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Publication Date
1974-10-01
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Prepared for the U. S. Atomic Energy Commission
under Contract W-7405-ENG-48
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STRUCTURE OF THE VERTEX FUNCTION IN FINITE QUANTUM ELECTRODYNAMICS

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October 1974

ABSTRACT

We study the structure of the renormalized electromagnetic current vertex, \( \tilde{T}_\mu(p,p+q,q) \), in finite quantum electrodynamics. Using conformal invariance we find that \( \tilde{T}_\mu(p,p,0) \) takes the simple form of \( Z_1 y_\mu \) when the external fermions are far off the mass shell. We interpret this result as an old theorem on the structure of the vertex function due to Gell-Mann and Zachariasen. We give the general structure of the vertex for arbitrary momentum transfer parametrically, and discuss how the Bethe-Salpeter equation and the Federbush-Johnson theorem are satisfied. We contrast the meaning of pointlike in a finite field theory with the meaning understood in the parton model. We discuss to what extent the condition \( Z_1 = 0 \), which may hold in conformal theories other than finite quantum electrodynamics, may be interpreted as a bootstrap condition.

I. INTRODUCTION

In recent years a serious effort has been made to try to understand the ultraviolet structure of local field theories. This requires a study of Green's functions far off the mass shell where ordinary perturbation techniques are generally unreliable, since this is the kinematic region which generates the divergences met order by order in perturbation theory. Johnson, Baker, and Willey¹ have tackled this problem vigorously and developed a finite theory of quantum electrodynamics (finite Q.E.D.) in which the divergences organize themselves away nonperturbatively. Their theory may now serve as a prototype for the sort of short-distance behavior we might expect in a realistic quantum field theory, and may be characterized by the concept of dynamical dimensions.² In their theory an eigenvalue condition for the coupling constant is required in order to make \( Z_3^{-1} \) finite with the photon propagator being asymptotically canonical. The electron propagator acquires an anomalous dimension which can be removed in the generalized Landau gauge (the finite gauge) to make \( Z_2^{-1} \) finite; and the composite mass operator \( \bar{\psi}\psi \) must acquire a negative anomalous dimension in order to make the bare mass vanish so that mass renormalization is then also finite. Theories of this type are also known to display asymptotic conformal invariance,³ and hence conformal invariance may prove useful in limiting the forms of the Green's functions of the theory, and provide nonperturbative asymptotic information about the theory.

In this work we shall use conformal invariance to construct the vertex functions of finite Q.E.D. to obtain their structure. Such information is useful in itself, and it will also enable us to determine the nature of the pointlike behavior of the dressed electron in a finite renormalizable field theory. As we shall see this behavior is very different from the meaning of pointlike understood in the parton model. Our approach will also help in studying to what extent the condition \( Z_1 = 0 \) is a bootstrap condition for the vertex in
theories where it is expected to hold. We shall present our main results in this introduction and leave the details of the calculation to Sec. II. For completeness we shall also discuss in Sec. II how the Bethe-Salpeter equation and the Federbush-Johnson theorem are satisfied in finite Q.E.D. in the vector current sector, and also make a similar analysis for the axial-vector vertex, $\tilde{F}_{\mu\nu}(p,p+q,q)$.

The most convenient starting point is to introduce the canonical commutator

$$\left[ j_0(x), \psi(y) \right] \delta(x_0 - y_0) = 0 \delta'(x - y) \psi(x). \quad (1)$$

In the finite theory this commutator is not destroyed by renormalization (except that there may be a c-number anomaly), so that the renormalized current is still canonical. Moreover in the finite gauge the electron is also canonical so that the conformal ansatz for the renormalized connected unamputated vertex function $\tilde{G}_\mu(x,z,y)$ in coordinate space gives

$$\langle \Omega | T \left( j_\mu(x) j_\nu^*(z) \right) | \Omega \rangle \quad \text{in coordinate space gives}^4$$

$$= A \left( \frac{\hat{x} - \hat{z}}{y - z} \right) \gamma_\mu \left( \frac{\hat{x} - \hat{z}}{y - z} \right) + B \frac{\left( \hat{x} - \hat{y} \right)}{(x - y)^2(y - z)^2(z - x)^2}$$

$$\times \left\{ \frac{(z_\mu - x_\mu)}{(z - x)^2} + \frac{(y_\mu - z_\mu)}{(y - z)^2} \right\}. \quad (2)$$

