Title
Measurement of the time-dependent CP asymmetry in $B^0 \rightarrow K^{*0} \gamma$ decays

Permalink
https://escholarship.org/uc/item/4vb870jt

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Publication Date
2007-12-01

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072009-2
MEASUREMENT OF TIME-DEPENDENT CP ASYMMETRY

072009-3
We present updated measurements of time-dependent CP asymmetries in fully reconstructed neutral $B$ decays containing a charmonium meson. The measurements reported here use a data sample of $(465 \pm 5) \times 10^6 Y(4S) \rightarrow BB$ decays collected with the BABAR detector at the PEP-II asymmetric energy $e^+e^-$ storage rings operating at the SLAC National Accelerator Laboratory. The time-dependent CP asymmetry parameters measured from $J/\psi K^0_S$, $J/\psi K^0_L$, $\psi(2S) K^0_S$, $\eta K^0_S$, $\eta_c K^0_S$, and $J/\psi K^*(892)^0$ decays are: $C_f = 0.024 \pm 0.020$(stat) $\pm 0.016$(syst) and $-\eta_f S_f = 0.687 \pm 0.028$(stat) $\pm 0.012$(syst).

\[
\text{A}_{CP}(\Delta t) = \frac{g_+ - g_-}{g_+ + g_-} = (1 - 2w)S_f \sin(\Delta m_d\Delta t),
\]

and $S_f = -\eta_f \sin2\beta$. If the assumption that $C_f = 0$ is relaxed, then $S_f = -\eta_f \sqrt{1 - C_f^2} \sin2\beta$.

In a previous publication [5], we reported time-dependent CP asymmetries in terms of the parameters $\sin2\beta$ and $|\lambda_f|$. In this paper, we report results in terms of $S_f$ and $C_f$ to be consistent with other time-dependent CP asymmetry measurements. We reconstruct $B^0$ decays to the final states $J/\psi K^0_S$, $J/\psi K^0_L$, $\psi(2S) K^0_S$, $\eta_c K^0_S$, and $J/\psi K^*(892)^0$ with $K^*(892)^0 \rightarrow K^0_S \pi^\pm$. The $J/\psi K^0_S$ final state is CP-even and the $J/\psi K^0_L$ final state is an admixture of CP-even and CP-odd amplitudes. The remaining final states are CP-odd. The CP-even and odd amplitudes in $B^0 \rightarrow J/\psi K^{*0}$ decays can be separated in an angular analysis [7]. In this analysis, we average over the angular information resulting in a dilution of the measured CP asymmetry by a factor $1 - 2R_\perp$, where $R_\perp$ is the fraction of the $L = 1$ contribution. In Ref. [7] we have measured $R_\perp = 0.233 \pm 0.010$(stat) $\pm 0.005$(syst), which gives an effective $\eta_f = 0.504 \pm 0.033$ after acceptance corrections for $f = J/\psi K^{*0}$. In addition to measuring a combined $S_f$ and $C_f$ for the $CP$ modes described above, we measure $S_f$ and $C_f$ for each final state $f$ individually. We split the $J/\psi K^0_S$ mode into samples with either $K^0_S \rightarrow \pi^+ \pi^- \pi^0 \pi^0$ or $\pi^+ \pi^- \pi^0$. We also combined the $J/\psi K^0$ channel with $K^0$, either a $K^0_S$ or $K^0_L$. Compared to our previous publication [5], the current analysis contains $82 \times 10^6$ additional $BB$ decays and improved track reconstruction algorithms have been applied to the entire data set.
II. THE DATA SET AND BABAR DETECTOR

The results presented in this paper are based on data collected with the BABAR detector at the PEP-II asymmetric energy $e^+e^-$ storage rings [8] operating at the SLAC National Accelerator Laboratory. At PEP-II, 9.0 GeV electrons and 3.1 GeV positrons collide at a center-of-mass energy of 10.58 GeV, which corresponds to the mass of the $Y(4S)$ resonance. The asymmetric energies result in a boost from the center-of-mass (CM) frame to the laboratory of $\beta \gamma = 0.56$. The data set analyzed has an integrated luminosity of 425.7 fb$^{-1}$ corresponding to $(465 \pm 5) \times 10^9 BB$ pairs recorded at the $Y(4S)$ resonance.

The BABAR detector is described in detail elsewhere [9]. Surrounding the interaction point is a five-layer, double-sided silicon vertex tracker (SVT), which measures the impact parameters of charged particle tracks in both the plane transverse to, and along the beam direction. A 40-layer drift chamber surrounds the silicon vertex tracker and provides measurements of the momenta for charged particles. Charged hadron identification is achieved through measurements of particle energy loss in the tracking system and the Cherenkov angle obtained from a detector of internally reflected Cherenkov light. A CsI(Tl) electromagnetic calorimeter (EMC) provides photon detection, electron identification, and $\pi^0$ reconstruction. The aforementioned components are enclosed by a solenoid magnet, which provides a 1.5 T magnetic field. Finally, the flux return of the magnet (IFR) is instrumented in order to allow discrimination of muons from pions. For the most recent 211.7 fb$^{-1}$ of data, a portion of the resistive plate chambers in the IFR has been replaced by limited streamer tubes [10].

