Title
THE VARIATION OF THE MOYER MODEL PARAMETER, H₀, WITH PRIMARY PROTON ENERGY

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Author
Liu, Kuei-Lin.

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Liu Kuei-Lin, Graham R. Stevenson, Ralph H. Thomas, and Simon V. Thomas

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THE VARIATION OF THE MOYER MODEL PARAMETER, $H_0$, WITH PRIMARY PROTON ENERGY

Liu Kuei-Lin
Institute of High Energy Physics, Academia Sinica, Beijing
People's Republic of China

Graham R. Stevenson
CERN
Geneva, Switzerland

Ralph H. Thomas
Lawrence Berkeley Laboratory and School of Public Health
University of California
Berkeley, California 94720 USA

and

Simon V. Thomas
California State University, Fresno, California 93740 USA
and
Lawrence Berkeley Laboratory, University of California,
Berkeley, California 97420 USA

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"Experience Joined With Common Sense, to Mortals is a Providence"

From,
"The Spleen"
Matthew Green
1696-1737
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ABSTRACT

Experimental values of the Moyer Model Parameter $H_0$ were summarized and presented as a function of proton energy, $E_p$. The variation of $H_0(E_p)$ with $E_p$ was studied by regression analysis. Regression Analysis of the data under log-log transformation gave a best value for the exponent $m$ of $0.77 \pm 0.26$, but a t-test did not reject $m = 1$ ($p \leq 20$ percent). Since $m = 1$ was not excluded, and a Fisher's F-test did not exclude linearity, a linear regression analysis was performed. A line passing through the origin was not rejected (Student's t-test, $p = 30$ percent) and has the equation: $H_0(E_p) = (1.61 \pm 0.19) \times 10^{-13} \text{Sv}\cdot\text{m}^2/\text{GeV}$ to be compared with a value of $(1.65 \pm 0.21) \times 10^{-13} \text{Sv}\cdot\text{m}^2/\text{GeV}$ published by Stevenson et al. (St 82).
1. INTRODUCTION

High-energy proton accelerators must be surrounded by substantial shielding to protect, for example: operators; scientists; hospital staff and members of the general public (Pa 73). If the accelerator is above ground this shielding generally consists of concrete or steel structures and the cost of providing shielding can be a significant fraction of the total cost of the facility. Some larger accelerators have been constructed underground in order to reduce shielding costs but even then excavation costs are high and the depth below ground level must be optimized.

Detailed shield design using sophisticated Monte-Carlo radiation transport calculations can be expensive and is used only for the final design stages of a large proton accelerator. Semi-empirical calculations are often employed in the preliminary design phase.

Moyer was one of the first to use such a technique in the design of the shielding for the Bevatron (Mo 62). The basic equation used by Moyer to calculate the radiation intensity, H, outside a shield around a target bombarded by protons is:

\[ H = \frac{1}{r^2} H_0 \exp(-\beta \theta) \exp(-d/\lambda) \]  

(1)

where the symbols \( r \), \( \theta \), and \( d \) are explained in Figure 1. The angular distribution parameter, \( \beta \), and the attenuation length, \( \lambda \), are well determined both by theoretical and experimental means (Le 72, Pa 73, St 82).

The parameter \( H_0 \), which is the subject of this paper, may be determined empirically. (It does, in fact, have a complex relation to several parameters related to particle production in the target and their transport through the shield.)

Recently Stevenson et al., have summarized experimental determinations of \( H_0 \) at several different proton accelerators (St 82).
2. **EXPERIMENTAL DATA**

Table 1 summarizes the experimental determinations of the Moyer Parameter, \( H_0 \), discussed by Stevenson et al. (St 82) and, in addition, includes the value reported by Routti and Thomas (Ro 69), as corrected by Stevenson et al. These data are plotted in Figures 2 and 3. Table 1 also indicates the original sources of these data. The value of \( H_0 \) reported are clearly independent, having largely been obtained at different accelerators by different experimenters and by different experimental techniques. The group of data obtained at the CERN PS in 1966 (Gi 68) were obtained by separate experiments.