Here $A$ and $B$ are arbitrary functions of the coupling constant with all of the space-time structure specified by the conformal group alone. (Equation (2) may not be the complete $\tilde{G}_\mu$ because of difficulties met in trying to make gauge invariance compatible with conformal invariance. Extra form factors may be required, and this point will be discussed in the appendix.) Since the current is canonical the vacuum polarization is fixed by conformal invariance to have the form

$$\tilde{\Pi}_{\mu\nu}(z) = \langle \Omega | T(j_\mu(z) j_\nu(0)) | \Omega \rangle \quad \text{in coordinate space gives}$$

$$= 4f(a) \frac{\left( g_{\mu\nu} \frac{Z_1}{Z_0^2} - z_\mu z_\nu \right)}{z^3}$$

so that it is automatically transverse. (This is a well-known feature of conformal invariance which only happens to Lorentz vector operators of dimension 3.) Here $f(a)$ is the function whose vanishing makes $Z_3^{-1}$ finite. Thus from the Schwinger-Dyson equation

$$\tilde{\Pi}_{\mu\nu}(x - z) = \lim_{y \to x} Z_1 \text{Tr} \gamma_\mu \tilde{G}_\nu(x,z,y) \quad (4)$$

(where the limit is taken symmetrically) we obtain a relation between the form factors of Eq. (2), viz.

$$Z_1 \left( A - B \frac{Z_1}{Z_0^2} \right) = f(a) \quad (5)$$

so that there is effectively only one form factor at the eigenvalue.

Now that we have Eq. (2) explicitly it is then in principle straightforward to Fourier transform the vertex, but in practice we can only do this in parametric form since the integrals do not appear to be reducible to named functions. However, for zero momentum transfer (with $p^2$ serving as an infrared cut-off) an analytic
solution will be obtained in Sec. II which when combined with the Ward identity leads us to an amputated vertex which satisfies

$$\lim_{p^2 \to -\infty} \Gamma_{\mu}(p,p,o) = Z_1 \gamma_\mu$$

(6)

a remarkably simple form. This appears to be an old result of Gell-Mann and Zachariasen\(^9\) which says that the asymptotic behavior of the vertex function as the momentum goes to infinity is the same as that of \(Z_1\) as the cut-off goes to infinity (\(Z_1^{-1}\) being finite in our case). Note that this is not the same as another relation which has appeared in the literature,

$$\lim_{q^2 \to -\infty} \tilde{\Gamma}_\mu(p^2 = m^2, (p+q)^2 = m^2, q) = Z_1 \gamma_\mu$$

(7)

which would be the asymptotic behavior of the on-shell form factor. There is some confusion in the literature between Eqs. (6) and (7). Källen's famous wrong proof\(^10\) that one of the renormalization constants in Q.E.D. had to be infinite was based on the validity of Eq. (7), and it is not too clear from reading Ref. 9 as to which asymptotic limit these authors had in mind. However, we only need recall that \(Z_1\) is a renormalization constant which multiplicatively renormalizes a product of fields at the same point and is hence related to short-distance behavior, so that it must be determined by a Green's function in which at least two legs go far off the mass shell. This is analogous to the relation for the off-shell propagator

$$\lim_{p^2 \to -\infty} S^{-1}(p) = Z_2 \gamma\cdot$$

(8)

and hence Eq. (6) says that \(Z_1\) is given in a similar manner by inserting a current carrying zero momentum into the off-shell propagator and then shortening the distance between the two fermions so as to trap the insertion. Equation (7) is a priori unreasonable (but could still presumably hold in particular dynamical situations) since the asymptotic behavior of the on-shell form factor is not given by the short-distance limit (it is an infrared problem). It was of course realized that even if Eq. (7) were to be true in some theories it could not be expected to hold in Q.E.D. since \(Z_1\) is gauge dependent (unless of course \(Z_1\) just happened to vanish in all gauges). This in fact started the controversy which led to the development of finite Q.E.D., and in a sense we have come full circle historically by deriving Eq. (6) (instead of Eq. (7)) as a consequence of the finiteness of the theory, so that Eq. (6) is consistent with all the renormalization constants being finite.

Though we cannot determine the on-shell form factor we can bound its asymptotic behaviour by inserting the form factor into the discontinuity of the vacuum polarization (which is in fact zero since the vanishing of \(f(a)\) removes the discontinuity from Eq. (3)). This then gives\(^11\)

$$\lim_{q^2 \to -\infty} \tilde{\Gamma}_\mu(p^2 = m^2, (p+q)^2 = m^2, q) = 0 .$$

(9)

We mention this to contrast with the parton model. In the parton model Eq. (7) is assumed to hold (with \(Z_1\) nonzero) rather than Eqs. (6) and (9), so that it uses a different interpretation of the word pointlike than that which applies to the electron. (We are not aware
of any derivation of Eq. (7) in the perturbatively asymptotically
free non-Abelian case\textsuperscript{12} either where Eq. (6) will still hold since
the theory is canonical (up to logarithms), since asymptotic freedom
does not apply outside the light cone). We mention this to indicate
that the parton model goes a lot further than the light-cone algebra,
and that the use of Eq. (7) in it (particularly when extended to
timelike $q^2$) may be unwarranted.

