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Electron Conductivity in High Magnetic Fields by the Test-Particle Technique

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The test-particle formalism of Rostoker is applied to electrons which move only in one dimension through randomly distributed, singly charged ions. This model is appropriate to the limit of infinite magnetic field. The test particle equations are solved to find the conductivity $\sigma(\vec{k}, \omega)$ for arbitrary wave number $\vec{k}$, neglecting electron-electron correlations. In the long-wavelength limit $\vec{k}=0$ our result agrees with previous work. We then find explicit expressions for $\sigma(\omega)$, using degenerate statistics for the electrons. Contributions emerge both from excitation of undamped normal modes and from nonresonant individual particle scattering. For finite magnetic fields the cyclotron motion of the electrons is important, and we explicitly analyze the corrections to the one-dimensional model when ω is comparable to ω_c. Further corrections for higher frequencies or lower magnetic fields may be obtained by a straightforward procedure.

I. INTRODUCTION

There are many treatments[1–9] of high-frequency conductivity for electrons which are scattered by stationary, randomly distributed ions. A completely classical analysis has been given by Oberman, Ron, and Dawson,[1] using the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy in the high-density plasma limit. A one-dimensional model of a Lorentz plasma was studied by Shanny, Dawson, and Greene,[2] and a computer "experiment" was performed on it. The effect of the presence of a constant magnetic field on the classical treatment of Ref. 1 was considered by Oberman and Shure.[3]

The quantum-mechanical problem was solved by Oberman and Ron[4] and by Wolman and Ron[5] for an electron gas in a magnetic field. Their approach was a kinetic one, much like the previous references. Rostoker[6,7] has developed a test particle formalism for dealing with many-body problems, and it has been extended to quantum systems by Rensink.[10] In this paper, we show how the test-particle method may be used to calculate the high-frequency conductivity of a one-dimensional model for the electrons in the limit of infinite magnetic field. Our calculations are based on the classical equations of motion given in Ref. 7. However, the results are also valid for quantum systems,[4] and we explicitly evaluate the conductivity for a zero-temperature electron gas containing static ions. In the long-wavelength high-frequency limit, we find a resonant contribution due to undamped electron plasma modes in the unperturbed system, as well as a nonresonant or individual particle contribution. The corrections to this result for finite magnetic fields are also analyzed.

II. THE ONE-DIMENSIONAL MODEL

In our model the electrons are scattered by a random distribution of stationary, singly charged ions. The problem is made one-dimensional by assuming that the electrons move only in straight lines parallel to some arbitrarily chosen direction. This would correspond to the situation in which a strong uniform magnetic field is applied to the system, thus constraining the electrons to move only along field lines. In this case, the "scattering" is simply a modification of the electron motion along the field lines due to the randomly distributed ions.

The basic idea of the test-particle treatment is that a system of interacting charged particles behaves approximately like a collection of statistically independent "dressed" test particles. A dressed test particle is a composite charge made up of a bare particle and the polarization charge that it induces in the system. The essential piece of information in this analysis is the response of the system to a single test particle; i.e., the probability for finding a particle with velocity $\vec{v}$ and position $\vec{x}$, given that there is a test particle with velocity $\vec{v}'$ at position $\vec{x}'$. With this information one can calculate the collisional contribution to the electrical conductivity.

For applied electric fields of the form $\exp[i(\vec{k}\cdot\vec{x} - \omega t)]$ the induced current density is related to the electric field in the system by

$$\vec{J}(\vec{k}, \omega) = \overleftrightarrow{\sigma}(\vec{k}, \omega)\cdot\vec{E}(\vec{k}, \omega)\ , \tag{1}$$

where $\overleftrightarrow{\sigma}(\vec{k}, \omega)$ is the conductivity tensor for wave vector $\vec{k}$ and frequency ω. In this paper, we consider only longitudinal fields and currents so we have a scalar relation defining the longitudinal conductivity,

$$J_L(\vec{k}, \omega) = \sigma_L(\vec{k}, \omega) E_L(\vec{k}, \omega)\ . \tag{2}$$

To find σ_L, we calculate the electron current density to first order in the electric field. The longitudinal current density is given by

$$J_L(\vec{k}, \omega) = -ne\int d\vec{v}(\vec{k}\cdot\vec{v}/|\vec{k}|)f(\vec{k}, \vec{v}, \omega), \tag{3}$$

where $f(\vec{k}, \vec{v}, \omega)$ is the Fourier space transform and Laplace time transform of the one-particle distribution function

$$f(\vec{x}, \vec{v}, t) = \left\langle \sum_{i=1}^{N} \delta[\vec{x} - \vec{x}_i(t)]\,\delta[\vec{v} - \vec{v}_i(t)] \right\rangle . \tag{4}$$

For our model system, Rostoker's equation[7] governing f may be written

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z} - \frac{e}{m}E(\vec{x}, t)\frac{\partial}{\partial v}\right) f(\vec{x}, \vec{v}, t) = -\frac{n_i e^2}{m} \times\int \left(\frac{\partial}{\partial z}\frac{1}{|\vec{x} - \vec{x}'|}\right)\frac{\partial}{\partial v} G_{eI}(\vec{x}, \vec{v}; \vec{x}', \vec{v}'; t)d\vec{x}'d\vec{v}'. \tag{5}$$

