selected primary mode and some asynchronous secondary modes; here again internal resonance does not take place because of the lack of one of the two essential requirements (phase-matching). This fact translates into the very small value associated of modal amplitude associated with the only synchronous mode, and the relatively higher values associated to the asynchronous secondary modes.

Figure 4.20 – Selected primary mode propagating at 80 kHz in the AREMA 136 RE rail web. (a) Contour plot of out-of-plane displacement field. (b) Vector plot of in-plane displacement field.
Figure 4.21 – Synchronous although non-resonant secondary mode propagating at 160 kHz in the AREMA 136 RE rail web. (a) Contour plot of out-of-plane displacement field. (b) Vector plot of in-plane displacement field.

Figure 4.22 – Complex power transfer distribution through the volume (top) and through the surface (bottom) between non-resonant primary and secondary modes propagating in the rail web in the (80–160) kHz frequency range.
Figure 4.23a and Figure 4.23b illustrate the selected primary and secondary modes, respectively. Figure 4.23c plots the modal amplitude results for the propagative secondary modes present at 160 kHz.

Figure 4.23 – Non-resonant combination of modes propagating in the AREMA 136 RE rail web. (a) Selected primary mode at 80 kHz. (b) Phase-matched (synchronous) although non-resonant secondary mode at 160 kHz. (c) Modal amplitude plot for propagative secondary modes.
4.3.2 Resonant combination

In this case a flexural vertical mode was selected as primary excitation (Figure 4.24). The results of the nonlinear SAFE analysis disclosed the presence of some synchronous secondary modes with one in particular (Figure 4.25), exhibiting a slightly different flexural vertical behavior, able to verify both requirements.

Figure 4.24 – Selected primary mode propagating at 80 kHz in the AREMA 136 RE rail head. (a) Contour plot of out-of-plane displacement field. (b) Vector plot of in-plane displacement field.

Figure 4.25 - Resonant secondary mode propagating at 160 kHz in the AREMA 136 RE rail head. (a) Contour plot of out-of-plane displacement field. (b) Vector plot of in-plane displacement field.
This time the complex power transfer through volume and surface of the rail (Figure 4.26) is such to generate internal resonance and a nonlinear double harmonic growing linearly with distance.

![Complex power transfer distribution through the volume (top) and through the surface (bottom) between resonant primary and secondary modes propagating in the rail head in the (80-160) kHz frequency range.](image)

Likewise the previous case, Figure 4.27a and Figure 4.27b display the selected modes, while Figure 4.27c spotlights the very high value of modal amplitude related to the secondary resonant mode; small amplitude values associated to the other synchronous modes, for which power transfer is absent, are also shown in the same figure. The obtained results point up an optimal combination of primary and secondary wave fields able to maximize the nonlinear response of the waveguide. Furthermore, it is worth noticing how the selected primary mode is not only able to produce a resonant condition, but also very attractive in terms of practical excitability by a piezoelectric transducer.
4.3.3 Resonant web flexural modes

In view of a wayside installation for the proposed system (potentially able to monitor rail Neutral Temperature and buckling conditions without interfering with trains
transit), waveguide modes with energy propagation confined in the rail web with little foot motion are optimal candidates. In this way the sensitivity of the proposed technique to boundary conditions (tie-to-tie variations) is minimized. Hence the nonlinear numerical analysis was developed next at higher frequencies. As evident from Figure 4.16, beyond (150-200) kHz the phase-velocity dispersion curves become extremely complicated and a massive amount of propagative modes coexists. Consequently, the complexity and the numerical size of the governing eigensystem are dramatically increased (thousands of eigenvalues need to be calculated at each frequency).

![Finite element mesh adopted for the nonlinear analysis of the AREMA 136 RE rail at higher frequencies.](image)

In order to streamline the computational demand of this process without compromising the precision of the results (thanks to using cubic finite elements), a
slightly coarser mesh, created via the same advancing front algorithm used in the previous cases, has been adopted (Figure 4.28).

The internal resonance analysis has been developed between 200 kHz (primary mode) and 400 kHz (double harmonic). Several complex propagative modes were discovered in this frequency range. Two of them are shown in Figure 4.29.

![Figure 4.29 – Complex waveguide modes propagating at relatively high frequencies (200 kHz) in the AREMA 136 RE railroad track. (a) Contour plot of out-of-plane displacement field for complex mode 1. (b) Vector plot of in-plane displacement field for complex mode 1. (c) Contour plot of out-of-plane displacement field for complex mode 2. (d) Vector plot of in-plane displacement field for complex mode 2.](image-url)
A particular propagating mode at 200 kHz showing a strong energy concentration in the rail web was selected as input for the nonlinear internal resonance analysis (Figure 4.30). The analysis revealed the presence of a resonant secondary mode propagating at 400 kHz (Figure 4.31). It is characterized by a similar displacement field and it exhibits strong energy concentration in the web area like the fundamental mode at 200 kHz.

Figure 4.30 - Selected primary mode propagating at 200 kHz in the AREMA 136 RE rail web. (a) Contour plot of out-of-plane displacement field. (b) Vector plot of in-plane displacement field.

Figure 4.31 - Resonant secondary mode propagating at 400 kHz in the AREMA 136 RE rail web. (a) Contour plot of out-of-plane displacement field. (b) Vector plot of in-plane displacement field.
Figure 4.32 illustrates these combinations of modes in the phase-velocity dispersion curve while Figure 4.33 displays the complex power transfer through the volume and the external surface of the rail.

![Phase Velocity Dispersion Curve](image)

**Figure 4.32** – Combination of synchronous primary and secondary modes propagating in the (200-400) kHz frequency range selected for internal resonance analysis.

![Power Transfer Distribution](image)

**Figure 4.33** – Complex power transfer distribution through the volume (top) and through the surface (bottom) between resonant primary and secondary modes propagating in the rail web in the (200-400) kHz frequency range.
Figure 4.34 - Resonant combination of modes propagating at relatively high frequencies in the AREMA 136 RE rail web. (a) Selected primary mode at 200 kHz. (b) Resonant secondary mode at 400 kHz. (c) Modal amplitude plot for secondary propagative modes.

The secondary modal amplitude plot emphasizes how strong is the predominance of a single secondary mode at 400 kHz (the resonant one) when compared to all the other modes propagating at the same frequency. The inset shows small modal amplitude values
associated with some synchronous propagating modes. They match (numerically) the phase velocity of the primary mode but their particular power transfer characteristics do not generate internal resonance.

The obtained favorable combination of resonant modes propagating at relatively high frequency and, most importantly, concentrating the energy in the rail web constituted a pivotal result in the development of the Neutral Temperature/Buckling detection system. The little interaction with the rail head eliminated the effects of residual stresses and changes in geometry (wear) of the waveguide. The little interaction with the rail foot, instead, eliminated effects of the rail supports (tie-to-tie variation problem).

As introduced in Chapter 2, the efficiency of a nonlinear NDE/SHM technique dramatically relies on the knowledge of the particular mode to excite in terms of modeshape and frequency to generate internal resonance and a nonlinear response which is cumulative and, consequently, grows with distance. Failing in this task inevitably leads to an inefficient approach where the nonlinear response, although present, is small (second harmonic amplitude is bounded and oscillates with distance) and most likely shadowed by other causes.

Guided by the results discussed above, a full 3D nonlinear finite element model has been analyzed in ABAQUS commercial code in order to explore numerically the evolution of the system nonlinearity as a function of the stress level acting in the rail. This was accomplished considering both the effect of a mechanical pretension (developed in the field before laying the rail to conveniently shift its Neutral Temperature) and a thermal variation uniformly applied to the waveguide volume.
4.4 ABAQUS 3D Finite Element simulations

4.4.1 Introduction

This section presents the results of a series of numerical simulations carried out in ABAQUS commercial finite element package. They were aimed at predicting the variability of the nonlinear parameter $\beta$ with stress level acting in the rail. The interrogating waveguide mode was selected accordingly to the results obtained in the previous section in order to produce internal resonance. Both ABAQUS Implicit and Explicit solvers (Dassault-Systèmes, 2011) were invoked exploiting their full potential. In particular, ABAQUS/Standard (implicit) was used to apply different prestress levels to the rail, while ABAQUS/Explicit was employed to analyze the guided wave propagation phenomena. The former uses a stiffness-based solution technique that is conditionally stable, while the latter uses an explicit integration algorithm, based on a central difference method, that is unconditionally stable and particularly efficient when dealing with very large models and very fast dynamic events.

As detailed in the following paragraphs, the analyses were particularly challenging because both mechanical and thermal effects had to be considered and, especially, extremely fast transient dynamic effects had to be correctly captured with an appropriate finite element mesh. These requirements were tackled performing all the calculations on a Dell Precision T5500 workstation featuring 12 Intel processors (24 threads), 48 GB of RAM, Solid-State drives and NVIDIA Quadro FX 1700 GPU.
4.4.2 Geometry

A 52-cm long section of a AREMA 136 RE railroad track (section details in Figure 4.11) was considered for the finite element analysis. It is depicted in Figure 4.35.

4.4.3 Material

Assumed rail steel material properties are:

- \( \rho = 7800 \text{ kg/m}^3 \) (density)
- \( E = 209 \text{ GPa} \) (Young’s Modulus)
- \( \nu = 0.3 \) (Poisson’s ratio)
- \( \alpha = 1.23\times10^{-5} \text{ m/m °C} \) (coefficient of thermal expansion)
- \( A = -340 \text{ GPa} \) (Landau-Lifshitz third order \( A \) constant)
- \( B = -646.667 \text{ GPa} \) (Landau-Lifshitz third order \( B \) constant)
- $C = -16.667$ GPa (Landau-Lifshitz third order $C$ constant)

Damping effects were neglected. ABAQUS material library does not include Landau-Lifshitz hyperelastic material formulation. Therefore, at this stage this particular model was implemented using a general Polynomial hyperelastic formulation with auxiliary test data generated using Eq. (2.53) as input source and classical nonlinear effects (due to the material nonlinearity) were explored. It is worth noticing that in Chapter 5 a new constitutive model specifically formulated for nonlinear axially constrained waveguides will be presented. At the present, studies are ongoing to implement this new formulation in ABAQUS using a special User Defined Material Subroutine.

### 4.4.4 Spatial resolution

It is well-known that the accuracy of finite element simulations strongly relies on temporal and spatial resolution of the analysis. For instance, both the integration time step and the finite element size are affected by the maximum frequency of interest in the dynamic problem. Exploiting the optimal combination of resonant web-modes discussed in Section 4.3, for the present case 200 kHz was considered as frequency of the interrogating signal and the second harmonic at 400 kHz was considered as maximum frequency to be explored.

The size of the finite element is typically imposed by the smallest wavelength to be analyzed. Different rules concerning this aspect have been proposed over the years. A good spatial resolution generally requires a minimum of 8 nodes per wavelength (Datta
and Kishore, 1996), although some studies recommend a more stringent condition of 20 nodes per wavelength (Moser et al., 1999).

Based on material properties defined in Section 4.4.3, the material longitudinal and shear bulk waves velocities can be calculated according to the well-known formulae:

\[ c_L = \left( \frac{E(1-v)}{\rho(1+\nu)(1-2\nu)} \right)^{\frac{1}{2}} = 6005.83 \text{ m/sec} \]  

\[ c_T = \left( \frac{E}{2\rho(1+\nu)} \right)^{\frac{1}{2}} = 3210.25 \text{ m/sec} \] (4.2)

Assuming \( f_{max} = 400 \text{ kHz} \), the wavelength of a shear wave propagating in the material can be calculated as \( \lambda_T = c_T f_{max} = 8.03 \text{ mm} \). In order to meet the spatial resolution criterion of having \( n = 8 \) nodes per wavelength, the recommended element size can be evaluated as \( L_{max} = \frac{\lambda_T}{(n-1)} = 1.15 \text{ mm} \). It is worth noticing that in reality \( \lambda_T \) could be larger than the smallest wavelength encountered at the maximum frequency \( f_{max} \) since some particular waveguide modes can exhibit phase velocities \( c_{ph} < c_T \). However, the shear wave velocity has been assumed in the present work as accurate enough to describe the smallest wavelength of the model and, therefore, define the typical finite element dimension.

The actual mesh was developed using 8-node linear hexaedral elements with Reduced Integration (Dassault-Systèmes, 2011). Exploiting the invariability of the cross-sectional features along the rail running direction, the mesh was first deployed in the cross-section plane and then it was extruded along the wave propagation direction. A typical element dimension of 1.5 mm (slightly bigger than the value suggested above) was employed for the rail head and rail web. In order to contain the model computational
demands inside reasonable limits, the mesh was progressively deteriorated moving towards the rail foot area. This process reduced dramatically the total model size without compromising the results. In fact, the coarser zone is away from the rail web, which is, instead, the area of particular interest where waveforms are acquired and finally post-processed. The mesh in the rail cross-section plane is represented in Figure 4.36. Figure 4.37 and Figure 4.38 illustrate mesh quality via Aspect Ratio and Jacobian (Onate, 2009), respectively.

