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THE IRREDUCIBILITY OF THE PRIMAL COHOMOLOGY OF THE THETA DIVISOR OF AN ABELIAN FIVEFOLD

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Abstract. We prove that the primal cohomology of the theta divisor of a very general principally polarized abelian fivefold is an irreducible Hodge structure of level 2.

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Introduction

Let $A$ be a principally polarized abelian variety of dimension $g \geq 4$ with smooth theta divisor $\Theta$. By the Lefschetz hyperplane theorem and Poincaré Duality (see, e.g., [IW14]) the cohomology of $\Theta$ is determined by that of $A$ except in the middle dimension $g - 1$. The primitive cohomology of $\Theta$, in the sense of Lefschetz, is

$$H^{g-1}_{pr}(\Theta) := \ker \left( H^{g-1}(\Theta, \mathbb{Z}) \xrightarrow{\cup \theta|_{\Theta}} H^{g+1}(\Theta, \mathbb{Z}) \right).$$

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The primal cohomology of $\Theta$ is defined as (see [IW14] and [ITW])

$$K := \text{Ker}(j_\ast : H^{g-1}(\Theta, \mathbb{Z}) \longrightarrow H^{g+1}(A, \mathbb{Z}))$$

where $j : \Theta \rightarrow A$ is the inclusion. This is a Hodge substructure of $H^{g-1}_{pr}(\Theta, \mathbb{Z})$ of rank $g! - \frac{1}{g+1} \binom{2g}{g}$ and level $g - 3$ while the primitive cohomology $H^{g-1}_{pr}(\Theta, \mathbb{Z})$ has full level $g - 1$.

The primal cohomology is therefore a good test case for the general Hodge conjecture. The general Hodge conjecture predicts that $K_{\mathbb{Q}} := K \otimes \mathbb{Q}$ is contained in the image, via Gysin pushforward, of the cohomology of a smooth (possibly reducible) variety of pure dimension $g - 3$ (see [IW14]). This conjecture was proved in [IS95] and [ITW] in the cases $g = 4$ and $g = 5$. When $g = 4$, it also follows from the proof of the Hodge conjecture in [IS95] that for $(A, \Theta)$ generic, $K$ is an irreducible Hodge structure (isogenous to the third cohomology of a smooth cubic threefold). When $g = 5$, the cohomology of the variety whose cohomology contains $K$ is no longer irreducible and the irreducibility of $K$ no longer follows from the proof of the Hodge conjecture.

Our main result is the somewhat unexpected (see [KW, 2.9])

**Theorem 0.1.** For a very general ppav $A$ of dimension 5 with smooth theta divisor $\Theta$. The primal cohomology $K$ of $\Theta$ is an irreducible Hodge structure of level 2.

As explained in [IW14], the above theorem considerably simplifies the proof of the Hodge conjecture in [ITW]: it is no longer necessary to show that the image of the Abel-Jacobi map in [ITW] contains all of $K$, only that it intersects $K$ non-trivially.

If $A$ is replaced by a projective space and $\Theta$ by a smooth hypersurface, then the primitive and the primal cohomology coincide. The primitive cohomology of a general hypersurface is irreducible (see, e.g., [Lam81, 7.3]).

Our strategy, explained below, for proving Theorem 0.1 is to use the Mori-Mukai proof [MM83] of the unirationality of $A_5$.

Let $T$ be an Enriques surface and

$$f : S \longrightarrow T$$

the K3 étale double cover corresponding to the canonical class (which is 2-torsion) $K_T \in \text{Pic}(T)$. Let $H$ be a very ample line bundle on $T$ with $H^2 = 10$. A general element in the linear system $|H| \cong \mathbb{P}^5$ is a smooth curve of genus 6 and such smooth curves are parametrized by the Zariski open subset $|H| \setminus D$, where $D$ is the dual variety of the embedding of $T$ in $|H|^*$. For each element
u ∈ |H| \ D, we obtain a nontrivial étale double cover $D_u := f^{-1}(C_u) \to C_u$. Associating to such a cover its Prym variety $P(D_u, C_u)$ defines a morphism from $|H| \setminus D$ to $A_5$:

\[ \begin{array}{c}
|H| \setminus D \\
\downarrow \rho_H \\
\downarrow \pi \\
A_5 \\
\end{array} \]

\[ \begin{array}{c}
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
R_6 \end{array} \]

Mori and Mukai [MM83] showed that as we vary $(T, H)$ in moduli, the family of maps $P_H$ dominates $A_5$.

The ppav $(A, \Theta)$ with singular theta divisor form the Andreotti-Mayer divisor $N_0$ in $A_5$ ([Bea77]). The divisor $N_0$ has two irreducible components $\theta_{null}$ and $N'_0$ ([Deb92],[Mum83]) (as divisors, $N_0 = \theta_{null} + 2N'_0$). The theta divisor of a general point $(A, \Theta) \in \theta_{null}$ has a unique node at a two-torsion point while the theta divisor of a general point in $N'_0$ has two distinct nodes $x$ and $-x$.

The primal cohomologies of the theta divisors form a variation of (polarized) Hodge structures over $U := |H| \setminus (D \cup P^{-1}_H(N_0))$. Inspired by [Lam81, 7.3], we prove Theorem 0.1 via a detailed study of the monodromy representation

\[ \rho : \pi_1(U) \to Aut(\mathbb{K}_Q, \langle , \rangle) \]

where $\langle , \rangle$ is the natural polarization on $\mathbb{K}_Q$ induced by the intersection pairing on $H^4(\Theta, \mathbb{Q})$.

1. Prym varieties associated to a Lefschetz pencil

1.1. A pencil of double covers. We denote by

\[ \tau : S \to S \]

the fixed point free covering involution such that $S/\tau \cong T$. By [Nam85, Prop. 2.3] the invariant subspace of the involution $\iota^*$ acting on the Néron Severi group $NS(S)$ is equal to $f^*(NS(T))$. Since the pullback

\[ f^* : NS(T) \to NS(S) \]

is injective, we deduce that $f^*(NS(T))$ is a rank 10 primitive sublattice in $NS(S)$. It follows that the Picard number of $S$ is greater than or equal to 10. By [Nam85, Prop. 5.6], when $T$ is general in moduli,

\[ NS(S) = f^*NS(T). \]

**Hypothesis:** Throughout this paper, we will assume $T$ satisfies (1.1).
Suppose $l \cong \mathbb{P}^1 \subset |H|$ is a Lefschetz pencil, i.e., it is transverse to the dual variety $D$, hence the singular curves of the pencil consist of finitely many irreducible nodal curves. Denote $\tilde{T} := Bl_{10}T$ (resp. $\tilde{S} := Bl_{20}S$) the blow-up of $T$ (resp. $S$) along the base locus of $l$ (resp. $f^*l$). We obtain a family of étale double covers parametrized by $l$:

$$
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\tilde{f}} & \tilde{T} \\
\downarrow{\pi'} & & \downarrow{\pi} \\
$ & \xrightarrow{\pi} & \end{array}
$$

**Proposition 1.1.** There are 42 singular fibers in the family $\tilde{T} \xrightarrow{\pi} l$.

**Proof.** We use the formula

$$
\chi_{top}(\tilde{T}) = \chi_{top}(T) + 10 = \chi_{top}(\mathbb{P}^1)\chi_{top}(C) + N,
$$

where $C$ is a smooth fiber in the pencil and $N$ is the number of singular fibers. We obtain $N = 42$. $\square$

Denote $C_t$ the fiber over $t \in l$ of $\pi$ and $D_t$ the corresponding étale double cover in $\tilde{S}$ and $\{s_i \in l : i = 1, ..., 42\}$ the 42 points where $\pi$ is singular.