We use a right-handed coordinate system with the $z$ axis along the electron beam direction and the $y$ axis upward. Unless otherwise stated, kinematic quantities are calculated in the laboratory rest frame. We use Monte Carlo (MC) simulated events generated with the BABAR simulation based on GEANT4 [11] for detector responses and EvtGen [12] for event kinematics to determine signal and background characteristics, optimize selection criteria, and evaluate efficiencies.

III. RECONSTRUCTION OF $B$ CANDIDATES

We select two samples of events in order to measure the time-dependent $CP$ asymmetry parameters $S_f$ and $C_f$: a sample of signal events used in the extraction of the $CP$ parameters ($B_{CP}$) and a sample of fully reconstructed $B$ meson decays to flavor eigenstates ($B_{flav}$). The $B_{CP}$ sample consists of $B^0$ decays to $J/\psi K^0_S$, $J/\psi K^0_S$, $\psi(2S)K^0_S$, $\eta$, $K^0_S$, $\chi_c1K^0_S$, and $J/\psi K^*(892)^0$, where $K^{*0}$ decays to $K^0\pi^0$. The $B_{flav}$ sample consists of $B^0$ decays to $D^{(*)-}(\pi^+, \rho^+, \eta^*)$ final states. We use the $B_{flav}$ sample to determine the dilution (mistag probability) and the resolution function, discussed in Sec. V. We assume that the interference between the $CP$ side and the tag side reconstruction is negligible and therefore that the dilution and resolution parameters are the same for the $B_{flav}$ and $B_{CP}$ samples. We also select a sample of fully reconstructed charged $B$ meson decays to $J/\psi K^+$, $\psi(2S)K^+$, $\eta$, $K^+$, and $J/\psi K^*(892)^+$, where $K^{*+}$ decays to $K^+\pi^0$ or $K_S^0\pi^+$, to use as a control sample.

The event selection is unchanged from that described in Ref. [5]. $J/\psi$ and $\psi(2S)$ mesons are reconstructed via their decays to $e^+e^-$ or $\mu^+\mu^-$ final states. At least one of the leptons is required to pass a likelihood particle identification algorithm based on the information provided by the EMC, the IFR, and from ionization energy loss measured in the tracking system. We require the invariant mass of the muon pair $m(\mu^+\mu^-)$ to be in the mass range 3.06–3.14 GeV/$c^2$ for $J/\psi$ or 3.636–3.736 GeV/$c^2$ for $\psi(2S)$ candidates. For $J/\psi \rightarrow e^+e^-$ and $\psi(2S) \rightarrow e^+e^-$ decays, where the electron may have radiated bremsstrahlung photons, part of the missing energy is recovered by identifying neutral clusters with more than 30 MeV lying within 35 mrad in the polar angle and 50 mrad in azimuth of the electron direction projected onto the EMC. The invariant mass of $e^+e^-$ pairs is required to be within 2.95–3.14 GeV/$c^2$ for $J/\psi$ candidates, or 3.436–3.736 GeV/$c^2$ for $\psi(2S)$ candidates.

We also construct $\psi(2S)$ mesons in the $J/\psi \pi^+\pi^-$ final state, where the $J/\psi$ candidate is combined with a pair of oppositely-charged tracks assumed as pions with no particle identification applied, and the pion pair-invariant mass between 400 MeV/$c^2$ and 600 MeV/$c^2$. Candidates with 3.671 GeV/$c^2 < m(J/\psi \pi^+\pi^-) < 3.701$ GeV/$c^2$ are retained.

The $\chi_{c1}$ candidates are reconstructed in the $J/\psi \gamma$ final state. The photon candidates are required to have an energy greater than 100 MeV but less than 2 GeV, and, when combined with other photons, not to form a $\pi^0$ candidate with invariant mass 120 MeV/$c^2 < m(\gamma\gamma) < 150$ MeV/$c^2$. The invariant mass of the $\chi_{c1}$ candidate is required to be between 3.477 GeV/$c^2$ and 3.577 GeV/$c^2$. Mass constraints are applied in the fits to improve the determinations of the energies and momenta of the $J/\psi$, $\psi(2S)$, and $\chi_{c1}$ candidates.

We reconstruct the $B^0 \rightarrow \eta K_S^0$ mode using the $\eta \rightarrow K_S^0 K^+\pi^-$ decay. We exploit the fact that the $\eta$ decays predominantly through a $K\pi$ resonance at around 1.43 GeV/$c^2$ and a $K_S^0 K$ resonance close to the threshold. We require that $m(K_S^0\pi^-)$ or $m(K^+\pi^-)$ is within the mass range of 1.26 GeV/$c^2$ and 1.63 GeV/$c^2$, or 1.0 GeV/$c^2 < m(K_S^0\pi^-) < 1.4$ GeV/$c^2$.