Though this paper is concerned with finite Q.E.D. we would
like to discuss here the closely related conformal bootstrap developed
by Migdal and Polyakov. Our discussion will give us some new
information about the bootstrap nature of the ansatz of conformal
invariance with anomalous dimensions, and we will also see (in Sec.
II) why finite Q.E.D. is not a bootstrap of this type. We shall
analyze the discussion given by Migdal\textsuperscript{3} of the $i\bar{\psi}\gamma_5\psi$: theory
under the ansatz of conformal invariance with anomalous dimensions.
Let $d_M$ and $d_F$ be the anomalous dimensions of the meson and the
fermion respectively. (In this theory there is no gauge so that the
usual positivity requirement leads to $d_F > 3/2$, $d_M > 1$). If we
renormalize the theory at some mass point, $\mu$, then the wave function
renormalization constants are given by

$$Z_F = \left( \frac{\Lambda^2}{\mu^2} \right)^{\frac{3}{2}-d_F}, \quad Z_M = \left( \frac{\Lambda^2}{\mu^2} \right)^{1-d_M} \tag{10}$$

so that both $Z_F$ and $Z_M$ vanish. Let us introduce $Z_\theta^{1/2}$ which
renormalizes $i\bar{\psi}\gamma_5\psi$: so that

$$Z_\theta = \left( \frac{\Lambda^2}{\mu^2} \right)^{3-d_\theta} \tag{11}$$

where $d_\theta$ is the anomalous dimension of the pseudoscalar composite.

We introduce $Z_p$ which renormalizes $\Gamma_p$, the vertex made by inserting
$i\bar{\psi}\gamma_5\psi$: into the inverse fermion propagator, which then satisfies

$$Z_p = Z_p Z_\theta^{-\frac{1}{2}} \sim \left( \frac{\Lambda^2}{\mu^2} \right)^{\frac{1}{4} \left( 4-d_M-d_\theta \right)} \tag{12}$$

Now since we are in an infrared-stable theory we require an eigenvalue
condition for the bare coupling constant so that there will in fact
be conformal invariance with anomalous dimensions. Thus

$$\frac{1}{\lambda_0} = \frac{Z_F Z_M^{1/2}}{Z_p} \sim \left( \frac{\Lambda^2}{\mu^2} \right)^{\frac{1}{4} \left( 4-d_M-d_\theta \right)} \tag{13}$$

is finite, so that we obtain a consistency condition

$$d_M + d_\theta = 4 \quad \tag{14}$$

But $d_M > 3$, so we see that $d_\theta < 3$, so that $i\bar{\psi}\gamma_5\psi$: has to have a
dimension less than canonical. (Exactly the same situation is met
in discussing mass renormalization in finite Q.E.D. where the
dimension of $\bar{\psi}\psi$: is required to be less than canonical\textsuperscript{13,14}.)

Now $d_F > 3/2$, so that from Eq. (12) we conclude that $Z_p = 0$.

Now $\tilde{\Gamma}_p$ satisfies the Bethe-Salpeter equation

$$\tilde{\Gamma}_p(p,p+q,q) = Z_p \gamma_5 + \int d^4 k \tilde{\Gamma}_p(k,k,q,q) \tilde{S}(k) \tilde{K}(p,k,q) \tilde{S}(k+q) \tag{15}$$

where the renormalized kernel is built out of the dressed propagators
and the dressed 3-point function $\langle \Omega | \bar{\psi}(x) \phi(z) \psi(y) | \Omega \rangle$ ($= \tilde{g}_\phi$).

The conformal ansatz for $\tilde{\Gamma}_p$ (unamputated) gives\textsuperscript{3}
This $\tilde{r}_P(x,z,y)$ (together with a similar structure for $\tilde{G}_\phi$) then reproduces itself self-consistently in Eq. (15) when $Z_p = 0$. Thus the bare vertex which has canonical dimension is eliminated, and the ansatz of conformal invariance with anomalous dimensions reproduces itself with the convergence of the integration in Eq. (15) achieved since $d_0 < 3$. Hence we recognize the condition $d_0 < 3$ as the consistency condition that the conformal bootstrap be implementable.

