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THE COMPUTATION OF THE MEAN PROPERTIES OF TURBULENT FLOWS BY THE METHOD OF COARSE GRAINING

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ABSTRACT

The method of coarse graining for the computation of the mean properties of turbulent flows is presented. A new derivation of the method is given which attempts to carefully elucidate the connection between the physical processes of turbulent flow and the mathematical expressions that are used to approximate them. In particular, a new general turbulent transport law is derived and a new relatively rigorous closure relation is given. The method is applied to a study of the turbulent flow in a channel, after special consideration is given to the formation of proper boundary conditions. The variation of the mean flow field with Reynolds number is investigated. The computed mean velocity profiles show good agreement with experiment and the computed friction law reveals a drag crisis at low Reynolds numbers and a bifurcation separating laminar and turbulent solutions at a critical Reynolds number near that found experimentally.
I. INTRODUCTION

An excellent summary of most of the established numerical methods for computing the mean properties of turbulent flows has been given by Reynolds (1976). Though the theoretical and computational aspects of each of the methods considered by Reynolds vary considerably from one to another, on the whole these methods are based on just two fundamental approaches toward closing the equations of motion. One technique is to close the mean momentum equation by approximating the Reynolds stresses using an eddy diffusivity model, e.g. Patankar and Spalding (1970), Cebeci and Smith (1974), Smagorinsky, et. al. (1965), Deardorff (1970), Saffman (1974) and Schumann (1975). The other, and more recent approach, is that which attempts to compute both the mean velocity field and the Reynolds stresses by modeling the Reynolds stress equations themselves, e.g. Launder, et al. (1975), Hanjalic and Launder (1976) and Lumley and Khajeh-Nouri (1974).

The physical processes of turbulent flow which are modeled as part of the formulation of closure in these methods are not, as yet, clearly understood. This has made it difficult to determine if a particular analytical expression that is used in one of these methods is an accurate approximation to the physical process it is intended to represent. This uncertainty has led to great difficulty in selecting a preferred approach to use, especially in view of the generally checkered performances given by these methods when applied to various turbulent flow situations. It is our intent in this article to present an alternative to these methods which relies on relatively rigorous mathematical representations of
clearly understandable physical processes in a turbulent flow. Furthermore, in applications it shows promise of giving consistently useful approximations of turbulent flow.

The method in question is an extensively revised formulation of the method of coarse graining that was developed by Chorin (1974). Our derivation of this method is intended to correct and improve the theoretical arguments and computational formulas used in it. We will take particular care to elucidate the connection between the physical processes of the turbulent flow and the analytical expressions that are used to approximate them numerically.

In the method of coarse graining,* the averaged state of the turbulent fluid is described through use of the mean and variance of the vorticity field. A closure to the equations for these properties is based on the use of two simple ideas about the dynamics of turbulent flow. The first idea, which is used in the derivation of a turbulent transport law for the vorticity and squared vorticity, is that of the conservation of the vorticity along the trajectories of fluid particles in a two-dimensional high Reynolds number flow. This property of a flow was used previously by Taylor in his Vorticity Transport theory (Taylor (1915), (1932)) and in fact our transport law will have some resemblance to the transport law he derived. The second basic property of the turbulent motion which is used is that of the limited distance over which the correlation between the vorticities at two points in a turbulent flow will be significant.

*For convenience we will continue to address our current method by the term "coarse graining." In fact, the present derivation of this approach does not incorporate the use of the coarse grained approximation of the turbulent flow field which was an integral part of the original derivation of the method by Chorin.
This property is used to compute the Reynolds stresses, which are an integral part of the transport law, in terms of the global distribution of the variance of the vorticity field.

To provide a stringent test of the revised method we will apply it to an investigation of the turbulent flow in a channel. We have made some small use of the experimental data to allow us to compute accurate solutions to this problem. This has consisted in the determination of two parameters appearing in our boundary conditions by using the measurements of the flow at one particular Reynolds number. One parameter is obtained using the value of the friction coefficient and the other by adjusting its value until the computed and experimental velocity profiles match. By following this procedure we have been able to make an honest investigation of the flow at all other Reynolds numbers. At other Reynolds numbers for which experimental data exists we were able to accurately compute the measured velocity profiles and friction coefficients. We also found that we were able to reproduce many of the qualitative features of the turbulent flow in a channel, such as a drag crisis at low Reynolds numbers and a critical Reynolds number below which our solutions collapse to the laminar flow and above which they are turbulent.

The theory and computations reported here presently apply to two dimensional flow. The application of our method to this case represents an instructive first step in establishing its usefulness since this flow is a non-trivial model problem whose physical features and mathematical difficulties are either equivalent to, or are simpler versions of, those encountered in three dimensions.

The not insignificant success we have in computing the mean properties of a flow in a channel indicates that it is at least partially valid to attempt to compute this real mean flow field using a two-dimensional method.
However, we must take care to delineate the important effects of two-dimensionality. In particular we should point out that we will be using various turbulent scales, e.g., the Lagrangian integral time scale, which will also have meaning in a three-dimensional flow. However, we cannot assume that they must have the same values in both two and three dimensions.

Our attitude will be to subject this method to a full accounting as to the validity of its physical assumptions and mathematical arguments. We will indicate the various current deficiencies to this approach and how they limit its effectiveness. Each of these limiting factors, however, will be seen to be a sharply defined problem which may possibly be resolved at a later time. Thus we hope to show that coarse graining is evolutionary in nature and may be improved in the future at such time as any of these outstanding problems are solved.

The remainder of this paper is divided into four parts. In Section II we present a complete and self-contained derivation of the method of coarse graining. This will consist in the formulation of a closed system of equations which may be solved numerically to determine the evolution of the mean vorticity field, $\overline{\zeta}$, and vorticity variance, $\overline{\xi'^2}$. $\xi = \overline{\zeta} + \xi'$ is the component of vorticity orthogonal to the plane of motion.

In Section III we apply the method of coarse graining to the channel problem and discuss the formation of a set of difference equations. In particular we must make an extensive analysis of the flow in the wall region of the channel in order to derive correct boundary conditions to the equations of motion. In Section IV we discuss the results of
of our computations of the flow in a channel, and in section V we summarize the favorable and unfavorable aspects of the method of coarse graining in its present form and the outlook for its improvement in the future.

Unless stated otherwise, all variables or functions that follow are assumed to be dimensionless. A characteristic length and velocity of the mean flow are used to perform this scaling and the Reynolds number appearing in the equations of motion is formed from them.
II. DERIVATION OF THE METHOD OF COARSE GRAINING

We will provide a closure to the exact equations for the mean vorticity field, $\overline{\zeta}$, and vorticity variance $\overline{\zeta^2}$, which we will henceforth call $\zeta$, i.e. $\overline{\zeta^2} = \zeta$. The exact equations for $\overline{\zeta}$ and $\zeta$ are derived from the vorticity equation. These equations contain the turbulent fluxes of $\zeta$ and $\zeta^2$, i.e., $u_i^1 \zeta$ and $u_i^1 \zeta^2$. To close the system of equations we must express these fluxes in terms of the $\overline{\zeta}$ and $\zeta$ fields. We will accomplish this by using a general transport law that will be derived for this purpose. This law is valid in any number of dimensions for any quantity which is passively convected in a turbulent flow. It will be in the form of an expansion in (roughly) the Lagrangian integral time scale and will to first order be a simple mean gradient approximation. To complete the closure of these equations we will show how the velocity moments $\overline{u_i^1 u_j^1}$, which are contained in the transport law may be computed from the $\zeta$ field.

The derivation of the method will now follow in four parts. In the first we derive our transport law, in the next two we derive equations for $\overline{\zeta}$ and $\zeta$ respectively, and in the final part we complete the closure of our equations by deriving a relationship between $\zeta$ and $\overline{u_i^1 u_j^1}$.

II.1 Turbulent Transport Law

We will derive a general turbulent transport law which is valid in any number of dimensions for any quantity $\phi$ which is passively convected in a velocity field $u$, i.e. obeys the relation
\[ \frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + u \cdot \nabla \phi = 0 \quad (1) \]

Let \( x(t) \equiv x(x_0, t) \) represent the trajectory of a fluid particle known to be at a position \( x_0 \) at the time \( t_0 \), i.e. \( x(t_0) = x_0 \). \( x(t) \) satisfies the integral equation

\[ x(t) = x_0 + \int_{t_0}^{t} u(s) ds \quad (2) \]

where \( u(s) = u(x(s), s) \). Using \( x(t) \) we may write (1) in the equivalent form

\[ \phi(x(t), t) = \phi(x_0, t_0) \quad (3) \]

If \( x(t_0 - \tau) = a \) where \( \tau \) is a short time interval, then (2) implies that

\[ a = x_0 - L \quad (4) \]

where \( L \equiv \int_{t_0 - \tau}^{t_0} u(s) ds. \)
We wish to approximate $u_i^\phi$ at a point $x_0$ at time $t_0$. Using (3) evaluated at $t_0 - \tau$ we have

$$u_i^\phi \equiv u_i^\phi(a, t_0 - \tau) = u_i^\phi(a, t_0 - \tau) + u_i^\phi'(a, t_0 - \tau)$$

where quantities are presumed to be evaluated at $x_0$, $t_0$ unless indicated otherwise. We now will discuss each of the terms on the far right in (5) in detail.

The first term may be written as $u_i^\phi(x_0 - L, t_0 - \tau)$ after replacing $a$ by (4). Making a Taylor's expansion of $\phi$ about $x_0$, $t_0$ we obtain

$$u_i^\phi(a, t_0 - \tau) = -u_i^L_i \frac{\partial \phi}{\partial x_i} + \tau u_i^L_j \frac{\partial^2 \phi}{\partial x_j \partial t} + \frac{1}{2} u_i^L_i L_j \frac{\partial^2 \phi}{\partial x_j \partial x_k} + O(\tau^3).$$

To place this expression in usable form we must examine the quantities $u_i^L_i$ and $u_i^L_i L_j$. From this point on we will refrain from indicating the presence of terms of order higher than $\tau^2$, and will drop them without comment as they arise in the ensuing equations. These higher order terms may be computed also, though beyond the point we have gone this becomes a rather tedious undertaking.

Let us consider the term $u_i^L_i L_j$. Define $\overline{L}_j = \int_{t_0}^{t_0 - \tau} U_j(s)ds$ and

$L'_j = \int_{t_0 - \tau}^{t_0} u_j^i(s)ds$ so that $L_j = \overline{L}_j + L'_j$. Substituting for $L_j$ in $u_i^L_i L_j$ we find
Using the definition of \( L_j \) the second term on the right becomes

\[
\overline{u_i' L_j} = \overline{u_i' L_j} + \overline{u_i' L_j'}.
\]  

(7)

Using the definition of \( L_j \) the second term on the right becomes

\[
\overline{u_i' L_j} = \mathbb{E} \left[ u_i' \int_{t_0}^{t_0} u_j(s)ds \right] = \overline{u_i' u_j} \int_{t_0}^{t_0} R_{ij}(s-t_0)ds
\]

\[
= \overline{u_i' u_j} \int_{0}^{T} R_{ij}(s)ds
\]

(8)

where

\[
R_{ij}(s) = \frac{u_i'(t_0)u_j'(t_0+s)}{u_i'(t_0)u_j'(t_0)}
\]

\( R_{ij}(s) \) is a Lagrangian auto-correlation function, a special case of which was originally defined by Taylor (1921), see also Hinze (1959) p. 47.

The final step in obtaining (8) required making the assumption that \( R_{ij}(-t) = R_{ij}(t) \) for \( 0 \leq t \leq \tau \) which holds if the turbulence may be considered to be approximately stationary over a time period of \( O(\tau) \).