The electron number density, charge, and mass are given by n, $-e$, and m, respectively. The ion number density is n_i. The electrons are free to move only in the z direction and we write $v_z = v$. The high-frequency electric field E is the sum of the applied field and the induced field in the z direction. The electron-ion correlation function G_{eI} is related to a conditional probability P by

$$G_{eI}(\vec{x}, \vec{v}; \vec{x}', \vec{v}'; t) = \delta(\vec{v}')P(\vec{x}', \vec{v}' | \vec{x}, \vec{v}, t) \tag{6}$$

for the case of fixed ($\vec{v}' = 0$) scattering centers (ions). We have neglected electron-electron correlations. The equations for P are

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z} - \frac{e}{m}E(\vec{x}, t)\frac{\partial}{\partial v}\right) P(\vec{x}', 0 | \vec{x}, \vec{v}, t) = -\frac{e}{m}\frac{\partial}{\partial v} f(\vec{x}, \vec{v}, t)\frac{\partial}{\partial z}\phi(\vec{x}\ |\vec{x}', t) , \tag{7}$$

$$\left(\frac{\partial}{\partial \vec{x}}\right)^2 \phi(\vec{x}|\vec{x}', t) = -4\pi e\delta(\vec{x} - \vec{x}') + 4\pi ne\int P(\vec{x}', 0 | \vec{x}, \vec{v}, t)d\vec{v} . \tag{8}$$

The potential ϕ is that of an individual ion plus its "polarization cloud" due to Coulomb interaction with the electrons.

III. LINEAR RESPONSE

We expand P, f, and ϕ in powers of the electric field $E(\vec{x}, t)$. For a homogeneous system the unperturbed electron distribution f_0 is a function of velocity only. The zeroth-order terms of Eqs. (7) and (8) are

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z}\right) P_0(\vec{x}' | X, t) = -\frac{e}{m}\frac{\partial f_0(\vec{v})}{\partial v}\frac{\partial}{\partial z}\phi_0(\vec{x}|\vec{x}', t) , \tag{9}$$

$$\left(\frac{\partial}{\partial \vec{x}}\right)^2 \phi_0(\vec{x}|\vec{x}', t) = -4\pi e\delta(\vec{x} - \vec{x}') + 4\pi ne\int P_0(\vec{x}' | X, t)d\vec{v} , \tag{10}$$

where $X = (\vec{x}, \vec{v})$. To first order in $E(\vec{x}, t)$,

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z}\right)f_1(X, t) - \frac{e}{m}E(\vec{x}, t)\frac{\partial f_0(\vec{v})}{\partial v} = -\frac{n_i e^2}{m}\int\left(\frac{\partial}{\partial z}\frac{1}{|\vec{x} - \vec{x}'|}\right)\frac{\partial}{\partial v}P_1(\vec{x}' | X, t)d\vec{x}' , \tag{11}$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z}\right) P_1(\vec{x}' | X, t) - \frac{e}{m}E(\vec{x}, t)\frac{\partial}{\partial v}P_0(\vec{x}' | X, t) = -\frac{e}{m}\left(\frac{\partial f_0(\vec{v})}{\partial v}\frac{\partial}{\partial z}\phi_1(\vec{x}|\vec{x}', t) + \frac{\partial}{\partial v}f_1(X, t)\frac{\partial}{\partial z}\phi_0(\vec{x}|\vec{x}', t)\right) , \tag{12}$$

$$\left(\frac{\partial}{\partial \vec{x}}\right)^2 \phi_1(\vec{x}|\vec{x}', t) = 4\pi ne\int P_1(\vec{x}' | X, t)d\vec{v} . \tag{13}$$

At this point, we Fourier decompose in $\vec{k}$, $\vec{q}$, and ω:

$$E \propto E(\vec{k}, \omega)e^{i\vec{k}\cdot\vec{x} - i\omega t}; \quad P_0, \phi_0 \propto P_0(\vec{q}, \vec{v}), \phi_0(\vec{q}, \vec{v})e^{i\vec{q}\cdot(\vec{x} - \vec{x}')};$$

$$f_1 \propto f_1(\vec{k}, \vec{v}, \omega)e^{i\vec{k}\cdot\vec{x} - i\omega t}; \quad P_1, \phi_1 \propto P_1(\vec{k}, \vec{q}, \vec{v}, \omega), \phi_1(\vec{k}, \vec{q}, \omega)e^{i\vec{k}\cdot\vec{x} + i\vec{q}\cdot(\vec{x} - \vec{x}') - i\omega t} .$$

The zeroth-order equations immediately yield the shielded ion potential and the static conditional probability:

$$\phi_0(\vec{q}) = 4\pi e/q^2 D(q, 0) , \tag{14}$$

$$P_0(\vec{q}, \vec{v}) = \frac{e}{m}q_z\frac{\partial f_0(\vec{v})}{\partial v}(i\nu - q_z v)^{-1}\phi_0(\vec{q}) . \tag{15}$$

The dielectric function is

$$D(\vec{q}, \omega) = 1 + \omega_p{}^2/q^2\int q_z\frac{\partial f_0(\vec{v})}{\partial v}(\omega + i\nu - q_z v)^{-1}d\vec{v} , \tag{16}$$

where ν is an infinitesimal positive quantity which insures that the perturbation goes to zero as $t \to -\infty$.