<table>
<thead>
<tr>
<th>Primary Proton Energy, ( [E_p] )</th>
<th>Moyer Parameter, ( [H_0(E_p)] )</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>(GeV)</td>
<td>(Sv m(^2))</td>
<td></td>
</tr>
<tr>
<td>7.4</td>
<td>1.4 10^{-12}</td>
<td>Sh69, St69</td>
</tr>
<tr>
<td>7.4</td>
<td>2.1 10^{-12}</td>
<td>Sh69, St69</td>
</tr>
<tr>
<td>10.0</td>
<td>0.96 10^{-12}</td>
<td>Ho 66</td>
</tr>
<tr>
<td>13.7</td>
<td>2.5 10^{-12}</td>
<td>Gi 68</td>
</tr>
<tr>
<td>13.7</td>
<td>3.1 10^{-12}</td>
<td>Gi 68</td>
</tr>
<tr>
<td>21.0</td>
<td>1.6 10^{-12}</td>
<td>Ho 79</td>
</tr>
<tr>
<td>23.0</td>
<td>3.5 10^{-12}</td>
<td>Ma 79</td>
</tr>
<tr>
<td>25.5</td>
<td>3.3 10^{-12}</td>
<td>Gi 68</td>
</tr>
<tr>
<td>25.5</td>
<td>5.0 10^{-12}</td>
<td>Gi 68</td>
</tr>
<tr>
<td>25.5</td>
<td>6.6 10^{-12}</td>
<td>Ro69, St82</td>
</tr>
<tr>
<td>30.0</td>
<td>3.4 10^{-12}</td>
<td>Aw 70</td>
</tr>
</tbody>
</table>
3. **Energy Variation of $H_0$**

It is important to understand the variation of $H_0$ with proton energy, both so that the experimental determinations of $H_0$ at various proton energies may be combined to permit accurate interpolation and perhaps, more importantly, to allow extrapolation to higher energies. Such a need arose, for example in the design of shielding for the 50 GeV Beijing Proton Synchrotron (Ch 80, Li 79).

Since the principle use of the Moyer Model is in the calculation of transverse shielding, we are interested in the global production of neutrons at large angles to the interaction target, as determined outside substantial shielding. At energies below 1 GeV there is evidence that the global production of neutrons is roughly proportional to neutron energy (for a summary see Pat 73). If an exponential variation of the form:

$$H_0 = cE_p^m$$  \((2)\)

is assumed a value of $m = 1$ sets an upper limit to the variation of neutron production with proton energy and this is therefore a conservative assumption for extrapolating the experimental determinations of $H_0$ to higher energies.

There has been some speculation in the literature as to the value of the coefficient $m$. Lindenbaum pointed out that the production of shower particles varied as $E^{0.25}$ and suggested a value of $m = 0.50$ for fast nucleons, intermediate between that for shower particles and low energy neutrons (Li 61, Pa 73). The data obtained from Monte-Carlo calculations of the hadron cascades generated in matter by high-energy protons suggest a value of $m = 0.75$ (Fe 72).

Until recently there were insufficient experimental data to empirically investigate the relationship between $H_0$ and $E_p$, but the experimental data of Table 1 now make this possible. The experimental data are shown plotted on log-log graph paper in Figure 2 and on linear paper on Figure 3. The regression analysis of these data is the subject of this note.
Fig. 2
Fig. 3
4. DATA ANALYSIS

The number of data points severely limit the analysis and our purpose here is to show that the assumption of linearity between the random variables \( H_p (E_p) \) and \( E_p \) is not excluded by the experimental data. The progression of statistical tests used was:

(a) Fisher's F-test of the hypothesis of linearity of the data under log-log transformation.
(b) Regression analysis to determine the best value of the coefficient \( m \).
(c) Student's t-Test of log-log transformed data for \( m = 1 \).
(d) Regression analysis under the assumption \( H_0 (E_p) = a + bE_p \).
(e) Student's t-Test for \( a = 0 \).
(f) Linear regression analysis with fit forced through the origin: \( H_0 (E_p) = b'E_p \).
(g) Analysis of variance techniques to calculate 95 percent confidence bands to regression lines.

4.1 REGRESSION ANALYSIS OF LOG-LOG TRANSFORMED DATA

A Fisher's F-test (F_70, S_n 80) of the log-log transformed data does not reject linearity and a regression analysis of the data gives:

\[
H_0 (E_p) = 3.07 E_{0.769}^p
\]

with

\[ S_m = \pm 0.257 \]

A Student's t-test does not reject \( m = 1.0 \) (p=20 percent).