**Proposition 1.2.** For any $t \in l$, the étale double cover $D_t$ of $C_t$ is an irreducible curve.

**Proof.** Suppose $D_t$ is reducible for some $t$. If $C_t$ is smooth, $D_t$ must be the trivial cover. If $C_t$ has one node, $D_t$ is either the trivial cover or the Wirtinger cover. In either case, the involution $\iota$ permutes the two components $D_t^1$ and $D_t^2$ of $D_t$. By (1.1), the class of $D_t^i$ in $NS(S)$ is $\iota$ invariant, thus $D_t^1$ and $D_t^2$ have the same class in $NS(S)$ and $H = 2D_t^1$. However, since $H^2 = 10$, the class of $H$ in $NS(T)$ is not 2-divisible, a contradiction. $\square$

**Corollary 1.3.** For a singular fiber $C_{s_i} = C_{pq} := \frac{C}{(p-q)}$ in the pencil $l$, the étale double cover $D_{s_i} := D_{pq}$ is obtained by glueing $p_i$ with $q_i$ for $i = 1, 2$ on a nontrivial étale double cover $D$ of $C$, where $p_i, q_i \in D$ are the inverse images of $p, q \in C$ respectively.

**Proof.** The étale double cover $D_{pq}$ of $C_{pq}$ is determined by a 2-torsion point in $Pic^0(C_{pq})$. The statement follows immediately from the irreducibility of $D_{s_i}$ and the exact sequence

$$
1 \longrightarrow \mathbb{Z}_2 \longrightarrow Pic^0(C_{pq})_2 \xrightarrow{\nu^*} Pic^0(C)_2 \longrightarrow 0,
$$
where \( \nu : C \to C_{pq} \) is the normalization map and the kernel of \( \nu^* \) is generated by the point of order 2 corresponding to the Wirtinger cover.

1.2. The compactified Prym variety. We describe the compactified Prym variety for the cover \( D_{pq} \to C_{pq} \) as in Corollary 1.3. The semiabelian part \( G_{pq} \) of the Prym variety is the identity component \( \ker^0(Nm_{pq}) \), of \( \ker(Nm_{pq}) \subset \text{Pic}^0(D_{pq}) \) in the following commutative diagram with exact rows and columns.

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & \mathbb{C}^* & \ker(Nm_{pq}) & \ker(Nm) & \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
1 & (\mathbb{C}^*)^2 & \text{Pic}^0(D_{pq}) & \to \text{Pic}^0(D) & \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
1 & \mathbb{C}^* & \text{Pic}^0(C_{pq}) & \to \text{Pic}^0(C) & \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0.
\end{array}
\]

It follows immediately that the group scheme \( G_{pq} \) is a \( \mathbb{C}^* \)-extension of the Prym variety \( (B, \Xi) := \text{Prym}(D, C) \):

\[
1 \to \mathbb{C}^* \to G_{pq} \to B \to 0.
\]

Let \( p : P^\nu \to B \) be the unique \( \mathbb{P}^1 \) bundle containing \( G_{pq} \) and write \( P^\nu \setminus G_{pq} = B_0 \amalg B_\infty \), where \( B_0 \) and \( B_\infty \) are the zero and infinity sections of \( P^\nu \).

The compactified ‘rank one degeneration’ \( P \) is constructed as follows (c.f. [Mum83, \S 1]).

1. On \( P^\nu \), we have \( B_0 - B_\infty \sim_{\text{lin}} p^{-1}(\Xi - \Xi_b) \) for a unique \( b \in B \). Thus

\[
B_0 + p^{-1}\Xi_b \sim_{\text{lin}} B_\infty + p^{-1}\Xi.
\]

2. Let \( L^\nu := \mathcal{O}_{P^\nu}(B_0 + p^{-1}\Xi_b) \). Then \( L^\nu|_{B_0} \cong \mathcal{O}_B(\Xi) \) and \( L^\nu|_{B_\infty} \cong \mathcal{O}_B(\Xi_b) \). Via the Leray spectral sequence for \( p \), we see that \( h^0(P^\nu, L^\nu) = 2 \) and \( B_0 + p^{-1}\Xi_b, B_\infty + p^{-1}\Xi \) span \( |L^\nu| \).

3. The compactified Prym variety \( P \) is constructed from \( P^\nu \) by identifying the zero section \( B_0 \overset{p}{\cong} B \) with the infinity section \( B_\infty \overset{p}{\cong} B \) via translation by \( b \in B \). We also denote \( \nu : P^\nu \to P \) the normalization morphism.
(4) The line bundle $L''$ descends to a line bundle $L$ on $P$, i.e., $ν^*L \cong L''$. The linear system $|L|$ is a point.

(5) The theta divisor $Υ \subset P$ is the unique divisor in $|L|$.

**Remark 1.4.** The $\mathbb{P}^1$ bundle $P'' \to B$ contains an open subset $P'' \setminus B_\infty$ (resp. $P'' \setminus B_0$), which is isomorphic to the total space of $N_{B_0}P'' \cong \mathcal{O}_{B_0}(B_0) \cong \mathcal{O}_B(Ξ - Ξ_b)$ (resp. $\mathcal{O}_B(Ξ_b - Ξ)$). We conclude that $P'' \cong \mathbb{P}(\mathcal{O}_B(Ξ - Ξ_b) \oplus \mathcal{O}_B(Ξ_b - Ξ)).$ In particular $G_{pq} \to B$ and $P'' \to B$ are **topologically trivial** $\mathbb{C}^*$ and $\mathbb{P}^1$ bundles, respectively.

**Proposition 1.5.** For a general rank one degeneration, the normalization $Υ''$ of the theta divisor is isomorphic to $Bl_{Ξ \cap Ξ_b} B \subset P''$, the theta divisor $Υ \subset P$ is obtained from $Υ''$ by identifying the proper transforms of $Ξ$ and $Ξ_b$.