The decay channels $K^+\pi^-$, $K^+\pi^-\pi^0$, $K^+\pi^+\pi^-\pi^0$, and $K_S^0\pi^+\pi^-$ are used to reconstruct $B^0$, while $D^-$ candidates are selected in the $K^+\pi^-\pi^0$ and $K_S^0\pi^-$ modes. We require that the $D^0$ and $D^-$ candidate invariant mass is within $\pm 3\sigma$ of their respective nominal mass, where $\sigma$ is the uncertainty calculated for each candidate. A mass-
constrained fit is then applied to the $D^0$ and $D^-$ candidates satisfying these requirements. We form $D^*$ candidates in the decay $D^+ \rightarrow D^0 \pi^+$ by combining a $D^0$ with a pion that has momentum greater than 70 MeV/c. The $D^*$ candidates are required to have $m(D^0 \pi^+)$ within $\pm 1.1$ MeV/c$^2$ of the nominal $D^{*+}$ mass for the $D^0 \rightarrow K^+ \pi^- \pi^0$ mode and $\pm 0.8$ MeV/c$^2$ for all other modes.

For the $J/\psi K^0_S$ decay, we use both $K^0_S \rightarrow \pi^- \pi^+$ and $K^0_S \rightarrow \pi^0 \pi^0$ candidates. For other $B$ decay modes we only use $K^0_S \rightarrow \pi^- \pi^-$. Candidates in the $K^0_S \rightarrow \pi^- \pi^+$ mode are selected by requiring an invariant $\pi^- \pi^-$ mass, computed at the vertex of the two oppositely-charged tracks, between 472.67 MeV/c$^2$ and 522.67 MeV/c$^2$. We further apply a mass constraint fit to the $K^0_S$ candidates before combining them with charmonium candidates to form $B^0$ candidates. Neutral pion candidates, in the mass range 100–155 MeV/c$^2$, are formed from two $\gamma$ candidates from the EMC. Pairs of $\pi^0$ are combined to construct $K^0_S \rightarrow \pi^0 \pi^0$ candidates. The minimum energy is required to be 30 MeV for $\gamma$, 200 MeV for $\pi^0$, and 800 MeV for $K^0_S$ candidates. To select $K^0_S$ candidates, the $\pi^0 \pi^0$ invariant mass is restricted to the region between 470 MeV/c$^2$ and 550 MeV/c$^2$.

Candidates for $K^*_{1L}$ are identified in the EMC and IFR detectors as reconstructed clusters that cannot be associated with any charged track in the event. As the energy of $K^*_{1L}$ cannot be measured well, the laboratory momentum of the $K^*_{1L}$ is determined by its flight direction and the constraint that the invariant mass of the $J/\psi K^*_{1L}$ system has the known $B^0$ mass. For events with multiple $J/\psi K^*_{1L}$ candidates, a hierarchy is imposed where the highest energy EMC cluster for multiple EMC combinations, or the IFR cluster with the largest number of layers for multiple IFR combinations, is selected. In case both EMC and IFR combinations are found, the EMC combination is chosen because of its better angular resolution.

We reconstruct $K^{*0}$ candidates in the $K^0_S \pi^0$ mode, while $K^{*+}$ candidates are reconstructed in the $K^+ \pi^0$ and $K^0_S \pi^+$ modes. The invariant mass of the two daughters is required to be within $\pm 100$ MeV/c$^2$ of the nominal $K^*$ mass.

The $\rho^+$ candidates are reconstructed in the $\pi^+ \pi^0$ final state, where the $\pi^+ \pi^0$ mass is required to lie within $\pm 150$ MeV/c$^2$ of the nominal $\rho^+$ mass. Candidates in the decay mode $a^+_1 \rightarrow \pi^+ \pi^- \pi^+$ are reconstructed by combining three charged tracks with pion mass assumption, and restricting the three-pion invariant mass to lie between 1.0 and 1.6 GeV/c$^2$.

Events that pass the selection requirements are refined using kinematic variables. For the $J/\psi K^0_S$ mode, the difference $\Delta E$ between the candidate’s CM energy and the beam energy in the CM frame, $E_{\text{beam}}$, is required to satisfy $|\Delta E| < 80$ MeV. For all other categories of events, we require $|\Delta E| < 20$ MeV and the beam-energy substituted mass $m_{\text{IES}} = \sqrt{(E_{\text{beam}})^2 - (p_B^*)^2}$ to be greater than 5.2 GeV/c$^2$, where $p_B^*$ is the $B$ momentum in the CM frame. When multiple $B$ candidates (with $m_{\text{IES}} > 5.2$ GeV/c$^2$) are found in the same event, the candidate with the smallest value of $|\Delta E|$ is selected.