For \( \tau \) large enough, say \( \tau \geq \tau^* \), \( R_{ij}(\tau) \) is approximately zero, implying that \( \int_{0}^{\tau} R_{ij}(s)ds, \tau \geq \tau^* \), is independent of \( \tau \). Defining the Lagrangian integral time scale \( T_{ij} = \int_{0}^{\infty} R_{ij}(s)ds \), (8) becomes
if \( \tau \geq \tau^* \). Note that no summation is implied in (9) or in similar relations to follow which involve \( T_{ij} \) or other time constants still be to defined. Note also that it is possible that \( T_{ij} \) varies in space or time in a particular flow if the turbulence is nonuniform or nonstationary.

Returning to the first term on the right side of (7) we have

\[
\overline{u_i' \overline{L}_j} = T_{ij} \overline{u_i' u_j'}
\]

(9)

and Taylor's expansion of \( U_j(s) \) about \((x_0, t_0)\) yields

\[
\overline{u_i' L_j} = E \left[ u_i' \int_{t_0-\tau}^{t_0} U_j(s)ds \right]
\]

and Taylor's expansion of \( U_j(s) \) about \((x_0, t_0)\) yields

\[
\overline{u_i' \overline{L}_j} = E \left[ u_i' \tau U_j + u_i' \int_{t_0-\tau}^{t_0} (x_k(s)-x_k)ds \frac{\partial U_j}{\partial x_k} + u_i' \int_{t_0-\tau}^{t_0} (s-t_0)ds \frac{\partial U_j}{\partial t} \right]
\]

\[
= \frac{\partial U_j}{\partial x_k} E \left[ u_i' \int_{t_0-\tau}^{t_0} ds(x_k(s)-x_k) \right]
\]

(10)
where $x_k$ is the kth component of $x_0$. Through the use of (2) for $x_k(s)-x_k$, the integral in (10) becomes

$$
\int_{t_0}^{t_0-\tau} (x_k(s) - x_k) \, ds = - \int_{t_0-\tau}^{t_0} ds' (s' - t_0 + \tau) u_k(s') .
$$

Substituting (11) in (10) after replacing $u_k(s')$ by $U_k(s') + u_k'(s')$ gives

$$
\overline{u_{iL}} = - \overline{u_{i} u_k'} \frac{\partial U_j}{\partial x_k} \int_0^\tau (\tau-s)R_{ik}(s) \, ds .
$$

In light of our assumption on $\tau$, (12) becomes

$$
\overline{u_{iL}} = \overline{u_{i} u_k'} (S_{ik} - \tau T_{ik}) \frac{\partial U_j}{\partial x_k} .
$$

where $S_{ik} = \int_0^\infty sR_{ik}(s) \, ds$. Combining (9) and (13) yields the result

$$
\overline{u_{iL}} = \overline{u_{i} u_k'} T_{ij} + \overline{u_{i} u_k'} (S_{ik} - \tau T_{ik}) \frac{\partial U_j}{\partial x_k} .
$$
The analysis of \( u_{ij}^1 L_{jk} \) is considerably longer than that of \( u_{ij}^1 \), so we will only quote the result: To third order

\[
\begin{aligned}
\overline{u_{ij}^1 L_{jk}} &= \tau u_{ij}^1 u_{ij}^1 T_{ik} + \tau u_{ik}^1 u_{ij}^1 T_{ij} + u_{ik}^1 u_{ij}^1 T_{ijk} \\
&= \int ds_1 \int ds_2 \left( \int \overline{u_{ij}^1 u_{ij}^1} \right) ds_1 ds_2
\end{aligned}
\]  

(15)

where

\[
T_{ijk} = \int ds_1 \int ds_2 R_{ijk}(s_1,s_2)
\]

and

\[
\overline{u_{ij}^1 u_{ij}^1u_{ik}^1 u_{ij}^1} = \overline{u_{ij}^1 u_{ij}^1u_{ij}^1 u_{ij}^1}.
\]

The derivation of (15) has made use of the relation \( R_{ijk}(-s_1,-s_2) = R_{ijk}(s_1,s_2) \) which holds for turbulence which is approximately stationary over the time period \( \tau \).

In case of \( \overline{u_{ij}^1 u_{ij}^1u_{ik}^1} = 0 \) as will occur if \( i=j=k \) and \( u_i^1 \) is symmetrically distributed, then the last term in (15) should read

\[
\int_{t_0}^{t_0+\tau} \int_{t_0}^{t_0+\tau} \overline{u_{ij}^1 u_{ij}^1u_{ij}^1 u_{ij}^1} ds_1 ds_2.
\]

However, it is most reasonable that under these conditions \( \overline{u_{ij}^1 u_{ij}^1u_{ij}^1 u_{ij}^1} \) will be negligible for all \( s_1, s_2 \).
This follows since \( u_i^ju_j^l(s_1)u_k^l(s_2) = 0 \) when \( s_1 = s_2 = t_0 \) and thus \( u_i^t, u_j^t \) and \( u_k^t \) are most correlated, and it is also equal to zero when \( s_1 \) and/or \( s_2 \) are not near \( t_0 \) and hence \( u_i^t \) and \( u_j^t(s_1) \) and/or \( u_k^t(s_2) \) are most uncorrelated. Therefore we will assume that the final term in (15) is actually zero when \( u_i^ju_j^t = 0 \) and will ignore the fact that \( T_{ijk} \) is undefined in this case. A similar argument applies to the terms in (14) involving \( u_i^ju_j^t \), i.e. if at a particular point \( u_i^ju_j^t = 0 \).

We will suppose that because of our condition as to the magnitude of \( \tau \) it is safe to presume that \( u_i^t \) at \((x_0^t, t_0)\) is uncorrelated with \( \phi^t \) at \((a, t_0 - \tau)\), i.e. \( u_i^t(\phi^t(a, t_0 - \tau) = 0 \). Though we cannot give a rigorous proof that this follows from our requirement that \( u_i^t(t_0)u_j^t(t_0 - \tau) = 0 \), it is indeed plausible in view of it, since velocity-velocity correlations are among the most enduring in a turbulent flow.

Substituting our results (14) and (15) into (6) and invoking the assumption we have just made we attain the general transport law:

\[
\overline{u_i^t} = T_{ij} \overline{u_i^tu_j^t} \quad (S_{ik} - \tau T_{ik}) \frac{\partial u_j^t}{\partial x_k} \frac{\partial \phi^t}{\partial x_j} + \\
\tau T_{ij} \overline{u_i^tu_j^t} \left[ \frac{\partial^2 \phi}{\partial x_j \partial t} + U_k \frac{\partial^2 \phi}{\partial x_j \partial x_k} \right] + \tau \overline{u_i^tu_j^t} T_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \quad (16)
\]

what is valid through second order in \( \tau \) for any function \( \phi \) satisfying (1).

The apparent dependence of the right side on the parameter \( \tau \) is only formal. In actuality, the larger we pick \( \tau \) the more terms in the expansion on the right are non-negligible, with the result that their sum is constant.
An important feature of this transport law is that it takes into account the often neglected fact, (see Corrsin (1974)), that the length and time scales of the fluctuating motion are frequently comparable to those of the mean flow field itself. This is clear since those terms on the right which are the product of the average of a fluctuating quantity and a derivative of the $\overline{\phi}$ field are an approximation to the variation of $\overline{\phi}$ over a distance and/or time selected by the coefficient of the $\overline{\phi}$ derivative. If the term is non-negligible then the variation of the $\overline{\phi}$ field is significant over this distance and/or time.

The transport law (16) is most useful in flow situations where it is permissible to keep only its lower order terms, for then we have the simple mean gradient transport law

$$\overline{u_i\phi} = -T_{ij} \overline{u_i u_j} \frac{\partial \overline{\phi}}{\partial x_j} \quad \ldots (17)$$

In a steady unidirectional mean velocity field $U(y)$ in which $\overline{\phi} = \overline{\phi}(y)$, such as in the channel, the second order terms in (16) are identically equal to 0. For this type of flow the transport law in the form (17) is especially valid.

II.2 The Mean Vorticity Equation

The mean vorticity equation is obtained by averaging the vorticity equation. In a two-dimensional flow the relation for $\overline{\zeta}$ is:
which is seen to contain the mean turbulent flux of vorticity \( \overline{u_i u_j} \).

For large Reynolds number two-dimensional flow at points sufficiently removed from any boundaries we may suppose that the vorticity satisfies (1) to good approximation. Therefore we may legitimately apply the expression (17) to \( \overline{u_i u_j} \). The closed form of the mean vorticity equation will then be

\[
\frac{\partial \overline{\zeta}}{\partial t} = -\overline{u \cdot \nabla \zeta} - \nabla \cdot \overline{u \cdot \zeta} + \frac{1}{R} \nabla^2 \zeta
\]

which is only valid in the interior of the flow domain.

II.3 The Equation for the Vorticity Variance

An exact relation for \( \zeta \) may be obtained by multiplying the vorticity equation by \( \xi \), averaging and then subtracting \( \overline{\zeta} \) times the exact equation for \( \overline{\zeta} \). We get:

\[
\frac{\partial \xi}{\partial t} = -\overline{u \cdot \nabla \xi} - 2\overline{u' \cdot \xi} \overline{\zeta} - \nabla \cdot \overline{u' \xi}^2 + \frac{1}{R} \nabla^2 \xi - \frac{2}{R} \left( \frac{\partial \xi}{\partial x_i} \right)^2
\]
where the first term on the right contributes to the convection of \( \zeta \) by the mean vorticity field, the next to the production of \( \zeta \) from the squared mean vorticity field, the next to the transport of \( \zeta \) due to the turbulent motion, the next to the molecular diffusion of \( \zeta \) and the last one to the viscous dissipation of \( \zeta \).

Since we have presumed that \( \xi \) satisfies (1) to good approximation so too must \( \xi^2 \), and therefore we may apply (17) for \( \xi^2 \) also. Using the identity \( \xi^{12} = \xi^2 - 2\xi \xi' - \overline{\xi^2} \) we will find that

\[
\overline{u_i \xi^{12}} = -T_{ij} \overline{u_i u_j} \frac{\partial \xi}{\partial x_j}
\]

and this may be used in (19).

To place (19) into useful form we must also approximate the last term representing dissipation. This is easily done by introducing Taylor vorticity microscales \( \lambda_i \), \( i = 1, 2 \) defined by

\[
\lambda_i^2 = \frac{-2}{\frac{\partial^2}{\partial x_i^2} R(0)}
\]

where \( R \) is the Eulerian vorticity correlation function:

\[
R(r) = \frac{\overline{\xi'(x) \xi'(x+r)}}{\overline{\xi^{12}(x)}}
\]
Using $\lambda_i$ one may show that for locally homogeneous turbulence

$$\frac{(\partial \xi')^2}{(\partial \xi')_{x_i}} = -\frac{\partial^2 R}{(0)\xi} = \frac{2\xi}{\lambda_i^2},$$

Defining $\lambda_d^2 = \frac{\lambda_1^2 \lambda_2^2}{2(\lambda_1^2 + \lambda_2^2)}$ and using our expressions for $\overline{u_i'\xi}$ and $\overline{u_i'\xi}^2$, the equation for $\xi$ becomes

$$\frac{\partial \xi}{\partial t} = -\nabla \cdot \nabla \xi + 2 T_{ij} \overline{u_i' u_j'} \frac{\partial \xi}{\partial x_i} \frac{\partial \xi}{\partial x_j}$$

$$+ \frac{\partial}{\partial x_i} \left( 1 \frac{\partial \xi}{R \partial x_i} + T_{ij} \overline{u_i' u_j'} \frac{\partial \xi}{\partial x_j} \right) - \frac{2\xi}{\lambda_d^2},$$

which again is not valid near boundaries.