**Proof.** Let $σ_0$, $σ_∞$ be elements of $H^0(P'', L'')$, such that $div(σ_0) = B_0 + p^{-1}Ξ_b$ and $div(σ_∞) = B_∞ + p^{-1}Ξ$. After rescaling, we may assume, under the natural identification $B_0 \cong B \cong B_∞$, that $σ_0|_{B_∞}$ and $σ_∞|_{B_0}$ differ by translation by $b$. Then $σ_0 + σ_∞$ descends to a section of $L$. Since $(σ_0 + σ_∞)|_{B_0}$ vanishes precisely on $Ξ$ and $(σ_0 + σ_∞)|_{B_∞}$ vanishes precisely on $Ξ_b$, we conclude that for $u \in B \setminus (Ξ \cap Ξ_b)$, $0 \neq (σ_0 + σ_∞)|_{p^{-1}(u)} \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$. Thus $Υ'' := div(σ_0 + σ_∞)$ maps one-to-one to $B$ away from $Ξ \cap Ξ_b$. On the other hand, the base locus of the pencil $|L''|$ is clearly $p^{-1}(Ξ \cap Ξ_b)$. Thus $Υ'' = m[Bl_{Ξ \cap Ξ_b} B]$, for some integer $m$, as divisors in $P''$. Since $(σ_0 + σ_∞)|_{B_0}$ is reduced, $m = 1$. □

2. **Numerical calculations**

The family of compactified Prym varieties defines a morphism $ρ : l \to \tilde{A}_5$ where $\tilde{A}_5$ is the partial compactification of $A_5$ parametrizing ppav $(A, Θ)$ of dimension 5 and their rank 1 degenerations. This space is a quasi-projective variety and is essentially the blow-up of the open set $A_5 \amalg A_4$ in the Baily-Borel compactification $A_5^*$ along its boundary $A_4$ ([Igu67]). The coarse moduli space $\tilde{A}_5$ is the union of $A_5$ and a divisor $∆$ parametrizing rank 1 degenerations. Mumford [Mum83] computed the class of the closure of $θ_{null}$ and $N'_0$ in $\tilde{A}_5$ to be

\[(2.1) \quad [θ_{null}] = 264λ - 32δ,\]

\[(2.2) \quad [N'_0] = 108λ - 14δ,\]

\[(2.3) \quad [N_0] = [θ_{null}] + 2[N'_0] = 480λ - 60δ,\]
Lemma 2.1. The degree of $\rho^*\lambda$ is 6.

Proof. The pull-back of the Hodge bundle $\Lambda$ to $l$ fits in the exact sequence

$$0 \longrightarrow \pi_*\omega_{\tilde{T}/l} \longrightarrow \pi'_*\omega_{\tilde{S}/l} \longrightarrow \rho^*\Lambda \longrightarrow 0,$$

where $\omega_{\tilde{T}/l}$ and $\omega_{\tilde{S}/l}$ are the relative dualizing sheaves. Thus $c_1(\rho^*\lambda) = c_1(\pi'_*\omega_{\tilde{S}/l}) - c_1(\pi_*\omega_{\tilde{T}/l})$. We directly compute that the relative dualizing sheaf $\omega_{\tilde{T}/l} = K_{\tilde{T}} \otimes \pi^*K_{l}^{-1}$ has self intersection number $(\omega_{\tilde{T}/l})^2 = 30$. Applying Mumford’s relation [ACG11, Chapter 13.7] on $M_6$, we see that $c_1(\pi_*\omega_{\tilde{T}/l}) = 30 + 42 \cdot 12 = 6$. Similarly, we compute $c_1(\pi'_*\omega_{\tilde{S}/l}) = 12$ and therefore $c_1(\rho^*\lambda) = 6$. \qed

Corollary 2.2. In the pencil $l$, there are 240 fibers with theta divisor singular at a unique two-torsion point and 60 fibers with theta divisor singular at two points.

Proof. We directly compute $l \cdot [\theta_{\text{null}}] = l \cdot (264\lambda - 32\delta) = 240$ and $l \cdot [N'_0] = l \cdot (108\lambda - 14\delta) = 60$. It follows from [SV90, Lemma B] that all these points occur with multiplicity 1 in the intersection $l \cap N_0$ for a generic choice of $l$. \qed

To summarize, we have a family of (compactified) Prym varieties and theta divisors associated to the pencil

$$\Theta \longrightarrow A \longrightarrow l.$$

This family has 240 fibers where theta has a single node, 60 fibers where theta has two nodes, and 42 fibers where theta is as in Proposition 1.5. Furthermore, we have

Proposition 2.3. The total spaces $A$ and $\Theta$ are smooth.

Proof. We show that the tangent spaces to $A$ and $\Theta$ have dimension 6 and 5 respectively everywhere. Let $p \in A_t$, resp. $p \in \Theta_t$, be a point of the fiber of $A \to l$, resp. $\Theta \to l$, at $t \in l$. If $A_t$ is smooth at $p$, it follows from [ITW, Proposition 3.1] that, for a generic choice of $l$, both $A$ and $\Theta$ (when $p \in \Theta$) are smooth at $p$. Assume therefore that $A_t$ is singular at $p$. In such a case, it follows from the description of $\Theta_t$ in Proposition 1.5 that, if $p \in \Theta$, $\Theta_t$ is also singular at $p$. By the description of $A_t$ in Section 1.2, resp. $\Theta_t$ in Proposition 1.5, the tangent space to $A_t$ at $p$, resp. $\Theta_t$ at $p$, has dimension 6, resp. 5. We therefore need to show that the tangent space to the total space $A$, resp. $\Theta$, is equal to the tangent space of the fiber. The tangent space to the fiber is the kernel of the differential of the map $A \to l$, resp. $\Theta \to l$. Since the map $\Theta \to l$ is the scheme-theoretic restriction
of the map $A \to l$, we need to show that the differential of the map $A \to l$ is 0 at $p$ to obtain the smoothness of $A$ at $p$ and also of $\Theta$ at $p$ when $p \in \Theta$.

The total space $A$ is the inverse image of the generic line $l \subset |H|$ in the relative Prym variety $P_H \to |H|$ constructed in [AFS15]. By [AFS15, Prop. 3.10, Prop. 4.4, Prop. 5.1], the singular locus of $P_H$ lies above a union of lines or points $m_i$ in $|H|$. We can therefore assume that $l$ does not meet any of the $m_i$. Furthermore, since all pull-backs are scheme-theoretic and all fibers reduced, the restriction of the differential of $P_H \to |H|$ to $A$ is the differential of the projection $A \to l$. The rank of the differential of $P_H \to |H|$ is not maximal at $p$, i.e., its image is a proper subspace of the tangent space of $|H|$ at $t$. Since $l$ is generic, the tangent space of $l$ at $t$ intersects this image in 0. Therefore the differential of $A \to l$ is 0 at $p$. \qed

### 3. General facts about the Clemens-Schmid exact sequence

We briefly review some general facts about the Clemens-Schmid exact sequence. We will apply the general theory in this section to compute the local monodromy representations near the degenerate theta divisors in the pencil.

#### 3.1. The Clemens-schmid exact sequence

Let

\[
\begin{array}{cccc}
Y_0 & \longrightarrow & \mathcal{Y} & \longrightarrow & Y_t \\
\downarrow & & \downarrow & & \downarrow \\
\{0\} & \longrightarrow & V & \longrightarrow & \{t\}
\end{array}
\]

be a one-parameter semistable degeneration (i.e., the total space $\mathcal{Y}$ is smooth and the central fiber $Y_0$ is reduced with simple normal crossing support) over a small disk $V$, and $0 \neq t \in \partial V$ a general point. The total space $\mathcal{Y}$ deformation retracts to $Y_0$. For such a family, the image of the monodromy representation

\[\rho : \pi_1(V \setminus \{0\}, t) \longrightarrow GL(H^\bullet(Y_t))\]

is generated by a unipotent operator $T : H^\bullet(Y_t) \to H^\bullet(Y_t)$, i.e. $(T - Id)^k = 0$ for some integer $k$ [Lan73]. Thus

\[N := \log T := (T - Id) - \frac{1}{2}(T - Id)^2 + \frac{1}{3}(T - Id)^3 + ...\]

is nilpotent.