The equations to first order in E become

$$-i(\omega-k_z v)f_1(\vec{k},\vec{v},\omega)-\frac{e}{m}E(\vec{k},\omega)\frac{\partial f_0(\vec{v})}{\partial v}=\left(\frac{n_i e^2}{m}\right)\int\frac{d\vec{q}}{(2\pi)^3}\frac{iq_z}{q^2}\frac{\partial}{\partial v}P_1(\vec{k},\vec{q},\vec{v},\omega)\,, \tag{17}$$

$$-i[\omega-(k_z+q_z)v]P_1(\vec{k},\vec{q},\vec{v},\omega)-\frac{e}{m}E(\vec{k},\omega)\frac{\partial}{\partial v}P_0(\vec{q},\vec{v})$$
$$=-i\frac{e}{m}\left(\frac{\partial f_0(\vec{v})}{\partial v}(k_z+q_z)\phi_1(\vec{k},\vec{q},\omega)+\frac{\partial}{\partial v}f_1(\vec{k},\vec{v},\omega)q_z\phi_0(\vec{q})\right)\,, \tag{18}$$

$$-|\vec{k}+\vec{q}|^2\phi_1(\vec{k},\vec{q},\omega)=4\pi ne\int P_1(\vec{k},\vec{q},\vec{v},\omega)d\vec{v}\,. \tag{19}$$

Using Eqs. (18) and (19),

$$\phi_1(\vec{k},\vec{q},\omega)=\frac{-i\omega_p^{\,2}}{(\vec{k}+\vec{q})^2D(\vec{k}+\vec{q},\omega)}\int\frac{1}{\omega-(k_z+q_z)v}\frac{\partial}{\partial v}[E(\vec{k},\omega)P_0(\vec{q},v)-iq_z\phi_0(\vec{q})f_1(\vec{k},\vec{v},\omega)]d\vec{v}\quad. \tag{20}$$

Equations (17), (18), and (20) yield an integral equation for f_1 of some complexity. For simplicity, we shall consider an iterative solution. If we neglect the collisions the right-hand side of Eq. (17) vanishes and we immediately find

$$f_1^{(0)}(\vec{k},\vec{v},\omega)=\frac{e}{m}\frac{\partial f_0(\vec{v})}{\partial v}(\omega-k_z v)^{-1}iE(\vec{k},\omega)\,. \tag{21}$$

Using Eqs. (15) and (21) in Eq. (20), we find

$$\phi_1^{(0)}(\vec{k},\vec{q},\omega)=\frac{-i\omega_p^{\,2}(e/m)q_z\phi_0(\vec{q})E(\vec{k},\omega)}{|\vec{k}+\vec{q}|^2D(\vec{k}+\vec{q},\omega)}\int d\vec{v}\frac{1}{\omega-(k_z+q_z)v}\frac{\partial}{\partial v}\frac{\omega-(k_z+q_z)v}{(\omega-k_z v)(i\nu-q_z v)}\frac{\partial f_0}{\partial v}\quad. \tag{22}$$

Integrating by parts, we may write $\phi_1^{(0)}$ in the form

$$\phi_1^{(0)}(\vec{k},\vec{q},\omega)=\frac{-i\omega_p^{\,2}(k_z+q_z)E(\vec{k},\omega)}{nq^2|\vec{k}+\vec{q}|^2q_z\omega^2D(\vec{q},0)D(\vec{k}+\vec{q},\omega)}[H(\vec{k}+\vec{q},\omega)-H(\vec{k},\omega)-H(\vec{q},0)]\,, \tag{23}$$

where $H(\vec{k},\omega)=k_z k^2[D(\vec{k},\omega)-1]\,.$ (24)

This relation for $\phi_1^{(0)}$ enables us to write Eq. (18) for $P_1^{(0)}$ as

$$P_1^{(0)}(\vec{k},\vec{q},\vec{v},\omega)=\{-ie\omega_p^{\,2}E(\vec{k},\omega)/mn[\omega-(k_z+q_z)v]q^2D(\vec{q},0)\}$$
$$\times\left[\frac{\partial}{\partial v}\left(\frac{(\omega-k_z v-q_z v)}{(\omega-k_z v)v}\frac{\partial f_0}{\partial v}\right)+\frac{(k_z+q_z)^2(\partial f_0/\partial v)[H(\vec{k}+\vec{q},\omega)-H(\vec{k},\omega)-H(\vec{q},0)]}{q_z\omega^2|\vec{k}+\vec{q}|^2D(\vec{k}+\vec{q},\omega)}\right]\quad. \tag{25}$$

This is the conditional probability for the first step of our iteration. The next step is to substitute $P_1^{(0)}$ into the right-hand side of Eq. (17). This gives the iterated solution $f_1^{(1)}$ the perturbed distribution function which is linear in both the electric field and the scattering potential:

$$f_1^{(1)}(\vec{k},\vec{v},\omega)=\frac{-\omega_p^{\,2}(n_i/n)}{4\pi(\omega-k_z v)}\int\frac{d\vec{q}}{(2\pi)^3}\frac{q_z}{q^2}\frac{\partial}{\partial v}P_1^{(0)}(\vec{k},\vec{q},\vec{v},\omega)\,. \tag{26}$$