We calculate the proper time difference $\Delta t$ between the two $B$ decays from the measured separation $\Delta z$ between the decay vertices of $B_{\text{rec}}$ and $B_{\text{tag}}$ along the collision ($z$) axis [13]. The $z$ position of the $B_{\text{rec}}$ vertex is determined from the charged daughter tracks. The $B_{\text{tag}}$ decay vertex is determined by fitting tracks not belonging to the $B_{\text{rec}}$ candidate to a common vertex, including constraints from the beam spot location and the $B_{\text{rec}}$ momentum [13]. Events are accepted if the calculated $\Delta t$ uncertainty is less than 2.5 ps and $|\Delta t|$ is less than 20 ps. The fraction of signal MC events satisfying such a requirement is 95%.

IV. B MESON FLAVOR TAGGING

A key ingredient in the measurement of time-dependent $CP$ asymmetries is the determination of whether the $B_{\text{rec}}$ was a $B^0$ or a $\bar{B}^0$ at the time of $\Delta t = 0$. This “flavor tagging” is achieved with the analysis of the decay products of the recoiling $B$ meson $B_{\text{tag}}$. The overwhelming majority of $B$ mesons decay to a final state that is flavor-specific, i.e., only accessible from either a $B^0$ or a $\bar{B}^0$. The purpose of the flavor-tagging algorithm is to determine the flavor of $B_{\text{tag}}$ with the highest efficiency $\epsilon_{\text{tag}}$ and lowest probability $w$ of assigning the wrong flavor. It is not necessary to fully reconstruct $B_{\text{tag}}$ in order to determine its flavor.

The figure of merit for the performance of the tagging algorithm is the effective tagging efficiency

$$Q = \epsilon_{\text{tag}}(1 - 2w)^2,$$

which is related to the statistical uncertainty $\sigma_S$ and $\sigma_C$ in the coefficients $S_f$ and $C_f$ through

$$\sigma_{S,C} \propto \frac{1}{\sqrt{Q}}.$$  

(4)

The tagging algorithm we employ [5,13] analyzes tracks on the tag side to assign a flavor and associated probability to $B_{\text{tag}}$. The flavor of $B_{\text{tag}}$ is determined from a combination of nine different tag signatures, such as isolated primary leptons, kaons, and pions from $B$ decays to final states containing $D^*$ mesons, and high momentum charged particles from $B$ decays. The properties of those signatures are used as inputs to a single neural network that is trained to assign the correct flavor to $B_{\text{tag}}$. The output of this neural network then is divided into seven mutually exclusive categories. These are (in order of decreasing signal purity) Lepton, Kaon I, Kaon II, KaonPion, Pion, Other, and Notag. The events with the neural network output $|\mathcal{N}| > 0.8$ are defined as a Lepton category, if they are also accompanied by an isolated primary lepton; otherwise they are categorized as a Kaon I tag. For the other five tag categories (Kaon II, KaonPion, Pion, Other, and
TABLE I. Efficiencies $\epsilon_i$, average mistag fractions $w_i$, mistag fraction differences between $B^0$ and $\bar{B}^0$ tagged events $\Delta w_i$, and effective tagging efficiency $Q_i$ extracted for each tagging category $i$ from the $B_{\text{flav}}$ sample.

<table>
<thead>
<tr>
<th>Category</th>
<th>$\epsilon_i$ (%)</th>
<th>$w_i$ (%)</th>
<th>$\Delta w_i$ (%)</th>
<th>$Q_i$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lepton</td>
<td>8.96 + 0.07</td>
<td>2.8 + 0.3</td>
<td>0.3 + 0.5</td>
<td>7.98 + 0.11</td>
</tr>
<tr>
<td>Kaon I</td>
<td>10.82 + 0.07</td>
<td>5.3 + 0.3</td>
<td>-0.1 + 0.6</td>
<td>8.65 + 0.14</td>
</tr>
<tr>
<td>Kaon II</td>
<td>17.19 + 0.09</td>
<td>14.5 + 0.3</td>
<td>0.4 + 0.6</td>
<td>8.68 + 0.17</td>
</tr>
<tr>
<td>KaonPion</td>
<td>13.67 + 0.08</td>
<td>23.3 + 0.4</td>
<td>0.4 + 0.7</td>
<td>3.91 + 0.12</td>
</tr>
<tr>
<td>Pion</td>
<td>14.18 + 0.08</td>
<td>32.5 + 0.4</td>
<td>5.1 + 0.7</td>
<td>1.73 + 0.09</td>
</tr>
<tr>
<td>Other</td>
<td>9.54 + 0.07</td>
<td>41.5 + 0.5</td>
<td>3.8 + 0.8</td>
<td>0.27 + 0.04</td>
</tr>
<tr>
<td>All</td>
<td>74.37 + 0.10</td>
<td></td>
<td></td>
<td>31.2 + 0.3</td>
</tr>
</tbody>
</table>

Notag) the outputs of the neutral network are required to satisfy: $0.6 < |NN| < 0.8$, $0.4 < |NN| < 0.6$, $0.2 < |NN| < 0.4$, $0.1 < |NN| < 0.2$, and $|NN| < 0.1$, respectively.

