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A Theorem on the Potential of a Double Layer

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1. Introduction

We study the potential $D_u$ of a double layer with density $\mu \in C^1(S)$ on a closed LYAPUNOV surface $S$ [1]:

$$D_u(P) = \int_{S} \mu(q) \frac{\partial}{\partial n} \left( \frac{1}{|P-q|} \right) \, dS_q.$$  \hspace{100pt} (1)

The letter $P$ denotes (the coordinate vector of) a point in $R_+$ or $R_-$, the interior and exterior regions defined by $S$. The letters $p$ and $q$ are (the coordinate vectors of) surface points. $R$ is a bounded but not necessarily connected region and $n_q$ is the inner unit normal at $q$. The kernel in Eq. (1),

$$K(q;P) = \frac{\partial}{\partial n_q} \left( \frac{1}{|P-q|} \right) = \frac{n_q \cdot (P-q)}{|P-q|^3} = \frac{\cos \theta}{|P-q|^2},$$

is the potential in the point $P$ of an electric dipole with dipole moment $n_q$ located at $q$. $K$ is analytic in a neighborhood of $S \times S$ except at $P=p=q$, where it has a weak singularity:

$$|K(q;P)| < \frac{L}{|P-q|^{2-\alpha}} \quad (L>0, 0<\alpha \leq 1).$$  \hspace{100pt} (2)

$L$ and $\alpha$ are the constants in the LYAPUNOV condition for the surface $S$:

$$\theta = \cos^{-1}(n_p \cdot n_q) \leq L |P-q|^{\alpha} \quad (0 \leq \theta \leq \pi/2).$$  \hspace{100pt} (3)

It is well known that the limiting values of $D_u$ on either side of the
surface $S$ are different in general [1]:

$$\lim_{\varepsilon \to 0} D_u(p \pm \varepsilon n_p) = (D_u)_\pm(p) = D_u(p) \pm \pi \mu(p) (\varepsilon > 0). \quad (4)$$

The purpose of this note is to establish the following theorem describing a similar discontinuous behaviour of the tangential derivatives [2]:

**THEOREM:** For a closed and bounded LYAPUNOV surface $S$ and a density $\mu \in C^1(S)$ the tangential derivatives of $(D_u)_\pm$ admit the following representation:

$$\nabla(D_u)_\pm(p) = \int_S [\mu(q) - \mu(p)] \nabla K(q;p) dS \pm 2\pi \nabla \mu(p). \quad (5)$$

The integral contains a weak singularity and is therefore absolutely and uniformly convergent.

The theorem is motivated by an application in potential aerodynamics, where it can be used to compute the tangential velocity of an inviscid fluid on the surface of a rigid, impermeable body $R_+$ (Section 3). Known expressions corresponding to Eq.(5) are not suited for numerical implementation [1, p.76] since $\nabla \mu$ occurs under the integral sign, or they contain a Cauchy principal value integral [3, p.140] which introduced additional errors [2, p.9-10].

Similar applications of Eq.(5) arise in superconductivity (Meissner effect [4]) and in electrostatics.
2. Proof of the theorem

We apply the surface-gradient operator $\vec{\nabla}$ to both sides of Eq. (4) and note that, by assumption, $\vec{\nabla} \mu \in C(S)$:

$$\vec{\nabla}(D\mu)_{\pm}(p) = \vec{\nabla}D\mu(p) \pm 2\pi \vec{\nabla} \mu(p).$$  \hspace{1cm} (6)

The integral at the right side may be written as

$$\vec{\nabla}D\mu(p) = \vec{\nabla} \left[ \int [\mu(q) - \mu(p)] K(q;p) dS_p + \vec{\nabla} \mu(p) \int K(q;p) dS_q \right].$$ \hspace{1cm} (7)

Since $S$ is a closed surface, the integral of the kernel is equal to $2\pi$ [1, p.36], yielding $2\pi \vec{\nabla} \mu(p)$ for the second term in (7). In the first term we interchange differentiation and integration:

$$\vec{\nabla} \left[ \int [\mu(q) - \mu(p)] K(q;p) dS_p \right] = \int \left[ [\mu(q) - \mu(p)] K(q;p) \right] dS_q.$$ \hspace{1cm} (8)

