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TopBot: A Study of Passive Stability Via Momentum Biased Heavy Top Dynamics Coupled with Hopping to Produce a Unique Form of Locomotion

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TopBot: A Study of Passive Stability Via Momentum Biased Heavy Top Dynamics Coupled with Hopping to Produce a Unique Form of Locomotion

A Thesis submitted in partial satisfaction of the requirements for the degree Master of Science in Mechanical Engineering by Jaron Clinton Scott

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2014
The Thesis of Jaron Clinton Scott is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2014
DEDICATION

TopBot is dedicated to those patient souls that have helped along the path of its conception, construction, and analysis. Professor Bewley’s ongoing support, the MAE 171B team that investigated every whim of TopBot’s design space, lab persons that lent a hand, Nick Morozovsky for opening the door and helping me after my long absence, and my mother, Ronna Scott, who relentlessly saw to the completion of the project. Without their support and the support of many others, I would not have had the opportunity to work on, much less complete this project/journey.
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ABSTRACT OF THE THESIS

TopBot: A Study of Passive Stability Via Momentum Biased Heavy Top Dynamics
Coupled with Hopping to Produce a Unique Form of Locomotion

by

Jaron Clinton Scott

Master of Science in Mechanical Engineering

University of California, San Diego, 2014

Professor Thomas Bewley

The purpose of TopBot is to explore two things; top dynamics to passively stabilize a system, and a jumping mechanism as a form of locomotion for a robot. As the field of robotics is becoming a more staple part of everyday life it is used in more instances of solving problems as well as exploring the world and universe around us; it is necessary to begin to look into non-traditional techniques of locomotion to fully exploit the wide range of advantages robots can bring us. The advent of smaller and faster processing units for robots has often led to using the same standard forms of locomotion
with faster calculation speeds and more intricate solution structures to improve performance. This system can however only improve performance to a point. The purpose of TopBot is to instead take a detailed look at a naturally occurring form of stability and exploit its benefits to stabilize a system as well as create a unique form of locomotion. This approach will help expand the solution set attainable from robotics in general.
INTRODUCTION

TopBot was a project created to explore passive stability and open-loop controls as a method of locomotion. The passive stability of top dynamics would be exploited as a way to keep TopBot righted while a hopping mechanism would be explored to allow for a form of locomotion. This allows for not only a computationally simple form of stability but could also be used in instances of lower gravity planets as a form of space exploration where hopping from location to location could provide an energy efficient alternative to rolling or stepping over harsh terrain.

A traditional top is spun to a maximum rotational speed and then set onto a surface to precess and nutate while the spin speed degrades over time as shown in Figure 1. Conversely, TopBot could be comprised of a “body” and a “rotor” attached to said body. The idea being, the rotor would be spun relative to the body and if the rotor provides enough momentum, it could keep the body and rotor upright with a type of pseudo-top dynamics. This preliminary concept is shown in Figure 2.
Figure 1. Typical Top Spinning about its own axis while nutating and precessing about a fixed frame.

Figure 2 then shows that TopBot could be outfitted with a hopping mechanism that would allow TopBot to jump in any direction by exploiting its nutation and precession angles. This would then constitute TopBot’s locomotive ability. When it landed, much like dropping a top from an elevated surface, TopBot would then return to its precession and nutation.
Figure 2. TopBot concept as a non-spinning body with an attached spinning rotor. A hopping mechanism then shows its ability to jump as a form of locomotion.

The appeal of TopBot is in its capability to operate with passive stability while maintaining agility. This allows for TopBot to remain upright without the complex controls or actuation mechanisms that are seen in similar walking/jumping robots that rely on computationally heavy control schemes. The single touch down point also allows TopBot to maintain a small footprint in whatever surface it is resting on. This would allow TopBot to maintain agility on uneven surfaces.

TopBot will be laid out in the following manner: First the dynamics will be described in respect to the assumptions and the equations used to construct TopBot. Then the design of the robot will be described. Although TopBot required an iterative design approach to meet the necessary metrics to conform to the assumptions listed, the design process will be described as a two-step process for brevity; initial and final design. Then the physical testing process will be described along with the numerical
analysis results. After the two are compared issues will be highlighted and “next steps” will be discussed.
DYNAMICS

The Dynamics of TopBot will be laid out by first looking at the assumptions that will be placed on the system and then evaluating the discrete modes of operation: top mode, the jumping process, description of in air motion.

ASSUMPTIONS

Below are Assumptions for TopBot that apply to its various modes of operation.

1. For TopBot operating as a top on the ground, the transverse and axial inertias of the rotor and body are both assessed as operating through the fixed origin, $O$, at the base of the foot and are perpendicular to the axial inertias. First of all, this constraint applies to how the origin of the system is described; therefore this assumption is reasonable since the location of the origin can be described at will. Secondly, this also requires a no slip condition so that the origin is unable to move in space. This is a reasonable design constraint that can be resolved by using high friction material to fix the foot of TopBot to the floor.

2. When TopBot is acting as a top on the ground, the distance from the origin, $O$, to the center of mass of the body and rotor will be determined by $L^b$ and $L'$ respectively. Further it will be assumed that $L^b = L' = L$. This is a strict design constraint of the robot but ensures that the dynamics of TopBot are greatly simplified.

3. The inertias of TopBot while it is operating as a top on the ground can be described such that the rotor exhibits axially symmetric transverse inertias and therefore $I'_{xx} = I'_{yy} = I'_{zz}$ and therefore the axial inertia of the rotor $I'_{zz}$ can be
simply described as $I'$. The body on the other hand will not display this characteristic so $I_{xx}^b \neq I_{yy}^b$. These can be seen as design constraints on the system. The rotor can simply be made to exhibit the mentioned characteristics while the body itself will have many point masses that will make it extremely difficult to maintain total axially symmetric transverse inertias. Therefore it would be unreasonable to impose this restriction on the design.

4. No matter the mode of operation, it will also be assumed that the nutation angle, $\theta$, and the rate of nutation, $\dot{\theta}$, as well as the precession angle, $\psi$, and the rate of precession, $\dot{\psi}$, are the same for the body and rotor. This requires physically constraining the rotor to the body such that their axial inertias reside about the same axis.

5. For TopBot as a free body spinning in space. It will be assumed that Assumptions 1 and 2 will still hold except the origin, $O$, will be at the center of mass. This change in origin will allow the system to be viewed such that the only external forces of the system, the jumping force of the foot and the gravitational force, will act through the center of mass which can be described as the origin. As with Assumption 1, this only requires changing the description of the system and is therefore reasonable.

6. Based on Assumption 3, a definition of the angular momentum, $\vec{H}_G$, can be created and described. Since the only external forces of the system will be operating through the center of mass this angular momentum can be said to be constant through flight. This allows us to describe the free body motion of TopBot based on the initial conditions of flight after TopBot leaves the ground.
7. The speed of rotation of the body, $\dot{\phi}$, is going to be very small compared to the speed of rotation of the ring, $\dot{\phi}'$. This applies to TopBot both on the ground and in flight. While TopBot is on the ground, $\dot{\phi}$ can be designed to be exactly zero by physically constraining it from spinning relative to the ground. Then while TopBot is airborne $\dot{\phi}$ can be approximated as zero since the amount of time the system spends in the air will be shown to be very short; only allowing enough time for the body to spin up to a still negligible speed.

8. The angular velocity of the nutation angle, $\dot{\theta}$, is zero. This also means that all higher dimensional forms of $\theta$ will be zero as well. This assumption is made because it can be assumed that the rotor speed can be varied if needed to maintain a particular nutation angle.

9. When TopBot jumps it is assumed that the actuator used for jumping will act in a linear fashion. This is a matter of selecting a jumping mechanism that will accommodate a linear actuation.

The given coordinate system for TopBot assumes that the transverse inertia, $I'$, travels through the origin $O$ and is perpendicular to the axial inertia, $I$. It also assumes that the transverse inertia is symmetric for any axis through the origin $O$ and perpendicular to the axial inertia. While this assumption of symmetry is very restrictive, it is fair to use as a design constraint. Deviating too far from this axial symmetry will cause TopBot to have an undesirable non-constant precession rate $\dot{\psi}$ and a non-constant angle of nutation $\theta$.