The performance of this algorithm is evaluated using the $B_{\text{flav}}$ sample. The final state of the $B_{\text{flav}}$ sample can be classified as mixed or unmixed depending on whether the reconstructed flavor eigenstate $B_{\text{flav}}$ has the same or opposite flavor as the tagging $B$. After taking the mistag probability into account, the decay rate $g_{\pm, B^0} (g_{\pm, \bar{B}^0})$ for a neutral $B$ meson to decay to a flavor eigenstate accompanied by a $B^0$ ($\bar{B}^0$) tag can be expressed as

$$g_{\pm, B^0}(\Delta t) \propto [(1 - \Delta w_i) \pm (1 - 2w_i) \cos(\Delta m_{B^0} \Delta t)],$$

$$g_{\pm, \bar{B}^0}(\Delta t) \propto [(1 + \Delta w_i) \pm (1 - 2w_i) \cos(\Delta m_{\bar{B}^0} \Delta t)],$$

where the $\pm$ sign in the index refers to mixed ($-$) and unmixed ($+$) events; the index $i$ denotes the $i$th tagging category. The performance of the tagging algorithm is summarized in Table I. The events in the Notag category contain no flavor information, so carry no weight in the time-dependent analysis. They are excluded from further analysis. The total effective tagging efficiency is measured to be $(31.2 \pm 0.3)\%$.

V. LIKELIHOOD FIT METHOD

We determine the composition of our final sample by performing simultaneous fits to the $m_{\text{ES}}$ distributions for the full $B_{\text{CP}}$ and $B_{\text{flav}}$ samples, except for the $J/\psi K_{S}^0$ sample for which we extract the $K_{S}^0$ momentum by using the $B^0$ mass constraint and fit the $\Delta E$ distribution. We then perform a simultaneous maximum likelihood fit to the $\Delta t$ distribution of the tagged $B_{\text{CP}}$ and $B_{\text{flav}}$ samples to measure $S_f$ and $C_f$.

We define a signal region of $5.27 < m_{\text{ES}} < 5.29$ GeV$/c^2$ ($|\Delta E| < 10$ MeV for $J/\psi K^0_S$), which contains 15481 candidate events of a $B_{\text{CP}}$ sample that satisfy the tagging and vertexing requirements (see Table II). The signal $m_{\text{ES}}$ distribution for the full $B_{\text{CP}}$ and $B_{\text{flav}}$ samples, except for the $J/\psi K^0_S$ sample, is described by a Gaussian function. The background $m_{\text{ES}}$ distribution is modeled by an ARGUS threshold function [14], where a shape parameter is allowed to vary in the fit. For the decay modes of $J/\psi K_{S}^0$, $\psi \omega K_{S}^0$, $\chi_{c1} K_{S}^0$, $J/\psi K^{*0}$, and $B_{\text{flav}}$, we use simulated events to estimate the fractions of background events that peak in the $m_{\text{ES}}$ signal region ($m_{\text{ES}} > 5.27$ GeV$/c^2$) due to cross feed from other decay modes. We describe this component with a Gaussian function having the same mean and width as the signal and refer to it as the peaking background because if neglected, it would lead to an overestimate of the signal yield. The peaking background is less than 1% in the decay of $B^0 \rightarrow J/\psi K_{S}^0$, and at the level of a few percent in most other decay modes. The only exception is the decay of $B^0 \rightarrow J/\psi K^{*0}$, where the peaking background level is about 13%. MC simulations show that it consists of 44% of $B^+\rightarrow \chi_c K^0_S$ decays, 32% of $B^0 \rightarrow \chi_c K^0_S$ decays, and 24% of other $B^0$ decays. For the $\eta_{c} K^{*0}$ mode, the cross feed fraction is determined from a fit to the $m_{K^{*0}}$ and $m_{ES}$ distributions in data. For the $J/\psi K^0_S$ decay mode, the signal $\Delta E$ distribution is determined from MC simulated events. The sample composition, effective $\eta_f$, and $\Delta E$ distribution of the individual background sources are determined either from simulation (for $B \rightarrow J/\psi X$) or from the $m_{c\tau}$ sidebands in data (for non-J/ψ background). Figure 1 shows the distributions of $m_{\text{ES}}$ obtained...
independent scale factors, and where
\[ J = \frac{C_2}{C_1} + \frac{C_3}{C_0} \]

FIG. 1. Distributions for \( B_{CP} \) and \( B_{flav} \) candidates satisfying the tagging and vertexing requirements: (a) \( m_{ES} \) for the final states \( J/\psi K^0_S, \psi(2S)K^0_S, \chi_{c1}K^0_S, \) and \( \eta_cK^0_S \); (b) \( \Delta E \) for the final state \( J/\psi K^0_L \); (c) \( m_{ES} \) for \( J/\psi K^{*0}(K^{*0} \to K^0\pi^0) \); and (d) \( m_{ES} \) for the \( B_{flav} \) sample. In each plot, the shaded region is the estimated background contribution.

for the \( B_{CP} \) and \( B_{flav} \) events, and \( \Delta E \) obtained for the \( J/\psi K^0_L \) events.