For details see the analysis in Appendix 1.
4.2 **LINEAR REGRESSION ANALYSIS**

Since \( m = 1 \) is not excluded, and a Fisher's F-Test does not exclude linearity a linear regression analysis assuming \( H_0(E_p) \) was performed giving:

\[
H_0(E_p) = 5.22 \times 10^{-13} + (1.37 \times 10^{-13})E_p
\]

The estimated variance of the intercept, \( S_a \), is:

\[
S_a = \pm 9.86 \times 10^{-13}
\]

Details of the regression analysis are given in Appendix 2. A Student's t-test does not reject \( a = 0 \) (\( p = 30 \% \)).

4.3 **LINEAR REGRESSION ANALYSIS FORCED THROUGH THE ORIGIN**

Finally, a regression analysis with the line forced through the origin:

\[
H_0(E_p) = b'E_p
\]

gives:

\[
\hat{b'} = 1.608 \times 10^{-13}
\]

\[
S_{\hat{b'}} = \pm 0.47 \times 10^{-13}
\]

For details of the calculation see Appendix 3.

Figure 3 shows the experimental data, the three lines obtained by regression analysis and the 95 percent confidence band to the lines obtained by log-log and linear regression analysis.
5. **CONCLUSIONS**

The experimental data may be fitted by straight lines either in linear or log-log transformation. The best value of the coefficient, $m$, is $0.77 \pm 0.26$, but linearity is not excluded by a Student's $t$-test ($p = 20$ percent). A straight line forced through the origin has a slope $(1.61 \pm 0.19) \times 10^{-13}$ Sv·m$^2$/GeV, to be compared with a value of $(1.65 \pm 0.21) \times 10^{-13}$ Sv·m$^2$/GeV given by Stevenson et al. (St 82).*

6. **ACKNOWLEDGMENTS**

The authors would like to thank their colleagues who have provided the data discussed in this paper. We gratefully acknowledge the advice and encouragement of Prof. Chang Wen-Yu of The Institute of High Energy Physics, Beijing; K. Goebel of CERN, and W.D. Hartsough of the Lawrence Berkeley Laboratory during the preparation of this paper. One of us (S.V.T.) would like to thank Prof. J. Coffey, of the Department of Information Systems and Decision Sciences, California State University, Fresno, for his instruction in Statistical Methods, to M. Simmons of the Computer Center, Lawrence Berkeley Laboratory for much useful advice and encouragement and to the Computer Center at LBL for a Summer Studentship which made this work possible.

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*In coherent SI units: the slope of the straight line forced through the origin is $(1.00 \pm 0.12) \times 10^{-3}$ Sv·m$^2$/J, to be compared with a value of $(1.03 \pm 0.13) \times 10^{-3}$ Sv·m$^2$/J given by Stevenson et al. (St 82).
REFERENCES


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APPENDIX 2. Linear Regression analysis and hypothesis testing ........... 21
APPENDIX 3. Regression analysis with line forced through the origin... 27
APPENDIX 1. REGRESSION ANALYSIS AND HYPOTHESIS TESTING USING A LOG-LOG TRANSFORMATION

In this appendix the data of Table A1-1 are first tested for the hypothesis of linearity using a Fisher's F-Test; the data are then subjected to regression analysis to determine the equation of the line fitted to the log-log transformed data; finally, the hypothesis that the estimated slope of the regression line is consistent with the value $m = 1$ is tested using a Student's $t$-test.

A1.1. LOG-LOG TRANSFORMATION

As discussed in the main body of the test there are good reasons for first investigating the regression of $\log_{10} H_0(E_p)$ on $\log_{10} E_p$ of the data shown in the scatter-diagram of Fig. 2. In this appendix, for simplicity, the variables $\log_{10} E_p$ and $\log_{10}^{10^{13}} (H_0(E_p))$ will be designated by $x$ and $y$ respectively.
A1.2. **NUMERICAL DATA**

Table A1-1. Numerical data for determination of regression line and for test of linearity log-log transformed data.