![Figure 4.36 – Finite element mesh of AREMA 136 RE rail cross-section.](image)

From Figure 4.37 and Figure 4.38 it is evident how the mesh quality slightly deteriorates just in the rail foot and in curved transition areas between rail head and rail web.
Figure 4.37 – Finite element mesh quality. Aspect ratio distribution in the rail cross-section.

Figure 4.38 – Finite element mesh quality. Jacobian distribution in the rail cross-section.
Salient features of the resulting 3D FE model are:

- 966264 nodes
- 898300 8-node linear hexaedral elements
- 2898792 total DOFs

Figure 4.39 shows the full 3D mesh employed for the analysis.

4.4.5 Temporal resolution

Numerical instability is a critical aspect of explicit numerical simulations that happens when the integration time step is not small enough to correctly track the dynamic event evolution. Failure in meeting this requirement results in unstable solutions characterized by unrealistic displacement fields that usually oscillate with increasing amplitudes. The total energy balance will also change significantly.

Figure 4.39 – 3D Finite Element model of AREMA 136 RE rail.
The smallest edge length of the entire mesh is approximately $L_{\text{min}} = 0.75$ mm. In transient FE simulations, a valid rule to meet time resolution requirements consists in using a minimum of 20 points per cycle at the highest frequency (Bartoli et al., 2005). It is also recommended to adopt a time step small enough to avoid that the longitudinal bulk waves travel across the smallest spatial resolution in one step (Datta and Kishore, 1996). Furthermore, since the time transient response will serve as a basis for the time Fourier transform process (described in the post-processing phase), in order to satisfy Shannon’s principle and avoid aliasing, a sampling frequency $f_s$ at least twice the highest frequency excited must be chosen. For the present case this condition reads:

$$f_s \geq 2 \cdot 400kHz$$

The aforementioned three conditions can be expressed as:

$$\Delta t \leq \min \left\{ \frac{1}{(20f_{\text{max}})} = 0.125E-06, \frac{L_{\text{min}}}{c_t} = 0.126E-06, \frac{1}{f_{s,800kHz}} = 1.25E-06 \right\}$$

Based on Eq. (4.4), the integration time step for the Explicit part of the analysis was conservatively set equal to $\Delta t = 1E-07$ sec. This limit clearly does not affect the implicit part of the analysis (where the preload and thermal stresses are applied to the rail) thanks to its unconditional stability.
4.4.6 Boundary conditions

Axial constraints were applied to the front and rear faces of the rail FE model to fix displacements along Z direction, as depicted in Figure 4.40.

![Figure 4.40 – Axial constraints applied to the Finite Element model front and rear faces.](image)

4.4.7 Analysis protocol

The nonlinear parameter $\beta$ was evaluated launching a specific interrogating waveguide mode into the rail web and post-processing the simulated waveforms received on a sensor point along the wave propagation direction in the frequency domain. In order to track the evolution of $\beta$ with the stress state acting in the rail, the analysis above was performed in five different scenarios described below:

- **SCENARIO 1**: rail is unstressed and this represent the Neutral Temperature state;
- **SCENARIO 2**: rail is pretensioned imposing a uniform displacement to one of its extreme faces, and fixing the other face. The displacement magnitude was such to produce a Neutral Temperature of 90 °F.

- **SCENARIO 3**: rail is pretensioned imposing a uniform displacement to one of its extreme faces, and fixing the other face. The displacement magnitude was such to produce a Neutral Temperature of 120 °F.

- **SCENARIO 4**: starting from the final pretensioned state of Scenario 2, both faces of the rail are axially constrained and a thermal variation was superimposed and applied uniformly to the whole rail volume. The amplitude of this variation was such to produce a final state in which the rail is precompressed and the absolute value of the stress amplitude is equivalent to the one present at the end of Scenario 2.

- **SCENARIO 5**: starting from the final pretensioned state of Scenario 3, both faces of the rail are axially constrained and a thermal variation was superimposed and applied uniformly to the whole rail volume. The amplitude of this variation was such to produce a final state in which the rail is precompressed and the absolute value of the stress amplitude is equivalent to the one present at the end of Scenario 3.

Displacement and thermal variation amplitudes to produce the particular states described above were estimated using Eq. (1.1) with rail steel material properties provided in Section 4.4.3 and geometrical features depicted in Figure 4.11. The values associated with each analysis scenario are:
**Scenario 1**: no mechanical and/or thermal distortions are applied to the model in this case. Rail is left unstressed.

**Scenario 2**: a displacement of 0.110 mm is imposed on the rail along the axial direction (Z).

**Scenario 3**: a displacement of 0.217 mm is imposed on the rail along the axial direction (Z) and a uniform thermal variation of 62 °F is applied to the rail volume.

**Scenario 4**: a displacement of 0.110 mm is imposed on the rail along the axial direction (Z) and a uniform thermal variation of 62 °F is applied to the rail volume.

**Scenario 5**: a displacement of 0.217 mm is imposed on the rail along the axial direction (Z) and a uniform thermal variation of 122 °F is applied to the rail volume.

The preload scenarios above are graphically described in Figure 4.41.

The analysis protocol consisted in using ABAQUS/Standard implicit code to calculate stresses and strains arising in the rail model at the end of each preload scenario. These final states were then imported into ABAQUS/Explicit code and implemented as predefined fields (Dassault-Systèmes, 2011).
In agreement with the numerical predictions on resonant combinations of modes discussed in Section 4.3.3, the explicit analysis involved the generation of a 10-cycle toneburst centered at 200 kHz (Figure 4.42) on a point of the rail web and the acquisition of the travelling waveform on another point of the rail web, as illustrated in Figure 4.43.

Figure 4.42 – Toneburst signal generated to interrogate the rail waveguide in the explicit analysis step (unitary amplitude used for representation purposes).

Figure 4.43 – Schematic of the explicit numerical simulation layout ($L = 52$ cm). Guided wave propagation is triggered conveying a 10-cycle windowed sinusoidal signal into the rail at the transmitter node location. Waveforms are acquired at the receiver node location.
A nominal amplitude (larger than a realistic force imposed in an actual structural component during an NDE/SHM assessment) was adopted. This should not compromise the final results. A separate study should consider the effect of different input energy.

To avoid numerical singularities, the load was implemented as pressure on the surface of the four finite elements surrounding the transmitter node (Figure 4.44).

![Figure 4.44 – Details of the applied load conditions.](image)

Each explicit analysis was performed with multi-cores multi-threads support (Figure 4.45). The workstation used in the present work required approximately 8 hours for each run.
4.4.8 Results

Complex dynamic features of guided wave propagation phenomena developing in the rail are illustrated in Figure 4.46 through Figure 4.51 as Von Mises Equivalent Stress contour plots for several successive time instants. For the sake of brevity, results of the explicit analysis on the unstressed rail only (Scenario 1) are presented. In the plots different scale factor were employed to represent the displacement field to conveniently emphasize the evolution of the wave propagation process.

The stress distribution clearly indicates how complicated the propagation of guided wave in a geometrically complex waveguide such as a railroad track is. Several waveguide modes coexist. Faster longitudinal modes precede slower transverse modes
during the propagation. From the plots it is also clear how complex the evolution of the wavefront is.

Figure 4.46 – Contour plot of Von Mises Equivalent Stress after 8E-06 sec.

Figure 4.47 - Contour plot of Von Mises Equivalent Stress after 2.4E-05 sec.
Figure 4.48 – Contour plot of Von Mises Equivalent Stress after 4.8E-05 sec.

Figure 4.49 - Contour plot of Von Mises Equivalent Stress after 6.4E-05 sec.
Figure 4.50 – Contour plot of Von Mises Equivalent Stress after 8E-05 sec.

Figure 4.51 - Contour plot of Von Mises Equivalent Stress after 1E-04 sec.
Acceleration waveforms resulting from the explicit analysis were collected at the receiver node (Figure 4.43) for all the five scenarios. Once post-processed with a Fast Fourier Algorithm the nonlinear parameter $\beta$ was evaluated and plotted for the considered cases. The final result corroborates the theoretically predicted variability of $\beta$ with the stress level acting in the waveguide and is presented in Figure 4.52. A U-shape trend was found reinforcing the idea of using the nonlinear parameter to track the stress level of the rail and pinpointing its neutral state (corresponding to Neutral Temperature) as minimum of the curve $\beta$ vs. Load.

![Figure 4.52 – Nonlinear parameter $\beta$ plotted against the preload state imposed to the rail model during the preliminary implicit analysis step.](image-url)
4.4.9 Conclusions

The results of the analyses carried out using ABAQUS code corroborated theoretical expectations and were pivotal in predicting the evolution of the nonlinear parameter $\beta$ as a function of the thermal stress acting in the rail. Implicit and Explicit solvers were conveniently combined to model the preload phase and the wave propagation phase, respectively. Final results suggested a U-shape trend for $\beta$ vs. stress with a minimum at the neutral state corresponding to the rail Neutral Temperature.

These findings highlighted the potential of the nonlinear parameter and reinforced the idea of using it as an indicator in order to track the rail neutral temperature in-situ in a nondestructive manner.

In the following chapter a novel physical model will be presented. It is based on fundamental concepts of molecular dynamics (interatomic potential) and will be employed to explain the origin of nonlinear wave propagation in waveguides under constrained thermal expansion.

4.5 Acknowledgements

This chapter, in part, has been published in the Mathematical Problems in Engineering Journal, Nucera, Claudio; Lanza di Scalea Francesco; (2012). The title of this paper is *Higher Harmonic Generation Analysis in Complex Waveguides via a Nonlinear Semi-Analytical Finite Element Algorithm*. The dissertation author was the primary investigator and primary author of this paper.
Chapter 5

Nonlinear thermo-elastic model for axially constrained waveguides

5.1 Introduction

Nonlinear phenomena arising in guided wave propagation have been classical treated using Acoustoelasticity (Egle and Bray, 1976) and Finite Amplitude Wave theory (de Lima and Hamilton, 2003). According to these theories, finite strains (and similarly finite amplitude waves) constitute a requirement for the occurrence of nonlinearity. Despite an initial pretension that could introduce finite deformation in the waveguide, in continuous welded rails the appearance of nonlinear effects should be explored from a different standpoint. These waveguides, in fact, are axially constrained because of the welds and they do not generally experience finite deformations, apart from the initial pretension imposed to conveniently shift their Neutral Temperature.

As detailed in next Chapter 6, nonlinear effects in terms of higher harmonic generation clearly appear in these particular structures when they are subjected to constrained thermal variations. When the rail experiences temperature changes, the structure cannot globally deform because of the boundaries but, at the same time, lattice
particles acquire an increasing energy of vibration (proportional to temperature) in agreement with classical theories of material science (Tilley, 2004).

A new constitutive model aimed at describing mathematically nonlinear phenomena in wave propagation along constrained waveguides under thermal variations is presented in this chapter. After a brief theoretical treatment, the proposed model is validated via experimental tests performed on a steel block. Results of this validation are also presented here.

### 5.2 Mie and Lennard-Jones interatomic potentials

Gustav Mie was one of the pioneers in the study of lattice properties and atomic interactions. At the beginning of the 20\textsuperscript{th} century he developed a very general mathematical framework and introduced an interatomic potential able to efficiently describe a broad variety of materials (Mie, 1903). Considering a couple of lattice particles, this potential reads:

\[
V_{\text{MIE}}(r) = \left( \frac{n}{n-m} \right) \left( \frac{n}{m} \right)^{\frac{m}{n-m}} \left[ \left( \frac{q}{r} \right)^n - \left( \frac{q}{r} \right)^m \right]
\]  

(5.1)

where \( r \) is the interatomic distance, \( w \) is the so-called “potential well depth”, \( q \) is the “van der Waals radius”, \( n \) and \( m \) are coefficients. The van der Waals radius represents the interatomic distance at which the interatomic potential is null while the potential well depth quantifies the strength of the interaction between the two atoms.

Mie potential consists of two components, a steep repulsive part (first term inside the brackets in Eq. (5.1)), and a smoother attractive part. A schematic illustration of this
potential with indication of $w$ and $q$ and attractive and repulsive branches highlighted is provided in Figure 5.1.

![Interatomic Potential Model](image)

**Figure 5.1 – Interatomic potential model proposed by Gustav Mie (Mie, 1903).**

The interatomic force exerted reciprocally by the two atoms can be calculated simply deriving the potential above with respect to the interatomic distance $r$. The result is:

$$F_{MIE}(r) = \frac{dV_{MIE}}{dr} = \frac{n \left( \frac{n}{m} \right)^{\frac{m}{n-m}} w \left[ -\left( \frac{q}{r} \right)^n + \left( \frac{q}{r} \right)^m \right]}{n-m}$$

(5.2)

Depending on the values of coefficients $n$ and $m$, several alternative formulations have been proposed to describe the interatomic potential over the years. One of the most widely used, especially in molecular dynamics, was proposed by Sir John Edward
Lennard-Jones in 1924 (Jones, 1924a, b, c). In this particular model the coefficients $n$ and $m$ take the values of 12 and 6, respectively. Using these values Eqs. (5.1) and becomes:

$$V_{LJ} = 4w\left[\left(\frac{q}{r}\right)^{12} - \left(\frac{q}{r}\right)^6\right]$$  \hspace{1cm} (5.3)

$$F_{LJ} = 4w\left[-\frac{12\sigma^{12}}{r^{13}} + \frac{6\sigma^6}{r^7}\right]$$  \hspace{1cm} (5.4)

It is worth mentioning that the Lennard-Jones model is not the most reliable representation of the interatomic potential, but its use is widespread due to its computational expediency.

