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Abstract

The paper shows that competitive forces in club economies lead to admissions prices that can be decomposed as linear prices on externality-producing attributes, where each member pays the same amount per unit attribute contributed. The externalities prices are sufficient to cover the costs of services provided within the club.

This paper extends joint work with Greg Engl. I thank him for his contributions and take sole responsibility for any mistakes.
1 Introduction

The key feature of a club economy is that agents form groups to confer externalities on each other. Optimal clubs balance the positive externalities, such as sharing the costs of public goods provided within the club, against the negative externalities, such as those that arise from congestion or the externality-producing characteristics of other members. Admissions to clubs can be priced like any other private goods, and similar decentralization theorems hold. For early discussions of this subject see Buchanan (1965), Ellickson (1973, 1979), Berglas (1976), Berglas and Pines (1980), Bewley (1981). For more recent discussions focussed on scale problems and core/competitive equivalence in club economies with one private good, as well as the special issues that arise when members of a club care about the ‘types’ of other members as well as their numbers, see Scotchmer and Wooders (1987a,b), Scotchmer (1994,1996), Conley and Wooders (1994), Brueckner (1994), and Epple and Romer (1993). For an exposition and resolution of the problems that arise with complementarities between club goods and private goods, see Gilles and Scotchmer (1996).

The above investigations assume that agents have ‘types’, and that agents care about both the types and numbers of other members of a club. Admissions prices to clubs are different for agents of different types. The latter reflects the idea that crowding externalities vary according to the individual’s characteristics, and admissions prices should reflect the crowding externalities imposed. However if the latter is true, admissions prices should be even more narrowly focussed: One would expect them to reflect only those characteristics that affect other members’ utilities or that affect costs of providing club goods. In addition, one might expect the admissions prices to be linear on the agents’ characteristics, in the same way that prices are linear in other competitive markets. Those hypotheses underlie Example 4 of Engl and Scotchmer (1996) (henceforth ES), and is the subject of this paper.

In their Example 4, ES present a club model in which admissions prices are linear
prices on agents' characteristics. In equilibrium each member of a club pays (is paid) the same incremental admissions price for each unit of an externality-producing attribute he contributes. However the club economies described in ES use two somewhat restrictive assumptions; namely a superadditivity assumption on attributes (not necessarily on agents) and a homogeneity assumption on preferences. This paper discusses how far their results can be extended to club economies with public goods where these assumptions are not required.

Section 2 describes club economies where agents are identified with their characteristics, some of which produce externalities. Section 3 describes games derived from club economies. ES showed that if their superadditivity assumption on attributes holds, then payoffs in the core of a suitably large derived game can be decomposed as a linear sum on the agents' attributes, called a 'hedonic payoff'. An example below shows that without the superadditivity assumption the linear decomposition may be impossible. ES also showed that admissions prices to clubs can be decomposed as a linear sum on the attributes. The example below shows that this property also depends on superadditivity in attributes. However Section 4 shows that admissions prices to equilibrium clubs can be decomposed as a linear sum on externality-producing attributes even without the superadditivity assumption. Section 5 presents a core/competitive convergence result, and Section 6 interprets externality pricing as a variant of the 'Henry George Theorem'.

2 The Club Economy

We assume that agents' types are indexed in a set \( N = \{1, \ldots, \eta\} \), and each type of agent, say \( i \in N \), is characterized by his attributes \((\alpha_0^i, \alpha^i)\), \( \alpha_0^i \in \mathbb{R}, \alpha^i \in \mathbb{R}_+^{\mathbb{T}} \setminus \{0\} \). The \( 0^{th} \) attribute is the endowment of a quasilinear good. As is customary in models with transferable utility (see Aumann and Shapley (1974)) we will permit negative consumption of the quasilinear good in order to preserve the transferability of utility, and hence there
is no loss in assuming that endowments \( \{\alpha_i^j\} \) are zero. The other attributes \( \alpha^i \) might represent for example, generosity, level of education, and inclination to smoke.

We index replicas of the set of agents by \( m = 1, 2, \ldots \). We describe a set of agents by \( mP \equiv (mN, (\alpha^i)_{i \in \mathbb{N}}) \) where \( mN \) is the index set for the agents, \( mN = \{1_1, \ldots, 1_m, 2_1, \ldots, \eta_m\} \). For a given \( i \in \mathbb{N} \) all agents with indices \( r, r \in \{1, \ldots, m\} \), have the same attributes \( \alpha^i \). Let \( \alpha \) represent a \( T \times \eta \) matrix with columns \( (\alpha^1, \ldots, \alpha^\eta) \).

A coalition is \( S \subset mN, S \neq \emptyset \). A collection of coalitions, say \( C \), is a partition of the set of agents \( mN \) if \( \cup_{S \in C} S = mN \) and \( S, S' \in C \Rightarrow S \cap S' = \emptyset \).

Let \( N^S_i \in \mathbb{Z}^T_+ \) represent the numbers of agents of each type in the coalition \( S \), i.e., \( N^S_i, i = 1, \ldots, \eta \), is the number of agents in \( S \) with attributes \( \alpha^i \). The attributes of a club or coalition \( S \) are simply the sum of its members' attributes, namely \( \alpha N^S \in \mathbb{R}^T_+ \).

Following Mas-Colell (1980), I avoid linear structure on the space of public goods. Instead I assume that there is a finite set of possible public projects \( \mathcal{Y} \) that coalitions can produce.\(^1\) The cost of producing \( y \in \mathcal{Y} \) in a club with attributes \( A \) is given by a function \( c : \mathcal{Y} \times \mathbb{R}^T_+ \rightarrow \mathbb{R}_- \). Let \( c(y, A) \equiv (\frac{\partial c(y, A)}{\partial A_1}, \ldots, \frac{\partial c(y, A)}{\partial A_T}) \) if the derivative exists. We say that the \( r \)th attribute produces cost externalities if \( c \) varies with \( A_r \); that is, \( c \) takes on different values for different values of \( A_r \) at some fixed \( (y, A_1, \ldots, A_{r-1}, A_{r+1}, \ldots, A_T) \).