This chosen coordinate system also assumes that all external reaction forces will be translated through the fixed origin, $O$. 
DERIVATIONS FOR DYNAMICS IN TOP MODE

The TopBot system in top mode will be looked at in two parts, the body of the robot and the rotor spinning relative to the body. The Lagrange function, $L = T - V$, will be used to obtain the equations of motion where $T$ describes the kinetic energy of the system and $V$ represents the potential energy. In order to arrive at the Lagrange equations, that will describe the motion of the robot, the angular velocity equations must be found for both the body and rotor so that the equation for kinetic energy can be used. By using the assumptions 1, 4 and 7, we arrive at the angular velocity equations given as equations (2.1) and (2.2)

$$\vec{\omega}^b = -\left(\dot{\psi} \sin \theta\right) \vec{T} + \dot{\theta} \vec{T}' + \left(\dot{\psi} \cos \theta\right) \vec{k}'$$ (2.1)

$$\vec{\omega}' = -\left(\dot{\psi} \sin \theta\right) \vec{T} + \dot{\theta} \vec{T}' + \left(\dot{\psi} \cos \theta + \dot{\phi}'\right) \vec{k}'$$ (2.2)

Here the body will have precession $\psi$ about the $Z$ axis, nutation will be defined as $\theta$ about the $y'$ axis and the spin rate of the body will be $\phi$ about the shared $z$ and $z'$
axis. The rotor is then described in a similar way but has the additional spin rate \( \dot{\phi}' \) relative to the spin rate of the body.

The Lagrange function will then be looked at in two parts. First, the combined kinetic energy of the body and rotor together as shown in equation (2.3). Then, the potential energy will be shown in a similar way; shown in equation (2.4).

\[
T = T^b + T' \tag{2.3}
\]

\[
V = V^b + V' \tag{2.4}
\]

Now using the general equation for kinetic energy where \( T = \frac{1}{2} I \omega^2 \), where \( I \) represents the inertia of a body, the kinetic energy of the body and rotor can be stated in equations (2.5) and (2.6) respectively using Assumption 3 to simplify the kinetic energy equation of the rotor.

\[
T^b = \frac{1}{2} I^b_{xx} \left\| \ddot{\alpha}_x^b \right\|^2 + \frac{1}{2} I^b_{yy} \left\| \ddot{\alpha}_y^b \right\|^2 + \frac{1}{2} I^b_{zz} \left\| \ddot{\alpha}_z^b \right\|^2 \tag{2.5}
\]

\[
T' = \frac{1}{2} I' \left( \left\| \ddot{\alpha}_x' \right\|^2 + \left\| \ddot{\alpha}_y' \right\|^2 + \left\| \ddot{\alpha}_z' \right\|^2 \right) + \frac{1}{2} I' \left\| \ddot{\alpha}_r' \right\|^2 \tag{2.6}
\]

Here we have broken up the equation to account for the transverse inertias represented by \( I' \) and the axial inertias represented by \( I \). For the body, the angular velocity equation (2.1) can be broken into its axial components and can be plugged into the body’s kinetic energy equation (2.5). The notation will be further un-encumbered by representing the body inertias as follows: \( I^b_{xx} = I_1, I^b_{yy} = I_2 \) and \( I^b_{zz} = I_3 \). By doing this, it is found that
\[ T^b = \frac{1}{2} \left[ \dot{\psi}^2 I_1 \sin^2 \theta + \dot{\theta}^2 I_2 + \ddot{\psi} I_3 \cos^2 \theta \right]. \quad (2.7) \]

The kinetic energy for the rotor is then found in a similar way and given below in equation (2.8).

\[ T^r = \frac{1}{2} \left[ \dot{\psi}^2 I'' \sin^2 \phi + \dot{\phi}^2 I' + \psi^2 I'' \cos^2 \phi + 2\dot{\psi} \dot{\phi} I' \cos \phi + \dot{\phi}^2 I' \right] \quad (2.8) \]

The two equations are then combined using equation (2.3) to obtain

\[ T = \frac{1}{2} \left\{ \dot{\psi}^2 \left( I_1 + I'' \right) + \cos^2 \phi \left( I_3 + I'' - I_1 - I'' \right) \right\} + \dot{\theta}^2 \left( I_2 + I' \right) + \dot{\phi} \left( 2\dot{\psi} \cos \phi + \dot{\phi} I' \right). \quad (2.9) \]

The potential energies of the system were then examined where \( V = mgh \) and \( m \) is the mass, \( g \) is the gravity acting on the system and \( h \) is the height of the center of mass to the ground. The potential energy of the body and rotor respectively are given in equation (2.10).

\[ V^b = m^b g L^b \cos \theta \]
\[ V^r = m^r g L^r \cos \theta \quad (2.10) \]

Here Assumption 2 determines that the length from the center of mass of the body and rotor are equal. The two potential energy equations were then also be combined as in equation (2.4) which yielded:

\[ V = \left( m^b + m^r \right) L g \cos \theta. \quad (2.11) \]

In the case of potential energy of the system in top mode, it is clear that the system has more potential energy when the system has a smaller angle of nutation.
After the kinetic energies and potential energies of the system were found, they were combined to give the Lagrange function of the system. In its simplified form, the Lagrange function is:

\[
\mathcal{L} = \frac{1}{2} \dot{\psi}^2 \left[ \left( I_1 + I'' \right) + \cos^2 \theta \left( I_3 + I' - I_1 - I' \right) \right] \\
+ \frac{1}{2} \dot{\theta}^2 \left( I_2 + I'' \right) + 2 \cos \theta \left( \dot{\phi} \dot{\psi} I' - \left( m^b + m^r \right) Lg \right) + \dot{\phi}^2 I'
\]  

(2.12)

Next it was determined which coordinates did not appear in the Lagrange function. These coordinates would make up the set of cyclic generalized coordinates which were ignored since the Lagrange equation does not depend upon them explicitly. They are \( \psi \) and \( \phi' \). The one non-cyclic generalized coordinate left in the system is the nutation angle \( \theta \). The equation of motion for the one non-cyclic generalized coordinate is given by:

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0
\]

(2.13)

Here the left side is set equal to zero which supposes that there will be no unaccounted for outside forces acting against the system in relation to the nutation angle. This suggests the ideal “lossless” case. By assessing the Lagrange function with respect to equation (2.13), equation (2.14) was created.

\[
\left( I_2 + I'' \right) \ddot{\theta} = \dot{\psi}^2 \sin \theta \cos \theta \left( I_1 + I' - I_3 - I' \right) + \sin \theta \left( \left( m^b + m^r \right) Lg - \dot{\phi} \dot{\psi} I' \right)
\]

(2.14)

Then Assumption 8 was used to set \( \ddot{\theta} \) equal to zero. This yielded:

\[
0 = \dot{\psi}^2 \sin \theta \cos \theta \left( I_1 + I' - I_3 - I' \right) + \sin \theta \left( \left( m^b + m^r \right) Lg - \dot{\phi} \dot{\psi} I' \right)
\]

(2.15)
Here $\sin \theta$ shows up in both of the right hand side equations. Since a value of $\sin \theta = 0$ corresponds to a 90 degree nutation angle, it was assumed that this nutation angle will never be reached so both sides were divided by $\sin \theta$ without a loss of generality. Then, a solution for the angular velocity of precession was found by solving equation (2.15) as the following quadratic equation:

$$
\dot{\psi} = \frac{\dot{\phi}' I' \pm \sqrt{(\dot{\phi}' I')^2 - 4\left(I_1 + I' - I_3 - I'\right)\cos \theta \left(m^b + m'\right)Lg}}{2\left(I_1 + I' - I_3 - I'\right)\cos \theta}
$$

(2.16)

Upon assessing this equation, it is important to note that a precession speed can only be found for a family of values of $\dot{\phi}'$ that satisfies the radicand in equation (2.16) without yielding an imaginary number. The solution to this radicand is given in equation (2.17) where positive values of $\dot{\phi}'$ give clockwise rotation and negative values would give counter-clockwise.

$$
|\dot{\phi}'| \geq \frac{1}{I'} \sqrt{4\left(I_1 + I' - I_3 - I'\right)\cos \theta \left(m^b + m'\right)Lg}
$$

(2.17)

For the cyclic generalized coordinates, $\psi$ and $\phi'$, the Lagrange function in equation (2.12) was assessed such that:

$$
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\psi}} \right) - \frac{\partial L}{\partial \psi} = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\psi}} \right) \Rightarrow \frac{d}{dt} \left( \frac{\partial T}{\partial \psi} \right) = \beta_{\psi}
$$

$$
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi'}} \right) - \frac{\partial L}{\partial \phi'} = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi'}} \right) \Rightarrow \frac{d}{dt} \left( \frac{\partial T}{\partial \phi'} \right) = \beta_{\phi'}
$$

