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Ribbon Schur functions and permutation patterns

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Chair

University of California, San Diego

2008
DEDICATION

To my brilliant family.

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ABSTRACT OF THE DISSERTATION

Ribbon Schur Functions and Permutation Patterns

by

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Professor Jeffrey Remmel, Chair

For a permutation \( \sigma \in S_n \), we define: \( \text{Des}(\sigma) = \{ i : \sigma_i > \sigma_{i+1} \} \) and \( \text{des}(\sigma) = |\text{Des}(\sigma)| \). Also, if \( \alpha_1, \ldots, \alpha_k \in S_n \), then we shall write \( \text{comdes}(\alpha_1, \ldots, \alpha^k) = |\bigcap_{i=1}^k \text{Des}(\alpha_i)| \).

\[
\sum_{n=0}^{\infty} \frac{u^n}{(n!)^2} \sum_{(\sigma, \tau) \in S_n \times S_n} x^{\text{comdes}(\sigma, \tau)} = \frac{1-x}{-x+J(x-1)}, \quad \text{where} \quad J(u) = \sum_{n \geq 0} \frac{u^n}{n!^2}.
\]

The central theme of this dissertation is to develop methods for extending results on generating functions for permutation statistics in several different ways. For example, in Chapter 2, we shall find the generating function analogous to Carlitz’s, except that we extend the results to sum over permutations that contain a given finite descent set. In essence, we are forcing the permutations in our sum to have descents at given positions. This result requires proving new identities for ribbon Schur functions \( Z_\alpha \). In Chapter 3, we extend the work of Mendes [30] on \( k \)-alternating permutations to start and end with any number of elements mod \( k \). In particular, we define a new class of symmetric functions and associated identities to achieve this result. In Chapter 4, we combine the results of Chapters 2 and 3 to permutations with repeating descent positions, except we force rises at a finite number of places. In Chapter 5, we again extend the results of Remmel and Mendes [33] to more general patterns, by which we mean those permutations which have repeating patterns of descents and rises. In order to do this, we must
modify the homomorphisms used previously and also modify the labeling system in this machinery.

We also present some combinatorial results on bases for the space of symmetric functions. In Chapter 6, we find a combinatorial interpretation for the coefficients of the dual basis to \( \{ Z_\lambda : \lambda \text{ is a partition of } n \} \) expanded in terms of the monomial symmetric functions. We introduce these results in more detail in the next subsections.
Chapter 1

Introduction and Background

1.1 Introduction

There has been a long line of research, [8], [9], [6], [24], [25], [33], [31], [37], [41], [30], which shows that a large number of generating functions for permutation statistics can be obtained by applying homomorphisms defined on the ring of symmetric functions $\Lambda$ to simple symmetric function identities. For example, the $n$-th elementary symmetric function, $e_n$ and the $n$-th homogeneous symmetric function, $h_n$, are defined by the generating functions

$$E(t) = \sum_{n \geq 0} e_n t^n = \prod_i (1 + x_i t) \quad (1.1.1)$$

and

$$H(t) = \sum_{n \geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t}. \quad (1.1.2)$$

We let $P(t) = \sum_{n \geq 0} p_n t^n$ where $p_n = \sum_i x_i^n$ is the $n$-th power symmetric function. For any partition, $\mu = (\mu_1, \ldots, \mu_\ell)$, we let $h_\mu = \prod_{i=1}^\ell h_{\mu_i}$, $e_\mu = \prod_{i=1}^\ell e_{\mu_i}$, and $p_\mu = \prod_{i=1}^\ell p_{\mu_i}$. Now it is well known that

$$H(t) = 1/E(-t) \quad (1.1.3)$$

and

$$P(t) = \frac{\sum_{n \geq 1} (-1)^{n-1} n e_n t^n}{E(-t)}. \quad (1.1.4)$$
A surprisingly large number of results on generating functions for various permutation statistics in the literature and large number of new generating functions can be derived by applying homomorphisms on $\Lambda$ to simple identities such as (1.1.3) and (1.1.4).

Let $S_n$ denote the symmetric group and write $\sigma \in S_n$ in one line notation as $\sigma = \sigma_1 \cdots \sigma_n$. In this section, we shall consider the following statistics for a permutation $\sigma \in S_n$,

$$\begin{align*}
    \text{Des}(\sigma) &= \{i : \sigma_i > \sigma_{i+1}\} \\
    \text{Rise}(\sigma) &= \{i : \sigma_i < \sigma_{i+1}\} \\
    \text{des}(\sigma) &= |\text{Des}(\sigma)| \\
    \text{rise}(\sigma) &= |\text{Rise}(\sigma)| \\
    \text{inv}(\sigma) &= \sum_{i<j} \chi(\sigma_i > \sigma_j) \\
    \text{coinv}(\sigma) &= \sum_{i<j} \chi(\sigma_i < \sigma_j),
\end{align*}$$

where $\chi(A)$ is 1 if $A$ is true and 0 if $A$ is false. Also, if $\alpha^1, \ldots, \alpha^k \in S_n$, then we shall write $\text{comdes}(\alpha^1, \ldots, \alpha^k) = |\bigcap_{i=1}^k \text{Des}(\alpha^i)|$. We note that these definitions also make sense for any sequence $\sigma = \sigma_1 \cdots \sigma_n$ of natural numbers. We shall use standard notations for $q$ and $p, q$ analogues. That is, we let

$$\begin{align*}
    [n]_q &= 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{1-q}, \\
    [n]_q! &= [n]_q[n-1]_q \cdots [1]_q, \\
    [n]_q^k &= \frac{[n]_q!}{[k]_q! [n-k]_q!}, \\
    [n]_q^0 &= 1. \\
\end{align*}$$

Similarly we can define $p, q$-analoga of all these formulas by replacing $[n]_q$ by $[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + p^1q^{n-2} + q^{n-1} = \frac{p^n-q^n}{p-q}$ in the formulas. Then all of the following results can be proved by applying a suitable homomorphism to the identity (1.1.3).

1. $\sum_{n=0}^\infty \frac{u^n}{n!} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} = \frac{1-x}{1-x+e^{u(x-1)}}$.

2. (Carlitz 1970) [11]

$$\sum_{n=0}^\infty \frac{u^n}{(n!)^2} \sum_{(\sigma,\tau) \in S_n \times S_n} x^{\text{comdes}(\sigma,\tau)} = \frac{1-x}{1-x+J(u(x-1))}.$$

3. (Stanley 1976) [39]

$$\sum_{n=0}^\infty \frac{u^n}{[n]!} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} q^{\text{inv}(\sigma)} = \frac{1-x}{1-x+e_q(u(x-1))}.$$
4. (Stanley 1976) [39]

\[ \sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} q^{\text{coinv}(\sigma)} = \frac{1-x}{-x+E_q(u(x-1))}. \]

5. (Fedou and Rawlings 1995) [14]

\[ \sum_{n=0}^{\infty} \frac{u^n}{[n]_q ![n]_p} \sum_{(\sigma, \tau) \in S_n \times S_n} x^{\text{comdes}(\sigma, \tau)} q^{\text{inv}(\sigma)} p^{\text{inv}(\tau)} = \frac{1-x}{-x+J_{q,p}(u(x-1))}. \]

6. (Garsia and Gessel 1979) [17]

\[ \sum_{n \geq 0} \frac{t^n}{[n]_q ![n]_p} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} t^{\text{maj}(\sigma)} q^{\text{inv}(\sigma)} = \sum_{k \geq 0} x^k e_q(-tr^0) \cdots e_q(-tr^k), \]

where \( J(u) = \sum_{n \geq 0} \frac{u^n}{n!} \), \( e_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q ![n]_p} q^{\binom{n}{2}} \), \( E_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q ![n]_p} \), \( J_{q,p}(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q ![n]_p} q^{\binom{n}{2}} p^{\binom{n}{2}} \), and \( (x, q)_n = (1-xq^0) \cdots (1-xq^{n-1}) \).

One of the main goals of this dissertation is to present methods for extending these results (1.1.5) in several different ways. We also present a some combinatorial results on bases for the space of symmetric functions. In Chapter 2, we use ribbon Schur functions to find generating functions similar to the ones above, except we may sum over permutations that contain a given finite descent set. In essence, we are forcing the permutations in our sum to have descents at given positions. This result requires proving new identities for ribbon Schur functions. In Chapter 3, we extend the work of Mendes [30] on \( k \)-alternating permutations to start and end with any number of elements mod \( k \). In particular, we define a new class of symmetric functions and associated identities to achieve this result. In Chapter 4, we combine the results of 2 and 3 to permutations with repeating descent positions, except we can force rises at a finite number of places. In Chapter 5, we again extend the results of Remmel and Mendes [33] to more general patterns, by which we mean those permutation who have repeating patterns of descents and rises. In order to do this, we must modify the homomorphisms used previously and also modify the labeling system in this machinery. In Chapter 6, we find a combinatorial
interpretation for the coefficients of the dual basis to $Z_\lambda$ expanded in terms of the monomial symmetric functions. We introduce these results in more detail in the next subsections.

1.1.1 Introduction to Chapter 2

The main goal of Chapter 2 is to give a uniform method to find generating functions similar to 1.1.5 involving the sum over all permutations $\sigma \in S_n$ such that $S \subseteq \text{Des}(\sigma)$, where $S$ is some fixed finite subset of the positive numbers \{1, 2, \ldots\}. For example, our methods give a systematic way to find the following generating functions for any $S \subset \{1, 2, \ldots\}$.

\[ 1) \sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{\sigma \in S_n, S \subseteq \text{Des}(\sigma)} x^\text{des}(\sigma). \]

\[ 2) \sum_{n=0}^{\infty} \frac{u^n}{(n!)^2} \sum_{(\sigma, \tau) \in S_n \times S_n, S \subseteq \text{Comdes}(\sigma, \tau)} x^\text{comdes}(\sigma, \tau). \]

\[ 3) \sum_{n=0}^{\infty} \frac{u^n}{n!q^n} \sum_{\sigma \in S_n, S \subseteq \text{Des}(\sigma)} x^\text{des}(\sigma) q^{\text{inv}(\sigma)}. \]

\[ 4) \sum_{n=0}^{\infty} \frac{u^n}{n!q^n!} \sum_{\sigma \in S_n, S \subseteq \text{Des}(\sigma)} x^\text{des}(\sigma) q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}. \]

\[ 5) \sum_{n=0}^{\infty} \frac{u^n}{n!q^n!p^n} \sum_{(\sigma, \tau) \in S_n \times S_n, S \subseteq \text{Comdes}(\sigma, \tau)} x^\text{comdes}(\sigma, \tau) q^{\text{inv}(\sigma)} p^{\text{inv}(\tau)}. \]

In fact, we shall show how to find a common generalization of all these results. If $\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)})$ is a sequence of permutations in $S_n$, then we define

\[ \text{Comdes}(\Sigma) = (\bigcap_{i=1}^{L} \text{Des}(\sigma^{(i)})) \]

and

\[ \text{comdes}(\Sigma) = |\text{Comdes}(\Sigma)|. \]

Suppose we are given two sequences of variables, $Q = (q_1, \ldots, q_L)$ and $P = \ldots$. 
$(p_1, \ldots, p_L)$. Then for any $m$, we let

$Q^m = q_1^m \cdots q_L^m$, \hspace{1em} P^m = p_1^m \cdots p_L^m,$

$[n]_Q = \prod_{i=1}^L [n]_{q_i}$, \hspace{1em} $[n]_{P,Q} = \prod_{i=1}^L [n]_{p_i,q_i}$,

$[n]_Q! = \prod_{i=1}^L [n]_{q_i}!$, \hspace{1em} $[n]_{P,Q}! = \prod_{i=1}^L [n]_{p_i,q_i}!$,

$\left\lfloor \lambda_1, \ldots, \lambda_k \right\rfloor_Q = \prod_{i=1}^L \left\lfloor \lambda_1, \ldots, \lambda_k \right\rfloor_{q_i}$,

$\left\lfloor \lambda_1, \ldots, \lambda_k \right\rfloor_{P,Q} = \prod_{i=1}^L \left\lfloor \lambda_1, \ldots, \lambda_k \right\rfloor_{p_i,q_i}$,

$Q^{\text{inv}}(\Sigma) = \prod_{i=1}^L q_i^{\text{inv}(\sigma^{(i)})}$, and

$P^{\text{coinv}}(\Sigma) = \prod_{i=1}^L p_i^{\text{coinv}(\sigma^{(i)})}.

\text{We let}

\[\text{exp}(t, Q, P) = \sum_{n \geq 0} \frac{t^n Q^{\binom{n}{2}}}{n! P^{\binom{n}{2}}}.\] \hspace{1em} (1.1.6)

Then for any finite set $S$, we shall show how to compute the generating function

$F^L_S(x, Q, P) = F^L_S(x, q_1, \ldots, q_L, p_1, \ldots, p_L) =
\sum_{n \geq 0} \frac{t^n}{[n]_{P,Q}!} \sum_{\Sigma \in S^n, S \subseteq \text{Comdes}(\Sigma)} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}$ \hspace{1em} (1.1.7)

by applying a certain ring homomorphism defined on the ring of symmetric functions $\Lambda$ to a symmetric function identity involving ribbon Schur functions.

We will also give methods to compute two closely related generating functions. That is, given a sequence of permutations $\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)})$ in $S_n$, define

$\text{oneRise}(\Sigma) = \{1, \ldots, n-1\} - \text{Comdes}(\Sigma)$ and \hspace{1em} (1.1.8)

$\text{Comrise}(\Sigma) = \cap_{i=1}^L \text{Rise}(\sigma^{(i)})$. \hspace{1em} (1.1.9)
Then for any pair of disjoint sets $S$ and $T$ of positive integers, we shall give methods to compute

$$
\phi_{(S,T)}(x, P, Q) = \sum_{\Sigma \subseteq \text{Comdes}(\Sigma), S \subseteq \text{Comdes}(\Sigma), T \subseteq \text{Comrise}(\Sigma)} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}. \quad (1.1.10)
$$

and

$$
\psi_{(S,T)}(x, P, Q) = \sum_{\Sigma \subseteq \text{Comdes}(\Sigma), S \subseteq \text{Comdes}(\Sigma), T \subseteq \text{Comrise}(\Sigma)} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}. \quad (1.1.11)
$$

### 1.1.2 Introduction to Chapter 3

Let $i, j, k,$ and $n$ be nonnegative integers satisfying $k \geq 2$, $0 \leq i \leq k - 1$, and $1 \leq j \leq k$. A permutation with descent set equal to $\{i + k, i + 2k, \ldots, i + nk\}$ will be called a permutation with a $k$-regular descent pattern. Let $E_{i+j,k}^{0,j,k}$ be the number of such permutations in $S_{i+nk+j}$.

In the special case where $k = 2$, $i = 0$, and $j = 2$, $E_{2n+2}^{0,2,2}$ is the number of permutations in $S_{2n+2}$ with descent set $\{2, 4, \ldots, 2n\}$. These are the classical even alternating permutations. André [1, 2] proved that

$$
1 + \sum_{n \geq 0} \frac{E_{2n+2}^{0,2,2}}{(2n + 2)!} t^{2n+2} = \sec t.
$$

Similarly, $E_{2n+1}^{0,1,2}$ counts the number of odd alternating permutations and

$$
\sum_{n \geq 0} \frac{E_{2n+1}^{0,1,2}}{(2n + 1)!} t^{2n+1} = \tan t.
$$

These numbers are also called the Euler numbers. When $k \geq 0$, $E_{kn+j}^{0,j,k}$ are called generalized Euler numbers [27]. There are well-known generating functions for $q$-analogues of the generalized Euler numbers; see Stanley’s book [40], page 148. Various divisibility properties of the $q$-Euler numbers have been studied in [3, 4, 15] and of the generalized $q$-Euler numbers in [19, 38]. Prodinger [35] also studied $q$-analogues of the number $E_{2n+1}^{1,2,2}$ and $E_{2n+2}^{1,1,2}$. 
Our goal is to find and refine generating functions for $E_{i+kn+j}^{i,j,k}$. This will be done by applying ring homomorphisms to symmetric function identities. This technique of understanding permutation enumeration through symmetric function identities further advances an already well-documented line of research [8, 25, 30, 32, 33, 41].

In [30], Mendes used similar methods to derive the generating functions for $p, q$-analogues of the generating functions for the even and odd alternating permutations. In this chapter, we significantly generalize his methods and the results of some of the authors mentioned above.

Let $C_{i+kn+j}^{i,j,k}$ denote the set of permutations $\sigma \in S_{i+kn+j}$ with $\text{Des}(\sigma) \subseteq \{i, i+k, \ldots, i+nk\}$ and $C_{i+kn+j}^{i,j,k} = |C_{i+kn+j}^{i,j,k}|$. Similarly, let $E_{i+kn+j}^{i,j,k}$ denote the set of permutations $\sigma \in S_{i+kn+j}$ with $\text{Des}(\sigma) = \{i, i+k, \ldots, i+nk\}$ so that $E_{i+kn+j}^{i,j,k} = |E_{i+kn+j}^{i,j,k}|$. Lastly, for $\sigma \in S_{i+kn+j}$, let $\text{Ris}_{i,k}(\sigma) = \{s : 0 \leq s \leq n \text{ and } \sigma_{i+sk} < \sigma_{i+sk+1}\}$ and $\text{ris}_{i,k}(\sigma) = |\text{Ris}_{i,k}(\sigma)|$. Then $E_{i+kn+j}^{i,j,k}$ is the number of $\sigma \in C_{i+kn+j}^{i,j,k}$ such that $\text{Ris}_{i,k}(\sigma) = \emptyset$. To generalize the results of Mendes, we will find the generating function for

$$\sum_{n \geq 0} \frac{(i+kn+j)!}{(i+kn+j)!} \sum_{\sigma \in E_{i+kn+j}^{i,j,k}} x^{\text{ris}_{i,k}(\sigma)}.$$  

Setting $x = 0$ in (3.1.1) will give the generating function for $E_{i+kn+j}^{i,j,k}$. We will also find $p, q$-analogues of such generating functions.

To obtain the generating function for (3.1.1), we introduce a new class of symmetric functions $p_{n,\alpha_1,\alpha_2}$ which depend on two weight functions $\alpha_1$ and $\alpha_2$. Our results will follow by applying a ring homomorphism to a symmetric function identity involving $p_{n,\alpha_1,\alpha_2}$’s. In fact, our methods provide vast extensions of (3.1.1). Moreover, our extension will contain as special cases all of the generating functions for the $q$-Euler and generalized $q$-Euler numbers in the papers mentioned above [40].

In order to fully extend our results, suppose $\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)})$ is a sequence of permutations with $\sigma^{(i)} \in C_{i+kn+j}^{i,j,k}$. Define

$$\text{Comris}_{i,k}(\Sigma) = \{s : 0 \leq s \leq n \text{ and for all } 1 \leq t \leq L, \sigma_{i+sk}^{(t)} < \sigma_{i+sk+1}^{(t)}\}$$
and let $comris_{i,k}(\Sigma) = |Comris_{i,k}(\Sigma)|$. As in subsection 1.1.1, for two sequences of indeterminates, $Q = (q_1, \ldots, q_L)$ and $P = (p_1, \ldots, p_L)$, let

$$Q^m = q_1^m \cdots q_L^m, \quad P^m = p_1^m \cdots p_L^m,$$

$$[n]_{P,Q} = \prod_{i=1}^L [n]_{p_i,q_i}, \quad [n]_{P,Q'} = \prod_{i=1}^L [n]_{p_i,q_i},$$

$$Q^{\text{inv}}(\Sigma) = \prod_{i=1}^L q_i^{\text{inv}(\sigma(i))}, \quad P^{\text{coinv}}(\Sigma) = \prod_{i=1}^L p_i^{\text{coinv}(\sigma(i))}, \quad \text{and}$$

$$\left[ \frac{n}{\lambda_1, \ldots, \lambda_k} \right]_{P,Q} = \prod_{i=1}^L \left[ \frac{n}{\lambda_1, \ldots, \lambda_k} \right]_{p_i,q_i}.$$

In addition, we set

$$e_{P,Q,k}(t) = \sum_{n \geq 0} \frac{t^{kn}}{[kn]_{P,Q}!} \quad \text{and} \quad e_{P,Q,k}^{(j)}(t) = \sum_{n \geq 0} \frac{t^{kn}}{[k(n-1) + j]_{P,Q}!}.$$  

Our first generalization of (3.1.1) is found when $i = 0$ and $j = k$. In this case, we will show that

$$1 + \sum_{n \geq 1} \frac{t^{kn}}{[kn]_{P,Q}!} \sum_{\Sigma \in (c_{kn}^{0,i,k})^L \comris_{i,k}(\Sigma)} Q^{\text{inv}}(\Sigma) P^{\text{coinv}}(\Sigma) = \frac{1 - x}{-x + e_{P,Q,k}(t(x - 1)^{1/k})}. \quad (1.1.13)$$

The next case is when $i = 0$ and $1 \leq j \leq k - 1$. Here, we will show that

$$\sum_{n \geq 1} \frac{t^{kn}}{[k(n-1) + j]_{P,Q}!} \sum_{\Sigma \in (c_{kn}^{0,i,k})^L \comris_{i,k}(\Sigma)} Q^{\text{inv}}(\Sigma) P^{\text{coinv}}(\Sigma) = -\frac{e_{P,Q,k}^{(j)}(t(x - 1)^{1/k})}{-x + e_{P,Q,k}(t(x - 1)^{1/k})}. \quad (1.1.14)$$

Lastly, in the case in the case where $1 \leq i, j \leq k - 1$, we will prove

$$\sum_{n \geq 2} \frac{t^{kn}}{[i + k(n-2) + j]_{P,Q}!} \sum_{\Sigma \in (c_{kn}^{i,j,k})^L \comris_{i,k}(\Sigma)} Q^{\text{inv}}(\Sigma) P^{\text{coinv}}(\Sigma) = \sum_{n \geq 2} \frac{(n-1)x^{n-2}P^{(i+k(n-2)+j)}}{[i + k(n-2) + j]_{P,Q}!} \sum_{n \geq 2} \frac{(n-1)x^{n-2}P^{(i+k(n-2)+j)}}{[i + k(n-2) + j]_{P,Q}!}$$

$$+ \frac{e_{P,Q,k}^{(i)}(t(x - 1)^{1/k}) e_{P,Q,k}^{(j)}(t(x - 1)^{1/k})}{(1 - x)(-x + e_{P,Q,k}(t(x - 1)^{1/k}))}. \quad (1.1.16)$$
Thus we will extend the current knowledge on this subject in three important ways:

- Our generating functions contain more information about our permutations. Whereas before in the research on the tangent, secant, and q-analogues, only the inversion statistic was enumerated, we extend these to coinversions as well.

- Our generating functions contain the results of [1, 2, 40], but with a different method of proof.

- Our generating functions are for permutations with descent set \( \{i, i+k, \ldots, i+kn\} \), extending the work of Mendes with descent set \( \{k, 2k, \ldots, kn\} \).

\[1.1.3\] Introduction to Chapter 4

Let \( \mathbb{N} \) denote the set of natural numbers and \( S \) be a set \( S \subseteq \mathbb{N} \). Then one can also ask for generating functions for the set of permutations \( \sigma \) of \( S_n \) such that

\[ S \cap \{1, \ldots, n\} \subseteq \text{Des}(\sigma) \quad \text{or} \quad S \cap \{1, \ldots, n\} = \text{Des}(\sigma). \]

The main goal of this chapter is to outline some methods that will allow us to find generating functions like six listed above for permutations \( \sigma \) of \( S_n \) such that \( S \cap \{1, \ldots, n\} = \text{Des}(\sigma) \) where \( S \) is of the form \( k\mathbb{N} \setminus T \) where \( T \) is finite set and \( k\mathbb{N} = \{k, 2k, 3k, \ldots\} \).

\[1.1.4\] Introduction to Chapter 5

In this chapter, we will modify pieces of the machinery described in the chapters above. These modifications will give us a large number of new generating functions, which we group into 3 sections based on the modifications involved.

The three types of modifications are:

1. Nonconsecutive descents between the largest elements between repeating patterns,
2. Consecutive descents for patterns whose smallest element is at the end, and
3. Consecutive descents for patterns whose largest element is at the end.

1.1.5 Introduction to Chapter 6

A zigzag or ribbon is a connected skew diagram that contains no $2 \times 2$ boxes. Given a composition $\beta = (\beta_1, \ldots, \beta_k)$, we let $Z_\beta$ denote the skew Schur function corresponding to the zigzag shape whose row lengths are $\beta_1, \ldots, \beta_k$ reading from top to bottom. For each $n$, the set $\{Z_\lambda\}_{\lambda \vdash n}$ is a basis for $\Lambda_n$, the space of homogeneous symmetric functions of degree $n$. In this chapter, we investigate some characteristics of the dual basis of $\{Z_\lambda\}_{\lambda \vdash n}$ relative to the Hall inner product which we denote by $\{DZ_\lambda\}_{\lambda \vdash n}$. We give a combinatorial interpretation for the coefficients in the expansion of $DZ_\lambda$ in terms of the monomial symmetric functions $\{m_\mu\}_{\mu \vdash n}$ as a certain signed sum of paths in the partition lattice under refinement. We shall show that in many cases, we can give an explicit formulas for the coefficients $a_{\mu,\lambda} = DZ_\lambda |_{m_\mu}$. In addition, we give explicit formulas for the coefficients that arise in the expansion of $DZ_\lambda$ in terms of Schur functions for several special cases. As an application, we obtain combinatorial interpretations for the coefficients in the expansion of Schur functions and general ribbon Schur functions in terms of ribbon Schur functions indexed by partitions.

1.2 Background

1.2.1 Symmetric Functions

A symmetric polynomial $p$ in the variables $x_1, \ldots, x_N$ is a polynomial over a field $F$ of characteristic 0 with the property that $p(x_1, \ldots, x_N) = p(x_{\sigma_1}, \ldots, x_{\sigma_N})$ for all $\sigma = \sigma_1 \cdots \sigma_N \in S_N$. A symmetric function in the variables $x_1, x_2, \ldots$ may be thought of as a symmetric polynomial in an infinite number of variables. Let $\Lambda$ be the ring of all symmetric functions (a more formal definition of $\Lambda$ may be
found in [28]). The previously defined elementary symmetric functions $e_n$ and the homogeneous symmetric functions $h_n$ are both elements of $\Lambda$.

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an integer partition; that is, $\lambda$ is a finite sequence of weakly increasing nonnegative integers. Let $\ell(\lambda)$ denote the number of nonzero integers in $\lambda$. If the sum of these integers is $n$, we say that $\lambda$ is a partition of $n$ and write $\lambda \vdash n$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$. The well-known fundamental theorem of symmetric functions says that \{ $e_\lambda : \lambda$ is a partition $\}$ is a basis for $\Lambda$. Similarly, if we define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$, then \{ $h_\lambda : \lambda$ is a partition $\}$ is also a basis for $\Lambda$.