### 5.3 Closed-form derivation of the Average Bonding Distance curve

From Figure 5.1, apart from the bottom of the potential well, each energy value $V_i$ corresponds to two interatomic distances, obtained intercepting the interatomic potential curve with the horizontal axis at $V = V_i$. One point lies on the repulsive branch while the other lies on the attractive branch (Figure 5.2). This couple of points is associated with two corresponding interatomic forces, one attractive and the other repulsive. The net result of the coexistence of these forces leads to a condition of equilibrium represented by the points on the average bonding distance curve. The atoms vibrate around these equilibrium states proportionally to the energy stored by the system. Referring to Figure 5.2, starting from a generic energy level $V_0$ and increasing progressively the energy dispensed to the system in form of heat (increasing the temperature), its representative
status point moves vertically towards bigger energy values $V_1$, $V_2$ and so forth. In these states there is an associated atomic energy of vibration which increases progressively with temperature as well. As a consequence, an instantaneous variation of potential and kinetic energy takes place, leaving the total energy unchanged.

![Figure 5.2 – Lennard-Jones interatomic potential model with equilibrium points and intercepts for three different energy levels highlighted.](image)

Next Figure 5.3 depicts exemplary plots of interatomic potential and interatomic force with equilibrium points highlighted assuming $w = 40$ kJ/mol and $q = 4$ Angstroms. The interatomic force is zero when the interatomic potential reaches its minimum at $r = r_0$, it is positive (attraction) and exponentially tending to 0 for interatomic distances $r > r_0$ (lattice particles gradually departing), and, finally, it is negative (repulsion) and asymptotically tending to $-\infty$ for interatomic distances $r < r_0$ (lattice particles gradually...
nearing). The latter clearly represents just a theoretical limit to guarantee compatibility conditions and continuity of matter.

Figure 5.3 – Lennard-Jones interatomic potential and interatomic force curves with equilibrium positions highlighted.

The salient aspect to be emphasized at this stage is the different physical behavior between attractive branch and repulsive branch. This difference leads to asymmetry in the interatomic potential curve with respect to the vertical axis at $r = r_0$ (passing through the minimum of energy). Because of the above asymmetry, a temperature increase generates two main effects: a proportional increase in the atomic vibrations and a shift of the interatomic distance of equilibrium towards the right in the plot. This fact explains thermal expansion in solids under a positive thermal variation and, conversely, thermal contraction under a negative thermal variation. If the interatomic potential curve was
parabolic (symmetric with respect to the vertical axis passing through its minimum) consequently the average bonding distance curve would have been a vertical line denoting invariability of the equilibrium points with increasing temperature. This theoretical case would be characterized by absence of thermal expansion/contraction, which does not correspond to reality.

The asymmetry behavior introduced above is generally referred as “Anharmonicity of the interatomic potential”. This anharmonicity is pivotal for the present treatment involving a solid (rail waveguide) which is axially constrained and thermally stressed. In this way, the dispensing of energy in form of heat translates into the acquisition of a potential which, in agreement with the discussion above, is nonlinear. Once this potential is correctly introduced in the general form of the elastic potential, higher harmonic generation in thermo-elastic constrained waveguides can be analytically described following an approach in line with the one classically used for acoustoelasticity and finite amplitude wave theory.

Considering specifically a CWR rail (the discussion, however, stands for any other constrained waveguide subjected to thermal variations), with a given starting energy level $V_1$ corresponding to an initial temperature $T_1$, the application of a positive thermal variation brings the system to a higher energy level $V_2$ corresponding to a temperature $T_2$. Being the solid constrained, in this process it stores an energy potential that evolves in a nonlinear fashion with temperature. Figure 5.4 illustrates the scenario above.

Referring to this figure, the two atoms are basically constrained to maintain the interatomic distance $r^*$ (corresponding to the initial energy level) due to the presence of
the axial constrains. Therefore, dispensing energy to the system does not change the equilibrium position (it remains at $r^*$) and the waveguide acquires an amount of energy proportional to the yellow area in figure that evolves nonlinearly with temperature.

![Figure 5.4 - Lennard-Jones interatomic potential with indication of the energy acquired by the interacting atoms in a constrained waveguide when temperature is increased from $T_1$ to $T_2$.](image)

The closed-form expression for the Lennard-Jones average bonding distance curve is derived below. Considering Eq. (5.3), the following change of variable is introduced:

$$x = \left(\frac{q}{r}\right)^6$$  \hspace{1cm} (5.5)

Using this new independent variable the interatomic potential in Eq. (5.3) can be formulated as:

$$V_{LJ}(x(q,r), w) = 4w(x^2 - x) = 4wx(x-1)$$  \hspace{1cm} (5.6)
The result is a quadratic form that can be conveniently written as (from now on $V_{LJ}$ is substituted by $V$ for simplicity):

$$x^2 - x - \frac{V}{4w} = 0 \quad (5.7)$$

The two roots of Eq. (5.7) are:

$$x_1 = \frac{1 + \sqrt{1 + \frac{V}{w}}}{2}, \quad x_2 = \frac{1 - \sqrt{1 + \frac{V}{w}}}{2} \quad (5.8)$$

Using again Eq. (5.5) the interatomic distance $r$ can be expressed as function of $x$ as:

$$r = \frac{q}{\sqrt[6]{x}} = qx^{\frac{1}{6}} \quad (5.9)$$

Consequently the two solutions in Eq. (5.8), expressed in terms of interatomic distance $r$, are:

$$r_1 = q \left( \frac{1 + \sqrt{1 + \frac{V}{w}}}{2} \right)^{\frac{1}{6}}, \quad r_2 = q \left( \frac{1 - \sqrt{1 + \frac{V}{w}}}{2} \right)^{\frac{1}{6}} \quad (5.10)$$

Solutions in Eq. (5.10) provide the two interatomic distances $r_1$ and $r_2$ intercepting the interatomic potential curve $V(r)$ for any value of interatomic potential $V$ and for a given material (fixing the value of its $w$ and $q$), as shown in Figure 5.5. The correctness of the obtained solutions can be quickly tested by calculating the interatomic distances corresponding to $V = 0$. Substituting this value in Eq. (5.10) one obtains $r_1 = q$ and $r_2 = \infty$, in accordance with the definition of van der Waals radius $q$ and the exponential tendency to zero of the interatomic potential with atoms gradually departing (Figure 5.1).
Figure 5.5 – Interatomic distances $r_1$ and $r_2$ corresponding to the line intercepting the interatomic potential curve for a generic potential value $V$ and related equilibrium point at $(r_1+r_2)/2$.

The average bonding distance curve can be obtained as geometric locus of midpoints between attractive and repulsive branches. The final result is:

$$r_{ABD} = \frac{r_1 + r_2}{2} = \frac{q}{2} \left[ \left( \frac{1 + \sqrt{1 + \frac{V}{w}}}{2} \right)^{\frac{1}{6}} + \left( \frac{1 - \sqrt{1 + \frac{V}{w}}}{2} \right)^{\frac{1}{6}} \right]$$

$$= \frac{1}{2^{\frac{5}{6}}} \left[ \left( \frac{1}{1+\sqrt{\frac{V+w}{w}}} \right)^{\frac{1}{6}} + \left( \frac{1}{1-\sqrt{\frac{V+w}{w}}} \right)^{\frac{1}{6}} \right] q$$

In Eq. (5.11) the subscript $ABD$ stands for Average Bonding Distance.
5.4 Proposed nonlinear constrained thermo-elastic waveguide model

In Eq. (5.11) the average bonding distance curve is formulated as \( r(V) \). However, in order to implement this energy contribution into the classical elastic potential energy framework and develop a new constitutive model, it is necessary to calculate the inverse function of Eq. (5.11), namely \( V(r) \). It can be noticed from Eq. (5.11) that the calculation of the inverse function, in practical terms, consists in solving an equation of twelfth order. Finding a closed-form solution for polynomials of fifth order and beyond is a long-standing mathematical problem and is generally very challenging (in fact not even possible in the majority of cases) (King, 2009). This preamble explains the vast employment of numerical methods to solve approximately the original problem with sufficient accuracy.

In the present work, MATLAB Curve Fitting Toolbox was used to calculate the average bonding distance curve expressed as \( V(r) \). Assuming exemplary values for \( q \) and \( w \) (clearly the validity of the present approach is not compromised by this assumption), namely \( q = 4 \) and \( w = 40 \), the following cubic interpolation curve was obtained (\( r \) is used in place of \( r_{ABD} \) to simplify the notation from this point on):

\[
V(r) = ar^3 + br^2 + cr + d = -1.63E-06r^3 + 3.594E-04r^2 - 3.112E-02r + 5.661
\]  

(5.12)

The precision of the interpolating function is graphically assessed in Figure 5.6. The following mathematical analysis follows the framework classically used for finite
amplitude wave theory (Kundu, 2004), conveniently modified to implement the proposed interatomic potential.

Figure 5.6 – Cubic interpolating function (formulated as \( V(r) \)) employed to invert the original average bonding distance curve (formulated as \( r(V) \)).

A 1D lattice comprising \( p \) atoms connected by nonlinear springs is considered. To characterize the treatment to constrained waveguides, the lattice is assumed to be axially fixed. Compared to traditional formulations, the novelty here interests a different spring elastic potential which takes into account the Lennard-Jones nonlinear interatomic potential (Eq. (5.12)). Assuming an infinitesimal deformation of the system from an initial equilibrium state (Figure 5.7) and introducing the nonlinear interatomic potential discussed above (only source of nonlinearity here), the overall elastic potential of the \( p \) particles can be expressed as:

\[
V = (V_0 + d) + \sum_p (k_i + c) \cdot \Delta u + \frac{1}{2!} \sum_p (k_2 + b) \cdot \Delta u^2 + \frac{1}{3!} \sum_p a \cdot \Delta u^3 + O(\Delta u^4) \quad (5.13)
\]
where $k_1$ is the first order elastic constant, $k_2$ is the second order elastic constant, $\Delta u$ is the displacement from the equilibrium condition, $a$, $b$, $c$ and $d$ are the coefficients of the cubic function interpolating the nonlinear Lennard-Jones potential (Eq. (5.12)).

The term $(V_0 + d)$ in Eq. (5.13) represents a constant that will vanish during the derivation of the equation of motion, as detailed in the following work.

Applying Newton’s second law to the $n$th particle, the differential equation governing its motion reads:

$$
\frac{d^2 u_n}{dt^2} = \frac{F_n}{m} = -\frac{dV}{du} = -(k_1 + c)\sum_p \frac{d}{du_n} (u_{p+1} - u_p) +
$$

$$
-\frac{1}{2!} (k_2 + b) \sum_p \frac{d}{du_n} (u_{p+1} - u_p)^2 - \frac{1}{3!} a \sum_p \frac{d}{du_n} (u_{p+1} - u_p)^3 + ...
$$

Eq. (5.14) can be simplified making use of the Dirac Delta function:

$$
\sum_p \left( \frac{du_{p+1}}{du_n} - \frac{du_p}{du_n} \right) = \delta_{p+1,n} - \delta_{p,n} = 0
$$

Substituting Eq. (5.15) into Eq. (5.14) leads to:

$$
\frac{d^2 u_n}{dt^2} = (k_2 + b) \left[ (u_{n+1} - u_n) - (u_n - u_{n-1}) \right] + \frac{1}{2} a \left[ (u_{n+1} - u_n)^2 - (u_n - u_{n-1})^2 \right] + ...
$$

It is possible to reformulate last equation in order to highlight the force exerted on the generic $n$th particle by particles $n+1$ and $n-1$:
\[ m \frac{d^2 u_n}{dt^2} = F_{n,n+1} - F_{n,n-1} = \left[ (k_1 + c) + (k_2 + b) \left( \frac{u_{n+1} - u_n}{h} \right) h + \frac{1}{2} a \left( \frac{u_{n+1} - u_n}{h} \right)^2 h^2 \right] + \]
\[ \left[ (k_1 + c) + (k_2 + b) \left( \frac{u_n - u_{n-1}}{h} \right) h + \frac{1}{2} a \left( \frac{u_n - u_{n-1}}{h} \right)^2 h^2 \right] + \ldots \]  

(5.17)

Auxiliary term \( h \) was introduced for convenience without altering the final result in view of the subsequent developments, described below.

Figure 5.8 - 3D Lattice of atoms connected by nonlinear springs before and after an infinitesimal deformation is imposed to the system (Kundu, 2004).