II.4 The Closure

Consider a general two dimensional turbulent flow in a domain $D$ with boundary $\partial D$. If we define a stream function $\psi'$ from $u' = \psi'_y$ and $v' = -\psi'_x$, then $\xi'$ and $\psi'$ are related through

$$\nabla^2 \psi' = -\xi'$$

with the boundary condition that $\psi' = 0$ on $\partial D$. The solution of this Poisson equation may be written as
\[
\psi'(x) = - \int_D G_r(x;x') \xi'(x') dx
\]

where \( G_r(x;x') \) is the Greens function associated with the Laplacian \( \nabla^2 \psi' \). Differentiating this expression we obtain, e.g.

\[
u'(x) = - \int_D G(x;x') \xi'(x') dx'
\]

(21)

where \( G(x;x') \equiv \frac{\partial G_r}{\partial y} (x;x') \).

To relate \( \bar{u}'^2 \) to \( \xi \) we square (21), average and then get

\[
\bar{u}'^2 = \int_D dx' G(x;x') \int_D dx'' G(x;x'') \bar{\xi}'(x') \bar{\xi}(x'')
\]

Reintroducing the Eulerian vorticity correlation function defined previously we have
\[
\overline{u'^2} = \int_{D} d\tilde{x}' G(x;\tilde{x}') \zeta(\tilde{x}') \int_{D} d\tilde{x}'' G(x;\tilde{x}'') R(\tilde{x}',\tilde{x}'-\tilde{x}')
\]

where we have now explicitly indicated the dependence of \( R(\tilde{r}) \) on \( \tilde{x} \) by writing it as \( R(\tilde{x}',\tilde{r}) \). Introducing the change in coordinates \( \tilde{x}'' = \tilde{x}' + \tilde{r} \) we get

\[
\overline{u'^2} = \int_{D} d\tilde{x}' G(x;\tilde{x}') \zeta(\tilde{x}') F(\tilde{x}')
\]

(22)

where

\[
F(\tilde{x}') \equiv \int_{D(\tilde{x}')} G(x;\tilde{x}'+\tilde{r}) R(\tilde{x}',\tilde{r}) d\tilde{r}
\]

and \( D(\tilde{x}') \equiv \{ \tilde{r}: \tilde{x}'+\tilde{r} \in D \text{ and } R(\tilde{x}'+\tilde{r}) \neq 0 \} \). At points in \( D \) not near \( \partial D \) it is clear that \( D(\tilde{x}') \) will be approximately a circular domain \( |\tilde{r}| < \xi \) for some length \( \xi \) centered about the point \( \tilde{x}' \).

In practice we will wish to compute \( \overline{u'^2} \) at a point \( \tilde{x} \) situated with respect to a grid as depicted in Figure 1, where \( D \) has been partitioned into \( N \) boxes, \( D_i, i = 1, \ldots, N \) of equal area \( h^2 \). The two boxes adjacent to the point \( \tilde{x} \) are called \( D_1 \) and \( D_2 \). Letting \( \tilde{x}_i \) represent the center of
the box $D_i$ and $\zeta_i = \zeta(x_i)$ we will derive an appropriate quadrature to the exact integral relation (22). This relation will allow us to determine $u^{\mp}(x)$ from the values of the $\zeta_i$, $i = 1, \ldots, N$. The $\zeta_i$ will also naturally appear when we derive a difference approximation to the $\zeta$ equation (20), in Section III.2.

Our first step in approximating (22) is to see how $F(x')$ may be computed. $G(x:x')$ may be written as

$$G(x:x') = G_1(x:x') + G_2(x:x')$$

(23)

where

$$G_1(x:x') = \frac{1}{2\pi} \frac{(y-y')}{(x-x')^2 + (y-y')^2}$$

(24)

$G_2 = 0$ in unbounded domains and in simple bounded domains (such as a channel) it represents the contribution of image vortices to $G$. In all cases $G_2$ is a smooth function of $x'$ if $x \not\in \partial D$. The correlation function $R(x',r)$ is in general unknown, thus precluding the possibility of computing $F(x')$ exactly. However if $x'$ is such that $x \not\in D(x')$ then $G(x:x'+r)$ behaves nicely for all $r \in D(x')$. Defining an area scale

$$L(x') \equiv \int_{D(x')} R(x',r)dr$$

a good approximation to $F(x')$ will be
on the condition that \( D(\zeta') \) is not too large.

Since we do not know of any experimental determinations of \( R(\tau) \) we can only speculate as to the conditions under which we expect \( D(\zeta') \) to be relatively small. Clearly \( R(\tau) \) is related to the size and frequency of occurrence of coherent vortical structures in a turbulent flow. In confined flows (such as a channel) which are strongly influenced by the presence of boundaries, the fluid will be a mixture of the many small vorticies that are generated at the walls and the larger coherent structures they coalesce into as they move away from them. We surmise that in this kind of flow situation it is safe to presume that \( D(\zeta') \) is not unduly large. In contrast, we may suppose that this assumption will be poor in exterior flows, such as a free shear layer, where the flow is dominated by a few large vortices. We will base our continued development of the closure relation on the assumption that \( D(\zeta') \) is small enough to insure that (25) is a good approximation.

For those values of \( \zeta' \) for which \( \zeta' \in D(\zeta') \), we may only assert with confidence that

\[
F(\zeta') \approx F'(\zeta') + G_2(\zeta' : \zeta') L(\zeta')
\]  

(26)
where

\[ F'(x') = \int_{D(x')} G_1(x:x'+r) R(x',r) \, dr. \quad (27) \]

The relations (25) and (26) will be better the further \( x' \) is from the boundaries or the point \( x \).

If we use the approximation (25) for all \( x' \notin D_1 UD_2 \) and (26) for \( x' \in D_1 UD_2 \) then (22) may be written, after breaking up the integral over \( D \) into a sum of integrals over the \( D_i \), as

\[
\overline{u^2} = \int_{D_1 UD_2} G_1(x:x') F'(x') \zeta(x') \, dx' + \int_{D_1 UD_2} G_1(x:x') G_2(x:x') L(x') \zeta(x') \, dx' + \]

\[
+ \int_{D_1 UD_2} G_2(x:x') F'(x') \zeta(x') \, dx' + \int_{D_1 UD_2} G_2(x:x') L(x') \zeta(x') \, dx' + \]

\[
\sum_{i=3}^N \int_{D_i} G_2^2(x:x') \zeta(x') L(x') \, dx'. \quad (28) \]

The integrals contained in the sum in the last term on the right side may be approximated as \( G_2^2(x:x_i) \zeta_i L_i h^2 + O(h^4) \) where \( L_i \equiv L(x_i) \). Similar approximations also apply for the next to last term on the right.
The second term on the right of (28) is $\approx 0$ since

$$\int_{D_1 U D_2} G_1(x:x')G_2(x:x')L(x')\zeta(x')dx' \approx G_2(x:x)L(x)\zeta(x) \int_{D_1 U D_2} G_1(x:x')dx' = 0$$

due to the antisymmetry of $G_1(x:x')$. The third integral is also $\approx 0$ because of the approximate antisymmetry of $F'(x')$. A careful analysis will show that, in fact, when $x$ is near the boundary both the second and third terms on the right of (28) will have small non-zero values due to the variation of the function $R(x',\tau)$ near the wall. We will assume that these terms are no larger than the truncation errors of $O(h^4)$ introduced earlier and thus may be ignored.

To complete our approximation of $u^{1/2}$ we now have to consider the first term on the right side. This integral may be simplified slightly by removing $\zeta(x')$ from the integrand and replacing it with the factor $(\zeta_1 + \zeta_2)/2$ outside of the integral. We are then left with the evaluation of

$$I = \int_{D_1 U D_2} G_1(x:x')F'(x')dx' .$$

(29)
To estimate this integral we have no choice but to compute $F'(x')$ explicitly, i.e., we must select a particular form for $R(x', r)$ and then evaluate the integral in (27). Thus suppose that for all $x'$

$$R(x', r) = \begin{cases} 1 & |r_1| < l & |r_2| < l \\ 0 & \text{otherwise} \end{cases}$$

(30)

i.e., $R(x', r)$ is unity in a square domain of area $4l^2$ surrounding the point $x'$. We may determine $l$ by requiring that $4l^2 = L_i$, $i = 1$ or 2.

More elaborate approximations to $R$ may be chosen with a subsequent increase in the labor needed to evaluate $F'(x')$ but with the offsetting benefit of greater accuracy. However, we may suppose that the choice (30) will permit the full flavor of what is involved here mathematically to be represented. If $x'$ is near a boundary then (30) should, in principle, be modified so as to reflect the reduced size or skewness of the domain $D(x')$; however we will not concern ourselves with this detail.

Though the use of (30) will permit an analytical determination of $F'(x')$ to be made, the exact computation of (29) will remain intractable. We may however compute an approximation to (29) as follows: Make a change of coordinates in (29) to get the explicit expression

$$I = \frac{1}{2\pi} \int_{-h/2}^{h/2} dy' \int_{-h}^{h} dx' \frac{y'}{x'^2 + y'^2} F'(x + x')$$

(31)

where

$$F'(x + x') = \frac{1}{2\pi} \int_{-\ell}^{\ell} dr' \int_{-\ell}^{\ell} dr \frac{y' + r_2}{(x' + r_1)^2 + (y' + r_2)^2}$$

(32)

From symmetry it follows that $F'(x) = 0$. If $F'(x + x')$ is expanded in Taylor series about $x$ and placed in (31) we get
\[ I = \frac{\partial F'}{\partial y} (x) C_1 h^2 + O(h^4) \]

where

\[ C_1 = \frac{2}{\pi} \int_0^{\frac{1}{2}} dy \int_0^1 dx \frac{y^2}{x^2 + y^2} = \frac{1}{2\pi} (1 + \pi/4 - 5/2 \tan^{-1} \frac{1}{2}) = 0.099675421 \]

The computation of \( \frac{\partial F'}{\partial y} (x) \) is easily done by bringing the \( y' \) derivative inside the integral in (32) and then changing it to a \( r_2 \) derivative. One finds that \( \frac{\partial F'}{\partial y} (x) = \frac{1}{2} \).

Assembling all of our partial results together we see that (28) then becomes

\[ \overline{u'^2} = (\zeta_1 + \zeta_2) h^2 C + \sum_{i=1}^{2} G_2^2 (x: x_1) L_i \zeta_i h^2 + \sum_{i=3}^{N} G_2^2 (x: x_i) L_i \zeta_i h^2 \quad (33) \]

where \( C = C_1 / 4 \). The non-local character of this relation is self-evident.

As we have seen during its derivation, this relation may be further refined both by using more accurate quadrature, e.g. taking into account the variation of the \( \zeta \) field, and by replacing the crude model (30) for \( R(x', r) \) by a more realistic one.

The relation (33) differs in two important ways from that used by Chorin (1974). First of all the contribution of the local \( \zeta \) field to \( \overline{u'^2} \) is now included. Secondly, the statistical property of the vorticity field that is used in the derivation of (33) contains within it the well defined area scale \( L_i \). This scale gives an indication of the extent of the flow domain over which the vorticity field will be highly correlated.
In the derivation of the closure given by Chorin an hypothesis as to the statistical independence of the circulations of disjoint regions of a certain size was used. The proper size of these regions was undetermined, however, and this added some arbitrariness to the computed solutions. It is now clear that these regions should be proportional to $L_i$.

