Title
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Permalink
https://escholarship.org/uc/item/5ct8666b

Journal
Physics of Fluids, 7(4)

ISSN
00319171

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Publication Date
1964

DOI
10.1063/1.1711228

Peer reviewed
Test Particle Method in Kinetic Theory of a Plasma

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(Received 22 November 1963)

A test particle of coordinates \( X = (x, v) \) is surrounded by a shield cloud of field particles of coordinates \( X' \) characterized by a conditional probability function \( P(X, X'; t) \). A relationship has been found between this function, the one-particle function \( f(X, t) \) and the two-particle correlation function \( G(X, X'; t) \). It is

\[
G(X, X'; t) = f(X)P(X | X'; t) = f(X)P(X' | X; t) + ne \int dX'' f(X'', t)P(X'' | X)P(X'' | X'; t).
\]

The first two terms indicate that each of the two particles involved is a test particle as well as part of the shield cloud of the other particle. The last term corresponds to the two particles shielding a third particle. This relation has been established without solving explicitly for anything and has none of the usual restrictions such as spatial homogeneity, adiabatic time behavior, etc., usually necessary for obtaining explicit solutions. It is useful because the problem of kinetic theory is reduced to determining \( P \) which involves only the Vlasov equation. In addition, superposition principles for fluctuations, etc., are apparent at the outset.

I. INTRODUCTION

We consider a gas of charged particles interacting only through Coulomb forces. The system may be described by the Liouville equation

\[
\frac{\partial}{\partial t} + \sum_{i=1}^{N} v_{i} \cdot \frac{\partial}{\partial x_{i}} - \frac{e}{m} \left[ F_{\text{ext}}(X, t) + \sum_{i=1}^{N} \frac{1}{|x_{i} - x_{j}|} \frac{\partial}{\partial v_{i}} f_{i} \right] D(X, t) = 0, \tag{1}
\]

where \( X_{i} = (x_{i}, v_{i}) \) the position and velocity of the \( i \)th particle, and \( X = (X_{1}, X_{2}, \ldots X_{N}) \). \( F_{\text{ext}}(X, t) = E_{\text{ext}}(x_{i}, t) + (1/c) v_{i} \cdot B_{\text{ext}}(x_{i}, t) \) where \( E_{\text{ext}} \) and \( B_{\text{ext}} \) are externally applied fields. Infinite mass randomly distributed ions are assumed leaving only electrons of charge \(-e\) and mass \( m \). This restriction is easily relaxed.

The system can also be described by the Bogoliubov–Born–Green–Kirkwood–Yvon hierarchy which is obtained by taking moments of Eq. (1)

\[
\frac{\partial}{\partial t} + \sum_{i=1}^{N} v_{i} \cdot \frac{\partial}{\partial x_{i}} - \frac{e}{m} \left[ F_{\text{ext}}(X, t) \right]
+ \sum_{i=1}^{N} \frac{1}{|x_{i} - x_{j}|} \frac{\partial}{\partial v_{i}} f_{i}
- \frac{ne^{2}}{m} \sum_{i=1}^{N} \int \frac{1}{|x_{i} - x_{j}|} \frac{\partial f_{i+1}}{\partial v_{i}} dX_{j} = 0, \tag{2}
\]

where

\[
f_{1}(X_{1}, \ldots X_{i}; t) = V' \int D(X_{i}) dX_{i+1} \cdots dX_{N}.
\]

It has previously been shown that Eq. (2) can be solved approximately by expanding in a parameter \( g = 1/nL_{n}^{a} \) where \( n \) is the density and \( L_{n} \) is the Debye length. Alternately one may expand in the discreteness parameters \( e, m \), and \( 1/n \) considered as being of the same order. To first order one finds

\[
f_{1}(X_{1}, \ldots X_{i}; t) = \prod_{i=1}^{N} f(X_{i}) + \frac{1}{2} \sum_{i=1}^{N} \left[ \prod_{i \neq k}^{N} f(X_{k}) \right] G(X_{i}, X_{i}; t), \tag{3}
\]

provided that \( f(X_{i}) \) and \( G(X, X'; t) \) satisfy the following equations