All the concepts discussed for the 1D lattice of atoms can be easily extended to the 3D case (Figure 5.8). In this scenario, everything that was applicable for the \( n \)th particle can be used for the \( n \)th plane. In order to simplify the treatment without any loss in generality, the resulting equation of motion will be characterized to the case of 1D longitudinal bulk waves along direction \( x_1 \) (therefore the following subscripts). The same reasoning could be applied to derive the governing equations for more general cases.

Introducing the unit surface \( S_1 \), perpendicular to axis \( x_1 \), the equation of motion for the \( n \)th plane becomes:
\[
\frac{m \, d^2 u_{1,n}}{S_1 \, dt^2} = \frac{F_{1,n+1}}{S_1} - \frac{F_{1,n-1}}{S_1} = \frac{(k_1 + c)}{S_1} (1-1) + \\
+ \frac{(k_2 + b)h_1}{S_1} \left[ \frac{u_{1,n+1} - u_{1,n}}{h_1} \right] - \left( \frac{u_{1,n} - u_{1,n-1}}{h_1} \right) + \\
+ \frac{1}{2} \frac{a h^2_1}{S_1} \left[ \left( \frac{u_{1,n+1} - u_{1,n}}{h_1} \right)^2 - \left( \frac{u_{1,n} - u_{1,n-1}}{h_1} \right)^2 \right] + \ldots
\] (5.18)

The passage from the discrete system to its continuum counterpart it is simply required to let the term \( h_1 \) tend to zero in Eq. (5.18). Exploiting the definition of derivative, namely:

\[
\lim_{h_1 \rightarrow 0} \frac{u_{1,n+1} - u_{1,n}}{h_1} = \lim_{h_1 \rightarrow 0} \frac{u_1(x_1 + h_1) - u_1(x_1)}{h_1} = \frac{\partial u_1}{\partial x_1} \] (5.19)

Eq. (5.18) can be rearranged as:

\[
\frac{d^2 u_{1,n}}{dt^2} = \frac{F_1(x_1)}{S_1} - \frac{F_1(x_1-h_1)}{S_1} = \sigma_{11} (x_1) - \sigma_{11} (x_1-h_1) = \\
= \frac{(k_1 + c)}{S_1} (1-1) + \frac{(k_2 + b)h_1}{S_1} \left[ \frac{\partial u_1}{\partial x_1} \right]_{x_1} - \left( \frac{\partial u_1}{\partial x_1} \right)_{x_1-h_1} + \\
+ \frac{1}{2} \frac{a h^2_1}{S_1} \left[ \left( \frac{\partial u_1}{\partial x_1} \right)^2 \right]_{x_1} - \left( \frac{\partial u_1}{\partial x_1} \right)^2_{x_1-h_1} + \ldots
\] (5.20)

At this stage three new elastic coefficients of first, second and third order are introduced. They conveniently combine the influence of the classical elastic potential with the new nonlinear interatomic potential. These coefficients are defined as:

\[
\overline{C_1} = \frac{(k_1 + c)}{S_1} \\
\overline{C_2} = \frac{(k_2 + b)h_1}{S_1} \\
\overline{C_3} = \frac{a h^2_1}{S_1}
\] (5.21)
Eq. (5.20) can be now divided by $h_1$ and the resulting expression is further manipulated, letting $h_1$ tend again to zero. Taking into account the new elastic coefficients in Eq. (5.21) and the fact that $m/(S_1 h_1) = \rho$, the final result of the above process is:

$$\rho \frac{\partial^2 u_t}{\partial t^2} = \frac{\partial \sigma_{11}}{\partial x_1} = C_2 \frac{\partial^2 u_t}{\partial x_1^2} + C_3 \left( \frac{\partial u_t}{\partial x_1} \right) \frac{\partial^2 u_t}{\partial x_1^2} + ...$$ (5.22)

The equation above represents the nonlinear partial differential equation governing the propagation of a longitudinal bulk wave in 1D for thermo-elastic axially constrained solid. In light of the above result, two new definitions are introduced for the bulk wave velocity and the nonlinear parameter, respectively:

$$V_i = \sqrt{\frac{C_2}{\rho}} = \sqrt{\frac{(k_2 + b) h_1}{\rho S_i}} \text{ WAVE SPEED}$$

$$\gamma_1 = -\frac{C_3}{C_2} = -\frac{ah_1}{(k_2 + b)} \text{ NONLINEAR PARAMETER}$$ (5.23)

It is clear from Eq. (5.23) that thermal changes coupled with axial constraints affect the speed of the propagating wave and the nonlinearity of the system via the coefficients $a$ and $b$. These are function of the material through its interatomic potential curve.

Using the definitions from Eq. (5.23), Eq. (5.22) can be conveniently arranged as:

$$\frac{\partial^2 u_t}{\partial t^2} = V_i \left[ 1 - \gamma_1 \left( \frac{\partial u_t}{\partial x_1} \right) \right] \frac{\partial^2 u_t}{\partial x_1^2}$$ (5.24)

In analogy with the classical finite amplitude wave formulation (Kundu, 2004), the constitutive equation of the system becomes:
\[
\sigma_{ij} = C_{ij}^1 + C_{ij}^2 \left[ \frac{\partial u_i}{\partial x_j} - \frac{1}{2} \gamma_i \left( \frac{\partial u_i}{\partial x_i} \right)^2 \right] + \ldots
\]  

(5.25)

In tensor notation, Eqs. (5.24)-(5.25) read:

\[
\rho \ddot{u}_i = \left[ C_{ijkl}^{ij} + C_{ijklmn}^{ij} u_{m,n} \right] u_{k,ji}
\]

(5.26)

\[
\sigma_{ij} = C_{ij}^1 + C_{ijkl}^1 u_{k,i} + C_{ijklmn}^1 u_{k,i} u_{m,n}
\]

(5.27)

where comma denotes derivation.

It is worth mentioning that the development in Eqs. (5.26)-(5.27) has been stopped to include only terms up to first order nonlinearity (quadratic nonlinearity). A similar approach is envisioned to extend the present formulation to higher-order nonlinearities.

In line with the discussion in Chapter 2, the solution of Eq. (5.24) is calculated using the perturbative approach. Therefore the solution of the governing equation can be written as:

\[
u_i(x,t) = u_i^{(1)}(x,t) + u_i^{(2)}(x,t)
\]

(5.28)

where \( u_i^{(1)} \) represents the linear part of the solution and \( u_i^{(2)} \) the nonlinear part of the solution, with \( u_i^{(2)} \ll u_i^{(1)} \) (perturbation condition).

It is assumed that at \( x_1 = 0 \) only a pure sinusoid exists. It constitutes the input signal and is defined as:

\[
u_i(x,t) = A \cos(\omega t)
\]

(5.29)

Basically, Eq. (5.29) represents the boundary condition of Eq. (5.24). The linear part of the solution can be evaluated solving the following linear second order partial differential equation:
\[
\frac{\partial^2 u_1^{(i)}}{\partial t^2} = \frac{1}{V_1^2} \frac{\partial^2 u_1^{(i)}}{\partial x_1^2}
\] (5.30)

whose solution is:

\[
u_1^{(i)} = A_i \cos(kx_1 - \omega t)
\] (5.31)

Using Eq. (5.31), the nonlinear part of the solution can be calculated solving the following linear equation:

\[
\frac{\partial^2 u_1^{(2)}}{\partial t^2} = \frac{1}{V_1^2} \frac{\partial^2 u_1^{(2)}}{\partial x_1^2} - \frac{1}{\gamma_1} \frac{\partial u_1^{(1)}}{\partial x_1} \frac{\partial^2 u_1^{(1)}}{\partial x_1^2}
\] (5.32)

Eq. (5.32) can be manipulated through the following definitions:

\[
\frac{\partial u_1^{(1)}}{\partial x_1} = -A_i k \sin(kx_1 - \omega t)
\]

\[
\frac{\partial^2 u_1^{(1)}}{\partial x_1^2} = -A_i k^2 \cos(kx_1 - \omega t)
\] (5.33)

\[
\sin(\alpha) \cos(\beta) = \frac{1}{2} \sin(\alpha + \beta) + \frac{1}{2} \sin(\alpha - \beta)
\]

Substituting the above definitions in Eq. (5.32) leads to:

\[
\frac{\partial^2 u_1^{(2)}}{\partial t^2} = \frac{1}{V_1^2} \frac{\partial^2 u_1^{(2)}}{\partial x_1^2} - \frac{1}{2} \frac{1}{\gamma_1} \frac{\partial u_1^{(1)}}{\partial x_1} A_i \sin 2(kx_1 - \omega t)
\] (5.34)

The governing equation for the nonlinear part is a linear inhomogeneous second order partial differential equation in analogy with the classical approach. The main difference when compared to Eq. (2.97) is that \(V_1^2\) and \(\gamma_1\) appear in place of \(c_L^2\) and \(\beta\), respectively.

In order to solve Eq. (5.34), the following d’Alembert solution is assumed:

\[
u_1^{(2)} = f(x_1) \sin 2(kx_1 - \omega t) + g(x_1) \cos 2(kx_1 - \omega t)
\] (5.35)
Substituting Eq. (5.35) into Eq. (5.34) and developing all the derivations in tensor form results in:

\[
\left(-\frac{4\omega^2 f}{V_1^2}\right)\sin 2(kx_1 - \omega t) + \left(-\frac{4\omega^2 g}{V_1^2}\right)\cos 2(kx_1 - \omega t) = \\
= \left(f_{,xx} - 4k^2 f - 4k g_{,x} - \frac{1}{2} \gamma_i k^3 A_i^2\right)\sin 2(kx_1 - \omega t) + \\
+ \left(4kf_{,x} + g_{,xx} - 4k^2 g\right)\cos 2(kx_1 - \omega t)
\]  

(5.36)

The definition of phase velocity for longitudinal bulk waves for the present case reads:

\[
\bar{V}_1 = \frac{\omega}{k}
\]  

(5.37)

Therefore, by equating the \(sin\) and \(cos\) coefficients on the right-hand side and left-hand side, the following system of two second order ordinary differential equations is obtained:

\[
f_{,xx} - 4k g_{,x} - \frac{1}{2} \gamma_i k^3 A_i^2 = 0
\]

\[
g_{,xx} + 4kf_{,x} = 0
\]  

(5.38)

Enforcing the functions \(f\) and \(g\) and their derivatives to be null at \(x_1 = 0\) (where just the pure sinusoid exists in accordance to the boundary conditions), the final solution was simply obtained using Mathematica symbolic package (Wolfram, 1996), as shown below:
The assembled final solution of the proposed nonlinear model is:

\[
    u_i = u_i^{(1)} + u_i^{(2)} = A_i \cos(kx_i - \omega t) - \frac{1}{8} \overline{\gamma}_i k^2 A_i^2 x_i \sin 2(kx_i - \omega t) \tag{5.39}
\]

The form of this solution is similar to the classical nonlinear solution for longitudinal bulk waves in solids with quadratic nonlinearity (de Lima and Hamilton, 2003). However, in the present case the proposed nonlinear interatomic potential is implemented and its contribution to the final solution appears through the nonlinear term \( \overline{\gamma}_i \), as defined in Eq. (5.23).

In the following paragraph, the nonlinear model just discussed is experimentally validated on a steel block subjected to heating cycles. Nonlinear ultrasonic measurements are taken at each temperature level and the effect of the interatomic potential in constrained waveguides is explored comparing the results obtained with and without axial constraints applied to the block.

In Section 5.6 an alternative theoretical formulation is proposed. A new system of partial differential equations is derived in closed-form starting from a slightly different Mie interatomic potential model.
5.5 Validation test – nonlinear bulk waves in a steel block

A series of experimental tests was conducted on a steel block in order to validate the proposed constitutive model. The influence of axial constraints (in presence of thermal variations, therefore thermal stresses) in terms of nonlinear effects in longitudinal bulk wave propagation was inspected. A steel block was used as the medium for wave propagation. It was specifically shaped to accommodate a high-temperature heating tape (Figure 5.9).

![Technical drawing with annotations of the steel block specimen.](image)

The heating tape is a flexible silicone rubber belt able to withstand temperatures from -35 °C to 230 °C. It was wrapped around the block and used to increase its temperature from ambient (22 °C) to 80 °C in a series of progressive steps via an attached
controller. A Tegam 871A handheld thermometer was installed on the top surface of the block to keep track of its temperature during the thermal test.

An Olympus C606-RB Centrascan Composite Protective Face Transducer with center frequency of 2.25 MHz was employed as transmitter. An Olympus C606-RB Centrascan Composite Protective Face Transducer with center frequency of 5.00 MHz was used as receiver on the other side of the block to gain sensitivity at the double harmonic. A high-temperature delay line was installed between the surface of the two sensors and the side surfaces of the steel block.

A National Instrument PXI-1010 DAQ system (described in detail in next Chapter 6) was used to generate a 10-cycle toneburst and acquire the response signals. The frequency of the interrogating signal was swept between 1.5 MHz and 2.5 MHz with a step of 0.25 MHz. At each temperature level, the first arrival in the received signals at each frequency was post-processed using a Fast Fourier Algorithm and nonlinear parameter was evaluated. This was finally plotted against temperature.