In practice we need to compute $\bar{v}^{1/2}$ at points situated with respect to the grid as represented by $x_0$ in Figure 1. Following the same steps as was used to approximate $\bar{u}^{1/2}$ we will find an expression for $\bar{v}^{1/2}$ identical to (33) except that each $G(x:x_i)$ is replaced by $H(x:x_i) \equiv -\frac{\partial G_r}{\partial x} (x:x_i)$.

$\bar{u}'\bar{v}'$ may also be computed using this method. However, one finds that for sufficiently regular domains such as a channel, $\bar{u}'\bar{v}' = 0$. The simplest explanation of this mistaken result, is that it is a consequence of supposing that the fluid motion (e.g. in the channel) is purely two dimensional. In view of the fact [see Tennekes and Lumley (1972) p. 41] that the largest contribution to $\bar{u}'\bar{v}'$ comes from vorticity lying above and below and parallel to the plane of motion and aligned in the direction of the mean rate of strain it is not surprising that we compute $\bar{u}'\bar{v}' = 0$. The anomaly found in the computation of $\bar{u}'\bar{v}'$ leads us to suspect that our computations of $\bar{u}^{1/2}$ and $\bar{v}^{1/2}$ will also be affected by the restriction of two dimensionality. We will have to take this suspicion under consideration later when we assess the results of our numerical computations.
We now have a complete specification of the method of coarse graining. This consists of the solution of difference approximations to (18) and (20). The mean velocity field is computed from the mean vorticity field by first computing a discrete approximation to the stream function \( \psi \) numerically, by inverting a difference approximation to Poisson's equation \( \nabla^2 \psi = -\xi \). The mean velocity field is then found from difference approximations of \( U = \frac{\partial \psi}{\partial y} \) and \( V = -\frac{\partial \psi}{\partial x} \). \( \overline{u'^2} \) is computed from (33) and \( \overline{v'^2} \) from a similar relation. In those domains for which our method of computing \( \overline{u'v'} \) does not have it identically equal to zero, it may be computed in a manner similar to \( \overline{u'^2} \) and \( \overline{v'^2} \). However, the value of \( \overline{u'v'} \) will be quite small when it isn't identically 0, so perhaps the wisest course of action at this time is to drop the terms containing \( \overline{u'v'} \) from the equations of motion (18) and (20), in view of the added labor needed to compute this small contribution.
III. ADAPTING THE METHOD OF COARSE GRAINING TO THE FLOW IN A CHANNEL

We will presume that the flow in the channel is fully developed, i.e. the mean flow properties are uniform up and downstream (the x direction) and steady. Consequently the mean velocity field $U$ depends solely on the coordinate $y$ spanning the channel and is symmetrical about the centerline of the channel. If lengths are scaled using the channel width, $2D$, then it follows from the assumptions on $U$ that $V = 0$, $\xi(y) = -\xi(1-y)$ and $\zeta(y) = \zeta(1-y)$.

Velocities will be scaled by $U_m$, the average mass flow velocity, so that

$$\int_0^1 U(y)dy = 1 . \quad (34)$$

If we define a stream function $\Psi$ from the relation $U = 1 + \frac{\partial \Psi}{\partial y}$, and if $\Psi = 0$ at $y = 0$, then (34) and the symmetry condition imply that $\Psi(y) = -\Psi(1-y)$.

Since conditions are uniform in the x direction, we will only have to solve for the values of $\xi$, $\zeta$ and $\Psi$ at one x position along the channel, which we choose to be $x = 0$. The set of points $[0,(j-\frac{1}{2})h], j = 1, ..., M$ with $h = 1/M$ forms a staggered grid, on which we will define discrete approximations $\xi_j$, $\zeta_j$ and $\Psi_j$ to $\xi((j-\frac{1}{2})h)$, $\zeta((j-\frac{1}{2})h)$ and $\Psi((j-\frac{1}{2})h)$ respectively. These grid functions will be related by difference equations, the solution of which for $\xi_j$ and $\zeta_j$ are found by allowing $\xi_j$ and $\zeta_j$ to
depend on time, and integrating the equations until a steady time independent solution is found. A superscript 'n' will refer to the time step \( n\Delta t \) where \( \Delta t \) is the interval of time between integration steps.

In Section III.1 we derive our difference equations for the \( \bar{\zeta}_j \) and in III.2 for \( \xi_j \). In Section III.3 we show how the \( \psi_j \) may be computed from the \( \bar{\zeta}_j \) and in IIIA we adopt our general closure relation to the flow in the channel by giving an explicit formula for \( \bar{v}^{12} \) at those points at which its value is required.

III.1 Mean Vorticity Equation

The mean vorticity equation for the flow in a channel is

\[
0 = \frac{\partial}{\partial y} \left( \frac{1}{R} \frac{\partial \bar{\zeta}}{\partial y} + \bar{v} \bar{\zeta} \right)
\]

which says that the total flux of vorticity, i.e. that due to molecular and turbulent causes, is constant across stream. The magnitude of this constant flux may be seen from the momentum equation to be \(-\lambda\), where \( \lambda \) is the friction coefficient defined by

\[
\lambda = \frac{\tau^*}{2 \rho U_m^2} = -\frac{\partial P}{\partial x}
\]

where \( \tau^* = \mu \frac{\partial U^*}{\partial y^*} \bigg|_{y^*=0} \) is the dimensioned shear stress at the wall and \( \frac{\partial P}{\partial x} \) is the dimensionless pressure gradient.
In the core region of the channel, our transport law applies so that (35) may be written as

\[0 = \frac{\partial}{\partial y} \left( \frac{1}{R} + T_{22} \bar{v}'^2 \right) \frac{\partial \bar{\xi}}{\partial y} . \tag{37}\]

If we assume that the core region contains the first grid point at \(h/2\) then we may make the straightforward difference approximation to (37):

\[
\frac{\bar{\xi}_{j}^{n+1} - \bar{\xi}_{j}^{n}}{\Delta t} = \left( \frac{1}{R} + T_{22} \bar{v}'_{j}^2 \right) \left( \frac{\bar{\xi}_{j+1}^{n} - \bar{\xi}_{j}^{n}}{h^2} \right) - \left( \frac{1}{R} + T_{22} \bar{v}'_{j-1}^2 \right) \left( \frac{\bar{\xi}_{j}^{n} - \bar{\xi}_{j-1}^{n}}{h^2} \right) \tag{38}\]

for \(j = 2, \ldots, M/2\), where \(\bar{v}'_{j}^2 \equiv \bar{v}'^2(jh)\), and where we use the symmetry condition to compute \(\bar{\xi}_{M/2+1} = -\bar{\xi}_{M/2}\), thus avoiding the need to write difference equations for \(j > M/2\). Notice that the term \(\partial \bar{\xi}/\partial t\) is approximated on the left side of (38) so as to permit iteration to the steady state solution.

For the grid point \(j=1\) we cannot use (37) because of the breakdown of the transport law next to the wall, but we may use the exact relation (35). Since the total mean flux of vorticity is constant across the boundary layer in spite of the fact that \(\bar{\xi}_y\) and \(\bar{v}'\bar{\xi}\) may individually vary greatly, we may difference (35) between \(y = 0\) and \(h\) without
creating a large truncation error. At \( y = h \) the transport law is valid, so we may use the approximation used in (38) for the total flux here. At \( y = 0 \) we must compute \( \frac{1}{R} \frac{\partial \bar{\xi}}{\partial y}(0) \) which is thus the boundary condition to the mean vorticity equation.

Since \( \bar{\xi}_y(0) = -U''(0) \), equation (36) and the momentum equation give

\[
\frac{1}{R} \bar{\xi}_y(0) = \lambda.
\]

The \( y \) momentum equation implies that \( \frac{\partial \bar{p}}{\partial x} \) is independent of \( y \) so that integrating the \( x \) momentum equation across the channel gives

\[
\lambda = \frac{U'(0)}{R_e} \tag{39}
\]

where \( R_e \equiv R/2 \). We will use (39) to compute \( \lambda \) and hence \( \frac{1}{R} \bar{\xi}_y(0) \).

The linear law of the wall is the experimental observation that the variation of the mean velocity field across the viscous sublayer is essentially linear. This implies that we may state, \( U'(0) = \frac{U(\delta)}{\delta} \) where \( \delta \) is a point within the viscous sublayer. Using this relation we can write the \( \bar{\xi} \) equation for \( j = 1 \) as

\[
\frac{\bar{\xi}^{n+1}_1 - \bar{\xi}^n_1}{\Delta t} = \left( \frac{1}{R} + T_{22} \frac{\nu}{h^2} \right) \left( \frac{\bar{\xi}^n_2 - \bar{\xi}^n_1}{h^2} \right) - \frac{2}{R h} \frac{U(\delta)}{\delta} \tag{40}
\]

where we are left with devising a means of computing \( \delta \) and \( U(\delta) \).

The fact that we have assumed our transport law to be valid at \( y = h \) implies that the boundary layer is entirely contained within the first grid box. It is necessary to use such a coarse grid because a finer one would require us to approximate the vorticity flux \( \bar{v}^T \bar{\xi} \) within the wall region where our transport law doesn't apply. Unfortunately, our coarse grid also makes it difficult for us to easily compute \( U \) at a particular
point $\delta$ because of the delicate and rapid variation of $U$ within the small boundary region.

To overcome these difficulties we have devised a crude method of obtaining a pair of values $[\delta, U(\delta)]$ that requires the evaluation of one empirical constant from the experimental data. This approach should be considered to be provisional until such time as our transport law is generalized to include the effect of viscosity and thereby permit the use of a fine grid near the boundaries.

Our method of computing $\delta$ relies on the construction shown in figure 2 which is a crude approximation of the behavior of the stream function in the region near the wall. In this figure we have drawn a straight line with slope -1 leaving the origin and a parabolic arc through the values of the computed stream function at the three grid points closest to the wall, i.e. $y = h/2$, $3h/2$, and $5h/2$. The line leaving the origin satisfies the boundary conditions that the stream function must satisfy at the wall. We have found that the point of intersection of these two lines always occurs in the viscous sublayer and so we have let $\delta$ be this point.

After some simple algebra, one may ascertain that

$$\delta = \frac{1 + B - \sqrt{(1+B)^2 - 4AC}}{2A}$$  \hspace{1cm} (41)
where

\[ A = \frac{1}{2h^2} (\hat{\psi}_1 - 2\hat{\psi}_2 + \hat{\psi}_3) \]

\[ B = \frac{1}{h} (-2\hat{\psi}_1 + 3\hat{\psi}_2 - \hat{\psi}_3) \]

and

\[ C = \frac{15}{8} \hat{\psi}_1 - \frac{5}{4} \hat{\psi}_2 + \frac{3}{8} \hat{\psi}_3 \]

As we have suggested without more information about the flow within the wall region the evaluation of \( U \) at the value of \( \delta \) given in (41) can only be done crudely. We have found nonetheless that the following artifice works surprisingly well as we shall see later: We know that \( U(y) = 1 + \psi_y \), so to compute \( U(\delta) \) we must find an approximation to \( \psi_y(\delta) \). We can estimate the order of magnitude of \( \psi_y(\delta) \) from the slope at the point \( \delta, (=B + 2A\delta) \), of the parabolic curve used to compute \( \delta \), and then suppose that \( \psi_y(\delta) \) is proportional to this slope. Therefore, introducing a parameter \( C_1 \), we have

\[ U(\delta) = 1 + C_1(B + 2A\delta). \]  \hspace{1cm} (42)
We will assume that $c_1$ is a constant independent of Reynolds number, even though in reality it may very well have some slight dependence on it. We do not believe that this assumption will introduce major errors into our computations since $c_1$ is only one of several factors, i.e. $A$, $B$ and $\delta$, whose variation with $R$ contributes to the change of $U(\delta)$ with $R$.