Let $\Lambda$ denote the ring of symmetric functions over infinitely many variables $x_1, x_2, \ldots$. It follows that \{ $e_0 = 1, e_1, e_2, \ldots$ $\}$ is an algebraically independent set of generators for $\Lambda$. Let $\Lambda_n$ denote the space of homogeneous symmetric functions of degree $n$ over infinitely many variables $x_1, x_2, \ldots$. Then $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$. We let $\ell(\lambda)$ denote the number of parts of $\lambda$. It is well-known that \{ $h_\lambda : \lambda \vdash n$, $e_\lambda : \lambda \vdash n$, and $p_\lambda : \lambda \vdash n$ $\}$ are all bases of $\Lambda_n$, see [28].

We let $F_\lambda$ denote the Ferrers diagram of $\lambda$. If $\lambda = (\lambda_1 \leq \cdots \leq \lambda_k)$ and $\mu = (\mu_1 \leq \cdots \leq \mu_m)$ are partitions where $m \leq k$ and $\lambda_{k-i} \geq \mu_{m-i}$ for all $i = 0, \ldots, m-1$, then we let $F_{\lambda/\mu}$ denote the skew shape that results by removing the cells of $F_\mu$ from $F_\lambda$. For example, Figure 6.2 pictures the skew diagram $(1, 2, 3, 3)/(1, 2)$ on the left. A column-strict tableau $T$ of shape $\lambda/\mu$ is any filling of $F_{\lambda/\mu}$ with natural numbers such that the entries in each row are weakly increasing from left to right and the entries in each column are strictly increasing from bottom to top. We define the weight of $T$ to be $w(T) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ where $\alpha_i$ is the number of times that $i$ occurs in $T$. For example, on the right of Figure 6.2, we have pictured a column strict tableau of shape $(1, 2, 3, 3)/(1, 2)$ and weight $x_1^2 x_2 x_3 x_4^2$. Then the skew Schur function $s_{\lambda/\mu} = \sum_T w(T)$ where the sum runs over all column strict tableau of shape $\lambda/\mu$. We define a ribbon (or zigzag) shape to be a connected skew shape that contains no $2 \times 2$ array of boxes. Ribbon (or zigzag) Schur functions
are the skew Schur functions with a ribbon shape and they can be indexed by compositions. A composition \( \beta = (\beta_1, \ldots, \beta_k) \) of \( n \), denoted \( \beta \models n \), is a sequence of positive integers such that \( \beta_1 + \cdots + \beta_k = n \). Given a composition \( \beta = (\beta_1, \ldots, \beta_k) \), we let \( Z_\beta \) denote the skew Schur function corresponding to the zigzag shape whose row lengths are \( \beta_1, \ldots, \beta_k \) reading from top to bottom. For example Figure 6.1 shows the zigzag shape corresponding to the composition \( (2, 3, 1, 4) \). We let \( \lambda(\beta) \) denote the partition that arises from \( \beta \) by arranging its parts in weakly increasing order and \( \ell(\beta) \) denote the number of parts of \( \beta \). For example, if \( \beta = (2, 3, 1, 2) \), then \( \lambda(\beta) = (1, 2, 2, 3) \). We also define \( \text{shape}(\beta) = \lambda/\nu \) where \( F_\beta = F_{\lambda/\nu} \) and \( \lambda = (\beta_1, \beta_1 + \beta_2 - 1, \ldots, \beta_1 + \cdots + \beta_k - (k - 1)) \). It is also well-known [23] that

\[ \{Z_\lambda : \lambda \models n\} \text{ is a basis for } \Lambda_n. \]

### 1.2.2 Transition matrices

In this subsection, we shall present the combinatorics of the transition matrices between various bases of symmetric functions that will be needed for our methods.

Since the elementary symmetric functions \( e_\lambda \) and the homogeneous symmetric functions \( h_\lambda \) are both bases for \( \Lambda \), it makes sense to talk about the coefficient of
the homogeneous symmetric functions when written in terms of the elementary symmetric function basis. This coefficient has been shown to equal the size of a certain set of combinatorial objects. A rectangle of height 1 and length $n$ chopped into “bricks” of lengths found in the partition $\lambda$ is known as a brick tabloid of shape $(n)$ and type $\lambda$. One brick tabloid of shape (12) and type (1, 1, 2, 3, 5) is displayed below.

Let $\mathcal{B}_{\lambda,n}$ denote the set of all $\lambda$-brick tabloids of shape $(n)$ and let $B_{\lambda,n} = |\mathcal{B}_{\lambda,n}|$. Through simple recursions stemming from (1.1.3), Eğecioğlu and Remmel proved in [12] that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} B_{\lambda,n} e_\lambda.$$  \hspace{1cm} (1.2.1)

Eğecioğlu and Remmel [12] also proved that

$$h_\mu = \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} B_{\lambda,\mu} e_\lambda$$  \hspace{1cm} (1.2.2)

where $B_{\lambda,\mu}$ is the number of $\lambda$-brick tabloids of shape $\mu$. For example, the (1, 1, 2, 2)-brick tabloids of shape (2, 4) are pictured in Figure 1.3.

More generally, let $\mathcal{B}_{\lambda,\mu}$ denote the set of $\lambda$-brick tabloids of shape $\mu = (\mu_1, \ldots, \mu_k)$. Suppose that $R$ is a ring and we are given any sequence of $\bar{u} = (u_1, u_2, \ldots)$ of elements of $R$. Then for any brick tabloid $T \in \mathcal{B}_{\lambda,\mu}$ we let $(b_1, \ldots, b_k)$ denote the lengths of the bricks which lie at the right end of the rows of $T$ reading from top to bottom and we set $w_{\bar{u}}(T) = u_{b_1} \cdots u_{b_k}$. We then set $w_{\bar{u}}(B_{\lambda,\mu}) = \sum_{T \in \mathcal{B}_{\lambda,\mu}} w_{\bar{u}}(T)$. For example if $u = (1, 2, 3, \ldots)$, then $w_u(T) = w(T)$ is just the product of the lengths of the bricks that lie at the end of the rows of $T$. We have given $w(T)$ for each of the brick tabloids in Figure 1.3. This given, we can define a new family of symmetric functions $p_{\bar{u}}^\lambda$ as follows. First we let $p_{\bar{u}}^\lambda = 1$ and

$$p_{\lambda,n}^{\bar{u}} = \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} w_{\bar{u}}(B_{\lambda,(n)}) e_\lambda$$  \hspace{1cm} (1.2.3)
for $n \geq 1$. Finally if $\mu = (\mu_1, \ldots, \mu_k)$ is a partition of $n$, we set $p_{\mu}^{\vec{u}} = p_{\mu_1}^{\vec{u}} \cdots p_{\mu_k}^{\vec{u}}$. The functions $p_{\mu}^{\vec{u}}$ were first introduced in [25] and [33]. It follows from results of Eğecioğlu and Remmel [12] that if $u = (1, 2, 3, \ldots)$, then $p_{n}^{\vec{u}}$ is just the usual power symmetric function $p_{n}$. Thus we call $p_{n}^{\vec{u}}$ a generalized power symmetric function.

Mendes and Remmel [31, 33] proved the following:

\begin{align}
\sum_{n \geq 1} p_{n}^{\vec{u}} t^{n} &= \frac{\sum_{n \geq 1} (-1)^{n-1} u_{n} e_{n} t^{n}}{E(-t)} \quad \text{and} \\
1 + \sum_{n \geq 1} p_{n}^{\vec{u}} t^{n} &= 1 + \frac{\sum_{n \geq 1} (-1)^{n} (e_{n} - u_{n} e_{n}) t^{n}}{E(-t)}
\end{align}

(1.2.4) (1.2.5)

Note if we take $\vec{u} = (1, 1, \ldots)$, then (1.2.4) becomes

\begin{align}
1 + \sum_{n \geq 1} p_{n}^{\vec{u}} t^{n} &= 1 + \frac{\sum_{n \geq 1} (-1)^{n-1} e_{n} t^{n}}{\sum_{n \geq 0} (-1)^{n} e_{n} t^{n}} = \frac{1}{\sum_{n \geq 0} (-1)^{n} e_{n} t^{n}} = 1 + \sum_{n \geq 1} h_{n} t^{n}
\end{align}

which implies $p_{n}^{(1,1,\ldots)} = h_{n}$. Other special cases for $\vec{u}$ give well-known generating functions. For example, by taking $u_{n} = (-1)^{k} \chi(n \geq k + 1)$ for some $k \geq 1$, $p_{n}^{\vec{u}}$ is the Schur function corresponding to the partition $(1^{k}, n)$.

A rim hook of $\lambda$ is a connected sequence of cells, $h$, along the northeast boundary of $F_{\lambda}$ which has a ribbon shape and such that if we remove $h$ from $F_{\lambda}$, we are left
with the Ferrers diagram of another partition. More generally, \( h \) is a rim hook of a skew shape \( \lambda/\mu \) if \( h \) is a rim hook of \( \lambda \) which does not intersect \( \mu \). We say that \( h \) is a \textit{special rim hook} of \( \lambda/\mu \) if \( h \) starts in the cell which occupies the north-west corner of \( \lambda/\mu \). We say that \( h \) is a \textit{transposed special rim hook} of \( \lambda/\mu \) if \( h \) ends in the cell which occupies the south-east corner of \( \lambda/\mu \).

A \textit{special rim hook tabloid} (transposed special rim hook tabloid) of shape \( \lambda/\mu \) and type \( \nu \), \( T \), is a sequence of partitions \( T = (\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \lambda^{(k)} = \lambda) \), such that for each \( 1 \leq i \leq k \), \( \lambda^{(i)}/\lambda^{(i-1)} \) is a special rim hook (transposed special rim hook) of \( \lambda^{(i)} \) and the weakly increasing rearrangement of \( (\lambda^{(1)}/\lambda^{(0)}, \ldots, \lambda^{(k)}/\lambda^{(k-1)}) \) is equal to \( \nu \). We show an example of a special rim hook tabloid and a transposed special rim hook tabloid of shape \( (4, 5, 6, 6)/(1, 3, 3) \) in Figure 6.5. We define the sign of a special rim hook \( h_i = \lambda^{(i)}/\lambda^{(i-1)} \) to be \( sgn(h_i) = (-1)^{r(h_i)-1} \) where \( r(h_i) \) is the number of rows that \( h_i \) occupies. Likewise we define the transposed sign of a transposed special rim hook to be \( t-\text{sgn}(h_i) = (-1)^{c(h_i)-1} \) where \( c(h_i) \) is the number of columns that \( h_i \) occupies. Let \( SRHT(\nu, \lambda/\mu) \) \((t-SRHT(\nu, \lambda/\mu))\) equal the set of special rim hook tabloids (transposed special rim hook tabloids) of type \( \nu \) and shape \( \lambda/\mu \). If \( T \in SRHT(\nu, \lambda/\mu) \), we let \( sgn(T) = \prod_{H \in T} sgn(H) \). If \( T \in t-SRHT(\nu, \lambda/\mu) \), then \( t-sgn(T) = \prod_{H \in T} t-sgn(H) \). For \( |\lambda/\mu| = |\nu| \), we let

\[
K^{-1}_{\nu, \lambda/\mu} = \sum_{T \in SRHT(\nu, \lambda/\mu)} sgn(T) \quad \text{and} \quad (1.2.6)
\]

\[
TK^{-1}_{\nu, \lambda/\mu} = \sum_{T \in t-SRHT(\nu, \lambda/\mu)} sgn(T). \quad (1.2.7)
\]
Then Eğecioğlu and Remmel [13] proved that

\[ s_{\lambda/\mu} = \sum_{\nu} K_{\nu,\lambda/\mu}^{-1} h_{\nu} \quad \text{and} \quad s_{\lambda/\mu} = \sum_{\nu} T K_{\nu,\lambda/\mu}^{-1} e_{\nu}. \]  

(1.2.8)
Chapter 2

Generating functions for permutations which contain a given descent set.

The outline of this chapter is as follows. Our main goals are to find the following generating functions (see 1.1.7, 1.1.10, and 1.1.11):

\[
F^L_S(x, Q, P) = F^L_S(x, q_1, \ldots, q_L, p_1, \ldots, p_L) = \sum_{n \geq 0} \frac{t^n}{[n]!_{P, Q}} \sum_{\Sigma \in S^L_S, S \subseteq \text{Comdes}(\Sigma)} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)},
\]

(2.0.1)

\[
\phi_{(S,T)}(x, P, Q) = \sum_{S \subseteq \text{Comdes}(\Sigma), T \subseteq \text{oneRise}(\Sigma)} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)},
\]

(2.0.2)

and

\[
\psi_{(S,T)}(x, P, Q) = \sum_{S \subseteq \text{Comdes}(\Sigma), T \subseteq \text{Comrise}(\Sigma)} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}.
\]

(2.0.3)

In section 2.1, we shall derive our key identity involving ribbon Schur functions which will be used to derive our expression for \( F^L_S(x, Q, P) \). In section 2.2, we
shall give our methods for finding the generating functions $F_L(x, Q, P)$ and give some examples. Finally, in section 2.3, we shall give our methods to compute the generating functions $\phi(S, T)(x, P, Q)$ and $\psi(S, T)(x, P, Q)$ and give some examples.

### 2.1 An identity for ribbon Schur functions

Let $\alpha = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$ be a composition. Then we let $\alpha^{(0)} = \alpha$, $\alpha^{(k)} = \emptyset$ and $\alpha^{(j)} = (\alpha_k, \ldots, \alpha_{j+1})$ for $j = 1, \ldots, k-1$. For example, if $\alpha = (3, 2, 1, 3)$, then $\alpha^{(0)} = (3, 2, 1, 3)$, $\alpha^{(1)} = (3, 2, 1)$, $\alpha^{(2)} = (3, 2)$, $\alpha^{(3)} = (3)$, and $\alpha^{(4)} = \emptyset$. We let $(\alpha, n)$ denote the composition that results by adding an extra part of size $n$ at the end of $\alpha$, i.e. $(\alpha, n) = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1, n)$. Let $Z_{\emptyset} = 1$.

The main goal of this section is to prove the following identity for ribbon Schur functions.

**Theorem 2.1.1.**

$$
\sum_{n \geq 1} Z_{(\alpha, n)} t^{n+|\alpha|} = \frac{\sum_{j=0}^{k-1} (-1)^j Z_{\alpha^{(j)}} t^{|\alpha^{(j)}|}}{E(-t)} + \left( (-1)^{k-1} + \sum_{j=1}^{k} (-1)^{j-1} \sum_{r=1}^{\alpha_{j-1}} Z_{(\alpha^{(j)}, r)} t^{r+|\alpha^{(j)}|} \right).
$$

For example, suppose $\alpha = (3, 2, 1, 3)$. Then Theorem 2.1.1 becomes

$$
\sum_{n \geq 1} Z_{(\alpha, n)} t^{n+|\alpha|} = \frac{Z_{(3, 2, 1, 3)} t^9 - Z_{(3, 2, 1)} t^6 + Z_{(3, 2)} t^5 - Z_{(3)} t^3 + 1}{E(-t)} + (-1 + (Z_{(3, 2, 1, 2)} t^8 + Z_{(3, 2, 1, 1)} t^7) + (Z_{(3, 1)} t^4) - (Z_{(2)} t^2 + Z_{(1)} t)).
$$

This example helps explain how to think of the right-hand side of (2.1.1). The numerator of the term $\frac{\sum_{j=0}^{k-1} (-1)^j Z_{\alpha^{(j)}} t^{|\alpha^{(j)}|}}{E(-t)}$ is just the alternating sum of the $Z_{\alpha^{(j)}} t^{|\alpha^{(j)}|}$’s where the first term $Z_{\alpha} t^{|\alpha|} = Z_{\alpha^{(0)}} t^{|\alpha^{(0)}|}$ starts with a plus sign. For each $1 \leq j \leq k-1$, the ribbon shapes that appear in $\sum_{r=1}^{\alpha_{j-1}} Z_{(\alpha^{(j)}, r)} t^{r+|\alpha^{(j)}|}$ consists of the ribbon shapes that one can obtain from the ribbon shape corresponding to
$(\alpha_k, \ldots, \alpha_{j+1}, \alpha_j)$ by removing at least one but not all of the squares at the end of the last row. We call these the auxiliary ribbon shapes derived from $\alpha^{(j-1)}$. In our example, if we start with the ribbon shape $\alpha^{(0)} = (3, 2, 1, 3)$ as pictured in the top of Figure 2.1, then the auxiliary ribbon shapes derived from $\alpha^{(0)}$ are the two ribbon shapes pictured at the bottom of Figure 2.1. Note that if $\alpha_j = 1$, then there are no auxiliary shapes derived from $\alpha^{(j-1)}$. Thus the second term in (2.1.1) consists of alternating signs of the generating functions of auxiliary shapes derived from the $\alpha^{(j-1)}$’s for $j = 1, \ldots, k$. Moreover, the term $(-1)^{k-1}$ which appears at the start of the second term can be thought of as the term which would be derived from the ribbon shape $\alpha^{(k-1)}$, which is just a single row $(\alpha_k)$, by removing all the squares and leaving us with $Z_\emptyset = 1$.

![Figure 2.1: The auxiliary ribbon shapes derived from the ribbon shape (3, 2, 1, 3).](image)

We note that in the special case where $\alpha = (1^k)$, there are no auxiliary shapes so that we obtain

$$
\sum_{n \geq 1} s_{(1^k, n)} t^{k+n} = \frac{\sum_{j=0}^{k} (-1)^j Z_{(1^{k-j})} t^{k-j}}{E(-t)} + (-1)^{k-1}
$$
\[
\begin{align*}
&= (-1)^k \sum_{j=0}^{k-1} (-1)^j e_j t^j \
&= \frac{(-1)^k E(-t)}{E(-t)} - \frac{(-1)^k E(-t)}{E(-t)}
\end{align*}
\]

\[
\begin{align*}
&= -(-1)^k \sum_{j \geq k+1} (-1)^j e_j t^j \
&= \frac{\sum_{j \geq k+1} (-1)^{j-1} (-1)^k e_j t^j}{E(-t)}
\end{align*}
\]

which is just the special case of (1.2.4) when \( u_n = (-1)^k \chi(n \geq k + 1) \) since \( p_n(-1)^k \chi(n \geq k + 1) \) is the Schur function corresponding to the partition \((1^k, n)\).

**Proof of Theorem 2.1.1.**

Proof. We start by using the expansion \( s_{\lambda/\mu} = \sum_{\nu} K_{\nu,\lambda/\mu}^{-1} h_{\nu} \). In the case where \( \lambda/\mu \) corresponds to the ribbon shape \( \alpha = (\alpha_k, \ldots, \alpha_1) \), we can classify the special rim hook tabloids by the length of the last special rim hook. For example, a typical special rim hook in the case where \( \alpha = (3, 2, 4, 5, 3) \) is pictured in Figure 2.2. Since in a special rim hook tabloid of ribbon shape \( \alpha = (\alpha_k, \ldots, \alpha_1) \) each of the rim hooks must start on the left hand border, it follows that the rim hook which ends in the lower-most square must cover the last \( j \) rows for some \( j \in \{1, \ldots, k\} \).

Now suppose that \( H \) is the last rim hook pictured in Figure 2.2 and we consider the sum \( \sum_{\mu} \sum_{T \in F(\mu, H)} sgn(T) h_{\mu} \) where \( F(\mu, H) \) is the set of special rim hook tabloids of type \( \mu \) and ribbon shape \( \alpha = (3, 2, 4, 5, 3) \) such that the last special rim of \( T \) is \( H \). Then it is easy to see that since the filling of the rim hooks in the first three rows of ribbon shape \( \alpha = (3, 2, 4, 5, 3) \) is arbitrary that this sum will equal

\[
sgn(H) h_{|H|} \sum_{\nu \vdash 9} \sum_{T \in SRHT(\nu, \gamma/\delta)} sgn(T) h_{\nu}
\]

where \( \gamma/\delta \) is just the skew shape corresponding to the ribbon shape \((3, 2, 4)\) so that this sum is just \( sgn(H) h_{|H|} Z_{(3,2,4)} \).

It follows that if we classify the special rim hook tabloids \( T \) of the ribbon shape \((\alpha, n)\) by the number \( j \) of rows in the ribbon shape corresponding to \( \alpha \) that the
Figure 2.2: A special rim hook tabloid of the ribbon shape (3, 2, 4, 5, 3).

last rim hook of \( T \) covers, then we obtain that

\[
Z_{(\alpha,n)} = \sum_{j=0}^{k} (-1)^j Z_{t|\alpha(j)} h_{n+\alpha_1+\ldots+\alpha_j}. \tag{2.1.2}
\]

Thus

\[
\sum_{n\geq 1} Z_{(\alpha,n)} t^{n+|\alpha|} = \sum_{j=0}^{k} (-1)^j Z_{t|\alpha(j)} \sum_{n\geq 1} h_{n+\alpha_1+\ldots+\alpha_j} t^{n+\alpha_1+\ldots+\alpha_j}
\]

\[
= \sum_{j=0}^{k} (-1)^j Z_{t|\alpha(j)} (H(t) - \sum_{r=0}^{\alpha_1+\ldots+\alpha_j} h_r t^r)
\]

\[
= \sum_{j=0}^{k} (-1)^j Z_{t|\alpha(j)} \left( \frac{1}{E(-t)} - \sum_{r=0}^{\alpha_1+\ldots+\alpha_j} h_r t^r \right)
\]

\[
= \sum_{j=0}^{k} (-1)^j Z_{t|\alpha(j)} \left( \frac{1}{E(-t)} - \frac{Z_{t|\alpha} + \sum_{j=1}^{k} (-1)^j Z_{t|\alpha(j)} \sum_{r=0}^{\alpha_1+\ldots+\alpha_j} h_r t^r}{E(-t)} \right). \tag{2.1.3}
\]

Now consider the sum

\[
Z_{t|\alpha} + \sum_{j=1}^{k} (-1)^j Z_{t|\alpha(j)} \sum_{r=0}^{\alpha_1+\ldots+\alpha_j} h_r t^r. \tag{2.1.4}
\]

It is an easy consequence of the Littlewood-Richardson Rule that for any composition \( \beta = (\beta_t, \ldots, \beta_1) \),

\[
Z_\beta h_r = Z_{(\beta,r)} + Z_{(\beta_t, \ldots, \beta_2, \beta_1+r)}. \tag{2.1.5}
\]
Thus we see that (2.1.4) is equal to
\[
\begin{align*}
&\sum_{j=0}^{k} (-1)^j Z_{\alpha(j)} t^{[\alpha(j)]} + \sum_{j=1}^{k} (-1)^j Z_{\alpha(j)} t^{[\alpha(j)]} \sum_{r=1}^{\alpha_1+\cdots+\alpha_j} h_r t^r = \\
&\sum_{j=0}^{k} (-1)^j Z_{\alpha(j)} t^{[\alpha(j)]} + \\
&\sum_{j=1}^{k} \sum_{r=1}^{\alpha_1+\cdots+\alpha_j} (-1)^j t^{[\alpha(j)]+r} (Z_{(\alpha_k,\ldots,\alpha_{j+1},r)} + Z_{(\alpha_k,\ldots,\alpha_{j+2},\alpha_{j+1}+r)}). \tag{2.1.6}
\end{align*}
\]

We can organize the $Z_\beta$'s that appear in (2.1.6) by the number of parts $s$ of $\beta$.

For $s = 0$, there is one term $(-1)^k Z_{\alpha(k)} = (-1)^k$.

For $1 \leq s < k$, we obtain the terms
\[
\begin{align*}
&(-1)^{k-s} Z_{(\alpha_k,\ldots,\alpha_{k-s+1})} t^{[\alpha(k-s)]} + \\
&(-1)^{k-s+1} \sum_{r=1}^{\alpha_1+\cdots+\alpha_{k-s+1}} Z_{(\alpha_k,\ldots,\alpha_{k-s+2},r)} t^{\alpha_k+\cdots+\alpha_{k-s+2}+r} + \\
&(-1)^{k-s} \sum_{r=1}^{\alpha_1+\cdots+\alpha_{k-s}} Z_{(\alpha_k,\ldots,\alpha_{k-s+2},\alpha_{k-s+1}+r)} t^{\alpha_k+\cdots+\alpha_{k-s+1}+r} = \\
&(-1)^{k-s+1} \sum_{r=1}^{\alpha_{k-s+1}-1} Z_{(\alpha_k,\ldots,\alpha_{k-s+2},r)} t^{[\alpha(k-s+1)]+r}. \tag{2.1.7}
\end{align*}
\]

For $s = k$, we have the terms
\[
Z_{(\alpha_k,\ldots,\alpha_1)} t^{[\alpha]} - \sum_{r=1}^{\alpha_1} Z_{(\alpha_k,\ldots,\alpha_2,r)} t^{\alpha_k+\cdots+\alpha_2+r} = -\sum_{r=1}^{\alpha_1-1} Z_{(\alpha_k,\ldots,\alpha_2,r)} t^{\alpha_k+\cdots+\alpha_2+r}. \tag{2.1.8}
\]

Combining these cases together, we see that (2.1.4) is
\[
(-1)^k + \sum_{j=1}^{\alpha_j-1} (-1)^j \sum_{r=1}^{\alpha_j} Z_{(\alpha(j),r)} t^{[\alpha(j)]+r}. \tag{2.1.9}
\]

Combining (2.1.9) with (2.1.3) yields (2.1.1).
2.2 Methods for Computing Generating Functions

In this section, we shall describe how we can use ribbon Schur functions to compute various generating functions over sets of permutations which contain a given descent set. In particular, we shall give methods to compute

\[ F^L_S(x, Q, P) = \sum_{n \geq 0} \left[ n \right]_{P, Q} \sum_{\Sigma \in SL_n, S \subseteq Comdes(\Sigma)} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}. \]  

Our method proceeds in three steps. First, for any composition \( \alpha = (\alpha_1, \ldots, \alpha_k) \) of \( n \), define \( h_\alpha = h_{\alpha_1} \cdots h_{\alpha_k} \) and

\[ \text{Set}(\alpha) = \{ \alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_k \}. \]

Then for \( \sigma \in S_n \), we define

\[ \text{Des}_\alpha(\sigma) = \text{Des}(\sigma) - \text{Set}(\alpha), \]
\[ \text{des}_\alpha(\sigma) = |\text{Des}_\alpha(\sigma)|, \]
\[ \text{Rise}_\alpha(\sigma) = \text{Rise}(\sigma) \cup \text{Set}(\alpha), \text{ and} \]
\[ \text{rise}_\alpha(\sigma) = |\text{Rise}_\alpha(\sigma)|. \]

If \( \Sigma = (\sigma_1, \ldots, \sigma_L) \in SL_n \), then we let

\[ \text{Comdes}_\alpha(\Sigma) = \cap_{i=1}^L \text{Des}_\alpha(\sigma_i), \]
\[ \text{comdes}_\alpha(\Sigma) = |\text{Comdes}_\alpha(\Sigma)|, \]
\[ \text{oneRise}_\alpha(\Sigma) = \{1, \ldots, n\} - \text{Comdes}_\alpha(\Sigma), \text{ and} \]
\[ \text{onerise}_\alpha(\Sigma) = |\text{oneRise}_\alpha(\Sigma)|. \]

Define a homomorphism \( \xi \) from the ring of symmetric functions \( \Lambda \) to the polynomial ring \( Q(q_1, \ldots, q_L, p_1, \ldots, p_L)[x] \) by setting

\[ \xi(e_n) = \frac{(1 - x)^{n-1} Q^n}{\left[ n \right]_{P, Q}}. \]  

(2.2.2)
Then our first step is to prove the following result which is a generalization of Beck and Remmel’s result [6].