The test was performed twice. In one case the block was placed on two rollers and left unconstrained so that it could freely expand under the effect of temperature variations. In the other test the block was axially constrained using two specially designed steel L-brackets.

The role of these structural elements was crucial in actually blocking axial deformations during the heating process. Without this requirement the validation results would have been of marginal importance. For these reasons, the L-brackets were designed to efficiently contain the deformations due to thermal variations during the heating cycle.
Figure 5.10 – Topology optimization process performed to obtain the final bracket shape adopted in the experiments to constrain the block.

Figure 5.11 – Technical drawing with annotations for the L-bracket final design.
Starting from a preliminary shape design (consisting in a simple prismatic block) which was assumed to be the maximum physical extent of the component, a topology optimization algorithm implemented in ABAQUS finite element commercial code was used to determine a new material distribution inside the volume of the bracket. The solver was set to limit the maximum displacement to 30 μm under the effect of an increase in temperature equal to 80 °C (in reality the specimen experience an increase of 58 °C as mentioned before) and this constitutes the optimization constraint. This threshold was considered small enough to assume rigid boundaries for the block.

The optimized shape was then refined so that it could be easily manufactured and a circular hole was placed in the middle of the bracket front face to accommodate the sensors. This adopted shape was finally validated repeating the finite element analysis again with the block fixed on both sides and subjected to a thermal variation of 80 °C. Results were satisfactory showing a maximum displacement of 28.7 μm. Figure 5.10 and Figure 5.11 represent the stages of the optimization performed to obtain the final bracket shape and the technical drawing detailing its geometry, respectively.

Figure 5.12 and Figure 5.13 provide an overall view of the experimental setup for both constrained and unconstrained configurations. A thermal camera was also used during the heating cycle to ensure that the heating belt was effectively able to produce a uniform temperature distribution in the specimen volume. A screenshot acquired during the heating process is depicted in Figure 5.14.
Figure 5.12 – Experimental setup used for the unconstrained steel block test.

Figure 5.13 - Experimental setup used for the constrained steel block test.
Selected results are presented in the following for two input frequency, namely 1.75 MHz and 2 MHz. The first arrival was isolated in the received signals, as depicted in Figure 5.15 and Figure 5.16.
Once the first arrival was post-processed using a Fast Fourier Transform algorithm, the nonlinear parameter $\beta$ was evaluated for both constrained and unconstrained cases at each temperature level. In order to highlight the influence of the constraints on the wave propagation, results are compared for both test scenarios and both input frequencies in Figure 5.17.

It can be seen from these plots that when the block is freely expandable, no clear trend is observed for nonlinear parameter $\beta$ vs. temperature curve. However, in presence of the L-bracket nonlinear effects (quantified via the nonlinear parameter $\beta$) come into play and, most importantly, they actually evolve following a very regular trend with increasing temperature. This experimental evidence confirms theoretical predictions in accordance with the proposed nonlinear constitutive model for constrained waveguides.
Figure 5.17 – Nonlinear parameter $\beta$ vs. temperature for unconstrained and constrained tests and for two representative input frequencies, namely 1.75 MHz and 2 MHz.

5.6 Alternative formulation – closed-form approach

Several variants of the interatomic potential model proposed by Gustav Mie (Eq. (5.1)) have been put forward in literature assuming different combinations of values for coefficients $n$ and $m$. In this section an alternative formulation is explored. It leads to a system of partial differential equations governing the nonlinear wave propagation in
thermo-elastic constrained solids. The extension to the guided wave propagation problem is trivial (stress-free boundary conditions need to be applied at the solid outer surface).

Supposing to describe the repulsive branch via \( n = 4 \) and the attractive branch via \( m = 2 \), Mie interatomic potential and interatomic force become:

\[
V_{4-2}(r) = 4w\left[\left(\frac{q}{r}\right)^4 - \left(\frac{q}{r}\right)^2\right]
\]

\[
F_{4-2}(r) = 4w\left[-\frac{q^4}{r^5} + \frac{2q^2}{r^3}\right]
\]

The subscripts in Eq. (5.40) refers to the assumed values for coefficients \( n \) and \( m \).

When compared to Lennard-Jones formulation (Eqs. (5.3)-(5.4)), the present model slightly varies the slopes of the attractive and repulsive branches because of the different exponents.

In accordance with the mathematical development discussed in Section 5.4, the average bonding distance curve function is calculated using the following variable substitution:

\[
x = \left(\frac{q}{r}\right)^2
\]

Using this new variable the same quadratic polynomial detailed in Eq. (5.7) is obtained. Following the same steps as in Section 5.4, the final result here is:

\[
r(V) = \frac{q}{2}\left[\left(\frac{1 + \sqrt{1 + \frac{V}{w}}}{2}\right)^{\frac{1}{2}} + \left(\frac{1 - \sqrt{1 + \frac{V}{w}}}{2}\right)^{\frac{1}{2}}\right]
\]
When compared to the previous case, it can be noticed that the biggest difference is the function shown in Eq. (5.42) that is analytically invertible in closed-form. This was done using Mathematica. As a result, the new interatomic potential expressed as function of the average bonding distance curve reads:

\[ V(r) = \frac{-2r^2wq^2 - wq^4 - wq^3\sqrt{4r^2 + q^2}}{2r^4} \]  

\((5.43)\)

Implementing the interatomic potential (5.43) into the general elastic potential (in analogy to Eq. (5.13)) and following the path drew in Section 5.4, the following constitutive equation valid for 1D longitudinal bulk wave is obtained:

\[ \sigma_{11} = (\lambda + 2\mu) \frac{\partial u_1}{\partial x_1} + \frac{2wq^2}{\left( \frac{\partial u_1}{\partial x_1} \right)^3} + \frac{2wq^4}{\left( \frac{\partial u_1}{\partial x_1} \right)^5} + \frac{6wq^3}{\left( \frac{\partial u_1}{\partial x_1} + q^2 \right)^{\frac{3}{2}}} + \frac{2wq^5}{\left( \frac{\partial u_1}{\partial x_1} \right)^5 \left[ 4 \left( \frac{\partial u_1}{\partial x_1} \right)^2 + q^2 \right]^\frac{1}{2}} \]  

\((5.44)\)

On the right-hand side of Eq. (5.44), the first term side represents the linear part of the longitudinal stress while the remaining terms arise from the assumed nonlinear interatomic potential.

Developing the spatial derivative of Eq. (5.44), the equation of motion is obtained as:
\[ \rho \ddot{u}_i = \frac{\partial \sigma_{i1}}{\partial x_1} = (\lambda + 2\mu) \frac{\partial^2 u_i}{\partial x_1^2} - \frac{6wq^2}{\left( \frac{\partial u_1}{\partial x_1} \right)^4} \frac{\partial^2 u_1}{\partial x_1^2} - \frac{10wq^4}{\left( \frac{\partial u_1}{\partial x_1} \right)^6} \frac{\partial^2 u_1}{\partial x_1^2} + \]

\[ \frac{24wq^3}{\left( \frac{\partial u_1}{\partial x_1} \right)^2} \frac{\partial^2 u_i}{\partial x_1^2} - \frac{18wq^3}{\left( \frac{\partial u_1}{\partial x_1} \right)^4} \left[ \frac{4 \left( \frac{\partial u_1}{\partial x_1} \right)^2 + q^2}{\partial x_1^2} \right] \]

\[ \frac{8wq^5}{\left( \frac{\partial u_1}{\partial x_1} \right)^4} \frac{\partial^2 u_i}{\partial x_1^2} - \frac{10wq^5}{\left( \frac{\partial u_1}{\partial x_1} \right)^6} \left[ \frac{4 \left( \frac{\partial u_1}{\partial x_1} \right)^2 + q^2}{\partial x_1^2} \right] \]  

Eq. (5.45) can be conveniently rearranged making use of tensor notation and introducing the following auxiliary terms:

\[ a = \lambda + 2\mu \]
\[ b = -6wq^2 \]
\[ c = -10wq^4 \]
\[ d = -24wq^3 \]
\[ e = -18wq^3 \]
\[ f = -8wq^5 \]
\[ g = 10wq^5 \]

\[ A = \sqrt{4 \left( \frac{\partial u_1}{\partial x_1} \right)^2 + q^2} \]  

The final result is:

\[ \rho \ddot{u}_i = au_{1,xx} + b \frac{u_{1,xx}}{\left( u_{1,x} \right)^4} + c \frac{u_{1,xx}}{\left( u_{1,x} \right)^6} + d \frac{u_{1,xx}}{A^3 \cdot (u_{1,x})^2} + \]

\[ e \frac{u_{1,xx}}{A \cdot (u_{1,x})^4} + f \frac{u_{1,xx}}{A^3 \cdot (u_{1,x})^4} + g \frac{u_{1,xx}}{A \cdot (u_{1,x})^6} \]  

The solution of Eq. (5.47) is object of on-going studies.
5.7 Physical interpretation

The model proposed in the present chapter pinpoints the anharmonicity of the interatomic potential as the source of nonlinearity in solids under constrained thermal expansions. On the other side, the classical framework of nonlinear elasticity, discussed in Chapter 2, explains the origin of nonlinear effects with finite deformations and associated material nonlinearity. In CWR rails, both mechanical pretension and constrained thermal expansion take place. A combination of the two models seems to be the appropriate mathematical framework to efficiently predict nonlinear effects in CWR rails and explain the U-shape trend predicted by the simulations carried out in ABAQUS (Chapter 4). The author’s physical interpretation of nonlinear mechanisms in CWR is illustrated in Figure 5.18.

Figure 5.18 – Schematic illustration of nonlinear effects in guided waves propagating in CWR rails.
5.8 Conclusions

A new physical model, based on interatomic potential, was discussed in this chapter. It explains the origin of nonlinear wave propagation under constrained thermal expansion. In fact, where the classical physics of nonlinear wave propagation assumes finite strains, the case at hand of constrained thermal expansion is, instead, characterized by infinitesimal (ideally zero) strains. Theoretical predictions were corroborated and validated experimentally for longitudinal bulk waves propagating in a steel block that was constrained and subjected to thermal excursions. Implementation of the proposed model into ABAQUS commercial code via a specialized User Defined Material Subroutine is object of ongoing work. Once successfully coded and validated, this material model will be used to refine the numerical simulations discussed in Section 4.4, properly accounting for constrained thermal variations influence on the nonlinear parameter $\beta$.

5.9 Acknowledgements

This chapter, in part, will be submitted for publication to the Journal of the Acoustical Society of America, Nucera, Claudio; Lanza di Scalea, Francesco; (2012). The running title of this paper is *Nonlinear Wave Propagation in Constrained Solids Subjected to Thermal Loads*. The dissertation author will be the primary investigator and primary author of this paper.
Chapter 6

**RAIL-NT System Development**

### 6.1 Introduction

The present chapter describes the development of a rail inspection prototype aimed at nondestructively assessing rail Neutral Temperature and incipient buckling conditions. The practical system implementation is discussed first. Then the results of several large-scale proof-of-concept experimental tests are presented in order to corroborate numerical findings and theoretical predictions described in Chapters 4-5. Finally a prototype system is presented and illustrated in detail.

### 6.2 **RAIL-NT System implementation**

The proposed system (defined *RAIL-NT*) is designed to work by nondestructively measuring the nonlinearity arising in ultrasonic guided waves propagating along the rail running direction, at specific guided wave modes and guided wave frequencies. As discussed in Chapter 5, the ultrasonic nonlinearity strongly depends on the level of thermal stress acting in the rail as well as on finite amplitude strains, contact and other classical origins widely investigated in the past. In turn, the level of thermal stress, $\sigma$, in addition to the rail temperature $T$, provides with the Neutral Temperature $T_N$ through Eq. (1.1).
From a practical standpoint, the ultrasonic nonlinearity is measured via higher-harmonic generation. In this case an ultrasonic monochromatic signal generated in the rail at a fixed frequency, $f$ (fundamental frequency) generates nonlinearity that manifests itself in the generation of multiple harmonics of $f$, e.g. $2f$ (second harmonic), $3f$ (third harmonic), $nf$ ($n^{th}$ harmonic). The ultrasonic relative nonlinear parameter, $\beta'$ can be then evaluated simply from the amplitude in the frequency domain of these harmonics, normalized by the amplitude of the fundamental frequency $A(f_1)$, as

$$\beta' = \frac{A(nf)}{A(f)} \quad \text{for} \quad n = 2, 3, ..., N$$

(5.48)

In another version, the relative nonlinear parameter $\beta'$ can be computed by normalizing to the first power of the fundamental, i.e.

$$\beta' = \frac{A(nf)}{A(f)} \quad \text{for} \quad n = 2, 3, ..., N$$

(5.49)

Normally the amplitudes $A$ are simply the Fourier Transform magnitude values of the received signals at the corresponding frequencies.

Alternatively the nonlinearity of the waveguide (rail) can be quantified measuring the modulation of the interrogating guided wave mode by a low-frequency vibration (Guyer and Johnson, 1999; Van den Abeele et al., 2000a; Van den Abeele et al., 2000b; Van Den Abeele et al., 2001). This technique is referred in literature as Nonlinear Wave Modulation Spectroscopy (NWMS) and is schematically illustrated in Figure 6.1.