It follows from the work of Clemens-Schmid [Cle77], [Sch73] and Steenbrink [Ste76] that one can define mixed Hodge structures on $H^\bullet(Y_t)$, $H^\bullet(\mathcal{Y})$ and $H_\bullet(\mathcal{Y})$ such that we have an exact sequence
of mixed Hodge structures (with suitable weight shifts)

\[ H_{2n+2-m}(\mathcal{Y}) \xrightarrow{\alpha} H^m(\mathcal{Y}) \xrightarrow{i_!^*} H^m(\mathcal{Y}_t) \xrightarrow{N} H^m(\mathcal{Y}_t) \xrightarrow{\beta} H_{2n-m}(\mathcal{Y}) \]

where \( n \) is the relative dimension of the fibration, \( \alpha \) is the composition

\[ H_{2n+2-m}(\mathcal{Y}) \xrightarrow{PD} H^m(\mathcal{Y}, \partial \mathcal{Y}) \xrightarrow{i_{\ast}^*} H^m(\mathcal{Y}), \]

and \( \beta \) is the composition

\[ H^m(\mathcal{Y}_t) \xrightarrow{PD} H^m_{2n-m}(\mathcal{Y}_t) \xrightarrow{i_{\ast}^*} H^m_{2n-m}(\mathcal{Y}). \]

Here ‘PD’ stands for Poincaré duality. The mixed Hodge structure on \( H^\bullet(Y_0) \) is not the usual pure Hodge structure but rather the ‘limit mixed Hodge structure’ (c.f. Section 3.3). We use the notation \( H^\bullet(Y_t)_{\text{lim}} \) to distinguish it from the pure Hodge structure.

3.2. The weight filtrations on \( H^m(\mathcal{Y}) \) and \( H_m(\mathcal{Y}) \). Denote

\[ H^m := H^m(\mathcal{Y}) \cong H^m(\mathcal{Y}_0), \]
\[ H_m := H_m(\mathcal{Y}) \cong H_m(\mathcal{Y}_0). \]

Recall from [Mor84, p. 103] that there is a Mayer-Vietoris type spectral sequence abutting to \( H^\bullet(Y_0) \) with \( E_1 \) term

\[ E_1^{p,q} = H^q(Y_0^{[p]}). \]

Here \( Y_0^{[p]} \) is the disjoint union of the codimension \( p \) strata of \( Y_0 \), i.e.,

\[ Y_0^{[p]} := \coprod_{i_0, \ldots, i_p} Z_{i_0} \cap \ldots \cap Z_{i_p} \]

where the \( Z_{i_j} \) are distinct irreducible components of \( Y_0 \).

The differential \( d_1 \)

\[ E_1^{p,q} \xrightarrow{d_1} E_1^{p+1,q} \]
\[ H^q(Y_0^{[p]}) \xrightarrow{d_1} H^q(Y_0^{[p+1]}) \]

is the alternating sum of the restriction maps on all the irreducible components. By [Mor84, p. 103] this sequence degenerates at \( E_2 \).

The weight filtration is given by

\[ W_k H^m := \bigoplus_{p+q=m, q \leq k} E_\infty^{p,q} = \bigoplus_{p+q=m, q \leq k} E_2^{p,q}. \]
Therefore the weights on $H^m$ go from 0 to $m$ and
$$Gr_k H^m \cong E_2^{m-k,k} = \frac{\text{Ker}(d_1 : H^k(Y_0^{|m-k|}) \to H^k(Y_0^{|m-k+1|}))}{\text{Im}(d_1 : H^k(Y_0^{|m-k-1|}) \to H^k(Y_0^{|m-k|}))}.$$ 

There is also a weight filtration on $H_m$:
$$W_{-k} H_m := (W_{k-1} H^m)^\perp$$
under the perfect pairing between $H^m$ and $H_m$. With this definition,
$$Gr_{-k} H_m \cong (Gr_k H^m)^\vee.$$

### 3.3. The limit mixed Hodge structure $H^m(Y_t)_{\text{lim}}$

The weight filtration associated to the nilpotent operator $N$ has the following form,
$$0 \subset W_0 \subset W_1 \subset \ldots \subset W_{2m} = H^m(Y_t).$$

We refer to [Mor84, pp. 106-109] for the precise definition of the monodromy weight filtration and only summarize the properties we need here.

In the applications in this paper, the nilpotent operator $N$ satisfies
$$N^2 = 0.$$ 

Thus the monodromy weight filtration satisfies the following
$$W_k = 0 \text{ for } k \leq m - 2,$$
$$W_{m-1} = \text{Im}(N),$$
$$W_m = \text{Ker}(N),$$
$$W_k = H^m(Y_t) \text{ for } k \geq m + 1.$$ 

Let $K^m_t := \text{Ker}(N) \subset H^m(Y_t)$ be the monodromy invariant subspace. It inherits an induced weight filtration from $H^m(Y_t)$. The graded pieces of $H^m(Y_t)_{\text{lim}}$ thus satisfy
$$Gr_m H^m(Y_t)_{\text{lim}} \cong Gr_m K^m_t \cong \frac{\text{Ker}(N)}{\text{Im}(N)}$$
$$Gr_{m+1} H^m(Y_t)_{\text{lim}} \cong Gr_{m+1} H^m(Y_t)_{\text{lim}} \cong Gr_{m+1} K^m_t \cong \text{Im}(N).$$

The weight filtrations on $H^m$ and $K^m_t$ are related by the Clemens-Schmid exact sequence. Below are the basic facts we will use (see [Mor84, pp. 107-109])
(1) $i_t^*$ induces an isomorphism

$$Gr_k H^m \overset{\cong}{\longrightarrow} Gr_k K_t^m$$  \hspace{1cm} \text{for } k \leq m - 1.$$  

(2) There is an exact sequence

$$0 \longrightarrow Gr_{m-2} K_t^{m-2} \longrightarrow Gr_{m-2n-2} H_{2n+2-m} \overset{\alpha}{\longrightarrow} Gr_m H^m \longrightarrow Gr_m K_t^m \longrightarrow 0.$$  

The limit Hodge filtration on $H^m(Y_t)_\text{lim}$ is given by ([Mor84], [Sch73])

$$F_p^\infty = \lim_{z \rightarrow \infty} \exp(-zN)F^p(z)$$

where $f : U' \rightarrow U \setminus \{0\}$, $f(z) = e^{2\pi iz}$ is the universal cover of the punctured disk and $F^p$ is the usual Hodge filtration on $H^m(Y_f(z))$ on the fixed underlying space $H^m(Y_t)$.

4. LOCAL MONODROMY REPRESENTATIONS NEAR $N_0$

4.1. **Local monodromy near $\theta_{null}$**. The local monodromy representation on the cohomology of the theta divisor near a general point $(A_0, \Theta_0) \in \theta_{null}$ is given by the classic Picard-Lefschetz formula. Fix a point $p_0 \in l \cap \theta_{null}$ and pick a small disk $U \subset l$ containing $p_0$. We have a family of theta divisors with smooth total space $\Theta_U$ (see Proposition 2.3):

$$\Theta_0 \longrightarrow \Theta_U \hspace{1cm} p_0 \longrightarrow U.$$  

The local monodromy representation on the cohomology of a general fiber $\Theta_t$ for $t \in U \setminus \{p_0\}$

$$\rho : \pi_1(U \setminus \{p_0\}, t) \longrightarrow GL(H^k(\Theta_t))$$

is trivial when $k \neq 4$. When $k = 4$, the Picard-Lefschetz formula (see, for instance, [Voi03, p. 78]) shows that $\rho(\pi_1(U \setminus \{p_0\}, t))$ is generated by

$$T_U : H^4(\Theta_t) \longrightarrow H^4(\Theta_t)$$

$$\alpha \quad \mapsto \quad \alpha - \langle \alpha, \gamma \rangle \gamma$$

where $\langle , \rangle$ is the intersection product on $H^4(\Theta_t)$, and $\gamma \in H^4(\Theta_t)$ is the class of the vanishing 4-sphere with $\langle \gamma, \gamma \rangle = 2$.