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>( E_0 ) (GeV)</th>
<th>( H_0(E_p) ) ((S_y m^2))</th>
<th>( x_{ij} )</th>
<th>( y_{ij} )</th>
<th>( \bar{y}_i )</th>
<th>( n_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>7.4</td>
<td>14\times10^{-13}</td>
<td>0.8692</td>
<td>1.1461</td>
<td>1.2342</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>7.4</td>
<td>21</td>
<td>1.3222</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>10.0</td>
<td>9.6</td>
<td>1.0000</td>
<td>0.9823</td>
<td>0.9823</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>13.7</td>
<td>25</td>
<td>1.1367</td>
<td>1.3979</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>13.7</td>
<td>31</td>
<td>1.4914</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>21.0</td>
<td>16</td>
<td>1.3222</td>
<td>1.2041</td>
<td>1.2041</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>23.0</td>
<td>35</td>
<td>1.3617</td>
<td>1.5441</td>
<td>1.5441</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>25.5</td>
<td>33</td>
<td>1.4055</td>
<td>1.5185</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>25.5</td>
<td>50</td>
<td>1.6990</td>
<td>1.6790</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>25.5</td>
<td>66</td>
<td>1.8195</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>30.0</td>
<td>34</td>
<td>1.4771</td>
<td>1.5315</td>
<td>1.5315</td>
<td>1</td>
</tr>
</tbody>
</table>

\( x = \log_{10}E_p \)

\( y = \log_{10}10^{13} \cdot H_0(E_p) \)
A1.3. **SUMMATIONS**

The basic summations of the data of Table Al-1 required for regression analysis and hypothesis testing are:

\[
\begin{align*}
\sum x &= \sum \sum x_{ij} = 13.3923 \\
\sum y &= \sum \sum y_{ij} = 15.6566 \\
\sum x^2 &= \sum \sum (x_{ij})^2 = 16.81418195 \\
\sum y^2 &= \sum \sum (y_{ij})^2 = 22.88768108 \\
\sum xy &= \sum \sum (x_{ij}y_{ij}) = 14.45320977 \\
N &= \sum n_i = 11 \\
\bar{x} &= 1.21748188 \\
\bar{y} &= 1.423327273
\end{align*}
\]

A1.4. **CALCULATIONS**

A1.4.1. **Regression Line Parameters**

The estimated values of constant, \( \hat{k} \), and slope, \( \hat{m} \), (Eq. 2) are given by:

\[
\hat{m} = \frac{\sum xy - \frac{1}{N} \sum \sum x \sum y}{\sum x^2 - \frac{1}{N} (\sum x)^2} = 0.768886239 \\
\hat{k} = \frac{\sum y - \hat{m} \sum x}{N} = 0.487219357
\]  

(A1-1)

A1.4.2. **Error Variance and Standard Error on \( \hat{m} \)** are given by:

\[
\begin{align*}
S^2_{y|x} &= \frac{1}{N - 2} [\sum y^2 - \hat{\kappa} \sum xy - \hat{m} \sum x y] = 0.0335698066 \\
S^2_m &= \frac{S^2_{y|x}}{x^2 - N(x)^2} = 0.065913594 \\
S_m &= 0.2567
\end{align*}
\]  

(A1-2)  

(A1-3)
A1.4.3. **Analysis of Variance (Partitioning Sums of Squares)**

Total sum of squares = $\sum y^2 - \frac{1}{N} (\sum y)^2 = 0.60322$

Regression sum of squares = $\hat{m}^2 [\sum x^2 - \frac{1}{N} (\sum x)^2] = 0.30136 \quad (A1-4)$

Residual sum of squares = Total SS - Regression SS = 0.30136

Within group sum of squares = $\sum y^2 - \sum_{i} \frac{1}{n_i} (\sum_{ij} y_{ij})^2 = 0.06578$

About regression SS = Residual SS - Within SS = 0.23608

A.5. **TEST FOR LINEARITY**

The experimental data of Table A1-1 have more than one determination of the y-values ($H_0(E_p)$) at energies of 7.4, 13.7 and 25.5 GeV and the assumption of linearity of the log-log transformed data may be tested.

Table A1-1 presents the data grouped into 7 compartments and the summations necessary for the test of linearity are given in Section A1.3. Table A1-2 summarises the analysis of variance data. Linearity may be tested using a Fisher's F-test. The null hypothesis, $HYP(0)$, is:

$HYP(0)$: $E(y|x) = k + mx$ or regression is linear
$HYP(1)$: not linear

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>0.30136</td>
<td>1</td>
<td>0.30136</td>
</tr>
<tr>
<td>About Regression</td>
<td>0.23608</td>
<td>$k-2 = 5$</td>
<td>0.04722</td>
</tr>
<tr>
<td>Within Group</td>
<td>0.06578</td>
<td>$N-k = 4$</td>
<td>0.01644</td>
</tr>
<tr>
<td>Residual</td>
<td>0.30186</td>
<td>$N-2 = 9$</td>
<td>0.03354</td>
</tr>
<tr>
<td>Total</td>
<td>0.60322</td>
<td>$N-1 = 10$</td>
<td></td>
</tr>
</tbody>
</table>