Migdal in his paper did not actually discuss the equation for $\tilde{r}_P$ but rather that for $\tilde{G}_\phi$ which is also homogeneous when $Z_p = 0$. This equation is then nonlinear in $\tilde{G}_\phi$ and is thus a bootstrap equation for the dressed $\tilde{G}_\phi$ 3-point function. This equation also looks like a typical homogeneous bound state equation in which the meson could have been a pole in the off-shell fermion anti-fermion scattering amplitude. Since the kernel is itself made up by exchanging the same meson the situation looks like a bound-state bootstrap. Hence it is often stated that the vanishing of the renormalization constants $Z_p, Z_p,$ and $Z_M$ gives a bound-state bootstrap. Moreover this approach is usually coupled with an analogue of Eq. (7) (see e.g. Ref. 15) where the implied fast falling on-shell form factor is interpreted as a compositeness condition. Now of course if there are bound states in the fermion anti-fermion scattering amplitude their vertices will satisfy homogeneous equations at the pole. So let us suppose there is a pseudoscalar bound state in the conformal theory. Such a bound state would have to appear as a pole in $\tilde{r}_P$ which symbolically behaves as

$$\tilde{r}_P \sim \frac{Z_p Y_5}{1 - \int \bar{S} K \bar{S}}$$

(17)

However, we are in a situation in which $Z_p = 0$, and thus $\tilde{r}_P$ cannot contain a pole. Hence the existence of homogeneous equations is a necessary but not sufficient condition that there be bound states. (The sufficient condition is that the off-shell bound state vertex function satisfy an inhomogeneous equation. 16) Thus in the example discussed by Migdal we see that the condition $Z_p = 0$ actually prevents the meson from being a bootstrapped bound state, and in his model $i\bar{\psi} \gamma_5 \psi$: is the source of the meson field instead of the meson being a pole in $\tilde{r}_P$. Thus though the conformal bootstrap leads to self-consistent equations for the dressed vertices and to an eigenvalue condition for the (bare) coupling constant it contains no bound states, and hence it appears to us that the conformal bootstrap cannot provide a field-theoretic realization of the S-matrix bootstrap.

II. CALCULATION OF THE VERTEX FUNCTION

We shall proceed directly to Fourier transform Eq. (2). The $A$ term is simply done and gives

$$\tilde{G}_{\mu \lambda} = \frac{A}{4F} Y_\mu \frac{1}{(p + k)}$$

(18)

The second term can be handled by noting that the Fourier transform of $f(z - x) g(x - y) h(y - z)$ is given as a Feynman diagram by
F.T. = \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) \tilde{g}(k + p) \tilde{h}(k - q), \quad (19)

so that we obtain (after a Euclidean rotation)

\tilde{\tilde{g}}_{\mu B} = \frac{B}{4\pi^2} \int d^4k \frac{(k + q)(2p_\mu - q_\mu)}{(k + p)^2(k - q)^2k^2}. \quad (20)

We evaluate this integral using Feynman parameters. If we introduce

\begin{align}
D &= \alpha(1 - \alpha)p^2 + \beta(1 - \beta)q^2 + 2\alpha\beta p \cdot q \\
C_{\alpha\mu} &= -4\alpha(1 - \alpha)p_\alpha p_\mu - 28(1 - 2\alpha)q_\alpha q_\mu \\
- 2p_\alpha q_\mu (1 - \alpha)(1 - 2\beta) - 4p_\mu q_\alpha \beta
\end{align} \quad (21)

then

\tilde{\tilde{g}}_{\mu B} = \frac{BY_\alpha}{4} \int_0^1 \int_0^{1 - \alpha} 4\delta \left\{ \frac{C_{\alpha\mu}}{D} + \frac{\alpha C_{\alpha\mu}}{2B^2} \right\}. \quad (22)

It does not appear possible to integrate Eq. (22) right out analytically so we keep it in this form. Further simplification is possible at \( q_\mu = 0 \) however, and yields

\tilde{\tilde{g}}_{\mu B}(p, p, q) = \frac{BY_\alpha}{4} \left\{ \frac{g_{\alpha\mu}}{p^2} - \frac{2p_\alpha p_\mu}{p^4} \right\}. \quad (23)

We now amputate the fermion legs using Eq. (8). (This definition of \( Z_2 \) may not necessarily coincide with the definition that \( Z_2 \) is the residue at the pole of the unrenormalized propagator, since a conformal theory has infrared problems, but will suffice for our purposes.) Thus we obtain

\begin{align}
\tilde{\tilde{g}}_{\mu B}(p, p, q) &= \frac{Z_2^2(A - B)}{4} \gamma_{\mu} \\
\tilde{\tilde{f}}_{\mu B}(p, p, q, o) &= \frac{Z_2}{4} (A - B) \gamma_{\mu} \quad (24)
\end{align}

whose nontrivial content is that there is no \( \bar{\psi} \gamma_{\mu} \psi \) term at\( q_\mu = 0 \). However, the renormalized Ward identity in differential form gives

\begin{align}
\frac{Z_2}{Z_1} \gamma_{\mu} \tilde{\tilde{f}}_{\mu B}(p, p, q, o) &= \gamma_{\mu} \frac{\partial}{\partial p_{\mu}} \bar{\psi} \gamma_{\mu} \psi \\
\text{so that} \\
Z_1 &= (A - B)Z_2^2 \quad (25)
\end{align}