We will have to show that this may be done in spite of the singularity at $p=q$. Then Eq. (7) becomes:

$$\vec{\nabla}D\mu(p) = \int \left[ [\mu(q) - \mu(p)] K(q;p) \right] dS_q + 2\pi \vec{\nabla} \mu(p)$$

$$= \int \left[ - \vec{\nabla} \mu(p) K(q;p) + [\mu(q) - \mu(p)] \vec{\nabla} K(q;p) \right] dS_q + 2\pi \vec{\nabla} \mu(p)$$

$$= -2\pi \vec{\nabla} \mu(p) + \int [\mu(q) - \mu(p)] \vec{\nabla} K(q;p) dS_q + 2\pi \vec{\nabla} \mu(p)$$

$$= \int [\mu(q) - \mu(p)] \vec{\nabla} K(q;p) dS_q.$$
Inserting this expression in Eq. (6) yields the relations (5).

In order to prove Eq. (8), we consider the LYAPUNOV sphere centered at $p$ and adjust its radius $d$ to satisfy $Ld^a<1/2$. Then the portion $S_p$ of the surface $S$ inside the sphere intersects lines parallel to the normal $n_p$ in at most one point, and the intersection of $S_p$ with an arbitrary plane containing $n_p$ is a continuous curve. We define a local coordinate system with origin at $p$ by the orthonormal vectors $e_x, e_y, e_z=n_p$. Let $(S_p)_z$ be the projection of $S_p$ onto the tangential plane $<e_x,e_y>$ spanned by $e_x$ and $e_y$. Consider further the largest circular disc $(S^p_{\rho_m})$ with radius $\rho_m$ and center $p$ which is contained in $(S_p)_z$. Finally, the portion of $S_p$ inside the circular cylinder with basis $(S^p_{\rho_m})$ and axis $e_z$ is denoted by $S^p_{\rho_m}$. Each point $q\in S^p_{\rho_m}$ may now be represented by cylindrical coordinates $(\rho,\phi,\hat{z})$:

$$q=(x,y,z)=(\rho\cos\phi,\rho\sin\phi,\hat{z}(\rho,\phi)) \quad (0\leq\rho\leq\rho_m, 0\leq\phi<2\pi).$$

Since $S^p_{\rho_m}\subset S$, the function $\hat{z}$ has the properties

$$\hat{z}, \frac{\partial^2 \hat{z}}{\partial \rho^2}, \frac{\partial^2 \hat{z}}{\partial \rho \partial \phi} \in C([0,\rho_m] \times [0,2\pi)).$$  \hspace{1cm} (9a,b,c)

In a next step, we define in Eq. (8):

$$f(q;p)=[u(q)-u(p)]K(q;p),$$

and observe that $f, \nabla f \in C((S-S^p_{\rho_m})\times(S-S^p_{\rho_m}))$. Therefore it remains to show that

$$\int_{S^p_{\rho_m}} f(q;p) dS_q = \int_{S^p_{\rho_m}} f(q;p) dS_q.$$  \hspace{1cm} (10)
It is sufficient and convenient to prove the equality of the projections of these vectors with respect to all possible directions $e_0$:

$$e_0 = (x_0^e, y_0^e, z_0^e) = (\cos \phi_0, \sin \phi_0, 0) \quad (0 \leq \phi_0 < 2\pi).$$

For a differentiable function $A$ on $S$ such a projection may be written as

$$e_0 \cdot \nabla A(p) = \lim_{p_0 \to p} \frac{A(p_0) - A(p)}{\sigma_0} = \frac{dA(p_0)}{d\sigma_0} \bigg|_{p_0 = p}, \quad (11)$$

where $p_0 \in S^p \cap <e_0, e_z>$,

and $p = (x_0, y_0, z_0) = (\rho_0 \cos \phi_0, \rho_0 \sin \phi_0, \hat{z}(\rho_0, \phi_0)) \quad (0 \leq \rho_0 \leq \rho_m)$.

