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Authors
Bierter, Willy
Morrison, Harry L.

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Willy Bierter and Harry L. Morrison

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MICROSCOPIC DERIVATION OF THE LOW-LYING EXCITATION SPECTRUM
OF AN INTERACTING BOSE SYSTEM

Willy Bierter
Lawrence Radiation Laboratory
University of California
Berkeley, California 94720

and

Harry L. Morrison
Lawrence Radiation Laboratory
University of California
Livermore, California 94550

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Summary. - The low-lying excitation spectrum of an interacting Bose system is derived directly from the N-body microscopic Hamiltonian expressed in terms of local densities and currents as quantum-mechanical coordinates.
Long ago Feynman (1) derived an important relationship between
the elementary excitation spectrum of the low-lying states of a Bose
system and its correlation function for the density fluctuations:

$$\omega(k) = k^2/2mS(k).$$

This result was obtained by Feynman as a consequence
of a special choice of the wave function and a subsequent variational
calculation. Although the range of validity of the Feynman wave function
is not quite clear (2), this result is supported by the saturated sum-
rule argument of Pines (3), and the hydrodynamic derivation of
Pitaevskii (4). Among these derivations, that of Pitaevskii is
simplest and most concise.

Recently, however, doubt has been cast on the validity of the
local velocity operator upon which the Landau theory of quantum hydro-
dynamics is based (5). Thus the consistency of the Pitaevskii theory
is likewise open to question, since its dynamical variables are those
of the Landau theory. The purpose of this note is to present an N-body
microscopic derivation of the Feynman result, which is in the spirit
of the Pitaevskii theory, but not subject to the objectionable use of
a quantum-mechanical velocity field.

As a motivational preliminary to a theory of hadron dynamics,
Dashen and Sharp (6) have demonstrated that the nonrelativistic quantum
mechanics of a system of N spinless identical particles may be
described completely and exactly by a theory in which the dynamical
variables are the local density and currents. Unlike previous theories
which are based on a density description of the system (7), no
The dynamical variable canonically conjugate to the density operator is introduced (8).

The expression for the Hamiltonian of a system of $N$ identical spinless bosons (of unit mass) interacting through a local two-body potential is, in the language of second quantization, given by

$$H = \frac{1}{2} \int d^3x \, \nabla \psi^+(x) \cdot \nabla \psi(x) ,$$

$$+ \frac{1}{2} \int \int d^3x \, d^3y \, \psi^+(x) \psi^+(y) V(|x-y|) \psi(x) \psi(y) .$$

The field operators $\psi^+(x)$ and $\psi(x)$ satisfy the usual canonical commutation relations:

$$[\psi^+(x), \psi^+(y)] = [\psi(x), \psi(y)] = 0 ,$$

$$[\psi(x), \psi^+(y)] = \delta(x - y) .$$

In the second-quantized formalism, the local density and current operators are given by

$$\rho(x) = \psi^+(x) \psi(x) ,$$

$$J(x) = \frac{1}{2i} [\psi^+(x) \nabla \psi(x) - \nabla \psi^+(x) \psi(x)] .$$

Although $\rho(x)$ and $J(x)$ are not canonically conjugate variables, together they provide a set of quantum-mechanical coordinates that is complete (9) and they satisfy the following algebra:
Using the crucial identities

\[ \nabla \rho(x) + 2i \dot{J}(x) = 2 \psi^+(x) \nabla \psi(x), \]

\[ \nabla \dot{\rho}(x) - 2i J(x) = 2 \dot{\psi}^+(x) \psi(x), \]

one may write the Hamiltonian (1) in terms of the observable quantities \( \rho(x) \) and \( \dot{J}(x) \) in the form

\[
(5) \quad H = \frac{1}{8} \int d^3x \left[ \nabla \rho(x) - 2i \dot{J}(x) \right] \frac{1}{\rho(x)} \left[ \nabla \rho(x) + 2i \dot{J}(x) \right] \\
+ \frac{1}{2} \iint d^3x \ d^3y \ \rho(x) \ V(|x-y|) \ \rho(y). \]