The expression for the longitudinal current is

$$J^{(1)}(\vec{k},\omega)=-ne\int d\vec{v}(k_z v/k)f_1(\vec{k},\vec{v},\omega)$$
$$=\frac{n_i e\omega_p^{\,2}\omega}{4\pi k}\int\frac{d\vec{q}}{(2\pi)^3}\frac{q_z}{q^2}\int d\vec{v}\frac{1}{\omega-k_z v}\frac{\partial}{\partial v}P_1^{(0)}(\vec{k},\vec{q},\vec{v},\omega)=\sigma^{(1)}(\vec{k},\omega)|\vec{E}(\vec{k},\omega)|\,. \tag{27}$$

Integrating by parts over v, and using

$$E(\vec{k},\omega)=(k_z/k)|\vec{E}(\vec{k},\omega)| \quad , \tag{28}$$

we obtain for the conductivity

$$\sigma^{(1)}(\vec{k},\omega)=\frac{n_i i\omega_p{}^6 k_z{}^2\omega}{n^2 16\pi^2 k^2}\int\frac{d\vec{q}}{(2\pi)^3}\frac{1}{q^4 D(\vec{q},0)}\int d\vec{v}\,\frac{\partial f_0}{\partial v}$$
$$\times\left[\frac{(k_z+q_z)^2[H(\vec{k}+\vec{q},\omega)-H(\vec{k},\omega)-H(\vec{q},0)]}{\omega^2(\omega-k_zv)^2(\omega-k_zv-q_zv)|\vec{k}+\vec{q}|^2 D(\vec{k}+\vec{q},\omega)}-\frac{q_z}{v(\omega-k_zv)^3}\left(\frac{2k_z}{\omega-k_zv}+\frac{k_z+q_z}{\omega-k_zv-q_zv}\right)\right] . \tag{29}$$

This is our result for the one-dimensional conductivity, ignoring the electron-electron correlation.

IV. LONG-WAVELENGTH LIMIT

Several authors have given expressions for $\sigma(\omega)$ in the long-wavelength limit, $k\to 0$, when the electrons were allowed to move in three dimensions. For our one-dimensional model

$$\lim_{k\to 0}\sigma^{(1)}(\vec{k},\omega)=\frac{i\omega_p{}^2}{4\pi\omega}\left[1-\frac{e^2}{m\omega^2}\left(\frac{n_i}{n}\right)\int\frac{d\vec{q}}{(2\pi)^3}\frac{q_z{}^2}{q^2}\left(\frac{1}{D(\vec{q},0)}-\frac{1}{D(\vec{q},\omega)}\right)\right] . \tag{30}$$

Since $D(\vec{q},0)$ is real, the real part of $\sigma^{(1)}(\omega)$, which is the dissipative portion, is given by

$$\mathrm{Re}\sigma^{(1)}(\omega)=-(e^2\omega_p{}^2 n_i/4\pi n m\omega^3)(2\pi)^{-3}\int d\vec{q}(q_z{}^2/q^2)\,\mathrm{Im}[D(q,\omega)]^{-1} . \tag{31}$$

Equations (30) and (31) agree with the results given in Refs. 3–5 for uncorrelated ions.

V. THE DEGENERATE ELECTRON GAS

A particular application for Eq. (31) is the problem of the degenerate Fermi gas in a high magnetic field. In the limit $\omega_c \gg \omega_p$, the electrons move much more readily along the field lines than perpendicular to them. This suggests a one-dimensional problem. The electrons will encounter the charged lattice sites and, in order to be scattered, must change their momenta along $\vec{B}$. For this problem we must use the quantum dielectric function which is given by[4,11]

$$D(\vec{q},\omega)=1+(4\pi e^2/q^2)$$
$$\times\sum_{n=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\int_{-\infty}^{+\infty}dv_z\int_0^{\infty}v_\perp dv_\perp\int_0^{2\pi}d\theta$$
$$\times\frac{e^{i(n-l)\theta}J_n(q_\perp v_\perp/\omega_c)J_l(q_\perp v_\perp/\omega_c)}{\omega-q_zv_z-n\omega_c}$$
$$\times[f(\vec{v}+\hbar\vec{q}/2m)-f(\vec{v}-\hbar\vec{q}/2m)] . \tag{32}$$

The normalization on the electron velocity distribution is

$$n=\int d\vec{v}f(\vec{v}) , \tag{33}$$

and J_n is the Bessel function of order n. The subscripts $\perp$ and z denote components perpendicular and parallel to the direction of the magnetic field.