**Assumption 1** *There exists \( y_0 \in \mathcal{Y} \) such that \( c(y_0, \cdot) \equiv 0 \).*

A club is a pair \( (y, A) \in \mathcal{Y} \times \mathbb{R}^T_+ \), where \( y \) is the project it produces and \( A \) is an aggregate vector of attributes representing the attributes of its members.

\(^1\)This idea has recently been revived by Diamantaras and Gilles (1996) and Diamantaras, Gilles and Scotchmer (1996). Without a linear structure on the space of public projects, one cannot assume that preferences are monotonic in public goods. Consumers may disagree on their ranking of mutually exclusive public projects, and their ranking may depend on the private goods consumed. Other results in club theory that avoid monotonicity and convexity are Scotchmer (1994), Proposition 4.1, and Gilles and Scotchmer (1996, Theorems 1 and 2), who also avoid convexity of preferences.
Following ES, Example 4, I assume that there are functions $\phi^t : \mathcal{Y} \times \mathbb{R}_+^T \to \mathbb{R}_+, \ t = 1, \ldots, T$, such that the utility of an agent with characteristics $\alpha^t$ (excluding consumption of the quasilinear good) is $\alpha^t \cdot \phi(y, \alpha^t \alpha^s)$, where $N_i^s > 0$. Except for quasi-linear utility and the fact that there are a finite number of types of agents, there is nothing restrictive in this structure, as one can always expand the list of attributes and the number of functions $\phi^t$ in order to differentiate preferences. The preferences have the property that the sum of utilities available to a coalition depends only on the sum of its attributes.

We say that the $t^{th}$ attribute produces consumption externalities if $\phi$ varies with $A_t$; that is, for some fixed $(y, A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_T)$, $\phi$ takes on different values for different values of $A_t$. If $\phi$ is differentiable, this means that $\frac{\partial \phi^k(y, A)}{\partial A_t} \neq 0$ for some $k \in \{1, \ldots, T\}$. The $t^{th}$ attribute may also affect its owner’s utility directly, as when $\phi^t > 0$ and $\alpha_t^j > 0$. Some attributes, such as smoking, affect both one’s own utility and others’ utility.

We say that types $i$ and $j$ have the same externality-producing attributes if for all $z \in \mathcal{Y}$ and $N \in \mathbb{Z}_+^n$ such that $N_i > 0$, $\phi(z, \alpha N - \alpha^i + \alpha^j) = \phi(z, \alpha N)$ and $c(z, \alpha N - \alpha^i + \alpha^j) = c(z, \alpha N)$, and vice versa when $N_j > 0$.

We say that an attribute produces externalities if it produces either cost of consumption externalities.

**Assumption 2** The values of $\frac{1}{|A|} A \cdot \phi(y, A) - c(y, A)$, for $A \in \mathbb{R}_+^T$, $y \in \mathcal{Y}$, are bounded.

A club economy is $(mP, c, \phi)$. A club structure for the economy $(mP, c, \phi)$, say $K$, is $\{ (y, S) | S \in C, y \in \mathcal{Y} \}$, where $C$ is a partition of the set of agents $mN$.

For the case that $\phi$ is differentiable, define a function $E : \mathcal{Y} \times \mathbb{R}_+^T \to \mathbb{R}^T$ by $E(y, A) = (E(y, A)_1, \ldots, E(y, A)_T)$, where $E(y, A)_t = A \cdot \frac{\partial \phi}{\partial A_t}(y, A)$. The value $E(y, A)_t$ is
the marginal externality imposed on a club \((y, A)\) by an additional unit of the \(i\)th attribute. The value of \(E(y, A)\) is 0 if the \(i\)th attribute does not produce externalities in consumption.

3 The Game Derived from the Club Economy

We now derive a game \((mP, Q)\) from the club economy \((mP, c, \phi)\) in the natural way, namely with a superadditive characteristic function \(Q : Z^m_+ \rightarrow \mathbb{R}_+\) defined by

\[
Q(N) = \sup \left\{ \sum_{\sigma \in \Sigma} \hat{Q}(N^\sigma) \mid \sum_{\sigma \in \Sigma} N^\sigma = N, \quad N^\sigma \in Z^m_+, \text{ each } \sigma \in \Sigma \right\}
\]

where \(\hat{Q} : \mathbb{R}_+^m \rightarrow \mathbb{R}_+\) is defined by

\[
\hat{Q}(N) = \max_{\{y \in Y\}} \{ (\alpha N) \cdot \phi(y, \alpha N) - c(y, \alpha N) \}
\]

For each \(S \subseteq mN\), \(Q(N^S)\) is the maximum total payoff available to that coalition. By Assumption 1, \(\hat{Q}\) (hence \(Q\)) is nonnegative, and by Assumption 2, the values of \(\frac{1}{|N|} \hat{Q}(N)\) are bounded.

A payoff in the game \((mP, Q)\) is a vector \(U = (U^i)_{i \in mN}\) such that \(\sum_{i \in mN} U^i \leq Q(m1)\). A coalition \(S\) can block \(U\) if \(\sum_{i \in S} U^i < Q(N^S)\). A payoff \(U\) is in the core of the game \((mP, Q)\) if no coalition can block it. We also say that a payoff \(U\) is in the core of the economy \((mP, c, \phi)\) if it is in the core of the derived game \((mP, Q)\), and is an equal-treatment payoff if \(U^{r_i}\) has the same value for all \(r = 1, \ldots, m\), for each \(i \in N\). We will refer to an equal-treatment payoff by a vector \(W = (W^1, \ldots, W^n)\).

We now restate for a special case the convergence result of ES. To state this theorem we must define \(\Delta = \{n \in \mathbb{R}_+^m \mid \sum_{i=1}^m n_i = 1\}\) and \(q : \Delta \rightarrow \mathbb{R}_+\) by extending the following
values to the entire simplex:

\[ q(n) = \sup_{r > 0} \left\{ \frac{Q(rn)}{r} \mid (rn) \in \mathbb{Z}^n_+ \right\} \]

The function \( q \) indicates the maximum per-capita utility available when the relative numbers of agents with different attributes are given by \( n \in \Delta \).