(2.18)
Where \( \frac{\partial L}{\partial \psi} \) and \( \frac{\partial L}{\partial \phi'} \) are both necessarily zero and neither the \( \psi \) or \( \phi' \) term arise in the equation for potential energy so it was only necessary to take the partial derivative of the kinetic energy function instead of the full Lagrange function. The \( \beta \) terms represent the respective conjugate inertias of the system, which are conserved. This led to the following set of coupled equations:

\[
\psi \left(I_1 + I''\right) + \cos^2 \theta \left(I_3 + I' - I_1 \right) \psi + \dot{\phi}' I' \cos \theta = \beta_\psi
\]

\[
\psi I' \cos \theta + \dot{\phi}' I' = \beta_\psi
\]

These equations were solved as in equation (2.20) in terms of the conjugate inertias. Since there are no time dependent terms in the equations, the spin speed of the rotor and the precession rate of the system are shown to be based on the initial conditions of the system.

\[
\begin{bmatrix}
\psi \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\left(I_1 + I''\right) + \cos^2 \theta \left(I_3 - I_1 - I''\right)} & -\cos \theta \\
-\cos \theta & \frac{\left(I_1 + I''\right) + \cos^2 \theta \left(I_3 - I_1 - I''\right)}{\left(I_1 + I'\right) + \cos^2 \theta \left(I_3 - I_1 - I'\right)}
\end{bmatrix} \begin{bmatrix}
\beta_\psi \\
\beta_\phi
\end{bmatrix}
\]

(2.20)

**DERIVATION OF DYNAMICS FOR IN AIR MOTION**

To determine the in air dynamics of TopBot, first the angular velocity equations for a free body in space were evaluated. Then the equations were applied to both the body and rotor separately with their respective assumptions applied. Finally, the initial conditions of flight were used to combine these equations so that the angular velocities of the system were determined.
To begin, an arbitrary body in space was evaluated in terms of its inertias and angular momentas. The angular velocity was then described as follows:

\[
\vec{\omega} = (-\psi \sin \theta \cos \phi + \dot{\theta} \sin \phi) \hat{T} + (-\psi \sin \theta \sin \phi + \dot{\theta} \cos \phi) \hat{J} + (\psi \cos \theta + \dot{\phi}) \hat{K} \tag{2.21}
\]

This general equation was assessed to describe the motion of the body and rotor respectively. The results are shown in equation (2.22).

\[
\vec{\omega}^* = (-\psi \sin \theta \cos \phi + \dot{\theta} \sin \phi) \hat{T} + (-\psi \sin \theta \sin \phi + \dot{\theta} \cos \phi) \hat{J} + (\psi \cos \theta + \dot{\phi}) \hat{K},
\]

\[
\vec{\omega}' = (-\psi \sin \theta \cos \phi + \dot{\theta} \sin \phi) \hat{T} + (-\psi \sin \theta \sin \phi + \dot{\theta} \cos \phi) \hat{J} + (\psi \cos \theta + \dot{\phi} + \dot{\phi}') \hat{K} \tag{2.22}
\]

Here, the term \( \dot{\phi}' \) represents the relative spin of the rotor about the \( \hat{K} \) axis with respect to the body of the system. Then by additively combining the angular velocities of equation (2.22) and the inertias of the system a representation of the system’s angular momentum was obtained as in equation(2.23).

\[
\vec{H}_G = (I'_{xx} \omega_x^* + I^b_{xx} \omega_x^b) \hat{T} + (I'_{yy} \omega_y^* + I^b_{yy} \omega_y^b) \hat{J} + (I'_{zz} \omega_z^* + I^b_{zz} \omega_z^b) \hat{K} \tag{2.23}
\]

By applying the before mentioned assumptions 4 and 5, this equation was reduced to equation(2.24). In this equation a change of notation was also used in an attempt to unencumber the equations. Here we see that \( I_1 = I^b_{xx}, \ I_2 = I^b_{yy} \) and \( I_3 = I^b_{zz} \).

\[
\vec{H}_G = \omega_x (I'_{xx} + I_1) \hat{T} + \omega_y (I'_{yy} + I_2) \hat{J} + (I'_{zz} \omega_z^* + I_3 \omega_z^b) \hat{K} \tag{2.24}
\]

It is also necessary to represent the vector form of the angular momentum of the system. This is done in the following equation:

\[
\vec{H}_G = -(H_G \sin \theta \cos \phi) \hat{T} + (H_G \sin \theta \sin \phi) \hat{J} + (H_G \cos \theta) \hat{K} \tag{2.25}
\]
Equations for $\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$ and $\ddot{\phi}$ were then obtained by independently equating the components of the angular momentum equations (2.24) and (2.25). This relation of components results in the following:

\[
\begin{align*}
(I'' + I_1)(-\dot{\psi} \sin \theta \cos \phi + \dot{\theta} \sin \phi) &= -(H_G \sin \theta \cos \phi) \\
(I'' + I_2)(\dot{\psi} \sin \theta \sin \phi + \dot{\theta} \cos \phi) &= (H_G \sin \theta \sin \phi) \\
I' \dot{\psi} \cos \theta + [(I' + I_3) \dot{\phi} + I' \ddot{\phi}] + I_3 \dot{\psi} \cos \theta &= (H_G \cos \theta). 
\end{align*}
\]

This is unfortunately a set of three equations with four unknowns. By setting

\[
[(I' + I_3) \dot{\phi} + I' \ddot{\phi}] = \Phi \text{ a solution can be found that relates } \dot{\phi} \text{ and } \ddot{\phi}. \text{ This set then leads to:}
\]

\[
\begin{align*}
\psi &= H_G \left( \frac{1}{I'' + I_1} + \sin^2 \phi \left( \frac{1}{I'' + I_2} - \frac{1}{I'' + I_1} \right) \right) \\
\dot{\theta} &= H_G \sin \theta \sin \phi \cos \phi \left( \frac{1}{I'' + I_2} - \frac{1}{I'' + I_1} \right) \\
\Phi &= (I' + I_3) \dot{\phi} + I' \ddot{\phi} = H_G \cos \theta \left( 1 - (I' + I_3) \left( \frac{1}{I'' + I_1} + \sin^2 \phi \left( \frac{1}{I'' + I_2} - \frac{1}{I'' + I_1} \right) \right) \right).
\end{align*}
\]

Here it is shown that some design considerations arise. By making the transverse inertias of the body as close to equal as possible, the term $\frac{1}{I'' + I_2} - \frac{1}{I'' + I_1}$ would go to zero in each equation allowing for slower less chaotic in air motion. However, in order to not lose generality, the equations will retain the difference in inertia. However, by taking advantage of assumption 6, where $\dot{\phi}$ can be set to zero, $\ddot{\phi}$ can be explicitly solved for in terms of measurable angles and inertial values. This and a few trigonometric simplifications led to the set of equations in (2.28).
\[ \ddot{\psi} = H_G \left( \frac{\sin^2 \phi + \cos^2 \phi}{I'' + I_2} \right) \]
\[ \dot{\theta} = H_G \left( \frac{1}{I'' + I_2} - \frac{1}{I' + I_1} \right) \sin \theta \sin \phi \cos \phi \]
\[ \phi' = \frac{H_G}{I'} \cos \theta \left( 1 - (I' + I_3) \left( \frac{\sin^2 \phi + \cos^2 \phi}{I'' + I_2} \right) \right) \]

From these equations, it is apparent that only one value of either precession, nutation or rotor spin speed need be known in order to solve for the constant angular momentum. From there, the other values can be solved for when the respective angles are known.