We suppose now that our system is at zero temperature and that the magnetic field is strong enough to ensure that only the lowest Landau level is occupied; i.e.,

$$B>B_0,\quad B_0=(\hbar c/2e)(4\pi^2 n)^{2/3} . \tag{34}$$

The electron velocity distribution for this situation is[11]

$$f_0(\vec{v})=\frac{nm}{2\pi\hbar v_F\omega_c}e^{-mv_\perp{}^2/\hbar\omega_c}H(v_F{}^2-v_z{}^2) , \tag{35}$$

where $H(x)$ is the Heaviside step function,

$$H(x)=1,\ x>0;\quad H(x)=0,\ x<0 . \tag{36}$$

The Fermi velocity v_F along the magnetic field lines is related to the field-dependent Fermi energy $E_F(B)$ by[11,12]

$$\tfrac{1}{2}mv_F{}^2=E_F-\tfrac{1}{2}\hbar\omega_c ,$$
$$E_F(B)=\tfrac{1}{2}\hbar\omega_c[1+2(B_0/B)^3] . \tag{37}$$

Note that v_F is inversely proportional to B in this high-field limit. When $f_0(\vec{v})$ is substituted in Eq. (32) one obtains the dielectric function[11]

$$D(\vec{q},\omega)=1+\frac{m\omega_p{}^2 e^{-\frac{1}{2}q_\perp{}^2a^2}}{2\hbar q^2 q_z v_F}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}q_\perp{}^2a^2\right)^n}{n!}$$
$$\times\left(\ln\left|\frac{\omega^2-(n\omega_c+q_zv_F+\hbar q_z{}^2/2m)^2}{\omega^2-(n\omega_c-q_zv_F+\hbar q_z{}^2/2m)^2}\right|\right.$$

$$+\pi i H[\omega^2-(n\omega_c-q_z v_F+\hbar q_z{}^2/2m)^2]$$
$$-\pi i H[\omega^2-(n\omega_c+q_z v_F+\hbar q_z{}^2/2m)^2]\Big), \quad (38)$$

where $a^2=\hbar/m\omega_c$.

A. $\omega\ll\omega_c$

For frequencies ω which are small compared with the cyclotron frequency ω_c, we need only the $n=0$ term in the dielectric function since the remaining terms bring in higher powers of $(1/\omega_c)$. Separating the real and imaginary parts of Eq. (38) we have[13]

$$D(\vec{q},\omega)=D_1(\vec{q},\omega)+iD_2(\vec{q},\omega)\ , \quad (39)$$

$$D_1(\vec{q},\omega)=1+(m\omega_p{}^2/2\hbar q^2 q_z v_F)\times\ln\left|\frac{\omega^2-(q_z v_F+\hbar q_z{}^2/2m)^2}{\omega^2-(q_z v_F-\hbar q_z{}^2/2m)^2}\right|, \quad (40)$$

$$D_2(\vec{q},\omega)=\frac{\pi m\omega_p{}^2}{2\hbar q^2 q_z v_F}, \quad q_1(\omega)<q_z<q_2(\omega), \quad (41)$$

where q_1 and q_2 are given by

$$\omega=q_1 v_F+\hbar q_1{}^2/2m, \quad \omega=-q_2 v_F+\hbar q_2{}^2/2m. \quad (42)$$

The real part of the conductivity from Eq. (31) is

$$\mathrm{Re}\,\sigma(\omega)=\left(\frac{n_i}{n}\right)(e^2\omega_p{}^2/8\pi^3 m\omega^3)\times\int_0^\infty q_\perp dq_\perp\int_0^\infty dq_z[q_z{}^2/(q_\perp{}^2+q_z{}^2)]\times\frac{D_2(q_z,q_\perp{}^2,\omega)}{[D_1(q_z,q_\perp{}^2,\omega)]^2+[D_2(q_z,q_\perp{}^2,\omega)]^2}. \quad (43)$$

Since D_2 is nonzero only for $q_1<q_z<q_2$, one might conclude that this gives the only contribution to the conductivity. However, if both the real and imaginary parts of $D(\vec{q},\omega)$ vanish, there is an additional "resonant" contribution. Near these resonance points, which define undamped normal modes of the unperturbed system, we can write $D(\vec{q},\omega)$ in the form

$$D(\vec{q},\omega)\approx(\omega-\omega_{\vec{q}}+i\nu)\left.\frac{\partial D}{\partial\omega}\right|_{\omega=\omega_{\vec{q}}}, \quad \nu\to 0\ , \quad (44)$$

or, in terms of q_z,

$$D(q_z,q_\perp{}^2,\omega)\approx[q_z-q_0(q_\perp{}^2,\omega)-i\nu]\left.\frac{\partial D}{\partial q_z}\right|_{q_z=q_0}, \quad (45)$$

where $q_0(q_\perp{}^2,\omega)$ is given by

$$D(q_0,q_\perp{}^2,\omega)=0\ . \quad (46)$$

Then we make use of the Plemjl formula and obtain

$$\mathrm{Im}\frac{1}{D(\vec{q},\omega)}\approx\left(\left.\frac{\partial D}{\partial q_z}\right|_{q_z=q_0}\right)^{-1}\mathrm{Im}\frac{1}{q_z-q_0-i\nu}=\pi\delta(q_z-q_0)\left(\left.\frac{\partial D}{\partial q_z}\right|_{q_z=q_0}\right)^{-1}. \quad (47)$$