**Corollary to Theorem 1, Engl and Scotchmer 1996:** (The core converges to an equal-treatment payoff in large games.) Suppose that Assumptions 1 and 2 hold, that \( (mP, Q) \) is a game with player set \( mP = (mN, (\alpha^i)_{i \in N}) \), and that \( q \) is differentiable at \( \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \). Let \( \gamma \in (0, 1] \), \( \delta > 0 \) be given. Then there exists \( m^* \) such that if

1. \( m \geq m^* \)
2. \( |S| \geq \gamma m, \text{ where } S \subset mN \).
3. \( (U^i)_{i \in N} \) is in the core of \( (mP, Q) \),
4. \( W \) is an equal-treatment payoff in the core of \( (mP, Q) \)

then \( \| N^S \cdot W - \sum_{i \in S} U^i \| < \delta |N^S| \).

The payoffs in the core converge to an equal-treatment payoff which is the gradient to \( q \) at \( \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \), which we assume to exist. Differentiability holds at almost all points in the simplex because superadditivity implies that \( q \) is concave (ES, Lemma A1).

ES show convergence also for approximate cores. The theorem to which this is a corollary is for the "hedonic core" in which core payoffs are decomposed as a linear function of players' attributes. Equal-treatment is implied, but players' equal-treatment payoffs may have additional structure: Under a superadditivity assumption on attributes described below, core payoffs are described by a linear function on attributes.

\[ \text{Since } Q \text{ is defined on integer vectors, } q \text{ is only defined on the rational values in the simplex } \Delta. \text{ ES Corollary 4 and Footnote 8 argue that there is a unique extension to the entire simplex.} \]
4 Externality Pricing in Competitive Equilibrium

This section shows that

- Competitive payoffs are in the equal-treatment core of the economy.
- Equilibrium admissions prices are anonymous linear prices on externality-producing attributes.
- If the game derived from the economy "exhausts blocking opportunities", every payoff in the equal-treatment core of the economy is a competitive payoff.
- Core payoffs converge to competitive payoffs in large games.

Let \((\pi^i)_{i \in \mathbb{N}}, \pi^i : \mathcal{Y} \times \mathbb{R}_+^T \to \mathbb{R}\) be a price system, where \(\pi^i(y, A)\) is the lump-sum price that an agent with attributes \(\alpha^i\) must pay for admission to a club \((y, A)\). A competitive equilibrium for the economy \((mP, c, \phi)\) is an ordered pair \((\pi, K)\), where \(K\) is a club structure for the economy, such that

1. No consumer could increase utility by buying admission to any other club:
\[
\alpha^i \cdot \phi(y, \alpha N^S) - \pi^i(y, \alpha N) \geq \alpha^i \cdot \phi(z, \alpha N) - \pi^i(z, \alpha N)
\]
for all \(i \in \mathbb{N}, (y, S) \in K\) such that \(N_i^S > 0\), and all \((z, N) \in \mathcal{Y} \times \mathbb{R}_+^T\).

2. Equilibrium clubs make zero profit, and no club could make positive profit:
\[
\sum_{i \in \mathbb{N}} N_i \pi^i(z, \alpha N) - c(z, \alpha N) \leq \sum_{i \in \mathbb{N}} N_i^S \pi^i(y, \alpha N^S) - c(y, \alpha N^S) = 0
\]
for all \((y, S) \in K, (z, N) \in \mathcal{Y} \times \mathbb{R}_+^T\).

Competitive equilibrium is defined such that agents with the same attributes receive the same utility. We say that the competitive equilibrium achieves the equal-treatment payoff \(W\) if for each \(i \in \mathbb{N}, W^i = \alpha^i \cdot \phi(y, \alpha N^S) - \pi^i(y, \alpha N^S)\) where \((y, S) \in K\) and \(N_i^S > 0\).
We say that an equal-treatment payoff $W$ is a competitive payoff of the economy if some competitive equilibrium achieves it.

We first show that competitive payoffs are in the core.

**Lemma 1** Suppose that $(\pi, K)$ is a competitive equilibrium for $(mP, c, \phi)$, and achieves an equal-treatment payoff $W$. Then

1. $N \cdot W \geq \hat{Q}(N)$ for all $N \in \mathbb{R}_+^n$.

2. $N^S \cdot W = \hat{Q}(N^S) = (\alpha N^S) \cdot \phi(y, \alpha N^S) - c(y, \alpha N^S)$ for all $(y, S) \in K$.

**Proof:** (1) For each $i \in \mathbb{N}$, let $(y^i, S^i) \in K$ be such that $N^i > 0$. Given $N \in \mathbb{R}_+^n$, for every $z \in \mathcal{Y}$,

$$N \cdot W = \sum_{i \in \mathbb{N}} N_i \left[ \alpha^i \cdot \phi(y^i, \alpha N^i) - \pi^i(y^i, \alpha N^i) \right] \geq \sum_{i \in \mathbb{N}} N_i \alpha^i \cdot \phi(z, \alpha N) - \sum_{i \in \mathbb{N}} N_i \pi^i(z, \alpha N) \geq (\alpha N) \cdot \phi(z, \alpha N) - c(z, \alpha N),$$

where the first inequality follows from condition 1. of the definition of competitive equilibrium, and the second inequality follows from condition 2. Hence $N \cdot W \geq \hat{Q}(N)$ for all $N \in \mathbb{R}_+^n$.

(2) The first equality holds because Lemma 1(1) holds, and because $\sum_{(y, S) \in K} N^S \cdot W = \sum_{(y, S) \in K} \hat{Q}(N^S)$. The second equality holds because otherwise $\sum_{(y, S) \in K} \hat{Q}(N^S) > \sum_{(y, S) \in K} (\alpha N^S) \cdot \phi(y, \alpha N^S) - c(y, \alpha N^S)$, which contradicts that the payoff $W$ is achieved by the competitive allocation.