**JUMPING EQUATIONS TO GOVERN VERTICAL HEIGHT OF JUMP**

The dynamics of jumping were assumed to be relatively simple for TopBot. By assumption 9, it is supposed that an actuator existed that could supply a constant force to make TopBot jump. Then, the equations to govern TopBot's jumping were relatively simple and given in equation (2.29).

\[
\begin{bmatrix} z(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 1 & \frac{(f - mg)t^2/2m}{(f - mg)t/m} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z(t_f) \\ v(t_f) \end{bmatrix} \left( g(t-t_f)^2/2 \right) \quad t \in [0,t_f]
\]
\[
\begin{bmatrix} z(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z(t_f) \\ v(t_f) \end{bmatrix} \left( g(t-t_f)^2/2 \right) \quad t \in (t_f, \infty)
\]

The equation is based on a simple body jumping in a gravitational field \( g \) where the jumping force \( f \) and gravity are assumed to act in line through the center of mass. Here, \( z \) represents position in the vertical position, \( v \) the velocity along that axis, and \( m \) is the aggregate mass of the body and rotor. Lastly, \( t \) represents the amount of time the
jumping force is applied and $t_j$ is the total elapsed time from when the actuator begins actuating to when it finishes its actuation process.
DESIGN OF TOPBOT

In order to verify the equations of motion empirically, TopBot was built as a physical test apparatus. This required creating a multimodal robot that would operate in all ways previously discussed while maintaining all the assumptions described. This required extensive work in finding a way to constrain a spinning rotor within a body, developing a jumping mechanism, finding a source of onboard power, appropriate sensing equipment, and finally a form of computing power to make all the components work together. The design process of incorporating all of these aspects began by looking at two areas; various hopping mechanisms that could give a ball park of height values for an estimated system mass and also a general body design that could properly constrain the ring and all the appropriate mechanisms that TopBot had to hold in order to operate. Very quickly, this led to an iterative process. It involved minimizing mass to make jumping more feasible while making sure to maintain top dynamics and subsequently, appropriate inertia values according to equation (2.17). The radicand had to remain non-imaginary but could not grow too large to make sure an achievable rotor spin speed could be maintained.

A ballpark mass of 5kg was initially chosen so that an idea of the actuation time could be obtained from equation (2.29) to make TopBot Jump. Distributing this mass in accordance with equation (2.17) then would lead to the shape that TopBot would take.

CANDIDATE JUMPING MECHANISMS

Equation (2.29) was used to give an idea of the actuation time and force needed to make TopBot jump 10 inches (.254 meters) with the use of a constant force during actuation. This calculation was executed in MATLAB and a plot of the results are shown
in Figure 4 (Appendix A-1. Jump Height and Velocity). It can be seen that the .1s actuation time with a 155N constant force can achieve a jump height of approximately .35m. This is approximately .1m higher than necessary. Here it is also good to note that the actuation distance is approximately .1m. This is a reasonable ballpark distance for an actuator to function within.

![Graph of Jump Height and Velocity](image)

**Figure 4.** Jump Height and Velocity According to an Applied Constant Force over a .1s Interval.

From this initial data, four jumping mechanisms were explored for TopBot; cold rocket, mass spring, solenoid actuation, pneumatic actuation. These four forms of operation
gave a wide variety of working mediums that could be constrained within a compact space.

COLD ROCKET

The idea of a cold rocket is to expel non-combusted fluid from the end of a nozzle such that the mass transfer out the bottom of the rocket nozzle can force the system upward (The idea of a combustion rocket was ruled out for safety reasons). For this a working fluid of $CO_2$ was selected. This working fluid was selected due to its ability to readily be found in small 800psi cartridges. This allows for a low profile and the ability to change out cartridges as needed without the use of a second tank or external source.

Cold rocket equations were used to find a suitable nozzle throat diameter for a converging diverging nozzle, the amount of time the rocket would have to be actuated, and the mass flow rate necessary for the lift. These numbers also led to the amount of $CO_2$ necessary to make TopBot jump the target 10 inches (0.25m). The simulation was performed in MATLAB and the results are shown in Figure 5 (Appendix 1-2).
Figure 5. Thrust Necessary for 10 in (.25m) Vertical Hop with associated properties.

Since a big concern with TopBot is weight and size management, it is advantages to choose a thrust value that would allow for a short actuation time. In the top left of the figure, it is shown that the amount of $CO_2$, in grams used, to make TopBot jump does not change considerably for thrust values over 250N. The $CO_2$ used per hop is shown to remain around 25-26g. From here it can be seen in the top left figure that an actuation time would be very small, approximately 50ms. Subsequently, for a thrust value of 250N, it is shown that a mass flow rate of approximately .5kg/s and a nozzle throat diameter of 1.4 cm. Here some troubles begin to arise with the cold rocket application on TopBot. First, the short actuation time of 50ms requires a very high resolution from electronics in order to ensure such a small time. Secondly and more
importantly, the mass required per hop is approximately 26g. This large amount of $CO_2$ used per hop would be difficult to store in order to allow for several actuations.

**MASS SPRING**

The idea of using a spring as a jumping mechanism for TopBot suggested the use of a compression spring that would be compressed while TopBot was not jumping and then a release mechanism would be used to release the spring and make TopBot jump. Subsequently a loading mechanism would need to recoil the spring and await the next hop.

Using the values already obtained in Figure 4. If a target jump height of .254 meters is to be reached, the jumping force of 155 N is a good starting point for finding an applicable spring for this application. For this analysis, Hooke’s Law can be applied as in equation(3.1), so long as it is assumed that the spring force is constant over the entire length of its displacement.

$$F = -kx$$ (3.1)

Here we see the spring force is given by $F$, the spring constant is $k$, and the displacement is given by $x$. In fact a spring with a spring constant of $k = 1550 \, N/m$ is very obtainable with a displacement of $x = .1m$; which would correspond to a spring force of $F = 155N$.

The problems with a spring mass jumping system did not occur with the parameters as in with the cold rocket approach. Instead, the physical implementation quickly became an issue. No design was able to make it past a concept generation phase without substantial issues. The amount of time it would take to reset the spring
into a mode ready for jumping was a critical point. Making a quick loading time for the spring meant using a heavier motor which made the TopBot system heavier. The non-symmetric shape that the jumping mechanism would have to take was also an issue since symmetry is crucial for top dynamics. While these issues created great complications in the design, it was still crucial to explore a spring mass system since it had the initial allure of being a simple solution.

**PNEUMATIC ACTUATION**

The jumping mechanism that was finally selected for TopBot was a pneumatic actuation system. More specifically; a single action pneumatic actuator with a spring return was selected. This meant that the pneumatic actuator, when at rest, is in a compressed state. Upon actuation, with the use of a working fluid, the actuator would extend. The working fluid could then be evacuated from the actuator and a spring within the device would return the actuator back to its resting compressed state. Pneumatic actuation was a serious design consideration because of its ability to efficiently use a working fluid without much loss. As will be discussed, there are several mechanisms that go into making pneumatic actuation happen but they can be arranged in any number of ways by simply attaching them together using flexible tubing so that the working fluid can travel between the various stages.

The various aspects of the pneumatic actuation system consisted of: a mechanism to store the working fluid onboard TopBot, some type of pressure regulation to ensure that the fluids working pressure was within spec of the devices chosen and finally a mechanism to release the working fluid into the actuator.

To begin with, a pneumatic actuator was found that met the requirements laid out in Figure 4. It was found that a .75in bore actuator with a 3in stroke length at 100psi
would provide 35lbf or 155.7N (McMaster-Carr). This would be enough for applications with TopBot. Next it was necessary to find components that could work within the 100psi specification with $CO_2$ as the working fluid. A combination component was found at Genuine Innovations that acted as both a holder for the $CO_2$ cartridge and an adjustable pressure regulator that would down regulate the $CO_2$ canisters 800psi to 100psi (Genuine Innovations). To make sure that the pneumatic actuator could be activated at 100psi enough times to allow for several jumps of TopBot, a MATLAB program was written based on the expansion of $CO_2$ (Appendix 1-3). The program revealed that with a constant temperature of the working fluid, the actuator should be able to activate 13 times before the tank pressure dropped below 200psi.

Finally, a solenoid had to be selected to control air flow and allow for controlled actuation windows. The SMC NVKF332V-6D—01T was selected. This solenoid is a 3-way solenoid with one normally closed path. It is able to handle 100psi while operating at a reasonable 12V with a 10ms response time to actuation. (SMC Pneumatics Inc.) A light weight nylon-manual on/off ball valve was also selected (McMaster-Carr) to allow for easy shutoff of the system which conserved $CO_2$ and gave a level of safety by having a manual off in case the electric solenoid failed. A Flow chart of the pneumatic system is shown in Figure 6.

Figure 6. Flow Chart of Pneumatic Actuation.
OVERALL ROBOT DESIGN

As discussed previously, the design process of TopBot was an iterative one that took careful balance of various mass properties and part selection that allowed for minimum-energy requirements while maintaining the top dynamics discussed previously. For brevity, the design process is shown as a two stage process, a preliminary design and subsequently the final design that followed.