Thus we can identify two distinct contributions to the conductivity. First, there is the resonant part due to the excitation of undamped normal modes of frequency ω; and second, there is a nonresonant "individual particle" contribution. This is shown qualitatively in Fig. 1 where we plot $\mathrm{Im}[-1/D(\vec{q},\omega)]$ versus q_z. To demonstrate further that the two contributions are really distinct we have shown in Fig. 2 the region of (q_z,ω) space where D_2 is nonzero (the shaded region) and we also plot the plasmon dispersion curve $D_1=0$. Note that the plasmon branch of $D(q,\omega)$ does not enter the damping region even if $\omega\gg\omega_p$. Thus for every ω there is an undamped normal mode which can resonantly absorb energy from an applied field. This is a peculiar feature of the electron gas in a strong magnetic field[14] and is exact within the self-consistent field approximation. The resonant and nonresonant parts of the conductivity are evaluated in Appendix A. The result is

$$\mathrm{Re}\,\sigma(\omega)_{\mathrm{res.}}=\frac{n_i}{n}\frac{e^2\omega_p{}^2}{48\pi^2 m v_F{}^3}, \quad \omega<\omega_p$$
$$=\frac{n_i}{n}\frac{e^2\omega_p{}^2}{48\pi^2 m v_F{}^3}\times[1-(1-\omega_p{}^2/\omega^2)^{3/2}], \quad \omega>\omega_p\ , \quad (48)$$

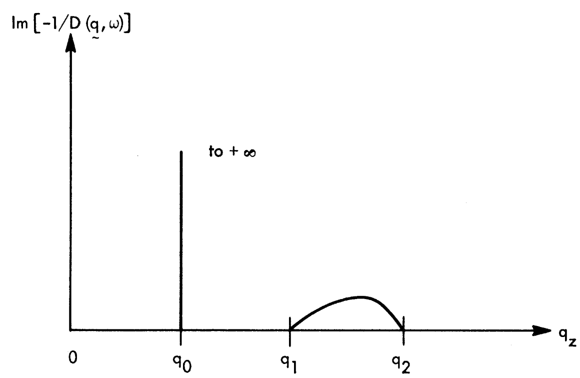

FIG. 1. Resonant and nonresonant contributions to the imaginary part of $[-1/D(\vec{q},\omega)]$ versus wave number.

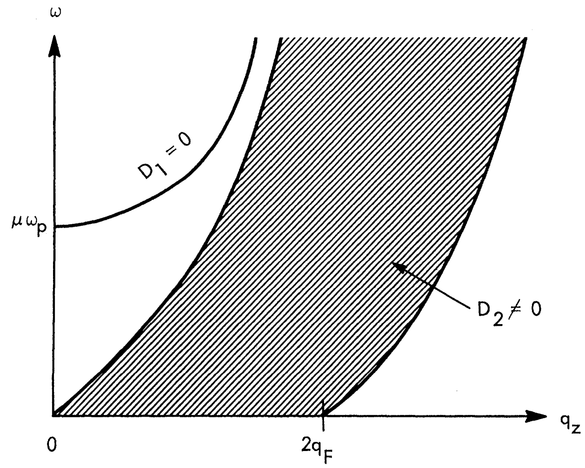

FIG. 2. Dispersion curve and damping region for the dielectric function when $\omega_p \ll \omega_c$ with $\mu = q_z/q$ fixed and $\hbar q_F = mv_F$.

$$\mathrm{Re}\,\sigma(\omega)_{\text{nonres.}} = \frac{n_i}{n} \frac{e^2 m \omega_p{}^2 v_F}{8\pi^2\hbar^2\omega^2} . \tag{49}$$

For $\omega \gg \omega_p$ the ratio of the resonant to the nonresonant contribution is $(\hbar\omega_p/2mv_F{}^2)^2$, and since $v_F \to 0$ as $B \to \infty$ we conclude that the resonant contribution will be dominant in high magnetic fields.

The role of the ions in the conductivity problem is twofold. First, they act as scattering centers for the density waves induced by an external field, thus supplying the momentum necessary to convert a $(0, \omega)$ density wave into a $(\vec{q}, \omega)$ normal mode density wave. Second, they scatter individual electrons via a frequency-dependent shielded Coulomb potential $[D^{-1}(\vec{q}, 0) - D^{-1}(\vec{q}, \omega)]V(\vec{q})$.

An interesting feature of this quantum-mechanical calculation is the fact that it has not been necessary to introduce any arbitrary cutoffs on the integral over $\vec{q}$ in the expression for $\sigma(\omega)$. This is in contrast to classical calculations[8] and has previously been noted in other quantum treatments[9] of the conductivity problem.

B. $\omega \gtrsim \omega_c$

The expression for the conductivity, Eq. (31), has been derived by using the reactive approximation, Eq. (21), for f_1. As shown in Ref. 3 this approximation fails when $\omega \approx \omega_c$ if $E(\omega_c)$ has components perpendicular to B. A perpendicular component, $E_\perp(\omega_c)$, will give rise to a static electric field in a reference frame rotating with the gyrating particles; resistive effects must then be included in order to limit the exchange of energy between $E_\perp(\omega_c)$ and the particles. In this section we analyze the conductivity only for electric fields parallel to B, so Eq. (31) remains valid in the vicinity of $\omega \approx \omega_c$.

