Title
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ON EIGENVECTORS OF NILPOTENT LIE
ALGEBRAS OF LINEAR OPERATORS

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Abstract

We give a condition ensuring that the operators in a nilpotent Lie algebra of
linear operators on a finite dimensional vector space have a common eigenvector.

Introduction

Throughout this paper $V$ is a vector space of positive dimension over a field $f$ and $g$ is a nilpotent Lie algebra over $f$ of linear operators on $V$. An element $u \in V$ is an eigenvector for $S \subseteq g$ if $u$ is an eigenvector for every operator in $S$. If $V$ has a basis $(e_1, \ldots, e_n)$ representing each element of $g$ by an upper triangular matrix, then $e_1$ is an eigenvector for $g$. Such a basis exists when $f$ is algebraically closed and $g$ is solvable (Lie’s Theorem), and also when every element of $g$ is a nilpotent operator (Engel’s Theorem). Our results are further conditions guaranteeing existence eigenvectors.

The minimal and characteristic polynomials of a linear operator $A$ on $V$ are denoted respectively by $\pi_A, \mu_A \in f[t] = \text{the ring of polynomials over } f,$ written $\#S$.

Let $k$ be a Galois extension field of $f$ of degree $d := [k : f]$, and define $M \subseteq \mathbb{N}$ to be the additive monoid generated by zero and the prime divisors $d$.

Consider the conditions:

(C1) $\mu_A$ splits in $k$ for every $A \in g$

(C2) $\dim V \notin M$

Our main result is:

Theorem 1 If (C1) and (C2) hold then $g$ has an eigenvector.
The proof is preceded by some applications.

When (C1) holds, Theorem shows that there is an eigenvector in every invariant subspace whose dimension is not in $M$. This is exploited to yield the following two results:

**Corollary 2** If a nilpotent Lie algebra of linear operators on $\mathbb{R}^n$ does not have an eigenvector, every nontrivial invariant subspace has odd dimension.

**Proof** When $f$ is the real field $\mathbb{R}$ and $k$ is the complex field $\mathbb{C}$, $M$ consists of the positive even integers. □

**Corollary 3** Let (C1) hold. Assume $g$ preserves a direct sum decomposition $V = \bigoplus_i W_i$, and let $D \subset \mathbb{N}$ denote the set of dimensions of the subspaces $W_i$.

(i) If $g$ does not have an eigenvector then $D \subset M$.

(ii) If $V' \subset V$ is a maximal subspace spanned by eigenvectors of $g$ then $\dim(V') \geq \#\{D \setminus M\}$.

**Proof** Assertion (i) follows from Theorem □. To prove (ii) order the $W_i$ so that $W_1, \ldots, W_m$ are the only summands whose dimensions are not in $M$. For each $j \in \{1, \ldots, m\}$ we choose an eigenvector $e_j \in W_j$ by Theorem □. The $e_j$ are linearly independent and belong to $V'$ by maximality of $V'$, whence (ii). □

**Example 4**

Assume $n \notin M$ and let $\alpha \in f[t]$ be a monic polynomial that splits in $k[t]$. Denote by $\mathcal{A}(\alpha)$ the set of $n \times n$ matrices $T$ over $f$ such that $\alpha(T) = 0$. Then every pairwise commuting family $T \subset \mathcal{A}(\alpha)$ has an eigenvector in $f^n$. This follows from Theorem □ applied to the Lie algebra $g$ of linear operators on $f^n$ generated by $T$. Being abelian, $g$ can be triangularized over $k$, hence (C1) holds.

**Example 5**

The assumption that $n \in M$ is essential to Theorem □. For instance, take $f = \mathbb{R}$, $k = \mathbb{C}$, $V = \mathbb{R}^2$. The abelian Lie algebra of $2 \times 2$ of real skew symmetric matrices does not have an eigenvector in $\mathbb{R}^2$.

**Example 6**

The hypothesis of Theorem □ cannot be weakened to $g$ being merely solvable. For a counterexample with $f = \mathbb{R}, k = \mathbb{C}$, take $g$ to be the solvable 3-dimensional real Lie algebra with basis $(X, U, V)$ such that $[X, U] = -V$, $[X, V] = U$, $[U, V] = 0$.