One checks immediately that

$$T_U^2 = Id.$$  

(4.1)
4.2. **Local monodromy near** $N'_0$. Next we fix a point $p_0 \in l \cap N'_0$ and a small disk $U \subset l$ containing $p_0$. The central fiber $\Theta_0$ of the family $\Theta_U$ has two ordinary double points $x$ and $-x$.

If we make a degree two base change $V \to U$ ramified at $p_0$:

$$
\begin{align*}
\Theta_V &\longrightarrow \Theta_U \\
V &\longrightarrow U,
\end{align*}
$$

then blow up the two singular points of $\Theta_V$, we obtain a family

$$
\begin{align*}
\tilde{\Theta}_0 &\longrightarrow \tilde{\Theta}_V \\
\tilde{p}_0 &\longrightarrow V,
\end{align*}
$$

where the central fiber $\tilde{\Theta}_0 = \Theta'_0 \cup Q_1 \cup Q_2$ is reduced with simple normal crossing support. Here $\Theta'_0$ is the blow-up of $\Theta_0$ at the two singular points and $Q_1 \cong Q_2$ are smooth quadric 4-folds. The double loci $\Theta'_0 \cap Q_1$ and $\Theta'_0 \cap Q_2$ are smooth quadric 3-folds.

Since $V \to U$ is a degree 2 ramified cover, the local monodromy operator $T_V$ for the family $\tilde{\Theta}_V \to V$ is equal to $T^2_U \in GL(H^4(\Theta_t))$.

**Proposition 4.1.** Notation as above, $T_V = T^2_U = Id \in GL(H^4(\Theta_t))$.

**Proof.** Since the central fiber $\tilde{\Theta}_0 = \Theta'_0 \cup Q_1 \cup Q_2$ only has a double locus, we have

$$Gr_kH^4(\Theta_t) = 0$$

for $k \neq 3, 4, 5$. Since $H^3(\Theta'_0 \cap Q_1) \oplus H^3(\Theta'_0 \cap Q_2)) = 0$, we conclude

$$Gr_3H^4(\Theta_t) \cong Gr_3H^4(\Theta_t) \cong Gr_3H^4(\tilde{\Theta}_0) = \text{Im}(N_V) = 0,$$

where $N_V := \log T_V = 0$. Therefore $T_V = Id$. \qed

5. **Local monodromy near the boundary** $\Delta$

Near the boundary $\Delta$, the family of Prym varieties $A_U \to U$ parametrized by a small disk $U \subset l$ has smooth general fiber $(A_t, \Theta_t)$ and central fiber $(P, \Upsilon)$ as in Proposition 1.5. We use the Clemens-Schmid exact sequence to compute the monodromy action.
5.1. The semi-stable reduction. Making a ramified base change $V \to U$ of order 2 of the family

\[
\begin{array}{ccc}
A_V & \longrightarrow & A_U \\
\downarrow & & \downarrow \\
V & \longrightarrow & U,
\end{array}
\]

and then blowing up the singular locus $P \setminus G_{pq}$ of $A_V$, we obtain a family $\tilde{A}_V \to V$.

**Proposition 5.1.** The central fiber $\tilde{A}_0$ of the family $(\tilde{A}_V, \tilde{\Theta}_V) \to V$ is the union of two copies $P'_1$ and $P'_2$ of $P'$, with $B_0 \subset P'_1$ identified with $B_\infty \subset P'_2$ via the identity map and $B_\infty \subset P'_1$ identified with $B_0 \subset P'_2$ via translation by $b$. The intersection $P'_1 \cap P'_2 = B_{0\infty} \cup B_\infty$ is the disjoint union of two copies of $B$.

**Proof.** Clearly the main component $P'_1 \cong P'$. We will show the exceptional divisor $P'_2$ is also isomorphic to $P'$. In the semistable family $\tilde{A}_V \to V$, we have

\[
N^{\nu}_{B_{0\infty}/P'_1} \cong N^{\nu}_{B_{0\infty}/P'_2}.
\]

Therefore $P'_2$ contains the total space of $\mathcal{O}_B(\Xi_b - \Xi) \cong \mathcal{O}_{B_0}(\Xi_b) \cong N_{B_{0\infty}/P'_2} = P'_2 \setminus B_{0\infty}$ as a Zariski open subset. Apply the same argument to $B_{\infty}$, we see that $P'_2$ also contains the total space of $\mathcal{O}_B(\Xi - \Xi_b) \cong N_{B_{0\infty}/P'_2} = P'_2 \setminus B_{\infty}$ as an open subset. We conclude that $P'_2 \cong \mathbb{P}_B(\mathcal{O}_B(\Xi - \Xi_b) \oplus \mathcal{O}_B(\Xi_b - \Xi)) \cong P'$. The statement about the gluing follows from the fact that after contracting $P'_2$, the infinity and zero sections of $P'_1$ are identified via translation by $b$. \qed

**Corollary 5.2.** The central fiber $\tilde{\Theta}_0$ of the family $(\tilde{A}_V, \tilde{\Theta}_V) \to V$ is the union $\Upsilon' \cup Q_{\Xi}$, where $\Upsilon' = B_{|\Xi|} \cup B$ and the conic bundle $Q_{\Xi}$ is the restriction of $P''_2 \to B$ to $\Xi$. The intersection $\Upsilon' \cap Q_{\Xi} = \Xi_{0\infty} \cup \Xi_{\infty}$ is the disjoint union of two copies of $\Xi$.

**Proof.** Immediate. \qed

5.2. The weight filtration on $H^m(\tilde{A}_0)$. By Section 3.2 and Proposition 5.1, the weight filtration on $H^m(\tilde{A}_0)$ only has the following possibly nontrivial graded pieces

\[
Gr_mH^m(\tilde{A}_0) = \text{Ker}(d_1 : H^m(P'_1) \oplus H^m(P'_2) \to H^m(B_{0\infty}) \oplus H^m(B_{\infty}))
\]

and

\[
Gr_{m-1}H^m(\tilde{A}_0) = \text{Coker}(d_1 : H^{m-1}(P'_1) \oplus H^{m-1}(P'_2) \to H^{m-1}(B_{0\infty}) \oplus H^{m-1}(B_{\infty}))
\]
Proposition 5.3. We have
\[ Gr_m H^m(\tilde{A}_0) \cong H^{m-2}(B) \oplus H^m(P^\nu), \]
and
\[ Gr_{m-1} H^m(\tilde{A}_0) \cong H^{m-1}(B). \]

Proof. By Remark 1.4, \( P^\nu \to B \) is a topologically trivial \( \mathbb{P}^1 \) bundle. The statements then follow easily from Proposition 5.1 and the Künneth formula. \( \square \)

Corollary 5.4. The monodromy weight filtration on \( H^m(A_t)_{\text{lim}} \) satisfies
\[ Gr_{m+1} H^m(A_t)_{\text{lim}} \cong Gr_{m-1} H^m(A_t)_{\text{lim}} \cong H^{m-1}(B). \]
Furthermore, \( \dim \mathbb{C} Gr_m H^m(A_t)_{\text{lim}} = \binom{10}{m} - 2 \binom{8}{m-1} \).