We should mention that implicit in the derivation of (38) and (40) is the assumption that $T_{22}$ is constant across the channel. There is reason to believe that this is approximately true because measurements of various velocity integral scales and velocity microscales by Laufer (1950) and Comte-Bellot (1965), have shown that these are roughly constant across the core region, implying that the character of the turbulent motion does not vary greatly and hence neither does the other characteristic scales, such as $T_{22}$. In the sequel, we will invoke this same reasoning to justify assuming that the other turbulent scales such as $\lambda_1$ and $L_1$ are also constant across the channel.

Since it will be our intent to explore the variation of the mean flow field with Reynolds number, we must make the further assumption that $T_{22}$ and the other characteristic scales are sufficiently weak functions of $R$ that they may be considered to be approximately independent of $R$. Again, there is some indirect experimental evidence of the type just mentioned to justify this assumption. In any event, the extent to which this condition is violated will be reflected in the results of our computations.
III.2 The Equation for the Vorticity Variance

The equation for $\zeta$ in the core region of the channel is, from a reduction of (20):

$$0 = \frac{\partial}{\partial y} \left( \frac{1}{R} + T_{22} \overline{v'^2} \right) \frac{\partial \zeta}{\partial y} + 2T_{22} \overline{v'^2} \frac{\partial \overline{v'}}{\partial y} \left( \frac{\partial \overline{\omega}}{\partial y} \right)^2 - \frac{2\zeta}{\lambda_d^2 R}.$$ (43)

This may be easily approximated at the grid points $j = 2, ..., M/2$ as

$$\frac{\zeta_{j+1}^n - \zeta_j^n}{\Delta t} = \left( \frac{1}{R} + T_{22} \overline{v'^2}_j \right) \left( \frac{\zeta_{j+1}^n - \zeta_j^n}{h^2} \right) - \left( \frac{1}{R} + T_{22} \overline{v'^2}_j \right) \left( \frac{\zeta_j^n - \zeta_{j-1}^n}{h^2} \right)$$

$$+ T_{22} \overline{v'^2}_j \left( \frac{\zeta_{j+1}^n - \zeta_j^n}{h^2} \right)^2 + T_{22} \overline{v'^2}_{j-1} \left( \frac{\zeta_j^n - \zeta_{j-1}^n}{h^2} \right)^2 - \frac{2}{R\lambda_d^2} \zeta_j^n$$ (44)

where from symmetry we have $\zeta_{M/2+1}^n = \zeta_{M/2}^n$.

For the first grid point we should in principle use the exact equation

$$0 = \frac{\partial}{\partial y} \left( \frac{1}{R} \frac{\partial \zeta}{\partial y} - \overline{v'\zeta'^2} \right) - 2 \overline{v'\zeta} \frac{\partial \overline{v'}}{\partial y} - \frac{2}{R} \left( \frac{\zeta'^2}{x} + \frac{\zeta'^2}{y} \right).$$ (45)
as a basis for a difference equation because $v'\xi^1$ and $v'\xi^{1/2}$ cannot be modeled by our transport law in a thin region adjacent to the wall.

However to make a straightforward approximation to (45) by differencing it between 0 and $h$ is not possible because we will be faced with two very difficult problems. Namely, how to approximate $\frac{\partial \zeta}{\partial y}(0)$ and how to compute the cumulative effect of the term $-2v'\xi^2 \frac{\partial \zeta}{\partial y}$ in this region.

Rather than attempt to solve these quite formidable problems we will instead derive a difference equation for the first grid box by taking advantage of the fact that the region of inapplicability of the transport law is exceedingly thin.

Thus let $y''$ be a distance from the wall where our transport law begins to be valid. Furthermore assume that $Re$ is large enough so that $y'' \ll h$ (at the end of the following discussion we will consider the conditions on $Re$ such that this assumption holds). Then (43) applies in $(y'', h)$ and if we integrate (43) over this region and divide by $h \approx h - y''$ we will see that the only difficult term to approximate in the resulting expression is the total flux of $\zeta$ at $y = y''$, i.e.

$$\left(\frac{1}{R} + T_{22} v^{1/2}\right) \frac{\partial \zeta}{\partial y} \bigg|_{y''}.$$ 

This is hard to estimate because of the factor $\frac{\partial \zeta}{\partial y}(y'')$. However we will be able to show that $\frac{\partial \zeta}{\partial y}(y'') > 0$ and thus that it may be crudely approximated as
where $C_2$ is a parameter to be determined using the experimental data. Like our other parameters we assume that $C_2$ is independent of $R_e$. The equation for the first grid box will then be, using (46)

$$
\frac{\partial \zeta}{\partial y} (y'') \approx C_2 \frac{\zeta_1}{h}, \quad C_2 > 0
$$

(46)

where obvious approximations have been used to derive the remaining terms in this equation.

We now will show why we believe that $\frac{\partial \zeta}{\partial y} (y'') > 0$. The sign of $\frac{\partial \zeta}{\partial y} (y'')$ is related to the direction of the flux of $\zeta$ at $y''$ so to show that $\frac{\partial \zeta}{\partial y} (y'') > 0$ it suffices to establish that the flux of $\zeta$ at $y''$ is directed towards the wall. We will show that this holds by establishing the following facts:

(i) There exists a distance from the wall, $y'$, where $0 < y' < y''$ such that the region $(0, y')$ is one of pure loss of $\zeta$, i.e., no production of $\zeta$ occurs here.
(ii) Almost all of the loss of $\zeta$ (at sufficiently high Reynolds numbers) occurring in the channel takes place in $(0, y')$.

(iii) The region $(y', y'')$ is one of intense production of $\zeta$, yet its total contribution to the production of $\zeta$ is not larger than that occurring in $(y'', \frac{1}{2})$, because of its extreme thinness.

It is clear that all of these facts taken together force the conclusion that most of the $\zeta$ product in $(y'', \frac{1}{2})$ is lost in $(0, y')$ and hence the flux of $\zeta$ at $y''$ is towards the wall. We will establish each of the statements (i)-(iii) in turn. To derive (i) we first must consider the qualitative behavior of $\bar{\zeta}$ across the channel.

The linear law of the wall states that $U(y) \approx \lambda R_e y$ in the viscous sublayer. However as noted earlier $U''(0) = -2\lambda R_e$, thus a more precise statement of the linear law of the wall is that $U(y) = \lambda R_e (y - y^2)$. Differentiation of this relation shows that $\bar{\zeta}(y) = -\lambda R_e (1-2y)$ in the viscous sublayer. Furthermore, it is widely known that the mean velocity profiles in a turbulent flow are considerably flatter in the center of the channel than the parabolic velocity profile found in Poiseulle flow. Consequently $|\bar{\zeta}| = \left| \frac{\partial U}{\partial y} \right|$ must be smaller in the core region of the channel in a turbulent flow than it is in a laminar one. We also know from experiments that $+U_y = -\bar{\zeta}$ monotonically increases from zero at the center of the channel to the wall $y = 0$ where it is $\lambda R_e$. Combining these facts it is clear that $\bar{\zeta}$ should vary across the channel as depicted in Figure 3. The length scale near the wall has been greatly exaggerated.
The distribution of $\bar{\xi}_y$ across the channel may be construed from that of $\bar{\xi}$. Differentiation of $\bar{\xi}$ in the viscous sublayer shows that $\bar{\xi}_y = 2\lambda R_e$ in this region. Figure 3 then shows that $\bar{\xi}_y$ must increase immediately outside of the viscous sublayer reaching a maximum at the place where $\bar{\xi}$ has a point of inflection and then rapidly fall to the much smaller value it has throughout most of the core region. The distribution of $\bar{\xi}_y$ across the channel is plotted in Figure 4, (actually $\frac{1}{R} \bar{\xi}_y$ is shown, for later convenience). The point where $\bar{\xi}_y$ reattains the value it had at the wall is called $y'$.

A rearrangement of the x momentum equation gives the relation

$$\frac{1}{R} \bar{\xi}_y - v'\xi = \lambda$$

(48)

which was used previously to tell us that the constant mean flux of vorticity is $-\lambda$. In the present context it reveals that $v'\xi$ must have the form shown in Figure 4. In particular that $v'\xi = 0$ at $y = y'$, $v'\xi > 0$ for $y < y'$ and $v'\xi < 0$ for $y > y'$. We thus see that the term $-2v'\xi \frac{\partial \xi}{\partial y}$ in (45) which is ostensibly a production term, does in fact represent the loss of $\xi$ by reconversion to $\xi^2$ in the region $(0,y')$. It is also interesting to note that since the vorticity is negative in this half of the channel we may infer that the mean turbulent flux of vorticity is towards the wall for $y < y'$ and away from the wall for $y > y'$.

It is now easily shown that the region $(0,y')$ acts as a sink for the excess $\xi$ produced in the remainder of the channel (i.e. $(y', \frac{1}{2})$):
If we integrate (45) between $y = 0$ and $y'$ we see that the flux of $\zeta$ at $y'$ is

$$
\left( \frac{1}{R} \frac{\partial \zeta}{\partial y} + v' \xi^2 \right)_{y = y'} = - \int_0^{y'} 2v' \xi \xi_y \, dy - \int_0^{y'} \frac{2}{R} \left( \xi_x^2 + \xi_y^2 \right) \, dy - \left. \frac{1}{R} \frac{\partial \zeta}{\partial y} \right|_{y = 0} \quad (49)
$$

We may assume that $\frac{\partial \zeta}{\partial y}(0) > 0$. This is likely since $-\frac{\partial \zeta}{\partial y}(0)/R$ represents the rate of molecular diffusion of $\zeta$ at the surface of the wall, which our intuition suggests should be directed towards it. If $\frac{\partial \zeta}{\partial y}(0)$ were < 0 we would have the implausible result that $\zeta$ is created by the viscous motion at the wall. From (49) we see that the assumption on $\frac{\partial \zeta}{\partial y}(0)$ and the fact that $v' \xi > 0$ in $(0, y')$ implies that the flux of $\zeta$ at $y = y'$ is negative, i.e. towards the wall.

If (45) is now integrated between $y'$ and $\frac{1}{2}$ we obtain

$$
\left( \frac{1}{R} \frac{\partial \zeta}{\partial y} + v' \xi^2 \right)_{y = y'} = \int_{y'}^{\frac{1}{2}} -2v' \xi \xi_y \, dy - \int_{y'}^{\frac{1}{2}} \frac{2}{R} \left( \xi_x^2 + \xi_y^2 \right) \, dy
$$

where the first integral on the right represents the total production of $\zeta$ in $(y', \frac{1}{2})$ and the second term the total dissipation here. This relation shows that the total flux of $\zeta$ at $y'$ which was shown by (49) to be into the wall region is equal to the net production of $\zeta$ occurring in the region $(y', \frac{1}{2})$. 

