UCLA
Recent Work

Title
Risk and Performance Measures in Investment Portfolios

Permalink
https://escholarship.org/uc/item/5dw7x4z5

Author
Stokie, Michael D.

Publication Date
1982-10-01
RISK AND PERFORMANCE MEASURES IN INVESTMENT PORTFOLIOS

Michael D. Stokie
Graduate School of Management
University of California, Los Angeles 90024
and
Deskin University
Victoria 3217, Australia

October 1982
RISK AND PERFORMANCE MEASURES
IN INVESTMENT PORTFOLIOS

by

Michael D. Stokie*
Deakin University, Victoria 3217, Australia

This version is preliminary. Comments will be appreciated.

October 1982

* This paper was written during a period as Visiting Scholar at the Graduate School of Management, U.C.L.A. I am grateful to Richard Roll for helpful comments on an earlier draft.
1. Introduction

The securities market line (SML) is a device for portfolio performance evaluation in which systematic risk is measured by portfolio beta relative to some index m. Roll (1977, 1978, 1980) has shown that there are several problems with this approach. If m is mean-variance efficient all securities and portfolios will plot exactly on the SML and no performance distinctions can be made. On the other hand if m is not on the efficient frontier, deviations from the SML vary with the choice of m so that the performance evaluation is at best ambiguous. Furthermore beta or systematic risk is only an approximation to the actual non-diversifiable risk that can be achieved in a portfolio having a particular return level. If m is efficient, this approximation understates the minimal risk available and the approximation is less accurate for portfolios which are distant from m.

The use of beta as a risk measure is also criticised by Camp and Eubank (1981). They argue that the performance of a portfolio should be evaluated on a basis involving total risk instead of beta since the portfolio is bearing diversifiable risk in addition to its systematic risk, but the measure they suggest is also index dependent.

In this paper, a new portfolio risk measure is proposed. It is based on the portfolio variability in excess of the global minimum variance, and it leads to a performance measurement device which is not dependent on an index and avoids the problems associated with the SML approach.
The structure of the paper is as follows. The properties of systematic risk and its relationship with total and minimal risk measures are discussed in Section 2. The new risk and performance measures are introduced in Section 3 and a standard scale of units is defined. Properties of the performance ratio are developed in Section 4, and the results are summarised in Section 5.

2. Traditional Risk Measures

Consider a universe of risky assets with ex-post mean return vector $\mathbf{R}$ and variance-covariance matrix $\mathbf{V}$. Roll (1977) has defined the efficient set constants $a = \mathbf{R}'\mathbf{V}^{-1}\mathbf{R}$, $b = \mathbf{R}'\mathbf{V}^{-1}\mathbf{1}$, $c = \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}$

where $\mathbf{1}$ is the unit vector. Any ex-post efficient portfolio $\mathbf{P}$ with mean return $r_P$ and variance $\sigma_P^2$ plots on the efficient frontier

$$\sigma_P^2 - \sigma_0^2 = \frac{(r_P - r_0)^2}{\mathbf{K}^2} \tag{1}$$

where $\mathbf{K} = \{(ac - b^2)/c\}^{1/2}$, and $r_0 = b/c$ and $\sigma_0^2 = 1/c$ are the mean return and variance of the global minimum variance portfolio $\mathbf{P}_0$.

The risk of an individual security or portfolio having return $r_i$ can be represented in any of three ways. Firstly the standard deviation $\sigma_i$ gives the total risk. Without change of return level, this risk can be reduced by diversification to $\sigma_P$ where $\mathbf{P}$ is the efficient portfolio with $r_P = r_i$. This gives the non-diversifiable component of total risk and will be called minimal risk. The third risk measure is the beta or systematic risk $\beta_i\mathbf{M}^\top\mathbf{M}$ relative to some market index $\mathbf{M}$. If $\mathbf{M}$ is efficient, the systematic risk is the horizontal distance from $r_i$ on the return
axis to the tangent line drawn to the efficient frontier at $M$. It is apparent from Figure 1 that the systematic risk understates the minimal risk $\sigma_p$ that can be achieved at any return level except $r_M$, and the accuracy of using systematic risk as an approximation to minimal risk decreases as $r_i$ becomes more distant from $r_M$.