\[
\frac{\partial f_{i}}{\partial t} + \frac{v_{i}}{x_{i}} \cdot \frac{\partial f_{i}}{\partial x_{i}} - \frac{e}{m} \frac{\partial f_{i}}{\partial v_{i}} F_{\text{ext}}(X_{i}, t) \frac{\partial f_{i}}{\partial v_{i}}
= \frac{ne^{2}}{m} \int \frac{1}{|x_{i} - x'|} \frac{\partial}{\partial v} G(X, X'; t) dX', \tag{4}
\]

\[
\frac{\partial}{\partial t} G(X, X'; t) + \left[ O(X_{i}) + O(X'_{j}) \right] G(X, X'; t)
= \frac{e^{2}}{m} \frac{\partial}{\partial x_{i}} \frac{1}{|x_{i} - x'|} \left[ f(X_{i}) \frac{\partial f(X_{i})}{\partial v} - f(X_{i}) \frac{\partial f(X_{i})}{\partial v'} \right] \tag{5}
\]

where

\[
O(X_{i}) \text{ is an operator that involves differentiation and integration, i.e.,}
\]

\[
\frac{1}{n L_{n}^{a}} \frac{\partial}{\partial t} + \frac{1}{e} \frac{\partial}{\partial x_{i}} \left[ \frac{1}{|x_{i} - x'|} f(X_{i}) dX' \right]
= \frac{1}{n L_{n}^{a}} \frac{\partial}{\partial t} \frac{1}{e} \frac{\partial}{\partial x_{i}} \left[ \frac{1}{|x_{i} - x'|} f(X_{i}) dX' \right].
\]

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Our objective is to show the relationship between this problem and a test-particle problem. The test-particle problem can be formulated simply by assuming that there is an additional external electric field

\[ E_{i}(x, t) = \frac{\partial}{\partial x} \frac{e}{|x - x_{i}(t)|} \]

where \( x_{i}(t) \) is the test-particle orbit. Thus Eq. (1) becomes

\[
\begin{aligned}
    \left[ \frac{\partial}{\partial t} + \sum_{i=1}^{N} v_{i} \frac{\partial}{\partial x_{i}} - \frac{e}{m} \sum_{i=1}^{N} \frac{\partial F_{i}(x, t)}{\partial x_{i}} \right] \delta f(X) \\
    + \sum_{i=1}^{N} \frac{\partial}{\partial x_{i}} \left[ \frac{e}{m} \frac{\partial F_{i}(x, t)}{\partial v_{i}} \right] \delta D(X) \\
    = \frac{e^{2}}{m} \sum_{i=1}^{N} \frac{\partial}{\partial x_{i}} \frac{1}{|x - x_{i}(t)|} \frac{\partial \delta D}{\partial v_{i}} \tag{1'}
\end{aligned}
\]

The previous procedure for producing a chain of equations and solving it by expansion results in the following modifications of Eqs. (4) and (5):

\[
\begin{aligned}
    \frac{\partial}{\partial t} \delta f + v_{o} \frac{\partial}{\partial x} \delta f - \frac{e}{m} F_{M}(X_{0}, t) \frac{\partial}{\partial v} \delta f \\
    = \frac{ne^{2}}{m} \int \frac{\partial}{\partial x} \frac{1}{|x - x'|} \frac{\partial}{\partial v} \hat{G}(X, X'; t) dx' \\
    + \frac{e^{2}}{m} \frac{\partial f(X_{0}, t)}{\partial v} \frac{1}{|x - x_{0}(t)|} \frac{\partial \hat{G}(X, X'; t)}{\partial v_{0}} \tag{4'}
\end{aligned}
\]

\[
\begin{aligned}
    \frac{\partial}{\partial t} \hat{G}(X, X'; t) - \frac{e^{2}}{m} \frac{\partial}{\partial x} \frac{1}{|x - x_{0}(t)|} \frac{\partial \hat{G}(X, X'; t)}{\partial v_{0}} \\
    - \frac{e^{2}}{m} \frac{\partial}{\partial x} \frac{1}{|x - x_{0}(t)|} \frac{\partial \hat{G}(X, X'; t)}{\partial v_{0}} \\
    + [\hat{O}(X_{0}) + \hat{O}(X_{0})] \hat{G}(X, X', t) \\
    = \frac{e^{2}}{m} \frac{\partial}{\partial x} \frac{1}{|x - x'|} \left[ f(X', t) \frac{\partial \delta f}{\partial v} - f(X_{0}) \frac{\partial \delta f}{\partial v_{0}} \right] \tag{5'}
\end{aligned}
\]