Proof. By (3.5) and (3.6), \( Gr_{m+1} H^m(A_t)_{\text{lim}} \cong Gr_{m-1} H^m(A_t)_{\text{lim}} \cong Gr_{m-1} H^m(\tilde{A}_0) \) which is isomorphic to \( H^{m-1}(B) \) by Proposition 5.3. The second part follows from Sequence (3.7). \( \square \)

5.3. The weight filtration on \( H^m(\tilde{\Theta}_0) \). By Section 3.2 and Proposition 5.2, the weight filtration on \( H^m(\tilde{\Theta}_0) \) only has the following possibly nontrivial graded pieces
\[ Gr_m H^m(\tilde{\Theta}_0) = \text{Ker}(d_1 : H^m(\Upsilon^\nu) \oplus H^m(Q_{\Xi}) \to H^m(\Xi_0\infty) \oplus H^m(\Xi_{\infty 0})) \]
and
\[ Gr_{m-1} H^m(\tilde{\Theta}_0) = \text{Coker}(d_1 : H^{m-1}(\Upsilon^\nu) \oplus H^{m-1}(Q_{\Xi}) \to H^{m-1}(\Xi_0\infty) \oplus H^{m-1}(\Xi_{\infty 0})). \]

Proposition 5.5. For \( m \leq 4 \),
\[ Gr_m H^m(\tilde{\Theta}_0) \cong H^m(\tilde{\Theta}_0) \cong H^m(B) \oplus H^{m-2}(\Xi \cap \Xi_b) \oplus H^{m-2}(\Xi), \]
and for all \( m \),
\[ Gr_{m-1} H^m(\tilde{\Theta}_0) \cong H^{m-1}(\Xi). \]

Proof. By Corollary 5.2, \( H^m(\Upsilon^\nu) \cong H^m(B) \oplus H^{m-2}(\Xi \cap \Xi_b) \) and the restriction map \( H^m(\Upsilon^\nu) \to H^m(\Xi_0\infty) \) can be identified with the map
\[ H^m(B) \oplus H^{m-2}(\Xi \cap \Xi_b) \xrightarrow{(\iota^* \iota_*)} H^m(\Xi). \]
Thus the image of
\[ H^m(\Upsilon^\nu) \to H^m(\Xi_0\infty) \oplus H^m(\Xi_{\infty 0}). \]
is contained in the image of
\[ H^m(Q_\Xi) \cong H^m(\Xi) \oplus H^{m-2}(\Xi) \to H^m(\Xi_0) \oplus H^m(\Xi_\infty), \]
which is equal to the diagonal of \( H^m(\Xi_0) \oplus H^m(\Xi_\infty) \). Thus
\[ Gr_{m-1}H^m(\tilde{\Theta}_0) \cong H^{m-1}(\Xi). \]
Next we compute \( Gr_m H^m(\tilde{\Theta}_0) \subset H^m(\Upsilon^\nu) \oplus H^m(Q_\Xi) \). By the previous discussion, for any \( x \in H^m(\Upsilon^\nu) \), we can find \( y \in H^m(Q_\Xi) \) such that \( (x, y) \in Gr_m H^m(\tilde{\Theta}_0) \). Thus we have an exact sequence
\[ 0 \to H^{m-2}(\Xi) \to Gr_m H^m(\tilde{\Theta}_0) \to H^m(\Upsilon^\nu) \to 0 \]
Therefore, we have a noncanonical isomorphism
\[ Gr_m H^m(\tilde{\Theta}_0) \cong H^{m-2}(\Xi) \oplus H^m(\Upsilon^\nu) \cong H^m(B) \oplus H^{m-2}(\Xi_0) \oplus H^{m-2}(\Xi) \]

\[ \Box \]

**Corollary 5.6.** The monodromy weight filtration on \( H^m(\Theta_t)_{\text{lim}} \) satisfies
\[ Gr_{m+1} H^m(\Theta_t)_{\text{lim}} \cong Gr_{m-1} H^m(\Theta_t)_{\text{lim}} \cong H^{m-1}(\Xi). \]
Furthermore, \( \dim C Gr_m H^m(\Theta_t)_{\text{lim}} = h^m(\Theta_t) - 2h^{m-1}(\Xi) \).

**Proof.** Analogous to the proof of Corollary 5.4. \[ \Box \]

### 5.4. The vanishing cocycles near the boundary.
Let \( Z \to |H| \cong \mathbb{P}^5 \) be the 2-to-1 cover ramified exactly along \( \Gamma := D + \overline{P^{-1}(N_0)} \) and set \( X := \nu^{-1}l, \ U := Z \setminus \Gamma \). Note that \( Z \) exists since \( \Gamma \) has even degree by Proposition 1.1 and Corollary 2.2. The curve \( X \) is a 2-to-1 cover of \( l \) ramified along \( X \cap \Gamma \). After base change to \( X \) and blowing up the singular locus of each singular theta divisor, we obtain a family \((\tilde{A}, \tilde{\Theta})\) with general fiber \((A_t, \Theta_t)\).

\[ \Theta_t \xrightarrow{i_t} \tilde{\Theta} \]
\[ j_t \downarrow \quad j \downarrow \]
\[ A_t \xrightarrow{h_t} \tilde{A} \]
\[ \{t\} \longrightarrow X. \]

The total spaces of \( \tilde{A} \) and \( \tilde{\Theta} \) are smooth and the local pictures are described in Sections 4.1, 4.2 and 5.1.
For each $s_i$, $i = 1, ..., 42$, corresponding to the degeneration in Section 1 (also see Section 5.1), choose a small disk $V_i \ni s_i$ and pick a general point $t_i \in V_i$. Let $\gamma_i \subset X$ be a general path connecting $t$ with $t_i$. The family $\tilde{\Theta}|_{\cup \gamma_i}$ deformation retracts to $\Theta_t$. Thus we have induced \textbf{diffeomorphisms}

$$\psi_i : \Theta_t \rightarrow \Theta_{t_i}.$$ 

Over each $V_i$ we have the Clemens-Schmid exact sequences (3.1) for the degenerations of the abelian varieties and their theta divisors

\begin{equation}
\begin{array}{ccccccccc}
& & & & & & & & & \\
\longrightarrow & H^m(\tilde{\Theta}_{V_i}) & \overset{i^*_t}{\longrightarrow} & H^m(\Theta_{t_i})_{\text{lim}} & \overset{N_i}{\longrightarrow} & H^m(\Theta_{t_i})_{\text{lim}} & \overset{\beta_i}{\longrightarrow} & H_{10-m}(\tilde{\Theta}_{V_i}) & \longrightarrow \\
& & & & & & & & & \\
\downarrow j^*_t & & & & & & & & & \downarrow j^*_t \\
& & & & & & & & & \\
\longrightarrow & H^{m+2}(\tilde{A}_{V_i}) & \longrightarrow & H^{m+2}(A_{t_i})_{\text{lim}} & \longrightarrow & H^{m+2}(A_{t_i})_{\text{lim}} & \longrightarrow & H_{10-m}(\tilde{A}_{V_i}) & .
\end{array}
\end{equation}

Here $j^*_t : H^m(\tilde{\Theta}_{V_i}) \rightarrow H^{m+2}(\tilde{A}_{V_i})$ is defined to be the transpose of $j^* : H_{10-m}(\tilde{A}_{V_i}) \rightarrow H_{10-m}(\tilde{\Theta}_{V_i})$ under Poincaré duality and is a morphism of mixed Hodge structures [ITW, ??].