**Theorem 2.2.1.** For any composition \( \alpha = (\alpha_1, \ldots, \alpha_k) \) of \( n \),

\[
[n]_{P,Q}^! \xi(h_\alpha) = \sum_{\Sigma = (\sigma_1, \ldots, \sigma_L) \in S^n} x^{\text{comdes}_\alpha(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}. \tag{2.2.3}
\]

**Proof.** First we consider the case where \( \alpha \) is a partition of \( n \). Given a brick tabloid \( T \in B_{\mu,\alpha} \), let \( b_1, \ldots, b_{\ell(\mu)} \) be the sequence which records the lengths of the bricks in \( T \) where we read the rows from top to bottom and the bricks in each row from left to right. In such a situation, we shall write \( T = (b_1, \ldots, b_{\ell(\mu)}) \). For example, for the brick tabloid \( T \in B_{(1^2,2^2,3,4),(2,5,6)} \) pictured at the top of Figure 2.3, we would write \( T = (2,1,3,1,4,2) \).

![Figure 2.3: The brick tabloid \( T = (2,1,3,1,4,2) \) in \( B_{(1^2,2^2,3,4),(2,5,6)} \).](image)

Then we know that

\[
[n]_{P,Q}^! \xi(h_\alpha) = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\mu,\alpha} \xi(e_\mu) = \]

\[
[n]_{P,Q}^! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \sum_{T = (b_1, \ldots, b_{\ell(\mu)})} \prod_{i=1}^{\ell(\mu)} \frac{(1 - x)^{b_i - 1} Q^{b_i}}{[b_i]_{P,Q}^!} = \]

\[
\sum_{\mu \vdash n} \sum_{T = (b_1, \ldots, b_{\ell(\mu)})} \left[ \binom{n}{b_1, \ldots, b_{\ell(\mu)}} \right]_{P,Q}^! Q^{\sum_{i=1}^{\ell(\mu)} \binom{b_i}{2}} (x - 1)^{n-\ell(\mu)}. \tag{2.2.5}
\]
Fix a brick tabloid \( T = (b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu,n} \). We want to give a combinatorial interpretation to \( \left[ \begin{array}{c} n \\ b_1, \ldots, b_{\ell(\mu)} \end{array} \right]_{P,Q} \). Let \( DF(T) \) denote the set of all fillings of the cells of \( T \) with the numbers \( 1, \ldots, n \) so that the numbers decrease within each brick reading from left to right. We can then think of each such filling as a permutation of \( S_n \) by reading the numbers in rows from top to bottom and the numbers from left to right in each row. For example, the filled brick tabloid at the bottom of Figure 2.3 is an element of \( DF((2,1,3,1,4,2)) \) whose corresponding permutation is \( 6 \\ 3 \\ 7 \\ 10 \\ 8 \\ 5 \\ 12 \\ 11 \\ 4 \\ 2 \\ 1 \\ 13 \\ 9 \). Then we have the following lemma.

**Lemma 2.2.2.** Let \( T = (b_1, \ldots, b_{\ell(\mu)}) \) be a brick tabloid in \( B_{\mu,\alpha} \), then

\[
q^{\sum_{i=1}^\infty \left[ \begin{array}{c} i \\ b_1, \ldots, b_{\ell(\mu)} \end{array} \right]_{P,Q}} = \sum_{\sigma \in DF(T)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}. \tag{2.2.6}
\]

**Proof.** It follows from a result of Carlitz [11] that for positive integers \( b_1, \ldots, b_\ell \) which sum to \( n \),

\[
\left[ \begin{array}{c} n \\ b_1, \ldots, b_\ell \end{array} \right]_{P,Q} = \sum_{r \in R(1^{b_1}, \ldots, \ell^{b_\ell})} q^{\text{inv}(r)} p^{\text{coinv}(r)}
\]

where \( R(1^{b_1}, \ldots, \ell^{b_\ell}) \) is the set of rearrangements of \( b_1 \) 1’s, \( b_2 \) 2’s, etc. Consider a rearrangement \( r \) of \( 1^{b_1}, \ldots, \ell^{b_\ell} \) and construct a permutation \( \sigma_r \) by labeling the 1’s from right to left with \( 1, 2, \ldots, b_1 \), the 2’s from right to left with \( b_1 + 1, \ldots, b_1 + b_2 \), and in general the \( i \)’s from right to left with \( 1 + \sum_{j=1}^{i-1} b_j, \ldots, b_i + \sum_{j=1}^{i-1} b_j \). In this way, \( \sigma_r^{-1} \) starts with the positions of the 1’s in \( r \) decreasing order, followed by the positions of the 2’s in \( r \) in decreasing order, etc. For example, if \( T = (2,1,3,1,4,2) \in B_{(1^2,2^2,3,4,1),(2,5,6)} \) is the brick tabloid pictured at the top of Figure 2.3, then one possible rearrangement to consider is \( r = 5 \\ 5 \\ 1 \\ 5 \\ 3 \\ 1 \\ 2 \\ 3 \\ 6 \\ 3 \\ 5 \\ 4 \\ 6 \). Below we display \( \sigma_r \) and \( \sigma_r^{-1} \).

<table>
<thead>
<tr>
<th>( \sigma_r )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_r )</td>
<td>11</td>
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<td>9</td>
<td>6</td>
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<td>3</td>
<td>5</td>
<td>13</td>
<td>4</td>
<td>8</td>
<td>7</td>
<td>12</td>
</tr>
<tr>
<td>( \sigma_r^{-1} )</td>
<td>6</td>
<td>3</td>
<td>7</td>
<td>10</td>
<td>8</td>
<td>5</td>
<td>12</td>
<td>11</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>13</td>
<td>9</td>
</tr>
</tbody>
</table>
It is then easy to see that
\[
\text{coinv}(r) = \text{coinv}(\sigma_r) = \text{coinv}(\sigma_r^{-1}) \quad \text{and}
\]
\[
\left(\frac{2}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{2}{2}\right) + \left(\frac{4}{2}\right) + \left(\frac{2}{2}\right) + \text{inv}(r) = \text{inv}(\sigma_r) = \text{inv}(\sigma_r^{-1}).
\]
We can think of \(\sigma_r^{-1}\) as a filling of the cells of the brick tabloid \(T = (2, 1, 3, 4, 2)\) with the numbers 1, . . . , 13 such that the numbers within each brick are decreasing, reading from left to right. In fact, this filling is precisely the filling pictured at the bottom of Figure 2.3.

Thus in general, for any \(T = (b_1, \ldots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu,\alpha}\), the correspondence which takes \(r \in \mathcal{R}(1^{b_1}, \ldots, \ell^{b_{\ell(\mu)}})\) to \(\sigma_r^{-1}\) shows that
\[
q^{\Sigma_{i=1}^{\ell(\mu)} \left(\begin{array}{c} \frac{2}{2} \\ \frac{1}{2} \end{array}\right)} \left[ n \atop b_1, \ldots, b_{\ell(\mu)} \right]_{p,q} = \sum_{\sigma \in DF(T)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}
\]
as desired.

It follows that for any \(T = (b_1, \ldots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu,\alpha}\),
\[
Q^{\Sigma_{i=1}^{\ell(\mu)} \left(\begin{array}{c} \frac{2}{2} \\ \frac{1}{2} \end{array}\right)} \left[ n \atop b_1, \ldots, b_{\ell(\mu)} \right]_{p,Q} = \prod_{i=1}^{L} q_{i}^{\Sigma_{i=1}^{\ell(\mu)} \left(\begin{array}{c} \frac{2}{2} \\ \frac{1}{2} \end{array}\right)} \left[ n \atop b_1, \ldots, b_{\ell(\mu)} \right]_{p_{i},q_{i}} = \prod_{i=1}^{L} \sum_{\sigma(i) \in DF(T)} q_{i}^{\text{inv}(\sigma(i))} p_{i}^{\text{coinv}(\sigma(i))}. \quad (2.2.7)
\]
Thus we can interpret \(Q^{\Sigma_{i=1}^{\ell(\mu)} \left(\begin{array}{c} \frac{2}{2} \\ \frac{1}{2} \end{array}\right)} \left[ n \atop b_1, \ldots, b_{\ell(\mu)} \right]_{p,Q}\) as the sum of the weights of the set of fillings of \(T\) with \(L\)-tuples of permutations \(\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)})\) such that for each \(i\), the elements of \(\sigma^{(i)}\) are decreasing within each brick of \(T\) and we weight such a filling with \(Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}\). For example, if \(T = (2, 2, 3, 2, 4, 3) \in \mathcal{B}_{(2^3,3^2,4),(4,5,7)}\) and \(L = 3\), then such a filling of \(T\) is pictured in Figure 2.4. We can then interpret the term \((x - 1)^{n-\ell(\mu)}\) from (4.5) as taking such a filling and labeling each cell which is not at the end of a brick with either an \(x\) or \(-1\) and labeling each cell at the end of a brick with 1. Again, we have pictured such a labeling of the cells of
T in Figure 2.4. We shall call such an object \( O \) a *labeled filled brick tabloid*. We define the weight of \( O \), \( W(O) \), to be the product over all the labels of the cells times \( Q^{inv(\Sigma)} P^{coinv(\Sigma)} \) if \( T \) is filled with permutations \( \Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \). Thus for the object pictured in Figure 2.4,

\[
W(O) = (-1)^4 x^6 q_1^{inv(\sigma^{(1)})} q_2^{inv(\sigma^{(2)})} q_3^{inv(\sigma^{(3)})} p_1^{coinv(\sigma^{(1)})} p_2^{coinv(\sigma^{(2)})} p_3^{coinv(\sigma^{(3)})}.
\]

\[
\sigma^{(1)} = 6 \ 2 \ 16 \ 10 \ 14 \ 12 \ 7 \ 13 \ 4 \ 15 \ 5 \ 3 \ 1 \ 11 \ 9 \ 8
\]

\[
\sigma^{(2)} = 10 \ 6 \ 12 \ 4 \ 16 \ 13 \ 3 \ 7 \ 2 \ 9 \ 8 \ 5 \ 1 \ 15 \ 14 \ 11
\]

\[
\sigma^{(3)} = 16 \ 4 \ 8 \ 2 \ 15 \ 9 \ 7 \ 14 \ 11 \ 13 \ 10 \ 5 \ 3 \ 12 \ 6 \ 1
\]

**Figure 2.4**: A label filled brick tabloid of shape \((4, 5, 7)\).

We let \( LF(\alpha) \) denote the set of all objects that can be created in this way from brick tabloids \( T \) of shape \( \alpha \). It follows that

\[
[n] P Q !\xi(h_\alpha) = \sum_{O \in LF(\alpha)} W(O). \tag{2.2.9}
\]

Next we define an involution \( I : LF(\alpha) \to LF(\alpha) \). Given \( O \in LF(\alpha) \), read the cells of \( O \) in the same order that we read the underlying permutations and look for the first cell \( c \) such that either:

(i) \( c \) is labeled with \(-1\) or

(ii) \( c \) is at the end of end of brick \( b \), the cell \( c+1 \) is immediately to the right of \( c \) and starts another brick \( b' \), and each permutation \( \sigma^{(i)} \) decreases as we go from \( c \) to \( c+1 \).
If we are in case (i), then $I(O)$ is the labeled filled brick tabloid which is obtained from $O$ by taking the brick $b$ that contains $c$ and splitting $b$ into two bricks $b_1$ and $b_2$, where $b_1$ contains the cells of $b$ up to and including the cell $c$ and $b_2$ contains the remaining cells of $b$, and changing the label on $c$ from $-1$ to $1$. In case (ii), $I(O)$ is the labeled filled brick tabloid which is obtained from $O$ by combining the two brick $b$ and $b'$ into a single brick and changing the label on cell $c$ from $1$ to $-1$. Finally, if neither case (i) or case (ii) applies, then we let $I(O) = O$. For example, if we consider the labeled filled brick tabloid $O$ pictured in Figure 2.4, then $I(O)$ is pictured in Figure 2.5.

Figure 2.5: $I(O)$.

It is easy to see that if $I(O) \neq O$, the $W(I(O)) = -W(O)$ since we change the label on cell $c$ from $1$ to $-1$ or vice versa. Moreover, it is easy to check that $I^2$ is the identity. Thus $I$ shows that

$$[n]_{P,Q} \xi(h_\alpha) = \sum_{O \in LF(\alpha)} W(O) = \sum_{O \in LF(\alpha), I(O) = O} W(O). \quad (2.2.10)$$
Thus we must examine the fixed points of $I$. Clearly if $I(\mathcal{O}) = \mathcal{O}$, then $\mathcal{O}$ can have no cells which are labeled with $-1$. Also it must be the case that between any two consecutive bricks in the same row of $\mathcal{O}$, at least one of the underlying permutations $\sigma^{(i)}$ must increase. It follows that each cell $c$ which is not at the end of the brick in $\mathcal{O}$ is labeled with $x$ and each of the permutations $\sigma^{(i)}$ has a descent at $c$ so that $c \in \text{Comdes}(\Sigma)$. All the other cells of $\mathcal{O}$ are either at the end of brick which has another brick to its right in which case $c \notin \text{Comdes}(\Sigma)$ or $c$ is at the end of a row in which case $c \in \text{Set}(\alpha)$. All such cells have label 1 so that $W(\mathcal{O}) = x^{\text{comdes}_a(\Sigma)}Q^{\text{inv}(\Sigma)}P^{\text{coinv}(\Sigma)}$.

Now if we are given $\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \in S_n^L$, we can construct a fixed point of $I$ from $\Sigma$ by using $(\sigma^{(1)}, \ldots, \sigma^{(L)})$ to fill a tabloid of shape $\alpha$, then drawing the bricks so that the cells $c$ which end bricks are precisely the elements of $\text{oneRise}(\Sigma) \cup \text{Set}(\alpha)$. This shows that

$$\sum_{\mathcal{O} \in \mathcal{L}(\alpha), I(\mathcal{O}) = \mathcal{O}} W(\mathcal{O}) = \sum_{\Sigma \in S^L_n} x^{\text{comdes}_a(\Sigma)}Q^{\text{inv}(\Sigma)}P^{\text{coinv}(\Sigma)}$$

(2.2.11)

as desired.

Now suppose $\beta$ is an arbitrary composition which can be rearranged to the partition $\alpha$. Observe that the order in which we decided to read the rows of the brick tabloid $T \in \mathcal{B}_{\mu, \alpha}$ determined how we read the bricks and how we associated a sequence of permutations with a filled brick tabloid. That is, we ordered the rows $R_1, \ldots, R_{\ell(\alpha)}$ of $T$ by reading the rows from top to bottom. This, in turn, how determined how we ordered the bricks so that we could write $T = (b_1, \ldots, b_{\ell(\mu)})$ and determined how we associated a permutation with each labeled filled brick tabloid based on $T$. That is, we ordered the bricks by reading the bricks from left to right in each row and reading the rows in the order $R_1, \ldots, R_{\ell(\alpha)}$. Similarly, we read the cells in rows from left to right in row and we read the rows in order $R_1, \ldots, R_{\ell(\alpha)}$ to determine how we associated a permutation with a filled brick tabloid. Now suppose that we decide to order the rows as $R_{\tau_1}, \ldots, R_{\tau_{\ell(\alpha)}}$ for some permutation $\tau \in S_{\ell(\alpha)}$ such that $(|R_{\tau_1}|, \ldots, |R_{\tau_{\ell(\alpha)}}|) = \beta$, where $|R_i|$ denotes the length of row $R_i$. Then we would have a new way to order the bricks by reading the bricks from
left to right in each row and reading the rows in order \( R_{\tau_1}, \ldots, R_{\tau_{\ell(\alpha)}} \). Similarly, we would have a new way to associate a permutation with each labeled filled brick tabloid \( T \) of shape \( \alpha \) by reading the cells in rows from left to right and reading the rows in order \( R_{\tau_1}, \ldots, R_{\tau_{\ell(\alpha)}} \). Everything in the proof will be exactly as before, except that cells at the end of the rows \( R_{\tau_1}, \ldots, R_{\tau_{\ell(\alpha)}} \) would correspond to the positions in \( \text{Set}(\beta) \) rather than set \( \alpha \). In this way, we could show that

\[
[n]_P Q! \xi(h_\alpha) = \sum_{\Sigma = (\sigma_1, \ldots, \sigma_L) \in S_n^L} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}
\]  

(2.2.12)

for any composition \( \beta \) that rearranges to \( \alpha \). Since \( h_\beta = h_\alpha \) in this case, it follows that (2.2.3) holds for any composition \( \alpha \).

Next given a composition of \( n \), \( \alpha = (\alpha_1, \ldots, \alpha_k) \), let \( F_\alpha \) denote the ribbon shape corresponding to \( \alpha \) and \( Z_\alpha \) denote the ribbon Schur function corresponding to \( \alpha \). Then we have the following.

**Theorem 2.2.3.**

\[
[n]_P Q! \xi(Z_\alpha) = \frac{(1 - x)^{k-1}}{x^{k-1}} \sum_{\Sigma = (\sigma_1, \ldots, \sigma_L) \in S_n^L} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}
\]  

(2.2.13)

**Proof.** We can expand the \( Z_\alpha \) using (6.2.1) as

\[
Z_\alpha = S_{\lambda/\nu} = \sum_{\mu \vdash |\lambda/\nu|} h_\mu \sum_{T \in \text{SRHT}(\mu, \lambda/\nu)} \text{sgn}(T).
\]  

(2.2.14)

Applying \( \xi \) to both sides of (2.2.14)

\[
[n]_P Q! \xi(Z_\alpha) = \sum_{\mu \vdash |\lambda/\nu|} \xi(h_\mu) \sum_{T \in \text{SRHT}(\mu, \lambda/\nu)} \text{sgn}(T) = \\
\sum_{T \in \text{SRHT}(\text{shape}(\alpha))} \text{sgn}(T)[n]_P Q! \xi(h_\beta(T)) = \\
\sum_{T \in \text{SRHT}(\text{shape}(\alpha))} \text{sgn}(T) \sum_{\Sigma \in S_n^L} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}.
\]
where $\beta(T)$ is composition induced by reading the rim hooks in $T$ from top to bottom. Thus we can identify $[n]_{P,Q}!\xi(Z_\alpha)$ with the sum over filled special rim hook tabloids $F(T)$, where each cell of the underlying special rim hook tabloid is filled with an $L$-tuple of numbers in such a way that when we read the numbers in cells starting from the top, we get a sequence of permutations 

$$\Sigma_{F(T)} = (\sigma_{F(T)}^{(1)}, \ldots, \sigma_{F(T)}^{(L)}) \in S_n^L.$$ 

For example, consider the filled special rim tabloid $F(T)$ at the top left of Figure 2.6. Then $L = 2$, $\alpha = (3, 1, 3, 2)$, $\beta(T) = (4, 5)$, and the underlying pair of permutations $\Sigma = (\sigma^{(1)}, \sigma^{(2)})$ is

$$\sigma^{(1)} = 5 \ 9 \ 3 \ 7 \ 6 \ 8 \ 4 \ 2 \ 1 \quad \sigma^{(2)} = 6 \ 4 \ 1 \ 2 \ 8 \ 5 \ 3 \ 7 \ 9.$$ 

In this case $\text{Comdes}(\Sigma) = \{2, 5\}$ so we have put $x$'s on top of each of the cells. The weight of this configuration is then

$$\text{sgn}(T)_x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} = x^2 q_1^{26} q_2^{12} p_1^{10} p_2^{24}.$$ 

![Figure 2.6: Pairing of special rim hook tabloids of shape $F_{(3,1,3,2)}$.](image)
2.6 and \( F(T_4) \) is the filled special rim hook tabloid at the bottom left of Figure 2.6, then \( (F(T_3), F(T_4)) \) is such a pair.

Now suppose that \( (F(T), F(S)) \) is any such pair of special rim hook tabloids where the first rim hook of \( S \) ends in the first row and the first rim hook of \( T \) does not end the first row. It is easy to see that the sign of the underlying special rim hook tabloid \( T \) is \(-1\) raised to number of vertical segments that are part of rim hooks so that it must be the case that \( sgn(T) = -sgn(S) \). Let \( c = \alpha_1 \) be the cell at the end of the first row \( F_\alpha \). Now if \( c \notin \text{Comdes}(\Sigma) \), then it easy to see that \( c \) is not an element of either \( \text{Comdes}_\beta(S)(\Sigma) \) or \( \text{Comdes}_\beta(T)(\Sigma) \) so that

\[
x^{\text{comdes}_\beta(T)(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} = x^{\text{comdes}_\beta(S)(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}.
\]

Hence any two such filled special rim hook tabloids will contribute 0 to

\[
\sum_{T \in \text{SRHT}(\text{shape}(\alpha))} sgn(T) \sum_{\Sigma \in S^L_k} x^{\text{comdes}_\beta(T)(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}.
\]

For example, the pair \((F(T_1), F(T_2))\) pictured at the top of Figure 2.6 is such a pair. However if \( c \in \text{Comdes}(\Sigma) \), then \( c \in \text{Comdes}_\beta(T)(\Sigma) \), but \( c \notin \text{Comdes}_\beta(S)(\Sigma) \). In this case, it follows that \( sgn(T) = -sgn(S) \) and

\[
sgn(T)x^{\text{comdes}_\beta(T)(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} = (-x)sgn(S)x^{\text{comdes}_\beta(S)(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}.
\]

For example, the pair \((F(T_3), F(T_4))\) pictured at the bottom of Figure 2.6 is such a pair. In this situation, the pair \((F(T), F(S))\) will contribute \((1 - x)sgn(S)x^{\text{comdes}_\beta(S)(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}\) to the sum

\[
\sum_{T \in \text{SRHT}(\text{shape}(\alpha))} sgn(T) \sum_{\Sigma \in S^L_k} x^{\text{comdes}_\beta(T)(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}.
\]

It follows that we can replace the sum

\[
\sum_{T \in \text{SRHT}(\text{shape}(\alpha))} sgn(T) \sum_{\Sigma \in S^L_k} x^{\text{comdes}_\beta(T)(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}
\]
by the sum

\[(1 - x) \sum_{T \in H_1(\alpha)} sgn(T) \sum_{\Sigma \in S_{n,\{\alpha_1\}}} x^{\text{comdes}_\beta(T)(\Sigma)} Q^{\text{inv}\Sigma} P^{\text{coinv}\Sigma}(2.2.15)\]

where \(H_1\) is the set of special rim hook tabloids of shape \(\alpha\) whose first special rim hook is horizontal and \(S_{n,\{\alpha_1\}}\) is the set of all \(L\)-tuples of permutations \(\Sigma\) such that \(\alpha_1 \in \text{Comdes}(\Sigma)\). Then we can repeat the same argument on this class of special rim hook tabloids by examining such fillings in pairs with identical integer fillings and identical special rim hooks except that one has a break in the special rim hooks at the end of the second row and the other one does not. It will follow that we can replace the sum in (2.2.15) by

\[(1 - x)^2 \sum_{T \in H_2(\alpha)} sgn(T) \sum_{\Sigma \in S_{n,\{\alpha_1,\alpha_1+\alpha_2\}}} x^{\text{comdes}_\beta(T)(\Sigma)} Q^{\text{inv}\Sigma} P^{\text{coinv}\Sigma}(2.2.16)\]

where \(H_2\) is the set of special rim hook tabloids of shape \(\alpha\) whose first two special rim hook are horizontal and \(S_{n,\{\alpha_1,\alpha_1+\alpha_2\}}\) is the set of all \(L\)-tuples of permutations \(\Sigma\) such that \(\alpha_1, \alpha_1 + \alpha_2 \in \text{Comdes}(\Sigma)\).

Continuing on in this way, we can show that if \(\alpha\) has \(k\) parts, then \([n]_{P,Q}^1\xi(Z_\alpha)\) is equal to the sum

\[(1 - x)^{k-1} \sum_{T \in H_k(\alpha)} sgn(T) \sum_{\Sigma \in S_{n,\text{Set}(\alpha)}} x^{\text{comdes}_\beta(T)(\Sigma)} Q^{\text{inv}\Sigma} P^{\text{coinv}\Sigma}(2.2.17)\]

where \(H_k\) is the set of special rim hook tabloids of shape \(\alpha\) whose first \(k\) special rim hook are horizontal and \(S_{n,\text{Set}(\alpha)}\) is the set of all \(L\)-tuples of permutations \(\Sigma\) such that \(\text{Set}(\alpha) \subseteq \text{Comdes}(\Sigma)\). However there is only one special rim hook tabloid \(T\) whose first \(k\) special rim hooks are horizontal and, clearly, \(sgn(T) = 1\) and \(\beta(T) = \alpha\). Note that if \(\Sigma\) is such that \(\text{Set}(\alpha) \subseteq \text{Comdes}(\Sigma)\), then \(x^{\text{comdes}(\Sigma)} = x^{k-1} x^{\text{comdes}_\alpha(\Sigma)}\). Thus, it follows that

\[\quad [n]_{P,Q}^1\xi(Z_\alpha) =
(1 - x)^{k-1} \sum_{\Sigma \in S_{n,\text{Set}(\alpha) \subseteq \text{Comdes}(\Sigma)}} x^{\text{comdes}_\alpha(\Sigma)} Q^{\text{inv}\Sigma} P^{\text{coinv}\Sigma} =
\frac{(1 - x)^{k-1}}{x^{k-1}} \sum_{\Sigma \in S_{n,\text{Set}(\alpha) \subseteq \text{Comdes}(\Sigma)}} x^{\text{comdes}(\Sigma)} Q^{\text{inv}\Sigma} P^{\text{coinv}\Sigma}\]
as desired.

We can then combine Theorem 2.1.1 with Theorem 2.2.3 to obtain the following.

**Theorem 2.2.4.** Let \( \alpha = (\alpha_1, \ldots, \alpha_k) \) be a composition of \( n \) and let

\[
\overline{S\ell}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_k\}.
\]

Then

\[
\sum_{n \geq 1} \frac{t^{n+|\alpha|}}{n + |\alpha|}P,Q \sum_{\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \in S^k_n} x^{\comdes(\Sigma)}Q^{\inv(\Sigma)}P^{\coinv(\Sigma)} \tag{2.2.18}
\]

\[
= \frac{x^k}{(1 - x)^{k-1}} \xi \left( -\sum_{j=0}^{k} Z_{\alpha(j)} t^{|\alpha(j)|} + \sum_{j=1}^{k} (-1)^{j-1} \sum_{r=1}^{\alpha_j-1} Z_{(\alpha(j), r)} t^{r+|\alpha(j)|} \right)
\]

**Proof.** First observe that

\[
\frac{1}{\xi(E(-t))} = \frac{1}{1 + \sum_{n \geq 1} \frac{(1-x)^n Q^{(2)}}{[n]_P,Q} (-t)^n} \tag{2.2.19}
\]

\[
= \frac{1}{1 - x + \sum_{n \geq 1} \frac{(1-x)^n Q^{(2)}}{[n]_P,Q} (-t)^n}
\]

\[
= \frac{1 - x}{-x + \exp(t(x-1), P, Q)}
\]

where \( \exp(u, P, Q) = \sum_{n \geq 0} \frac{u^n Q^{(2)}}{[n]_P,Q} \).