Since we are only concerned with a first order calculation in \( g = e, m, \) or \( 1/n, \) some simplifications are permitted. Since \( \hat{G} \) is already first order, the second and third terms of Eq. (5') are second order and may be omitted. Furthermore, Eq. (4') differs from Eq. (4) only in the second term on the right which is clearly a first order quantity being due to only one test charge. Therefore we may assume

\[ \delta f(X_{0}) = f(X_{0}) + \delta f(X_{0}) \]

where \( \delta f(X_{0}) \) is a first-order quantity. Thus in Eq. (5') \( f(X_{0}) \) can everywhere be replaced by \( f(X') \). \( \hat{O}(X_{0}) \) differs from \( O(X_{0}) \) only in the fact that \( f(X_{0}) \) in Eq. (6) is replaced by \( f(X') \). Therefore \( \hat{O}(X_{0}) = O(X_{0}) \) in Eq. (5') and finally we deduce that to first order \( \hat{G}(X, X'; t) = G(X, X'; t) \). Equation (4') to first order in \( g \) becomes

\[
\frac{\partial}{\partial t} \delta f + e \frac{\partial}{\partial x} \delta f - \frac{e}{m} F_{M}(X_{0}, t) \frac{\partial}{\partial v} \delta f \\
- \frac{e}{m} \frac{\partial f(X_{0}, t)}{\partial v} \frac{\partial}{\partial v} \frac{1}{|x - x_{0}(t)|},
\]

where

\[
\delta F_{M}(X, t) = ne \int \frac{\partial}{\partial x} \frac{1}{|x - x'|} \delta f(X') dx'
\]

and we have made use of Eq. (4). The time dependence of \( \delta f(X_{0}) \) can more conveniently be expressed by introducing \( P(X_{0}, t) = \delta f(X_{0}) \) in which case

\[
\frac{\partial}{\partial t} P(X_{0}, t) = \frac{\partial}{\partial t} P(X_{0} | X_{0}) + v_{0} \frac{\partial}{\partial x_{0}} P + v_{0} \frac{\partial}{\partial v_{0}} P.
\]

The orbit \( x_{0}(t) \) has thus far not been specified. We are free to specify it as we please. Assume that it satisfies the equations

\[
\frac{dx_{0}}{dt} = v_{0}, \quad \frac{dv_{0}}{dt} = -\frac{e}{m} F_{M}(X_{0}, t)
\]

With these definitions the test particle problem can be stated as follows:

\[
\frac{\partial}{\partial t} P(X_{0} | X_{0}) + \left( v_{0} \frac{\partial}{\partial x_{0}} - \frac{e}{m} F_{M}(X_{0}, t) \frac{\partial}{\partial v_{0}} P(X_{0} | X_{0}) \right) \frac{dx_{0}}{dt} + n \int dX'' f(X'') \delta f(X') P(X'' | X_{0}) \tag{7}
\]

We now proceed to show that a relationship exists between the solution of Eq. (7) and the solution of Eq. (5). It is

\[ G(X, X'; t) = f(X_{0}) P(X | X') + f(X') P(X' | X_{0}) \]

which is valid to the same order of approximation that Eqs. (5) and (7) are valid. Equation (8) has the following physical interpretation. There are two particles involved in the correlation function \( G(X, X'; t) \). The first term corresponds to \( X' \) being a field particle and \( X \) a test particle with the proba-
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\[ \frac{\partial f(X_t)}{\partial t} P(X \mid X\prime t) + \frac{\partial f(X\prime t)}{\partial t} P(X\prime \mid X_t) + n \int dX'' \frac{\partial f(X''t)}{\partial t} P(X'' \mid X'0)P(X'' \mid X' t) + f(X_t) \frac{\partial}{\partial t} P(X \mid X\prime t) \]

\[ + f(X' t) \frac{\partial}{\partial t} P(X \mid X_t) + n \int dX'' f(X''t) \left[ \frac{\partial P(X'' \mid X_t)}{\partial t} P(X'' \mid X' t) + P(X'' \mid X_t) \frac{\partial P(X'' \mid X' t)}{\partial t} \right] \]

\[ + O(X_0)[f(X_0)P(X \mid X' t)] + f(X' t)O(X_0)P(X' \mid X_t) + O(X' t)[f(X_t')P(X' \mid X_t)] + f(X_t)O(X' t)PX \mid X' t \]

\[ + n \int dX'' f(X''t)[O(X_0)P(X'' \mid X' t)]P(X'' \mid X' t) + P(X'' \mid X_t)[O(X_t')P(X'' \mid X' t)] \]

\[ = \frac{\partial}{\partial t} \frac{1}{|x - x'|} \left[ f(X_t) \frac{\partial}{\partial v} f(X_t) - f(X_t') \frac{\partial}{\partial v} f(X_t') \right]. \]