The length $\sigma_0$ of the arc $\sigma$ between $p$ and $p_0$ on the continuous curve $S^p \cap <e_0, e_z>$ is given by

$$\sigma_0 = \sigma(\rho_0) = \rho_0 \left[1 + \left(\frac{\hat{z}''}{\hat{z}'}\right)^2\right]^{1/2} = \int_0^{\rho_0} \frac{d\sigma(\rho)}{d\rho} \, d\rho.$$

It follows from Eq.(9b) that $\sigma \in C^1([0, \rho_m])$ and, by construction, we have

$$\frac{d\sigma(p_0)}{dp_0} \bigg|_{p_0 = 0} = \frac{d\sigma(0)}{dp_0} = 1.$$

The projection (11) therefore assumes the form

$$e_0 \cdot \nabla A(p) = \left(\frac{\hat{A}(\rho_0, \phi_0)}{\hat{z}'} \frac{d\rho_0}{d\sigma_0}\right) \bigg|_{p_0 = 0} = \left(\frac{\hat{A}(\rho_0, \phi_0)}{\hat{z}'} \frac{d\rho_0}{d\sigma_0}\right) \bigg|_{p_0 = 0} = \left(\frac{\hat{A}(0, \phi_0)}{\hat{z}'} \frac{d\rho_0}{d\sigma_0}\right) \bigg|_{p_0 = 0}.$$
We denote by \( \hat{A} \) the integral on the left side of Eq.(10) which becomes

\[
\hat{A}(\rho_0, \phi_0) = \int f(q; p_0) \, dq \int_0^{2\pi} f(\rho, \phi; \rho_0, \phi_0) \, \rho \, d\rho \, d\phi
\]

The angle \( \theta \) satisfies the LYAPUNOV condition (3) with \( |p - q| = r = [\rho^2 + \rho^2(\rho, \phi)]^{1/2} \).

Note that \( \cos \theta > 7/8 \) since

\[
1 - \cos \theta = 2\left(1 - \frac{\theta^2}{2} + \ldots\right) < \frac{\theta^2}{2} \leq \frac{(L \rho \alpha)^2}{2} < \frac{(L \rho)^2}{2} < \frac{1}{8}.
\]

The singularity of \( \hat{g} \) in \( r_0 = |p_0 - q| = 0 \) is weaker than (2) because of \( v \in C^1(S) \):

\[
|\hat{g}| = |\hat{f}(\rho, \phi; \rho_0, \phi_0)| = \frac{\rho}{\cos \theta} = |\hat{u}(\rho, \phi; \rho_0, \phi_0)| \cdot |\hat{K}(\rho, \phi; \rho_0, \phi_0)| = \frac{\rho}{\cos \theta}
\]

\[
< (H r_0) \cdot \left(\frac{L}{r_0^{2-\alpha}}\right)(\frac{d}{7/8}) = \frac{8HLd}{7r_0^{1-\alpha}} \quad (0 < H < \infty).
\]

Define by \( \hat{B}(0, \phi_0) \) the projection of the right side of Eq.(10) onto \( e_0 \):

\[
\hat{B}(0, \phi_0) = \int_0^{2\pi} f(q; p) \, dq \int_0^{2\pi} \frac{\partial \hat{g}(\rho, \phi; 0, \phi)}{\partial \rho} \, d\rho \, d\phi.
\]

The following estimate holds for the integrand:
\[
\left| \frac{\partial g(\rho, \phi; 0, \phi_0)}{\partial \rho_0} \right| = \left| [\hat{A}(\rho, \phi) - \hat{A}(0, \phi_0)] - \frac{\partial \hat{u}(0, \phi_0)}{\partial \rho_0} \hat{k}(\rho, \phi; 0, \phi_0) \right| \frac{\rho}{\cos \theta}
\]

\[
< \left( H_r \left( \frac{7L}{r^{3-\alpha}} \right) + \frac{\partial \hat{u}(0, \phi_0)}{\partial \rho_0} \right) \frac{L}{r^{2-\alpha}} \right] \frac{8\rho}{7}
\]

\[
= \frac{D_0^\rho}{r^{2-\alpha}} = \frac{D_0 \cos^{2-\alpha} \beta}{\rho^{1-\alpha}} \quad (\rho=r \cos \beta).
\]

Note that \( D_0 > 0 \) is a finite number since \( \mu \in C^1(S) \) and the surface is bounded.