PRELIMINARY DESIGN PHASE

The preliminary design process revolved around equation (2.17). As discussed in the Dynamics section, this equation gives the necessary mass properties and spin speed relationships for the top motion to occur. This equation shows that ideally, to make a low energy system, TopBot should have a center of mass as low as possible with a minimal mass. Also, it would be ideal to maintain a balance of inertia such that the equation (4.1) holds.

\[
0 < I_1 + I'' - I_3 - I' < \varepsilon
\]

(4.1)

It is also important to note the denominator of equation (2.17) shows that the axial inertia of the rotor should be large to keep the equation small. In order to then maintain equation (4.1) the transverse inertia of the rotor should be smaller to maintain balance. This relationship gives great insight to the necessary shape of the rotor and body. It states that the rotor should be thin (small height \( h \)) and wide (large radius \( r \)), this allows for the axial inertia \( I' \) to be larger and as \( h \) gets smaller, the transverse inertia \( I'' \) to be smaller. To further exploit this, it was then noted that instead of a solid rotor modeled as a disk, with the axial inertia given by
The rotor could be modeled as a ring with an axial inertia given by:

\[ I = \frac{mr}{2}. \]  

(4.2)

In this equation it is possible to have a cylindrical rotor with an inner radius \( r_1 \) and outer radius \( r_2 \). This would give the increased axial inertia needed as well as make TopBot lighter by having less mass in the rotor. From the sum, \( m^b + m' \) under the radicand of Equation (2.17), it is apparent that the inertia of the body of TopBot would also have to be minimized. This design also allows for another less apparent benefit. By creating an opening in the middle of the rotor, parts of the body such as the pneumatic system could be mounted close to the axis of rotation. This also contributes to small inertia values by keeping the mass near the center of rotation.

Since energy was going to be needed to both spin the rotor and activate the pneumatic actuator already selected, it was decided that on board batteries were going to be needed to supply the energy. In order to maintain the inertial balance between the body of TopBot and the rotor, it was considered that the batteries should not be secured to the body but should instead be secured to the spinning rotor. This was in anticipation that weight would be added to the body in other ways, such as with the pneumatics assembly and the motors that would be used to spin TopBot's rotor. Then by placing the batteries in the rotor, the inertia balance previously discussed would be better followed. The preliminary design is shown in Figure 7.
Figure 7. Preliminary Frame and Rotor Design. Incorporates batteries inside the rotor to reduce the amount of weight both on the body and the overall system.

A problem arose with transferring power from the batteries in the spinning ring to the components in the body. This issue is a known problem with known solutions. Most common is the solution of implementing a ‘slip ring’ system that allows for energy transfer. Although this is a solvable system, it is difficult to implement for a system spinning as fast as a top when both reliability and price are concerns. One design is given by Honeybee electronics where polished gold contact wheels are used to transfer electricity across a spinning joint. (Robotics) This design is shown in Figure 8. Here, gold is used as a reliable material to transfer power across a discontinuous junction. Also, a very high precision level of machining is shown in the sprung wheels that allow for compliance. Therefore, although the solution is good it is well outside of the fiscal reach of TopBot.
Figure 8. Honeybee Robotics’s solution to the slipring problem. Polished gold sprung wheels to allow for constant contact with low loss of torque. (Robotics)

Since a reliable and cost effective solution was not obtainable for the transfer of power between the spinning rotor and the stationary body of TopBot, the concept of housing the batteries within the rotor was abandoned. This notion was further supported when it was discovered that the mass density of the candidate alkaline D battery was about \(0.0873 \text{ lbm/in}^3\), whereas the approximate density of the rotor would most likely be around \(0.289 \text{ lbm/in}^3\), since that is the approximate density for machine-able steels; which was the candidate material for the ring to help maximize its inertia. Therefore, adding the batteries into the ring would actually lower the rings axial inertia when compared to a ring strictly made of steel.

Next it was decided that “ball park” figures would be needed to move forward with the design. By numerically modeling the system according to Equation (2.17), with some approximate dimensions and mass values, it is possible to refine the design.
process and ensure that preliminary designs would operate within the constraints of the problem (Appendix A-4. Preliminary Spin Speed of Rotor). It is supposed that the body of TopBot could temporarily be modeled as a cylinder with a mass of 6.6lbm evenly distributed over a radius and height of 2.4in and 7in respectively. Then also the ring could be modeled as a ring with an inner and outer radius of 3in and 4in respectively with a height of .5in. Since a maximum axial inertia value was sought after for the rotor, steel was the most probable material to be selected to make the rotor which would lead to an overall rotor mass of 2.9lbm (1.32kg).

The generalized body shape was chosen based on the components of the pneumatic system that were already selected. This body size accounted for both extra space and more mass than necessary. This was used as a factor of safety in the design to allow for unpredicted weight increases.

FINAL DESIGN

A top-down isotropic view of the final design of TopBot is shown as a CAD representation in Figure 9 and Figure 10. The physical design space of TopBot proved to be one of the biggest constraints of the final design. It lead to a necessity to keep the body as compact and light as possible which required using the supportive “Top Plate” and “Bottom plates” not only as mounting points for the equipment on TopBot but also for the structure of the system.
Figure 9. Structure and Mechanics of TopBot.

In Figure 9 an isotropic top-down view of TopBot is shown with key structural features labeled. Here the structure is shown as a see through top plate and solid bottom plate. Between the plates is the rotor and bearing assemblies. Top and bottom collars are then used to keep the plates in place relative to the central pneumatic actuator labeled in Figure 10.

Figure 10 shows the assembly and orientation of the pneumatic actuation components, battery pack, and rotor drive motor. Air hoses and wiring were not modeled since the design space of TopBot allows for a very customizable routing.
In Figure 11 a cross-sectional view of TopBot is shown. Part (a) shows the general configuration while Part (b) shows a close up of the bearing assembly that both constrains the spinning ring and allows it to spin. A key feature of the rotor as shown is the tapered walls leading to the inner diameter. These tapered surfaces were machined each to a 30 degree angle with the horizontal. Corresponding delrin conical pads were made to contact the top and bottom of the ring. By using three of these assemblies evenly spaced around the rotor, the rotor would settle level and centered about the center axis. The conical pads were backed with steel plates to ensure that they would not deform. Behind the plates were then combination journal thrust bearings. These bearing would not only allow the rotor to spin at high speeds but would still allow large loads to be placed on them during the jumping and landing process.
In Figure 11 Part (c) the central axis of TopBot is shown. TopBot was itself built around a central pneumatic actuator. An aluminum sleeve was machined to fit around this actuator and be constrained by the lower nut fixture on the actuator. This allowed for a lower collar to be fastened on to the sleeve without pressing on the pneumatic actuator and damaging its internal components. Aluminum was selected over a stronger metal since hoop strength could be exploited with evenly distributed pressure of the collar. The lower plate was then slipped around the sleeve and rested on the collar. With the bearings and rotor then in place, the top plate was set on. The top collar was then fastened to sandwich the entire structure together. This designed allowed TopBot to be modular, such that a failed pneumatic actuator could easily be slid from the sleeve without disassembling the entire robot. A Top Collar design also allowed for pretension customization. As the delrin pads experienced run-in it would be necessary to re-sandwich the structure to take up gaps created from the run-in. All together this gave TopBot a mass of 8.657 lbm (3.927kg). Final physical data can be found in Appendix A-5. Here it is shown that the final values of TopBot’s physical characteristics do not strictly abide by Equation (4.1) despite best attempts during the design process. While this will lead to unstable motion of TopBot, it will still allow for data correlation between the physical model and the numerical model.
Figure 11. (a) Cross-sectional Side View of TopBot. (b) Bearing Assembly. (c) Pneumatic Actuator and Constraint System.
TESTING AND CORRELATION

The testing off TopBot occurred in two independent phases. First there was the testing of the jumping mechanism and then the testing of the precession rate of the robot. The idea behind the testing was to verify the equations of motion through empirical testing and numerical analysis.