Now substitute Eq. (7) to eliminate all expressions of the type \( \frac{\partial}{\partial t} P(X \mid X\prime t) \) + \( O(X' t)P(X \mid X' t). \) This produces the following transformation of Eq. (1):

\[ \frac{\partial f(X_t)}{\partial t} P(X \mid X\prime t) + \frac{\partial f(X\prime t)}{\partial t} P(X\prime \mid X_t) + n \int dX'' \frac{\partial f(X''t)}{\partial t} P(X'' \mid X'0)P(X'' \mid X' t) \]

\[ - f(X_t) \left[ v \frac{\partial}{\partial v} P(X \mid X' t) - \frac{e}{m} F_n(X_t) \frac{\partial}{\partial v} P(X \mid X' t) \right] \]

\[ - f(X' t) \left[ v' \frac{\partial}{\partial v} P(X \mid X_t) - \frac{e}{m} F_n(X' t) \frac{\partial}{\partial v} P(X' \mid X_t) \right] + O(X_0)[f(X_0)P(X \mid X' t)] \]

\[ + O(X' t)[f(X_t)P(X' \mid X_t)] + n \int dX'' f(X''t)P(X'' \mid X' t) \left[ -v'' \frac{\partial}{\partial v''} P(X'' \mid X_t) \right] \]

\[ + \frac{e}{m} F_n(X'' t) \frac{\partial}{\partial v''} P(X'' \mid X_t) + \frac{e^2}{m} f(X_t) \frac{\partial}{\partial v''} P(X'' \mid X_t) \]

\[ = n \int dX'' f(X''t)P(X'' \mid X_t) \]

\[ = \left[ v'' \frac{\partial}{\partial v''} - \frac{e}{m} F_n(X'' t) \frac{\partial}{\partial v''} \right] P(X'' \mid X_t) \]

\[ + n \int dX'' P(X'' \mid X_t)P(X'' \mid X' t) \]

\[ = O(X_t')[f(X_t)P(X' \mid X_t)] \]

\[ - O(X' t)[f(X_t)P(X' \mid X_t)] \]
Combine this with the second last expression and the result is
\[ \frac{dX'}{dt} P(X'' | X') P(X' | X') \]
\[ \left( \frac{v'' \cdot \partial}{\partial x} - \frac{e}{m} F_u(X') \cdot \frac{\partial}{\partial v} \right) \frac{f(X'')}{P(X'')} \]
\[ = n q_i \int \frac{\partial}{\partial x} \frac{q_i}{m} \frac{\partial}{\partial v} G_i(X, X'; t) \, dX'. \] (11)

The macroscopic field is
\[ F_u(X') = F_{u0}(X') \]
\[ - n \int \frac{\partial}{\partial x} \frac{q}{m} f_i(X') \, dX'. \]

The pair correlation function \( G_i(X, X'; t) \) has the symmetry property \( G_i(X, X'; t) = G_i(X', X; t) \) and satisfies the equation
\[ \frac{\partial}{\partial t} G_i(X, X'; t) + \sum_i O_{ii}(X) G_i(X, X'; t) \]
\[ + O_{ii}(X') G_i(X', X; t) = \frac{q_i q_i}{m_i} \frac{\partial}{\partial x} \frac{1}{|x - x'|} \( f_i(X') \cdot \frac{\partial f_i(X')}{\partial v} \), \] (12)

which is the generalization of Eq. (5). The operator \( O_{ii}(X) \) is given by
\[ O_{ii}(X) = \delta_{ii} \left[ v \cdot \frac{\partial}{\partial x} + \frac{q_i}{m_i} F_u(X') \cdot \frac{\partial}{\partial v} \right] \]
\[ - n q_i \frac{\partial f_i(X')}{\partial v} \frac{\partial}{\partial x} \frac{1}{|x - x'|} \] \( X' \),

where the function it operates on is to be placed in the curly brackets.