The Test-Statistic is given by:

$$F = \frac{\text{About Regression Mean Square}}{\text{Within Group Mean Square}}$$

$$= \frac{0.04722}{0.01644} = 2.90$$

The critical region with a level of significance, $\alpha = 0.05$, $N = 11$ and $k = 7$ is $F > F_{0.05,(5,4)} = 6.26$, and thus the data do not support rejection of the hypothesis that the relation between $x$ and $y$, $(\log_{10} H_{0}(E_p)$ and $\log_{10} E_p)$, is linear.

A1.6. EQUATION OF TRANSFORMED LINE

We have:

$$y = \hat{k} + \hat{m}x$$

(A1-10)
Remembering that
\[ y = \log_{10} 10^{13} H_0(E_p) \]
\[ x = \log_{10} E_p \]  \hspace{1cm} (A1-11)

Then:
\[ H_0(E_p) = (10^{-13} \cdot 10^k) E_p^m \]  \hspace{1cm} (A1-12)

Substituting the values for \( k \) and \( m \) we obtain:
\[ H_0(E_p) = (3.07 \times 10^{-13}) E_p^{0.769} \]  \hspace{1cm} (A1-13)

This line is drawn on Fig. 2. The confidence band for this regression line is calculated in the following section.

A1.7. CALCULATION OF CONFIDENCE BAND FOR REGRESSION LINE

The boundaries for a \((1 - \alpha)\) confidence level band may be calculated from:

Upper limit: \( \hat{y}_\ell + c \hat{S}_{\hat{y}_\ell} \) \hspace{1cm} (A1-14a)

Lower limit: \( \hat{y}_\ell - c \hat{S}_{\hat{y}_\ell} \) \hspace{1cm} (A1-14b)

Where \( c \hat{S}_{\hat{y}_\ell} \) is given by:
\[ c^2 \hat{S}_{\hat{y}_\ell}^2 = 2F_{2, N-2} \hat{S}_x^2 \left[ \frac{1}{N} + \frac{(x_i - \bar{x})^2}{\sum(x_i - \bar{x})^2} \right] \]  \hspace{1cm} (A1-15)
Substituting:

\[ F_{0.95(2,9)} = 4.26 \quad \text{for} \quad \alpha = 0.05 \]

\[ S^2_{y|x} = 0.03357 \]

\[ N = 11 \]

\[ \Sigma (x_i - \bar{x}) = 0.5093 \]

\[ \bar{x} = 1.217482 \]

We obtain:

\[ c \sqrt{S^2_{y|x}} = \sqrt{0.02598 + 0.561120(x - 1.21748)^2} \quad (A1.16) \]

Table A1-3 summarizes the values of lower and upper bounds calculated using Eqs. (A1-14), (A1-15) and (A1-16).

<table>
<thead>
<tr>
<th>( E_p ) (GeV)</th>
<th>( H_0(E_p) ) (Lower Limit) ( S_v \cdot m^2 \times 10^{13} )</th>
<th>( H_0(E_p) ) (Upper Limit) ( S_v \cdot m^2 \times 10^{13} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.03</td>
<td>25.9</td>
</tr>
<tr>
<td>2</td>
<td>4.02</td>
<td>26.5</td>
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<td>10.6</td>
<td>27.8</td>
</tr>
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<td>10</td>
<td>16.9</td>
<td>30.5</td>
</tr>
<tr>
<td>15</td>
<td>20.6</td>
<td>35.9</td>
</tr>
<tr>
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<td>23.5</td>
<td>45.7</td>
</tr>
<tr>
<td>30</td>
<td>25.0</td>
<td>75.0</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td>154</td>
</tr>
</tbody>
</table>
A1.8. **TEST OF HYPOTHESIS m = 1.0**

Having established that the experimental data under log-log transformation are consistent with the hypothesis of linearity, and having calculated the best value of slope, \( \hat{m} \), as 0.77 ± 0.26 we now proceed to test the hypothesis that this estimated value of \( \hat{m} \) differs from the upper limit value of 1.0 due only to random variations.

