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TEARING-MODE STABILITY OF A FORMING SPHEROMAK PLASMA

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ABSTRACT. The results of numerical calculations of $\Delta'$ for a class of equilibria typical of those encountered during the early formation stage of the S1 Spheromak are presented. The equilibrium plasma is assumed to be cylindrically symmetric and pressureless. It encloses a current-carrying perfect conductor (flux core) and is surrounded by a vacuum with zero longitudinal field. Stability boundaries in the space formed by the equilibrium parameters are mapped. The plasma is tearing-mode-stable provided $B_2/B \sigma$ at the flux core is below a certain critical value which depends on the equilibrium parameters. For typical equilibria, this critical value is $0.65$.

A method for producing a spheromak plasma in which both poloidal and toroidal currents are inductively transferred from a flux core to a surrounding plasma sleeve has been proposed [1] and successfully tested experimentally [2]. This formation method operates on a relatively slow time-scale so as to limit peak power requirements and improve the energy efficiency of the formation process. The success of the method suggests that the plasma progresses through a succession of meta-equilibria, each of which is stable, or, at most, weakly unstable, to instabilities that grow in times shorter than the formation time-scale $T_F$. In particular, to form the plasma on the time-scale $\tau_s < \tau_F \ll T_R$, where $\tau_s = T_A^3/\tau_A^2$ is the characteristic time for the tearing-mode instability [3], $\tau_F = \mu_0 \alpha^2/\eta$ is the resistive diffusion time, and $\tau_A = (\mu_0 \rho)^{1/2} a/B_0$ is the Alfven transit time, a sequence of equilibria stable to the tearing mode should exist.

In this letter, we assess the tearing-mode stability of a class of MHD equilibria encountered during the inductive formation of the S1 Spheromak. During the early stages of this formation process, the plasma is well represented by the following circular cylindrical system (Fig. 1): a perfectly conducting flux core of radius $a$ and periodicity length $2\pi R$ is surrounded by a pressureless plasma that fills the region $a < r < b$. A vacuum region extends from $r = b$ out to a conducting wall at $r = c$. No current flows outside of the plasma ($r > b$) although the flux core does, in general, carry a current. The current $\vec{J}$ within the plasma satisfies $\vec{J} = \nabla \times \vec{B}$ and $\vec{J}(r) = \alpha(r) \vec{B}(r)$. The function $\alpha(r)$ is chosen to be

$$\alpha(r) = \alpha_0 \{1 - (r-a)/[(b-a)]^3\}^{1/4}$$

where $\alpha_0$, $i$, and $j$ are 'free' parameters. With this choice of $\alpha(r)$, the equilibrium current, the longitudinal field $B_z$, and the safety factor $q = r B_z / \rho B \sigma$ vanish at the plasma edge. We consider only equilibria whose safety factors decrease monotonically between $r = a$ and $r = b$.

We determine the tearing-mode stability of these equilibria by using a procedure similar to the one described in Refs [4, 5]. The equilibrium equation in the plasma is

$$\frac{a^2 B_z}{dr^2} + \left[\frac{1}{r} - (1/a) \frac{d \alpha}{dr}\right] \frac{dB_z}{dr} + \alpha^2 B_z = 0$$

FIG. 1. Class of equilibria analysed for tearing-mode stability.
This is solved by expanding $B_z$ in power series about the regular singular point $r = b$. The radius of convergence of these series extends beyond $r = a$ to the origin. One of the two linearly independent solutions is rejected because it does not satisfy the boundary condition $B_z(b) = 0$. The retained series converges rapidly for $a/(b-a) < 0.5$ and is typically truncated after sixty terms. To assess the stability of this equilibrium, we subject it to a perturbed radial magnetic field,

$$\tilde{B}_z(r) = i\psi(r) \exp\{i(kz - m\theta)\}$$

(3)

where $k = n/R$ and $m$ and $n$ are integers, and analyse the stability of that mode. A necessary condition for tearing instability of a particular mode is that the mode rational surface $r_s$ lie in the plasma, i.e. that $q(r_s) = (m/n)$ for $a < r_s < b$. For the monotonically decreasing $q$-profiles considered here, an equivalent necessary condition for instability is that $q_a > m/n$. Everywhere except in the vicinity of such a mode rational surface, the function $\psi$ obeys the ‘infinite-conductivity equation’:

$$\frac{d}{dr}\left[\frac{H}{F} \frac{d\psi}{dr}\right] - \psi \left[\frac{g}{F} + \frac{d}{dr}\left(\frac{H}{F} \frac{d\psi}{dr}\right)\right] = 0$$

(4)

where

$$F = k^2 \frac{\dot{B}}{B} = kB_z - \frac{(m/r)B_\theta}{B_\theta}$$

$$H = r \frac{z}{(k^2 r^2 + m^2)}$$

$$g = \frac{(m^2 - 1)x z r^2 + \frac{k^2}{k^2 + m^2} + \frac{k^2}{k^2 + m^2} + \frac{(2kzB_z + mB_\theta)}{(k^2 r^2 + m^2)}}$$