The variables \( \rho(x) \) and \( \dot{J}(x) \) are natural collective variables for a description of oscillatory processes in systems consisting of large numbers of interacting particles (10). In order to study the oscillation spectrum, we set \( \rho(x) = \langle \rho \rangle + \beta(x) \), where \( \langle \rho \rangle \) is the equilibrium or ground state average of \( \rho(x) \) (which is a constant for
translationally invariant systems). With this substitution in (5), we expand to second order in \( \beta(x) \). One obtains

\[
H = \frac{1}{2} \frac{\partial}{\partial \rho} \int d^3x \left[ (\nabla \phi(x))^2 + 4(\phi(x))^2 \right]
+ \frac{1}{2} \int d^3x \int d^3y \left( \rho \phi(x) \right) v(|x-y|) \left( \rho \phi(x) \right)
\]

To eliminate \( J(x) \), we use the equation of continuity,

\[
\frac{\partial \phi}{\partial t} = -\nabla \phi \cdot \rho
\]

We then expand the density fluctuation \( \beta(x) \) into Fourier components:

\[
\rho(x) = \sum_{k \neq 0} e^{ikx} \beta(k) e^{ikx}
\]

where the \( k = 0 \) term is omitted due to particle conservation. Thus the Hamiltonian becomes, to second order in \( \rho(k) \),

\[
H = \frac{1}{2} \rho^2 \int d^3x d^3y v(|x-y|)
+ \frac{(2\pi)^3}{2} \sum_{k \neq 0} \left\{ \frac{k^2}{4 \rho} + V(k) \right\} |\beta(k)|^2 + \frac{1}{k^2 \rho} |\phi(k)|^2
\]

This is immediately recognized as the Hamiltonian for a system of independent harmonic oscillators with frequencies

\[
\omega(k)^2 = k^2 \rho \left( \frac{k^2}{4 \rho} + V(k) \right)
\]
Because the single particle kinetic energy is given by $T(k) = k^2 / 2$, one can write eq. (8) as

$$\omega(k) = \sqrt{T^2(k) + 2\langle \rho \rangle V(k) T(k)} \ .$$

Thus we obtain the Bogoliubov spectrum of elementary excitations as the frequency of the oscillators. The energy of each oscillator is related to its frequency by

$$E(k) = \omega(k) \left( n_k + \frac{1}{2} \right) \quad (n_k = 0, 1, 2, \ldots) .$$

The ground state energy of the Bose liquid is

$$E_0 = \frac{1}{2} \langle \rho \rangle^2 \int \int d^3x \ d^3\chi \ V(|\chi - \chi'|) + \sum_k \frac{1}{2} \omega(k) \ .$$

Since the mean value of the potential energy of an oscillator in a given state is half the mean value of the total energy of the oscillator in that state, we may write

$$\frac{1}{4} \omega(k) = \frac{(2\pi)^3}{2} \left( \frac{k^2}{\langle \rho \rangle} + V(k) \right) \langle |\beta(k)|^2 \rangle \ ,$$

where the bracket denotes a ground state average. With the help of eq. (8), we immediately obtain

$$\omega(k) = \frac{k^2}{2\Omega(k)} \ .$$
where

\[ S(\mathbf{k}) = (2\pi)^3 \left( |\beta(\mathbf{k})|^2 \right) \langle \rho \rangle \]

is the Fourier component of the density correlation function. Equation (10) is exactly Feynman's result.

Thus, we have shown that the small oscillation approximation to the Dashen-Sharp Hamiltonian (5) yields the Bogoliubov spectrum independently of any special assumptions about Bose condensation, and it further provides an alternative microscopic derivation of the Pitaevskii theory. It is interesting that the phonon spectrum, which we obtain in the limit of small \( k \) values, is independent of the presence of the Bose-condensed particles. This implies a sound velocity for these quanta which is the same above and below the \( \lambda \)-point. This property of the spectrum is in concert with the recent neutron scattering experiments of Woods (11) on liquid \(^4\text{He}\). For quanta having \( k < 0.38 \, \text{Å} \), Woods observes that the sound velocity is essentially independent of temperature through the \( \lambda \)-point.

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FOOTNOTES AND REFERENCES

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5. H. FROHLICH, Physica 34, 847 (1967); J. C. GARRISON, J. WONG, and H. L. MORRISON, Lawrence Radiation Laboratory Report UCRL-7 1472, to be published.


8. The existence of such an operator is in conflict with the positivity condition on the density; see Ref. 5.

9. Every operator that commutes with both $\rho(x)$ and $J(x)$ is a multiple of the identity.

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