The generalization of Eq. (13) is
\[ \frac{\partial}{\partial t} P_i(X | X') \]
\[ + \sum_i O_{ii}(X') P_i(X | X') + O_{ii}(X) P_i(X | X') \]
\[ = \frac{q_i q_i}{m_i} \frac{\partial}{\partial v} \frac{1}{|x - x'|} \] \( X' \).

As an illustration of Eq. (14) consider the case of thermal equilibrium. Assume that
\[ f_i(X) = \frac{(2\pi m_i)^{3/2} e^{-m_i/2\Theta}}{4\pi^{3/2} e^{3/2}}, \]
where \( m_i e^2 = \Theta \) and \( F_{u0}(X') = 0 \). Since \( f_i(v) \) is

III. GENERALIZATION TO INCLUDE IONS

If we use the label \( i \) to denote the species, \( q_i \) for the charge, and \( m_i \) for the mass, Eq. (4) becomes
\[ \frac{\partial f_i(X)}{\partial t} + v \cdot \frac{\partial f_i}{\partial x} + \frac{q_i}{m_i} \frac{\partial f_i}{\partial v} \]
\[ = \frac{n q_i}{m_i} \int \frac{\partial}{\partial x} \frac{q_i}{m} \frac{\partial}{\partial v} G_i(X, X'; t) \, dX'. \]
independent of \( x \), \( F_w(X_t) = 0 \). For this case Eq. (13) simplifies to

\[
\frac{\partial}{\partial t} P_{i}(X | X^t) + \left( v \cdot \frac{\partial}{\partial x} + v^r \cdot \frac{\partial}{\partial x^r} \right) P_{i}(X | X^t) = \frac{1}{m_i} \sum q_i \frac{\partial f_j}{\partial v^r} \cdot \frac{1}{|x - x^r|} \frac{1}{P_{i}(X | X^t)}.
\]

This can be solved by Fourier and Laplace transformation,

\[
P_{i}(X | X^t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi i} e^{i\lambda p} \int d\lambda e^{i\lambda (x - x^t)} P_{i}(k; v | v p),
\]

where \( \bar{P}_{i} \) satisfies

\[
[p + i \cdot (v' - v)] \bar{P}_{i} = 4\pi n \sum \frac{q_i q_j}{m_i} \frac{i k}{k^2} \frac{\partial f_j}{\partial v^r} + \int d\nu' \bar{P}_{i}(k; v | v', p) - \int d\nu' \bar{P}_{i}(k; v | v', p),
\]

This can, of course, be solved by dividing by \( p + i k \cdot (v' - v) \) and integrating. The solution is conveniently expressed in terms of the dielectric coefficient

\[
\epsilon(k, p) = 1 - \sum \frac{\omega_j^2}{k^2} \int \frac{d\nu' \bar{P}_{i}(k; v | v', p)}{p + i k \cdot (v' - v)} dv' = 1 + \sum \frac{\omega_j^2}{k^2} \int_0^\infty dt e^{-i\nu't^2} e^{-1/k^2 t^2}, \tag{16}
\]

where \( \omega_j^2 = 4\pi n q_j^2 / m_i \) is the plasma frequency of species \( j \). Thus the asymptotic solution is

\[
\bar{P}_{i}(k; v | v') = \lim_{p \to \infty} p \bar{P}_{i}(k; v | v', p) = -\frac{4\pi q_j q_i}{m_j m_i} \frac{(v' - v) f_i(v)}{k^2 \epsilon(k, -ik \cdot v)}, \tag{17}
\]

which describes the shielding of a particle in thermal equilibrium. Assuming

\[
\tilde{G}_{i}(X, X') = \frac{d \lambda}{(2\pi)^3} e^{i\lambda (x - x')} \tilde{G}_{i}(k; v, v'),
\]

the Fourier transform of Eq. (14) is

\[
\tilde{G}_{i}(k; v, v') = f_i(v) \bar{P}_{i}(k; v | v') + f_i(v') \bar{P}_{i}(k; v' | v) + n \int d\nu' \sum f_i(v') \bar{P}_{i}(k; v' | v'') P_{i}(k; v'' | v').
\]

After substituting Eq. (17) this becomes

\[
\tilde{G}_{i}(k; v, v') = -\frac{4\pi q_j q_i}{\Theta k^2} f_i(v) f_i(v')
\]