Then (2.2.18) can be derived from Theorems 2.1.1 and 2.2.3 by applying \( \xi \) to

\[
\frac{x^k}{(1 - x)^k} \sum_{n \geq 1} Z_{(\alpha, n)} t^{n+|\alpha|}.
\]

For example, suppose that we want to compute

\[
\sum_{n \geq 1} \frac{t^{n+|\alpha|}}{n + |\alpha|}P,Q \sum_{\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \in S^k_n} x^{\comdes(\Sigma)}Q^{\inv(\Sigma)}P^{\coinv(\Sigma)}. \tag{2.2.20}
\]
To have the left-hand side of (2.2.18) equal (2.2.20), we must choose $\alpha = (2, 2)$. Thus we must compute
\[
\frac{x^2}{(1 - x)^2} \xi \left( \frac{Z_{2,2} t^4 - Z_{2} t^2 + 1}{E(-t)} + (Z_{2,1} t^3 - (Z_1 t + 1)) \right). \tag{2.2.21}
\]

This necessitates computing the expansion of $Z_{(2,2)}$, $Z_{(2,1)}$, and $Z_2$ in terms of the elementary symmetric functions. One can easily list the transposed special rim hook tabloids by hand in each case. These are pictured in Figure 2.7.

![Figure 2.7: Transposed special rim hook tabloids for $Z_{(2,2)}$, $Z_{(2,1)}$, and $Z_2$.](image)

Thus it follows that from (6.2.1) that
\[
Z_{(2,2)} = e_1^2 e_2 - 2 e_1 e_3 + e_4, \\
Z_{(2,1)} = e_1 e_2 - e_3, \quad \text{and} \\
Z_2 = e_1^2 - e_2.
\]

Hence,
\[
\frac{x^2}{(1 - x)^2} \xi \left( \frac{Z_{2,2} t^4 - Z_{2} t^2 + 1}{E(-t)} + (Z_{2,1} t^3 - (Z_1 t + 1)) \right) = \tag{2.2.22}
\]
\[
\frac{x^2}{(1 - x)^2 E(-t) (e_1 e_2 - e_3)} \xi \left( (e_1^2 e_2 - 2 e_1 e_3 + e_4) t^4 - (e_1^2 - e_2) t^2 + 1 + ((e_1 e_2 - e_3) t^3 - e_1 t - 1) E(-t) \right) =
\]
Collecting terms, one can show that (2.2.22) is equal to

\[
\frac{x^2}{(1 - x)^2 \xi(E(-t))} \xi \left( (e^2 e_2 - 2e_1 e_3 + e_4) t^4 - (e^2_1 - e_2) t^2 + 1 + ((e_1 e_2 - e_3) t^3 - e_1 t - 1) (1 - e_1 t + e_2 t^2 - e_3 t^3) + ((e_1 e_2 - e_3) t^3 - e_1 t - 1) \sum e_n (-t)^n \right) =
\]

\[
\frac{x^2}{(1 - x)^2 \xi(E(-t))} \xi \left( e_4 t^4 + (e_1 e_2^2 - e_2 e_3) t^5 + (e_3^2 - e_1 e_2 e_3) t^6 + \sum_{n \geq 4} (-t)^{n+3} (e_3 e_n - e_1 e_2 e_n) + \sum_{n \geq 4} (-t)^{n+1} e_1 e_n - \sum_{n \geq 4} e_n (-t)^n \right).
\]

Collecting terms, one can show that (2.2.22) is equal to

\[
\frac{x^2}{(1 - x)^2 \xi(E(-t))} \left( (-t)^5 (e_2 e_3 - e_1 e_2^2 + e_1 e_4 - e_5) + t^6 (e_3^2 - e_1 e_2 e_3 + e_1 e_3 - e_6) + \sum_{n \geq 7} (-t)^n (e_3 e_{n-3} - e_1 e_2 e_{n-3} + e_1 e_{n-1} - e_n) \right) =
\]

\[
\frac{x^2}{(1 - x)^2 \xi(E(-t))} \times \left( \sum_{n \geq 5} (-t)^n (e_3 e_{n-3} - e_1 e_2 e_{n-3} + e_1 e_{n-1} - e_n) \right).
\]

Using (2.2.19), it follows that

\[
\sum_{n \geq 1} \frac{t^{n+4}}{[n + 4]_{P, Q}} \sum_{\Sigma = \{P, Q\}} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} =
\]

\[
\frac{x^2}{(1 - x)(-x + \exp(t(x - 1), P, Q))} \left( \sum_{n \geq 5} (-t)^n \left[ \frac{(1 - x)^{n-2} Q^{3+\binom{n-3}{2}}}{[3]_{P, Q} [n-3]_{P, Q}} - \frac{(1 - x)^{n-3} Q^{1+\binom{n-3}{2}}}{[2]_{P, Q} [n-3]_{P, Q}} + \frac{(1 - x)^{n-2} Q^{\binom{n-1}{2}}}{[n-1]_{P, Q}} - \frac{(1 - x)^{n-1} Q^{n}}{[n]_{P, Q}} \right] \right) =
\]

\[
\frac{x^2}{(-x + \exp(t(x - 1), P, Q))} \left( \sum_{n \geq 5} (-t)^n (1 - x)^{n-4} Q^{\binom{n-3}{2}} \frac{[n]_{P, Q}}{[n]_{P, Q}} \right) \times
\]

\[
\left[ (1 - x) Q^3 \binom{n}{3}_{P, Q} + Q^{2n-5} [n]_{P, Q} - (1 - x)^2 Q^{3n-6} - Q \left[ 1, 2, n - 3 \right]_{P, Q} \right).
\]
2.3 Extensions

In this section, we describe how to compute generating functions of the form

\[ \phi(S,T)(x, P, Q) = \sum_{\Sigma \in S^L} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} \]  

(2.3.1)

and

\[ \psi(S,T)(x, P, Q) = \sum_{\Sigma \in S^L} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} \]  

(2.3.2)

where \( S \) and \( T \) are disjoint sets of positive integers.

One can easily find an expression for (2.3.1) by inclusion-exclusion. That is, we claim

\[ \phi(S,T)(x, P, Q) = \sum_{S \subseteq U \subseteq S \cup T} (-1)^{|U - S|} \sum_{n \geq 0} \frac{t^n}{[n]_{P, Q}!} \sum_{\Sigma \in S^L} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} \]  

(2.3.3)

where we interpret the sum

\[ \sum_{\Sigma \in S^L} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} \]

to be 0 if there are no \( L \)-tuples \( \Sigma \in S^L_n \) such that \( U \subseteq \text{Comdes}(\Sigma) \). For example, if \( n \) is too small, it may be the case that there are no \( L \)-tuples \( \Sigma \in S^L_n \) such that \( U \subseteq \text{Comdes}(\Sigma) \). Clearly the right hand side of (2.3.3) equals

\[ \sum_{n \geq 0} \frac{t^n}{[n]_{P, Q}!} \sum_{\Sigma \in S^L_n} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} \times \sum_{S \subseteq U \subseteq S \cup T} (-1)^{|U - S|} \chi(U \subseteq \text{Comdes}(\Sigma)). \]

Then it easily follows from inclusion-exclusion that

\[ \sum_{S \subseteq U \subseteq S \cup T} (-1)^{|U - S|} \chi(U \subseteq \text{Comdes}(\Sigma)) \]  

(2.3.4)
is equal to 1 if $S \subseteq \text{Comdes}(\Sigma)$ and $T \cap \text{Comdes}(\Sigma) = \emptyset$ and 0 otherwise. Hence
\[
\phi(S,T)(x, P, Q) = \sum_{\Sigma \in \mathcal{S}_n} x^{|\text{comdes}(\Sigma)|} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}. \tag{2.3.5}
\]

For example, suppose that we wish to find the generating function for $L$-tuples of permutations $\Sigma$ such that $\{1, 3\} \subseteq \text{Comdes}(\Sigma)$ and $\{2\} \subseteq \text{oneRise}(\Sigma)$. Then
\[
\sum_{n \geq 1} \frac{t^{3+n}}{[3+n]_{P,Q}!} \sum_{\Sigma \in \mathcal{S}_n} x^{|\text{comdes}(\Sigma)|} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} = \tag{2.3.6}
\]
\[
\sum_{n \geq 1} \frac{t^{3+n}}{[3+n]_{P,Q}!} \sum_{\Sigma \in \mathcal{S}_n} x^{|\text{comdes}(\Sigma)|} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} - \sum_{n \geq 4} \frac{t^{3+n}}{[3+n]_{P,Q}!} \sum_{\Sigma \in \mathcal{S}_n} x^{|\text{comdes}(\Sigma)|} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}.
\]

We can apply Theorem 2.2.4 with $\alpha = (1,2)$ to conclude that
\[
\sum_{n \geq 1} \frac{t^{3+n}}{[3+n]_{P,Q}!} \sum_{\Sigma \in \mathcal{S}_n} x^{|\text{comdes}(\Sigma)|} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} = \tag{2.3.7}
\]
\[
\frac{x^2}{(1-x)^2} \xi \left( \frac{Z_{(1,2)} t^3 - Z_1 t + 1}{E(-t)} + Z_{(1,1)} t^2 - 1 \right) = \frac{x^2}{(1-x)^2 \xi(E(-t))} \xi \left( Z_{(1,2)} t^3 - Z_1 t + 1 + [Z_{(1,1)} t^2 - 1] E(-t) \right) = \frac{x^2}{(1-x)(-x + \exp(t(x-1), P, Q))} \xi \left( (e_1 e_2 - e_3) t^3 - e_1 t + 1 + [e_2 t^2 - 1] (1 - e_1 t + e_2 t^2 - e_3 t^3) + [e_2 t^2 - 1] \sum_{n \geq 4} (-t)^n e_n \right) = \frac{x^2}{(1-x)(-x + \exp(t(x-1), P, Q))} \times \xi \left( e_2^2 t^4 - e_2 e_3 t^5 + [e_2 t^2 - 1] \sum_{n \geq 4} (-t)^n e_n \right) =
\]
\[
\frac{x^2}{(1-x)(-x + \exp(t(x-1), P, Q))} \times \xi \left( \sum_{n \geq 4} (e_2e_{n-2} - e_n)(-t)^n \right) = \\
\frac{x^2}{(1-x)(-x + \exp(t(x-1), P, Q))} \sum_{n \geq 4} (-t)^n \xi(e_2e_{n-2} - e_n).
\]

Here we have used that \( Z_{(2,1)} = e_1e_2 - e_3 \) and \( Z_{(1,1)} = e_2 \). Similarly, we can apply Theorem 2.2.4 with \( \alpha = (1,1,1) \) to conclude that

\[
\sum_{n \geq 1} t^{3+n} \left[ 3 + n \right]_{P, Q} \sum_{\Sigma \in S_n^+} x^\text{comdes}(\Sigma) Q^\text{inv}(\Sigma) P^\text{coinv}(\Sigma) = \quad (2.3.8)
\]

\[
\frac{x^3}{(1-x)^3} \frac{\xi \left( \frac{Z_{(2,1,1)}t^3 - Z_{(1,1)}t^2 + Z_1t - 1}{E(-t)} + 1 \right)}{x^3} = \\
\frac{x^3}{(1-x)^3} \xi \left( \frac{Z_{(2,1,1)}t^3 - Z_{(1,1)}t^2 + Z_1t - 1 + E(-t)}{x^3} \right) = \\
\frac{x^3}{(1-x)^2(-x + \exp(t(x-1), P, Q))} \sum_{n \geq 4} (-t)^n \xi(e_n).
\]

It follows that (2.3.6) equals

\[
\frac{x^2}{(1-x)(-x + \exp(t(x-1), P, Q))} \times \\
\sum_{n \geq 4} \frac{1}{(1-x)} \xi(e_2e_{n-2}) - \frac{1}{(1-x)} \xi(e_n) - \frac{x}{(1-x)^2} \xi(e_n) = \\
\frac{x^2}{(1-x)^2(-x + \exp(t(x-1), P, Q))} \times \\
\sum_{n \geq 4} \frac{1}{(1-x)} \xi(e_2e_{n-2}) - \frac{1}{(1-x)^2} \xi(e_n) = \\
\frac{x^2}{(1-x)(-x + \exp(t(x-1), P, Q))} \times \\
\sum_{n \geq 4} \frac{1}{(1-x)} \frac{Q(1-x)}{[2]_{P, Q!}} \frac{Q^{(n-2)}(1-x)^{n-3}}{(n-2)_{P, Q!}} - \frac{1}{(1-x)^2} \frac{Q^{(n)}(1-x)^{n-1}}{[n]_{P, Q!}} =
\]
\[
\frac{x^2}{-x + e^{x}(t(x-1), P, Q)} \times \sum_{n \geq 4} \frac{(-t)^n(1-x)^{n-3}Q^{1 + (\frac{n-2}{2})}}{[n]_{P,Q}!} \left( \left[ \frac{n}{2} \right]_{P,Q} - Q^{2n-4} \right).
\]

Thus we have proved the following.

**Theorem 2.3.1.**

\[
\sum_{n \geq 1} \frac{t^{3+n}}{[3+n]_{P,Q}!} \sum_{\Sigma \in S_n \backslash \mathcal{F}_L \atop \{1,3\} \subseteq \text{Comdes}(\Sigma) \atop \{2\} \subseteq \text{Comris}(\Sigma)} x^{\text{comdes}(\Sigma)} Q^\text{inv}(\Sigma) P^\text{coinv}(\Sigma) = \sum_{n \geq 4} \frac{(-t)^n(1-x)^{n-3}Q^{1 + (\frac{n-2}{2})}}{[n]_{P,Q}!} \left( \left[ \frac{n}{2} \right]_{P,Q} - Q^{2n-4} \right).\tag{2.3.9}
\]

Next we will give a method to compute the generating functions of \(L\)-tuples of \(\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \in S_n^L\) such that \(\{1,3\} \subseteq \text{Comdes}(\Sigma)\) and \(\{2\} \subseteq \text{Comris}(\Sigma)\). Note that if \(L \geq 2\), then the condition that \(\{2\} \subseteq \text{Comris}(\Sigma)\) is different than the condition that \(\{2\} \subseteq \text{oneRise}(\Sigma)\) because the latter condition only asserts the \(\{2\} \in \text{Rise}(\sigma^{(i)})\) for some \(i\) rather than for all \(i\).

The basic idea is the following. We can apply the reasoning from Theorem 2.2.1 to

\[
[n]_{P.Q}! \xi(h_n) = \sum_{\mathcal{O} \in \mathcal{L}F(n)} W(\mathcal{O}).\tag{2.3.10}
\]

where \(\mathcal{L}F(n)\) denotes the set of all labeled filled brick tabloids \(T = (b_1, \ldots, b_k)\) of shape \((n)\) such that all the permutations are decreasing within each brick. Now we would like to modify things so that the following conditions are meet:

1. the first brick \(b_1\) is of length 4 or greater,

2. each permutation \(\sigma^{(i)}\) has the property that

\[
\sigma^{(i)}(1) > \sigma^{(i)}(2) < \sigma^{(i)}(3) > \sigma^{(i)}(4) > \cdots > \sigma^{(i)}(b_1)
\]

and is decreasing in all the other bricks, and
3. the labels on cells 1 and 3 are \( x \), the label on cell 2 is 1, and the remaining labels are as before.

We can accomplish this by replacing \( h_n \) in (2.3.10) by \( p_n^\mathcal{U} \) for an appropriate \( \mathcal{U} \).

To this end, assume \( n \geq 4 \) and let \( T_n \) be the set of permutations \( \sigma \in S_n \) such that \( \sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) > \cdots > \sigma(n) \). That is, \( T_n \) is the set of permutations such that \( \text{Rise}(\sigma) = \{2\} \). We want to compute

\[
\sum_{\sigma \in T_n} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}. \tag{2.3.11}
\]

It easy to see that if \( \sigma \in T_n \), then either (i) \( \sigma(3) = n \), or (ii) \( \sigma(3) = n - 1 \) in which case \( \sigma(1) \) is forced to be \( n \). In case (i), \( n \) gives rise to \( n - 3 \) inversions and 2 coinversions. The rest of the permutation consists of two decreasing sequences, one of length 2 and one of length \( n - 3 \) so that by (2.2.7), these give rise to a factor of \( q^{1+\binom{n-3}{2}}[n-1]_{p,q} \). Thus the total contribution from case (i) to (2.3.11) is \( q^{1+\binom{n-3}{2}}[n-1]_{p,q} \). In case (ii) \( n \) gives rise to \( n - 1 \) inversions and no coinversions and \( n-1 \) gives rise to \( n-3 \) inversions and 1 coinversion. The rest of the permutation consists of two decreasing sequences, one of length 1 and one of length \( n - 3 \) so that by (2.2.7), these give rise to a factor of \( q^{1+\binom{n-3}{2}}[n-1]_{p,q} = q^{\binom{n-3}{2}}[n-2]_{p,q} \).

Thus the total contribution from case (ii) to (2.3.11) is \( q^{n-1+\binom{n-2}{2}}[n-2]_{p,q} \). Thus

\[
\sum_{\sigma \in T_n} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} = q^{1+\binom{n-3}{2}}\left[\frac{n-1}{2}\right]_{p,q} + q^{n-1+\binom{n-2}{2}}[n-2]_{p,q}. \tag{2.3.12}
\]

It follows that

\[
\sum_{\Sigma=(\sigma^{(1)},\ldots,\sigma^{(L)}) \in T^L_n} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)} = \tag{2.3.13}
\]

\[
\prod_{i=1}^{L} \left( q_i^{1+\binom{n-3}{2}} p_i^{\left[\frac{n-1}{2}\right]}_{p_i,q_i} + q_i^{n-1+\binom{n-2}{2}} p_i^{[n-2]}_{p_i,q_i} \right) =
\]

\[
Q^{1+\binom{n-2}{2}} \left( \prod_{i=1}^{L} p_i^{\left[\frac{n-1}{2}\right]}_{p_i,q_i} + q_i^{n-2} p_i^{[n-2]}_{p_i,q_i} \right).
\]
This given, consider the sequence \( \vec{u} = (u_1, u_2, \ldots) \) where

\[
\begin{align*}
    u_n &= \frac{x^2Q^{-(\frac{n}{2})}}{(1-x)^3}Q^{1+\binom{n-2}{2}}
    \left( \prod_{i=1}^{L} \frac{p_i^2}{p_i^{2}} \binom{n-1}{2} + q_i^{n-2}p_i[n-2]_{p_i,q_i} \right).
\end{align*}
\]

if \( n \geq 4 \) and \( u_1 = u_2 = u_3 = 0 \). Then we have the following theorem.

**Theorem 2.3.2.** For \( n \geq 4 \),

\[
\begin{align*}
    [n]_{P,Q}! \xi(p^\vec{u}_n) &= \sum_{\forall i(\sigma^{(1)}) > \sigma^{(2)} < \sigma^{(3)} > \sigma^{(4)} } x_{\text{comdes}(\Sigma)}Q_{\text{inv}(\Sigma)}P_{\text{coinv}(\Sigma)}.
\end{align*}
\]

**Proof.** In this case, we shall consider only brick tabloids of shape \((n)\) so we shall write \( T = (b_1, \ldots, b_{\ell(\mu)}) \) if the size of the bricks in \( T \) are \( b_1, \ldots, b_{\ell(\mu)} \), reading from left to right. We interpreted \( p^\vec{u}_n \) as the sum over weighted brick tabloids \( T = (b_1, \ldots, b_\ell) \) where the weight of \( T \) is given by \( u_b \prod_i (-1)^{b_i-1}e_{b_i} \). That is, the last brick contributed an extra factor of \( u_b \). However, by the simple process of reversing the order of the bricks in each brick tabloid, we can also interpret \( p^\vec{u}_n \) as the sum over weighted brick tabloids \( T = (b_1, \ldots, b_\ell) \) where the weight of \( T \) is \( u_b \prod_i (-1)^{b_i-1}e_{b_i} \). That is, we can decide to the first brick contributes an extra factor of \( u_b \) rather than having the last brick contribute an extra factor of \( u_b \). It follows that

\[
\begin{align*}
    [n]_{P,Q}! \xi(p^\vec{u}_n) &= \sum_{\mu} (-1)^{n-\ell(\mu)}w_{\vec{u}}(B_{\mu,\ell(\mu)}) \xi(e_{\mu}) =
    \sum_{\mu} (-1)^{n-\ell(\mu)} \sum_{T = (b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu,\ell(\mu)}} u_b \prod_{i=1}^{\ell(\mu)} (1-x)^{b_i-1}Q^{(b_i)}_{(b_i)} \\
    &= \sum_{\mu, n} \sum_{T = (b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu,\ell(\mu)}} \left[ b_1, \ldots, b_{\ell(\mu)} \right]_{P,Q} Q^{n(\frac{b_i}{2})}_{\mu,\ell(\mu)} (x-1)^{n-\ell(\mu)} \times x^2Q^{-\binom{n}{2}}
    \left( \prod_{i=1}^{L} \frac{p_i^2}{p_i^{2}} \binom{b_1-1}{2} + q_i^{b_i-2}p_i[b_1-2]_{p_i,q_i} \right).
\end{align*}
\]
As in the proof of Theorem 2.2.1, for any \( T = (b_1, \ldots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n} \), we can interpret \( \mathcal{Q}_{\sum_{i=1}^{\ell(\mu)} \left[ b_i, \ldots, b_{\ell(\mu)} \right]} - \mathcal{P} \) as the set of fillings of \( T \) with \( L \)-tuples of permutations \( \Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \) such that for each \( i \), the elements of \( \sigma^{(i)} \) are decreasing within each brick of \( T \). We weight such a filling with \( \mathcal{Q}^{\text{inv}(\Sigma)} \mathcal{P}^{\text{coinv}(\Sigma)} \).

There is a factor of \( x^2(x - 1)^{n - \ell(\mu) - 3} \) in (2.3.17) arising from \( T \) which we interpret as taking such a filling and labeling cells 1 and 3 with \( x \), labeling cell 2 with 1, and labeling each remaining cell which is not at the end of a brick with either an \( x \) or \(-1\) and labeling each cell at the end of a brick with 1. Finally, given (2.3.13), we can interpret the extra factor, \( \mathcal{Q}^{-\left( b_1 \right)} \mathcal{Q}^{1+\left( b_1 - 2 \right)} \left( \prod_{i=1}^{L} p_i^2 \left[ b_1 - 1 \right] \right) + q_i^{b_1 - 2} p_i [b_1 - 2]_{p_i,q_i} \), as allowing us to replace the decreasing fillings of the first brick by all fillings where in each \( \sigma^{(i)} \)

\[
\sigma^{(i)}(1) > \sigma^{(i)}(2) < \sigma^{(i)}(3) > \sigma^{(i)}(4) > \sigma^{(i)}(5) > \cdots > \sigma^{(i)}(b_1).
\]

That is, by rearranging the elements in the first brick, we do not change the number of inversions and coinversions between elements that lie in the first brick and the rest of elements in any given \( \sigma^{(i)} \). Thus, we need only account for the difference between the inversions and coinversions for elements that lie in the first brick caused by going from a decreasing sequence in each row to a sequence such that

\[
\sigma^{(i)}(1) > \sigma^{(i)}(2) < \sigma^{(i)}(3) > \sigma^{(i)}(4) > \sigma^{(i)}(5) > \cdots > \sigma^{(i)}(b_1).
\]

Clearly, when all the elements that lie in the first brick form a decreasing sequence in each row, they contribute a factor of \( \mathcal{Q}^{-\left( b_1 \right)} \) to \( \mathcal{Q}^{\text{inv}(\Sigma)} \mathcal{P}^{\text{coinv}(\Sigma)} \). After we arrange the sequences, the elements in the first brick contribute

\[
\mathcal{Q}^{1+\left( b_1 - 2 \right)} \left( \prod_{i=1}^{L} p_i^2 \left[ b_1 - 1 \right] \right) + q_i^{b_1 - 2} p_i [b_1 - 2]_{p_i,q_i} \]

to \( \mathcal{Q}^{\text{inv}(\Sigma)} \mathcal{P}^{\text{coinv}(\Sigma)} \). Thus the factor

\[
\mathcal{Q}^{-\left( b_1 \right)} \mathcal{Q}^{1+\left( b_1 - 2 \right)} \left( \prod_{i=1}^{L} p_i^2 \left[ b_1 - 1 \right] \right) + q_i^{b_1 - 2} p_i [b_1 - 2]_{p_i,q_i}
\]
is exactly what we need to compensate for replacing the decreasing sequences in the first brick by sequences such that

\[ \sigma^{(i)}(1) > \sigma^{(i)}(2) < \sigma^{(i)}(3) > \sigma^{(i)}(4) > \ldots > \sigma^{(i)}(b_1) \]

for all \( i \).

We shall call an object \( \mathcal{O} \) created in this way a *did-labeled filled brick tabloid* where *did* is short for decrease-increase-decrease. An example of a *did*-labeled filled brick tabloid for \( L = 3 \) and \( n = 12 \) is given in Figure 2.8. Then we define the weight of \( \mathcal{O} \), \( W(\mathcal{O}) \), to be product over all the labels of the cells times \( Q_{inv(\Sigma)}P_{coinv(\Sigma)} \) if \( T \) is filled with permutations \( \Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \). Thus for the object pictured in Figure 2.8,

\[
W(\mathcal{O}) = (-1)^4 x^4 q_1^{inv(\sigma^{(1)})} q_2^{inv(\sigma^{(2)})} q_3^{inv(\sigma^{(3)})} p_1^{coinv(\sigma^{(1)})} p_2^{coinv(\sigma^{(2)})} p_3^{coinv(\sigma^{(3)})}.
\]

\[
\begin{array}{cccc|cccc|cccc|cccc}
 x & 1 & x & -1 & l & x & 1 & -1 & x & 1 \\
10 & 6 & 9 & 4 & 1 & 11 & 7 & 3 & 12 & 8 & 5 & 2 \\
 9 & 1 & 6 & 3 & 2 & 10 & 7 & 5 & 12 & 11 & 8 & 4 \\
12 & 5 & 10 & 6 & 4 & 9 & 3 & 1 & 11 & 8 & 7 & 2 \\
\end{array}
\]

\[
\sigma^{(1)} = 10 \hspace{1cm} 9 \hspace{1cm} 12 \\
\sigma^{(2)} = 6 \hspace{1cm} 1 \hspace{1cm} 5 \\
\sigma^{(3)} = 4 \hspace{1cm} 3 \hspace{1cm} 10
\]

Figure 2.8: A *did*-labeled filled brick tabloid of shape (12).