Thus finally we obtain

\tilde{\tilde{f}}_{\mu B}(p, p, q, o) = Z_1 \gamma_{\mu} \quad (27)

the advertised result. Also we note that in general\( \tilde{\tilde{f}}_{\mu B}(p, p^* q, q) \) will not have the simple form of Eq. (27) since the other form factors supplied by \( C_{\alpha\mu} \) will appear as well so that there will be terms of the form \( \bar{\psi} \gamma_{\mu} (\bar{\psi} + q) \) etc. In passing we also remark that \( \tilde{\tilde{f}}_{\mu B}(p^2 = 0, (p + q)^2 = 0, q) \) is found to be infrared divergent, as is expected in a massless theory. One further remark is to combine Eqs. (26) and (5) to give

\begin{align}
Z_1 A &= \frac{4}{3} \left\{ f(\alpha) - \frac{Z_1^2}{Z_2^2} \right\} \\
Z_1 B &= \frac{4}{3} \left\{ f(\alpha) - \frac{4Z_1^2}{Z_2^2} \right\}
\end{align} \quad (28)
Thus if we can go to the mass shell in Eq. (25) to recover the familiar \( z_1 = z_2 \) we can then deduce that \( z_1 A \) and \( z_1 B \) are necessarily nonvanishing at the eigenvalue. Thus \( z_1 \) does not vanish at the eigenvalue unless \( A \) is infinite (see comments in Ref. 10).

We would like to stress that Eq. (27) has been derived from the fact that \( J_\mu \) is canonical, and has not required the use of Eq. (5). In a sense we have been working with an abstracted theory in which we only look at Green's functions with external fermion lines and fermion composites such as the electromagnetic current, and have required this theory to display asymptotic conformal invariance.

(We recall that the Ward identity holds before we extend the theory to a local gauge). We can thus discuss the fermion sector in a non-perturbative and non-Lagrangian manner without an explicit photon. It is only when we ask what particular Lagrangian field theory is going to provide us with these Green's functions that we introduce the dimensionless \( e J_\mu A^\mu \) interaction by extending the theory to a local gauge and then make a graphical analysis. Then the equation of motion \( \Box A_\mu = e J_\mu \) essentially dictates that the photon must be introduced canonically if \( e \) is to be finite. Thus finite Q.E.D. may be thought of as a theory in which a global canonical current remains canonical after extension to a local gauge. Before we introduce the photon explicitly however, \( \tilde{\Gamma}_{\mu
u} \), which is defined by the fermion sector only, satisfies conformal invariance in Eq. (3) without requiring \( f(\alpha) = 0 \). We mention this to indicate that Eq. (27) may then hold in a more general context than finite Q.E.D. alone.

We turn now to the Bethe-Salpeter equation satisfied by \( \tilde{\Gamma}_\mu \),

\[
\tilde{\Gamma}_\mu(p,p+q,q) = Z_\mu \gamma_\mu + \int d^4k \tilde{\Gamma}_\mu(k,k+q,q) \tilde{S}(k) \tilde{K}(p,k,q) \tilde{S}(k+q) \tag{29}
\]