This integral vanishes because \( \epsilon(k, -ik'v') \) has no poles in the upper half of the \( v' \) plane, and \( \lim_{u \to 1} \epsilon(k, -ik'v') = 1 \) for \( \arg u' < \pi \). Writing out \( 1/(u' - u + i\delta) \) as \( P \epsilon(1/(u' - u - i\delta)) \), etc., and taking real and imaginary parts leads to the following integral dispersion relations

\[
\frac{1}{\pi} \int du' \frac{P}{u' - u - i\delta} e^{i\delta(\nu + \nu')}, \tag{17'}
\]

where \( P \) means the principal part. (The imaginary part of the quantity in brackets is not singular at \( u = 0 \), but the real part is.) Dispersion relation can be derived from the following integral:

\[
\int_{-\infty}^{\infty} \frac{du'}{u' - u + i\delta} \frac{1}{u' + i\delta} = 0.
\]
The point of this calculation is to show that verification of Eq. (14) or Eq. (8) directly from the analyticity properties and suggests unnecessary mathematical restrictions. This is the case for most of the mathematical manipulations necessary to get the results of kinetic theory into a form that can be interpreted physically. It is easier to carry them out before solving the problem as in the derivation of Eq. (8).

IV. FLUCTUATIONS

Consider for example the calculation of the ensemble average \( \langle A(x_t)B(x', t) \rangle \) where \( A(x_t), B(x_t) \) are any observables of the form

\[
A(x_t) = \sum_{i=1}^{N} a(x_i | x), \quad B(x, t) = \sum_{i=1}^{N} b(x_i | x). \tag{19}
\]

For example, if \( a(x_i | x) = \partial / \partial x_1 (\partial / \partial x - x_i | x) \) \( A(x_t) = E_r(x_t) \), the \( x \) component of the electric field. \( x_i = (x_i, v_i) \) are coordinates and velocities of the particles evaluated at time \( t \) and the sum is over all particles. The ensemble average is defined as

\[
\langle A(x_t)B(x', t) \rangle = \int dX D(X_t) \sum_{i=1}^{N} a(x_i | x)b(x_i | x') \tag{20}
\]

Integrations over all coordinates but two can be carried out with the result

\[
\langle A(x_t)B(x', t) \rangle = n \int f(X_t) a(x_1 | x)b(X_1 | x') \, dX_1 + n^2 \int f(X_t) f(X_2) a(x_1 | x)b(x_1 | x') \, dX_1 \, dX_2.
\]

The first term comes from \( i = j \) in the sum and the second from \( i \neq j \). Now substitute \( f_2(x_1, X_2; t) = f(X_t) f(X_2) + G(X_1, X_2; t) \) and Eq. (8) for \( G \).

\[
\langle A(x, t)B(x', t) \rangle = \langle A(x_t)B(x', t) \rangle + n \int f(X_t) a(x_1 | x)b(X_1 | x') \, dX_1
\]

\[
+ n^2 \int f(X_t) f(X_2) a(x_1 | x)b(x_1 | x') \, dX_1 \, dX_2.
\]

Instead of the bare-particle quantities \( a(x_1 | x), b(x_1 | x) \), define the corresponding quantities for quasiparticles,

\[
d(x_1 | x_t) = a(x_1 | x)
\]

\[
+ n \int a(x_2 | x) p(X_1 | x_t) \, dX_2,
\]

etc. and substitute into Eq. (21). Thus

\[
\langle A(x_t)B(x' t) \rangle = \langle A(x_t)B(x' t) \rangle
\]

\[
+ n \int f(X_t) a(X_1 | x)b(X_1 | x') \, dX_1
\]

\[
+ n^2 \int f(X_t) f(X_2) a(X_1 | x)b(X_1 | x') \, dX_1 \, dX_2.
\]

Thus we have established the principle of superposition of statistically independent quasiparticles under very general circumstances, i.e., for any observable expressible by Eq. (19) and for any time and space variation of \( f(X_t) \). Previously\(^3\) such relationships have been derived by obtaining explicit solutions for specific cases and manipulating them into the form of Eq. (23) after a great deal of algebra.

V. TWO-TIME DISTRIBUTION FUNCTIONS

In order to calculate correlation functions, it is necessary to introduce two-time distribution func-

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\(^{3}\) Some further examples of direct verification are given in the previous paper [N. Rostoker, Phys. Fluids 7, 479 (1964)].