We let \( DIDLF(n) \) denote the set of all objects that can be created in this way from brick tabloids \( T \) of shape \( (n) \). Then it follows that

\[
[n]_{P,Q} \mathcal{I}_e(p_n^\Omega) = \sum_{\mathcal{O} \in DIDLF(n)} W(\mathcal{O}). \tag{2.3.18}
\]

Then we define an involution \( I : DIDLF(n) \to DIDLF(n) \) exactly as we did before. That is, given \( \mathcal{O} \in DIDLF(n) \) read the cells of \( \mathcal{O} \) from left to right and
look for the first cell $c$ such that either

(i) $c$ is labeled with $-1$ or
(ii) $c$ is at the end of end of brick $b$, the cell $c+1$ is immediately to the right of $c$ and starts another brick $b'$, and each permutation $\sigma^{(i)}$ decreases as we go from $c$ to $c+1$.

If we are in case (i), then $I(O)$ is the did-labeled filled brick tabloid which is obtained from $O$ by taking the brick $b$ that contains $c$ and splitting $b$ into two bricks $b_1$ and $b_2$ where $b_1$ contains the cells of $b$ up to and including the cell $c$ and $b_2$ contains the remaining cells of $b$ and changing the label on $c$ from $-1$ to $1$. In case (ii), $I(O)$ is the did-labeled filled brick tabloid which is obtained from $O$ by combining the two bricks $b$ and $b'$ into a single brick and changing the label on cell $c$ from $1$ to $-1$. Finally, if neither case (i) or case (ii) applies, then we let $I(O) = O$. For example, we consider the did-labeled filled brick tabloid $O$ pictured in Figure 2.8, then $I(O)$ is pictured in Figure 2.9.

As before, $I$ is a weight-preserving sign-reversing involution, so that $I$ shows that

$$[n]_{P,Q} \xi(p^n) = \sum_{\mathcal{O} \in DI\text{DLF}(n)} W(\mathcal{O}) = \sum_{\mathcal{O} \in DI\text{DLF}(n), I(\mathcal{O}) = \mathcal{O}} W(\mathcal{O}).$$

Thus we must examine the fixed points of $I$. Clearly if $I(O) = O$, then $O$ can have no cells which are labeled with $-1$. Also it must be the case that between
any two consecutive bricks of $\mathcal{O}$, at least one of the underlying permutations $\sigma^{(i)}$ must increase. It follows that each cell $c$ which is not among the first three cells and which is not at the end of a brick in $\mathcal{O}$ is labeled with $x$ and each of the permutation $\sigma^{(i)}$ has a descent at $c$ so that $c \in \text{Comdes}(\Sigma)$. All the rest of the cells of $\mathcal{O}$ other than the last cell are at the end of a brick which has another brick to its right in which case $c \notin \text{Comdes}(\Sigma)$. All such cells have label 1 so that 

$$W(\mathcal{O}) = x^{\text{comdes}(\Sigma)}Q^\text{inv}(\Sigma)P^\text{coinv}(\Sigma).$$

Now if we are given $\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \in S_n^L$ such that for each $i$, $\sigma^{(i)}(1) > \sigma^{(i)}(2) < \sigma^{(i)}(3) > \sigma^{(i)}(4)$, then we can construct a fixed point of $I$ from $\Sigma$ by using $(\sigma^{(1)}, \ldots, \sigma^{(L)})$ to fill a tabloid of shape $\alpha$, then drawing the bricks so that the cells $c$ which end bricks are precisely the elements of $\text{oneRise}(\Sigma) \cap \{4, \ldots, n\}$. This shows that

$$\sum_{\mathcal{O} \in \text{DIDLF}(n), I(\mathcal{O}) = \mathcal{O}} W(\mathcal{O}) = \sum_{\Sigma \in S_n^L} x^{\text{comdes}(\sigma)}Q^\text{inv}(\Sigma)P^\text{coinv}(\Sigma)$$

$$\sigma^{(i)}(1) > \sigma^{(i)}(2) < \sigma^{(i)}(3) > \sigma^{(i)}(4) \quad \forall \quad i$$

as desired. 

Using Theorem 2.3.2, we can apply $\xi$ to both sides of the identity

$$\sum_{n \geq 1} p_n \bar{t}^n = \sum_{n \geq 1} (-1)^{n-1} u_n c_n t^n$$

$$E(-t)$$

to prove the following:

$$\sum_{n \geq 1} \frac{t^{3+n}}{[3+n]!} = \sum_{\Sigma \in S_{3+n}^L} x^{\text{comdes}(\Sigma)}Q^\text{inv}(\Sigma)P^\text{coinv}(\Sigma)$$

$$\forall (\sigma^{(i)}(1) > \sigma^{(i)}(2) < \sigma^{(i)}(3) > \sigma^{(i)}(4))$$
\[
\frac{x^2}{(1-x)^2(-x + \exp(t(x-1), P, Q))} \times \\
\sum_{n \geq 4} (-1)^{n-1} \frac{(1-x)^{n-1} Q^{(n)}[\frac{n}{2}]_{P,Q}}{Q^{(n)}[\frac{n}{2}]_{P,Q}} Q^{1+(\frac{n}{2})} \times \\
\left( \prod_{i=1}^{L} p_i^2 \left[ \frac{b_i - 1}{2} \right]_{p_i,q_i} + q_i^{b_i - 2} p_i [b_i - 2]_{p_i,q_i} \right) = \\
\frac{x^2}{(-x + \exp(t(x-1), P, Q))} \times \\
\sum_{n \geq 4} (-1)^{n-1} \frac{(1-x)^{n-3} Q^{1+(\frac{n}{2})}}{Q^{(n)}[\frac{n}{2}]_{P,Q}} \left( \prod_{i=1}^{L} p_i^2 \left[ \frac{b_i - 1}{2} \right]_{p_i,q_i} + q_i^{b_i - 2} p_i [b_i - 2]_{p_i,q_i} \right). \\
\]  

(2.3.20)

We note that it should be the case that (2.3.9) and (2.3.20) are equal when \( L = 1 \) since the condition that \( 2 \in \text{oneRise}(\Sigma) \) and \( 2 \in \text{Comrise}(\Sigma) \) is the same in that case. Indeed, the two expressions are the same because

\[
\begin{align*}
p_1^2 \left[ \frac{n - 1}{2} \right]_{p_1,q_1} + q_1^{n-2} p_1 [n - 2]_{p_1,q_1} &= \\
\frac{[n-1]_{p_1,q_1} [n-2]_{p_1,q_1}}{[2]_{p_1,q_1}} + q_1^{n-2} ([n-1]_{p_1,q_1} - q_1^{n-2}) &= \\
\frac{[n-1]_{p_1,q_1}}{[2]_{p_1,q_1}} (p_1^2 [n - 2]_{p_1,q_1} + q_1^{n-2} [2]_{p_1,q_1}) - q_1^{2n-4} &= \\
\frac{[n-1]_{p_1,q_1}}{[2]_{p_1,q_1}} [n]_{p_1,q_1} - q_1^{2n-4} &= \\
\frac{n}{2} \left[ \frac{1}{p_1,q_1} - q_1^{2n-4}. \right. 
\end{align*}
\]

Our second method can be applied to obtain generating functions for \( L \)-tuples of permutations \( \Sigma \) such that \( S \subseteq \text{Comdes}(\Sigma) \) and \( T \subseteq \text{Comrise}(\Sigma) \) where \( S \cup T = \{1, \ldots, m\} \) for some \( m \). That is, let \( D_{m+k}^{S,T} \) equal the set of permutations \( \sigma \in S_{n+k} \) such that \( S \cup \{m+1, \ldots, n+k-1\} = \text{Des}(\sigma) \) and \( T = \text{Rise}(\sigma) \). We let

\[
D_{m+k}^{S,T}(p, q) = \sum_{\sigma \in D_{m+k}^{S,T}} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}. \\
\]  

(2.3.21)

Then if we define \( \vec{u} = (u_1, u_2, \ldots) \) so that

(1) \( u_i = 0 \) if \( i \leq m \) and
(2) \( u_n = \frac{x^{|S|}}{(1-x)^m} \prod_{i=1}^{L} D_{n}^{S,T}(p_i, q_i) \) for \( n \geq m + 1 \),
we can use the same type of argument that we used to prove (2.3.20) to show that
for \( n \geq m + 1 \),
\[
[n]_{P, Q}! \xi(p_n) = \sum_{\Sigma \in S^k \subseteq \text{Comdes}(\Sigma)} \sum_{T \subseteq \text{Comrise}(\Sigma)} x^\text{comdes}(\Sigma) Q^\text{inv}(\Sigma) P^\text{coinv}(\Sigma) \tag{2.3.22}
\]
and \( \xi(p_n) = 0 \) if \( n \leq m \). It will then follow that
\[
\xi(\sum_{n \geq 1} p_n) = \sum_{n \geq m+1} \frac{t^n}{[n]_{P, Q}!} \sum_{\Sigma \in S^k \subseteq \text{Comdes}(\Sigma)} \sum_{T \subseteq \text{Comrise}(\Sigma)} x^\text{comdes}(\Sigma) Q^\text{inv}(\Sigma) P^\text{coinv}(\Sigma) \tag{2.3.23}
\]
so that we can find an expression for the right hand side of (2.3.23) by applying \( \xi \) to the identity (1.2.4).
Computing \( D_{n}^{S,T}(p, q) \) can be tedious, but, at least in small cases, it can by
done on a case by case basis. That is, suppose that \( n \geq m + 1 \) and the last rise
of permutation \( \sigma \in D_{n}^{S,T} \) occurs at position \( j \). Then \( \sigma_{j+1} \) will be greater than to
\( \sigma_j \) and \( \sigma_r \) for \( r = j + 2, \ldots, n \). It follows that \( \sigma_{j+1} \in \{n - j + 1, \ldots, n\} \).
Having chosen a value \( i \) for \( \sigma_{j+1} \), we can list the possible ways to distribute the numbers
\( i + 1, \ldots, n \) numbers in the first \( j - 1 \) places of \( \sigma \) and we will be left with a \( p,q \)-
multinomial coefficient for the rest of the values. For example, suppose \( m = 5, \)
\( S = \{2, 4, 5\} \), and \( T = \{1, 3\} \). Then the last rise in a \( \sigma \in D_{n}^{S,T} \) occurs at position
3 so that the value of \( \sigma_4 \) must either be \( n, n-1 \), or \( n-2 \). We have pictured the
three possibilities for permutations in \( D_{n}^{S,T} \) in Figure 2.10. Thus we have 3 cases.

**Case 1.** \( \sigma_4 = n \). Then \( n \) will cause 3 coinversions and \( n-4 \) inversions giving us a factor of \( p^3 q^{n-4} \). By Lemma 2.2.2, it follows that \( q \binom{n}{2} q \binom{3}{n-1} [n-3]_{p,q} \) equals
the sum \( q^{\text{inv}(\tau)} p^{\text{coinv}(\tau)} \) over all permutations \( \tau \in S_{n-1} \) where \( \tau_1 > \tau_2 > \tau_3 \) and
\( \tau_4 > \cdots > \tau_{n-1} \). Thus we can let \( \tau_4, \ldots, \tau_{n-1} \) be the values of \( \sigma_5, \ldots, \sigma_{n-1} \) respectively. Now we want to replace the decreasing sequence \( \tau_1 > \tau_2 > \tau_3 \) by either
\(\tau_2 \tau_1 \tau_3\) or \(\tau_3 \tau_1 \tau_2\) which means that we want to replace the factor \(q^{(3)}\) by

\[
q^{\text{inv}(\tau_2 \tau_1 \tau_3)} \cdot p^{\text{coinv}(\tau_2 \tau_1 \tau_3)} = q^{\text{inv}(\tau_3 \tau_1 \tau_2)} \cdot p^{\text{coinv}(\tau_3 \tau_1 \tau_2)} =
\]

\[pq^2 + p^2q = pq[2]_{p,q}.
\]

Thus we get a factor of \(pq[2]_{p,q} q^{(n-4)} \left[ \frac{n-1}{3} \right]_{p,q}\) from the possible ways of filling in the values \(\sigma_1, \sigma_2, \sigma_3\) and \(\sigma_5, \ldots, \sigma_n\). It follows that the sum of \(q^{\text{inv}(\sigma)} \cdot p^{\text{coinv}(\sigma)}\) over all permutations \(\sigma \in D_n^{S,T}\) with \(\sigma_4 = n\) is

\[
p^4qq^{(n-3)} \left[ \frac{n-1}{3} \right]_{p,q}.
\]

**Case 2.** If \(\sigma_4 = n - 1\), then one can easily see from Figure 2.10 that we must have \(\sigma_2 = n\). Then there will be a factor of \(pq^{n-2}\) due to the fact that \(\sigma_2 = n\) and a factor of \(p^2q^{n-4}\) due to the fact that \(\sigma_4 = n - 1\). Finally we can use Lemma 2.2.2 to see that we get a factor of \(q^{(n-4)} \left[ \frac{n-2}{1,1,n-4} \right]_{p,q}\) from the possible ways of filling in the values \(\sigma_1, \sigma_3\) and \(\sigma_5, \ldots, \sigma_n\). It follows that the sum of \(q^{\text{inv}(\sigma)} \cdot p^{\text{coinv}(\sigma)}\) over all permutations \(\sigma \in D_n^{S,T}\) with \(\sigma_4 = n - 1\) is

\[
p^3qq^{(n-2)} \left[ \frac{n-2}{1,1,n-4} \right]_{p,q}.
\]

**Case 3.** If \(\sigma_4 = n - 2\), then one can easily see from Figure 2.10 that we must have \(\sigma_2 = n\) and \(\sigma_1 = n - 1\). Then there will be a factor \(q^{n-2}\) due the fact that \(\sigma_1 = n - 1\), a factor of \(pq^{n-2}\) due to the fact that \(\sigma_2 = n\), and a factor of \(pq^{n-4}\) due to the fact that \(\sigma_4 = n - 2\). Finally we can use Lemma 2.2.2 to see that the we get a factor of \(q^{(n-4)} \left[ \frac{n-3}{1} \right]_{p,q}\) from the possible ways of filling in the values \(\sigma_3\) and \(\sigma_5, \ldots, \sigma_n\). It follows that the sum of \(q^{\text{inv}(\sigma)} \cdot p^{\text{coinv}(\sigma)}\) over all permutations \(\sigma \in D_n^{S,T}\) with \(\sigma_4 = n - 2\) is

\[
p^3qq^{(n-1)} \left[ n-3 \right]_{p,q}.
\]
It follows that

\[ D_n^{(2,4,5),\{1,3\}}(p, q) = \]

\[ p^4 q q^{(n-3)}[2]_{p,q} \left[ \frac{n-1}{3} \right]_{p,q} + p^3 q q^{(n-2)} \left[ \begin{array}{c} n-2 \\ 1, 1, n-4 \end{array} \right]_{p,q} + p^3 q q^{(n-1)}[n-3]_{p,q}. \]  

(2.3.27)

Finally suppose that \( S \cup T \) is not of the form \( \{1, \ldots, m\} \). Let \( m = \max(S \cup T) \) and \( \text{ptn}(S, T, m) \) be the set of all pairs of sets \( (U, V) \) such that \( U \cup V = \{1, \ldots, m\} \), \( U \cap V = \emptyset \), \( S \subseteq U \), and \( T \subseteq V \). Then it is easy to see that

\[ \mathcal{D}_n^{S,T} = \bigcup_{(U,V)\in\text{ptn}(S,T,m)} \mathcal{D}_n^{U,V}. \]  

(2.3.28)

Now if we are given an \( L \)-tuple \( \Gamma = \langle (U_1, V_1), \ldots, (U_L, V_L) \rangle \) of pairs from \( \text{ptn}(S, T, m) \), we let \( \text{Comdes}(\Gamma) = \{1, \ldots, m\} \cap \text{Comdes}(\sigma^{(1)}, \ldots, \sigma^{(L)}) \) where \( (\sigma^{(1)}, \ldots, \sigma^{(L)}) \) is any sequence of permutations from \( \mathcal{D}_n^{U_1,V_1} \times \cdots \times \mathcal{D}_n^{U_L,V_L} \) and \( \text{comdes}(\Gamma) = |\text{Comdes}(\Gamma)| \). For example, suppose that \( L = 2 \) and \( S = \{4, 5\} \) and \( T = \{1, 3\} \). Then there are two pairs in \( \text{ptn}(S, T, 5) \), \( (U_1, V_1) = (\{2, 4, 5\}, \{1, 3\}) \)
and \((U_2, V_2) = (\{4, 5\}, \{1, 2, 3\})\). In Figure 2.11, we have pictured the single configuration for elements in \(D_{U_2,V_2}\). That is, it is easy to see that \(\sigma_3 = n\) for all \(\sigma \in D_{U_2,V_2}\). Then it is easy to see that if \(((\sigma(1), \sigma(2)) \in D_{U_i,V_i} \times D_{U_j,V_j}, \text{Comdes}(\sigma(1), \sigma(2) ) \cap \{1, \ldots, 5\}\) equals \(\{2, 4, 5\}\) if \(i = j = 1\) and is equal to \(\{4, 5\}\) otherwise. Thus

\[
\text{comdes}(\Gamma) = \begin{cases} 
3 & \text{if } \Gamma = ((U_1, V_1), (U_1, V_1)), \\
2 & \text{if } \Gamma = ((U_1, V_1), (U_2, V_2)), \\
2 & \text{if } \Gamma = ((U_2, V_2), (U_1, V_1)) \text{ and,} \\
2 & \text{if } \Gamma = ((U_2, V_2), (U_2, V_2)).
\end{cases}
\]

Figure 2.11: The configuration for \(D_{n}\).

Now suppose that we define \(\vec{u} = (u_1, u_2, \ldots)\) so that

1. \(u_i = 0\) if \(i \leq m\) and
2. \(u_n = \sum_{\Gamma = ((U_1, V_1), \ldots, (U_L, V_L)) \in \text{ptn}(S,T,m)^L} \frac{x^{\text{comdes}(\Gamma)}}{(1-x)^m} \prod_{i=1}^{L} D_{n}^{U_i,V_i}(p_i, q_i)\) for \(n \geq m + 1\),

then we can use the same type of argument that we used to prove (2.3.20) to prove that for \(n \geq m + 1\),

\[
[n]_{P,Q} \xi(p_n^{\vec{u}}) = \sum_{\Sigma \in S_k, S \subseteq \text{Comdes}(\Sigma), T \subseteq \text{Comrise}(\Sigma)} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}
\tag{2.3.29}
\]

and \(\xi(p_n^{\vec{u}}) = 0\) if \(n \leq m\). Again it will be the case that

\[
\xi(\sum_{n \geq 1} p_n^{\vec{u} i^n}) = \sum_{n \geq m+1} \frac{t^n}{[n]_{P,Q}!} \sum_{\Sigma \in S_k, S \subseteq \text{Comdes}(\Sigma), T \subseteq \text{Comrise}(\Sigma)} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}
\tag{2.3.30}
\]

so that we can find an expression for the right hand side of (2.3.29) by applying \(\xi\) to the identity (1.2.4).
For example, suppose $S = \{4, 5\}$ and $T = \{1, 3\}$. It is easy to compute $D_{n}^{(4,5),\{1,2,3\}}(p, q)$. That is, one can see from Figure 2.11 that $\sigma_{4} = n$ if $\sigma \in D_{n}^{(4,5),\{1,2,3\}}$. Thus we will get a factor of $p_{4}^{3}q_{4}^{(n-4)}$ from the fact that $\sigma_{4} = n$. It is then easy to see that we get a factor of $p_{4}^{(\frac{3}{2})}q_{4}^{(n-1)}\binom{n-1}{3}_{p,q}$ from the ways to fill in the values of $\sigma_{1}, \sigma_{2}, \sigma_{4}, \ldots, \sigma_{n}$. Thus for $n \geq 6$,

$$D_{n}^{(4,5),\{1,2,3\}}(p, q) = p_{4}^{3}q_{4}^{(n-3)}\binom{n-1}{3}_{p,q}.$$  

(2.3.31)

Thus if $L = 2$, $S = \{4, 5\}$, and $T = \{1, 3\}$, then we must define $\bar{u} = (u_{1}, u_{2}, \ldots)$ so that $u_{i} = 0$ for $i = 1, \ldots, 5$ and

$$u_{n} = \frac{1}{(x-1)^{n}} \times \left(x^{3}D_{n}^{U_{1},V_{1}}(p_{1}, q_{1})D_{n}^{U_{1},V_{1}}(p_{2}, q_{2}) + x^{2}D_{n}^{U_{1},V_{1}}(p_{1}, q_{1})D_{n}^{U_{2},V_{2}}(p_{2}, q_{2}) + \right.$$  

$$x^{2}D_{n}^{U_{2},V_{2}}(p_{1}, q_{1})D_{n}^{U_{1},V_{1}}(p_{2}, q_{2}) + x^{2}D_{n}^{U_{2},V_{2}}(p_{1}, q_{1})D_{n}^{U_{2},V_{2}}(p_{2}, q_{2})\right)$$

for $n \geq 6$. Then we can show that for $n \geq 6$,

$$[n]_{P,Q}! \xi(p_{n}^{\bar{u}}) = \sum_{\Sigma \in S_{n}^{\text{comdes}(\Sigma)} \subseteq \text{Comdes}(\Sigma) \subseteq \text{Comrise}(\Sigma)} x^{\text{comdes}(\Sigma)}Q^{\text{inv}(\Sigma)}P^{\text{coine}(\Sigma)}$$  

(2.3.32)

and $\xi(p_{n}^{\bar{u}}) = 0$ if $n \leq 6$. Again it will be the case that

$$\xi(\sum_{n \geq 1} p_{n}^{\bar{u}}n^{t}) = \sum_{n \geq 6} \frac{t^{n}}{n} [n]_{P,Q}! \sum_{\Sigma \in S_{n}^{\text{comdes}(\Sigma)} \subseteq \text{Comdes}(\Sigma) \subseteq \text{Comrise}(\Sigma)} x^{\text{comdes}(\Sigma)}Q^{\text{inv}(\Sigma)}P^{\text{coine}(\Sigma)}$$  

(2.3.33)

so that we can find an expression for the right hand side of (2.3.33) by applying $\xi$ to the identity (1.2.4).

Finally, we should note that in a forthcoming paper, we shall give other methods where one can get generating functions for $L$-tuples of permutations with various conditions on their common descent sets. For example, in [34], we shall give methods for finding generating functions of $L$-tuples of permutations $\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \in S_{i}^{L}$ such that $\text{Comdes}(\sigma) \subseteq \{i, i + k, i + 2k, \ldots, i + nk\}$ or $\text{Comdes}(\sigma) = \{i, i + k, i + 2k, \ldots, i + nk\}$ where $k \geq 2$, $0 \leq i \leq k - 1$, and
$1 \leq j \leq k$. To compute such generating functions we need to define a new class of symmetric functions $p_n^{\vec{r}, \vec{s}}$ depending on two infinite sequences $\vec{r} = (r_1, r_2, \ldots)$ and $\vec{s} = (s_1, s_2, \ldots)$. Finally, we can combine the methods presented in section 4 with the methods of [34] to find generating functions of $L$-tuples of permutations $\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \in S_{i+nk+j}$ such that $S \subseteq \text{Comdes}(\sigma) \subseteq \{i, i+k, i+2k, \ldots, i+nk\}$ where $S$ is some fixed subset of $\{i, i+k, i+2k, \ldots\}$. The key idea is to define an analogue of the ribbon Schur function $Z_\alpha$ in terms of special rim hook tabloids in such a way that if $T$ is a special rim hook tabloid of shape $F(\alpha, n)$ whose special rim hooks have length $a_1, \ldots, a_k$, reading from top to bottom, then we weight $T$ by $\text{sgn}(T)p_{a_1}^{\vec{r}, \vec{s}}h_{a_2} \cdots h_{a_k}$ instead of $\text{sgn}(T)h_{a_1}h_{a_2} \cdots h_{a_k}$.

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Chapter 3

Permutations with \( k \)-regular descent patterns.

We find generating functions for the permutations in \( S_{i+nk+j} \) with descent set \( \{i, i+k, i+2k, \ldots, i+nk\} \) for integers \( i, j, k \) and \( n \) satisfying \( k \geq 2, 0 \leq i \leq k-1 \) and \( 1 \leq j \leq k \). These permutations are said to have \( k \)-regular descent patterns.

Figure 3.1: A \( k \)-regular descent pattern, with \( i = 1, k = 3, j = 2 \).

The generating functions are found by introducing homomorphisms on the ring of symmetric functions. For the most general results, we introduce a new class of symmetric functions \( p_{n,\alpha_1,\ldots,\alpha_r} \) that depend on \( r \) functions \( \alpha_i \). We also derive new identities between the series of \( p_{n,\alpha_1,\ldots,\alpha_r} \) and \( e_n \). The generating function identities follow from applying our homomorphisms to these new identities involving \( p_{n,\alpha_1,\alpha_2} \).
3.1 Introduction

Let $i, j, k,$ and $n$ be nonnegative integers satisfying $k \geq 2$, $0 \leq i \leq k - 1$, and $1 \leq j \leq k$. A permutation with descent set equal to $\{i + k, i + 2k, \ldots, i + nk\}$ will be called a permutation with a $k$-regular descent pattern. Let $E_{i+k+kn+j}^{i,j,k}$ be the number of such permutations in $S_{i+k+kn+j}$.

In the special case where $k = 2$, $i = 0$, and $j = 2$, $E_{2n+2}^{0,2,2}$ is the number of permutations in $S_{2n+2}$ with descent set $\{2, 4, \ldots, 2n\}$. These are the classical even alternating permutations. André [1, 2] proved that

$$1 + \sum_{n \geq 0} \frac{E_{2n+2}^{0,2,2}}{(2n + 2)!} t^{2n+2} = \sec t.$$ 

Similarly, $E_{2n+1}^{0,1,2}$ counts the number of odd alternating permutations and

$$\sum_{n \geq 0} \frac{E_{2n+1}^{0,1,2}}{(2n + 1)!} t^{2n+1} = \tan t.$$ 

These numbers are also called the Euler numbers. When $k \geq 0$, $E_{kn+1}^{0,j,k}$ are called generalized Euler numbers [27]. There are well-known generating functions for $q$-analogues of the generalized Euler numbers; see Stanley’s book [40], page 148. Various divisibility properties of the $q$-Euler numbers have been studied in [3, 4, 15] and of the generalized $q$-Euler numbers in [19, 38]. Prodinger [35] also studied $q$-analogues of the number $E_{2n+1}^{1,2,2}$ and $E_{2n+2}^{1,1,2}$.

Our goal is to find and refine generating functions for $E_{i+k+kn+j}^{i,j,k}$. This will be done by applying ring homomorphisms to symmetric function identities. This technique of understanding permutation enumeration through symmetric function identities further advances an already well-documented line of research [8, 25, 30, 32, 33, 41].

In [30], Mendes used similar methods to derive the generating functions for $p, q$-analogues of the generating functions for the even and odd alternating permutations. In this chapter, we significantly generalize his methods.