**Hypothesis:**

- **HYP(0):** \( m_0 = 1 \)
- **HYP(1):** \( m_0 < 1 \)

**Test statistic:**

\[
T = \frac{\hat{m} - m_0}{S_{\hat{m}}} \tag{A1-17}
\]

Substituting the values \( \hat{m} = 0.769 \) and \( S_{\hat{m}} = 0.257 \) from Section A1-4:

\[
T = \frac{0.769 - 1.0}{0.257} = -0.90
\]

The critical region at a level of significance of 5 percent, df = 9 is \( T < -2.26 \). Since the calculated value of \( T \) is not less than \(-2.26\), the null hypothesis is not rejected. The experimental data are not inconsistent with the value of \( b = 1.0 \). The probability of observing a value of \( T \) less than \(-0.90 \) (one tail test) is 20 percent.
APPENDIX 2. LINEAR REGRESSION ANALYSIS AND HYPOTHESIS TESTING

Having shown in Appendix 1 that linearity \( m = 1.0 \) is not excluded linear regression can be carried out. In this appendix the data are tested for linearity, subjected to regression analysis \( H_0(E_p) = \hat{a} + \hat{b}E_p \) and the hypothesis \( \hat{a} = 0 \) tested. For convenience, throughout this appendix the variables \( H_0(E_p) \) and \( E_p \) are denoted by \( y \) and \( x \) respectively.

A2.1. NUMERICAL DATA

Table A2-1 summarizes the experimental data drawn up into 7 groups (as was done in Table A1-1) for the original measurements given in Table 1.

Table A2-1. Numerical data for determination of regression line and for hypothesis testing.

<table>
<thead>
<tr>
<th>Group</th>
<th>Energy Parameter ( x ) (GeV)</th>
<th>Moyer Parameter ( y ) (Sv m(^2))</th>
<th>( n_j )</th>
<th>( \sum x_{ij} )</th>
<th>( \sum x_{ij}^2 )</th>
<th>( \sum y_{ij} )</th>
<th>( \sum (y_{ij})^2 )</th>
<th>( \sum x_{ij} y_{ij} )</th>
<th>( n_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.4</td>
<td>14x10^{-13}</td>
<td>14.8</td>
<td>109.52</td>
<td>35</td>
<td>637</td>
<td>259</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7.4</td>
<td>21x10^{-13}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10.0</td>
<td>9.6x10^{-13}</td>
<td>1</td>
<td>100</td>
<td>9.6</td>
<td>92.16</td>
<td>96</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>13.7</td>
<td>25x10^{-13}</td>
<td>27.4</td>
<td>375.38</td>
<td>56</td>
<td>1586</td>
<td>767.2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>13.7</td>
<td>31x10^{-13}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>21</td>
<td>16x10^{-13}</td>
<td>21</td>
<td>441</td>
<td>16</td>
<td>256</td>
<td>336</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>23</td>
<td>35x10^{-13}</td>
<td>23</td>
<td>529</td>
<td>35</td>
<td>1225</td>
<td>805</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>25.5</td>
<td>33x10^{-13}</td>
<td>76.5</td>
<td>1950.75</td>
<td>149</td>
<td>7945</td>
<td>3799.5</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>25.5</td>
<td>50x10^{-13}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>25.5</td>
<td>66x10^{-13}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>30</td>
<td>34x10^{-13}</td>
<td>30</td>
<td>900</td>
<td>34</td>
<td>1156</td>
<td>1020</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
A2.2. SUMMATIONS

The summation data required for regression analysis and hypothesis testing are:

\[ \Sigma x = \sum_{i,j} x_{ij} = 202.7 \]
\[ \Sigma y = \sum_{i,j} y_{ij} = 334.6 \times 10^{-13} \]
\[ \Sigma x^2 = \sum_{i,j} x_{ij}^2 = 4405.65 \]
\[ \Sigma y^2 = \sum_{i,j} y_{ij}^2 = 12897.16 \times 10^{-26} \]
\[ \Sigma xy = \sum_{i,j} x_{ij}y_{ij} = 7082.7 \times 10^{-13} \]
\[ N = \sum_{i} n_i = 11 \]
\[ \bar{x} = 18.4272723 \]
\[ \bar{y} = 30.418182. \]