Since the perturbed radial field at the flux core must vanish, one boundary condition is $\psi(a) = 0$. The boundary condition at the plasma-vacuum interface is deduced from continuity of pressure and continuity of the tangential electric field across the interface. For a pressureless plasma with $B_z(b) = 0$, these conditions imply that the perturbed fields satisfy $\tilde{B}_\theta(b) = \tilde{B}_\theta(b')$ and $\tilde{B}_z(b) = \tilde{B}_z(b')$. The vacuum fields are expressed in terms of modified Bessel functions, and $\tilde{B}_\theta$ is eliminated by using the Euler-Lagrange equations [7] yielding the boundary condition at $r = b$:

$$\phi' + \phi\left(\frac{(2k^2 b^2 + m^2)}{m b} \frac{B_z}{\dot{B}} + \frac{kb}{m} \frac{B_\theta}{\dot{B}} + \frac{k^2 b^2}{m^2} \frac{B_\theta}{\dot{B}}\right) = \frac{(k^2 b^2 + m^2)}{kb^2} \left[K_m'(kc)I_m(kb) - I_m'(kc)K_m(kb)\right]$$

(5)

To determine the tearing-mode stability; we solve this equation for $\psi$ in four distinct regions and match the solutions; the matching coefficients determine the tearing stability. First, the equation is solved for $\psi_-(r)$ and $\psi_+(r)$ in the ‘outer regions’, $a < r < r_s$ and $r_s < r < b$, using an accurate extrapolative differential-equation solver [6]. If $\psi$ vanishes only at the boundaries of these ‘outer regions’, the perturbation is a stable ideal-MHD mode [7], and we proceed with the resistive-MHD analysis. In the ‘intermediate regions’, $r = r_s \pm \epsilon$, we expand the functions $F$, $H$, and $g$ in Taylor series about $r = r_s$ and find that $\psi \sim A_+ \psi_L + B_+ \psi_S$, where $\psi_L$ and $\psi_S$ are of the form

$$\phi_L(x) = 1 + a_1 x \ln |x| + a_2 x^2 \ln |x| + a_3 x^2$$

(6)

$$\phi_S = x + b_1 x^2$$

(7)

where

$$x = r - r_s$$

and the $a_i$ and $b_i$ depend on the functions $F$, $H$, and $g$. The constants $A_+$ and $B_+$ are determined by matching the ‘outer’ solution on the left-hand side to the ‘intermediate’ solution on the left-hand side; $A_+$ and $B_+$ are found in the same fashion on the right-hand side. In practice, this matching is accomplished by incrementally reducing the value of $\epsilon$ until the ratio $B/A$ changes by one part in $10^{-4}$ when $\epsilon$ is halved. The value of $\Delta' \equiv (\psi' - \dot{\psi})/\psi = B_z/A_+ - B_+/A_+$ is then computed, the condition for tearing-mode stability being that $\Delta' \leq 0$. The values of $\Delta'$ calculated by following this procedure typically change by one part in $10^{-4}$, when the number of terms in the power series for $B_z$ is doubled or when the accuracy of the differential-equation solver is increased by a factor of ten.

J.B. Taylor [8] has developed a simple procedure for computing the stability for the special class of cylindrical equilibria that satisfy $\nabla \times \vec{B} = \mu \vec{B}$, $\mu = \text{const.}$ We may attempt to model the forming spheromak with Taylor equilibria by placing conductors at $r = a$ and at $r = b = c$, and by requiring
that $B_z(b) = 0$, but the resulting equilibria are not realistic since they have a finite jump in the poloidal current at $r = b$ and do not have a vacuum region. Although the results derived from Taylor's criterion are not physically meaningful, they do provide a convenient analytical check of our numerical calculations. For the boundary conditions assumed above, the axisymmetric solution $\nabla \times \vec{B} = \mu \vec{B}$ is given by

$$B_z(r) = \frac{J_0(\mu b)}{\gamma_0(\mu b)} Y_0(\mu r)$$

(8)

$$B_\theta(r) = \frac{J_1(\mu r)}{\gamma_0(\mu b)} Y_1(\mu r)$$

(9)

$$B_r(r) = 0$$

(10)

The eigenvalue equation for the perturbed modes is given by

$$f(\mu, k) = \frac{J_{m-1}(\gamma_a) + m/\gamma_a (\mu/k - 1)J_m(\gamma_a)}{Y_{m-1}(\gamma_a) + m/\gamma_a (\mu/k - 1)Y_m(\gamma_a)}$$

$$- \frac{J_{m-1}(\gamma_b) + m/\gamma_b (\mu/k - 1)J_m(\gamma_b)}{Y_{m-1}(\gamma_b) + m/\gamma_b (\mu/k - 1)Y_m(\gamma_b)} = 0$$

(11)

where $\gamma_a = a(\mu^2 - k^2)^{1/2}$ and $\gamma_b = b(\mu^2 - k^2)^{1/2}$. The minimum value of $\mu$ that satisfies this equation for $b = 2a$ is $\mu_{\text{crit}} = 2.91$, $ka = 0.51$, $m = 1$. According to Taylor's theory, this implies that axisymmetric equilibria with $\mu < \mu_{\text{crit}}$ are stable. We have calculated $\Delta'$ numerically for these Taylor equilibria and find excellent agreement between the stability criterion deduced from $\Delta'$ and the analytical result.