JUMPING MECHANISM

As discussed previously, a pneumatic actuation system was chosen for TopBot. The goal was to design a system that would abide by Figure 4. Using a camera recording at 60 frames per second it was possible to determine, through observation, that actuation of the pneumatic system took about 5.5-6 frames, which corresponds to .092-.1 seconds. For this analysis it is important to recall that Figure 4 corresponds to ideal theoretical conditions. In practice, TopBot’s final mass was 3.927kg with an actuator length of .076m. The expected jump height for these parameters turned out to be 13.9in. A final jump height was achieved of 10.5in. This gave a 32% error in the system. This was found to be reasonable since the system was not acting as ideal. The line loss and thermal loss of the CO2 system was substantial. Tubing diameter was increased from .125in to .25in to combat these effects but they remained persistent, not allowing the actuator to meet its 155N force in the length of actuation.
PRECESSION RATE TESTING

As previously discussed, the precession rate and nutation angle of TopBot was determined to be a function of the initial conditions and the maintained speed of rotation of the rotor. A numerical model was then made to model a time stepped representation of TopBot. Physical results were then compared to the model to determine how well the physical robot was able to mimic the model created or conversely how well the theory represented a physical system.

PHYSICAL TESTING

Testing TopBot for its properties of precession required collecting data on rotor speed, angle of precession, and angle of nutation. To accomplish this, a compass was attached to TopBot to establish an initial heading and provide a “zero” to measure unwanted body spin. An accelerometer was then used to measure the angle of nutation and determine the precession angle of the system (Appendix A-6. Nutation Angle from Accelerometer). Photographs of this process are shown in Figure 13.
Figure 13. Change in Nutation angle. (a) Initial position ~4 degrees. (b) After first rotation ~6 degrees. (c) After second rotation ~8 degrees. (d) After third rotation ~10.5 degrees. (e) After fifth rotation ~15 degrees.

Here an angle is drawn to show the angle of nutation after each \(360^\circ\) precession cycle. This shows the evolution of the nutation angle. By noting the time it takes to complete one full precession cycle, a period of precession was also determined. A summary in Table 1 shows the speed of the rotor, the period of rotation and the corresponding change in angle. A graph of the rotors speed is also shown in Figure 14. This figure shows the change of speed while the rotor is spun up, the speed during testing while TopBot is precessing, and the degradation of speed while TopBot spins down. It is also worth noting that as TopBot precessed about its axis it would dip occasionally do to the inevitable offset mass which was not aligned with its axis of actuation. While every precaution is taken to ensure that the center of mass in aligned with the actuator, part tolerances inevitably lead to an imperfectly centered mass and lack of axial symmetry.
**Table 1.** Change in Nutation angle. From initial position to 4\textsuperscript{th} full precession. Corresponding to Figure 13.

<table>
<thead>
<tr>
<th>Rotor Speed (rev/sec)</th>
<th>Number of Revolutions</th>
<th>Lap Time (sec)</th>
<th>Angle of Nutation (degree)</th>
<th>Change in angle (degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.68</td>
<td>initial</td>
<td>n/a</td>
<td>4</td>
<td>n/a</td>
</tr>
<tr>
<td>1</td>
<td>1.9</td>
<td>6</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.1</td>
<td>8</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.2</td>
<td>10.5</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>15</td>
<td>4.5</td>
<td></td>
</tr>
<tr>
<td><strong>Average:</strong></td>
<td><strong>2.05</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Sum:</strong></td>
<td><strong>8.2</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
NUMERICAL MODEL

To create a model of TopBot, Equations (2.14) and (2.19) were used. A summary of these equations is shown in Equation (5.1).

\[
\begin{align*}
\dot{\theta} &= \dot{\theta} \\
\ddot{\theta} &= \left[ \psi^2 \sin(\theta) \cos(\theta) \left( I_i + I'' - I_3 - I' \right) + \sin(\theta) \left( (m^2 + m') Lg - \dot{\phi}' I' \right) \right] / (I_i + I'') \\
\phi &= \left[ \begin{array}{c}
\frac{1}{(I_i + I'')} + \cos^2 \theta \left( I_3 - I_1 - I'' \right) \\
-I \cos \theta \left( I_i + I'' \right) + \cos^2 \theta \left( I_3 - I_1 - I'' \right)
\end{array} \right] \\
\dot{\phi} &= \left[ \begin{array}{c}
-I \cos \theta \left( I_i + I'' \right) + \cos^2 \theta \left( I_3 - I_1 - I'' \right) \\
-I' \left( I_i + I'' \right) + \cos^2 \theta \left( I_3 - I_1 - I'' \right)
\end{array} \right] \left[ \begin{array}{c}
\dot{\beta}_x \\
\dot{\beta}_y
\end{array} \right]
\end{align*}
\]

Using the RK4 method (Bewley) of modeling to create a time stepped representation of the system, it was possible to determine \( \theta, \dot{\theta}, \psi, \) and \( \phi' \) for the system at each time step \( t \). For each state listed, Equation (5.2) was used to step the system through the precession angle \( 0 \leq \psi \leq 8\pi \). This showed the evolution of the system and created a model to compare physical results to.

\[
\begin{align*}
f_1 &= f(x_n, t_n) \\
f_2 &= f\left(x_n + \left( \frac{h}{2} \right) f_1, t_n + \frac{h}{2} \right) \\
f_3 &= f\left(x_n + \left( \frac{h}{2} \right) f_2, t_n + \frac{h}{2} \right) \\
f_4 &= f(x_n + hf_3, t_{n+1}) \\
x_{n+1} &= x_n + h \left( \frac{1}{6} f_1 + \frac{1}{3} f_2 + \frac{1}{3} f_3 + \frac{1}{6} f_4 \right)
\end{align*}
\]
Using this system of modeling, MATLAB was used to create plots of the system as the precession angle evolved from $\psi = 0$ to $\psi = 8\pi$. The script used is shown in A-5.

RK4 Analysis of TopBot The results are shown in Figure 15.

**Figure 15.** States of the TopBot System.

In this figure, (a) shows the nutation angle $\theta$ as a semi-quadratic equation and the rate of change of the nutation angle, $\dot{\theta}$, is then shown to be near linear over each
$2\pi$ cycle of precession (b). It is then also shown that the rotor angle $\phi$, relative to the starting position, is represented by a linear equation showing that the speed of rotation $\dot{\phi}$ is held constant. Finally it is shown that the rate of change of the precession angle $\psi$ is constant by the linear equation of its position $\psi$ shown in (d). From Figure 15 it is also shown that the time taken for TopBot to precess from $0 \leq \psi \leq 8\pi$ (4 revolutions) is 6.96 sec. Comparing this to physical test results, where 4 revolutions took 8.2 sec, there is shown to be a 16% error in results. It should also be noted that TopBot is not in retrograde motion since the angle of precession and rotor spin angle are both changing in the positive direction relative to the body frame for both the physical test and the RK4 model. A second graphical representation is shown in Figure 16. Here the system is shown as decaying since as the precession grows the nutation angle also grows with it.
Figure 16. Combining $\theta$ and $\psi$ to create a top down two dimensional view of TopBot where $0 \leq \psi \leq 8\pi$.

Figure 17 shows a comparison of the nutation angles for the test data and the RK4 analysis in respect to the completed precession cycles. It shows both models starting at approximately a 4 degree angle, at zero completed rotations, and terminating near a 15 degree angle, at 4 completed rotations. It was determined that the test system had sources of error that kept the results from matching theory. The mass and inertial values used in the RK4 analysis were from the computer generated part model and were ideal values. The actual construction of TopBot however incurred real world tolerances that lead to off center masses and transverse inertias that were not perfectly equal. The
accelerometer used to measure the precession angle also relied on imperfect calibration that led to errors up to 1 degree.

**Figure 17.** Comparison of Test Data and RK4 Analysis for Nutation Angle at Completed Precession Cycles.
PROBLEM ISOLATION AND FUTURE CONSIDERATIONS

As Figure 16 demonstrates, TopBot’s angle of nutation will degrade with time. This is due to the unstable model when the inertial values do not abide by Equation (4.1). It does however show that the model can be used to create an approximation for the motion of TopBot. Another reality encountered by TopBot was the impossibility of making a fully axisymmetric body with actuation directly through the center of mass of the body. The inability to create this ideal situation eventually leads to eccentric movement causing TopBot to dip when the mass of the system is offset and not allowing it to fully recuperate the loss of energy from the dip.