A2.3. CALCULATIONS

A2.3.1. Regression Line Parameters

Substituting the appropriate sums into the regression line equations (see A1.4.1) we obtain:

\[ \hat{a} = 5.215990555 \times 10^{-13} \]
\[ \hat{b} = 1.367657148 \times 10^{-13} \]

A2.3.2. Error Variance and Estimated Variance of \( \hat{a} \) and \( \hat{b} \)

\[ S^2_{y|x} = \frac{1}{N - 2} \left[ \sum y^2 - \hat{a} \sum xy - \hat{b} \sum x^2 \right] = 162.7982531 \times 10^{-26} \quad \text{(A1-2)} \]

\[ S^2_{y|x} = 1.275924 \times 10^{-12} \]

\[ S^2_a = \frac{S^2_{y|x}}{\sum x^2 - N(\bar{x})^2} = 2.42822799 \times 10^{-27} \]

\[ S^2_a = 4.927700071 \times 10^{-14} \]
The estimated variance of $\hat{a}$, $S^2_{\hat{a}}$, is given by:

$$S^2_{\hat{a}} = S^2_y x \left[ \frac{1}{N} + \frac{x^2}{\sum (x_{ij} - \bar{x})^2} \right] = 9.725365962 \times 10^{-25}$$  \hspace{1cm} \text{(A2-1)}$$

$$S^2_{\hat{a}} = 9.86172701 \times 10^{-13}$$

A2.3.2. Analysis of Variance (Partitioning Sums of Squares)

The formulae for calculating the various sums of squares are given in Section A1.4.3.

Total sums of squares = $2.71924 \times 10^{-23}$
Regression sum of squares = $1.25405 \times 10^{-23}$
Residual sum of squares = $1.46518 \times 10^{-23}$
Within group sum of squares = $0.58717 \times 10^{-23}$
About regression sum of squares = $0.87801 \times 10^{-23}$

A2.4. TEST FOR LINEARITY

Table A2-2. Analysis of Variance.

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>$1.2540 \times 10^{-23}$</td>
<td>1</td>
<td>$1.2540 \times 10^{-23}$</td>
</tr>
<tr>
<td>About Regression</td>
<td>$0.8790 \times 10^{-23}$</td>
<td>$k-2 = 5$</td>
<td>$0.1756 \times 10^{-23}$</td>
</tr>
<tr>
<td>Within Group</td>
<td>$0.5872 \times 10^{-23}$</td>
<td>$N-k = 4$</td>
<td>$0.1468 \times 10^{-23}$</td>
</tr>
<tr>
<td>Residual</td>
<td>$1.4652 \times 10^{-23}$</td>
<td>$N-2 = 9$</td>
<td>$0.1628 \times 10^{-23}$</td>
</tr>
<tr>
<td>Total</td>
<td>$2.7192 \times 10^{-23}$</td>
<td>$N-1 = 10$</td>
<td></td>
</tr>
</tbody>
</table>
The null hypothesis, HYP(0), is that:

\[ HYP(0): E(y|x) = a + bx \]
\[ HYP(1): \text{Not linear} \]

The test statistic is given by:

\[ F = \frac{\text{ABOUT REGRESSION MEAN SQUARE}}{\text{WITHIN GROUP MEAN SQUARE}} = \frac{0.1756 \times 10^{-23}}{0.1468 \times 10^{-23}} = 1.20 \]

The critical region with a level of significance \( \alpha = 0.05 \), \( N = 11 \), \( k = 7 \) is \( F < F_{0.95(5,4)} = 6.26 \).

Since \( F < 6.26 \) the data do not support the rejection of the hypothesis of linearity.

A2.5. **CALCULATION OF CONFIDENCE BAND FOR REGRESSION LINE**

A2.5.1. **Calculation of 95 Percent Confidence Band**

The boundaries for the \((1 - \alpha)\) confidence level band are:

Upper Bound = \( y_\ell + CS_y \)

Lower Bound = \( y_\ell - CS_y \)

where:

\[ S_{y|\ell}^2 = S_y^2 \left[ \frac{1}{N} + \frac{(x_\ell - \bar{x})^2}{\sum(x_i - \bar{x})^2} \right] \]
\[ c^2 = 2 F_{2,N-2} \]

Substituting:

\[ S_y^2 | x = 1.62798 \times 10^{-24} \]

\[ N = 11 \]

\[ \Sigma(x_i - \bar{x})^2 = 610.441818 \]

\[ F_{2,9} = 4.26 \quad (a = 0.05) \]

\[ S_y^2 = 1.479982 \times 10^{-25} + 2.429219655 \times 10^{-27} (x_\alpha - 18.427272)^2 \]

\[ c^2 S_y^2 = 1.260946 \times 10^{-24} + 2.06843146 \times 10^{-26} (x_\beta - 18.427272)^2 \]

\[ (A2-2) \]

Table A2-3 summarizes values of the upper and lower limits of the 95 percent confidence band as a function of \( x \). These values are plotted on Fig. 3.