**Proposition 1**

*If $W$ is a competitive payoff for $(mP, c, \phi)$ then $W$ is in the core of the economy.*

---

This Lemma requires that conditions 1. and 2. of competitive equilibrium hold for all potential clubs $(y, \alpha N)$, $y \in \mathcal{Y}, N \in \mathbb{R}_+^n$, and not just for clubs $(y, \alpha N)$ with $N \leq (m, \ldots, m)$. See Gilles and Scotchmer (1996) for a discussion of this issue in a larger class of economies. With the weaker condition of equilibrium the two conclusions of the Lemma nevertheless hold provided a "small groups" condition holds. This was shown by Scotchmer and Wooders (1987b) and is restated in Scotchmer (1994).
Proof: Suppose to the contrary that there exists \( S \subset mN \) such that \( N^S \cdot W < Q(N^S) \). Then \( \sum_{\sigma \in \Sigma} N^\sigma \cdot W < \sum_{\sigma \in \Sigma} \hat{Q}(N^\sigma) \) for some collection \( (N^\sigma)_{\sigma \in \Sigma} \) such that \( \sum_{\sigma \in \Sigma} N^\sigma = N^S \). Hence \( N^\sigma \cdot W < \hat{Q}(N^\sigma) \) for some \( \sigma \subset mN \), which contradicts Lemma 1(1).

The following proposition characterizes equilibrium prices. Part (2) uses a superadditivity hypothesis on the function \( V : \mathbb{R}_+^T \rightarrow \mathbb{R}_+ \) defined as follows:

\[
V(A) = \max_{y \in Y} \left[ A \cdot \phi(y, A) - c(y, A) \right]
\]

\( V(A) \) is the total payoff available to a coalition with total attributes \( A \), and \( V(\alpha N) = \hat{Q}(N) \) for each \( N \in \mathbb{R}_+^T \). The characteristic function \( V \) is superadditive in the attributes if \( V(A) + V(A') \leq V(A + A') \) for all \( A, A' \in \mathbb{R}_+^T \).

Typically there will be many price systems that support an optimal or core allocation as a competitive equilibrium, but part (1) of the following proposition says that in any equilibrium, the admissions prices to equilibrium clubs can be understood as linear prices on the externality-producing attributes. Each member pays for the marginal externalities he imposes on other members. The second and third parts of the proposition refer to particular equilibria. By part (2), which was also shown in ES, Example 4, if feasible payoffs are superadditive in the attributes, then there is an equilibrium in which all admissions prices (not just prices to equilibrium clubs) are hedonic linear prices in the sense that each member pays the same price per attribute contributed. The hedonic price coefficients depend only on the externality-producing attributes. Part (3) says that there is another equilibrium in which the total price paid depends only on the externality-producing attributes. That is, two agents with the same externality-producing attributes pay the same price to each conceivable club. Part (3) is similar to a characterization of Conley and Wooders (1994), who show that admissions prices can depend only on what they call "crowding types". Their argument uses a small-groups assumption as in Scotchmer and Wooders (1987a,b).
Proposition 2 (Admissions prices are prices on externality-producing attributes.)

Suppose that $(\pi, K)$ is a competitive equilibrium for the economy $(mP, c, \phi)$, and achieves competitive payoff $W$. Then

1. (Externality Pricing in Equilibrium Clubs)
   If $\dot{Q}$ is differentiable at $N^S$ for each $(y, S) \in K$, then
   \[ \pi^i(y, \alpha N^S) = \alpha^i \cdot [-E(y, \alpha N^S) + \nabla c(y, \alpha N^S)], \text{ for all } (y, S) \in K \text{ if } N_i^S > 0. \]

2. (Hedonic Admissions Prices) Suppose that $V$ is superadditive in the attributes. There exists $p : \mathcal{Y} \times \mathbb{R}_+^T \to \mathbb{R}^T$ such that $((\alpha^i \cdot p)_{i \in N}; K)$ is a competitive equilibrium, and for each $z \in \mathcal{Y}$, $p(z, A) = p(z, A')$ if $A_t = A'_t$ for all attributes $t \in T$ that produce externalities, irrespective of the other attributes.

3. (Externality Pricing in all Clubs) There exists a competitive equilibrium $(\hat{\pi}, K)$ such that if type-$k$ and type-$j$ agents have the same externality-producing attributes, then $\hat{\pi}^k(z, \alpha N) = \hat{\pi}^i(z, \alpha N)$ for all $z \in \mathcal{Y}$, $N \in \mathbb{Z}_+^T$.

Proof of 1: By Lemma 1 and differentiability, $W$ is the gradient vector to $\dot{Q}$ at each $N^S$, $(y, S) \in K$. Hence

\[ W^i = \alpha^i \cdot [\phi(y, \alpha N^S) + E(y, \alpha N^S) - \nabla c(y, \alpha N^S)] \quad \text{if } N_i^S > 0. \quad (1) \]

But since $W^i = \alpha^i \cdot \phi(y, \alpha N^S) - \pi^i(y, \alpha N^S)$, the result follows.

Proof of 2: Let $W$ be the competitive payoff for the economy, so that the two conclusions of Lemma 1 hold. Then since $\dot{Q}(\alpha N) = V(\alpha N)$ for all $N \in \mathbb{R}_+^n$,

\[ N \cdot W \geq V(\alpha N) \text{ for all } N \in \mathbb{R}_+^n \text{ and } \]

(2)
\[ N^S \cdot W = V(\alpha N^S) \]

\[ = \alpha N^S \cdot \phi(y, \alpha N^S) - c(y, \alpha N^S) \quad \text{for all} \ (y, S) \in K \]

The attributes of the set of agents \( mN \) are \( m\alpha \cdot 1 = \sum_{(y,S) \in K} \alpha N^S \), where \( 1 = (1, \ldots, 1) \in Z_+ ^m \). Since \( V \) is superadditive, \( V(m\alpha \cdot 1) \geq \sum_{(y,S) \in K} V(\alpha N^S) \). Since \( \sum_{(y,S) \in K} W \cdot N^S = mW \cdot 1 \), it follows from (2) and (3) that \( mW \cdot 1 = V(m\alpha \cdot 1) \geq \sum_{(y,S) \in K} V(\alpha N^S) \). Using (2) and superadditivity of \( V \), for integers \( r \geq 1 \), \( V(rma \cdot 1) \leq rW \cdot 1 = rV(ma \cdot 1) \leq V(rma \cdot 1) \), hence

\[ V(rma \cdot 1) = rV(ma \cdot 1) = r \sum_{(y,S) \in K} V(\alpha N^S) \quad \text{for integers} \ r \geq 1 \]

