Submitted for publication

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THE EINSTEIN-PODOLSKY-ROSEN AND BELL PARADOXES

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November 1982
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AXIOMATIC SET THEORY AND PROPOSED RESOLUTION OF THE
EINSTEIN-PODOLSKY-ROSEN AND BELL PARADOXES

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ABSTRACT

A recent proposal for using axiomatic set theory
to resolve the Einstein-Podolsky-Rosen and Bell paradoxes
is examined and found to violate the critical locality
requirement.

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This work was supported by the Director, Office of
Energy Research, Office of High Energy and Nuclear
Physics, Division of High Energy Physics of the U.S.
Department of Energy under Contract DE-AC03-76SF00098.

Pitowsky\textsuperscript{1} has claimed to have resolved the Einstein-Podolsky-
Rosen and Bell paradoxes by means of a local model of spin correlations
based on axiomatic set theory. Mermin\textsuperscript{2} and MacDonald\textsuperscript{3} have apparently
refuted this claim: they have demonstrated that one key property of
Pitowsky's model, namely that it assigns to each electron a definite
spin-value of $+1/2$ or $-1/2$ in each direction, is by itself incom-
patible with the validity of the statistical predictions of quantum
theory. In particular, Mermin showed, essentially, that no combi-
nation of finite sequences of spin values $+1/2$ or $-1/2$ in three
specified direction can yield values of the associated spin-correlation
average values that lie simultaneously within certain finite distances
of the limiting values predicted by quantum theory: at least one of
the examined spin-correlation average values cannot approach the
large-$n$ limit predicted by quantum theory. Since experimental physics
deals with large yet finite numbers of experiments it seems apparent
that no argument based on abstract postulates concerning transfinite
numbers and operations can overturn the elementary numerical argument
of Mermin. Yet Pitowsky does not retract his claim.\textsuperscript{4}

To evaluate this situation two remarks about Pitowsky's arguments
may prove helpful.

Pitowsky's arguments are based on the axiom of choice and the
continuum hypothesis (or the weaker Martin's axiom). The deep results
of Gödel and Cohen showed that one can independently add either one, or
both, of these two axioms to the remaining standard axioms of set
theory and obtain alternative set theories that are self-consistent
if the original one is. The question thus arises: which of these
alternatives set theories, if any, is a suitable ingredient of physical
theory?
Important light is shed on this question by the fact that if one includes the two extra axioms then it is possible, by using the methods employed by Pitowsky, to prove (nonconstructively) the existence of a bivalent function \( f(x, y) = \pm 1/2 \) defined on the unit square \( S = \{(x, y); 0 \leq x \leq 1; 0 \leq y \leq 1\} \) that enjoys the following properties:

\[
\int_0^1 f(x, y) \, dx = 1/2 \quad \text{for every } y(0 \leq y \leq 1); \quad \int_0^1 f(x, y) \, dy = -1/2 \quad \text{for every } x(0 \leq x \leq 1); \quad f(a, b) = -f(b, a) \quad \text{for every } a \neq b.
\]

Let \((x_1, x_2, \ldots)\) and \((y_1, y_2, \ldots)\) be two infinite sequences satisfying \(0 \leq x_i \leq 1, 0 \leq y_i \leq 1\) (all \(i\)), and define \(f_n = n^{-1}(f(x_1, y_1) + \ldots + f(x_n, y_n))\). Then the law of large numbers entails the following two properties: (1), for any fixed sequence \((y_1, y_2, \ldots)\) the sequence \((f_1, f_2, \ldots)\) tends to 1/2 on a set of sequences \((x_1, x_2, \ldots)\) of measure one, where this measure is calculated by, for each \(i\), restricting \((x_1, y_1)\) to the line of fixed \(y_i\); (2), for any fixed sequence \((x_1, x_2, \ldots)\) the sequence \((f_1, f_2, \ldots)\) tends to -1/2 on a set of sequences \((y_1, y_2, \ldots)\) of measure one, where the measure is calculated by, for each \(i\), restricting \((x_1, y_1)\) to the line of fixed \(x_i\). If measure one corresponds to probability one then one concludes that the sequence \((f_1, f_2, \ldots)\) tends to +1/2 with probability one if the sequence \((y_1, y_2, \ldots)\) is considered fixed, but tends to -1/2 with probability one if the sequence \((x_1, x_2, \ldots)\) is considered fixed. Since the predicted limit of the sequence \((f_1, f_2, \ldots)\) should not depend on whether one imagines the \(x_i\)'s or \(y_i\)'s to be fixed one can conclude that the version of set theory that includes the two extra axioms does not always lead to sensible physical theories.

This conclusion does not necessarily entail that the particular theory developed by Pitowsky is not sensible. But Pitowsky attempts to circumvent Mermin's argument by using his model not as a literal model of the sequence of electrons in a particular experiment but rather as a basis for claiming that certain averages converge to their quantum-theoretical values with probability one. However, the example of the function \(f(x, y)\) discussed above shows that if the two extra axioms are accepted then the value to which an average converges with probability one can depend critically upon what restrictions are placed on the set of points that are used to calculate this probability.

In Pitowsky's model one considers first a sequence of spin functions, and then the subsequence \((s_1, s_2, \ldots)\) in which the spin is "up" in some direction \(y\). If \(w\) represents some second direction then the average value \(A_n(w, y) = n^{-1}(s_1(w) + \ldots + s_n(w))\) is shown to tend, in the large-\(n\) limit, to the quantum-theoretical value \(y \cdot w / 2\) on a set of directions \(w\) of measure one, where this measure is calculated by restricting the directions \(w\) to the circle \(c(y, \arccos y \cdot w)\).

The quantity \(A_n(w, y)\) depends on \(y\) exclusively through the fact that the sequence \((s_1, s_2, \ldots)\) is defined relative to \(y\). But the quantity that equals \(y \cdot w / 2\) in Pitowsky's theory depends on \(y\) in a second way: the explicitly specified measure is with respect to the set of directions \(w\) confined to the circle \(c(y, \arccos y \cdot w)\). In the context of the EPR-Bell experiment this explicit second dependence on \(y\) is a dependence on the far-away choice of measurement. Thus the procedures used to calculate the value
from the underlying local spin model introduce an explicit nonlocal element. It is the occurrence of this explicit nonlocal element in Pitowsky's calculational procedure that reconciles his theory with Bell's result that no local theory can reproduce the statistical predictions of quantum theory.

ACKNOWLEDGMENT

This work rests heavily on conversations with Professor P. Chernoff and Professor J. Feldman of the mathematics department, and has benefited from discussions with P. Eberhard, T. Trippe, and J. Karush. This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.

REFERENCES

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

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