<table>
<thead>
<tr>
<th>( E_0 ) (GeV)</th>
<th>( H_0 \times 10^{13} )</th>
<th>( cS_y \times 10^{13} )</th>
<th>Upper Limit ( y + cS_y ) ( \times 10^{13} )</th>
<th>Lower Limit ( y - cS_y ) ( \times 10^{13} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5.22</td>
<td>28.79</td>
<td>34.00</td>
<td>-23.57</td>
</tr>
<tr>
<td>5</td>
<td>12.07</td>
<td>22.34</td>
<td>34.39</td>
<td>-10.29</td>
</tr>
<tr>
<td>10</td>
<td>18.92</td>
<td>16.52</td>
<td>35.42</td>
<td>2.37</td>
</tr>
<tr>
<td>15</td>
<td>25.77</td>
<td>12.26</td>
<td>37.99</td>
<td>13.47</td>
</tr>
<tr>
<td>20</td>
<td>32.62</td>
<td>11.46</td>
<td>44.03</td>
<td>21.11</td>
</tr>
<tr>
<td>25</td>
<td>39.47</td>
<td>14.68</td>
<td>54.09</td>
<td>24.73</td>
</tr>
<tr>
<td>30</td>
<td>46.32</td>
<td>20.08</td>
<td>66.33</td>
<td>26.17</td>
</tr>
</tbody>
</table>
A2.6. TEST FOR $a = 0$

The working hypothesis is that the straight line passes through the origin (i.e., $a = 0$).

Student's t-test gives:

$$T = \frac{\hat{a} - a_0}{S_{\hat{a}}}$$  \hspace{1cm} (A2-2)

Substituting the values:

$$a_0 = 0 \text{ (null hypothesis)}$$
$$\hat{a} = 5.216 \times 10^{-13}$$
$$S_{\hat{a}} = 9.862 \times 10^{-13}$$

into Eq. (A2-2) we obtain:

$$T = 0.529$$

We reject the null hypothesis if

$$|T| > t_{\alpha, N-2} \quad (9 \text{ degrees of freedom}) = 2.262$$

0.529 $\leq$ 2.262 and thus the null hypothesis is not excluded ($p \approx 0.30$) and the experimental data are consistent with the hypothesis that they may be represented by a straight line passing through the origin.
APPENDIX 3. REGRESSION ANALYSIS WITH LINE FORCED THROUGH THE ORIGIN

In Appendix 2 it was shown that the experimental data are compatible with the hypothesis that the relationship between \( H_0(E_p) \) and \( E_p \) is of the form:

\[
H_0(E_p) = b'E_p
\]  \hspace{1cm} (A3-1)

In this appendix, as in Appendix 2, the variables \( H_0(E_p) \) and \( E_p \) will be denoted by \( y \) and \( x \).

For a line forced through the origin:

\[
b' = \frac{\sum xy}{\sum x^2}
\]  \hspace{1cm} (A3-2)

and substituting the values for the summations from Appendix 2:

\[
b = 1.60764 \times 10^{-13}
\]

The error variance is:

\[
S_y' = \frac{\sum y^2 - b'^2 \sum x^2}{N-2}
\]  \hspace{1cm} (A3-3)

which yields upon substitution:

\[
S_y' = 1.510295 \times 10^{-24}
\]

The estimated variance of \( b' \) is given by:

\[
\frac{S^2}{b'} = \frac{S_y'^2}{\sum x^2}
\]  \hspace{1cm} (A3-4)

Substitution yields:
\[
\hat{s}^2 = \frac{1.5107245 \times 10^{-24}}{4405.65} = 3.429073 \times 10^{-28}
\]
\[
\hat{s}_{b'} = 1.852 \times 10^{-14}
\]
This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory, or the Department of Energy.

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