We now consider statement (ii), i.e., we wish to show that most of the loss of $\zeta$ in the channel occurs by its reconversion to $\overline{\varepsilon}^2$ in the region $(0, y')$ and is not due to its dissipation by viscosity. Since $\lambda_d$ is unknown we cannot directly estimate the amount of $\zeta$ which is dissipated by viscosity. However, we may eliminate the possibility that this accounts for most of the loss of $\zeta$ by the following argument: If the viscous dissipation of $\zeta$ occurring through the term $-2\zeta/R\lambda^2_d$ is the dominant source of its loss, then the general rate of loss of $\zeta$ throughout the channel is roughly proportional to $\zeta/R$ (assuming $\lambda_d$ is constant). However, as (33) indicates, $\zeta$ is proportional to $v'^2$ while the relation $\lambda \approx -T_22v'^2\overline{\varepsilon}_y$ which holds in the core region implies that $v'^2 \sim \lambda/\overline{\varepsilon}_y$. We then see that $\zeta \sim \lambda/\overline{\varepsilon}_y$ and thus the total dissipation varies like $\lambda/\overline{\varepsilon}_y$. On the other hand, the rate of production of $\zeta$ is proportional to $\lambda\overline{\varepsilon}_y$ in the core region and $\lambda^2 R$ in the wall region (we will show this momentarily). Since it is known from experiments [e.g. Conte-Bellot (1965)] that the mean velocity profiles do not change dramatically when $Re$ is increased above some moderate value, say 25,000, then neither can $\overline{\varepsilon}_y$ change drastically with $Re$. However, a balance between total production and total dissipation must be established at every $Re$ so it is clear that if production is proportional to $\lambda^2 R$ and $\lambda\overline{\varepsilon}_y$ and dissipation to $\lambda/\overline{\varepsilon}_y$, then an unrealistically large change in $\overline{\varepsilon}_y$ must occur if $Re$ is significantly varied. Consequently, a major proportion of the $\zeta$ loss must occur in the region $(0, y')$.

Now let us establish point (iii) by comparing the total amount of production of $\zeta$ which comes from the regions $(y', y'')$ and $(y'', \frac{3}{2})$. The maximum rate of production of $\zeta$ occurs in $(y', y'')$. This may be seen from the fact that $-2v'\xi \overline{\varepsilon}_y$ is maximized for $0 \leq v' \xi < \lambda$, under the constraint
(48) when \( -\overline{v'\xi} = \overline{\xi_y}/\overline{R} = \lambda/2 \). The point where this occurs is shown in Fig. 4. The maximum production rate is \( \frac{1}{2}\lambda^2 \overline{R} \) which is considerably larger than the typical rate of production taking place in the core region, which is \( \approx \lambda/2h \) since \( \overline{v'\xi} \approx \lambda \) and \( \overline{\xi_y} \approx \frac{\lambda}{h} \), here.

In spite of the fact that the region \((y', y'')\) is one of intense production of \( \zeta \), the total amount of production of \( \zeta \) occurring here is not larger than the total production of \( \zeta \) occurring within \((y'', \lambda_2)\). This may be seen for the case when \( R_e = 57,000 \) by using some numbers (we assume that the same result holds for all other large Reynolds numbers). Suppose that the maximum rate of production in the wall region occurred throughout a region of thickness \( h/4 \), then a liberal estimate of the total produced here would be \( h/4 \lambda^2 R/2 \), which for \( R_e = 57,000 \) is \( \approx 0.012 \) using the value of \( \lambda = 0.0037 \) that has been found in experiments (Comte-Bellot, 1965). If the core region is assumed to be of width \( \lambda_2 \), then the total production in this zone is \( \lambda/2h\lambda_2 \approx 0.015 \), thus establishing our point.

We now have some justification for the use of equation (47) when \( y'' \ll h \). Note that the relative magnitudes of \( C_2 \) and \( 1/\lambda_d^2 \) in (47) decide the proportionate share of the dissipation of \( \zeta \) which occurs in the core region and the wall region. When we choose a value for \( \lambda_d \) later we will have to make certain that it permits most of the dissipation to occur near the wall.

We now estimate the dependence of the magnitude of \( y'' \) on \( R_e \) so we may see when our assumption \( y'' \ll h \) is meaningful. We may make such an estimate by first computing \( y' \) using an additional characterization of this point. By definition, \( \overline{v'\xi} = 0 \) at \( y = y' \). However, for the fully developed flow in the channel \( \overline{v'\xi} = -\frac{\partial}{\partial y} \overline{u'v'} \). Thus at the point \( y' \), \( \frac{\partial}{\partial y} \overline{u'v'} = 0 \), and so
is an extremal point of $u'v'$. An integration of the momentum equation yields $u'v' = \lambda(y - \frac{1}{2}) - \frac{\varepsilon}{R}$, so that $u'v'$ has the form shown in Fig. 3 which is a familiar result. It is then clear that $u'v'$ has a minimum at $y'$. This point has been measured in experiments of turbulent flow past walls and in channels and pipes. It is generally found [see Tennekes and Lumley (1973), p.161] that this minimum occurs at $y^* u_T / \nu \approx 30$ where $u_T \equiv \sqrt{\tau_w / \rho}$ is the friction velocity, $\tau_w$ is the dimensioned shear stress at the wall and $y^* = 2Dy$ is the dimensionless $y$ coordinate. After nondimensionalization we have $y' \approx 30 \sqrt{2/\lambda} / R$. For $Re = 57,000$ we may compute that $y' = 0.006$ which is $\ll$ the value $h = 0.0625$, which we use in our computations. However, as $Re \to 0$ this relation becomes progressively less true and in fact for $Re \approx 7500$ it appears from our computations of $\lambda$ that $h > y'$.

The point $y'$ also coincides with the lower limit of the range of $y$ values in which the logarithmic law of the wall holds [see Comte-Bellot, 1965]]. One may show that the region of rapid variation of $\overline{e_y}$ and the other mean quantities does not extend very far into the log law region. We may also expect that our transport law is valid where the gradient in the mean quantities are not large and thus we conclude that $y''$ is not very much larger than $y'$. Thus we will suppose that whenever $y' \ll h$ we also have $y' < y'' \ll h$. Consequently our estimates of the $Re$ for which $y' \ll h$ also apply roughly to $y''$. By computing $y'$ a posteriori for each $Re$ we may find out whether the assumption $y'' \ll h$ was valid for that computation.
III.3 Stream Function

The stream function \( \bar{\psi} \) may be determined from a known vorticity field, \( \bar{\xi} \), by solving the equation

\[
\frac{d^2 \bar{\psi}}{d\gamma^2} = -\bar{\xi}
\]  

(50)

with the boundary conditions \( \bar{\psi} = 0 \) at \( \gamma = 0 \) and 1. Similarly, the discrete stream function \( \bar{\psi}_j \) is computed from the \( \bar{\xi}_j \) by a finite difference analogue to (50). At the grid points \( j = 2, \ldots, M/2 \), which are distant from the boundary, we approximate (50) consistently as

\[
\frac{\bar{\psi}_{j+1} - 2\bar{\psi}_j + \bar{\psi}_{j-1}}{h^2} = -\bar{\xi}_j
\]

(51)

which is accurate to \( O(h^2) \).

When \( j = 1 \), (51) cannot be used because this would necessitate the use of a point outside of the flow domain. We may, however, use the approximation

\[
\frac{-5\bar{\psi}_1 + 2\bar{\psi}_2 - 1/5\bar{\psi}_3}{h^2} = -\bar{\xi}_1
\]

(52)

which may be easily verified to also be of second order accuracy in \( h \).

If equation (51) for \( j = 2 \) is multiplied by 1/5 and then added to (52) we get

\[
\frac{-24}{5} \frac{\bar{\psi}_1}{h^2} + \frac{8}{5} \frac{\bar{\psi}_2}{h^2} = -\bar{\xi}_1 - \frac{1}{5} \bar{\xi}_2
\]

(53)

The system of equations (51), \( j = 2, \ldots, M/2 \) and (52) have the same solution if (52) is replaced by (53). This latter system is tridiagonal and may
be easily solved using the standard algorithm.

Once the \( \psi_j \) are known we can find approximations \( U_j \) to \( U(jh) \), \( j=1,\ldots,M/2 \) from

\[
U_j = 1 + \frac{\psi_{j+1} - \psi_j}{h}.
\]

In Section IV we will compare the computed values of \( U_j \) with those found experimentally.

III.4 Computation of the Velocity Moment

In this section we will show how the \( \bar{v}_{j}^{1/2} \) may be computed using the technique of section II.4. We will evaluate \( \bar{v}_{j}^{1/2} \) at the point \((0,jh)\) shown in Fig. 5. The channel has been partitioned into boxes with \( M \) of them spanning the channel. It is seen that each \( \bar{v}_{j}^{1/2} \) is situated with respect to the grid in the same manner as was required for the development of section II.4. Thus, we may apply the results of that section to the present case and find that

\[
\bar{v}_{j}^{1/2} = (\zeta_j + \zeta_{j+1})Ch^2 + h^2 \sum_{k=1}^{M} \zeta_k L_k \left[ \sum_{k \neq 0} H^2(x_j;\bar{x}_{k\ell}) \right]
\]

where \( H(x_j;\bar{x}_{k\ell}) = -\partial G(x_j;\bar{x}_{k\ell})/\partial x \), \( L_k \equiv L(x_{k\ell}) \), \( x_j = (0,jh) \) and \( \bar{x}_{k\ell} = (x_k,y_\ell) = (kh,(\ell-\frac{1}{2})h) \). Note that \( H(x_j;\bar{x}_{0\ell}) = 0 \).

\( H(x_j;\bar{x}_{k\ell}) \) represents the \( y \) component of velocity induced at a point \( x_j \) due to a vortex of circulation one sitting at \( \bar{x}_{k\ell} \). For a vortex placed in a channel \( H(x_j;\bar{x}_{k\ell}) \) may be computed exactly by setting up an array of image vortices: plus vortices are situated at the points \((x_k,y_\ell+2m), \quad -\infty < m < +\infty\) and negative vortices at \((x_k,2m-y_\ell), \quad -\infty < m < +\infty\). The
velocity at any point \( x \) in the channel due to the vortex at \( x_{k\ell} \) is then the sum of the velocities induced at \( x \) as if no boundaries are present by this vortex plus all of the image vortices.

The velocity at \( x_{j} \) arising from this infinite collection of vortices may be written in closed form using the velocity field induced by a row of vortices, given in Lamb (1932). We have

\[
H(x_{j};x_{k\ell}) = -\frac{1}{4} \frac{\sinh(\pi x_{k})}{\cosh \pi x_{k} - \cos(\pi h - y_{\ell})} + \frac{1}{4} \frac{\sinh(\pi x_{k})}{\cosh \pi x_{k} - \cos(\pi h + y_{\ell})}
\]

the first term coming from the column of + vortices and the second from the - vortices.

The use of (54) in practice is much simplified if we define a function \( A(j,\ell) \) which gives the contribution to \( v_{j}^{T} \) from all the boxes with center at \( y = (\ell - \frac{1}{2})h \). Thus

\[
A(j,\ell) = \sum_{k \neq 0} H^{2}(x_{j};x_{k\ell})
\]

Using this function (54) becomes

\[
\bar{v}_{j}^{T} = h^{2}C(\zeta_{j} + \zeta_{j+1}) + h^{2} \sum_{\ell = 1}^{M} \zeta_{\ell} A(j,\ell)L_{\ell}
\]  \hspace{1cm} (55)

Since \( \zeta_{M+1-\ell} = \zeta_{\ell} \), (55) may be simplified further to

\[
\bar{v}_{j}^{T} = h^{2}C(\zeta_{j} + \zeta_{j+1}) + h^{2} \sum_{\ell = 1}^{M/2} B(j,\ell)\zeta_{\ell}L_{\ell}
\]  \hspace{1cm} (56)

where \( B(j,\ell) = A(j,\ell) + A(j,M+1-\ell) \). The \( M/2 \times M/2 \) array \( B(j,\ell) \) may be computed once and for all and stored, making the computation of \( \bar{v}_{j}^{T} \) a trivial operation at each time step.
IV. RESULTS OF COMPUTATION

Once values have been obtained for the constants $T_{22}$, $\lambda_d$, $L_1$, $C_1$ and $C_2$, we will then have a closed set of equations which may be solved for $\bar{\xi}_j$ and $\xi_j$, $j = 1, \ldots, M/2$. The solutions are computed by integrating in time from arbitrary initial conditions, $\bar{\xi}_j^0$, $\xi_j^0$ (except that at least one of $\xi_j^0$ must be $> 0$ for otherwise equations (44), (47), and (56) show that the solution cannot help but be laminar), until a time-independent solution is found. The algorithm proceeds by solving (51) and (53) for the $\bar{\psi}_j^n$ using the known values of $\bar{\xi}_j^n$. The $\bar{\psi}_j^n$ are then used to compute $\delta$ from (41) and $U(\delta)$ from (42). The $\xi_j^n$ are then used in (56) to compute the $\bar{\nu}_{ij}^2$. Finally the values of $\bar{\xi}_j^{n+1}$ and $\xi_j^{n+1}$ are computed from (38), (40), (44), and (47).