Let \( \Delta^T = \{ a \in \mathbb{R}_+ ^T \mid \sum_{t=1}^T a_t = 1 \} \). Define \( \nu : \Delta^T \to \mathbb{R}_+ \) by

\[ \nu(a) = \sup_{r > 0} \frac{V(ra)}{r} \]

Since \( V \) is superadditive \( \nu \) is concave (ES Lemma A1), hence we can let \( w \in \mathbb{R}^T \) satisfy \( w \cdot (\frac{ma \cdot 1}{|ma \cdot 1|}) = \nu(\frac{ma \cdot 1}{|ma \cdot 1|}) \) and \( w \cdot a \geq \nu(a) \) for all \( a \in \Delta^T \). It follows that

\[ w \cdot (\alpha N) \geq V(\alpha N) \quad \text{for all} \ N \in \mathbb{R}_+ ^n \]

To show that \( (w \cdot \alpha^i)_{i \in \mathbb{N}} \) is a competitive payoff, we first show that \( w \cdot (\alpha N^S) = V(\alpha N^S) \) for all \( (y, S) \in K \). Since \( W \cdot (\alpha N^S) \geq V(\alpha (r N^S)) \) for all \( r > 0 \), it follows that \( W \cdot N^S \geq \frac{V(\alpha(r N^S))}{r} \) for all \( r > 0 \), and hence \( \frac{W \cdot N^S}{|\alpha N^S|} \geq V(\frac{\alpha N^S}{|\alpha N^S|}) \). But since \( W \cdot N^S = V(\alpha N^S) \), it follows that

\[ V(\alpha N^S) = |\alpha N^S| \nu(\frac{\alpha N^S}{|\alpha N^S|}) \]

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Since \( v \) is concave and \( \sum_{(y,s) \in K} \alpha N^s = m\alpha \cdot 1 \), \( v\left(\frac{m\alpha \cdot 1}{|m\alpha \cdot 1|}\right) \geq \sum_{(y,s) \in K} \frac{|\alpha N^s|}{|m\alpha \cdot 1|} v\left(\frac{\alpha N^s}{|\alpha N^s|}\right) \).

If the inequality is strict, then there exists an integer \( r \geq 1 \) such that

\[
\frac{V(rm\alpha \cdot 1)}{|rm\alpha \cdot 1|} > \sum_{(y,s) \in K} \frac{|\alpha N^s|}{|m\alpha \cdot 1|} v\left(\frac{\alpha N^s}{|\alpha N^s|}\right) = \frac{1}{|m\alpha \cdot 1|} \sum_{(y,s) \in K} V(\alpha N^s)
\]

which contradicts (4). It follows that

\[
w \cdot \sum_{(y,s) \in K} \frac{\alpha N^s}{|m\alpha \cdot 1|} = w \cdot \left(\frac{m\alpha \cdot 1}{|m\alpha \cdot 1|}\right) = v\left(\frac{m\alpha \cdot 1}{|m\alpha \cdot 1|}\right) = \sum_{(y,s) \in K} \frac{|\alpha N^s|}{|m\alpha \cdot 1|} v\left(\frac{\alpha N^s}{|\alpha N^s|}\right) = \sum_{(y,s) \in K} V(\alpha N^s)
\]

Hence \( w \cdot (m\alpha \cdot 1) = \sum_{(y,s) \in K} w \cdot (\alpha N^s) = \sum_{(y,s) \in K} V(\alpha N^s) \). But since \( w \cdot \alpha N^s \geq V(\alpha N^s) \) for each \( (y,s) \in K \),

\[
w \cdot (\alpha N^s) = V(\alpha N^s) \quad \text{for all } (y,s) \in K \tag{7}
\]

By (5), (7), \( w \) is the gradient to \( V \) at \( (\alpha N^s) \), each \( (y,s) \in K \), hence, using the second equality in (3):

\[
w = \nabla V(\alpha N^s) = \phi(y, \alpha N^s) + E(y, \alpha N^s) - \nabla c(y, \alpha N^s), \text{ for each } (y,s) \in K
\]

Let \( p(y, \alpha N) = \phi(y, \alpha N) - w \) for each \( y \in Y, N \in \mathbb{R}_+^n \). These prices satisfy \( p(y, \alpha N) = p(y, \alpha N') \) if \( \alpha N \) and \( \alpha N' \) have the same externality-producing attributes, because then \( \phi(y, \alpha N) = \phi(y, \alpha N') \).

Condition 2. of competitive equilibrium holds because for all \( y \in Y, N \in \mathbb{R}_+^n \), (8) holds by (5) and (7) with equality at \( (y,s) \in K \).

\[
\sum_{i \in N} \hat{N}_i \alpha^i \cdot p(y, \alpha N) - e(y, \alpha N) = (\alpha N) \cdot (\phi(y, \alpha N) - w) - e(y, \alpha N) \leq 0 \tag{8}
\]

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Condition 1. of competitive equilibrium holds because agents with attributes \( \alpha^i \) receive utility \( w \cdot \alpha^i \) in every club \( (y, \alpha N) \):

\[
w \cdot \alpha^i = \alpha^i \cdot [\phi(y, \alpha N) - p(y, \alpha N)] = \alpha^i \cdot [\phi(y, \alpha N) - \phi(y, \alpha N) + w].
\]

**Proof of 3:** Let \((N^k)_{k \in K} \) be a partition of the set of types \( N \) such that if \( i, j \in N^k \), then \( \alpha^i \) and \( \alpha^j \) have the same externality-producing attributes, and if \( i \in N^k \), \( j \in N^{k'} \), then \( \alpha^i \) and \( \alpha^j \) have different externality-producing attributes. For each \( z \in \mathcal{Y}, N \in \mathbb{Z}^+_{\alpha} \), \( k \in K \), and \( i \in N^k \), let

\[
\hat{\pi}^i(z, \alpha N) = \max_{j \in N^k} (\alpha^j \phi(z, \alpha N) - W^j)
\]

The prices \( \hat{\pi} \) are constructed such that all agents with the same externality-producing attributes pay the same admissions prices to a given club.