![Diagram](image)

**Figure 1**: Risk Measures of Portfolio $i$ In Relation to the Ex-post Efficient Frontier and Index $M$.

The dependence of systematic risk (SR) on the index $M$ can be shown analytically as follows. Start with the covariance $\sigma_{iM}$ between an efficient portfolio $M$ and any other portfolio $i$:

$$\sigma_{iM} = \{a-b(r_i+r_M) + cr_i r_M\}/(ac-b^2). \quad (2)$$

Roll (1977, p.163) derived this expression for a pair of efficient portfolios but it still holds if only one of the portfolios is
efficient. Next use the corresponding expression for $\sigma_M^2$ to write $\text{SR} = \beta_{iM} \sigma_M = \sigma_{iM}/\sigma_M$ as a function of $r_M$, and restrict $M$ to the upper branch of the efficient frontier. Now

$$\frac{d(SR)}{d(r_M)} = \frac{(r_i - r_M)}{(ac - b^2)\sigma_M^3},$$

so $\text{SR}$ has a local maximum of $\sigma_p$ where $r_M = r_i$ provided $r_i > r_0$, and if $r_i < r_0$ there is an end-point maximum of $\sigma_0$ at $r_M = r_0$. As $r_M$ increases, $\text{SR}$ asymptotes to $\sqrt{\sigma_p^2 - \sigma_0^2}$ if $r_i > r_0$ and to $-\sqrt{\sigma_p^2 - \sigma_0^2}$ if $r_i < r_0$. This variation of $\text{SR}$ with the index $M$ is illustrated in Figure 2 and corresponds to similar patterns for beta in Roll (1978).

![Figure 2: Systematic Risk (SR) of Security i for Various Indices M on the Upper Branch Efficient Frontier](image)

We can also note the connections between the various risk measures. Total risk and systematic risk satisfy $\text{SR} = \rho_{iM} \sigma_i$ where $\rho_{iM}$ is the correlation coefficient between portfolio $i$ and $M$. Since $\sigma_i$ is fixed, this also underscores the dependence of $\text{SR}$ on $M$. For an efficient $M$ and any $i$, equations (1) and (2) lead to

$$\sigma_{iM} = \frac{(a-b(r_p + r_M) + cr_pr_M)}{(ac-b^2)}$$
$$= 1/c + \frac{(r_p - r_M)(r_M - r_0)}{K^2}$$
$$= \sigma_0^2 + \{(\sigma_p^2 - \sigma_0^2)(\sigma_M^2 - \sigma_0^2)\}^{1/2}.$$
This gives the relationship between systematic risk of any portfolio i and the corresponding minimal risk $\sigma_P$ as

$$SR = \left(\sigma_0^2 + \sigma_P^2 - \sigma_0^2\right)\frac{1}{\left(M_0 - \sigma_0^2\right)\sigma_M}, \quad (4)$$

which reduces to $SR = \sigma_P$ only if $M$ and $P$ coincide.

3. **Gamma Risk and the Performance Ratio $\lambda$**

3.1 **Definitions**

A new risk measure is now proposed. It will lead to a fixed measure of non-diversifiable risk that is not index-based, thereby avoiding the problems associated with beta or systematic risk, and which is preferred over minimal risk $\sigma_P$ because of its more convenient mathematical properties.

In a universe of risky assets the total variability $\sigma_i^2$ of any portfolio $i$ is the sum of a constant component $\sigma_0^2$ and a supplemental component $\left(\sigma_i^2 - \sigma_0^2\right)$. We will use this second component to represent the risk in portfolio $i$ additional to the risk associated with the minimum variance portfolio $P_0$. For any portfolio $i$, we define

$$\text{Supplemental Risk} = \sqrt{\sigma_i^2 - \sigma_0^2}. \quad (5)$$

At any return level, supplemental risk is minimised in the efficient portfolio $P'$ with $r_{P'} = r_i$. We define

$$\text{Minimal Supplemental Risk} = \sqrt{\sigma_P^2 - \sigma_0^2}. \quad (6)$$
Geometrically this minimal supplemental risk is represented by the horizontal distance from \( r_i \) on the return axis to the asymptote \( Y - r_0 = \pm XX \) of the efficient frontier hyperbola in mean-standard deviation space; when \( Y = r_i = r_p \), \( X = \sqrt{\sigma_p^2 - \sigma_0^2} \). This is illustrated by the line \( AB \) in Figure 3.