In this case a low-frequency vibration centered at $f_1$ and an interrogating high-frequency ultrasonic guided wave centered at $f_2$ are simultaneously conveyed into the rail. Apart from higher harmonic generation like the previous approach ($nf_1$ and $nf_2$ with $n = 2,$
3, ..., N), the Fourier transformation discussed in this part reveals “frequency mixing terms”, specifically the sum frequency \((f_1 + f_2)\) and the difference frequency \((f_1 - f_2)\). In this case the nonlinear parameter \(\beta'\) can be calculated from the amplitude of the frequency mixed term, \(A(f_1 + f_2)\) or \(A(f_1 - f_2)\), usually normalized by the higher of the two fundamental frequencies, \(f_2\):

\[
\beta' = \frac{A(f_1 + f_2)}{A(f_2)} \quad \text{or} \quad \beta' = \frac{A(f_1 - f_2)}{A(f_2)}
\]

(5.50)

As in the previous approach, the amplitudes \(A\) are simply extracted from the Fourier Transform magnitude values of the received signals at the corresponding frequencies.

![Schematic of nonlinear frequency mixing phenomenon.](image)

Higher harmonic generation is the preferred approach and has been considered in detail in the present dissertation. In a practical rail testing, two types of implementation of the proposed method could exist: a stationary “way-side” implementation and an “in-
motion” implementation. In the following these two flavors are briefly discussed. The stationary wayside installation is the one adopted in the present research.

6.2.1 Stationary wayside implementation

The wayside type represents the preferred installation type and will be discussed in detail in the subsequent section. In this typology the ultrasonic transducers are attached to a fixed position of the rail (either by gluing or by magnetic mounting). These would typically be piezoelectric transducers. The transducers can be attached to either the rail web (along the neutral axis of the rail) or to the rail head (on the field side or on the top of the head in the case of magnetic mounting for quick removal prior to the passing of a train).

These two potential installations are schematically depicted in Figure 6.2 and Figure 6.3 and illustrated as 3D renders in Figure 6.4 and Figure 6.5. The transmitting transducer generates a guided wave of specific wave mode and specific fundamental frequency $f$. The preferred generation signal is a high-voltage one (~ +/- 600 V typical) to highlight the ultrasonic nonlinearity due to the thermal stress. The receiving transducer receives the waves, performs a Fourier Transform in the frequency domain, and extracts the amplitudes of the fundamental frequency $f$ and those of the higher-harmonics (most typically the second harmonic $2f$).

The relative nonlinear parameter $\beta'$ is then calculated from the appropriate Eqs. (5.48) or (5.49). Concurrently, the rail temperature, $T$, is measured by either a thermocouple or a remote sensor (e.g. Infra-Red IR sensor). The measurement of $\beta'$ and $T$ are performed at different times of the day to allow the rail to go through the state of zero
stress or Neutral Temperature. The plot of $\langle \beta' \rangle$ vs. $T$ is then recorded: the minimum of that plot corresponds to the zero-stress point or Neutral Temperature.

![Diagram](image1)

Figure 6.2 - Schematic of nonlinear ultrasonic measurements to determine rail thermal stresses or rail Neutral Temperature. Wayside implementation on the rail web.

![Diagram](image2)

Figure 6.3 - Possible variations of the wayside implementation with sensor installation on the rail head.
In another possible version of the wayside implementation, the rail Neutral Temperature can be estimated “instantaneously” by a single $\beta'$ measurement at a single rail temperature $T$. In order to accomplish this goal, the curve of $<\beta'$ vs. $T>$ will likely
have to be previously calibrated for a given type of rail and given rail manufacturer. Since, as detailed in the next section, the $\beta'$ parameter cannot distinguish tension stress vs. compression stress (i.e. curve is symmetric around the minimum), a concurrent measurement of wave velocity by a simple time-of-flight measurement can be recorded using the same transducer. The wave velocity, which also changes with stress, will then indicate the sign of the stress, i.e. the side of the $<\beta'$ vs. $T>$ curve of that point of the rail.

Knowledge of the current thermal stress, $\beta'$, and temperature, $T$, will finally allow to determine the rail Neutral Temperature from Eq. (1.1).

### 6.2.2 In-motion implementation

In addition to the wayside “stationary” implementation, an “in-motion” implementation could be envisaged. In the “in-motion” implementation, the ultrasonic transducers are allowed to move relative to the rail. These transducers can be then either of a non-contact nature (Electro-Magnetic Acoustic Transducers – EMATs, laser-based, air-coupled transducers) or of a contact nature (wheel-based or sled-based transducers).

The wheel-based approach would be the most practical because common rail inspection systems are based on this solution. A possible configuration is shown in Figure 6.6 and Figure 6.7.
Figure 6.6 - Schematic of nonlinear ultrasonic measurements to determine rail thermal stresses or rail Neutral Temperature. In-motion implementation with ultrasonic wheel transducers.

Figure 6.7 – Possible in-motion system implementation.
The transducers would be oriented in the wheels at a specific angle that, through Snell’s law of refraction (Achenbach, 1973), generates a specific guided wave mode in the rail. As in the wayside implementation, the higher harmonic generation approach could be used to extract the nonlinear parameter beta related to the thermal stress. The results would be similar to those obtained in the wayside implementation, with a minimum of $\beta'$ values corresponding to the zero-stress temperature (or rail Neutral Temperature).

As in the wayside implementation, for an “instantaneous” indication of the rail Neutral Temperature from a single measurement point, the beta value could be correlated to a velocity value to determine the sign of the thermal stress at that specific section of rail. Calibration curves of $<\beta'$ vs. thermal stress>, previously determined on rail sections of different types and from different manufacturers, may need to be used for the determination of absolute thermal stress level.

6.3 Proof-of-principle experimental investigations

Large-scale Experimental investigations of CWR buckling occurrence firstly appeared in the 1930’s in Europe (Ammann and Gruenewald, 1932; Nemcsek, 1933). In these tests hydraulic jacks were used to induce compression forces in the rails of a track. In later track buckling research studies axial forces were induced by electrical heating. These test setups consisted of a track section whose movements were constrained at both ends by two heavy concrete piers (Bartlett, 1960; Birmann and Raab, 1960; Bromberg, 1966) or by locomotives that were placed on both ends of the test section (Nemesdy,
1960; Numata, 1960; Prud'homme and Janin, 1969). A comprehensive literature review on these early buckling detection experimental trials was provided by Kerr in 1978 (Kerr, 1978a).

The above studies have focused on the buckling behavior of CWR (vertical or horizontal buckling, effect of ties and ballast on buckling load, effect of imperfections). In an effort to confirm numerical results (Chapter 4) and theoretical predictions (Chapter 5), and pave the way for a practical system implementation, a unique test-bed has been constructed at UCSD Powell Structural Laboratories (Figure 6.8 - among the largest laboratories in the US for structural testing) under the sponsorship of a Federal Railroad Administration Office of Research and Development grant. In contrast to previous studies, the focus of the present experimental investigations is on effect of thermally-induced load on dynamic behavior of CWR. In particular, the thermal effect on nonlinear signatures of the rail response (higher harmonic generation) is of prominent importance here.

Figure 6.8 – Powell Structural Laboratories at University of California San Diego.
6.3.1 Experimental setup

The experimental setup features a 70-ft long full AREMA 136 RE continuous welded track and a freely expandable rail section (same type) placed in the middle and used to analyze the temperature influence alone (without mechanical and thermal stress). Materials for the test-bed and know-how for design and construction were donated in part by Burlington Northern Santa Fe Railway Company (BNSF - http://www.bnsf.com/). Volpe National Transportation Systems Center (http://www.volpe.dot.gov/) participated with technical advice.

Figure 6.9 - Large-scale experimental setup at UCSD Powell Structural Laboratories - rendered isometric view with descriptors.
Figure 6.10 – Large-scale experimental setup at UCSD Powell Structural Laboratories - rendered plan view.

Figure 6.11 - Large-scale experimental setup at UCSD Powell Structural Laboratories – technical drawings and details.
Figure 6.9 represents a rendered isometric view with descriptions of the various experimental layout components while Figure 6.10 provides a rendered plan view of the testing area. A technical illustration of the test layout is shown in Figure 6.11 through an elevation view, a plan view and a detail view on the test-bed cross-section. A laser positioning system was employed for sleepers alignment (Figure 6.12)

Figure 6.12 – Sleepers placement and alignment using a laser positioning system.

To cover the whole length of the test-bed, two rail sections on each side had to be jointed using exothermic welding. This process, depicted in Figure 6.13, employs an

Figure 6.13 – Exothermal welding of adjacent rail sections.
exothermic reaction of a copper thermite composition to heat the copper and permanently join the two adjacent rail sections.

Two post-tensioned concrete blocks were placed at the rail ends and used to apply an initial pretension level and hold the rail in place. After concrete cured, one of them (Figure 6.14) was post-tensioned and rigidly connected to the strong floor using steel rebars. Steel rollers were placed underneath the other concrete block so that it could slide (Figure 6.15).

Two hydraulic actuators (Vickers Actuator Products Inc., Decatur, AL) with 500 kips capacity and 48 inch stroke were installed against this sliding block (Figure 6.16) and were used to apply an initial pretension of 26.20 MPa (3.8 ksi) in the rails.

Figure 6.14 – Fixed post-tensioned concrete block. Rendered view, rebars layout and technical drawing.
The installation temperature was of 21.1 °C (70 °F). Therefore, taking into account material and geometrical properties of rail steel (described in Chapter 4) and using Eq. (1.1), the applied pretension was such to have the Neutral Temperature at around 32.2 °C (90 °F).
Special U-channels and end plates were welded at rail ends in order to improve the shear transfer in the concrete blocks. These structural elements are highlighted in next Figure 6.17.

A specially designed flexible rail switch heater wire produced by Thermal-Flex Systems Inc. (http://tflexsys.com/) was installed to simulate increasing compression load in the rail (Figure 6.18). The heating element is encased in a flexible, spiral fluted, watertight tube. The tube, or rail heater, is positioned at the neutral axis on the field side (external) of the rail. A controller (also shown in Figure 6.18) is used to vary the heating power of the system and develop different heating paths. The rail heater is then covered by a containment channel and the completed assembly is held in place by spring steel track clips (Figure 6.19).
Starting from the initial pretensioned state (at ambient temperature), several heating tests with various sensors layouts (discussed in the following) were performed
increasing progressively the rail temperature in successive steps via the heating system.

In this way, repeated ultrasonic nonlinear measurements were systematically recorded at each temperature step, passing through the neutral temperature state (Figure 6.20).

![Figure 6.20 – Thermal test protocol. Ultrasonic nonlinear features recorded at each measurement point during the heating cycle.](image)

Following this particular test protocol it was possible to efficiently analyze the evolution of the nonlinear response (in terms of nonlinear parameter $\beta'$) as a function of the state of thermal stress acting in the rail.

The rail was heavily instrumented in order to explore efficiently and exhaustively the full static and dynamic response of the track. More specifically, 48 self-temperature compensated strain gages (Ajovalasit, 2008) and 6 linear potentiometers were employed to monitor in real time strains and displacements, respectively. Temperature was recorded using 6 thermocouples installed at the rail neutral axis. An infrared thermal camera (Flir Systems A320 - [http://www.flir.com/US/](http://www.flir.com/US/)) was used to map graphically the temperature
distribution during heating cycles. Figure 6.22 shows an overview of the experimental setup.

Figure 6.21 – Experimental instrumentation description and layout.

Figure 6.22 – Overall view of the experimental setup (UCSD Powell Structural laboratories).
6.3.2 Data acquisition system

A National Instrument PXI-1010 Chassis was used as part of the data acquisition (DAQ) system. A schematic representation of its front and rear sides is provided in Figure 6.23 and Figure 6.24, respectively. The interrogating signals generally consisted in narrowband modulated sinusoids centered at a particular frequency using specific window functions. These waveforms were generated through a National Instrument PXI-5411 High-Speed Arbitrary Waveform Generator (Figure 6.25). It includes all the features of sweep generators and function generators and features a 40 MS/s waveform update rate, linking and looping capabilities and up to 8 million samples of standard waveform memory per channel.

Figure 6.23 – NI PXI-1010 chassis – Front view with descriptors.
A National Instrument PXI-5105 High-Speed Digitizer was installed in the chassis and served as acquisition module (Figure 6.26). This high-resolution digitizer is very convenient for the present application since it features eight 60 MS/s simultaneously sampled input channels with 12-bit resolution, 60 MHz bandwidth, and 16 MB of internal memory.
A high-power gated amplifier (RITEC GA2500) was used to increase the interrogating wave energy and, consequently, maximize the nonlinear response and improve the sensitivity to nonlinear parameters.
A 48 channels instrumentation cabinet was used to monitor and record all the data acquired by the array of sensors installed on the rails. Figure 6.27 shows the assembled DAQ system while Figure 6.28 depicts the control station and the instrumentation cabinet. A signal sampler (RITEC SS-40) with 40 dB of attenuation was installed after the high-power amplifier in order to monitor the amplified interrogating waveform in a separate oscilloscope (LeCroy WaveJet 314) (Figure 6.29). The arbitrary waveform generator served also a trigger for the oscilloscope and for the digitizer.
The instrumentation layout is schematically detailed in Figure 6.30. An ad-hoc software was programmed in LabVIEW environment (Figure 6.31) to control, execute and manage signal generation and acquisition processes.