Ordinarily in perturbation theory the finiteness of the left-hand side is achieved by a cancellation of the infinity of \( Z_1 \) against the integration over the kernel. In finite Q.E.D. we make the integration finite by fixing the gauge of the propagators of the internal photons. This is to be contrasted with the conformal bootstrap situation discussed previously in which the integration in Eq. (15) was made finite through anomalous dimensions, with the ansatz of conformal invariance with anomalous dimensions reproducing itself identically in Eq. (15). In finite Q.E.D. we have strictly canonical dimensions so that a bare vertex is compatible with conformal invariance in Eq. (29) with nonvanishing \( Z_1 \). Thus Eq. (29) is not a nonlinear integral equation for \( \tilde{\Gamma}_\mu \), so that there is no vertex bootstrap in finite Q.E.D. (unless \( Z_1 \) just happens to vanish at the eigenvalue). Moreover since we have an explicit bare vertex in Eq. (29) we must ask how the conformal ansatz of Eq. (2) reproduces itself in an inhomogeneous Bethe-Salpeter equation. The answer would appear to have to be that when we build the kernel out of \( \tilde{\Gamma}_\mu \) and then feed in Eq. (2) we will obtain altogether two coupled equations for the different projections such as \( \gamma_\mu \) and \( \gamma_\mu \) etc., and these two relations will self-consistently be Eqs. (5) and (26). (Recall again the connection between canonical dimensions (used in Eq. (2)) and current conservation (used in Eq. (25)).) Unfortunately we are unable to confirm this expected interplay between the \( A \) and \( B \) form factors since we only have Eq. (22) in parametric form. However, let us consider the dressed ladder approximation to Eq. (29) in which...
we replace the kernel by a single dressed photon coupled to the
dressed vertices of Eq. (2). If we only retain the A form factor
we see that Eq. (29) is not satisfied in the ladder approximation since
point couplings in the integration (which is now just like the usual
lowest order radiative correction to the vector vertex) generate a
nontrivial momentum dependence at arbitrary \( q_\mu \) which cannot be
reproduced by the remaining terms. Thus to reproduce this momentum
dependence we will need the B form factor as well, so that already
in the ladder approximation we will need an interplay between the
form factors. (It is conceivable that the A form factor could
satisfy Eq. (29) by itself when we use the exact kernel rather than
just the lowest order approximation. Since we do not know the exact
kernel we cannot eliminate this possibility.) This situation should
be contrasted with the dressed ladder approximation to Eq. (15) which
is satisfied by Eq. (16). We note that Eq. (16) does not correspond
to a point coupling unless \( d_F = \frac{3}{2}, d_\gamma = 3 \) (so that \( d_M = 1 \)) which
was already excluded. Thus \( \tilde{\Gamma}_p \) has a nontrivial momentum dependence
which is reproduced identically. The only case where the dressed
ladder approximation will not be satisfied is when both the meson and
its source are canonical, so that the coupling is a point coupling.
The only nontrivial theory in which both the meson and its source
are canonical is finite Q.E.D. (since \( Z_3 \) renormalizes the charge
by itself), so that this is the only case where the naive point-like
A form factor will not suffice. Thus we see that finite Q.E.D. is
not a vertex bootstrap theory, and we see the specific role played by
the additional B form factor.\(^1\)

Another interesting formula is obtained by inserting Eq. (27)
into Eq. (29), viz.

\[
Z_1 \int d^4k \gamma_\mu \tilde{S}(k) \tilde{K}(p,k,\omega) \tilde{S}(k) = 0
\]

which must be satisfied for arbitrary external \( p \). Thus we see that
the Bethe-Salpeter kernel is compact and square integrable in finite
Q.E.D., or at least its \( \gamma_\mu \) projection is compact, at \( q_\mu = 0 \). This
is to be contrasted with the ladder approximation, which does not
possess a compact kernel.\(^1\) We shall return to this point again below
in discussing the absence of Goldstone poles in the axial-vector
vertex, but remark only now that a compact kernel at the eigenvalue is
in fact welcome, since ultimately we would like to make contact with
conventional low energy quantum electrodynamics which should contain
a positronium bound state.

We discuss next the axial-vector vertex. We define \( A_5, B_5 \)
alogously to Eq. (2) by putting the \( \gamma_5 \) immediately to the right
of the \( \mu \) index. This leads to

\[
\Gamma_{\mu 5}(p,p,\omega) = \frac{Z_2(A_5 + B_5)}{4} \gamma_\mu \gamma_5
\]

The axial-vector Ward identity then yields

\[
\Gamma_{\mu 5}(p,p,\omega) = Z_A \gamma_\mu \gamma_5
\]

where \( Z_A \) renormalizes the axial-vector vertex. This time there is
no relation

\[
Z_A \left( A_5 + \frac{B_5}{4} \right) = f(\alpha)
\]
unless we invoke chiral symmetry to fix the "axial" vacuum polarization Schwinger term. Alternatively Eq. (33) should also be expected to follow from the consistency of the axial-vector vertex Bethe-Salpeter equation. However, there is a relation analogous to Eq. (30),

\[ Z_A \int d^4 k \, \gamma_\mu \gamma_5 \tilde{S}(k) \bar{\tilde{R}}(p, k, o) \tilde{S}(k) = 0. \quad (34) \]

Equation (34) is of interest since it explains why the existence of a nontrivial solution to the self-consistent equation for the fermion mass does not require the presence of a Goldstone pole in \( \tilde{\Gamma}_\mu^5 \). In finite Q.E.D. the electron mass is required to arise dynamically (so that \( m_0 \) is zero to solve the mass renormalization problem), and since Q.E.D. is a closed theory there should hopefully be no accompanying Goldstone boson. The authors of Ref. 1 noted that in the ladder approximation the \( \gamma_5 \) projection of the kernel was noncompact so that the bound-state problem was not of the Fredholm type. We see now that at the eigenvalue a possible pole in \( \tilde{\Gamma}_\mu^5 \) is avoided by Eq. (34) so that the \( \gamma_\mu \gamma_5 \) projection of the kernel can be compact and still not produce any Goldstone pole in \( \tilde{\Gamma}_\mu^5 \) even while it satisfies an inhomogeneous equation (similar in structure to Eq. (17) only with a nonzero \( Z_A \) and with a denominator equal to unity at \( q_\mu = 0 \)). These points will be explained in more detail elsewhere.\(^{14}\)