![Diagram showing minimal supplemental risk AB at return level \( r_i \).](image)

**Figure 3**: Minimal Supplemental Risk \( AB \) at Return Level \( r_i \)

The quality or level of efficiency of portfolio \( i \) is indicated by its proximity to \( P \). This will be measured by the ratio of minimal supplemental risk to supplemental risk and is called the performance ratio of security \( i \), denoted \( \lambda_i \). We define:

\[
\lambda_i = \begin{cases} 
\sqrt{\frac{\sigma_p^2 - \sigma_0^2}{\sigma_i^2 - \sigma_0^2}}, & r_i \geq r_0 \\
-\sqrt{\frac{\sigma_p^2 - \sigma_0^2}{\sigma_i^2 - \sigma_0^2}}, & r_i < r_0 
\end{cases}
\]  

(7)

The negative sign when \( r_i < r_0 \) indicates inferior performance because higher mean returns are available at lower minimal supplemental risk levels. Observe that \( \lambda_i \) is always in the interval \(-1 \leq \lambda_i \leq 1\) and that \( \lambda_i = 1 \) only if \( i \) is efficient.
3.2 Standard Scale of Units

The supplemental risk measure (5) is scale-dependent and it is helpful to introduce a standard scale of units. The efficient frontier (1) is uniquely specified by its vertex and any other point on it. We arbitrarily choose a point \( r_Z \) on the return axis below \( r_0 \) and from \( r_Z \) draw the tangent to the efficient hyperbola meeting it at point \( P_U \) with return \( r_U \) and standard deviation \( \sigma_U \) as shown in Figure 4.

![Figure 4: The Determination of a Unit Portfolio \( P_U \) Using Arbitrary \( r_Z < r_0 \).](image)

Substitution for \( P_U \) in (1) gives

\[
k^2 = \frac{(r_U - r_0)^2}{(\sigma_0^2 - \sigma_U^2)}
\]

which permits (1) to be written in a scale-independent form

\[
\frac{(r_P - r_0)}{(r_U - r_0)} \cdot k^2 = \frac{(\sigma_P^2 - \sigma_0^2)}{(\sigma_U^2 - \sigma_0^2)}
\]

The point \( P_U \) has determined a standard scale with the new units on the horizontal and vertical axes being \( \sqrt{\sigma_U^2 - \sigma_0^2} \) and \( r_U - r_0 \) respectively. \( P_U \) will be called the unit portfolio. The supplemental risk (5) of any portfolio \( i \) will be denoted by \( \gamma_i \) when converted to standard units, and is referred to as standard supplemental risk or more simply gamma-risk. So we define

\[
\text{Gamma Risk } \gamma_i = \left( \frac{\sigma_i^2 - \sigma_0^2}{\sigma_U^2 - \sigma_0^2} \right)^{1/2}.
\]
We can introduce a similar terminology

\[
\text{Standard Excess Return } = \frac{r_i - r_0}{r_i - r_0} \quad (10)
\]

to measure in standard units the return of portfolio \( i \) additional to the minimum-risk return \( r_0 \). The performance ratio \( \lambda \) is independent of scale and does not require restatement in standard units; it is related to the gamma risks of portfolio \( i \) and of the efficient portfolio \( P \) at the same return level by

\[
\gamma_P = \begin{cases} 
\lambda_i \gamma_i, & r_i \geq r_0 \\
-\lambda_i \gamma_i, & r_i < r_0 
\end{cases} \quad (11)
\]

Equation (8) shows that the standard excess return and the gamma risk are equal for efficient portfolios. Each choice of the positive parameter \( \gamma \) determines an efficient portfolio \( P_\gamma \) having \( \gamma \) standard units of excess return and of supplemental risk. This is illustrated in Figure 5 using mean-variance space; note that the points \( P_\gamma, P_0 \) and \( r_z \) are collinear in this space.