In Figure 6.30 both the possible wayside stationary implementations are included. However, as it has already been mentioned, in the present work the prototype development was focused on the web implementation. This solution (discussed in next section) is more convenient because it does not interfere at all with trains operation. For the sake of completeness, results for both possible wayside installations are reported,
highlighting the potential of the proposed technique, especially in view of a possible future in-motion implementation where, most likely, the rail head will be excited.

Figure 6.31 – LabVIEW program used to control waveform generation and signal acquisition phases. (a) Front panel. (b) Block diagram.

6.3.3 Experimental results for rail head implementation

In this specific layout, three ultrasonic transducers were installed at the top of the rail head, according to the layout in Figure 6.3c (with two receivers rather than one receiver). One of them was used as exciter and the other two as receivers at two different locations.
Numerical predictions discussed in section 4.3.2 were exploited in this experimental investigation. Therefore 80 kHz was chosen as primary frequency for the interrogating signal. One PAC (Physical Acoustics Corporation - www.pacndt.com) R6α and two R15α ultrasonic piezoelectric transducers were employed as transmitter and receivers, respectively, because of their particular frequency response spectrum (Figure 6.32 and Figure 6.33). From these figures it can be noticed how the frequency response spectrum was very beneficial for frequencies of 80 kHz (fundamental frequency) and 160 kHz (double harmonic).

Figure 6.32 – Physical Acoustics R6α ultrasonic transducer frequency response spectrum (Calibration based on ASTM E1106 in blue and Calibration based on ASTM E976 in red) with fundamental frequency range highlighted.

Figure 6.33 – Physical Acoustics R15α ultrasonic transducer frequency response spectrum (Calibration based on ASTM E1106 in blue and Calibration based on ASTM E976 in red) with second harmonic range highlighted.
The two ultrasonic receivers were installed on the rail at 25.5’ and 50’ away from the exciter using specially designed magnetic holders (Figure 6.34).

Several experimental tests were performed on this layout according to the aforementioned test protocol (Figure 6.20). After raw waveforms were acquired and processed in the frequency domain using a Fast Fourier Transform algorithm, the evolution of the nonlinear parameter (defined from now on simply $\beta$ instead of $\beta'$) with temperature (hence with longitudinal thermal strain) was evaluated. A typical result is illustrated in Figure 6.35. It corroborates numerical/theoretical predictions discussed...
before showing a “U-Shape” of the $\beta$ curve when plotted against longitudinal thermal strains measured in the rail by the strain gages. More importantly, this curve exhibits a minimum precisely corresponding to the zero-strain point as measured by the strain gages (or, equivalently, the zero-stress point or rail NT). The obtained result highlights the very promising sensitivity of the nonlinear parameter $\beta$ to thermal stress variations and its excellent suitability for tracking the neutral temperature in CWRs.

![Figure 6.35](image)

**Figure 6.35** – Nonlinear parameter $\beta$ (quantifying second harmonic generation) measured on the large-scale rail test-bed using the wayside configuration with transducers installed on the rail head.

Next Figure 6.36 stresses the importance of tracking the correct combinations of waveguide modes to gain efficiency when applying the proposed nonlinear guided waves technique. This figure displays the time-history responses for the two ultrasonic receivers and the relative nonlinear parameter curves plotted against longitudinal thermal strain. It can be noticed that despite the existence of numerous waveguide modes (isolated using different windows, labeled as $W_i$ with $i = 1…7$ in the same figure), just few of them (one in the present case) are generally able to meet internal resonance requirements and produce, consequently, a cumulative nonlinear response which grows with distance of
propagation. This cumulative effect is apparent in the present case. In fact, a factor slightly bigger than 2 correlates the maximum nonlinear parameter amplitude at the two different locations, being the further receiver installed at double the distance than the closer one.

![Figure 6.36](image)

**Figure 6.36** – Second harmonic generation and cumulative effect measured on the large-scale rail test-bed using the wayside configuration with transducers on the rail head. (a) Time-history signal for receiver #1 (25.5’ away from transmitter). (b) Nonlinear parameter curve against longitudinal thermal strain for receiver #1 (highlighted mode). (c) Time-history signal for receiver #2 (50’ away from transmitter). (d) Nonlinear parameter curve against longitudinal thermal strain for receiver #2 (highlighted mode).

### 6.3.4 Experimental results for rail web implementation

In view of a practical web implementation, the proposed technique was further investigated installing the transmitter and the receivers on the rail web at the neutral axis location, according to the positions shown in Figure 6.2. The two receivers were placed at 12.75’ and 25.5’ away from the transmitter, respectively (Figure 6.37).
The optimal combination of resonant waveguide modes discussed in section 4.3.3 was exploited in this test and 200 kHz was considered as fundamental frequency of the interrogating waveform. The Physical Acoustics R15 ultrasonic transducers were used for both transmission and reception. Their frequency response spectrum (depending on the adopted calibration procedure) shows a slight decrease in response amplitude at 200 kHz and 400 kHz (Figure 6.38). However, this was not a critical issue for the experimental investigation.
Experimental results confirmed the general trend previously discussed for the head installation. However, the web installation requires different waveguide modes and frequencies than the head installation. The selection of the correct modes to be used played a decisive role for this case too. Exemplary results are illustrated in Figure 6.39 in the same fashion as for the head installation case discussed before. It can be noticed how the nonlinearity associated with the third waveguide mode (highlighted in Figure 6.39a and Figure 6.39c) is very efficient in tracking the rail thermal stress state. Also with the present system implementation, the nonlinearity parameter $\beta$ evolves following a U-shape curve against longitudinal thermal strain and its minimum pinpoints very precisely the zero stress state (neutral temperature).

Figure 6.39 - Second harmonic generation measured on the large-scale rail test-bed using the wayside configuration with transducers on the rail web. (a) Time-history signal for receiver #1 (12.75’ away from transmitter). (b) Nonlinear parameter curve against longitudinal thermal strain for receiver #1 (highlighted mode). (c) Time-history signal for receiver #2 (25.5’ away from transmitter). (b) Nonlinear parameter curve against longitudinal thermal strain for receiver #2 (highlighted mode).
6.3.5 Temperature influence analysis – free rail test

Many operational parameters and environmental conditions can negatively affect the measurements and, consequently, the efficiency of the proposed methodology. Among them, temperature influence on waveguide material characteristics is the most critical. A recent study (Loveday and Wilcox, 2010) explored the sensitivity of guided wave modes to axial load and changes in the elastic modulus due to temperature. After a comprehensive analytical and computational treatment, the authors concluded that temperature influence on guided wave propagation properties (via elastic modulus changes) was one order of magnitude larger than the influence of axial load in terms of acoustoelastic effect (Egle and Bray, 1976).

In order to confirm the insensitivity of the proposed technique to temperature variations alone, an unconstrained (freely expandable along the running direction) AREMA 136 RE rail was installed in the middle of the two constrained rails (Figure 6.9). The web installation setup previously discussed was reproduced (Figure 6.40) and nonlinear measurements were performed during several heating cycles. The rail was supported by steel rollers. A layer of Teflon was guaranteed underneath each roller to conveniently reduce friction (Figure 6.41).

Results are presented in Figure 6.42. No clear trend is evident between nonlinear parameter $\beta$ and temperature or, proportionally, longitudinal thermal strain. This outcome indicates that $\beta$ is effectively related to thermal stress and not on side effects of temperature alone.
Figure 6.40 – Ultrasonic transducers installed on the free rail web.

Figure 6.41 – Support details for the free rail test.

Figure 6.42 – Results of experimental tests on unconstrained rail. Nonlinear parameter $\beta$ vs. temperature plots for both receivers.
6.3.6 Repeatability

Repeatability was assessed collecting experimental measurements at different locations in both rails. Selected results are presented in Figure 6.43.

![Figure 6.43 – Test results repeatability assessment. Nonlinear parameter curves evaluated at two different locations of the large-scale test-bed.](image)

Precision in the previous figure was calculated assuming the coefficient of thermal expansion for the rail steel as $\alpha = 6.45 \, \mu\varepsilon/\degree F$. The results above emphasize the reliability of the proposed nondestructive technique in efficiently and reliably pinpointing the rail NT. Nonlinear measurements, in fact, appear very similar and still precise when performed at different locations involving different rails.
6.3.7 Validation tests on a plate extracted from rail web

A series of additional experimental tests was carried out on a plate extracted from the web of an AREMA 136 RE railroad track (Figure 6.44) to further corroborate the proposed technique in a more controlled laboratory setup and inspect separately the influence of thermal and mechanical stresses on the nonlinearity of the response.

![Figure 6.44 – Geometrical details of the plate extracted from the rail web and used for validation tests.](image)

The web-plate was subjected to both mechanical and thermal tests. Furthermore, the influence of an initial mechanical pretension was studied. More specifically, four validation studies were performed to explore the conditions above. They are described in detail in the following sections. In terms of sensors, frequencies, DAQ system and other similar devices, the same instrumentation apparatus discussed before for the large-scale rail testing with web installation was employed also in these validation tests.
6.3.7.1 Validation test I – Mechanical stress only with pretension

In the first validation test, an MTS hydraulic tensile machine (force capacity = 500 kN, max operating pressure = 62 MPa and max pressure of installation = 69 MPa) was employed to apply a mechanical pretension to the plate specimen and to progressively change the load level (passing through the zero stress state) until reaching a compressive load level equal in magnitude to the initial pretension level. According to the test protocol, the MTS machine was used to apply axial load from 133.44 kN (30 kips) to -133.44 kN (-30 kips) in increments of 1 kips (4.45 kN) for a total of 61 steps. Nonlinear guided wave measurements were taken at each load level.

Figure 6.45 – Validation test I experimental layout (Mechanical stress only with pretension).
Figure 6.46 shows a typical result. In accordance to the classical finite amplitude wave theory, U-shape trends were found for nonlinear parameter vs. longitudinal strain curves, with a minimum in proximity of the rail Neutral Temperature.

6.3.7.2 Validation test II – Thermal stress only with pretension

In this case an initial mechanical pretension of 133.44 kN was applied again and the heating belt (described in section 5.4) was used to gradually build up thermal stress, passing through Neutral Temperature in analogy to Validation test I. Like the previous case, nonlinear ultrasonic measurements were acquired and processed at each temperature level. An exemplary result is depicted in Figure 6.48.
Validation test II experimental layout (Thermal stress only with pretension).

Figure 6.47

Nonlinear parameter $\beta$ vs. longitudinal thermal strain curve for Validation test II.

Figure 6.48
Theoretical Rail NT was calculated using Eq. (1.1) and considering rail steel material properties discussed before and the initial pretension state. In the present test nonlinearity arises from both the initial mechanical pretension and the evolution of the thermal stress, in agreement with the proposed theoretical constitutive model (Chapter 5).

The web-plate was finally tested in absence of initial mechanical pretension. Is so doing, the thermal stress influence on nonlinear guided wave propagation could be analyzed separately and a further confirmation of the proposed constitutive model was obtained. Experimental results are discussed in the following sections. In one case the plate was unconstrained and it could freely expand under thermal variations. In the last case the plate was axially constrained and nonlinear parameter evolution was studied during a final heating cycle.

6.3.7.3 Validation test III – Unconstrained plate without pretension

Here the web-plate was placed on three rollers and left axially unconstrained. The heating belt was deployed along a twisting pattern around the plate and was used to progressively increase its temperature from ambient (21 °C) to 80 °C. Nonlinear ultrasonic measurements were acquired every 3 °C. The experimental layout is represented in Figure 6.49.

Selected results considering a fundamental frequency of 240 kHz (second harmonic at 480 kHz) and a specific waveguide mode are illustrated in Figure 6.50. As expected, nonlinear parameter $\beta$ evolves following an irregular behavior with no clear trend with increasing temperature. In fact, nonlinear sources are absent in this particular case.
Figure 6.49 - Validation test III experimental layout (Unconstrained plate without pretension).

Figure 6.50 - Nonlinear parameter $\beta$ vs. longitudinal thermal strain curve for Validation test III.
6.3.7.4 Validation test IV – Axially constrained plate without pretension

In contrast to Validation test III, in this last experiment the axial deformation due to thermal changes induced by the heating belt is constrained by two rigid L-brackets (described in Section 5.4). The experimental setup is shown in Figure 6.51.