We now show that \((\hat{\pi}, K) \) is a competitive equilibrium. **Condition 2.** For each \( k \in K \), \( y \in \mathcal{Y}, N \in \mathbb{Z}^+_{\alpha} \), let \( \hat{\pi}(k, z, N) \in N^k \) be such that \( \hat{\pi}(k, z, N)(y, \alpha N) = \alpha^i \phi(z, \alpha N) - W^i \). For every \( z \in \mathcal{Y}, N \in \mathbb{Z}^+_{\alpha} \), let \( M(z, N) \in \mathbb{Z}^+_{\alpha} \) be the vector such that \( M(z, N)(i, k, z, N) = \sum_{i \in N^k} N_i \) and \( M(z, N)(i, k, z, N) = 0 \) if \( i \neq \hat{\pi}(k, z, N) \) for all \( k \in K \). Then the profit earned by a club \((z, \alpha N) \) is \( \sum_{i \in N} Ni \hat{\pi}(z, \alpha N) - c(z, \alpha N) = \sum_{i \in N} M(z, N)(i, \alpha M(z, N)) - W^i - c(z, \alpha M(z, N)) \leq Q(\alpha M(z, N)) - M(z, N) \cdot W \leq 0 \) by Lemma 1(1). Here we use the fact that \( \phi(z, \alpha N) = \phi(z, \alpha M(z, N)) \) and \( c(z, \alpha N) = c(z, \alpha M(z, N)) \), since \( \alpha N \) and \( \alpha M(z, N) \) have the same externality-producing attributes (but not the same other attributes).

**Condition 1.** For each \( i \in N \), \( z \in \mathcal{Y}, N \in \mathbb{Z}^+_{\alpha} \), \( \alpha^i \phi(z, \alpha N) - \hat{\pi}(z, \alpha N) \leq W^i \) by construction of \( \hat{\pi} \). We now show that for each \( (y, S) \in K \) such that \( N^S \geq 0 \), \( \alpha^i \phi(y, \alpha N^S) - \hat{\pi}(y, \alpha N^S) = W^i \). Otherwise, using condition 2., \( 0 \geq \sum_{i \in \mathbb{N}} N^S_i \hat{\pi}(y, \alpha N^S) - c(y, \alpha N^S) > \sum_{i \in \mathbb{N}} N^S_i [\alpha^i \phi(y, \alpha N^S) - W^i] - c(y, \alpha N^S) = Q(\alpha N^S) - N^S \cdot W \), here the last equality uses the second equality of Lemma 1(2). But this is a contradiction, since \( 0 = Q(\alpha N^S) - N^S \cdot W \).
by the first equality of Lemma 1(2).

ES used superadditivity of $V$ to show both that core payoffs in large games can be decomposed as a linear sum on players’ attributes called a ‘hedonic payoff’, and that the core can be supported as a competitive equilibrium with hedonic prices as in Proposition 2(2) above. However the example below shows that superadditivity of $V$ might not hold in club economies. The key feature of the example is that the two attributes licentiousness and religiousness conflict with each other rather than complement each other. That is, one individual with both attributes receives lower utility than the sum of utilities available to two individuals, each with a single attribute, provided they do not share their attributes in a club. The one individual with both attributes cannot separate them in different clubs, and must therefore live with the conflict. This reduces his utility. Superadditivity of $V$ excludes such conflict in attributes. If $V$ is homogeneous and concave, then it is superadditive, so concavity can also exclude such conflicts.

**Example:** The function $V$ might not be superadditive in the attributes. If not, it might be impossible to decompose the core payoffs as a linear sum on attributes.

Suppose the only public project is the costless one (which can be interpreted as “no public project”), so that agents form clubs only to share their characteristics. Suppose there are three attributes in addition to the quasilinear good, and the utility of a member with attributes $\alpha^i$ in a club $(y_0, A)$ is

$$\alpha^i \left[ \frac{A_2}{A_1 + A_3} \frac{1}{A_2 + A_3} + 1 \right]$$

Thus $\phi(y_0, A) = (\phi^1(y_0, A), \phi^2(y_0, A), \phi^3(y_0, A)) = (\left[ \frac{A_2}{A_1 + A_3} \frac{1}{A_2 + A_3} \right], 0, 0)$. The second and third attributes produce externalities, and they are at war with each other in the sense that if they are combined in one club (or one individual), utility is reduced. For example these two attributes could represent “religiousness” and “licitentiousness”. The characteristic function $V$, which is neither concave nor superadditive, is drawn in Figure
1, which is the simplex restricted to the the 2nd and 3rd attributes, and the first attribute has value $\frac{1}{2}$.

If there are three types of agents, with attributes $\alpha^1 = (\frac{1}{2}, \frac{1}{2}, 0)$, $\alpha^2 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $\alpha^3 = (\frac{1}{2}, 0, \frac{1}{2})$ as shown in Figure 1, then total utility is maximized if the three types of agents are segregated in three different types of clubs. It can be seen that superadditivity is violated by noticing that the type-2 agent cannot achieve as much payoff as if he could divide himself into half a type-1 agent and half a type-3 agent.

Equal-treatment core payoffs are $U = (\frac{1}{2}, \frac{2}{3}, \frac{1}{2})$, which cannot be represented by an hedonic payoff; that is, there does not exist $w = (w_1, w_2, w_3)$ such that $U^i = w_1 \alpha^i_1 + w_2 \alpha^i_2 + w_3 \alpha^i_3$ for $i = 1, 2, 3$.