Newcomb [7] has shown that if a linear pinch is stable to ideal-MHD perturbations for $m = 1$, $-\infty < n < \infty$, then it is stable to all ideal perturbations with $m > 1$. We find empirically that, for the equilibria we have studied, this comparison theorem is also valid for tearing-mode stability. Consequently, we present results only for $m = 1$.

We consider three different values of the current-shaping parameters $i$ and $j$:

Peaked: $\alpha = \alpha_o \left[1 - \left(\frac{r-a}{b-a}\right)^2\right]$

Rounded: $\alpha = \alpha_o \left[1 - \left(\frac{r-a}{b-a}\right)\right]$

Broad: $\alpha = \alpha_o \left[1 - \left(\frac{r-a}{b-a}\right)^3\right]$

FIG. 2. Comparison of equilibrium longitudinal current (a), azimuthal current (b), and safety factor $q(c)$ for three different values of the 'current-shaping' parameters $i$ and $j$. To facilitate comparison of profiles, the ordinates have been normalized.
In Fig. 2, representative marginally stable current and $'q'$-profiles are plotted for the three cases, 'peaked', 'rounded', and 'broad'. $b/a = 2.0$ in these graphs.

The effect on stability of the geometrical and current-shaping parameters is summarized in the nine curves that constitute Fig. 3. The longitudinal mode number $n_l$, the rotational transform $q$, and the normalized length of the cylinder $2 \pi R/a$ appear in the calculation of $\Delta'$ only in the combinations $ka = na/R$ and $nq$. The curve $\Delta' = 0$ (the boundary between regions of stability and instability) is therefore plotted in Fig. 3 as a function of $nqa$ and $ka$ with the geometrical parameter $b/a$ and the two current-shaping parameters $i$ and $j$ held fixed. The radius of the external conductor $c/a$ is effectively infinite in these diagrams.

A specific equilibrium ($b/a$, $c/a$, $R/a$, $\alpha_0$, $i$ and $j$ all specified) corresponds in Fig. 3 to a line that passes through the origin with a slope equal to the ratio of poloidal field to toroidal field, $B_0(r=a)/B_z(r=a)$. Mode rational surfaces with $m = 1$ and $k = n/R$ correspond to discrete points along that line. Points for which $nqa < 1$ are always stable since there is no corresponding mode rational surface. Points along the line for which $nqa > 1$ may or may not be stable, depending on the slope of the line and the details of the stability boundary. In every case plotted in Fig. 3, there exist equilibria with sufficiently small ratio of toroidal field to poloidal field that all modes are stable. Listed in Table I are the critical values of this ratio above which tearing-mode instability occurs. Also included in Table I are these critical values expressed in terms of the ratio of magnetic field pitch to the thickness of the current layer, $aB_z(r=a)/(b-a)B_0(r=a)$.

The results in Table I are in qualitative agreement with analytical results for the tearing mode in slab geometry. Furth, Killeen, and Rosenbluth [3] also found a critical value of pitch to thickness of the current layer above which instability occurs. For their 'smooth' current profile, $j \propto 1/cosh^2(y/a)$, this critical value is 1 while, for their 'broad' profile,

$$j = \begin{cases} 
1 & |y/a| < 1 \\
0 & |y/a| > 1 
\end{cases}$$

the critical value is 1.6. We note that the critical values of pitch-to-current layer thickness we have calculated for forming spheromak equilibria are, like the results of Ref. [3], of $O(1)$. Further, just as with the analytical results in slab geometry, forming spheromak equilibria with broader current profiles have larger critical coefficients of pitch-to-current.
layer thickness than equilibria with more narrow profiles. For the same value of \( aB_z(a)/(b-a)B_\theta(a) \), the broad current profiles have weaker current gradients than the more narrow profiles and are, therefore, more stable.

One of the equilibria (rounded, \( b/a = 2.5 \)) has an anomalously low critical value of toroidal field to poloidal field due to a narrow region of the plasma with unfavourable stability properties (the spike near \( nq_a = 1 \) in Fig.3c). In a real physical system with discrete values of \( k \), this narrow region is likely to lie between two modes and is thus of little consequence. A more realistic critical value of \( B_z/B_\theta \) for this equilibrium is, therefore, 0.46.

The position of the outer conductor \( c/b \) has only a slight effect on tearing-mode stability (Fig.4). If the conductor is placed at the plasma edge, it can sometimes stabilize an otherwise weakly unstable mode. For \( c/b \gtrsim 1.1 \), however, shell stabilization is ineffective.

These theoretical results have immediate relevance to the inductive formation method for producing a spheromak plasma of Refs [1, 2]. Since there exists a critical value of pitch-to-plasma width below which the plasma is stable to the tearing mode, it should be
possible, by operating below this critical value, to form the plasma on a time-scale that is fast only compared to the resistive time of the plasma. If the ratio of pitch-to-plasma width exceeds this critical value, it may still be possible to burst through this region by forming on a time-scale fast compared to the tearing-mode growth rate.

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