\[\text{Figure 5: Standard Excess Return and Gamma Risk of Efficient Portfolios in Mean-Variance Space}\]
For any portfolio $i$, the relationship between its return $r_i = r_p$, its risk level $\gamma_i$, and its performance ratio $\lambda_i$ is derived from (8), (9) and (11) as

$$r_i = r_0 + (r_U - r_0)\lambda_i \gamma_i.$$  \hfill (12)

This is merely a mathematical consequence of the definitions adopted here, but it might be helpful to observe the linear relationship between portfolio return and its standard supplemental risk $\gamma_p = \lambda_i \gamma_i$, which is analogous to return-beta linearity in the Capital Asset Pricing Model. The portfolio return is expressed in terms of a minimum return $r_0$, a risk premium $(r_U - r_0)\lambda_i$, and a risk level $\gamma_i$. The risk premium is variable and increases with the quality of the portfolio as measured by $\lambda_i$. As $\lambda_i$ approaches 1 the risk premium increases so that the highest reward is given for optimal investments. If the portfolio is of low quality the risk premium might even be negative.

3.3 Alternative Expressions for $\lambda$

The performance ratio $\lambda$ has been defined to measure the extent to which available risk diversification has been achieved. This is specified in (11) by $\lambda_i = \gamma_p / \gamma_i$. Some alternative expressions are now derived.

Since $P$ is efficient, we can use (8) and $r_p = r_i$ to rewrite (11) as

$$\lambda_i = ((r_i - r_0)/(r_U - r_0)) / \gamma_i.$$  \hfill (13)
Although (13) uses standard units involving \( r_U \) and \( \sigma_U \), it has the computational advantage that it does not require the calculation of \( \sigma_p^2 \) for each portfolio \( i \) when several performance ratios are being evaluated. It uses each \( r_i \) and \( \sigma_i \) directly and shows that for each portfolio:

\[
\text{Performance Ratio } \lambda = \frac{\text{Standard Excess Return}}{\text{Gamma Risk}}. \tag{14}
\]

One further expression for \( \lambda \) is available. Consider the efficient portfolio \( Q \) (upper branch) having the same risk level as portfolio \( i \) so that \( \sigma_i^2 = \sigma_Q^2 \). Using (8), equation (13) becomes

\[
\lambda_i = \frac{r_i - r_0}{r_Q - r_0}. \tag{15}
\]

This expresses \( \lambda \) as the ratio between the actual excess return of portfolio \( i \) and the maximum excess return available within the same risk class. In Figure 6, all portfolios in a particular risk class will plot somewhere on \( AB \). Only one portfolio in this risk class is optimal with \( \lambda = 1 \), and the quality of each portfolio is indicated by its \( \lambda \)-value.

![Figure 6: Performance Ratios of Portfolios in Risk Class AB](image-url)
3.4 Computational Considerations

The calculation of gamma risk and performance ratio $\lambda$ requires the return and variance for portfolios $P_0$ and $P_u$. From Roll (1977) we know that $r_0 = b/c$, $\sigma_0^2 = 1/c$, $r_u = (a-br_z)/(b-cr_z)$ and $\sigma_u^2 = (a-2br_u+cr_u^2)/(ac-b^2)$ where $r_z$ is an arbitrarily assigned number less than $r_0$. If $r_z$ is chosen as zero we get $r_u = a/b$ and $\sigma_u^2 = a/b^2$. Although $a, b, c$ are defined in terms of the mean return vector $R$, the unit vector $\lambda$ and the inverse of the covariance matrix $V$ it is not necessary to actually compute $V^{-1}$. Separately solve the linear systems $Vx = \lambda$ and $Vy = R$ for the vectors $x$ and $y$, and then $a = R'y$, $b = R'x$, $c = \lambda'x$.