In addition, equilibrium admissions prices might not be expressable as a linear function of attributes in which the coefficients depend only on the attributes of the club that produce externalities. To see this, introduce a fourth and fifth type of agent with attributes $\alpha^4 = (2, \frac{1}{2}, 0)$ and $\alpha^5 = (\frac{1}{10}, \frac{1}{2}, 0)$. Types-4 and -5 have the same externality-producing attributes as type-1, but type-4 receives four times as much utility from any club, and type-5 receives a fifth as much. The core allocation has types-2 and -3 in their own clubs, but types-1 and -4 and -5 can share a club. The core payoffs for the five types of players are $(\frac{1}{2}, \frac{2}{5}, \frac{1}{2}, 2, \frac{1}{10})$.

We will consider admissions prices to clubs of type $(y_o, (A_1, 3, 1))$, with arbitrary $A_1$. For each $N \in \hat{N}$, $\hat{N} = \{(4, 4, 0, 0, 0), (6, 0, 2, 0, 0), (0, 0, 2, 6, 0), (0, 0, 2, 0, 6)\}$, the attributes of the clubs $(y_o, \alpha N)$ are $(4, 3, 1), (4, 3, 1), (13, 3, 1), (2, 3, 1)$. If the hedonic price $p(y_o, (A_1, 3, 1)) \in \mathbb{R}^3$ depends only on externality-producing attributes, it should have the same value for all these clubs. On the contrary, there do not exist prices $(p_1, p_2, p_3)$ such that no agent prefers one of the clubs $(y_o, \alpha N)$, $N \in \hat{N}$, and all such clubs make nonpositive profit. The conditions of nonpositive profit are, $(\alpha N) \cdot p \leq 0$, $N \in \hat{N}$, namely
Figure 1
\[
\begin{align*}
4 \ p_1 + 3 \ p_2 + p_3 & \leq 0 \\
4 \ p_1 + 3 \ p_2 + p_3 & \leq 0 \\
13 \ p_1 + 3 \ p_2 + p_3 & \leq 0 \\
\frac{16}{10} \ p_1 + 3 \ p_2 + p_3 & \leq 0 
\end{align*}
\]

The conditions that none of the four types of players could receive more utility than their core payoff under the price system is:

\[
\begin{align*}
\frac{1}{2} \left[ \frac{A_2}{A_2 + A_3} \right] \left( \frac{1}{2} p_1 + \frac{1}{2} p_2 \right) & \leq \frac{1}{2} \\
\frac{1}{2} \left[ \frac{A_2}{A_2 + A_3} \right] \left( \frac{1}{2} p_1 + \frac{1}{4} p_2 + \frac{1}{4} p_3 \right) & \leq \frac{2}{5} \\
\frac{1}{2} \left[ \frac{A_2}{A_2 + A_3} \right] \left( \frac{3}{2} p_1 + \frac{1}{2} p_2 \right) & \leq \frac{1}{2} \\
2 \left[ \frac{A_2}{A_2 + A_3} \right] \left( 2 p_1 + \frac{1}{2} p_2 \right) & \leq 2 \\
\frac{1}{10} \left[ \frac{A_2}{A_2 + A_3} \right] \left( \frac{1}{10} p_1 + \frac{1}{2} p_2 \right) & \leq \frac{1}{10}
\end{align*}
\]

where \( \frac{A_2}{A_2 + A_3} = \frac{1}{2^{4+1}} = \frac{16}{19} \). One can check that these inequalities are mutually inconsistent for all \((p_1, p_2, p_3) \in \mathbb{R}^3\).

5 Core/Competitive Convergence

Proposition 1 states that competitive equilibrium is in the core. If the economy is "large" then, conversely, payoffs in the equal-treatment core are competitive payoffs. We define the relevant notion of large by defining the set of "blocking opportunities" for the economy \((mP, c, \phi)\):

\[
\Omega = \{ W \in \mathbb{R}^n_+ \mid N \cdot W < \hat{Q}(N) \text{ for some } N \in \mathbb{R}^n_+ \}
\]

The economy \((mP, c, \phi)\) exhausts blocking opportunities if for every \(W \in \Omega\) there exists \(S \subset mN\) such that \(\hat{Q}(N^S) > N^S \cdot W\).

Exhaustion of blocking opportunities means that every \(W \in \Omega\) can be blocked, and therefore an equal-treatment payoff \(W\) in the core of the game is on the upper boundary
of the set Ω. This is Claim 1 below. The proof of Proposition 3 observes that a feasible
payoff in the upper boundary of Ω is a competitive payoff. Thus Lemma 1 and Proposition
3 together imply that a feasible payoff is a competitive payoff if and only if it is a payoff in
the upper boundary of Ω. For a diagram of the set of Ω, see ES, Figure 6, and for the same
idea without restricting to transferable utility, see Gilles and Scotchmer (1996), Figure 1
and Theorem 2.

Unless one makes an extreme assumption such as that only a finite set of clubs
"matter\(^4\), exhaustion of blocking opportunities is a limit concept.

Proposition 3 (If the economy exhausts blocking opportunities, every alloca-
tion in the equal-treatment core is a competitive equilibrium.)

Suppose that

1. \((mP, c, \phi)\) exhausts blocking opportunities.

2. \(W\) is an equal-treatment payoff in the core of \((mP, c, \phi)\).

3. \(mW \cdot 1 = \sum_{(y,S) \in K} (\alpha N^S) \cdot \phi(y, \alpha N^S) - c(y, \alpha N^S)\) for some club structure \(K\) of the
economy, and \(\dot{Q}\) is differentiable at \(N^S\) for each \((y, S) \in K\).

Then

4. \(W\) is a competitive payoff for \((mP, c, \phi)\).