We conclude this section by noting how the Federbush-Johnson theorem\(^{19}\) is satisfied in finite Q.E.D. All that happens is that all matrix elements of the current \( J_\mu \) are purely real (i.e., just the equal time parts of the \( T \) products survive at the eigenvalue), while in the Wightmann functions \( J_\mu \) annihilates the vacuum since \( Z_3^{-1} \) is finite. In general the Federbush-Johnson theorem states that the n-point \( T \) products of a fundamental field will vanish if the imaginary part of the two point function vanishes. For a composite operator however, some real parts can survive even if its two point Wightmann function vanishes. This is because of the singularity at equal times that the \( T \) product possesses (i.e., the equal time commutator of \( J_\mu \) with \( \psi \) is independent of \( J_o \)). Thus in finite Q.E.D.

\[ \langle \Omega | T(\psi \mu J_\mu \bar{\psi})|\Omega \rangle \] vanishes, but not \( \langle \Omega | T(\psi \mu J_\mu \bar{\psi})|\Omega \rangle \), so that the Federbush-Johnson theorem does not entail the separate vanishing of \( A \) and \( B \). We mention this because the discussion of the Federbush-Johnson theorem is obscured by the lack of positivity in certain gauges in Q.E.D.\(^{20}\) Our point is that the real part of \( \tilde{\Gamma}_\mu \) is not obliged to vanish even if the Federbush-Johnson theorem can be extended to the charge sector in the finite gauge. Also \( \tilde{\Gamma}_\mu^\nu \) is purely real at the eigenvalue. However, the subtlety is that even a real \( \tilde{\Gamma}_\mu \) could contribute a logarithmic divergence to \( \tilde{\Gamma}_\mu^\nu \) through the Schwinger-Dyson equation to give \( \tilde{\Gamma}_\mu^\nu \) a discontinuity. Equation (5) stops this from happening by a cancellation between the form factors. Thus we need more than one form factor in order to satisfy the Federbush-Johnson theorem. Thus in conclusion we again remark that it is this interplay between the form factors of Eq. (2) which has enabled us to maintain consistency in the various situations that we have discussed throughout this work.
In this appendix we discuss briefly the interplay of gauge invariance and conformal invariance. As we noted in the introduction the transversality condition on $\pi^{\mu\nu}$ is only achieved if $J_{\mu}$ has dimension 3. Thus a canonical photon of dimension 1 would not be transverse. Adler and Abdellatif have resolved this conflict by introducing an extra 4-vector $b_{\mu}$, a gauge point, so that the conformal photon propagator $\left\langle \Omega \left| T(A_{\mu}(x) A_{\mu}(y)) \right| \Omega \right\rangle$ becomes a 3-point function in an arbitrary conformal gauge, $b_{\mu}$. Suppose we now try to construct the $\tilde{\alpha}_{\mu}$ vertex in perturbation theory by considering only graphs with a single continuous electron line dressed with free conformal photons. This $\tilde{\alpha}_{\mu}(x,z,y)$ will now depend on $b_{\mu}$ as well so that instead of the previous two form factors of Eq. (2) there will now be six independent form factors. Moreover since we now have four coordinates we can build harmonic ratios, so that the coefficients of these form factors will no longer be pure constants. By using the Wilson expansion Christ was able to eliminate these extra four unwanted form factors and recover Eq. (2) in the special gauge where $b_{\mu}$ is chosen to be $z_{\mu}$, the coordinate of the external current. This then leads us again to Eq. (24) in this particular gauge but not necessarily to Eq. (27) since in the presence of $b_{\mu}$ the Ward identity corresponds to a different kinematic configuration than that used for the short-distance Wilson expansion analysis. Thus in perturbation theory we obtain slightly less information than deduced in this paper. However, it is a moot point as to whether Christ's discussion should actually be applied when we are sitting at the eigenvalue. As we have indicated there is a nongraphical formulation of the theory which discusses the fermion sector alone without ever needing to introduce an explicit photon. In this approach we go straight to the eigenvalue and introduce Eqs. (1) and (2) without any need to introduce a gauge point at all, so that there are only two form factors and the Ward identity is relevant. This viewpoint, which merits further study, then implies that there are two formulations of finite Q.E.D., a perturbative graphical one summed loopwise with free photons, and a nonperturbative one at the eigenvalue, with perturbation theory not being a good guide to the structure of the theory at the eigenvalue itself.