\(^{N. Rostoker, Nucl. Fusion 1, 101 (1961).}\)
Before printing, for example, \( D_2(X, t; X'_{\tau}) \) \( dX \) \( dX' \) is the probability of finding the system in \((X, dX)\) at \( t \) and in \((X', dX')\) at \( t'. \) \( D_2(X,t;X'_{\tau}) \) satisfies the Liouville equation in \((X', t')\) and the initial condition 

\[
D_2(X,t;X'_{\tau}) = D(Xt) \delta(X - X').
\]

We can obtain a BBKGY chain for \( D \) and solve it approximately by expansion as in the case of \( D(Xt). \) Since this has been done previously\(^3\) we simply quote the results. For most calculations we require only two moments of \( D_2 \), namely

\[
W_{11}(X,t;X'_{\tau}) = V^2 \int D_2(X,t;X'_{\tau})
\]

\[
dX_2 \cdots dX_N \int dX' \int dX'' \cdots dX'_N,
\]

\[
W_{12}(X,t;X'_{\tau}) = V^2 \int D_2(X,t;X'_{\tau})
\]

\[
dX_2 \cdots dX_N \cdot dX'_2 \cdots dX'_N.
\]

In terms of these moments the autocorrelation function \( \langle A(xt)B(x'_{\tau}) \rangle \) is

\[
\langle A(xt)B(x'_{\tau}) \rangle = \int dX D_2(X,t;X'_{\tau}) \sum_{i=1}^N a(X_i | x) b(X'_i | x')
\]

\[
= \frac{n}{V} \int dX_1 \cdots dX_N \int dX' \int dX'' \cdots dX'_N \int dX' W_{11}(X_1,t;X'_1) W_{12}(X_1,t;X'_1).
\]

(24)

If the parameter expansion \( (g = e, m, \) or \( 1/n) \) is carried out as discussed in Sec. I, the equations for \( W_{11} \) and \( W_{12} \) are as follows:

\[
\left[ \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} - \frac{e}{m} F_m(X_1, t) \cdot \frac{\partial}{\partial v_{1i}} \right] W_{11}(X_1,t;X'_1) = 0
\]

(25)

and

\[
W_{11}(X_1,t;X'_1) = Vf_j(xt) \delta(X'_1 - X_1)
\]

is the initial condition.

\[
W_{12}(X_1,t;X'_1) = f(xt)f(x'_{\tau}) + G_{12}(X_1,t;X'_1),
\]

where

\[
\left[ \frac{\partial}{\partial t} + O(X'_1) \right] G_{12}(X_1,t;X'_1) = \frac{e^2}{m} \frac{\partial}{\partial v'^2} \int \frac{W_{11}(X_1,t;X'_1) dX'^1}{|x'_2 - x'_1|}
\]

(26)

and the initial condition is

\[
G_{12}(X_1,t;X'_1) = G(X_1, X'_1; 0).
\]

The formal solution of Eq. (26), as may be verified by direct substitution, is

\[
G_{12}(X_1,t;X'_1) = \frac{1}{V} \int dX_2 W_{11}(X_2,t;X'_2) P(x_1 | X'_2)
\]

\[
+ \frac{1}{V} \int dX_2 W_{11}(X_2,t;X'_2) P(X_2 | x_1,t)
\]

\[
+ \frac{n}{V} \int dX_2 dX'_2 W_{11}(X_2,t;X'_2) P(x_1 | X'_2).
\]

(27)

It is clear that this is a fairly obvious generalization of Eq. (8). Returning now to Eq. (24) and substituting Eq. (27), we obtain

\[
\langle A(xt)B(x'_{\tau}) \rangle = \frac{n}{V} \int dX_1 dX'_1 \int dX_2 dX'_2 W_{11}(X_1,t;X'_1)
\]

\[
\cdot a(X_1 | x) b(X'_1 | x') + \langle A(xt) \rangle \langle B(x'_{\tau}) \rangle
\]

\[
+ \frac{n^2}{V} \int dX_1 dX'_1 \int dX_2 dX'_2 W_{11}(X_1,t;X'_1) W_{12}(X_1,t;X'_1)
\]

\[
\cdot \cdot a(X_1 | x) b(X'_1 | x') P(X'_1 | X'_2) + \langle A(xt) \rangle \langle B(x'_{\tau}) \rangle
\]

\[
+ \frac{n^2}{V} \int dX_1 dX'_1 \int dX_2 dX'_2 W_{11}(X_1,t;X'_1) W_{12}(X_1,t;X'_1)
\]

\[
\cdot b(X'_1 | x') \int dX_1 a(X_1 | x) P(X_1 | x_1,t)
\]

\[
+ \frac{n^2}{V} \int dX_1 dX'_1 \int dX_2 dX'_2 W_{11}(X_1,t;X'_1) W_{12}(X_1,t;X'_1)
\]

\[
\cdot a(X_1 | x) P(X_1 | x_1,t) \int dX'_1 b(X'_1 | x') P(X'_1 | X'_2).
\]