We will take $M=16$ for all of our computations so that $h=0.0625$. The integration time step, $\Delta t$, must be chosen to be small enough so that the difference equations are stable. Since we only seek steady solutions, we will find out a posteriori from the fact of the convergence of our iterations that $\Delta t$ was not too large. However, as a guide in selecting an acceptable value of $\Delta t$ we will use the appropriate condition for an approximation such as (38) to the heat equation, viz

$$\frac{\Delta t}{h^2} \left( \frac{1}{R} + T_{22} \sup_j \bar{\nu}_{ij}^2 \right) < \frac{1}{2}.$$

The criterion for deciding that $\bar{\xi}_j$ and $\xi_j$ have converged to a solution of the steady state equations is that $\sup_j |\bar{\xi}_j^{n+1} - \bar{\xi}_j^n| < 10^{-10}$ and $\sup_j |\xi_j^{n+1} - \xi_j^n| < 10^{-10}$. We have found that the convergence is, in general, quite rapid requiring $< 3000$ iterations and less than one second of CDC
7600 computer time for most Reynolds numbers. However, when \( R_e \) is within the range of values separating the laminar and turbulent solutions, the rate of convergence is considerably slower, and as many as 40,000 iterations may be required to obtain a steady solution.

It is worthwhile before presenting our numerical results to recapitulate what the shortcomings are of the closed system of equations we hope to solve for the mean properties of the flow in a channel. These problems represent an a priori limit to the type of solutions we can hope to achieve. They are:

(i) The transport law introduces the Lagrangian integral time scale \( T_{22} \), which is as yet indeterminable from theory.

(ii) The dissipation term in the equation for \( \zeta \) introduces the Taylor vorticity microscales \( \lambda_1 \) and \( \lambda_2 \) which are a property of the small scale motion, and again, are not yet amenable to theoretical prediction.

(iii) The closure relation (56) contains the area scales \( L_1 \) whose values are unknown.

(iv) In deriving the local contribution of the \( \zeta \) field to \( \nu'^2 \) in (56), we have had to assume a crude form for the correlation function \( R(r) \).

(v) For Reynolds numbers near the critical value the distance \( y'' \) may be comparable to \( h \). This implies that our use of the transport law in deriving the difference equations for the two grid points closest to the wall is unjustified. Furthermore, the condition \( y'' \ll h \), which was used in the derivation of equation (47) for \( \zeta_1 \) is violated.
(vi) The rate of diffusion of \( \zeta \) into the region \( y < y'' \) has been modeled crudely in (47) and this has resulted in the introduction of a constant \( C_2 \) which must be determined empirically.

(vii) The boundary condition to the mean vorticity equation requires the evaluation of \( U \) at a point \( \delta \) within the viscous sublayer. This can only be accomplished at the expense of introducing an additional unknown parameter \( C_1 \).

The problems listed in (v)-(vii) are those that we hope may be solved one day by a more sophisticated treatment of the wall region. On the other hand, (i)-(iv) present much more fundamental difficulties and will require extensive new theoretical developments before they are resolved.

We will set \( T_{22} = 0.4 \) for all of our computations, a value which corresponds roughly to that of the integral scales that have been measured experimentally. Since for any particular Reynolds number \( \lambda \) must be \( \approx T_{22} v'^{1/2} \zeta_y \) in the core region of the channel, a fixed value of \( T_{22} \) implies that the order of magnitude of \( v'^{1/2} \) is determined. In principle, then, the value of \( T_{22} \) may be adjusted a posteriori so that our computed values of \( v'^{1/2} \) agree with those of experiments. However, as we will show later, the computed shape of \( v'^{1/2} \) is in serious disagreement with its measured profile, making it idle to adjust \( T_{22} \). The value \( T_{22} = 0.4 \) nevertheless will insure that \( v'^{1/2} \) is of the correct order of magnitude and we consider this sufficient for our purposes.

The choice of a value of \( \lambda_d \) (i.e. \( \lambda_1 \) and \( \lambda_2 \)) must be governed by the considerations discussed in section III.2, i.e., it must be such that most
of the loss of $\zeta$ will occur by its diffusion into the wall region to be reconverted to $\bar{\varepsilon}^2$. We have found that the choice $\lambda_d^2 = 0.025$ reasonably satisfies this requirement. Since this value implies that $\lambda_1 \approx \lambda_2 \approx 0.316$ we see that the real (i.e. experimental) value of $\lambda_d^2$ would most likely be considerably smaller than 0.025. This situation illustrates the point made in the introduction that we cannot presume the value of a turbulent scale such as $\lambda_d$ to be the same in a two and three dimensional flow.

There is at present no clear-cut means of determining proper values for the $L_i$. We expect that this is not of major consequence since the influence of any particular choice will be absorbed indirectly in the values we eventually obtain for $C_1$ and $C_2$. We therefore will take each $L_1 = h^2$, a value which has the feature of insuring that both the near and far terms in (56) make significant contributions to $\bar{v}'^2$.

One might expect that there should be a relationship between $\lambda_d$ and $L_i$ since they are both defined through the function $R(r)$. For example, it would seem that the condition $\lambda_1 \lambda_2 < L_i$ (which is violated by $L_1 = h^2$), should be satisfied. However, we cannot assume that $R(r)$ should resemble its three-dimensional counterpart, particularly in view of the special requirements placed on $\lambda_1$ and $\lambda_2$ by the two-dimensionality, so we will not insist that $L_i$ satisfy this condition.

The constants $C_1$ and $C_2$ will be evaluated using the experimental measurements of the turbulent flow at $Re = 57,000$ obtained by Comte-Bellot (1965). This is done as follows: The experiments of Comte-Bellot have shown that for $Re = 57,000$, $\lambda \approx 0.003 U_0^2$ where $U_0$ is the dimensionless centerline velocity. If we use this value of $\lambda$, in place of $\lambda = U(\delta)2/\delta R$ in (40), then for each value of $C_2$ the system of equations can be solved
numerically. By making a visual comparison of the experimental and computed mean velocity profiles for different values of $C_2$ we may then choose a value which gives a reasonably close fit. We have found such a value to be $C_2 = 1.25$. A comparison of these velocity distributions is shown in figure 6. Using the computed values of $\delta$, $U(\delta)$ and $U_0$ we may then compute $C_1$ from the identity $0.003 \frac{U^2}{U_0} = U(\delta)2/R\delta$. We have found that $C_1 = 3.593$.

We are now in a position to study the variation of the mean flow properties with Reynolds number over a sizeable part of its range. We will expect our numerical results to be less accurate the further removed $Re$ is from 57,000 because of the assumption that none of our parameters change with $Re$. A stringent test of just how serious these errors are is provided by comparing our computed solutions with the experimental results for $Re = 120,000$ and 230,000 which were also studied in detail by Comte-Bellot. For $Re = 120,000$, the experimentally determined value of $\lambda/U_0^2$ was 0.0026 and our computed result was 0.0024. Similarly for $Re = 230,000$, experiment found $\lambda/U_0^2 = 0.00206$ and our computed result was 0.00182. Figure 6 shows a comparison for each of these Reynolds numbers of the computed and experimentally determined mean velocity profiles. These curves are shown rearranged in figure 7 so as to allow a comparison to be made of the way in which the computed and experimental velocity profiles vary with Reynolds number.

Figure 8 shows our prediction of the friction law, i.e., the dependence of $\lambda$ on $Re$, for $Re$ up to 1,000,000. This result must be viewed with some caution in the lower range of Reynolds numbers in light of the questionable validity of our difference equations there. A drag crisis
at low Reynolds numbers is clearly evident in this figure where the value of $\lambda$ suddenly increases from the laminar friction law $\lambda = 6/R_e$ represented by the straight line on the left. A distinct bifurcation in the computed results is observed at $R_e \approx 6100$. This is a critical Reynolds number below which the solutions are laminar and above which they are turbulent. We cannot actually compute a smooth transition from the laminar solution to the turbulent ones because our method of computing $U$ and $\delta$ breaks down for the parabolic case. This could be easily corrected by a suitable artifice in our computer program but the only gain in doing this would be aesthetic. The collapse of our computed solutions to the laminar case is manifested quite clearly in practice and is equivalent to actually giving a prediction of the parabolic curve.

In figure 9 a family of computed velocity profiles is shown covering the full range of Reynolds numbers up to 1,000,000 that we have studied. For comparison, a plot of the parabolic velocity profile is included. Figure 10 shows the computed mean vorticity profiles for the same range of Reynolds numbers. The straight line evident in this picture is the vorticity profile for laminar flow.

The dependence of the $\zeta$ distribution on $R_e$ is shown in figure 11 while figure 12 shows a plot of $v'^{1/2}$ for $R_e = 57,000$. It is apparent from figure 11 that the magnitude of $\zeta$ increases with $R_e$ to a maximum of $R_e \approx 30,000$, and then decreases subsequently. The velocity correlation $v'^{1/2}$ has the same dependence on $R_e$ as does $\zeta$, so the curves of $v'^{1/2}$ also decrease with increasing $R_e$ above 30,000.

Figure 13 shows a plot of the total production of $\zeta$ occurring in the region $(y'', \frac{1}{2})$, curve (i), and the two sources of loss of this $\zeta$, versus
$Re$. Curve (ii) represents the loss of $\zeta$ by its diffusion into the wall region and curve (iii) represents that by viscous dissipation. It is seen that at high $Re$, most of the dissipation is by diffusion into the wall layer, while the viscous dissipation plays a minor role. At low $Re$ this situation is reversed. It may also be noticed that the maximum of curve (i) occurs near where $\zeta$ was maximized in its $Re$ dependence in our numerical solutions.
V. CONCLUSION

The application of the method we have derived herein to study the turbulent flow in a channel represents a stringent test of its theoretical and analytical foundations. From interpreting these numerical results we will be able to partially assess the worth of coarse graining as a means of obtaining useful and accurate predictions of the mean properties of turbulent flows. We will first review the evidence which supports the belief that coarse graining is an effective means of computing the mean properties of turbulent flows, and then summarize the major limitations that are currently impeding the usefulness of this approach, and finally, we will discuss the prospects for resolving them in the future.

The first demonstration of the effectiveness of this method was that after suitably adjusting our parameters we were able to compute the mean velocity (and vorticity) profiles with great accuracy. This is significant in more than just the negative sense that if it were not so we would be in trouble. The good agreement here suggests that the stretching terms which appear in the three-dimensional mean vorticity equation but not in its two-dimensional counterpart, do not make a very large contribution to the balance of $\bar{\xi}$ in the channel. Consequently we may consider the form of our transport law in the core region to be validated, since by elimination, turbulent transport is the only important physical process remaining to effect the balance of $\bar{\xi}$ there.