As ω increases from $\omega \ll \omega_c$ to $\omega \approx \omega_c$ then $n = 1$ term of the dielectric function becomes important and additional resonant and nonresonant contributions to the conductivity are present. The reason for this may be seen by examining the dielectric function in the vicinity of $\omega = \omega_c$. In this region we keep only the dominant $n = 1$ term in Eq. (38) and find

$$D_1(\vec{q}, \omega) \approx 1 + \frac{m\omega_p{}^2 q_\perp{}^2 a^2 \exp(-\frac{1}{2}q_\perp{}^2 a^2)}{4\hbar q^2 q_z v_F} \times \ln\left| \frac{\omega^2 - (\omega_c + q_z v_F + \hbar q_z{}^2/2m)^2}{\omega^2 - (\omega_c - q_z v_F + \hbar q_z{}^2/2m)^2} \right| , \tag{50}$$

$$D_2(\vec{q}, \omega) \approx \frac{\pi m\omega_p{}^2 q_\perp{}^2 a^2 \exp(-\frac{1}{2}q_\perp{}^2 a^2)}{4\hbar q^2 q_z v_F} , \quad \omega_1(q_z) < \omega < \omega_2(q_z) , \tag{51}$$

where $\omega_1(q_z) = \omega_c - q_z v_F + \hbar q_z{}^2/2m$,

$$\omega_2(q_z) = \omega_c + q_z v_F + \hbar q_z{}^2/2m . \tag{52}$$

In Fig. 3 we plot $D_1(\vec{q}, \omega) = 0$ to illustrate the new resonant mode at $\omega \approx (\omega_c{}^2 + \omega_p{}^2)^{1/2}$ and we also show the region where $D_2(\vec{q}, \omega) \neq 0$. For fixed $\omega > \omega_c$ we see that there are again two distinct parts to the integral over $\vec{q}$. The resonant and nonresonant contributions to $\mathrm{Im}\{1/D(\vec{q}, \omega)\}$ and hence to the conductivity can, in principle, be calculated exactly as before. Analytical difficulties prevent us from giving explicit expressions for this part of the conductivity at the present time. The resonant contribution is due to the excitation of the first off-axis cyclotron mode, and the nonresonant individual particle contribution arises from single electrons making transitions to the next higher Landau level.

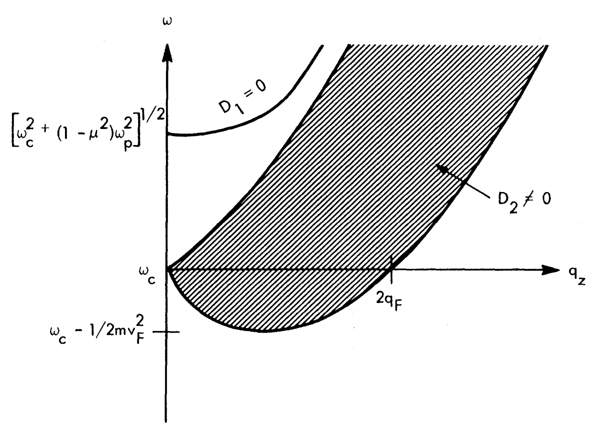

FIG. 3. Dispersion curve and damping region for the dielectric function near $\omega = \omega_c$ with $\mu = q_z/q$ fixed and $\hbar q_F = mv_F$.

As one goes to higher frequencies ω or weaker magnetic fields so that $\omega \gtrsim n\omega_c$ where n is an integer, then a similar picture of the resonant and nonresonant parts of the conductivity will be valid. In this paper we have calculated the conductivities arising from the $n=0$ and $n=1$ terms in the dielectric function. For arbitrary n, the behavior of $D(q, \omega)$ in the vicinity of $\omega = n\omega_c$ will be qualitatively similar to that shown in Fig. 3. If contributions from higher n are desired, they may be found by straightforward (though perhaps tedious) calculation using the dominant terms in the dielectric function.

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APPENDIX A.

To find the resonant contribution to $\sigma(\omega)$ we use the semiclassical expression for $D(\vec{q}, \omega)$ obtained by taking $\hbar \to 0$:

$$D(\vec{q}, \omega) \approx 1 - (q_z^2/q^2)[\omega_p^2/(\omega^2 - q_z^2 v_F^2)] \quad .$$

In spherical coordinates we have $q_z = \mu q$ and the zeros of D as a function of μ are given by

$$\mu_0^2(q) = \omega^2/(\omega_p^2 + q^2 v_F^2) \; .$$

For $\mu \approx \mu_0$ we have

$$D(\mu, q, \omega) \approx \left.\frac{\partial D}{\partial \mu}\right|_{\mu_0} [\mu - \mu_0 - i\nu]$$

$$= \frac{-2(\omega_p^2 + q^2 v_F^2)^{3/2}}{\omega \omega_p^2} (\mu - \mu_0 - i\nu) \; .$$

Since $|\mu| \leqslant 1$ we get a resonant contribution to $\sigma(\omega)$ only by requiring $\mu_0^2 \leqslant 1$; i.e.,

$$\omega_p^2 + q^2 v_F^2 \geqslant \omega^2 \; .$$

Then, from Eqs. (31) and (47), we have

$$\sigma(\omega)_{\text{res.}} = -(e^2 \omega_p^4 n_i / 16\pi^2 mn)$$

$$\times \int_{q_c}^{\infty} q^2 dq (\omega_p^2 + q^2 v_F^2)^{-5/2} \; ,$$

where $q_c = [(\omega^2 - \omega_p^2) v_F^{-2}]^{1/2}$, $\quad \omega \geqslant \omega_p$,

$q_c = 0$, $\quad \omega \leqslant \omega_p$.