Use has also been made of a Lemma which will be established after this proof, namely:

**Lemma:**

\[
|\nabla_p \hat{K}(q; p)| < \frac{7L}{r^{3-\alpha}} \quad (r \to 0).
\]  

(12)

Eq. (10) has now been reduced to the following scalar identity in \( \phi_0 \):

\[
\left. \frac{\partial \hat{A}(\rho_0, \phi_0)}{\partial \rho_0} \right|_{\rho_0 = 0} \equiv \hat{B}(0, \phi_0) \quad (0 \leq \phi_0 < 2\pi).
\]

Its validity is established if we can prove that

\[
\lim_{\rho_0 \to 0} I(\rho_0, \phi_0) \equiv \lim_{\rho_0 \to 0} \left\{ [\hat{A}(\rho_0, \phi_0) - \hat{A}(0, \phi_0)]/\rho_0 - \hat{B}(0, \phi_0) \right\} \equiv 0. \quad (13)
\]

In the integration domain \( (S^p_{\rho_m})_z \), we define a circular disc, with center \( p \) and radius \( 2\rho_0 \), which contains the point \( p_0 \) in its interior.
Then $I(\rho_0, \phi_0)$ may be written as

$$I(\rho_0, \phi_0) = \int_0^{2\pi} \int_0^{\rho_0} \left\{ \hat{g}(\rho, \phi; \rho_0, \phi_0) - \hat{g}(\rho, \phi; 0, \phi_0) \right\} / \rho_0 \, d\rho \, d\phi$$

$$- \int_0^{2\pi} \int_0^{\rho_0} \frac{\partial \hat{g}(\rho, \phi; \rho_0, \phi_0)}{\partial \rho} \, d\rho \, d\phi$$

$$+ \int_0^{2\pi} \int_0^{\rho_m} \left\{ \hat{g}(\rho, \phi; \rho_0, \phi_0) - \hat{g}(\rho, \phi; 0, \phi_0) \right\} / \rho_0 - \frac{\partial \hat{g}(\rho, \phi; \rho_0, \phi_0)}{\partial \rho} \right\} \, d\rho \, d\phi. \quad (14)$$

The first integral converges for all $\rho_0 > 0$ and the second integral is bounded by $2^{\alpha+1} - \pi D_0 \rho_0^\alpha$; both integrals vanish for $\rho_0 \to 0$.

We define the function $\hat{y}$ by $\hat{y}(\xi) = \hat{g}(\rho, \phi; \xi, \phi_0)$, $\xi \in [0, \rho_0]$. Note that $\rho \geq 2\rho_0$ and therefore $u \in C^1(S)$ implies $\hat{y} \in C^1([0, \rho_0])$. By the Mean Value Theorem there exists a constant $\theta$ ($0 < \theta < 1$) such that

$$\frac{\hat{y}(\rho_0) - \hat{y}(0)}{\rho_0} = \frac{d \hat{y}(\rho_0)}{d \xi}.$$

This leads to the estimate of the last integral in Eq. (14):

$$\int_0^{2\pi} \int_0^{\rho_m} \left| \frac{\partial \hat{g}(\rho, \phi; \rho_0, \phi_0)}{\partial \xi} - \frac{\partial \hat{g}(\rho, \phi; 0, \phi_0)}{\partial \xi} \right| \, d\rho \, d\phi \leq 2\pi (\rho_m - 2\rho_0) \epsilon(\rho_0, \phi_0).$$

$\epsilon(\rho_0, \phi_0)$ denotes an upper bound of the integrand which tends to zero with $\rho_0$ for all fixed $\phi_0 \in [0, 2\pi)$, by virtue of $\hat{y} \in C^1([0, \rho_0])$.