Proof:

\(^4\)Engl and Scotchmer 1996b and Scotchmer 1996 show in their respective Lemmas 1 that under such an
assumption finite games can exhaust blocking opportunities. ES show in Proposition 1 that even without
such an assumption, large games "almost" exhaust blocking opportunities, provided per-capita payoffs are
bounded.
Claim 1

1. If hypotheses 1, and 2. hold, then \( N \cdot W \geq \hat{Q}(N) \) for all \( N \in \mathbb{R}_+^n \).

2. If hypotheses 1, 2. and 3. hold, then
\[
N^S \cdot W = \hat{Q}(N^S) = (\alpha N^S) \cdot \phi(y, \alpha N^S) - c(y, \alpha^S) \text{ for each } (y, S) \in K.
\]

Proof of Claim: (1) If \( N \cdot W < \hat{Q}(N) \) for some \( N \in \mathbb{Z}_+^n \), then by hypothesis 1. there exists \( S \subset mN \) such that \( N^S \cdot W < \hat{Q}(N^S) \), which would contradict hypothesis 2. Part (2) follows from Claim 1(1) and hypotheses 2. and 3. \( \square \)

For each \( y \in \mathcal{Y}, N \in \mathbb{R}_+^n \) and \( i \in \mathcal{N} \), let \( \pi^i(y, \alpha N) = \alpha^i \cdot \phi(y, \alpha N) - W^i. \)

Condition 2. of competitive equilibrium holds because \( \sum_{i \in \mathcal{N}} N_i \pi^i(y, \alpha N) - c(y, \alpha N) = (\alpha N) \cdot \phi(y, \alpha N) - c(y, \alpha N) - N \cdot W \leq 0 \), where the inequality holds by Claim 1(1), and holds with equality at \( (y, S) \in K \) by Claim 1(2). Condition 1. holds because prices have been chosen so that agents of type-\( i \) receive utility \( W^i \) in whatever club they join. \( \square \)

Proposition 4 (Convergence of the Core to a Competitive Payoff) Let \( (mP, c, \phi) \) be an economy for which Assumptions 1 and 2 hold and competitive equilibrium exists. Consider a sequence of economies \( (r_{mP}, c, \phi), r = 1, 2, \ldots \). Suppose that \( q \) is differentiable at \( (\frac{1}{n}, \ldots, \frac{1}{n}) \). Let \( \gamma \in (0, 1], \delta > 0 \) be given. Then there exists \( r^* > 0 \) such that if

1. \( W \) is a competitive payoff for \((mP, c, \phi)\).

2. \( r > r^* \)

3. \( |S| \geq \gamma r \) where \( S \subset rmN \)

4. \( (U^i)_{i \in rmN} \) is a payoff in the core of \((rmP, c, \phi)\),

then
5. \( W \) is a competitive payoff for \((rmP, c, \phi)\)

6. \( |N^S \cdot W - \sum_{i \in S} U^i| < \delta|N^S| \).

**Proof:** First we prove 5. \( W \) is feasible for \((rmP, c, \phi)\) because it can be achieved by replicating the club structure that achieves \( W \) \( r \) times. By Lemma 1, \( \hat{Q}(N) \leq N \cdot W \) for all \( N \in \mathbb{R}^n_+ \), hence for all \( N^S, S \subseteq rmN \). It follows that \( W \) is in the core of \((rmP, Q)\). The proof of Proposition 3 then shows that there is a competitive equilibrium for \((rmP, c, \phi)\). Conditions 1. and 2. of Claim 1 hold by Lemma 1 instead of by hypotheses 1. and 2. of Proposition 3, and the remainder of the proof of applies.

Since \( W \) is in the equal-treatment core of \((rmP, c, \phi)\), each \( r \geq 1 \), part 6. follows as a restatement of the Corollary to Theorem 1 of ES stated above.

\[ \square \]

6 Conclusion

Most decentralization theorems in club theory use aggregate prices in which each agent's admissions price depends on his 'type' and on characteristics of the club. However the intuition behind decentralization is that agents are asked to pay for the externalities they impose on other members. Sometimes the externalities are positive, as when they share the costs of public goods, and sometimes negative, as when other members dislike some of their personal characteristics.

In their Example 4, ES presented a club model in which they showed that the decentralizing prices could be understood as hedonic prices on agents' characteristics: Each agent pays the same amount per unit attribute contributed, and the only attributes with nonzero prices in equilibrium clubs are those that impose externalities. In this paper I have relaxed the assumptions of superadditivity in attributes and homogeneity of preferences,
and included public projects. With these relaxations, the result holds only in equilibrium clubs.

The arguments above are of two types: a characterization of equilibrium admissions prices as linear prices on attributes, and in particular prices on externality-producing attributes, and an argument that core payoffs converge to competitive payoffs. However there is no argument that competitive equilibrium exists. Theorem 2 of Gilles and Scotchmer (1996), which applies to the economies described here, states that competitive equilibrium exists if and only if the economy has "efficient scale". In the notation above, the economy has efficient scale if and only if a payoff $W$ in the upper boundary of $\Omega$ is feasible. Scotchmer (1996) exalts two reasons that the economy might not have efficient scale. First, there might be economies of scale regarding the sizes of clubs, and second, even if per-capita payoffs in clubs are homogeneous of degree 1 in the size of the club, there might be a "composition" problem. However if blocking opportunities can be exhausted in a finite economy, then in a large economy a payoff on the boundary of $\Omega$ can be achieved for all but a small fraction of agents, and an approximate competitive equilibrium exists in the sense that all but a small fraction of agents are optimizing.

Finally, I close with a remark on the 'Henry George Theorem', which states that it is optimal to increase usage of a public facility until costs are just covered when each user pays for the externality he imposes. This is implied by Proposition 2 above, since equilibrium requires that the revenues collected through externality pricing are just equal to the costs of public goods. To see that the result mimics one that is well known, I recall the result for clubs with homogeneous members. The 'optimal' club size $n^*$ maximizes $U(n, y(n), w - \frac{C(y, n)}{n})$ where $y(n) = \arg \max \ U(n, y, w - \frac{C(y, n)}{n})$, and $w$ is each agent's endowment of private good. The admissions price $p(n^*)$ that supports this optimum and sustains zero profit is $p(n^*) = n^* \cdot \frac{-U_s(n^*, y(n^*), w - \frac{C(y(n^*), n^*)}{n^*})}{U_s(n^*, y(n^*), w - \frac{C(y(n^*), n^*)}{n^*}) + C_s(y(n^*), n^*)}$. That is, the price is equal to the crowding externality imposed by an agent plus his contribution to the cost of club goods.
References