An illustration of the measures proposed here is provided from the study by Stokie (1982) of monthly returns on leading Australian securities in several five year periods. For the period 1959–1963 the efficient set constants were $a = 1.460$, $b = 0.279$, $c = 0.3045$, so that $r_0 = 0.916$, $\sigma_0^2 = 3.284$ and if $r_z$ is taken as zero $r_u = 5.235$ and $\sigma_u^2 = 18.77$. An equally weighted market index had return $r_m = 1.33$ and variance $\sigma_m^2 = 8.94$ in this period. Using (9), (10) and (14) we have for portfolio $m$:

- Standard Risk = 0.60 units
- Standard Excess Return = 0.096 units
- Performance Ratio = 0.159

This is prima facie evidence that the index $m$ is not mean-variance efficient, but the statistical significance of this cannot be assessed because the sampling properties of $\lambda$ are not yet available.
4. Relationships Between Performance Ratios

4.1 Comparison of Performance Ratios of Two Portfolios

Two portfolios will be regarded as equivalent in quality if their performance ratios are equal: \( \lambda_1 = \lambda_2 \). This permits a comparison of the quality of different portfolios from varying risk-classes. The contour in mean-variance space of all portfolios with constant performance ratio \( \lambda \) will be a parabola, because if any such portfolio has gamma risk \( \gamma \) its standard excess return will be \( \lambda \gamma \) so that it plots on the parabola \( \gamma = \lambda \sqrt{\lambda} \).

Some typical contours are shown in Figure 7.

![Figure 7: Contours of Portfolios with Constant Performance Ratios](image)

This provides a useful device for comparing investment performance. The risk levels of separate portfolios can be measured by \( \gamma \) and their quality by the performance ratios \( \lambda \). Many portfolios might have the same quality \( \lambda \) even though they have different risk levels \( \gamma \). The best performers on a risk-adjusted basis will be those with \( \lambda \)-values closest to 1.
The problem of ambiguity in performance measures based on beta and a market index are avoided by using $\lambda$ in relation to the ex-post efficient frontier. It will be of interest to compare risk levels and performance ratios of individual portfolios with those of a market index, but these measures are not dependent on the index in the way that beta is.

4.2 Combinations of Several Securities

Suppose $n$ securities with returns $r_i$, gamma risks $\gamma_i$ and performance ratios $\lambda_i$ are combined with weights $w_i$ to form a portfolio $g$. Expressions for the gamma risk $\gamma_g$ and performance ratio $\lambda_g$ of the portfolio $g$ in terms of its component securities are now derived. It will be convenient to adopt the notation

$$\gamma_{ij} = (\sigma_{ij} - \sigma_0)/(\sigma_0 - \sigma_0)$$

where $\gamma_{ii} = \gamma_i^2$. Now the gamma risk of $g$ satisfies

$$\gamma_g^2 = \frac{(\sigma_g - \sigma_0)^2}{(\sigma_0 - \sigma_0)} = \frac{\sum w_i \sum w_j \sigma_{ij} - \sigma_0^2}{(\sigma_0 - \sigma_0)}.$$

Using $1 = (\sum w)^2 = \sum w_i w_j$ gives

$$\gamma_g^2 = \frac{\sum w_i \sum w_j \gamma_{ij} - \sigma_0^2 \sum w_i w_j}{(\sigma_0 - \sigma_0)} = \sum w_i w_j \gamma_{ij}.$$

So we have

$$\gamma_g = (\sum w_i w_j \gamma_{ij})^{1/2}.$$  \hfill (17)

To derive an expression for $\lambda_g$, start by using (12) to write portfolio return as

$$r_g = \sum w_i r_i$$

$$= r_0^{+}(r_U - r_0) \sum w_i \lambda_i \gamma_{ii}.$$
But from (12),
\[ r_g = r_0^+ (r_U - r_0) \lambda g y_g \]
so that we have
\[ \lambda g y_g = \sum w_i \lambda_i y_i . \]
Using (17) the performance ratio of portfolio \( g \) is
\[ \lambda_g = \sum w_i \lambda_i y_i / \{ \sum w_i w_j y_{ij} \}^{1/2} . \] (18)

5. Summary

The use of systematic risk to measure rewardable portfolio risk in assessing investment performance is of dubious validity because of the influence of the market index on portfolio beta values. This paper proposes a new measure of portfolio risk that is independent of the market index and is based on the portfolio's variability of returns supplemental to the global minimum variance. A measure of portfolio performance which uses the proximity of the portfolio to the ex-post efficient frontier is also introduced, together with a standard scale of units to facilitate comparison between portfolios.

The next stage will be to develop the statistical properties of the risk and performance measures, probably using simulation, and apply these in empirical studies of various securities and portfolios.
References