The \( \Delta t \) distributions of the \( B_{CP} \) sample are modeled by Eq. (1) and those of the \( B_{flav} \) sample by Eq. (5). The \( \Delta t \) distributions for the signal are convolved with a resolution function common to both the \( B_{flav} \) and \( B_{CP} \) samples, modeled by the sum of three Gaussian functions [13], called the core, tail, and outlier components, which can be represented as a function of the reconstruction uncertainty \( \delta t = \Delta t - \Delta t_{true} \) as follows:

\[
R(\delta t; \sigma_{\Delta t}) = f_{core} h_G(\delta t; \delta_{core} \sigma_{\Delta t}, S_{core} \sigma_{\Delta t}) + f_{tail} h_G(\delta t; \delta_{tail} \sigma_{\Delta t}, S_{tail} \sigma_{\Delta t}) + f_{out} h_G(\delta t; \delta_{out}, S_{out}).
\]  

(6)

where

\[
h_G(\delta t; \delta, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(\delta t - \delta)^2}{2\sigma^2}\right).
\]  

(7)

and

\[ f_{core} + f_{tail} + f_{out} = 1. \]  

(8)

The widths (\( \sigma \)) of the core and tail components include two independent scale factors, \( S_{core} \) and \( S_{tail} \), to accommodate an overall underestimate or overestimate of the \( \Delta t \) measurement error \( \sigma_{\Delta t} \) for all events. The parameter \( S_{core} \) is free in the fit and its value is close to unity. The value of \( S_{tail} \) is derived from MC studies and fixed to be 3. Studies show that the measurement of \( C_f \) and \( S_f \) is not sensitive to the choice of the \( S_{tail} \) value. We later vary the \( S_{tail} \) value within a large region and assign the shift of the measured \( C_f \) and \( S_f \) values as the corresponding systematic uncertainties. We account for residual charm decay products included in the \( B_{tag} \) candidate vertex by allowing the core and tail Gaussian functions to have nonzero mean values (bias, \( \delta_{core} \neq 0 \) and \( \delta_{tail} \neq 0 \)). The bias (\( \delta_{core} \)) and width (\( S_{core} \)) of the core component are allowed to differ for the lepton-tagged and nonlepton-tagged events. We use common parameters for the tail component. In order to account for the strong correlations with other resolution parameters, the outlier component bias (\( \delta_{out} \)) and width (\( S_{out} \)) are fixed to 0 ps and 8 ps, respectively.

The \( \Delta t \) spectrum of the combinatorial background is described by an empirical distribution, consisting of components with zero and nonzero lifetimes (\( \tau_{bg} \)) that are convolved with a resolution function [13] distinct from that used for the signal. Here, we use a double-Gaussian distribution, which has components similar to the core and
We fix $P$ and that the fit returns reasonable estimates of the statistical component of the combinatorial background contains both mixed and unmixed events. Therefore, we allow the value of $\Delta m_d$ for this component ($\Delta m_{d,\text{bg}}$) to vary in the fit.

In addition to $S_f$ and $C_f$, there are 69 free parameters in the fit. For the signal, these are

(i) 7 parameters for the $\Delta t$ resolution: $\delta_{\text{core}}$, $S_{\text{core}}$ for the lepton-tagged and nonlepton-tagged events, $f_{\text{core}}$, $f_{\text{tail}}$, and $\delta_{\text{tail}}$;

(ii) 12 parameters for the average mistag fractions $w_i$ and the differences $\Delta w_i$ between $B^0$ and $\bar{B}^0$ mistag fractions for each tagging category;

(iii) 1 parameter for the small difference between $B^0$ and $\bar{B}^0$ reconstruction efficiency [13]; and

(iv) 6 parameters for the small difference between $B^0$ and $\bar{B}^0$ tagging efficiencies in each tagging category [13].

The background parameters that are allowed to vary are

(i) 24 mistag fraction parameters: $w_i$ and $\Delta w_i$ of each tagging category for background components with zero and nonzero lifetime, respectively;

(ii) 3 parameters for the $\Delta t$ resolution: $\delta_{\text{core}}$, $S_{\text{core}}$, and $f_{\text{core}}$;

(iii) 4 parameters for the $B_{\text{flav}}$ time dependence: 2 parameters for the fraction ($f_{\text{prompt}}$) of a zero lifetime component for the lepton-tagged and nonlepton-tagged events, $\tau_{bg}$ and $\Delta m_{d,\text{bg}}$;

(iv) 8 parameters for possible $CP$ violation in the background, including the apparent $CP$ asymmetry of nonpeaking events in each tagging category;

(v) 1 parameter for possible direct $CP$ violation in the $\chi_{c1} K^0_S$ background coming from $J/\psi K^{*0}$, and

(vi) 3 parameters for possible direct $CP$ violation in the $J/\psi K^{*0}$ mode, coming from $J/\psi K^{*0}_{L}$, $J/\psi K^{*0}_{S}$, and the remaining $J/\psi$ backgrounds.