Further analysis can still be performed to determine the feasibility of other systems using a similar modeling technique. While this iteration of TopBot was not successful, it does show that TopBot in general can be explored using the given equations of motion. It would be highly recommended to make a smaller scale non-hopping proof of concept model to explore the top dynamics relative to varying rotor spin speeds. This would lead to the ability to not only study a stable system but also one that exhibits other forms of motion such as cuspidal and looping precession. System Robustness could then also be tested by adding and subtracting mass from the system to determine its effect on the system and dynamics. This would create another level of confidence in the model/test relationship.
A-1. Jump Height and Velocity

function [D] = funkyfunc2(F0)
% close all
x = 0; %initialize
v = 0; %initialize
tf = .1; %time to exert force
N = 57; %total number of time steps
dt = .01; %time step
m = 5; %kg
g = 9.8; %m/s^2
hd = .3; %m target height
f = abs(F0(1))
for i = 1:N;
    t(i) = i*dt;
    if (i*dt <= tf);
        x(i) = (f-m*g)*(t(i))^2/(2*m);
        v(i) = (f-m*g)*t(i)/m;
    else
        x(i) = x(tf/dt)+t(i-tf/dt)*v(tf/dt)-g*((t(i)-t(tf/dt))^2/2);
        v(i) = v(tf/dt)-g*(t(i)-t(tf/dt));
    end
end
D = abs(hd-x(length(x)));
A-2. Cold Rocket Thrust

% Cold Thrust Rocket Calculations closeall;
clear; clc;

% Variables % Robot and robot performance parameters
m = 5; % Robot mass, m kg
h = 0.254; % Desired hop height, m kg
g = 9.8; % Local gravity, m/s^2

% CO2 gas properties
R = 188.9; % CO2 gas constant, [J/kg-K]
gamma = 1.3; % CO2 ratio of specific heats, []

% Gas system info
p0 = 2*689e3; % Tank pressure, Pa
pe = 101.325e3; % Exit pressure, Pa - ATMOSPHERIC
Temp0 = 298; % Tank temperature, K
rho0 = p0/(R*Temp0); % Tank density, kg/m^3
cp = gamma*R/(gamma-1); % CO2 specific heat

% FUNCTIONS %%%%%%%%
% Throat density
rhoThroat = @(rho0) rho0*(2/(gamma+1))^(gamma/(gamma-1));
% Throat pressure
pThroat = @(p0) p0*(2/(gamma+1))^(gamma/(gamma-1));
% Throat temperature
TempThroat = @(Temp0) Temp0*(2/(gamma+1));
% Speed of sound, m/s
aSound = @(Temp) sqrt(gamma.*R.*Temp);
% Mach number at exit
Mache = @(p0,pe) sqrt( ( (p0./pe)^((gamma-1)./gamma) - 1 )*(2./(gamma-1)) )
% Mass flow rate, kg/s
mFlow = @(rho,u,A) rho.*u.*A;
% Temperature from local Mach number
TempM = @(M) Temp0./(1+(gamma-1)/2.*M.^2);
% Nozzle Area Ratio (A/A*) from local Mach number
NozzleAreaRatio = @(M) sqrt(1./M.^2*((2./(gamma+1)).*(1+(gamma-1)./2.*M.^2)).^((gamma+1)/(gamma-1)));
% Thrust times % Calculates time to apply given thrust to achieve desired height
% Accounts for gravity
Ttime = @(T) sqrt( (2.*g.*h) / ( (T./m-g).*T/m ) );

% CALCULATIONS %%%%%%%%
Tempt = TempThroat(Temp0);
Thrusts = [50:1:1000]; % Everything parameterized by Thrusts, N
TTimes = Ttime(Thrusts); % Times to apply thrusts
Me = Mache(p0,pe); % Exit Mach number
Tempe = TempM(Me); % Exit Temperature
NARatio = NozzleAreaRatio(Me); % Nozzle output to throat ratio
ut = aSound(Tempt); % Throat at Mach 1
rhot = rhoThroat(rho0); % Throat density
ue = Me*aSound(Tempre); % Exit velocity
ueCheck = sqrt(2*cp*(Temp0-Tempre)); % Check exit velocity using energy analysis
mdots = Thrusts./ue; % Determine mass flow rate from Thrust = mdot*ue
Ats = mdots./rhot./ut; % Throat areas
Dts = sqrt(Ats/pi)*2; % Throat diameters
mperhops = mdots.*TTimes; % Mass CO2 per hop
disp(sprintf('Tank CO2 density: %f kg/m^3',rho0))
disp(sprintf('Throat CO2 density: %f kg/m^3',rhot))
disp(sprintf('Throat velocity: %f m/s',ut))
disp(sprintf('Exit Mach number: %f',Me))
disp(sprintf('Exit velocity: %f m/s',ue))
disp(sprintf('Exit temperature: %f K (%f °C)',Tempe,Tempe-273))
disp(sprintf('Throat temperature: %f K (%f °C)',Tempt,Tempt-273))
disp(sprintf('Nozzle area ratio: %f',NARatio))
%h=figure(6);

subplot(2,2,1);
plot(Thrusts,TTimes*1000);
title('Time to Apply Thrust vs Thrust');
xlabel('Thrust, N');ylabel('Time to Open Nozzle, ms');
a = 300;
b = find(Thrusts >= a,1);
%markPt(h, a, TTimes(b)*1000, 300, 40) % Marks a point on the graph

subplot(2,2,2);
plot(Thrusts,mperhops*1000);
title('CO2 Mass Per Hop vs Thrust');
xlabel('Thrust, N');ylabel('CO2 Mass Per Hop, g');
a = 300; b = find(Thrusts >= a,1);
%markPt(h, a, mperhops(b)*1000, 300, 15.5)

subplot(2,2,3);
plot(Thrusts,mdots);
title('Mass Flow Rate vs Thrust');
xlabel('Thrust, N');ylabel('Mass Flow Rate, kg/s');
a = 300; b = find(Thrusts >= a,1);
%markPt(h, a, mdots(b), 300, 0.45)

subplot(2,2,4);
plot(Thrusts,Dts*100);
title('Nozzle Throat Diameter vs Thrust');
xlabel('Thrust, N');ylabel('Nozzle Throat Diameter, cm');
a = 300; b = find(Thrusts >= a,1);
%markPt(h, a, Dts(b)*100, 300, 1.4)
A-3. Number of CO2 Canister Shots

\%P_1*V_1 = n*R*T, \: P_2*V_2 = n*R*T

\begin{align*}
P_1 &= 800 \: \text{Pressure of tank (psi)} \\
V_1 &= 3 \: \text{Volume of tank (in}^3) \\
P_2 &= 100 \: \text{Pressure of actuator (psi)} \\
V_2 &= (\tfrac{.75}{2}^2*3.14*3) \: \text{Volume of actuator (in}^3)
\end{align*}

\text{poofs} = 0

\textbf{while } P_1 > P_2 + 100 \\
\quad P_1 = P_1 - \frac{P_2*V_2}{V_1} \\
\quad \text{poofs} = \text{poofs} + 1
\textbf{end}

\% \text{shaft} \_\text{length} = 2.5 \%(\text{in})
\% \text{actuator} \_\text{body} = 5.75 \%(\text{in})
\% \text{actuator} \_\text{length} = \text{shaft} \_\text{length} + \text{actuator} \_\text{body} \%(\text{in})

A-4. Preliminary Spin Speed of Rotor

\textbf{function } [\gamma \_\text{dot}] = \text{topbot} \_\text{rotor} \_\text{size}
\text{\%use metric}
\text{\%-----Initialization (in standard units)}
\quad g = 9.8 \: \text{m/s}^2 \\
\quad \theta = \pi/6 \: \text{newtation angle degrees} \\
\quad L = .1016 \: \text{length to CM (m)}

\begin{align*}
m_b &= 3 \: \text{mass of body (kg)} \\
r_b &= .06 \: \text{radius of body} \\
h_b &= .177 \: \text{height of body}
\end{align*}

\begin{align*}
r_1 &= .1542/2 \: \text{inner radius} \\
r_2 &= .2032/2 \: \text{outer radius} \\
h_r &= .0127 \: \text{height of ring} \\
\rho_{\text{steel}} &= 7600 \: \text{kg/m}^3 \\
m_r &= \pi*(r_2^2-r_1^2)*h_r*rho_{\text{steel}} \: \text{mass of ring (kg)}
\end{align*}

\text{\%-----Inertias from CM}
\begin{align*}
J_{b} &= m_b*r_b^2/2 \\
J_{bt} &= m_b/12*(3*r_b^2+h_b^2) \\
J_f &= m_r/2*(r_1^2+r_2^2) \\
J_{rt} &= m_r/12*(3*(r_2^2+r_1^2)+h_r^2)
\end{align*}