Let $C_{i+k+kn+j}^{i,j,k}$ denote the set of permutations $\sigma \in S_{i+k+kn+j}$ with $\text{Des}(\sigma) \subseteq \{i, i + k, \ldots, i + nk\}$ and $C_{i+k+kn+j}^{i,j,k} = \left| C_{i+k+kn+j}^{i,j,k} \right|$. Similarly, let $E_{i+k+kn+j}^{i,j,k}$ denote the set of
permutations $\sigma \in S_{i+kn+j}$ with $Des(\sigma) = \{i, i+k, \ldots, i+nk\}$ so that $E_{i+kn+j}^{i,j,k} = |C_{i+kn+j}^{i,j,k}|$. Lastly, for $\sigma \in S_{i+kn+j}$, let $Ris_{i,k}(\sigma) = \{s : 0 \leq s \leq n \text{ and } \sigma_{i+sk} < \sigma_{i+(s+1)k}\}$ and $ris_{i,k}(\sigma) = |Ris_{i,k}(\sigma)|$. Then $E_{i+kn+j}^{i,j,k}$ is the number of $\sigma \in C_{i+kn+j}^{i,j,k}$ such that $Ris_{i,k}(\sigma) = \emptyset$. To generalize the results of Mendes, we will find the generating function for

$$\sum_{n \geq 0} (i+kn+j)! \sum_{\sigma \in C_{i+kn+j}^{i,j,k}} x^{ris_{i,k}(\sigma)}.$$  (3.1.1)

Setting $x = 0$ in (3.1.1) will give the generating function for $E_{i+kn+j}^{i,j,k}$. We will also find $p,q$-analogues of such generating functions.

To obtain the generating function for (3.1.1), we introduce a new class of symmetric functions $p_{n,\alpha_1,\alpha_2}$ which depend on two weight functions $\alpha_1$ and $\alpha_2$. Our results will follow by applying a ring homomorphism to a symmetric function identity involving $p_{n,\alpha_1,\alpha_2}$’s. In fact, our methods provide vast extensions of (3.1.1). Moreover, our extension will contain as special cases all of the generating functions for the $q$-Euler and generalized $q$-Euler numbers in the papers mentioned above [40].

In order to fully extend our results, suppose $\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)})$ is a sequence of permutations with $\sigma^{(i)} \in C_{i+kn+j}^{i,j,k}$. Define

$$Comris_{i,k}(\Sigma) = \{s : 0 \leq s \leq n \text{ and for all } 1 \leq t \leq L, \sigma_{i+sk}^{(t)} < \sigma_{i+(s+1)k}^{(t)}\}$$

and let $comris_{i,k}(\Sigma) = |Comris_{i,k}(\Sigma)|$. As in subsection 1.1.1, for two sequences of indeterminates, $Q = (q_1, \ldots, q_L)$ and $P = (p_1, \ldots, p_L)$, let

$$Q^m = q_1^m \cdots q_L^m, \quad P^m = p_1^m \cdots p_L^m,$$

$$[n]_{P,Q} = \prod_{i=1}^L [n]_{p_i,q_i}, \quad [n]_{P,Q}! = \prod_{i=1}^L [n]_{p_i,q_i}!,$$

$$Q^{\text{inv}}(\Sigma) = \prod_{i=1}^L q_i^{\text{inv}(\sigma^{(i)})}, \quad P^{\text{coinv}}(\Sigma) = \prod_{i=1}^L p_i^{\text{coinv}(\sigma^{(i)})}, \quad \text{and}$$

$$\left[\begin{array}{c} n \\ \lambda_1, \ldots, \lambda_k \end{array}\right]_{P,Q} = \prod_{i=1}^L \left[\begin{array}{c} n \\ \lambda_1, \ldots, \lambda_k \end{array}\right]_{p_i,q_i}.$$
In addition, we set
\[ e_{P,Q,k}(t) = \sum_{n \geq 0} \frac{t^{kn} P^{\binom{kn}{2}}}{[kn]_{P,Q}!} \quad \text{and} \quad e_{P,Q,k}^{(j)}(t) = \sum_{n \geq 1} \frac{t^{kn} P^{\binom{kn-1+j}{2}}}{[k(n-1)+j]_{P,Q}!}. \]

Our first generalization of (3.1.1) is found when \( i = 0 \) and \( j = k \). In this case, we will show that
\[ 1 + \sum_{n \geq 1} \frac{t^{kn}}{[k(n-1)+j]_{P,Q}!} \sum_{\Sigma \in (c_{kn}^{0,k,k})} x^{comris_{0,k}(\Sigma)} Q^{inv(\Sigma)} P^{coinv(\Sigma)} = \frac{1-x}{-x + e_{P,Q,k}(t(x-1)^{1/k})}. \]

(3.1.2)

The next case is when \( i = 0 \) and \( 1 \leq j \leq k-1 \). Here, we will show that
\[ \sum_{n \geq 1} \frac{t^{kn}}{[k(n-1)+j]_{P,Q}!} \sum_{\Sigma \in (c_{kn}^{0,j,k})} x^{comris_{0,k}(\Sigma)} Q^{inv(\Sigma)} P^{coinv(\Sigma)} = \frac{-e_{P,Q,k}^{(j)}(t(x-1)^{1/k})}{-x + e_{P,Q,k}(t(x-1)^{1/k})}. \]

(3.1.3)

Lastly, in the case in the case where \( 1 \leq i, j \leq k-1 \), we will prove
\[ \sum_{n \geq 2} \frac{t^{kn}}{[i+k(n-2)+j]_{P,Q}!} \sum_{\Sigma \in (c_{i+k(n-2)+j}^{i,j,k})} x^{comris_{i,k}(\Sigma)} Q^{inv(\Sigma)} P^{coinv(\Sigma)} = \sum_{n \geq 2} \frac{(n-1)x^{n-2}P^{\binom{i+k(n-2)+j}{2}}t^{kn}}{[i+k(n-2)+j]_{P,Q}!} + \frac{e_{P,Q,k}^{(i)}(t(x-1)^{1/k})e_{P,Q,k}^{(j)}(t(x-1)^{1/k})}{(1-x)((-x + e_{P,Q,k}(t(x-1)^{1/k}))}. \]

(3.1.5)

In section 3.2, we provide the necessary background on symmetric functions and introduce our new symmetric functions \( p_{n,\alpha_1,\ldots,\alpha_r} \) and identities. In section 3.3, we shall deal with the cases where \( i = 0 \) and prove (3.1.2) and (3.1.3). Finally, in section 3.4, we will deal with the cases where \( 1 \leq i \leq k \) and \( 1 \leq j \leq k \) and prove (3.1.5).

### 3.2 New Symmetric Functions \( p_{n,\alpha_1,\ldots,\alpha_r} \)

In this section we give some necessary background on symmetric functions needed for our proofs of (3.1.2), (3.1.3), and (3.1.5), and we introduce our new
symmetric functions $p_{n,\alpha_1,\ldots,\alpha_r}$. We also derive an analogous relationship to the $H(t) = \frac{1}{E(t)}$ relationship that has been so fruitful thus far.

A symmetric function $p_{n,\nu}$ has a relationship with $e_\lambda$ which is analogous to the relationship between $h_n$ and $e_\lambda$. It was first introduced in [25] and [30]. Let $\nu$ be a function which maps the set of nonnegative integers into the field $F$. Recursively define $p_{n,\nu} \in \Lambda_n$ by setting $p_{0,\nu} = 1$ and letting

$$p_{n,\nu} = (-1)^{n-1}\nu(n)e_n + \sum_{k=1}^{n-1}(-1)^k e_k p_{n-k,\nu}$$

for all $n \geq 1$. By multiplying series, this means that

$$\left( \sum_{n \geq 0} (-1)^n e_n t^n \right) \left( \sum_{n \geq 1} p_{n,\nu} t^n \right) = \sum_{n \geq 1} \left( \sum_{k=0}^{n-1} p_{n-k,\nu} (-1)^k e_k \right) t^n = \sum_{n \geq 1} (-1)^{n-1}\nu(n)e_n t^n,$$

where the last equality follows from the definition of $p_{n,\nu}$. Therefore,

$$\sum_{n \geq 1} p_{n,\nu} t^n = \frac{\sum_{n \geq 1} (-1)^{n-1}\nu(n)e_n t^n}{\sum_{n \geq 0} (-1)^n e_n t^n}$$

or, equivalently,

$$1 + \sum_{n \geq 1} p_{n,\nu} t^n = \frac{1 + \sum_{n \geq 1} (-1)^n (e_n - \nu(n)e_n) t^n}{\sum_{n \geq 0} (-1)^n e_n t^n}. \quad (3.2.1)$$

When taking $\nu(n) = 1$ for all $n \geq 1$, (6.4) becomes

$$1 + \sum_{n \geq 1} p_{n,1} t^n = 1 + \frac{\sum_{n \geq 1} (-1)^{n-1} e_n t^n}{\sum_{n \geq 0} (-1)^n e_n t^n} = \frac{1}{\sum_{n \geq 0} (-1)^n e_n t^n} = 1 + \sum_{n \geq 1} h_n t^n$$

which implies $p_{n,1} = h_n$. Other special cases for $\nu$ give well-known generating functions. For example, if $\nu(n) = n$ for $n \geq 1$, then $p_{n,\nu}$ is the power symmetric function $\sum_i x_i^n$. By taking $\nu(n) = (-1)^k \chi(n \geq k+1)$ for some $k \geq 1$, $p_{n,(-1)^k\chi(n \geq k+1)}$ is the Schur function corresponding to the partition $(1^k, n)$.

This definition of $p_{n,\nu}$ is desirable because of its expansion in terms of elementary symmetric functions. The coefficient of $e_\chi$ in $p_{n,\nu}$ has a nice combinatorial
interpretation similar to that of the homogeneous symmetric functions. Suppose
is a brick tabloid of shape \((n)\) and type \(\lambda\) and that the final brick in \(T\) has length \(\ell\). Define the weight of a brick tabloid \(w_\nu(T)\) to be \(\nu(\ell)\) and let

\[
w_\nu(B_\lambda,n) = \sum_{T \text{ is a brick tabloid of shape } (n) \text{ and type } \lambda} w_\nu(T).
\]

When \(\nu(n) = 1\) for \(n \geq 1\), \(B_\lambda,n\) and \(w_\nu(B_\lambda,n)\) are the same. By the recursions found in the definition of \(p_{n,\nu}\), it may be shown that

\[
p_{n,\nu} = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} w_\nu(B_\lambda,n) e_\lambda
\]
in almost the exact same way that (1.2.1) was proved in [12].

Suppose that we are given \(r\) functions \(\alpha_i : \mathbb{P} \to R\) where \(R\) is some field. We write \(T = (b_1, \ldots, b_k)\) if \(T\) is brick tabloid of shape \((n)\) where \(n = b_1 + \cdots + b_k\) and the sizes of the bricks are \(b_1, \ldots, b_k\) as we read from left to right. If \(T = (b_1, \ldots, b_k)\) where \(k \geq r\), we define

\[
w_{\alpha_1, \ldots, \alpha_r}(T) = \alpha_1(b_1) \cdots \alpha_r(b_r) \prod_{i=1}^{k} (-1)^{b_i-1}.
\]

Let \(B_n = \bigcup_{\lambda \vdash n} B_{\lambda,n}\) denote the set of all brick tabloids of shape \((n)\). Define \(p_{n;\alpha_1, \ldots, \alpha_r}\) by

\[
p_{n;\alpha_1, \ldots, \alpha_r} = \sum_{T=(b_1, \ldots, b_k) \in B_n, k \geq r, b_i \geq 1} w_{\alpha_1, \ldots, \alpha_r}(T) \prod_{i=1}^{k} e_{b_i}.
\]

We can partition the brick tabloids \(T = (b_1, \ldots, b_k) \in B_n\) with \(k \geq r\) into two classes. First, there is the class of brick tabloids where \(k = r\) and, second, there is the class of brick tabloids where \(k > r\). This second class can be further classified by the size of the last brick to give the following recursion:

\[
p_{n;\alpha_1, \ldots, \alpha_r} = \sum_{a_1 + \cdots + a_r = n, a_i \geq 1} \prod_{i=1}^{r} \alpha_1(a_i) (-1)^{a_i-1} e_{a_i} + \sum_{k=1}^{n-r} (-1)^{k-1} e_k p_{n-k;\alpha_1, \ldots, \alpha_r}.
\]
It follows that
\[
\sum_{k=0}^{n-r} (-1)^k e_k p_{n-k; \alpha_1, \ldots, \alpha_r} = \sum_{a_i + \ldots + a_r = n \atop a_i \geq 1} \prod_{i=1}^r \alpha_i (a_i) (-1)^{a_i-1} e_{a_i}
\]
which, in turn, implies that
\[
\left( \sum_{k \geq 0} (-1)^k e_k t^k \right) \left( \sum_{n \geq r} p_{n; \alpha_1, \ldots, \alpha_r} t^n \right) = \prod_{i=1}^r \left( \sum_{n \geq 1} (-1)^{n-1} \alpha_i (n) e_n t^n \right).
\]
Therefore,
\[
\sum_{n \geq r} p_{n; \alpha_1, \ldots, \alpha_r} t^n = \frac{\prod_{i=1}^r \left( \sum_{n \geq 1} (-1)^{n-1} \alpha_i (n) e_n t^n \right)}{\sum_{k \geq 0} (-1)^k e_k t^k}.
\]
This is an analogous equation to equation (6.4).

### 3.3 The case where \( i = 0 \)

In this section, we show why (3.1.2) and (3.1.3) are true. We will start with the situation where \( j = k \). Fix \( k \geq 2 \). Define a homomorphism \( \xi_k \) from the ring of symmetric functions \( \Lambda \) to the polynomial ring \( \mathbb{Q}(q_1, \ldots, q_L, p_1, \ldots, p_L)[x] \) by setting
\[
\xi_k(e_j) = 0 \text{ if } j \not\equiv 0 \text{ mod } k \text{ and } \xi_k(e_{kn}) = \frac{(-1)^{kn-1} (x - 1)^{n-1} \mathbf{P}(k^n)}{[kn]_{\mathbf{P}, \mathbf{Q}}!},
\]
otherwise.

**Theorem 3.3.1.** If \( n \not\equiv 0 \text{ mod } k \), then \( \xi_k(h_n) = 0 \). Otherwise,
\[
[kn]_{\mathbf{P}, \mathbf{Q}}! \xi_k(h_{kn}) = \sum_{\Sigma = (\sigma_1, \ldots, \sigma_L) \in (C_{kn}^0, k)^L} \mathbf{x}_{\text{comris}, k}(\Sigma) \mathbf{Q}^{\text{inv}}(\Sigma) \mathbf{P}^{\text{coinv}}(\Sigma).
\]

**Proof.** If \( \lambda = (\lambda_1, \ldots, \lambda_t) \) is a partition of \( n \), we let \( k \lambda = (k \lambda_1, \ldots, k \lambda_t) \). Similarly, if \( T = (b_1, \ldots, b_t) \in \mathcal{B}_{\mu,n} \), then we let \( kT = (kb_1, \ldots, kb_t) \in \mathcal{B}_{k \mu,kn} \).

The relationship between the symmetric functions \( h_n \) and \( e_\lambda \) gives
\[
\xi_k(h_n) = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\mu,n} \xi_k(e_\mu). \tag{3.3.1}
\]
If \( n \) is not equivalent to 0 mod \( k \), then every \( \mu = (\mu_1, \ldots, \mu_{\ell(\mu)}) \) on the right-hand side of (3.3.1) must contain a part \( \mu_i \) which is not equivalent to 0 mod \( k \) and hence \( \xi_k(e_\mu) = 0 \). Thus \( \xi_k(h_n) = 0 \) in this case. Similarly, if \( \mu \) is a partition of \( kn \) where \( n \geq 1 \), then \( \xi(e_\mu) = 0 \) unless \( \mu \) consists entirely of parts which are equal to 0 mod \( k \). Thus in the expansion of \( \xi_k(h_{kn}) \), we can restrict ourselves to partitions \( \mu \) of the form \( k\lambda \) where \( \lambda \) is a partition of \( n \).

We have that

\[
[kn]_{P,Q}! \xi_k(h_{kn}) = [kn]_{P,Q}! \sum_{\mu \vdash n} (-1)^{kn-\ell(\mu)} B_{kn,kn} \xi(e_\mu)
\]

\[
= [kn]_{P,Q}! \sum_{\mu \vdash n} (-1)^{kn-\ell(\mu)}
\times \sum_{T=(b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu,n}} \prod_{i=1}^{\ell(\mu)} (-1)^{kb_i-1} (x-1)^{b_i-1} \left[kb_i\right]_{P,Q}^{(kb_i)}
\]

\[
= \sum_{\mu \vdash n} \sum_{T=(b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu,n}} \left[n \atop kb_1, \ldots, kb_{\ell(\mu)}\right]_{P,Q}^{kn} \prod_{i=1}^{\ell(\mu)} \left(kb_i^2\right) (x-1)^{n-\ell(\mu)}.
\]

Fix a brick tabloid \( T = (b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu,n} \). We want to give a combinatorial interpretation to \( p^{\sum_{i=1}^{\ell(\mu)} \left(b_i\right)[n \atop b_1, \ldots, b_{\ell(\mu)}]}_{P,Q} \). Let \( IF(T) \) denote the set of all fillings of the cells of \( T = (b_1, \ldots, b_{\ell(\mu)}) \) with the numbers 1, \ldots, \( n \) so that the numbers increase within each brick reading from left to right. We then think of each such filling as a permutation of \( S_n \) by reading the numbers from left to right in each row. For example,

\[
\begin{array}{cccccccccc}
4 & 6 & 12 & 1 & 5 & 7 & 8 & 10 & 11 & 2 & 3 & 9
\end{array}
\]

is an element of \( IF(3,6,3) \) whose corresponding permutation is 4 6 12 1 5 7 8 10 11 2 3 9.

**Lemma 3.3.2.** If \( T = (b_1, \ldots, b_{\ell(\mu)}) \) is a brick tabloid in \( B_{\mu,n} \), then

\[
p^{\sum_{i=1}^{\ell(\mu)} \left(b_i\right)[n \atop b_1, \ldots, b_{\ell(\mu)}]}_{P,Q} = \sum_{\sigma \in IF(T)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}.
\]
Proof. It follows from a result of Carlitz [11] that for positive integers $b_1, \ldots, b_\ell$ which sum to $n$,
\[
\begin{bmatrix} n \\ b_1, \ldots, b_\ell \end{bmatrix}_{p,q} = \sum_{r \in \mathcal{R}(b_1, \ldots, b_\ell)} q^{\text{inv}[r]} p^{\text{coinv}[r]}
\]
where $\mathcal{R}(b_1, \ldots, b_\ell)$ is the set of rearrangements of $b_1$ 1’s, $b_2$ 2’s, etc.

Consider a rearrangement $r$ of $b_1, \ldots, b_\ell$ and construct a permutation $\sigma_r$ by labeling the 1’s from left to right with 1, 2, \ldots, $b_1$, the 2’s from left to right with $b_1+1, \ldots, b_1+b_2$, and in general the $i$’s from left to right with $1+\sum_{j=1}^{i-1} b_j, \ldots, b_i+\sum_{j=1}^{i-1} b_j$. In this way, $\sigma_r^{-1}$ starts with the positions of the 1’s in $r$ in increasing order, followed by the positions of the 2’s in $r$ in increasing order, etc. For example, if $T = (2, 1, 3, 1, 4, 1) \in B_{(1,1,1,2,3,4)}$ is below

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
5 & 5 & 1 & 5 & 3 & 1 & 2 & 3 & 6 & 3 & 5 & 4 \\
8 & 9 & 1 & 10 & 4 & 2 & 3 & 5 & 12 & 6 & 11 & 7 \\
3 & 6 & 7 & 5 & 8 & 10 & 12 & 1 & 2 & 4 & 11 & 9 \\
\end{array}
\]
then one possible rearrangement to consider is $r = 5 5 1 5 3 1 2 3 6 3 5 4$. Below we display $\sigma_r$ and $\sigma_r^{-1}$.

\[
\begin{array}{cccccccccccc}
r & = & 5 & 5 & 1 & 5 & 3 & 1 & 2 & 3 & 6 & 3 & 5 & 4 \\
\sigma_r & = & 8 & 9 & 1 & 10 & 4 & 2 & 3 & 5 & 12 & 6 & 11 & 7 \\
\sigma_r^{-1} & = & 3 & 6 & 7 & 5 & 8 & 10 & 12 & 1 & 2 & 4 & 11 & 9 \\
\end{array}
\]
We can think of $\sigma_r^{-1}$ as a filling of the cells of the brick tabloid $T = (2, 1, 3, 1, 4, 1)$ with the numbers 1, \ldots, 12 such that the numbers within each brick are increasing, reading from left to right, pictured below.

\[
\begin{array}{cccccccccccc}
3 & 6 & 7 & 5 & 8 & 10 & 12 & 1 & 2 & 4 & 11 & 9 \\
\end{array}
\]

It is then easy to see that
\[
\binom{2}{2} + \binom{1}{2} + \binom{3}{2} + \binom{1}{2} + \binom{4}{2} + \binom{1}{2} + \text{coinv}(r) = \text{coinv}(\sigma_r) = \text{coinv}(\sigma_r^{-1})
\]
and $\text{inv}(r) = \text{inv}(\sigma_r) = \text{inv}(\sigma_r^{-1})$. 

In general, for any $T = (b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu,\alpha}$, the correspondence which takes $r \in \mathcal{R}(b_1, \ldots, \ell(b))$ to $\sigma_r^{-1}$ shows that

$$\sum_{\sigma \in IF(T)} q^{inv(\sigma)} p^{coinv(\sigma)},$$

thereby completing the proof of the lemma.

It follows that for any $T = (b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu,n}$,

$$\prod_{i=1}^{\ell(\mu)} \left[ \sum_{q=1}^{k_b} \right] \prod_{i=1}^{\ell(\mu)} \left[ \sum_{q=1}^{k_b} \right] = \prod_{i=1}^{\ell(\mu)} \left[ \sum_{q=1}^{k_b} \right]$$

Thus we can interpret $\prod_{i=1}^{\ell(\mu)} \left[ \sum_{q=1}^{k_b} \right]$ as the sum of the weights of the set of fillings of $kT$ of $L$-tuples of permutations $\Sigma = (\sigma(1), \ldots, \sigma(L))$ such that for each $i$, the elements of $\sigma(i)$ are increasing within each brick of $kT$ and where the weight of such a filling is $Q^{inv(\Sigma)} P^{coinv(\Sigma)}$. For example, if $T = (2, 1, 2) \in B_{(1,2^2),5}$, $k = 3$, and $L = 3$, then such a filling of $kT$ is pictured below.

| $\sigma(1)$ | 5 | 8 | 9 | 11 | 13 | 15 | 4 | 7 | 10 | 1 | 2 | 3 | 6 | 12 | 14 |
| $\sigma(2)$ | 1 | 2 | 7 | 10 | 11 | 12 | 4 | 9 | 14 | 3 | 5 | 6 | 8 | 13 | 15 |
| $\sigma(3)$ | 3 | 5 | 9 | 10 | 13 | 14 | 1 | 4 | 11 | 2 | 6 | 7 | 8 | 12 | 15 |

We order the cells of such a filled brick tableau from left to right. We can interpret the term $(x - 1)^{n - \ell(\mu)}$ as taking such a filling and labeling the cells of the form $sk$ which are not at the end of a brick with either an $x$ or $-1$ and labeling each cell at the end of a brick with 1. This was done in the previous figure. Objects $O$ constructed in this way will be called labeled filled brick tabloids. We define the weight of $O$, $W(O)$, to be the product over all the labels of the cells times $Q^{inv(\Sigma)} P^{coinv(\Sigma)}$ if $T$ is filled with permutations $\Sigma = (\sigma(1), \ldots, \sigma(L))$. Thus for the object above,

$$W(O) = (-1) x q_1^{inv(\sigma(1))} q_2^{inv(\sigma(2))} q_3^{inv(\sigma(3))} p_1^{coinv(\sigma(1))} p_2^{coinv(\sigma(2))} p_3^{coinv(\sigma(3))}.$$
Let $\mathcal{L}F^{(k)}(kn)$ denote the set of all objects that can be created in this way. It follows that

$$[kn]_{P,Q}!\xi (h_{kn}) = \sum_{O \in \mathcal{L}F^{(k)}(kn)} W(O).$$

To finish the proof, let us define an involution $I : \mathcal{L}F^{(k)}(kn) \rightarrow \mathcal{L}F^{(k)}(kn)$. Given $O \in \mathcal{L}F^{(k)}(kn)$, read the cells of $O$ from left to right looking for the first cell $kc$ for which either

(i) $kc$ is labeled with $-1$, or

(ii) $kc$ is at the end of end of brick $b$, the cell $kc + 1$ is immediately to the right of $kc$ and starts another brick $b'$, and each permutation $\sigma^{(i)}$ increases as we go from cell $kc$ to $kc + 1$.

If we are in case (i), then $I(O)$ is the labeled filled brick tabloid which is obtained from $O$ by taking the brick $b$ that contains $kc$ and splitting $b$ into two bricks $b_1$ and $b_2$ where $b_1$ contains the cells of $b$ up to and including the cell $kc$ and $b_2$ contains the remaining cells of $b$ and the label on $kc$ is changed from $-1$ to $1$.

If in case (ii), $I(O)$ is the labeled filled brick tabloid which is obtained from $O$ by combining the two bricks $b$ and $b'$ into a single brick and changing the label on cell $kc$ from $1$ to $-1$. If neither case (i) or case (ii) applies, then we let $I(O) = O$. For example, the image of the brick tabloid depicted above under $I$ is below:

<table>
<thead>
<tr>
<th>$\sigma^{(1)}$</th>
<th>5</th>
<th>8</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>4</th>
<th>7</th>
<th>10</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>12</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma^{(2)}$</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>4</td>
<td>9</td>
<td>14</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>13</td>
<td>15</td>
</tr>
<tr>
<td>$\sigma^{(3)}$</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>10</td>
<td>13</td>
<td>14</td>
<td>1</td>
<td>4</td>
<td>11</td>
<td>2</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>12</td>
<td>15</td>
</tr>
</tbody>
</table>

If $I(O) \neq O$, then $W(I(O)) = -W(O)$ since we change the label on cell $c$ from $1$ to $-1$ or vice versa. Moreover, $I$ is an involution. Thus, $I$ shows that

$$[kn]_{P,Q}!\xi (h_{kn}) = \sum_{O \in \mathcal{L}F^{(k)}(kn)} W(O) = \sum_{O \in \mathcal{L}F^{(k)}(kn), I(O) = O} W(O).$$

We are therefore led to examine the fixed points under $I$. If $I(O) = O$, then $O$ can have no cells which are labeled with $-1$. Also it must be the case that between
any two consecutive bricks at least one of the underlying permutations \(\sigma^{(i)}\) must decrease. Each cell \(kc\) which is not at the end of the brick in \(O\) is labeled with \(x\) and each of the permutations \(\sigma^{(i)}\) has a rise at \(kc\) so that \(kc \in Comris_{0,k}(\Sigma)\). All the other cells of the form \(kc\) in \(O\) except the last cell are at the end of a brick which has another brick to its right in which case \(kc \notin Comris_{0,k}(\Sigma)\). All such cells have label 1 so that \(W(O) = x^{comris_{0,k}(\Sigma)}Q^{inv(\Sigma)}P^{coinv(\Sigma)}\).