The effective value of $|\lambda_f|$ for the non-$J/\psi$ background is fixed from a fit to the $J/\psi$-candidate sidebands in $J/\psi K^{*0}_{L}$. We fix $\tau_{B^0} = 1.530$ ps and $\Delta m_d = 0.507$ ps$^{-1}$ [3]. The determination of the mistag fractions and $\Delta t$ resolution function parameters for the signal is dominated by the $B_{\text{flav}}$ sample, which is about 10 times larger than the $CP$ sample.

VI. LIKELIHOOD FIT VALIDATION

We perform three tests to validate the fitting procedure. The first of these tests consists of generating ensembles of simulated experiments from the probability density function and fitting each simulated experiment. We determine that the fitted values of $S_f$ and $C_f$ parameters are unbiased, and that the fit returns reasonable estimates of the statistical uncertainties by verifying the distribution of the pull $P$ on a parameter $O$, given by $P = (O_{\text{fit}} - O_{\text{gen}})/\sigma(O_{\text{fit}})$, is consistent with a Gaussian centered about zero with a width of one. The quantity $O_{\text{fit}}$ is the fitted value, with a fitted error of $\sigma(O_{\text{fit}})$, and $O_{\text{gen}}$ is the generated value.

The second test involves fitting simulated signal events that include the full BABAR detector simulation. For each decay mode, we divide the signal MC sample to many data-sized samples, fit them one by one, and then examine the distribution of the fitted results. We make sure that the $P$ distributions for these signal-only simulated experiments are consistent with a Gaussian distribution centered at zero with a width of one.

The third test is to perform null tests on control samples of neutral and charged $B$ events where $S_f$ and $C_f$ should be very small or zero. The parameters $S_f$ and $C_f$ are consistent with zero for the charged $B$ sample of the $J/\psi K^{\pm}$, $\psi(2S)K^{\pm}$, $\chi_{c1} K^{\pm}$, and $J/\psi K^{*\pm}$ final states. For the neutral $B_{\text{flav}}$ sample, we find that the $S_f$ and $C_f$ parameters slightly deviate from zero at approximately twice the statistical uncertainty (see Table II). The deviation of $S_f$ from zero is consistent with the directly measured $CP$ asymmetry $S \sim -2\sin(2\beta + \gamma)\cos(\delta) \leq 0.04$ [15] in $B^0 \to J^{(*)} \pi^{\pm} h^{\mp}$ [16] due to interference from doubly CKM-suppressed decays, where $\gamma = \arg(-V_{ub}V_{cb}^*)/(V_{cd}V_{cb}^*)$, $\delta$ is the strong phase difference between CKM-favored and doubly CKM-suppressed amplitudes, and $r \sim 0.02$ is the ratio of the two amplitudes. Considering this expected $CP$ asymmetry in the $B_{\text{flav}}$ sample and systematic uncertainties (at $\sim 1\%$ level), we conclude that our analysis is free of pathological behaviors.

VII. RESULTS

The fit to the $B_{CP}$ and $B_{\text{flav}}$ samples yields $-\eta_f S_f = 0.687 \pm 0.028$ and $C_f = 0.024 \pm 0.020$, where the errors are statistical only. The correlation between these two parameters is $+0.1\%$. We also performed the fit using sin2$\beta$ and $|\lambda_f|$ as fitted parameters, and found sin2$\beta = 0.687 \pm 0.028$ and $|\lambda_f| = 0.977 \pm 0.020$. The correlation between the fitted sin2$\beta$ and $|\lambda_f|$ parameters is $-0.14\%$. Figure 2 shows the $\Delta t$ distributions and asymmetries in yields between events with $B^0$ and $\bar{B}^0$ tags for the $\eta_f = -1$ and $\eta_f = +1$ samples as a function of $\Delta t$, overlaid with the projection of the likelihood fit result. Figure 3 shows the time-dependent asymmetry between unmixed and mixed events for hadronic $B$ candidates with $m_{B_{\text{IS}}} > 5.27$ GeV/c$^2$. We also perform a fit in which we allow different $S_f$ and $C_f$ values for each charmonium decay mode, a fit to the $J/\psi K_0^{*0}(\pi^+ \pi^- + \pi^0 \pi^0)$ mode, and a fit to the $J/\psi K^0(L_{S} + L_{S}')$ sample. The results for some of these studies are shown in Fig. 4. We split the data sample by run period and by tagging category. We perform the $CP$ measurements on control samples with no expected $CP$ asymmetry. The results of these fits are summarized in Table II.
The dominant systematic uncertainties on $S_f$ are summarized in Tables III and IV. The dilution due to flavor tagging can be different between $B_{CP}$ and $B_{	ext{flav}}$ events. We study this effect by comparing the results in large samples of simulated $B_{CP}$ and $B_{	ext{flav}}$ events. The uncertainties due to $\Delta t$ resolution functions for both signal and background components are estimated by varying the fixed parameters and by using alternative models. We also vary the peaking background fractions based on estimates derived from simulation, and vary the CP content of the background over a wide range to estimate the effect due to our limited knowledge of background properties. The uncertainties in the $J/\psi K^0_S$ sample are studied by varying the compositions of the signal and background, by modifying the $\Delta E$ probability density function based on studies performed with the $J/\psi K^0_S$ control sample, and by varying the branching fractions of the background modes and their CP asymmetries. Other sources of uncertainty such as the values of the physics parameters $\Delta m_d$, $\Gamma_B$, $\Delta \Gamma_d/\Gamma_d$, the beam spot and detector alignment, and other fixed parameters, are studied by varying them according to their world averages, the calibration, and the statistical uncertainty, respectively.