Denote $\nabla^m_i := \psi_i^* \text{Ker } \beta_i = \psi_i^* \text{Im}(N_i) = \psi_i^* \text{Gr}_{m-1}H^m(\Theta_{t_i})_{\text{lim}} \subset H^m(\Theta_t)_{\text{lim}}$.

**Proposition 5.7.** The space $\nabla_i$ is the space of 'local vanishing $m$-cocycles', i.e., cohomology classes whose Poincaré dual vanishes in $\Theta_{V_i}$.

**Proof.** This follows immediately from the definition of $\beta_i$ in (3.3). \qed

By Corollary 5.6, we have

$$\text{Im}(N_i) = \text{Gr}_{m-1}H^m(\Theta_{t_i})_{\text{lim}} \overset{i^*_t}{\cong} \text{Gr}_{m-1}H^m(\tilde{\Theta}_{V_i}) \cong H^{m-1}(\Xi).$$

When $m = 4$, we can further rewrite the above isomorphisms as

\begin{equation}
\text{Gr}_3H^4(\Theta_{t_i})_{\text{lim}} \cong H^3(\Xi) \cong H^3(B) \oplus \mathbb{H}'_i \cong j^*_t \text{Gr}_3H^4(A_{t_i})_{\text{lim}} \oplus \mathbb{H}'_i,
\end{equation}

where $\mathbb{H}'_i \subset H^3(\Xi)$ is the primal cohomology of $\Xi$ in $B$, which is 10-dimensional. Let $\mathbb{H}_i \subset \nabla^4_i \subset H^4(\Theta_t)$ be the image of $\mathbb{H}'_i$ under the composition

$$H^3(B) \oplus \mathbb{H}'_i \cong \text{Gr}_3H^4(\Theta_{t_i})_{\text{lim}} \subset H^4(\Theta_{t_i})_{\text{lim}} \overset{\psi_i^*}{\rightarrow} H^4(\Theta_t).$$

\section{Global monodromy}

Let $H^m(\Theta_{t_{\text{var}}}) := \text{Ker}(i_{t*} : H^m(\Theta_t) \rightarrow H^{m+2}(\tilde{\Theta}))$ and $H^m(A_{t_{\text{var}}}) := \text{Ker}(h_{t*} : H^m(A_t) \rightarrow H^{m+2}(\tilde{A}))$ be the variable cohomology of $\Theta_t$ in $\tilde{\Theta}$ and $A_t$ in $\tilde{A}$, respectively.
6.1. The primal cohomology and the variable cohomology. The next four propositions describe the variable middle cohomology $H^4(\Theta)_{\text{var}}$ and its relation with the primal cohomology $\mathbb{K}_{\ell}$.

**Proposition 6.1.** The variable cohomology $H^m(\Theta)_{\text{var}}$ is equal to $\sum_{i=1}^{42} \mathbb{V}^m_i$.

**Proof.** By Equation (4.1) and Proposition 4.1, when the theta divisor has one or two nodes, the local monodromy representation is trivial after we make a base change of order 2. Thus from the Clemens-Schmid sequence, there is no ‘local vanishing cocycles’ near these singular theta divisors. Therefore the space of vanishing cocycles is generated by the ‘local vanishing cocycles’ near $\Theta_s$, $i = 1, \ldots, 42$. \qed

**Proposition 6.2.** The pull-back maps $i^*_s : H^4(\tilde{\Theta}) \to H^4(\Theta)$ and $(j \circ i)_* : H^4(\tilde{A}) \to H^4(\Theta)$ have the same image. As a consequence, $H^4(\Theta)_{\text{var}} = (\text{Ker}(j \circ i)_*) : H^4(\Theta) \to H^8(\tilde{A})$.

**Proof.** Choose another general point $u \neq t$ in $X$. Write $W := X \setminus \{u\}$, and $(\tilde{A}_W, \tilde{\Theta}_W) := (p^{-1}(W), (p \circ j)^{-1}(W))$.

Consider the Gysin sequence

\[
\begin{array}{cccccccc}
H^4(\tilde{A}) & \longrightarrow & H^4(\tilde{A}_W) & \overset{\text{Res}}{\longrightarrow} & H^3(A_u) & \overset{h_u*}{\longrightarrow} & H^5(\tilde{A}) & \longrightarrow \\
\downarrow j^* & & \downarrow j^*_W & & \cong & & \downarrow j^* & \\
H^4(\tilde{\Theta}) & \longrightarrow & H^4(\tilde{\Theta}_W) & \overset{\text{Res}}{\longrightarrow} & H^3(\Theta_u) & \overset{i^*_u*}{\longrightarrow} & H^5(\tilde{\Theta}) & \longrightarrow \\
\end{array}
\]

where $\text{Res}$ denotes Griffiths’ residue map. We claim that $j^*_W : H^k(\tilde{A}_W) \to H^k(\tilde{\Theta}_W)$ is an isomorphism for $k \leq 4$ and injective for $k = 5$ (this is the Lefschetz hyperplane theorem in a slightly modified setting). To this end, apply the long exact sequence of singular cohomology of the pair $(\tilde{A}_W, \tilde{\Theta}_W)$. The relative cohomology $H^k(\tilde{A}_W, \tilde{\Theta}_W)$ is isomorphic to $H_{12-k}(\tilde{A}_W \setminus \tilde{\Theta}_W)$ [Voi03, p. 33].

Note that $\Theta$ is p-ample, and therefore $\Theta + kA_u$ is ample in $\tilde{A}$ for some $k > 0$. We conclude that the open set $\tilde{A}_W \setminus \tilde{\Theta}_W = \tilde{A} \setminus (\Theta \cup A_u)$ is affine, thus has the homotopy type of a CW-complex of real dimension 6. Therefore $H_{12-k}(\tilde{A}_W \setminus \tilde{\Theta}_W) = 0$ for $k \leq 6$, which implies the claim.

By Proposition 6.1 and Corollaries 5.4 and 5.6, $H^3(A_u)_{\text{var}} := \text{Ker}(h_u*) \cong H^3(\Theta_u)_{\text{var}}$, thus by the Gysin sequence and the fact that $j^*_W$ is an isomorphism when $k = 4$, the restriction map $H^4(\tilde{\Theta}) \to H^4(\tilde{\Theta}_W)$ has the same image as the composition $H^4(\tilde{A}) \to H^4(\tilde{A}_W) \overset{j^*_W}{\longrightarrow} H^4(\tilde{\Theta}_W)$. Taking the restriction map from $H^4(\tilde{\Theta}_W)$ to $H^4(\Theta)$, the first statement follows immediately.