This gives

$$\sigma(\omega)_{\text{res.}} = \frac{e^2 \omega_p^2 n_i}{48\pi^2 m v_F^3 n} \, , \qquad \omega < \omega_p$$

$$= \frac{e^2 \omega_p^2 n_i}{48\pi^2 m v_F^3 n}$$

$$\times [1 - (1 - \omega_p^2/\omega^2)^{3/2}] , \qquad \omega \geqslant \omega_p$$

$$\to \frac{e^2 \omega_p^4 n_i}{32\pi^2 m\omega^2 v_F^3 n} \, , \qquad \omega \gg \omega_p \; .$$

To get the nonresonant part of the conductivity we need to evaluate

$$I = \int_0^\infty q_\perp dq_\perp \int_{q_1}^{q_2} dq_z (q_z^2/q^2)$$

$$\times \frac{D_2(\vec{q}, \omega)}{[D_1(\vec{q}, \omega)]^2 + [D_2(\vec{q}, \omega)]^2} \; ,$$

with q_1, q_2, D_1, and D_2 given by Eqs. (39)–(42). If $\hbar\omega \gg m v_F^2$ then for q_z in the interval (q_1, q_2) we can use the approximate forms

$$D_1(\vec{q}, \omega) \approx 1 + \Lambda(q_\perp^2) \ln\left(\frac{q_z - q_1}{q_2 - q_z}\right) \; ,$$

$$D_2(\vec{q}, \omega) \approx \pi\Lambda(q_\perp^2) \; ,$$

where

$$\Lambda(q_\perp^2) = m\omega_p^2 (q_\perp^2 + q_0^2) 2\hbar q_0 v_F$$

and $q_0^2 = 2m\omega/\hbar$.

We make a change of variable from q_z to x:

$$x = \ln[(q_z - q_1)/(q_2 - q_z)] \; ,$$

and the required integral becomes

$$I = \int_0^\infty q_\perp dq_\perp [q_0^2/(q_\perp^2 + q_0^2)]$$

$$\frac{2mv_F}{\hbar} \int_{-\infty}^{+\infty} dx \frac{e^x}{(e^x+1)^2} \frac{\pi\Lambda}{(\pi\Lambda)^2 + (1+\Lambda x)^2} \; .$$

The exponential factors ensure that the main contribution to the integral comes from $x \leqslant 1$. Thus we can further make the approximation

$$\frac{\pi\Lambda}{(\pi\Lambda)^2 + (1+\Lambda x)^2} \approx \frac{\pi\Lambda}{(\pi\Lambda)^2 + 1}$$

which is valid for both large and small Λ. This enables us to write

$$I = (\pi m^2 \omega_p^2 q_0 / \hbar^2)$$

$$\times \int_0^\infty \frac{q_\perp dq_\perp}{(\pi m \omega_p^2 / 2\hbar q_0 v_F)^2 + (q_\perp^2 + q_0^2)^2}$$

$$= \frac{\pi m v_F q_0^{\,2}}{2\hbar}\left[1 - \frac{2}{\pi}\tan^{-1}\left(\frac{2\hbar q_0^{\,3} v_F}{\pi m \omega_p^{\,2}}\right)\right].$$

In the limit of high magnetic fields, the Fermi velocity goes to zero and we have

$$I \approx \pi m^2 v_F \omega/\hbar^2 ,$$

$$\sigma(\omega)_{\text{nonres.}} = \frac{e^2 m \omega_p^{\,2} v_F n_i}{8\pi^2\hbar^2\omega^2 n} .$$

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Critical Velocity of a Superflowing Liquid-Helium Film Using Third Sound*†

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An investigation of the critical velocity of the helium film is made by impressing a third-sound wave (surface ripple with van der Waals restoring force) on a superflowing film. The velocities of the wave are measured traveling upstream and downstream, and the superfluid critical velocity is deduced as $V_c = \frac{1}{2}(U_{3\,\text{down}} - U_{3\,\text{up}})$.

The critical velocity at the lowest temperature (1.25°K) as a function of the film height is found to be close to $\hbar/2md$, where d is the film thickness. This is compared with existing theories and is found compatible with the velocity necessary for the stability of a vortex-line-image pair with a separation of $2d$. Disturbing effects produced by the method used to excite the third sound make it impossible to measure the critical velocity at high temperatures. Arguments are advanced for believing that these disturbing effects do not invalidate the low-temperature results. Since the third-sound wave is formed by the longitudinal oscillation of the superfluid component, the observation of the wave in a film which is flowing at the critical velocity indicates the possibility of film velocities in excess of V_c for measured times $T \sim 10$ msec, where T is half a third-sound period. This implies that there is a relaxation time τ for the creation of vortices where $\tau > T$. From the data, an upper limit is placed on the difference in thickness between the flowing film and a static film.

I. INTRODUCTION

The critical velocity of the liquid-helium film, V_c, flowing under a gravitational potential has heretofore been measured by examining the critical transfer rate $\sigma = (\rho_s/\rho)V_c d$, where ρ_s/ρ is the