Now if we are given \(\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \in (C_{kn}^{0,k,k})^L\), we can construct a fixed point of \(I\) from \(\Sigma\) by using \((\sigma^{(1)}, \ldots, \sigma^{(L)})\) to fill a tabloid of shape \((kn)\), then drawing the bricks so that the cells \(kc\) which end bricks are precisely the cells \(kc\) where one of the permutations \(\sigma^{(i)}\) decreases from \(kc\) to \(kc + 1\). This shows that

\[
\sum_{O \in \mathcal{L}^{(k)}(kn), I(O) = O} W(O) = \sum_{\Sigma \in (C_{kn}^{0,k,k})^L} x^{comris_{0,k}(\Sigma)}Q^{inv(\Sigma)}P^{coinv(\Sigma)}.
\]

This completes the proof of the theorem. \(\square\)

To find the generating function in equation (3.1.2), we apply the homomorphism \(\xi_k\) to both sides of the identity

\[
1 + \sum_{n \geq 1} h_n t^n = H(t) = \frac{1}{E(-t)}
\]

to find that

\[
1 + \sum_{n \geq 1} \frac{t^{kn}}{[kn]P,Q!} \sum_{\Sigma \in (C_{kn}^{0,k,k})^L} x^{comris_{0,k}(\Sigma)}Q^{inv(\Sigma)}P^{coinv(\Sigma)} = \frac{1}{\xi_k(E(-t))}. \tag{3.3.2}
\]

Now

\[
\xi_k(E(-t)) = 1 + \sum_{n \geq 1} (-t)^{kn}(-1)^{kn-1}(x - 1)^{n-1}P_{(2)}^{(kn)}[kn]P,Q!
\]

\[
= \frac{1}{(1 - x)} \left( (1 - x + \sum_{n \geq 1} \frac{t^{kn}(x - 1)^nP_{(2)}^{(kn)}}{[kn]P,Q!}) \right)
\]

\[
= \frac{1}{(1 - x)} \left( -x + \epsilon_{P,Q,k}(t(x - 1)^{1/k}) \right). \tag{3.3.3}
\]

Combining (3.3.2) and (3.3.3) gives the following theorem.
Theorem 3.3.3. For \( k \geq 2 \),
\[
1 + \sum_{n \geq 1} \frac{t^{kn}}{[kn]_{P,Q}!} \sum_{\Sigma \in (C_{kn}^{0,k,k})^L} x^{comris_{0,k}(\Sigma)} Q^{inv(\Sigma)} P^{coinv(\Sigma)} = \frac{1 - x}{-x + e_{P,Q,k}(t(x-1)^{1/k})}
\]
and
\[
1 + \sum_{n \geq 1} \frac{t^{kn}}{[kn]_{P,Q}!} \sum_{\Sigma \in (\mathcal{E}_{kn}^{0,k,k})^L} Q^{inv(\Sigma)} P^{coinv(\Sigma)} = \frac{1}{e_{P,Q,k}(t(-1)^{1/k})}.
\]

Next, suppose that \( k \geq 2 \) and \( 1 \leq j \leq k - 1 \). Define a function \( \nu \) on the positive integers by setting \( \nu(j) = 0 \) if \( j \not\equiv 0 \mod k \) and
\[
\nu(kn) = \frac{[kn]_{P,Q} \downarrow_{k-j}}{P^{(k-j)kn-(k-j+1)}}
\]
where for any \( 1 \leq s \leq n \), \( [n]_{P,Q} \downarrow_s = [n]_{P,Q}[n-1]_{P,Q} \cdots [n-s+1]_{P,Q} \). This definition of \( \nu(kn) \) is designed so that
\[
\nu(kn) \xi_k(e_{nk}) = (-1)^{kn-1} \frac{(x-1)^{n-1}P^{(kn)}_{(2)}}{[kn]_{P,Q}!} \frac{\prod_{s=1}^{k-j}kn-j}{\prod_{s=1}^{k-j}kn-j} = (-1)^{kn-s} \frac{(x-1)^{n-1}P^{(k(n-1)+j)}}{[k(n-1)+j]_{P,Q}!}.
\]

Theorem 3.3.4. For any \( k \geq 2 \) and \( 1 \leq j \leq k - 1 \), \( \xi_k(p_{j,\nu}) = 0 \) if \( j \not\equiv 0 \mod k \) and for all \( n \geq 1 \),
\[
[k(n-1)+j]_{P,Q}!p_{kn,\nu} = \sum_{\Sigma \in (C_{(n-1)+j,k})^L} x^{comris_{0,k}(\Sigma)} Q^{inv(\Sigma)} P^{coinv(\Sigma)}.
\]

Proof. The relationship between \( p_{n,\nu} \) and the elementary symmetric functions gives
\[
\xi_k(p_{n,\nu}) = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \nu(B_{\mu,\nu}) \xi(\epsilon_{\mu}). \tag{3.3.4}
\]

If \( n \) is not equivalent to 0 \( \mod k \), then every \( \mu = (\mu_1, \ldots, \mu_{\ell(\mu)}) \) on the right-hand side of (3.3.4) must contain a part \( \mu_i \) which is not equivalent to 0 \( \mod k \) and hence \( \xi_k(\epsilon_{\mu}) = 0 \). Thus \( \xi_k(p_{n,\nu}) = 0 \) in this case. Similarly, if \( \mu \) is a partition of \( kn \) where \( n \geq 1 \), then \( \xi(\epsilon_{\mu}) = 0 \) unless \( \mu \) consists entirely of parts which are equal to 0.
mod $k$. Thus, in the expansion of $\xi_k(h_{nk})$, we can restrict ourselves to partitions $\mu$ of the form $k \lambda$ where $\lambda$ is a partition of $n$. Therefore,

$$[k(n-1) + j]_{P,Q} \xi_k(p_{kn,\nu})$$

$$= [k(n-1) + j]_{P,Q} \sum_{\mu \vdash n} (-1)^{kn-\ell(\mu)} \nu(B_{k\mu, kn}) \xi(e_{k\mu})$$

$$= [k(n-1) + j]_{P,Q} \sum_{\mu \vdash n} (-1)^{kn-\ell(\mu)} \sum_{(b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu, n}} \nu(k b_{\ell(\mu)}) \xi(e_{k b_{\ell(\mu)}})$$

$$\times \prod_{i=1}^{\ell(\mu)-1} \frac{(-1) \nu_{b_i} \mu_{b_i}^{-1}(x-1)^{b_i-1} P(k b_i)}{[k b_i] P_{Q}!}$$

$$= [k(n-1) + j]_{P,Q} \sum_{\mu \vdash n} (-1)^{kn-\ell(\mu)}$$

$$\times \sum_{(b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu, n}} (-1)^{kb_{\ell(\mu)}-1} (x-1)^{b_{\ell(\mu)}-1} P(k b_{\ell(\mu)}^{-1} + j)$$

$$\times \prod_{i=1}^{\ell(\mu)-1} \frac{(-1) \nu_{b_i} \mu_{b_i}^{-1}(x-1)^{b_i-1} P(k b_i)}{[k b_i] P_{Q}!}$$

$$= \sum_{\mu \vdash n} \sum_{(b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu, n}} (x-1)^{n-\ell(\mu)}$$

$$\times \left[ \frac{kn-k+j}{kb_1 \cdot \ldots \cdot kb_{\ell(\mu)-1}, k(b_{\ell(\mu)}-1) + j} \right] P(k b_{\ell(\mu)}^{-1} + j + \sum_{i=1}^{\ell(\mu)-1} (k b_i))_{P,Q}$$

By Lemma 3.3.2, we can interpret

$$\left[ \frac{kn-k+j}{kb_1 \cdot \ldots \cdot kb_{\ell(\mu)-1}, k(b_{\ell(\mu)}-1) + j} \right] P(k b_{\ell(\mu)}^{-1} + j + \sum_{i=1}^{\ell(\mu)-1} (k b_i))_{P,Q}$$

as the set of fillings of the brick tabloid $U = (kb_1, \ldots, kb_{\ell(\mu)}-1, k(b_{\ell(\mu)}-1) + j)$ of $L$-tuples of permutations $\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)})$ such that for each $i$, the elements of $\sigma^{(i)}$ are increasing within each brick of $U$ and we weight such a filling with $Q_{\text{inv}(\Sigma)} P_{\text{coinv}(\Sigma)}$. In fact, we shall think of $U$ as the brick tabloid $kT = (kb_1, \ldots, kb_{\ell(\mu)})$ where the last $k-j$ cells of the last brick are blank.

Again, order the cells of such a filled brick tabloid from left to right. Interpret the term $(x-1)^{n-\ell(\mu)}$ as taking such a filling and labeling the cells of the form $sk$
which are not at the end of a brick with either an \( x \) or \(-1\) and labeling each cell at the end of a brick with 1. Below is an example of such a tabloid.

<table>
<thead>
<tr>
<th>( \sigma^{(1)} )</th>
<th>( \sigma^{(2)} )</th>
<th>( \sigma^{(3)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
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<td>7</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

Call such an object \( \mathcal{O} \) a labeled filled brick tabloid and define the weight of \( \mathcal{O} \), \( W(\mathcal{O}) \), to be product over all the labels of the cells times \( Q_{\text{inv}}(\Sigma)P_{\text{coinv}}(\Sigma) \) if \( T \) is filled with permutations \( \Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \). Thus for the object pictured above,

\[
W(\mathcal{O}) = (-1)x^\text{inv}\sigma_1 q^\text{inv}\sigma_2 q_3^\text{inv}\sigma_3 p_1^\text{coinv}\sigma_1 p_2^\text{coinv}\sigma_2 p_3^\text{coinv}\sigma_3.
\]

We let \( \mathcal{L}F^{(k,j)}(kn) \) denote the set of all objects that can be created in this way from brick tabloids \( T \) in \( \mathcal{B}_n \). Then it follows that

\[
[k(n - 1) + j]_{\mathcal{P}, \mathcal{Q}} \xi_k(p_{kn, \nu}) = \sum_{\mathcal{O} \in \mathcal{L}F^{(k,j)}(kn)} W(\mathcal{O}).
\]

We now define an involution \( I : \mathcal{L}F^{(k,j)}(kn) \rightarrow \mathcal{L}F^{(k,j)}(kn) \) exactly as we did in the proof of Theorem 3.3.1. That is, given \( \mathcal{O} \in \mathcal{L}F^{(k,j)}(kn) \), read the cells of \( \mathcal{O} \) from left to right looking for the first instance of either:

(i) \( kc \) is labeled with \(-1\), or

(ii) \( kc \) is at the end of end of brick \( b \), the cell \( kc + 1 \) is immediately to the right of \( kc \) and starts another brick \( b' \), and each permutation \( \sigma^{(i)} \) increases as we go from \( kc \) to \( kc + 1 \).

If in case (i), then \( I(\mathcal{O}) \) is the labeled filled brick tabloid which is obtained from \( \mathcal{O} \) by taking the brick \( b \) that contains \( kc \) and splitting \( b \) into two bricks \( b_1 \) and \( b_2 \) where \( b_1 \) contains the cells of \( b \) up to and including the cell \( kc \) and \( b_2 \) contains the remaining cells of \( b \) and changing the label on \( kc \) from \(-1\) to 1. If in case (ii), \( I(\mathcal{O}) \) is the labeled filled brick tabloid which is obtained from \( \mathcal{O} \) by combining the two bricks \( b \) and \( b' \) into a single brick and changing the label on cell \( kc \) from 1 to \(-1\).
If neither case (i) or case (ii) applies, take $I(O) = O$. As an example, the image of the above figure under this map is below:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
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<th>13</th>
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</thead>
<tbody>
<tr>
<td>$\sigma^{(1)}$</td>
<td>5</td>
<td>8</td>
<td>9</td>
<td>11</td>
<td>12</td>
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<td>4</td>
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<td>2</td>
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</tr>
<tr>
<td>$\sigma^{(2)}$</td>
<td>1</td>
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<td>8</td>
</tr>
<tr>
<td>$\sigma^{(3)}$</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>10</td>
<td>12</td>
<td>13</td>
<td>1</td>
<td>4</td>
<td>11</td>
<td>2</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

Then we argue exactly as in Theorem 3.3.1 that

$$
\sum_{O \in \mathcal{L}(^{(k,j)}(kn), I(O) = O)} W(O) = \sum_{\Sigma \in (c_{k(n-1)+j}^{0,j,k})^L} x^{comris_{0,k}(\Sigma)} Q^{inv(\Sigma)} P^{coinv(\Sigma)},
$$

as desired.

We can now apply $\xi_k$ to the relationship between $p_{n,\nu}$ and the elementary symmetric functions to arrive at (3.1.3). We have

$$
\xi_k(\sum_{n \geq 1} (-1)^{n-1} \nu(n) e^t_n t^{kn}) = \sum_{n \geq 1} (-1)^{kn-1} \nu(kn) \xi_k(e^{kn}_t) t^{kn}
$$

$$
= \sum_{n \geq 1} (-1)^{kn-1} (-1)^{kn-1} (x-1)^{n-1} p^{(k(n-1)+j)}_{kn} t^{kn} [k(n-1) + j]_{P,Q}!
$$

$$
= \frac{1}{(x-1)} \sum_{n \geq 1} (x-1)^n p^{(k(n-1)+j)}_{kn} t^{kn} [k(n-1) + j]_{P,Q}!
$$

$$
= \frac{1}{(x-1)} e^{(j)}_{P,Q,k}(t(x-1)^{1/k}).
$$

Therefore, we have that

$$
\sum_{n \geq 1} \xi_k(p_{kn,\nu}) t^{kn} = \sum_{n \geq 1} \frac{t^{kn}}{[k(n-1) + j]_{P,Q}!} \sum_{\Sigma \in (c_{k(n-1)+j}^{0,j,k})^L} x^{comris_{0,k}(\Sigma)} Q^{inv(\Sigma)} P^{coinv(\Sigma)}
$$

$$
= \frac{1}{(x-1)} e^{(j)}_{P,Q,k}(t(x-1)^{1/k})
$$

$$
= \frac{-e^{(j)}_{P,Q,k}(t(x-1)^{1/k})}{-x + e^{(j)}_{P,Q,k}(t(x-1)^{1/k})},
$$

and we have arrived at the following theorem.
Theorem 3.3.5. For any $k \geq 2$ and $1 \leq j \leq k - 1$,
\[
\sum_{n \geq 1} \frac{t^{kn}}{[k(n-1) + j]P_{j,k}Q!} \sum_{\Sigma \in (c_{k(n-1)+j}^0)^L} x^{\text{comris}_0,k}(\Sigma)Q^{\text{inv}(\Sigma)}P^{\text{coinv}(\Sigma)}
\]
\[
= \frac{-e_{P,Q,k}(t(x-1)^{1/k})}{-x + e_{P,Q,k}(t(x-1)^{1/k})}
\]
and
\[
\sum_{n \geq 1} \frac{t^{kn}}{[k(n-1) + j]P_{j,k}Q!} \sum_{\Sigma \in (c_{k(n-1)+j}^{j,k})^L} Q^{\text{inv}(\Sigma)}P^{\text{coinv}(\Sigma)} = \frac{-e_{P,Q,k}(t(-1)^{1/k})}{e_{P,Q,k}(t(-1)^{1/k})}.
\]

Theorem 3.3.5 also gives us the generating functions for permutations in $C_{k(n-1)+j}^{j,k}$ where $1 \leq j \leq k - 1$. That is, given $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let

$$
\sigma^c = (n+1 - \sigma_1) \cdots (n+1 - \sigma_n) \quad \text{and} \quad \sigma^r = \sigma_n \cdots \sigma_1.
$$

Given $\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \in S_n^L$, we let $(\Sigma^c)^r = (\tau^{(1)}, \ldots, \tau^{(L)}) \in S_n^L$ where for each $i$, $\tau^{(i)} = ((\sigma^{(i)})^c)^r$. Then $\Sigma \in (C_{k(n-1)+j}^{0,j,k})^L$ if and only if $(\Sigma^c)^r \in (C_{k(n-1)+j}^{j,k})^L$ and $\text{comris}_0,k(\Sigma) = \text{comris}_{j,k}((\Sigma^c)^r)$. Thus the only cases that we have left are the generating functions for permutations in $C_{k(n-1)+j}^{i,j,k}$ where $1 \leq i, j \leq k - 1$. Such generating functions are considered in the next section.

### 3.4 The case $i \neq 0$ and $j < k$

The goal of this section is to prove the identity in (3.1.5). To do this, we will use the symmetric functions $p_{n, \beta_1, \beta_2}$ where $\beta_1(j) = 0$ if $j \neq 0 \mod k$ for $n \geq 2$, and otherwise

$$
\beta_1(kn) = \frac{[kn]_{P,Q} \downarrow_{k-i}}{P^{(k-i)kn} - \frac{(k+i+1)!}{2}}, \quad \beta_2(kn) = \frac{[kn]_{P,Q} \downarrow_{k-j}}{P^{(k-j)kn} - \frac{(k+j+1)!}{2}}.
$$

We have defined $\beta_1(kn)$ and $\beta_2(kn)$ so that for $n \geq 2$

\[
\beta_1(kn)\xi_k(e_{nk}) = (-1)^{kn-1} \frac{(x-1)^{n-1}P^{(kn)!}_{(kn)}}{[kn]_{P,Q} \downarrow_{k-i} P^{\Sigma_{s=1}^{k-1} \xi_{kn-s}}} \times (-1)^{kn-1} \left( - \frac{(x-1)^{n-1}P^{(kn+1)!}_{(kn+1)+i}}{[k(n-1) + i]_{P,Q}!} \right).
\]
and

$$\beta_2(kn)\xi_k(e_{nk}) = (-1)^{kn-1}(x-1)^{n-1}\mathbf{P}^{(kn)}_{k(n-2)+j}[kn]\mathbf{P}_{kn,s} \downarrow_{k-j}$$
$$= (-1)^{kn-1}(x-1)^{n-1}\mathbf{P}^{(k(n-1)+j)}[k(n-1)+j]\mathbf{P}_{kn,s}. \hspace{1cm} (3.4.2)$$

**Theorem 3.4.1.** For $k \geq 2$ and $1 \leq i, j \leq k-1$, $\xi_k(p_j,\beta_1,\beta_2) = 0$ if $j \not\equiv 0 \mod k$ or $j < 2k$. Otherwise, for $n \geq 2$,

$$[i + k(n-2) + j]_P \xi_k(p_{kn,\beta_1,\beta_2})$$
$$= ((n-1)x^{n-2} - x^{n-1})\mathbf{P}^{(i+k(n-2)+j)} + \sum_{\Sigma \in (i+k(n-2)+j)^L} x^{comris_{i,k}(\Sigma)} \mathbf{Q}^{inv(\Sigma)} \mathbf{P}^{coinv(\Sigma)}.$$

**Proof.** We have

$$\xi_k(p_{m,\beta_1,\beta_2}) = \sum_{T=(b_1, b_2, b_3, \ldots, b_s) \atop b_1 + \cdots + b_s = m} (-1)^{\sum_{r=1}^{s} b_r - 1} \beta_1(b_1) \beta_2(b_2) \prod_{r=1}^{s} \xi_k(e_{b_r}).$$

Thus if $m \not\equiv 0 \mod k$, then at least one $b_r \not\equiv 0 \mod k$ and hence $\xi_k(b_r) = 0$. Thus $\xi_k(p_{m,\beta_1,\beta_2}) = 0$. Similarly if $m < 2k$, then either $b_1 < k$ in which case $\beta_1(b_1) = 0$ or $b_2 < k$ in which case $\beta_2(b_2) = 0$ so again we can conclude that $\xi_k(p_{m,\beta_1,\beta_2}) = 0$. Similarly, if $\mu$ is a partition of $m = kn$ where $n \geq 2$, then all the $b_r$'s must be multiples of $k$. It follows that for $n \geq 2$,

$$[i + k(n-2) + j]_P \xi_k(p_{kn,\beta_1,\beta_2})$$
$$= [i + k(n-2) + j]_P \xi_k(p_{kn,\beta_1,\beta_2}) \sum_{T=(b_1, b_2, b_3, \ldots, b_s) \atop b_1 + \cdots + b_s = n} (-1)^{\sum_{r=1}^{s} kb_r - 1} \beta_1(kb_1) \beta_2(kb_2) \prod_{r=1}^{s} \xi_k(e_{kb_r}).$$

For each $T = (b_1, b_2, b_3, \ldots, b_s)$ appearing on the right-hand side of the above equation, we let $\bar{T} = (b_1, b_3, \ldots, b_s, b_2) = (c_1, \ldots, c_s)$. Then isolating a portion of the above equation,
\[ \beta_1(kb_1)\beta_2(kb_2) \prod_{r=1}^{s} \xi_k(e_{kb_r}) = \]
\[ (-1)^{kb_1-1} \frac{(x-1)^{kb_1-1}P^{(kb_1-1)+i}}{[k(b_1-1)+i]_{P,Q}!} \times \]
\[ (-1)^{kb_2-1} \frac{(x-1)^{kb_2-1}P^{(kb_2-1)+j}}{[k(b_2-1)+j]_{P,Q}!} \times \]
\[ \prod_{r=3}^{s} (-1)^{kb_r-1} \frac{(x-1)^{kb_r-1}P^{(kb_r)}}{[kb_r]_{P,Q}!}. \]

The factors of \((-1)\) above will cancel the \((-1)^\sum_{r=1}^{s} kb_r\) term. We can combine the denominators that appear in the right-hand side of the above equation with the \([i+k(n-2)+j]_{P,Q}!\) term to form the multinomial coefficient
\[ \left[ i + k(n-2) + j \right]_{P,Q} \frac{\xi_k(p_{kn,\beta_1,\beta_2})}{(x-1)^{\sum_{r=1}^{s} c_r-1}}. \]

It follows that
\[ [i+k(n-2)+j]_{P,Q}! \xi_k(p_{kn,\beta_1,\beta_2}) = \sum_{\tau=(c_1,\ldots,c_s) \atop c_1+\cdots+c_s=n,s\geq 2} \left[ i + k(n-2) + j \right]_{P,Q} \prod_{r=1}^{s} (x-1)^{c_r-1}. \]

By Lemma 3.3.2, we can interpret
\[ \left[ i + k(n-2) + j \right]_{P,Q} \frac{\xi_k(p_{kn,\beta_1,\beta_2})}{(x-1)^{\sum_{r=1}^{s} c_r-1}} \]
as the set of filling of the brick tabloid \( U = (i+k(c_1-1), kc_2, \ldots, kc_{s-1}, k(c_s-1) + j) \) of \( L \)-tuples of permutations \( \Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)}) \) such that for each \( i \), the elements of \( \sigma^{(i)} \) are increasing within each brick of \( U \) and we weight such a filling with \( Q^{\text{inv}}(\Sigma)P^{\text{coinv}}(\Sigma) \). In fact, we shall think of \( U \) as the brick tabloid \( kT = (kc_1, \ldots, kc_s) \) with the first \( k-i \) cells of the first brick blank and the last \( k-j \) cells of the last brick blank.

Again, order the cells of such a filled brick tabloid from left to right and interpret the term \( \prod_{r=1}^{s} (x-1)^{c_r-1} \) as taking such a filling and labeling the cells of the form \( sk \) which are not at the end of a brick with either an \( x \) or \(-1\) and labeling each cell at the end of a brick with 1. An example is pictured below:
Again, $\mathcal{O}$ is a labeled filled brick tabloid. We define the weight of $\mathcal{O}$, $W(\mathcal{O})$, to be product over all the labels of the cells times $Q^{\text{inv}}(\Sigma)P^{\text{coinv}}(\Sigma)$ if $T$ is filled with permutations $\Sigma = (\sigma(1), \ldots, \sigma(L))$.

Let $\mathcal{L}F^{(k,i,j)}(kn)$ denote the set of all objects that can be created in this way from brick tabloids $T = (c_1, \ldots, c_s)$ where $s \geq 2$ and $c_1 + \cdots + c_s = n$. Then it follows that

$$[i + k(n - 2) + j]p.Q^\xi_k(p_{kn,\beta_1,\beta_2}) = \sum_{\mathcal{O} \in \mathcal{L}F^{(k,i,j)}(kn)} W(\mathcal{O}).$$

Next we define an involution $I : \mathcal{L}F^{(k,i,j)}(kn) \rightarrow \mathcal{L}F^{(k,i,j)}(kn)$ which is a slight variation of our previous two involutions. That is, given $\mathcal{O} \in \mathcal{L}F^{(k,i,j)}(kn)$, read the cells of $\mathcal{O}$ in the same order that we read the underlying permutations and look for the first cell $kc$ such that either:

(i) $kc$ is labeled with $-1$ or
(ii) $kc$ is at the end of end of brick $b$, the cell $kc + 1$ is immediately to the right of $kc$ and starts another brick $b'$, and each permutation $\sigma(i)$ increases as we go from $kc$ to $kc + 1$.

If we are in case (i), then $I(\mathcal{O})$ is the labeled filled brick tabloid which is obtained from $\mathcal{O}$ by taking the brick $b$ that contains $kc$ and splitting $b$ into two bricks $b_1$ and $b_2$ where $b_1$ contains the cells of $b$ up to and including the cell $kc$ and $b_2$ contains the remaining cells of $b$ and changing the label on $kc$ from $-1$ to $1$. In case (ii), if $\mathcal{O}$ has at least three bricks, then $I(\mathcal{O})$ is the labeled filled brick tabloid which is obtained from $\mathcal{O}$ by combining the two bricks $b$ and $b'$ into a single brick and changing the label on cell $kc$ from $1$ to $-1$. However, if we are in case (ii) and $\mathcal{O}$ has exactly 2 bricks, then $I(\mathcal{O}) = \mathcal{O}$. Finally, if neither case (i) or case (ii)
applies, then we let $I(\mathcal{O}) = \mathcal{O}$. For example, the image of the above figure under $I$ is

\[
\begin{array}{cccccccccc}
\sigma^{(1)} & 5 & 8 & 9 & 11 & 12 & 4 & 7 & 10 & 1 \\
\sigma^{(2)} & 1 & 2 & 7 & 10 & 11 & 4 & 9 & 12 & 3 \\
\sigma^{(3)} & 3 & 5 & 9 & 10 & 12 & 1 & 4 & 11 & 2 \\
\end{array}
\]

This sign-reversing weight-preserving involution $I$ shows

\[
[i + k(n - 2) + j] \mathbf{p, q} \xi_k(p_{\alpha, \beta}, \delta_k) = \sum_{\mathcal{O} \in \mathcal{L}F^{(k,i,j)}(kn)} W(\mathcal{O}) = \sum_{\mathcal{O} \in \mathcal{L}F^{(k,i,j)}(kn), I(\mathcal{O}) = \mathcal{O}} W(\mathcal{O}).
\]

(3.4.3)

If $I(\mathcal{O}) = \mathcal{O}$, then $\mathcal{O}$ can have no cells which are labeled with $-1$. If $\mathcal{O}$ has at least 3 bricks, then it must be the case that between any two consecutive bricks of $\mathcal{O}$, at least one of the underlying permutations $\sigma^{(i)}$ must decrease. It follows that each cell $kc$ which is not at the end of the brick in $\mathcal{O}$ is labeled with $\sigma$ and each of the permutations $\sigma^{(i)}$ has a rise at $kc$ so that $kc \in Comris_{i,k}(\Sigma)$. All the other cells of the form $kc$ in $\mathcal{O}$ other than the last cell are at the end of brick which has another brick to its right in which case $kc \notin Comris_{i,k}(\Sigma)$. All such cells have label 1 so that $W(\mathcal{O}) = x^{comris_{i,k}(\Sigma)} \mathbf{p}^{\text{coinv}(\Sigma)} \mathbf{q}^{\text{inv}(\Sigma)}$.