The second statement follows from the fact that Gysin push-forward is the transpose of the pull-back map. \qed
Proposition 6.3. The primal cohomology $\mathbb{K}_t := \text{Ker}(j_t^* : H^4(\Theta_t) \to H^6(A_t))$ is contained in the variable cohomology $H^4(\Theta_t)_{\text{var}}$.

Proof. By Proposition 6.2, we have $H^4(\Theta_t)_{\text{var}} = (\text{Ker}(j \circ i)_* : H^4(\Theta_t) \to H^8(\tilde{A}))$, which implies $\mathbb{K}_t \subset H^4(\Theta_t)_{\text{var}}$. \hfill \Box

Proposition 6.4. The primal cohomology $\mathbb{K}_t$ is equal to $\sum_{i=1}^{42} \mathbb{H}_i$.

Proof. The morphism $j_* : H^4(\tilde{\Theta}_{V_i}) \to H^6(\tilde{A}_{V_i})$ in (5.1) is a morphism of mixed Hodge structures. The induced morphism

\[
\begin{array}{ccc}
Gr_3 H^4(\tilde{\Theta}_{V_i}) & \longrightarrow & Gr_5 H^6(\tilde{A}_{V_i}) \\
\cong & & \cong \\
H^3(\Xi) & \longrightarrow & H^5(B)
\end{array}
\]

is Gysin pushforward. By construction, $\mathbb{H}' \subset Gr_3 H^4(\tilde{\Theta}_{V_i}) \subset H^4(\tilde{\Theta}_{V_i})$ is contained in Ker$(j_*)$. Thus by sequence (5.1), $i^*_t \mathbb{H}' \subset \text{Ker}(j_{t*} : H^4(\Theta_{t_i}) \to H^6(A_{t_i}))$, or equivalently, $\mathbb{H}_i \subset \mathbb{K}_t$. It remains to show $\mathbb{K}_t \subset \sum_{i=1}^{42} \mathbb{H}_i$. To this end, pick any $\alpha \in \mathbb{K}_t$, by Proposition 6.1 and Equation (5.2), we can write $\alpha = \sum_{i=1}^{42} (x_i + y_i)$, where $x_i \in j^*_t H^4(A_t)$ and $y_i \in \mathbb{H}_i \subset \mathbb{K}_t$. From the direct sum decomposition

\[
H^4(\Theta_t) = j^*_t H^4(A_t) \oplus \mathbb{K}_t,
\]

we conclude $\sum_{i=1}^{42} x_i = 0$ and $\alpha \in \sum_{i=1}^{42} \mathbb{H}_i$. \hfill \Box

6.2. The proof of the main theorem. From now on we will abuse notation by considering $N_i$ in (5.1) as an endomorphism on $H^4(\Theta_t)$ via $\psi^*_i$ and then restricting it to $\mathbb{K}_t$. With the new notation, $N_i : \mathbb{K}_t \to \mathbb{K}_t$ satisfies

\[
(6.1) \quad N_i^2 = 0,
\]

\[
(6.2) \quad N_i(\mathbb{K}_t) = \mathbb{H}_i.
\]

Since the monodromy operator preserves the intersection product $\langle , \rangle$ on $\mathbb{K}_t$, $N_i$ also satisfies the equality

\[
(6.3) \quad \langle N_i(x), y \rangle + \langle x, N_i(y) \rangle = 0
\]

for any $x, y \in \mathbb{K}_t$.

Each $N_i$ induces a ‘limit mixed Hodge structure’ $\mathbb{K}_t^{i_{\text{lim}}}$ on $\mathbb{K}_t$ as in Section 3.3.
Lemma 6.5. We have $\cap_{i=1}^{42} \ker(N_i) = 0$.

Proof. Equation (6.3) implies that $\langle N_i(x), y \rangle = 0$ for any $x \in \mathbb{K}_t$ and $y \in \ker(N_i)$. Thus $\ker(N_i) \perp H_i$. Any element in $\cap_{i=1}^{42} \ker(N_i)$ is therefore perpendicular to all $H_i$, $i = 1, \ldots, 42$. The statement now follows immediately from Proposition 6.4 and the fact that the intersection product is nondegenerate. □

Lemma 6.6. With the notation of Section 5.4, all $H_i, i = 1, \ldots, 42$ are conjugate under the monodromy representation

$$\rho : \pi_1(V, t) \rightarrow \text{Aut}(\mathbb{K}_t, \langle , \rangle).$$

Proof. For any $i \neq j$, choose a path $\delta'$ in $l$ connecting $t_i$ and $t_j$. By perturbing $\delta'$, we can assume $\delta'$ does not intersect the inverse image of $N_0$. We can lift $\delta'$ to a path $\delta \subset X \cap V$ as a smooth section over $\delta'$ in the tubular neighborhood of the smooth locus $D^0$ of $D$ in $V$. A $C^\infty$-trivialization of the total space of the theta divisors over $\delta$ induces a map on cohomology, which sends $H^i_{\theta} \subset H^4(\Theta_i)$ to $H^j_{\theta} \subset H^4(\Theta_j)$. This precisely means that under the monodromy action, $\rho(\gamma_i \cdot \delta \cdot \gamma_j^{-1})$ sends $H_i$ to $H_j$. □

Proof. of Theorem 0.1. It suffices to show that for very general $t \in X \cap V$, $\mathbb{K}_t$ is an irreducible Hodge structure. Suppose $0 \subsetneq F_t \subset \mathbb{K}_t$ is a rational Hodge substructure, then $F_t$ is an invariant subspace under the action of the Mumford-Tate group $MT(\mathbb{K}_t)$. For very general $t$, $MT(\mathbb{K}_t)$ contains the identity component $I_V$ of the algebraic monodromy group $G_V$, i.e., the Zariski closure in $GL(\mathbb{K}_t)$ of the monodromy group $\rho(\pi_1(V))$, (c.f. [Sch11, Prop. 6]), thus by further passing to a finite étale cover $V'$ of $V$, we can assume $F_t$ is invariant under $\rho(\pi_1(V'))$. Therefore, we obtain a local subsystem $F_{V'} \subset \mathbb{K}_{V'}$ over $V'$.

Note that

$$I_{V'} = I_V,$$

since $I_{V'} \subset I_V$ is of finite index and $I_V$ is connected. Moreover, $T_i = \exp(N_i) \in I_V = I_{V'}$. (Because $T_i$ is in the image of the exponential map $\exp : gl(\mathbb{K}_t) \rightarrow GL(\mathbb{K}_t)$.) We conclude that $F_t$ is invariant under $T_i$ and therefore $N_i$. Each $N_i$ then induces a ‘limit mixed Hodge structure’ $F_{t \lim}$ on $F_t$.

By Lemma 6.5, for any $0 \neq x \in F_t$, $x \notin \ker(N_i)$ for some $i$, thus $0 \neq N_i(x) \in F_t \cap H_i = F_t \cap W_3^i \mathbb{K}_t = W_3^i F_t$. Since $H_i = W_3^i \mathbb{K}_t$ is an irreducible pure Hodge structure (follows from the main result of [IS95]), we conclude $H_i \subset F_t$. By Lemma 6.6, the $H_i$ are conjugate under the monodromy group $\pi_1(V)$, thus $H_i \subset F_t$ for all $i$ and, by Proposition 6.4, $F_t = \mathbb{K}_t$. □
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