\%from parallel axis theorem
\[ J_{bt} = J_{bt} + m_bL^2 \]
\[ J_{rt} = J_{rt} + m_rL^2 \]

\%-----gamma_dot calc
\[ I_{diff} = (J_{rt}+J_{bt}) - (J_r+J_b) \]
\[ \gamma_{dot} = 2\left(\frac{\pi}{J_{diff}}(m_r+m_b)gL\cos(\theta)\right)^{1/2}/(J_r)(60/(2\pi)) \%rpm \]

\textbf{A-5. RK4 Analysis of TopBot}

\texttt{function [theta, theta_dot_save, phi_p_save, psi_save] = RK4_Top_Bot}
\texttt{clear all, close all, clc}
\texttt{g = 9.8; \%m/s^2}
\texttt{theta_0 = 15*pi/180; \% newtation angle radians}
\texttt{phi_p_dot_0 = 62.8; \%rad/sec}
\texttt{L = .1016*.6; \%length to C (m)}
\texttt{m_b = 2.6; \%kg}
\texttt{m_r = 1.327;}
\texttt{J_{bt} = 0.0243; \%kgm^2}
\texttt{J_{rt} = 0.01489;}
\texttt{J_{rt} = 0.010906;}
\texttt{J_{rt} = 0.005469;}
\texttt{\%By parallel axis thrm}
\texttt{J_{bt} = J_{bt} + m_bL^2}
\texttt{J_{rt} = J_{rt} + m_rL^2}

\%psi dot calc
\texttt{a = 2*(J_{bt}+J_{rt}+J_{b}+J_{r})*cos(theta_0);}
\texttt{b = -phi_p_dot_0*J_r;}
\texttt{c = (m_b+m_r)*L^2;g;}
\texttt{psi_dot_pos = (-b + sqrt(b^2-4*a*c))/(2*a);}
\texttt{psi_dot_neg = (-b - sqrt(b^2-4*a*c))/(2*a);}
\texttt{if abs(psi_dot_pos)<abs(psi_dot_neg), psi_dot_0 = psi_dot_pos, else,...}
\texttt{ psi_dot_0 = psi_dot_neg, end}

\%Initialize States
\texttt{theta = theta_0;}
\texttt{theta_dot = 0;}
\texttt{phi_p = 0;}
\texttt{phi_p_dot = phi_p_dot_0;}
\texttt{psi = 0;}
\texttt{psi_dot = psi_dot_0;}
\%Beta terms for psi and phi
\( B_{\psi} = \psi_{dot} (J_{bt} + J_{rt}) + (\cos(\theta))^2 (J_b + J_r - J_{bt} - J_{rt}) \psi_{dot} \)

\( B_{\phi_{p}} = \psi_{dot} (J_{rt} \cos(\theta)) + \phi_{p \ dot} J_{r} \)

\%
\%

\%RK4
\n\% Set parameters
\n\% Create values to save data
\n\% Derivatives of the states
\n\% Theta dot
\n\% Theta double dot
\n\% Place Holder
\n\% Phi prime dot
\n\% Place Holder
\n\% Place Holder
\[ f_{C4} = -\cos(\theta + h \cdot f_{A3}) \cdot B_{\psi}/X + ((J_{brt} + J_{rʈ}) + (\cos(\theta + h \cdot f_{A3}))^2 \cdot (J_b + J_r - J_{brt} - J_{rʈ})) \cdot B_{\phi_p}/(J_r \cdot X); \]
\[ f_{D4} = B_{\psi}/X - \cos(\theta + h \cdot f_{A3}) \cdot B_{\phi_p}/X; \]

\%Step System States Forward
\begin{align*}
\theta &= \theta + h \cdot (f_{A1}/6 + (f_{A2} + f_{A3})/3 + f_{A4}/6); \\
\theta_{\dot{}} &= \theta_{\dot{}} + h \cdot (f_{B1}/6 + (f_{B2} + f_{B3})/3 + f_{B4}/6); \\
\phi_{p\dot{}} &= \phi_{p\dot{}} + h \cdot (f_{C1}/6 + (f_{C2} + f_{C3})/3 + f_{C4}/6); \\
\psi &= \psi + h \cdot (f_{D1}/6 + (f_{D2} + f_{D3})/3 + f_{D4}/6); \\
\end{align*}

\[ t = t + h; \]
\end{normalsize}

figure(1), hold on
title('Evolution of Nutation Angle')
xlabel('time (sec)'), ylabel('Angle (rad)')
plot(t_save, theta_save)
figure(2), hold on
title('Rate of Change of Nutation angle')
xlabel('time (sec)'), ylabel('Angular Velocity (rad/s)')
plot(t_save, theta_dot_save)
figure(3), hold on
title('Evolution of Rotor Angle')
xlabel('time (sec)'), ylabel('Angle (rad)')
plot(t_save, phi_p_save)
figure(4), hold on
title('Evolution of Precesion Angle')
xlabel('time (sec)'), ylabel('Angle (rad)')
plot(t_save, psi_save)
figure(5), hold on
title('Top Down View (XY-Plane) of Center of Mass')
xlabel('X (meters)'), ylabel('Y (meters)')
plot(L \cdot \sin(\theta_{\text{save}}) \cdot \cos(\psi_{\text{save}}), L \cdot \sin(\theta_{\text{save}}) \cdot \sin(\psi_{\text{save}}))

A-6. Nutation Angle from Accelerometer
#include <Wire.h>
//SDA Analog Pin 4
//SCL Analog Pin 5
//This will use the small angle approximation on the tilt and roll data to make one
//angle of newtation. All data will be given in degrees. This file should be seen as an
//extension to "tilt_actuated_compass".
int compassAddress = 0x32 >> 1;
int heading = 0; // variable to hold the heading angle
int tilt = 0;  // variable to hold the tilt angle
int roll = 0;  // variable to hold the roll angle
int abs_tilt = 0;
int abs_roll = 0;
int WLED = 13;
int nutation = 0;
byte responseBytes[6];  // for holding the sensor response bytes
void setup()
{
    delay(500);  //Wait at least 500 milli-seconds for device initialization
    Wire.begin();       // join i2c bus (address optional for master)
    Serial.begin(9600); // start serial communication at 9600bps
    pinMode(WLED, OUTPUT);
    digitalWrite(WLED, HIGH); // just turn ON the onboard LED
}
void loop()
{
    readSensor();  // read data from the HMC6343 sensor
    // Note that heading, tilt and roll values are in tenths of a degree, for example
    // if the value of heading is 1234 would mean 123.4 degrees, that's why the result
    // is divided by 10 when printing.
    abs_tilt = abs(tilt)/10;
    abs_roll = abs(90-abs(roll/10));
    nutation = acos(cos(abs_tilt*.0174)*cos(abs_roll*.0174))*60;
    //nutation = acos(sqrt(square(cos(abs_tilt*.0174))+square(cos(abs_roll*.0174))))*60;
    Serial.print("abs_tilt: ");
    Serial.print(abs_tilt, DEC);
    Serial.print("abs_roll: ");
    Serial.print(abs_roll, DEC);
    Serial.print("Nutation: ");
    Serial.print(nutation, DEC);
    Serial.print("Heading: ");
    Serial.println(heading / 10, DEC);
    // Serial.println(" Tilt: ");
    // Serial.println(tilt / 10, DEC);
    // Serial.println(" Roll: ");
    // Serial.println(roll / 10, DEC);
    delay(500);                   // wait for half a second
}
void readSensor()
{
    // step 1: instruct sensor to read echoes
    Wire.beginTransmission(compassAddress); // transmit to device
    // the address specified in the datasheet is 66 (0x42)
    // but i2c addressing uses the high 7 bits so it's 33
    Wire.write(byte(0x50)); // Send a "Post Heading Data" (0x50) command to the
    HMC6343
    Wire.endTransmission(); // stop transmitting
    // step 2: wait for readings to happen
    delay(2); // datasheet suggests at least 1 ms
    // step 3: request reading from sensor
    Wire.requestFrom(compassAddress, 6); // request 6 bytes from slave device #33
// step 4: receive reading from sensor
if (6 <= Wire.available()) // if six bytes were received
{
  for (int i = 0; i<6; i++) {
    responseBytes[i] = Wire.read();
  }

  heading = ((int)responseBytes[0] << 8) | ((int)responseBytes[1]); // heading MSB and LSB
}
BIBLIOGRAPHY