Another successful facet of our numerical predictions of the flow in a channel has been our accurate computation of the change in the mean flow properties with Reynolds number. This was shown in our comparison
of mean velocity profiles given in figure 6; in the friction law shown in figure 8, which has the proper qualitative form (e.g., the presence of a drag crisis) and in the computed decrease of $\sqrt{\bar{v}^2}$ with large $Re$, a phenomenon that has been observed experimentally by Comte-Bellot. The key to achieving these results has been to capture the proper change in the rate of dissipation of $\zeta$ with $Re$. This is the crucial element because our approximations to the other aspects of the turbulent flow did not involve the arbitrariness which we found to be inherent in the dissipation term. It follows that the conjecture that at high $Re$ most of the loss of $\zeta$ occurs by its diffusion into the wall layer to eventually be reconverted to $\zeta$, is reasonably accurate. Furthermore, we conclude that the method of coarse graining in its present form can give useful insight into the dependence of a turbulent flow on Reynolds number.

Though we have cautioned against interpreting our results at moderate Reynolds numbers too seriously because of the expected growth of the wall region to encompass most of the first grid box, we nonetheless may be encouraged by our computations in this range of $Re$. As mentioned, we have predicted a drag crisis in this region and, more dramatically, a bifurcation separating laminar and turbulent solutions at $Re \approx 6100$. The "collapse" of the turbulent solutions to laminar ones is clearly seen in figure 8 for the friction law and also in figure 9 which demonstrates that the velocity profiles do not make a smooth transition across the critical range of $Re$. The computed value for the critical Reynolds number, 6100, compares quite favorably with the value of 5850 which was observed experimentally by Kao and Park (1970). We may conclude that our equations of motion as they now stand incorporate a sufficiently realistic
accounting of the basic processes of the turbulent flow in a channel, that the balance of effects which determines whether the flow at a given $R_e$ will be laminar or turbulent is meaningfully modeled.

It is clear that the accuracy of our computed solutions rested to a large extent on our having experimental measurements available so that we could choose proper values for the constants $C_1$ and $C_2$. We could also have just merely guessed values for $C_1$ and $C_2$ (as we did for $\lambda_1$, $T_{22}$ and $L_1$) and computed solutions anyway. In this case we could only hope to compute crude approximations to the flow in a channel. In fact, for flow situations in which no experimental measurements exist, we have no choice but to seek crude solutions that give a qualitative description of the flow field. We have followed just such a procedure in computations of the turbulent flow generated within a two-dimensional cylinder by the compressive motion of a piston (Bernard, 1976). These computations revealed the creation of large vortical structures which are similar to those observed visually by Oppenheim et al (1976). These results supply more support for the belief that in its present form, coarse graining is an effective means of obtaining useful information about turbulent flows.

We now consider those factors which are responsible for the unfavorable numerical results or act to limit the usefulness of the method. These are generally of three types: those which accrue from the two-dimensionality of our theory and computations; those arising from the special nature of the turbulent flow adjacent to a boundary; and those which stem from the use of the various turbulent scales in the closure to the equations of motion. We now consider each of these problem areas in detail.
The effect of two-dimensionality is most obvious in our computation of the Reynolds stresses. We saw previously that $u'v'$ is identically equal to zero in a channel if computed by our approach. Furthermore, the predictions of $\overline{v'^2}$ as shown in figure 12 show a significant departure from the experiments of Comte-Bellot in which $\overline{v'^2}$ is approximately constant across the channel only dropping off to zero a short distance from the wall. The explanation for these phenomena, as previously noted, is readily supplied by the fact that contributions to these correlations from vorticity lying above, below and parallel to the plane of motion are excluded. Apparently the contributions of the whole fluctuating field must be taken into account to accurately compute these correlations.

Other effects of two-dimensionality are more subtle. In particular, there is a serious question about the relationship between the turbulent scales of motion such as $\lambda_d$ in two and three dimensions. This problem surfaced explicitly when choosing a value for $\lambda_d$. Because of the limited amount of production of $\zeta$ which could occur in two dimensions, we were forced to assign a value to $\lambda_d$ which insured that the dissipation rate would not be excessive. In three dimensions, however, the stretching and rotating of vortex filaments acts as an additional source of $\zeta$, implying that an entirely different restriction on the permissible amount of dissipation must hold. In fact, the greater rate of production of $\zeta$ in three-dimensional flow is consistent with a smaller (and more physically plausible) value of $\lambda_d$.

We have seen that our transport law is invalid in the region adjacent to the boundary where the viscosity of the fluid exerts a major influence on the fluid motion. As a consequence of this we have been unable to
approximate the detailed flow near the boundaries. This is evident in figures 6, 7, 9, 10, and 11. Of even graver consequence to the current effectiveness of coarse graining is that this deficiency has also led to the need to introduce the empirical parameters $C_1$ and $C_2$ into our boundary conditions. Since the accuracy of our computed solutions is tied to the proper values for these parameters (which do not have a precise physical meaning), this represents a major hindrance to the use of this method to make accurate predictions of a turbulent flow.

The final deficiency to the method of coarse graining has to do with the introduction of the turbulent scales, $\lambda_1$, $\lambda_2$, $T_1$, and $L_1$ into the equations of motion. In spite of the fact that each of these quantities has a precise definition as a statistical property of the turbulent motion, it is not yet possible to compute these parameters theoretically, so they must be measured by performing experiments. The attempt to measure them experimentally is complicated by the fact that, in general, these scales may vary both spatially and temporally.

Each of these factors which detract from the effectiveness of coarse graining exists as a separate well-defined problem. They may be investigated either separately or as a whole. The solution of any one will act to further the evolution of coarse graining into a powerful method of predicting the mean properties of turbulent flows. The prospect for removing each of the current limitations to the method is varied and we consider each one now.

Certainly listed amongst the most formidable problems is that of theoretical prediction of the turbulent scales $\lambda_1$, $\lambda_2$, $T_1$, and $L_1$. To compute these requires the development of a kinetic theory of the
small-scale fluctuating turbulent motion. We may imagine that the construction of such a theory would share many of the extreme difficulties faced in devising a kinetic model of the motion of a liquid or dense gas so as to predict the molecular viscosity. Avoiding this problem by devising a closure which does not incorporate such fundamental scales is probably not possible. In fact, we may conjecture that the need to use these parameters in the equations for the mean flow properties is as necessary as the need to use the viscosity in the Navier-Stokes equations themselves, i.e., they both serve to help express the way in which a small-scale random field effects the diffusion of a macroscopic quantity.

In most other methods of computing turbulent flows, various turbulent scales are also employed in the construction of a closure. In fact, the means by which these scales are computed has received a large share of the effort spent in developing these numerical methods. In some methods, empirical relationships or simple heuristical arguments are employed to compute these scales from the mean flow field (e.g., Crawford and Kays, (1975), Patankar and Spalding (1970), Cebeci and Smith (1974)). Such procedures may also be incorporated into the method of coarse graining. The advantages of doing this may not be significant, however, since as we saw in our computation of the flow in a channel, that with just the simple assumption that each of the scales was constant throughout the flow field, we were nonetheless able to compute accurate results. In recent times some (Saffman (1974), Hanjalic and Launder (1976)) have used conservation equations for the turbulent scales, but this practice is of such dubious physical merit that it is just as likely to add inaccuracies
to coarse graining as to remove them.

Developing a more sophisticated treatment of the flow near the boundaries is perhaps one of the most fertile areas for future investigation. In many ways this is the most interesting region of the turbulent flow and so has received the bulk of previous experimental investigation. Many of the basic features of the flow in this region are fully or partially understood (see Willmarth (1975)) and simply await a proper analytical synthesis which would permit an accurate and detailed computation of the flow in this region. For the method of coarse graining such a development would mean that the need for the assumptions that went into the derivation of the boundary conditions to the \( \zeta \) equation would be eliminated, in addition to the need to use the constants \( C_1 \) and \( C_2 \).

Experiments have shown that the flow near boundaries is three-dimensional in an essential way. Thus further improvements in our treatment of the boundary flow might go hand in hand with the generalization of coarse graining into a method for computing three-dimensional flows. The successful adaptation of coarse graining to three-dimensional flow entails fulfilling several requirements. First of all, a closed set of equations must be derived for the three components of the mean vorticity field, \( \bar{\zeta}_i \), and six components, \( \xi_{ij}^{\prime} \), of the vorticity moments. In particular this requires finding a suitable model describing the process of stretching and rotating of vorticity, so that the terms accounting for these effects in the equations of motion may be approximated. Furthermore, we must modify our transport law to include these effects on the transport of vorticity. Finally, the closure relation (56) must be considerably modified to reflect the contribution of all the moments \( \xi_{ij}^{\prime} \) to the velocity moments \( u_i u_j^{\prime} \).
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FIGURE CAPTIONS

Figure 1. The relationship between a point \( x \) where \( \overline{u'}^2 \) is to be evaluated in the numerical algorithm, to the partition of \( D \) into boxes \( D_i \). \( x_0 \) represents a similar point for \( \overline{v'}^2 \).

Figure 2. Construction used to derive the expression (41) for \( \delta \). (i) Parabolic arc through the points \( (h/2, \overline{\psi}_1), (3h/2, \overline{\psi}_2) \) and \( (5h/2, \overline{\psi}_3) \); (ii) Straight line \( \overline{\psi} = -y \).

Figure 3. The qualitative behavior of \( \overline{\xi} \) and \( \overline{u'v'} \) across the channel. The length scale near the wall \( y = 0 \) has been greatly exaggerated.

Figure 4. The qualitative behavior of \( 1/R \partial \overline{\xi}/\partial y \) and \( -\overline{\nu'}^2 \) across the channel. The sum of these two curves is equal to the constant \( \lambda \). The length scale near the wall \( y = 0 \) has been greatly exaggerated.

Figure 5. The grid for the channel flow.

Figure 6. Comparison of the computed and experimentally determined distribution of the mean velocity field across the channel. □, experimental data of Comte-Bellot; ×, computed. (a) \( R_e = 57,000 \); (b) \( R_e = 120,000 \); (c) \( R_e = 230,000 \).

Figure 7. Comparison of the nature of the Reynolds number dependence of the computed and experimentally determined mean velocity profiles. (a) Computed; (b) Experiments of Comte-Bellot.

Figure 8. Variation of the logarithm of the computed friction coefficient with the logarithm of the Reynolds number. -----, Poiseuille flow.

Figure 9. Dependence of the computed mean velocity profiles on Reynolds number.

Figure 10. Dependence of the computed mean vorticity profiles on Reynolds number.

Figure 11. Dependence of the computed vorticity variance with Reynolds number.

Figure 12. Computed distribution of \( \overline{v'}^2 \) across the channel for \( R_e = 57,000 \).

Figure 13. Dependence of \( \zeta \) budget for the region \( (y'', z) \) on the logarithm of the Reynolds number. (i) Production; (ii) Loss due to diffusion into the wall region; (iii) Viscous dissipation.
Fig. 2
Viscous sub-layer

\[ \xi = -6(1 - 2y) \]

(Poiseuille flow)

\[ \bar{u} \bar{v} R \]

Fig. 3
Viscous sublayer

\[ 1/R \frac{\partial \bar{\xi}}{\partial y} \]

\[ -v' \xi \]

Fig. 4
Fig. 6
Fig. 7

(a) U/U₀ vs. XBL

(b) U/U₀ vs. XBL

Re

230,000

120,000

57,000

-72-

XBL 772-271
Re
1,000,000
500,000
230,000

Re
57,000
15,000
6500

Poiseuille flow

Fig. 9
Fig. 10

Poiseuille flow

$Re$

1,000,000
230,000
57,000

15,000
6500
Fig. 11
Fig. 12

$R_e = 57,000$
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