![Figure 6.51 – Validation test IV experimental layout (Axially constrained plate without pretension).](image)

In Figure 6.52 results are presented for the same frequency and waveguide mode to highlight the difference with the previous case. It is clear how the presence of the boundaries translates into thermal stress which, in turn, generates nonlinearity in the response (according to the proposed theoretical model discussed in Chapter 5). This nonlinearity, as expected, increases quite smoothly with increasing temperature. The clear difference in trend between Figure 6.50 (unconstrained plate) and Figure 6.52 (constrained plate), confirms theoretical predictions and experimental findings discussed in previous sections.
6.4 Discussion

The results of the validation tests confirmed the suitability of the nonlinear parameter $\beta$ in effectively and efficiently mapping the zero stress state of structural elements subjected to thermal and/or mechanical stresses. Through Validation test III, it was also confirmed that temperature effects alone do not affect the proposed technique.
6.5 *RAIL-NT* prototype design

6.5.1 Introduction

This section describes the development stages and actual prototyping of the proposed rail inspection system (*RAIL-NT*) aimed at nondestructively determining the longitudinal forces (or stresses) in the rail as a function of changing rail temperature. The system is designed to be magnetically installed on the rail web, according to the layout shown in Figure 6.2. The prototype technology embeds theoretical predictions and computational results presented in Chapters 4-5, and experimental findings discussed in the first paragraphs of the present chapter.

6.5.2 Hardware

The *RAIL-NT* system prototype features several instrumentations which were assembled in analogy to the system employed for the proof-of-principle testing, presented in Section 6.3. The original setup concept was conveniently modified and optimized to gain portability in view of field deployment.

A National Instrument PXI-1033 5-slot chassis with integrated MXI-Express controller (Figure 6.53) was used to accommodate the DAQ system components. A schematic representation of its front and rear sides is provided in Figure 6.54 and Figure 6.55, respectively.

Like in the large-scale testbed experiments, a National Instrument PXI-5411 High-Speed Arbitrary Waveform (Figure 6.25) was used to generate the interrogating
waveforms and a National Instrument PXI-5105 High-Speed Digitized served as acquisition module (Figure 6.26).

Figure 6.53 – NI PXI-1033 chassis with integrated MXI-Express controller and 34-mm Express Card.

Figure 6.54 – NI PXI-1033 chassis – Front view with descriptors.
A Piezosystem EPA-104 amplifier (Figure 6.56) was installed to raise the energy of the interrogating signal. It is a single channel, high voltage (± 200 Vp), high current (± 200 mA), and high frequency (DC to 250 kHz).

A laptop computer was interfaced with the chassis using its integrated MXI-Express controller and a 34-mm express card and it was used to control and manage the data acquisition, storing and post-processing.

Two ultrasonic transducers were used as transmitter and receiver, respectively.
The two transducers were embodied into a specially designed holder. It features a perforated encasement to protect the assembly and guarantee sufficient air circulation (in order to avoid excessive heat inside the enclosed chamber that could negatively affect transducers functioning) and two magnetic holders to accommodate both transmitter and receiver and magnetically hold the prototype in place. Two rails were hollowed to allow receiver repositioning if needed. With this setup, the prototype can be easily and quickly installed on the rail web. Figure 6.57 illustrates a 3D rendered views of the RAIL-NT prototype front and rear side with descriptors. Figure 6.58 shows a picture of the actual prototype.
Figure 6.57 – 3D rendered view of RAIL-NT prototype front side.

Figure 6.58 – RAIL-NT prototype top view.
6.5.3 Software

The prototype operation is controlled by a special software programmed in LabVIEW. Three panels provide all the necessary settings for tuning the interrogating signal, digitizing the acquired signals and controlling the test execution.

A timing mechanism was also implemented so that unattended automatic testing could also be performed. This is a very useful feature especially in view of a potential testing protocol, involving the development of overnight measurement to experience a temperature variation sufficiently big to pass through neutral temperature.

6.5.4 System deployment

Once installed on the rail web, RAIL-NT system operation develops generating a high-power windowed toneburst tuned at a specific frequency in order to excite a specific waveguide mode (selected in light of preliminary wave propagation modeling using CO.NO.SAFE algorithm, as discussed in Chapter 4). The propagating signal is picked up by the ultrasonic receiver and post-processed in the laptop unit. Nonlinear parameter $\beta$ is then evaluated. This process is repeated at regular intervals under a thermal variation large enough to pass through rail NT. For this reason, a testing protocol involving overnight measurements is envisioned.

The experimental setup is schematically illustrated in Figure 6.59. Figure 6.60 presents a 3D rendered view of RAIL-NT prototype installed on a AREMA 136 RE railroad track.
Figure 6.59 – Schematic of RAIL-NT system setup.

Figure 6.60 – 3D Render of RAIL-NT prototype installed on rail web.
Figure 6.61 shows the RAIL-NT DAQ system assembled during the proof-of-principle experimental investigations. The actual RAIL-NT prototype installed on the experimental rail web is depicted in Figure 6.62.

Figure 6.61 – RAIL-NT prototype installed on the experimental rail (UCSD Powell Laboratories).

Figure 6.62 – RAIL-NT DAQ system assembled during proof-of-principle experimental tests.
A first field test for the proposed technology is being planned for the summer of 2012 in coordination with Federal Railroad Administration, BNSF Railway Company and Volpe National Transportation Systems Center. This test will be performed on a 50’ long experimental CWR section at Transportation Technology Center, Inc. (TTCI) in Pueblo, CO (Figure 6.63).

![Transportation Technology Center facility, Pueblo, CO.](image)

**Figure 6.63** – Transportation Technology Center facility, Pueblo, CO.

### 6.6 Conclusions

The experimental results obtained with the prototype in the Large-Scale Test-bed are extremely encouraging, showing an accuracy of only a few degrees for the determination of the rail NT. If confirmed in the field, this result could revolutionize the
way CWR are maintained to prevent rail buckling with respect to the thermal stress management problem.

A potential future vision of a field deployment for the proposed system could consist in a series of inspection devices installed on the rail web at distributed locations to create a sensor network. This array could perform continuous nonlinear measurements and map the rail Neutral Temperature for the various rail sections (Figure 6.64). In doing so, dangerous sections could be easily pinpointed and necessary remedial actions could be consequently taken to prevent buckling occurrence.

Figure 6.64 – Potential future vision of RAIL-NT system field deployment.
6.7 Acknowledgements

A Provisional Patent Application has been filed for the proposed inspection system on 11/10/2011 (USPTO #61/558353).

This chapter, in part, will be submitted for publication to the Structural Health Monitoring Journal, Nucera, Claudio; Lanza di Scalea, Francesco; (2012). The running title of this paper is Measurement of Neutral Temperature in Continuous Welded Rails: Results from UCSD Large-Scale Rail NT Test-bed. The dissertation author will be the primary investigator and primary author of this paper.
Chapter 7

Conclusions and future work

7.1 Review of the research work performed and summary of the novel contributions

The broader topic of this dissertation is nonlinear ultrasonic wave propagation. The use of nonlinear features in ultrasonic testing of materials and structures has recently gained increasing attention by the structural health monitoring and nondestructive evaluation communities. Nonlinear wave features (e.g. higher-harmonic generation) have shown greater sensitivity to structural conditions when compared to the more conventional linear ultrasonic features (amplitude, phase, velocity, etc...).

This dissertation focuses on nonlinearities arising in the case of ultrasonic guided waves that lend themselves to the monitoring of structural waveguides. A novel numerical framework is proposed. It combines a nonlinear semi-analytical finite element formulation with finite element preprocessors, solvers and postprocessors (CO.NOSAFE). This tool allows to predict favorable conditions of higher-harmonic guided wave generation (i.e. obeying synchronicity and nonzero power flux requirements) in complex waveguides. Several benchmark cases were studies by the CO.NOSAFE algorithm including: a viscoleastic isotropic plate, an elastic anisotropic
composite laminate, a reinforced concrete slab, and a railroad track. The last case is the principal application of the dissertation.

Continuously Welded Rail (CWR) is used in modern rail construction including high-speed rail transportation. The absence of expansion joints in these structures brings about the risk of breakage in cold weather and of buckling in warm weather due to the resulting thermal stresses. In fact, safety statistics data from the US Federal Railroad Administration (FRA) indicate rail buckling from uncontrolled thermal stresses as the leading cause of train accidents, within the track category, in recent years. Currently, no well-established method exists to properly monitor the rail thermal stresses in-situ. Of particular interest is the determination of the rail Neutral Temperature (NT), or the rail temperature where the thermal stress is zero.

The consideration of nonlinear wave features to monitor thermal stresses in solids has required the development of a new physical model that does not rely on finite strain conditions that are assumed by classical nonlinear wave studies. Instead, the origin of the nonlinearity was explained in this dissertation on the basis of interatomic potentials under varying temperature. These potentials suggest at least a cubic dependence on strain of the residual strain energy that is stored in the material due to the prevented thermal expansion. The cubic relation between strain energy and strain gives raise to second-harmonic generation of propagating elastic waves. This principle was validated experimentally for longitudinal bulk waves propagating in a steel block that was constrained and subjected to thermal excursions.

Following this theoretical development, the study was focused to the problem of the measurement of rail NT. For this case, CO.NO.SAFE models were developed for a
AREMA 136 RE rail in order to identify proper waveguide modes that exhibit nonlinear behavior under thermal stresses. Requirements of the desired modes were little interaction with the rail head and with the rail foot. The rail head was avoided to eliminate effects of residual stresses and changes in geometry (wear) of the waveguide. The rail foot was avoided to eliminate effects of the rail supports (the so-called tie-to-tie variation problem). Hence special nonlinear waveguide modes were identified with predominant motion of the rail web alone.

Hand-in-hand with the modeling study, an extensive set of experimental tests was conducted at UCSD Large-Scale Rail NT Test-bed. This facility, a one-of-a-kind 70-ft long track, allows to impose thermal loads in a highly controlled laboratory environment, and yet in a quite realistic manner. The Test-bed was instrumented with 48 strain gages, 6 thermocouples, 6 potentiometers, and an infrared camera to fully capture its behavior during the thermal cycles.

A prototype was designed, constructed and tested on the Large-scale Test-bed. The prototype consists of an ultrasonic transmitter and an ultrasonic receiver that are mounted on a case that is magnetically attached to the rail web for a wayside installation. The nonlinear parameter (higher-harmonic generation) of the selected ultrasonic guided modes is measured as a function of rail temperature. A minimum of the ultrasonic nonlinear parameter indicates precisely the rail NT (zero stress temperature). The accuracy of the rail NT measurement was found of only a few degrees. This is an excellent result that was consistently confirmed at several locations of the Test-bed.

These encouraging results have now led to the planning of a field test of the rail NT technology that is being organized by UCSD in close collaboration with the Federal
Railroad Administration, the Burlington Northern Santa Fe railroad, and the Transportation Technology Center in Colorado.

If the field tests are successful, this technology has certainly the potential to revolutionize the maintenance of CWR vis-à-vis the thermal stress problem. For example, knowledge of the current rail NT in-situ would allow railroads to take condition-based decisions, such as imposing slow-order mandates to trains in warm weather.

### 7.2 Recommendations for future studies

Identifying optimum guided wave modes for structural condition monitoring is a daunting task, particularly given the plurality of different wave modes propagating at high frequencies. The case of a railroad track is particularly complex given the shape of its cross-section. Consequently, there exists a plurality of combinations of primary modes and higher-harmonic modes, at various frequencies, that satisfy nonlinear internal resonance conditions in rails. The particular selection of guided modes used in rail NT wayside prototype was determined based on the CO.NO.SAFE models and experimentations. Additional work could be carried out to explore additional combinations of modes for enhanced sensitivity to the thermal stress. This could be achieved with an optimization-type study.

This dissertation concludes that it is possible to identify the rail NT by tracking the minimum of the nonlinear parameter measured at various rail temperatures. A natural extension of this result is the determination of the absolute level of thermal stress from one nonlinear measurement on the rail. This is a difficult task, but worth being
investigated further. In this case, the effects of residual stresses (although small in the rail web) should be considered. Also, the effects of the transducer-to-structure coupling should be taken into account (and compensated for) for an absolute measurement of stress. This study was limited to laboratory tests. The prototype developed here must be tested in the field under real-world conditions. Plans for future field tests are already being made at the time this dissertation was written.

Numerical Implicit/Explicit simulations performed using the ABAQUS code were a classical nonlinear material formulation to predict the evolution of the nonlinear parameter as function of the stress level acting in the waveguide. The implementation of a more precise material model (based on the novel formulation presented in Chapter 5) via a specialized User Defined Material Subroutine is currently under investigation.

The theoretical framework for quadratic nonlinear wave propagation was classically developed approximating the full strain energy representation (Eq. (2.49)) up to cubic displacement gradients, leading to the final expression (Eq. (2.50)). Very few studies where the nonlinear elastic wave propagation was predicted more realistically including displacement gradients up to the 4th order (Cattani and Rushchitskii, 2003) have been proposed in literature over the years. New phenomena, which could not be anticipated by the classical nonlinear formulation (third harmonic generation), were predicted, creating new possibilities for wave modeling. However, these studies were focused on nonlinear elastic waves propagating in unbounded media (bulk waves). Additional theoretical studies could be performed in order to extend the validity of this new analytical framework to nonlinear elastic waves propagating in waveguides (guided waves).
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