There is one other open question posed by the work of Schnitzer and Christ. At the eigenvalue we have the naive Wilson expansion

$$T(\psi(x) \bar{\psi}(o)) = Z^{-1}_2 \frac{\lambda}{4} \left\{ C_1(\alpha) j^1(o) + C_2(\alpha) \frac{x \cdot y}{x^2} j^\nu(o) \right\}$$

(A1)

plus additional terms which are irrelevant to this discussion. The consistency of this expansion with the structure of Eqs. (2) and (3) (the Crewther analysis) leads to the relations

$$Z_2 A = C_1(\alpha) f(\alpha)$$

$$Z_2 B = -C_2(\alpha) f(\alpha)$$

(A2)

Thus from Eqs. (28) (with $Z_1 = Z_2$) we deduce that the Wilson coefficients have to be infinite at the eigenvalue, which is not a desirable situation. (Anomalous infinities in the Wilson coefficients will also be obtained if the axial-vector current triangle anomaly is not renormalized at the eigenvalue.) We shall conclude this appendix by presenting a possibility which could remove the
undesirable infinities in Eq. (A2). In obtaining Eq. (A1) it is tacitly assumed that at the eigenvalue the naive Wilson expansion still holds and has the structure suggested by perturbation theory where \( j_\mu \) is given explicitly by the fermion bilinear \( :\bar{\psi} \gamma_\mu \psi: \).

However, in our discussion of Eqs. (1) and (2) we never needed to make such an identification for \( j_\mu \). Now in general the conformally invariant Green's functions of fermion lines and fermion composites do correspond precisely with the naive forms that would be obtained in a free massless fermion theory, except in the special case of Eq. (2). In a free fermion theory the coefficient \( B \) vanishes identically, whereas in finite Q.E.D. we need a nontrivial \( B \) form factor to maintain consistency as we discussed in this paper. This suggests that at the eigenvalue the current \( j_\mu \) introduced in Eq. (1) may be represented by a more complicated object then \( :\bar{\psi} \gamma_\mu \psi: \), so that there could be possible additional terms in Eq. (A1). Unfortunately for the moment we see no way of testing this proposal, and of determining whether or not the Wilson expansion is reliable in a finite field theory.

REFERENCES AND FOOTNOTES


4. In his original work Migdal (Ref. 3) presented three form factors. Various authors then noted that one of the set was superfluous.


10. See e.g. G. K"allen, Quantum Electrodynamics, Springer-Verlag, New York, 1972. In passing we point out that K"allen's discussion could not distinguish between \( Z_1 \) vanishing in the limit of infinite cut-off and \( Z_1 \) vanishing say because of some eigenvalue condition for the coupling constant. (It is not thought that this actually happens to \( Z_1 \) in finite Q.E.D.) Though both situations lead to an infinite \( Z_1^{-1} \), the latter situation actually corresponds to a finite theory. Thus there can be theories in which the cut-off can be removed without introducing ultraviolet divergences in which some of the renormalization constants are still zero. Such theories should still be thought of as finite. Also we note that it is unlikely that \( Z_1 \) could vanish because of infrared divergences since in a massless theory ultraviolet and infrared divergences are handled simultaneously because of dimensional reasons.


16. If the meson were a bound state of mass $m$ then the coupling of the off-shell meson to an off-shell fermion anti-fermion pair would satisfy a Bethe-Salpeter equation with an inhomogeneous term of the form $(q^2 - m^2)\gamma_5$.
17. Experience with the Thirring model indicates that the anomalous dimension of the fermion has no physical significance even in a theory which possesses only a global gauge invariance and has no local gauge at all. See e.g. G. F. Dell'Antonio, Y. Frishman, and D. Zwanziger, Phys. Rev. D6, 988 (1972). Thus we can build the whole of the fermion sector by postulating Eqs. (1) and (2) only as an ansatz without needing to refer to the photon at all.
18. In a recent paper F. Englert, J. M. Frère, and P. Nicoletopoulos, Nuovo Cimento 19A, 305 (1974), tried to set up a vertex bootstrap in Q.E.D. to obtain the condition $Z_1 = 0$. However, it appears that they only retained the $A$ form factor in the finite gauge so that $\tilde{F}_\mu$ had no momentum dependence. Thus we do not share their conclusion that $Z_1 = 0$ is consistent with Eq. (29) in the finite gauge.
20. There has been no explicit derivation that positivity actually is lost in covariant gauges. Positivity of the electron self-energy spectral function would require the anomalous dimension $\gamma_F(\alpha)$ to be positive semidefinite, and this constraint is satisfied in the finite gauge. It is possible then that only those gauges which have $\gamma_F(\alpha) < 0$ are not subject to the Federbush-Johnson theorem.
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