After making use of the definitions of \( a(X_1 | x_1), \) etc., for quasiparticles given by Eq. (22), this reduces to

\[
\langle A(xt)B(x'_{\tau}) \rangle = \frac{n}{V} \int dX_1 dX'_1 \int dX_2 dX'_2 W_{11}(X_1,t;X'_1)
\]

\[
\cdot a(X_1 | x_1) b(X'_1 | x'_{\tau}) + \langle A(xt) \rangle \langle B(x'_{\tau}) \rangle.
\]

(28)

The principle of superposition is thus obtained in a very general form.

In order to include finite-mass ions, Eq. (27) must be modified as follows

\[
G_{12}(X_1,t;X'_1) = \frac{1}{V} \int dX_2 W_{11}(X_2,t;X'_2) P_{s_2}(x_1 | X'_2)
\]

\[
+ \frac{1}{V} \int dX_2 W_{11}(X_2,t;X'_2) P_{s_2}(x_2 | X, t) +
\]

\[
+ \frac{n}{V} \int dX_2 dX'_2 W_{11}(X_2,t;X'_2) P_{s_2}(x_1 | X'_2).
\]

(29)
This is by now an obvious generalization of Eq. (14). Not all observables of interest are quite of the form of Eq. (19). For example, the electron density is

$$n_e(x,t) = \sum_{i=1}^{N} n_e(x_i | x),$$  \hspace{1cm} (30)

where $n_e(x_i | x) = \delta(x - x_i)$ and the sum is only over electrons. The quantity $\langle n_e(x,t)n_e(x',t') \rangle$ is of interest in connection with the scattering of electromagnetic waves. It can be conveniently expressed in terms of electron densities for two kinds of quasiparticles,

$$\hat{n}_{ie}(X_1 | x_t) = n_e(x_1 | x) + n \int n_e(x_2 | x) P_{ie}(X_1 | X_2 t) dX_2,$$  \hspace{1cm} (31)

$$\hat{n}_{ie}(X_1 | x_t) = n \int n_e(x_2 | x) P_{ie}(X_1 | X_2 t) dX_2.$$  \hspace{1cm} (32)

$\hat{n}_{ie}(X_1 | x_t)$ means the electron density due to an electron quasiparticle at $X_1$; $\hat{n}_{ie}(X_1 | x_t)$ is the electron density due to an ion quasiparticle at $X_1$. The subscripts $i, j$ run over $e, I$ to denote electrons and ions. The desired correlation function is

$$\langle n_e(x,t)n_e(x',t') \rangle = \frac{n}{V} \int dX_1 \int dX'_1 n_e(x_1 | x)n_e(x'_1 | x') W_e(e_1 t; X_1 t') + n^2 \int dX_1 \int dX'_1 n_e(x_1 | x)n_e(x'_1 | x') W_{ii}(X_1 t; X'_1 t').$$  \hspace{1cm} (33)

After substituting from Eqs. (29), (31), and (32), this is easily brought to the form

$$\langle n_e(x,t)n_e(x',t') \rangle = \langle n_e(x_t)n_e(x'_t) \rangle + \frac{n}{V} \sum_{i,e} \int dX_1 \int dX'_1 W_{ie}(X_1 t, X'_1 t') \cdot \hat{n}_{ie}(X_1 | x_t)\hat{n}_{ie}(X'_1 | x'_t),$$

where

$$\langle n_e(x_t) \rangle = n \int f_e(x_t)n_e(x_1 | x) dX_1.$$  \hspace{1cm} (34)

This result was previously obtained from the explicit solution for particular cases after a very considerable amount of algebra.  

ACKNOWLEDGMENTS

This research was supported by the Advanced Research Projects Agency, Department of Defense, under Project Defender and was monitored by the Air Force Weapons Laboratory under Contract Number AF29(601)-5338.