A Lie algebra $\beta$ over $f$ is supersolvable if the spectrum of the linear map $\text{ad} A : \beta \to \beta$ lies in $f$ for all $A \in \beta$. If $\beta$ is not supersolvable it need not have an eigenvector, as is shown by Example □. We don’t know if Theorem □ extends to supersolvable Lie algebras, except for the following special case:
Theorem 7  A supersolvable Lie algebra $\beta$ of linear transformations of $\mathbb{R}^3$ has an eigenvector.

Proof  Lacking an algebraic proof, we use a dynamical argument. Let $G \subset GL(3, \mathbb{R})$ be the connected Lie subgroup having Lie algebra $\beta$. The natural action of $G$ on the projective plane $\mathbb{P}^2$ of lines in $\mathbb{R}^3$ through the origin fixes some $L \in \mathbb{P}^2$. This follows from supersolvability because $\dim(\mathbb{P}^2) = 2$, the action on $\mathbb{P}^2$ is effective and analytic, and the Euler characteristic of $\mathbb{P}^2$ is nonzero (Hirsch & Weinstein [1]). The nonzero points of $L$ are eigenvectors for $\beta$.

Proof of Theorem 1

We rely on Jacobson’s Primary Decomposition Theorem [2, II.4, Theorem 5]. This states that $V$ has a $\mathfrak{g}$-invariant direct sum decomposition $\bigoplus V_i$ where each primary component $V_i$ has the following property: For each $A \in \mathfrak{g}$ the minimal polynomial of $A|V_i$ is a prime power in $f[t]$.

Condition (C2) implies the dimension of some primary component is $\not\in \mathbb{M}$. To prove Theorem 1 it therefore suffices to apply the following result to such a primary component:

Theorem 8  Assume (C1) and (C2). If $\pi_A$ is a prime power in $f[t]$ for each $A \in \mathfrak{g}$ then the following hold:

(a) $\pi_A(t) = (t - r_A)^n$, $r_A \in f$

(b) there is a basis putting $\mathfrak{g}$ in triangular form

Assertion (a) is equivalent to $\pi_A$ having a root $r_A \in f$. Therefore (a) follows from:

Lemma 9  Let $\alpha \in f[t]$ be a polynomial of degree $n$ that splits in $k[t]$. If $n \not\in \mathbb{M}$ then $\alpha$ has a root in $f$, and the sum of the multiplicities of such roots is $\not\in \mathbb{M}$.

Proof  Let $R \subset k$ denote the set of roots of $\pi$, and $R_j \subset R$ the set of roots of multiplicity $j$. The Galois group $\Gamma$ has order $[k : f]$ and acts on $R$ by permutations. The cardinality of each orbit divides $[k : f]$, and $R \cap f$ is the set of fixed points of this action. Each $R_j$ is a union of orbits, as is $R_j \setminus f$. It follows that $#(R_j \setminus f) \in \mathbb{M}$.

Let $k \leq n$ denote the sum of the multiplicities of the roots that are not in $f$. Then

$$k = \sum_{j=2}^n j \cdot #(R_j \setminus f)$$

Therefore $k \in \mathbb{M}$ because $\mathbb{M}$ is closed under addition. By hypothesis $n \not\in \mathbb{M}$, hence $n - k \not\in \mathbb{M}$ and $n - k > 0$. As $n - k$ is the sum of the multiplicities of the roots in $f$, the conclusion follows.

Now that (a) of Theorem 8 is proved, assertion (b) is a consequence of the following result:
**Lemma 10** Let \( \mathfrak{h} \) be a nilpotent Lie algebra of linear operators on \( V \). Assume that for all \( A \in \mathfrak{h} \) there exists \( r_A \in \mathfrak{f} \) such that \( \pi_A(t) = (t - r_A)^n \). Then \( V \) has a basis putting \( \mathfrak{h} \) in triangular form.

**Proof** Every \( A \in \mathfrak{h} \) can be written uniquely as \( r_AI + N_A \) with \( N_A \) nilpotent and \( I \) the identity map of \( V \). It is easy to see that the set comprising the \( N_A \) is closed under commutator brackets. Therefore \( V \) has a basis triangularizing all the \( N_A \) (Jacobson [2 II.2, Theorem 1’]), and such a basis triangularizes \( \mathfrak{h} \).

This completes the proof of Theorem 1.

**References**