The weak singularity of the integrand in Eq. (5) follows immediately from the Lemma and the assumption $u \in C^1(S)$. The theorem is proved.
Proof of the Lemma: For \( p \neq q \) the tangential derivative of \( K \) in Eq.(12) may be written as the projection of its gradient onto the tangential plane at \( p \):
\[
\nabla_p K(q;p) = \nabla_p K(q;p) - (n \cdot \nabla_p K(q;p))n_p
\]
\[
= \frac{n_{pq}}{r^3} + \frac{3K(q;p)}{r} [\cos(\gamma - \theta) n_p - e_{pq}],
\]  
(15)

where \( r = |p-q| \), \( e_{pq} = (p-q)/r \), and \( n_{pq} = n_q - (n \cdot n_q)n_q \). It is important to note that
\[
|n_{pq}| = \sin \theta.
\]  
(16)

Indeed, we have for fixed \( q \) and variable \( p \):
\[
n_{pq} + (n_q \cdot n_p) = n_q = \text{constant vector}.
\]

Moreover, \( n_{pq} \) forms a right angle with \( n_p \). Therefore \( n_{pq} \) describes a certain part of the surface of the sphere centered at \( q + n_q/2 \) with radius 1/2. This geometrical description leads immediately to Eq.(16). Further we have \( \theta < 1 \) for \( r \to 0 \), and since the terms in
\[
\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots
\]

have alternating signs, the sum of the terms following \( \theta \) is negative and hence \( \sin \theta < \theta \). Recalling the LYAPUNOV condition (3) we obtain finally
\[
|n_{pq}| < Lr^a. \text{ Using this estimate, we can project out the strong singularity } O(r^{-3}) \text{ of the normal component } n_p \cdot \nabla_p K(q;p) \text{ in Eq.(15):}
\]
\[
\left| \nabla_p K(q;p) \right| \leq \frac{|n_{pq}|}{r} + \frac{3}{r}|K(q;p)| \cdot |\cos(\gamma-\theta)n_p - e_{pq}|
\]
\[
< \frac{Lr^\alpha}{r^3} + \frac{3}{r}\left(\frac{L}{r^{2-\alpha}}\right)^2 = \frac{7L}{r^{3-\alpha}}.
\]

3. An application in potential aerodynamics

Potential flow in the region \( R_+ \) around a rigid, impermeable body \( R_- \) is described by the well-known Neumann problem for the disturbance potential \( \phi \in C^2(R_-) \) with the boundary condition \([1,2,6]\):

\[
\frac{\partial \phi_-(p)}{\partial n_p} = - \frac{\partial \phi_\infty(p)}{\partial n_p}.
\]

\( V_\infty = \nabla \phi_\infty \) is the velocity of the undisturbed flow (constant in most cases).

\( \phi_\infty \) is basically a harmonic function in a region containing \( R_+ \cup S \) in its interior \([2,5,10]\).

In order to avoid the discretization of the unbounded, 3-dimensional region \( R_- \), this problem is reformulated as an integral equation on the boundary \( S \) \([2,5,6,7,8,9,10,11]\). In one of these formulations \( \phi \) is represented by the potential of a double layer:

\[
\phi(P) = - \frac{1}{4\pi} D\phi_-(P) \quad (P \in R_-).
\] (17)

The total velocity potential \( \phi_- = \phi_\infty + \phi_- \) on \( S \) is the unique solution of the following integral equation on the boundary \( S \) \([2,5]\):
\[ \phi_-(p) = \frac{1}{4\pi} \int_{S} \left[ \phi_-(p) - \phi_-(q) \right] K(q;p) dS_q + \phi_\infty(p). \]  

(18)

An important field quantity in applications is the tangential fluid velocity \( V_\perp = \nabla \phi_\perp \) on \( S \). The exterior jump-relation (5) proved in Section 2 yields an integral formula for \( V_\perp \). Indeed, the density in Eq. (17) is \( \mu = -\phi_\perp /4\pi \) and it has been proved in [2] that \( \phi_\perp \in C^1(S) \):

\[ \nabla \cdot \phi_\perp = - \frac{1}{4\pi} \int_{S} \left[ \phi_-(q) - \phi_-(p) \right] \nabla_p K(q;p) dS_q + \frac{1}{2} \nabla \cdot \phi_\perp(p). \]

Addition of \( \nabla \cdot \phi_\perp(p) \) on both sides leads to

\[ V_\perp(p) = \frac{1}{2\pi} \int_{S} \left[ \phi_-(p) - \phi_-(q) \right] \nabla_p K(q;p) dS_q + 2[V_\infty(p) - (n \cdot V_\infty(p)) n_p]. \]

This expression serves as basis for numerical schemes in which the approximating solution of the integral equation (18) is inserted [2,5].

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