Despite the large amount of simulated signal events that included the full BABAR detector simulation, we can only validate the possible fit bias to be no more than certain precision. As a result, we assign a systematic uncertainty corresponding to any deviations and the statistical uncertainties of the mean values of the fitted $S_f$ and $C_f$ from the generated values as the possible fit bias (MC statistics).

The only sizable systematic uncertainties on $C_f$ are due to the CP content of the peaking backgrounds and due to the possible interference between the suppressed $b \rightarrow \bar{u}c\bar{d}$ amplitude with the favored $b \rightarrow c\bar{u}d$ amplitude for some tag-side $B$ decays [15]. The total systematic error on $S_f(C_f)$ is calculated by adding the individual systematic uncertainties in quadrature and is found to be 0.012 (0.016). The main sources of systematic uncertainty are listed in Tables III and IV.

For the $\eta_KK^0_S$ mode, we found $-\eta_fS_f = 0.925 \pm 0.160(\text{stat}) \pm 0.057(\text{syst})$, which has a significance of 5.4$\sigma$ standard deviations including systematic uncertainties. Our result is the first observation of CP violation in this mode.

**VIII. CONCLUSIONS**

We report improved measurements of the time-dependent CP asymmetry parameters. The results in this paper supercede those of our previous publication [5]. We report our measurements in terms of $C_f$ and $S_f$. We find

$$C_f = 0.024 \pm 0.020(\text{stat}) \pm 0.016(\text{syst}),$$

$$\eta_fS_f = 0.687 \pm 0.028(\text{stat}) \pm 0.012(\text{syst}),$$

providing an independent constraint on the position of the apex of the unitarity triangle [17]. Our measurements agree...
with previous published results [5,18] and with the theoretical estimates of the magnitudes of CKM matrix elements within the context of the SM [19]. We also report measurements of $C_f$ and $S_f$ for each decay mode in our $CP$ sample and for the combined $J/\psi K^0(K_S^0 + K_L^0)$ mode. $CP$ violation in $\eta K_S^0$ mode is established at the level of 5.4\sigma standard deviations including systematic uncertainties.

**ACKNOWLEDGMENTS**

We are grateful for the extraordinary contributions of our PEP-II colleagues in achieving the excellent luminosity and machine conditions that have made this work possible. The success of this project also relies critically on the expertise and dedication of the computing organizations that support BABAR. The collaborating institutions...
TABLE III. Main systematic uncertainties on $S_f$ and $C_f$ for the full $CP$ sample, and for the $J/\psi K^0$, $J/\psi K^0_\Sigma$, and $J/\psi K^0_\Sigma$ samples. For each source of systematic uncertainty, the first line gives the error on $S_f$ and the second line the error on $C_f$. The total systematic error (last row) also includes smaller effects not explicitly mentioned in the table.

<table>
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<th>Source/sample</th>
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<th>$J/\psi K^0_\Sigma$</th>
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TABLE IV. Main systematic uncertainties on $S_f$ and $C_f$ for the $J/\psi K^0_\Sigma(\pi^+\pi^-)$, $J/\psi K^0_\Sigma(\pi^0\pi^0)$, $\psi(2S)K^0_\Sigma$, $\chi_c K^0_\Sigma$, $\eta K^0_\Sigma$, and $J/\psi K^0_\Sigma(K^0 \to K^0_\Sigma \pi^0)$ decay modes. For each source of systematic uncertainty, the first line gives the error on $S_f$ and the second line the error on $C_f$. The total systematic error (last row) also includes smaller effects not explicitly mentioned in the table.

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gemeinschaft (Germany), the Istituto Nazionale di Fisica Nucleare (Italy), the Foundation for Fundamental Research on Matter (The Netherlands), the Research Council of Norway, the Ministry of Education and Science of the Russian Federation, Ministerio de Educación y Ciencia (Spain), and the Science and Technology Facilities Council (United Kingdom). Individuals have received support from the Marie-Curie IEF program (European Union) and the A. P. Sloan Foundation.

[4] See, for example, D. Kirkby and Y. Nir, p. 146 in Ref. [3].
[6] Charge conjugation is implied throughout this paper.
[17] See, for example, A. Ceccucci, Z. Ligetti, and Y. Sakai, p. 138 in Ref. [3].