Next consider the fixed points $\mathcal{O}$ of $I$ which have two bricks. Again $\mathcal{O}$ can have no cells labeled with $-1$. There are two cases here. If the last cell of first brick is $kc$ and at least one of the underlying permutations $\sigma^{(i)}$ decreases form $kc$ to $kc + 1$, then again it will be the case that $W(\mathcal{O}) = x^{comris_{i,k}(\Sigma)} \mathbf{p}^{\text{coinv}(\Sigma)} \mathbf{q}^{\text{inv}(\Sigma)}$. However, if each of the underlying permutations $\sigma^{(i)}$ increases from $kc$ to $kc + 1$, then it must be the case that each of the $\sigma^{(i)}$’s is the identity permutation. In this case, $W(\mathcal{O}) = x^{n-2} \mathbf{p}^{(i+k(n-2)+j)}$. Moreover, since $kc$ can be either 1, \ldots, $n - 1$, it follows that the contribution of $\text{Id} = (\sigma^{(1)}, \ldots, \sigma^{(L)})$ where each $\sigma^{(i)}$ is the identity permutation to (3.4.3) is $(n - 1)x^{n-2} \mathbf{p}^{(i+k(n-2)+j)}$. However the contribution of $\text{Id}$ to

\[
\sum_{\Sigma \in (C_{i,k}^{i,j,k})^L} x^{comris_{i,k}(\Sigma)} \mathbf{p}^{\text{coinv}(\Sigma)} \mathbf{q}^{\text{inv}(\Sigma)}
\]
is \(x^{n-1}P^{(i+k(n-2)+j)}\). It thus follows that \(\sum_{\mathcal{O}\in \mathcal{E}^{(k,i,j)}(kn)} W(\mathcal{O}) = 0\) is equal to \((n-1)x^{n-2}P^{(i+k(n-2)+j)} = x^{n-1}P^{(i+k(n-2)+j)} + \sum_{\Sigma \in (C_{kn}^{i,j})^L} x^{comris,i,k}(\Sigma) Q^{inv}(\Sigma) P^{coinv}(\Sigma)\) as desired.

We can now apply \(\xi_k\) to the identity

\[
\sum_{n \geq 1} p_{n,\beta_1,\beta_2} t^n = \frac{\left(\sum_{n \geq 1} (-1)^{n-1} \beta_1(n) e_n t^n\right) \left(\sum_{n \geq 1} (-1)^{n-1} \beta_2(n) e_n t^n\right)}{E(t)}.
\]

Using (3.4.1), we can find that

\[
\xi(\sum_{n \geq 1} (-1)^{n-1} \beta_1(n) e_n t^n) = \sum_{n \geq 1} (-1)^{kn-1} \beta_1(kn) \xi_k(e_n) t^{kn} = \sum_{n \geq 1} (-1)^{kn-1} (-1)^{kn-1} (x-1)^{n-1} P^{(k(n-2)+i)} t^{kn} \frac{1}{k(n-1) + i} P_{Q, Q}^1.
\]

\[
= \frac{1}{(x-1)} \sum_{n \geq 1} (x-1)^n t^{kn} P^{(k(n-2)+i)} \frac{1}{k(n-1) + i} P_{Q, Q}^1.
\]

\[
= \frac{1}{(x-1)} \text{e}^{(i)}_{P, Q, k}(t(x-1)^{1/k}) \tag{3.4.4}
\]

Similarly,

\[
\xi(\sum_{n \geq 1} (-1)^{n-1} \beta_2(n) e_n t^n) = \frac{1}{(x-1)} \text{e}^{(j)}_{P, Q, k}(t(x-1)^{1/k}). \tag{3.4.5}
\]

Combining (3.3.3), (3.4.4), and (3.4.5), we see that

\[
\sum_{n \geq 1} \xi_k(p_{n,\beta_1,\beta_2}) t^n = \sum_{n \geq 2} \frac{(n-1)x^{n-2}P^{(i+k(n-2)+j)} t^{kn}}{[i + k(n - 2) + j] P_{Q, Q}^1} - \sum_{n \geq 2} \frac{x^{n-1}P^{(i+k(n-2)+j)} t^{kn}}{[i + k(n - 2) + j] P_{Q, Q}^1} \times
\]

\[
\sum_{\Sigma \in (C_{kn}^{i,j})^{(kn-1)+j} L} x^{comris,i,k}(\Sigma) Q^{inv}(\Sigma) P^{coinv}(\Sigma)
\]

\[
= \frac{1}{(x-1)} \text{e}^{(i)}_{P, Q, k}(t(x-1)^{1/k}) \frac{1}{x-1} \text{e}^{(j)}_{P, Q, k}(t(x-1)^{1/k})
\]

\[
= \frac{1}{x-1} \frac{1}{(1-x)} (-x + \text{e}^{(j)}_{P, Q, k}(t(x-1)^{1/k}))
\]

\[
= \frac{\text{e}^{(j)}_{P, Q, k}(t(x-1)^{1/k}) \text{e}^{(j)}_{P, Q, k}(t(x-1)^{1/k})}{(1-x)((-x + \text{e}^{(j)}_{P, Q, k}(t(x-1)^{1/k}))},
\]
and we have therefore proved the following theorem, achieving the final goal set out in the introduction.

**Theorem 3.4.2.** For any \( k \geq 2 \) and \( 1 \leq i, j \leq k - 1 \),

\[
\sum_{n \geq 2} \frac{t^{kn}}{[i + k(n - 2) + j]_{P,Q}!} \sum_{\Sigma \in (E_{i,k}^{(n-2)} + j)^L} x^{\text{comris}_{i,k}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}
\]

\[
= \sum_{n \geq 2} \frac{x^{n-1}P^{(i+k(n-2)+j)} \ t^{kn}}{[i + k(n - 2) + j]_{P,Q}!} - \sum_{n \geq 2} \frac{(n-1)x^{n-2}P^{(i+k(n-2)+j)} \ t^{kn}}{[i + k(n - 2) + j]_{P,Q}!}
\]

\[
+ \sum_{n \geq 2} \frac{x^{n-1}P^{(i+k(n-2)+j)} \ t^{kn}}{[i + k(n - 2) + j]_{P,Q}!}
\]

\[= \sum_{n \geq 2} \frac{x^{n-1}P^{(i+k(n-2)+j)} \ t^{kn}}{[i + k(n - 2) + j]_{P,Q}!} - \sum_{n \geq 2} \frac{(n-1)x^{n-2}P^{(i+k(n-2)+j)} \ t^{kn}}{[i + k(n - 2) + j]_{P,Q}!}
\]

\[= \sum_{n \geq 2} \frac{x^{n-1}P^{(i+k(n-2)+j)} \ t^{kn}}{[i + k(n - 2) + j]_{P,Q}!}
\]

and

\[
\sum_{n \geq 2} \frac{t^{kn}}{[i + k(n - 2) + j]_{P,Q}!} \sum_{\Sigma \in (E_{i,k}^{(n-2)} + j)^L} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}
\]

\[
= \sum_{n \geq 2} \frac{x^{n-1}P^{(i+k(n-2)+j)} \ t^{kn}}{[i + k(n - 2) + j]_{P,Q}!} - \sum_{n \geq 2} \frac{(n-1)x^{n-2}P^{(i+k(n-2)+j)} \ t^{kn}}{[i + k(n - 2) + j]_{P,Q}!}
\]

\[= \sum_{n \geq 2} \frac{x^{n-1}P^{(i+k(n-2)+j)} \ t^{kn}}{[i + k(n - 2) + j]_{P,Q}!}
\]

A few remarks are in order. The Salié numbers count permutations \( \sigma \in S_n \) such that \( \text{Des}(\sigma) = \{2, 4, \ldots, 2k\} \) where \( 2k \leq n \). Carlitz [10] showed that the generating function of the Salié numbers is given by \( \cosh(t)/\cos(t) \). More recently, Prodinger [36] and Guo and Zeng [21] studied \( q \)-analogues of the Salié numbers. The methods of this chapter can also be used to prove analogues of (3.1.2), (3.1.3), and (3.1.5) for the set of permutations \( \sigma \in S_n \) such that \( \text{Des}\sigma \subseteq \{i+ks, i+k(s+1), \ldots, i+kt\} \) for some \( 0 \leq s < t \) where \( kt < n \). We will show in a forthcoming paper that we can obtain such generating functions by varying the weight functions described in this chapter in an appropriate manner.

In Chapter 2, we have shown how to find the generating functions for permutations which contain a given descent set by applying ring homomorphisms to symmetric function identities involving ribbon Schur functions. These methods give a systematic way to find the following generating functions for any \( S \subset \{1, 2, \ldots\} \):

1. \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{\sigma \in S_n, S \subseteq \text{Des}(\sigma)} x^{\text{des}(\sigma)} \)
2. \[ \sum_{n=0}^{\infty} \frac{u^n}{(n!)^2} \sum_{(\sigma,\tau)\in S_n \times S_n, S \subseteq Comdes(\sigma,\tau)} x^{comdes(\sigma,\tau)} \]

3. \[ \sum_{n=0}^{\infty} \frac{u^n}{[n]^q!} \sum_{\sigma \in S_n, S \subseteq Des(\sigma)} x^{des(\sigma)} q^{inv(\sigma)} \]

4. \[ \sum_{n=0}^{\infty} \frac{u^n}{[n]^q[p]!} \sum_{\sigma \in S_n, S \subseteq Des(\sigma)} x^{des(\sigma)} q^{inv(\sigma)} p^{coinv(\sigma)} \]

5. \[ \sum_{n=0}^{\infty} \frac{u^n}{[n]^q[p]!} \sum_{(\sigma,\tau)\in S_n \times S_n, S \subseteq Comdes(\sigma,\tau)} x^{comdes(\sigma,\tau)} q^{inv(\sigma)} p^{inv(\tau)} \]

It is also possible to derive similar analogues of (3.1.2), (3.1.3), and (3.1.5). That is, we can find generating function for \( L \)-tuples of permutations in \( C_{i+j+k}^{i,j,k} \) such that \( Comris_{i,k}(\Sigma) \) contains a given finite set of the form \( \{s_1k, s_2k, \ldots, s_rk\} \).

One can also get other regular patterns of descents in permutations by replacing the increasing fillings within bricks by more complicated patterns, which we will introduce in Chapter 5. Such ideas will be developed more fully in upcoming papers as well.

### 3.5 Acknowledgements

Chapter 3 contains some material as it will appear in the Proceedings of Permutation Patterns 2007, Mendes, A; Remmel, J; and Riehl, A; Cambridge University Press London Mathematical Society Lecture Notes, 2008. The dissertation author was a coauthor of this paper.
Chapter 4

Permutations with \( k \)-regular descent patterns except for a given set.

In this chapter, we describe by example a way to combine the methods of Chapters 2 and 3. Let \( \mathbb{N} \) denote the set of natural numbers and \( S \) be a set \( S \subseteq \mathbb{N} \). Then one can also ask for generating functions for the set of permutations \( \sigma \) of \( S_n \) such that \( S \cap \{1, \ldots, n\} \subseteq \text{Des}(\sigma) \) or \( S \cap \{1, \ldots, n\} = \text{Des}(\sigma) \). For example, if \( O \) equals the odd numbers, then saying \( S \cap \{1, \ldots, n\} = \text{Des}(\sigma) \) implies that \( \sigma \) is an up-down permutation or alternating permutations. It is well known that the exponential generating function of the set of alternating permutations of odd length is \( \tan(x) \) and the exponential generating function of the set of alternating permutations of even length is \( \sec(x) \). The main goal of this chapter is to outline some methods that will allow us to find generating functions like (1)-(5) for permutations \( \sigma \) of \( S_n \) such that \( S \cap \{1, \ldots, n\} = \text{Des}(\sigma) \) where \( S \) is of the form \( k\mathbb{N} \setminus T \) where \( T \) is finite set and \( k\mathbb{N} = \{k, 2k, 3k, \ldots\} \).

The outline of this chapter is as follows. In section 4.1, we shall explain, with an example, how to find generating functions for permutations \( \sigma \in S_n \) such that \( \text{Des}(\sigma) = k\mathbb{N} \cap \{1, \ldots, n\} \) based on the ideas of Mendes [30]. Then in section
4.2, we shall describe how we can combine methods from the previous chapters 2 and 5 to find generating functions for permutations $\sigma \in S_n$ such that $Des(\sigma) = (k\mathbb{N} \setminus T) \cap \{1, \ldots, n\}$ where $T$ is any finite subset of $k\mathbb{N}$.

### 4.1 Permutations that have descents at $k\mathbb{N}$

In this section, we shall use ideas of Mendes [30] to find the generating functions for permutations $\sigma \in S_n$ such that $i \in Des(\sigma)$ if and only if $i \equiv 0 \mod k$. We will give an example in the case where $k = 3$. Let $S_n^{(3)}$ denote the set of all permutations $\sigma$ of $S_n$ such that $i \in Des(\sigma)$ iff $i \in k\mathbb{N}$ and let $A_{n,3} = |S_n^{(3)}|.$

First we let $\xi^{(3)}$ denote the ring homomorphism which maps $\Lambda$ into the rationals $\mathbb{Q}$ defined by

\[
\begin{align*}
\xi^{(3)}(e_0) &= 1 \\
\xi^{(3)}(e_k) &= 0 \text{ if } k = 3n + 1 \text{ or } k = 3n + 2 \\
\xi^{(3)}(e_k) &= \frac{(-1)^{3n}}{3n!}(-1)^n \text{ if } k = 3n > 0.
\end{align*}
\]

We will also consider two weighting functions $\vec{u} = (u_1, u_2, \ldots)$ where $u_i = i$ for all $i$ and $\vec{v} = (v_1, v_2, \ldots)$ where $v_i = i(i-1)$ for all $i$.

Then we claim that we have the following result.

**Theorem 4.1.1.**

1. $(3n)!\xi^{(3)}(h_{3n}) = A_{3n,3}^n$ for $n \geq 1$ and $(3k)!\xi^{(3)}(h_k) = 0$ if $k$ is not equivalent to 0 mod 3.

2. $(3n + 1)!\xi^{(3)}(p_{3n+3}^\vec{v}) = A_{3n+1,3}$ for $n \geq 1$ and $(3k)!\xi^{(3)}(p_{k}^\vec{v}) = 0$ if $k$ is not equivalent to 0 mod 3.

3. $(3n + 2)!\xi^{(3)}(p_{3n+3}^\vec{u}) = A_{3n+2,3}$ for $n \geq 1$ and $(3k)!\xi^{(3)}(p_{k}^\vec{u}) = 0$ if $k$ is not equivalent to 0 mod 3.

**Proof.** We start by proving part (1). Using (1.2.2), we have that

\[
\xi^{(3)}(h_k) = \sum_{\mu \vdash k} (-1)^{k-\ell(\mu)} B_{\mu,k} \xi^{(3)}(e_\mu).
\]
Clearly, if \( k \) is not equivalent to 0 mod 3, then every partition \( \mu \) of \( k \) must have a part which is not equivalent of 0 mod 3 and hence \( \xi^{(3)}(e_\mu) = 0 \). Thus if \( k \) is not equivalent 0 mod 3, then \( \xi^{(3)}(h_k) = 0 \). A similar argument will show that if \( k \) is not equivalent to 0 mod 3, then \( \xi^{(3)}(p_k^\mathbb{Z}) = \xi^{(3)}(p_k^\mathbb{Z}) = 0 \).

Given a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of \( n \), we let \( 3\lambda = (3\lambda_1, \ldots, 3\lambda_k) \). Clearly the only partitions of \( 3n \) such that \( \xi^{(3)}(e_\mu) \neq 0 \) are partitions where \( \mu = 3\lambda \) for some partition \( \lambda \) of \( n \). Thus we can use (1.2.2) to conclude that

\[
(3n)!^3\lambda(3n) = (3n)! \sum_{\lambda \vdash n} (-1)^{3n-\ell(\lambda)} B_{3\lambda,(3n)} \xi^{(3)}(e_{3\lambda})
\]

\[
= (3n)! \sum_{\lambda \vdash n} (-1)^{3n-\ell(\lambda)} B_{3\lambda,(3n)} \prod_{i=1}^{\ell(\lambda)} (-1)^{3\lambda_i} (3\lambda_i)! (-1)^{\lambda_i}
\]

\[
= \sum_{\lambda \vdash n} B_{3\lambda,(3n)} \left( \begin{array}{c} 3n \\ 3\lambda_1, \ldots, 3\lambda_{\ell(\lambda)} \end{array} \right) \prod_{i=1}^{\ell(\lambda)} (-1)^{\lambda_i-1}. \tag{4.1.1}
\]

Next we interpret the left-hand side of (4.1.1) as weighted sum of objects. That is, given brick tabloids \( T \in B_{3\lambda,(3n)} \), we write \( T = (b_1, \ldots, b_{\ell(\lambda)}) \) if the sizes of the bricks in \( T \), reading from left to right, are \( b_1, \ldots, b_{\ell(\lambda)} \). Then clearly, we have \( \left( \begin{array}{c} 3n \\ 3\lambda_1, \ldots, 3\lambda_{\ell(\lambda)} \end{array} \right) = \left( \begin{array}{c} 3n \\ b_1, \ldots, b_{\ell(\lambda)} \end{array} \right) \) so that we can think of the binomial coefficient \( \left( \begin{array}{c} 3n \\ b_1, \ldots, b_{\ell(\lambda)} \end{array} \right) \) as choosing sets of size \( b_1, \ldots, b_{\ell(\lambda)} \) from \( \{1, \ldots, 3n\} \) to place in the bricks of \( T \) in such a way that the elements within a brick are in increasing order and each number is used only once. Thus a typical filled brick tabloid \( F \) that we would produce in the case where \( n = 5 \) is pictured in Figure 4.1. Then the factor \( \prod_{i=1}^{\ell(\lambda)} (-1)^{\lambda_i-1} \) allows us to place a \(-1\) in every third square in a brick except for the last cell.

We can then define the weight of the filled brick tabloid \( F \) to be the product of the \(-1\)'s on top of the cells in \( F \). Hence the filled brick tabloid in Figure 4.1 has weight 1. In this way, we can interpret \( (3n)!^3\lambda(3n) \) as the sum of the weights \( W(F) \) over the set \( \mathcal{F}BT_{3n} \) of all filled brick tabloids \( F = (B_1, \ldots, B_{\ell(\lambda)}) \) of shape \( 3n \) such that size of each brick \( B_i \) is a multiple of 3.

We can define a involution \( I \) on \( \mathcal{F}BT_{3n} \) as follows. We scan the filled brick tabloid \( F \) from left to right until either

(i) we find a cell which has a \(-1\) on top in which case we split the brick into two
bricks at that cell and change the $-1$ to $1$ or

(ii) we find two consecutive bricks $B_i$ and $B_{i+1}$ such that the elements in these
bricks form an increasing sequence reading from left to right in which case we
combine the two bricks into a single brick $B$ and place a $-1$ on top of the last cell
of $B_i$.

If both (i) and (ii) fail, then we define $I(F) = F$. For example, if $F$ is the filled
brick tabloid in Figure 4.1, then $I(F)$ is the filled brick tabloid pictured in Figure
4.2.

It is easy to see that $I$ is an involution which proves that

$$(3n)!ξ^{(3)}(h_{3n}) = \sum_{F ∈ FBT_{3n}, I(F) = F} W(F).$$

Thus we must examine the fixed points of $I$. Note that $I(F) = F$, then $F$ must
consist entirely of bricks of size 3 since any brick of size greater than 3 must have at
least one cell which has a $-1$ on top. Moreover there must be a decrease between
any two bricks. Thus $W(F) = 1$ and the permutation $σ(F)$ that we obtain by
reading the entries of $F$ from left to right must have $Des(σ_F) = \{3, 6, \ldots, 3(n-1)\}.
Hence $(3n)!ξ^{(3)}(h_{3n}) = A_{3n,3}$ as desired.

For part 2, we can use the same reasoning to show that
\begin{align*}
(3n + 1)! \xi^{(3)}(p_{3n+3}^\pi) &= (3n + 1)! \sum_{\lambda \vdash n+1} (-1)^{3n+3 - \ell(\lambda)} w_\pi(B_{3\lambda,(3n+3)}) \xi^{(3)}(e_{3\lambda}) \\
&= (3n + 1)! \sum_{\lambda \vdash n+1} (-1)^{3n - \ell(\lambda)} w_\pi(B_{3\lambda,(3n+3)}) \times \\
&\quad \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{3\lambda_i}(-1)^{\lambda_i}}{(3\lambda_i)!} \\
&= \sum_{\lambda \vdash n+1} w_\pi(B_{3\lambda,(3n+3)}) \frac{(3n + 1)!}{(3\lambda_1)! \cdots (3\lambda_{\ell(\lambda)})!} \times \\
&\quad \prod_{i=1}^{\ell(\lambda)} (-1)^{\lambda_i - 1}. 
\end{align*}

(4.1.2)

Once again we interpret the left-hand side of (4.1.2) as a weighted sum of objects. That is, if \( T = (b_1, \ldots, b_{\ell(\lambda)}) \), then

\[
\frac{(3n + 1)!}{(3\lambda_1)! \cdots (3\lambda_{\ell(\lambda)})!} = \frac{b_{\ell(\lambda)}(b_{\ell(\lambda)} - 1)}{b_1! \cdots b_{\ell(\lambda) - 1}! b_{\ell(\lambda)}!} = \left(\frac{3n + 1}{b_1, \ldots, b_{\ell(\lambda) - 1}, b_{\ell(\lambda)} - 2}\right).
\]

Thus, in this case, we can think of the binomial coefficient \( \binom{3n+1}{b_1, \ldots, b_{\ell(\lambda)}, b_{\ell(\lambda) - 2}} \) as choosing sets of size \( b_1, \ldots, b_{\ell(\lambda) - 1}, b_{\ell(\lambda)} - 2 \) to place in the bricks of \( T \) in such a way that the elements within a brick are in increasing order and the last two cells of the last brick are empty. Thus a typical filled brick tabloid \( F \) that we would produce in the case where \( n = 5 \) is pictured in Figure 4.3.

Then the factor \( \prod_{i=1}^{\ell(\lambda)} (-1)^{\lambda_i - 1} \) allows us to place a \(-1\) in every third square in a brick except for the last cell. We can then define the weight of the filled brick tabloid \( F \) to be the product of the \(-1\)’s on top of the cells in \( F \). Hence the filled brick tabloid in Figure 4.3 has weight \(-1\). In this way, we can interpret \((3n + 1)! \xi^{(3)}(p_{3n+3}^\pi)\) as the sum of the weights \( W(F) \) over the set \( \mathcal{F}BT_{3n+3} \) of all filled brick tabloids \( F = (B_1, \ldots, B_{\ell(\lambda)}) \) of shape \((3n + 3)\) such that the last two cells are empty and the size of each brick \( B_i \) is a multiple of 3. We can then use the exact same involution \( I \) in part (1) to prove that \((3n + 1)! \xi^{(3)}(p_{3n+3}^\pi) = A_{3n+1,3}.\)
Finally the proof a part (3) is similar to the proof of part (2) except in this case we will get filled brick tabloids $F = (B_1, \ldots, B_{\ell(\lambda)})$ of shape $(3n + 3)$ such that the last cell is empty and the size of each brick $B_i$ is a multiple of 3.

\[ \square \]

One can then apply $\xi^{(3)}$ to (1.1.3) and (1.2.4) to prove the following generating functions.

\[
\sum_{n \geq 0} A_{3n, 3}^{3n} \frac{t^{3n}}{(3n)!} = \frac{1}{1 + \sum_{n \geq 1} (-t)^{3n} \frac{t^{3n}}{(3n)!}}.
\]

\[
\sum_{n \geq 1} A_{3n-1, 3}^{3n-1} \frac{t^{3n-1}}{(3n - 1)!} = \frac{\sum_{n \geq 1} (-1)^{n-1} t^{3n-1} \frac{t^{3n-1}}{(3n-1)!}}{1 + \sum_{n \geq 1} (-t)^{3n} \frac{t^{3n}}{(3n)!}}.
\]

\[
\sum_{n \geq 1} A_{3n-2, 3}^{3n-2} \frac{t^{3n-2}}{(3n - 2)!} = \frac{\sum_{n \geq 1} (-1)^{n-2} t^{3n-2} \frac{t^{3n-2}}{(3n-2)!}}{1 + \sum_{n \geq 1} (-t)^{3n} \frac{t^{3n}}{(3n)!}}.
\]

### 4.2 Permutations that have descents at $k\mathbb{N} \setminus A$

In this section, we shall outline how we can obtain generating functions for the number of permutations $\sigma \in S_n$ such that $i \in \text{Des}(\sigma)$ if and only if $i \in k\mathbb{N} \setminus A$ where $A$ is some finite subset of $k\mathbb{N}$. We shall illustrate our example, in the case where $k = 3$ and $A = \{3, 9\}$. First, we let $S_n^{(3,A)}$ denote the set of all $\sigma \in S_n$ such that $i \in \text{Des}(\sigma)$ if and only if $i \in 3\mathbb{N} \setminus A$.

Now suppose that $\alpha = (\alpha_1, \ldots, \alpha_k)$ is composition of $n$. Then define $h_{3\alpha} = h_{3\alpha_1} \cdots h_{3\alpha_k}$ and $\text{Set}(3\alpha) = \{3\alpha_1, 3\alpha_1 + 3\alpha_2, \ldots, 3\alpha_1 + \cdots + 3\alpha_{k-1}\}$. First we define $T_n^{(3,\text{Set}(3\alpha))}$ by declaring that $\sigma \in S_n$ is in $T_n^{(3,\text{Set}(3\alpha))}$ if and only if

\[
(3\mathbb{N} \setminus \text{Set}(3\alpha)) \cap \{1, \ldots, n - 1\} \subseteq \text{Des}(\sigma) \subseteq 3\mathbb{N} \cap \{1, \ldots, n - 1\}.
\]
Let \( \vec{u} = (u_1, u_2, \ldots) \) and \( \vec{v} = (v_1, v_2, \ldots) \) where \( u_i = i \) and \( v_i = i(i-1) \) for all \( i \). Then we claim that we have the following.

**Theorem 4.2.1.** If \( \alpha = (\alpha_1, \ldots, \alpha_k) \) is composition of \( n \), then

1. \((3n)!\xi^{(3)}(h_{3\alpha}) = |T_{3n}^{(3, \text{Set}(3\alpha))}|.\)
2. \((3n + 3s + 1)!\xi^{(3)}(h_{3\alpha} p_{3s+3}) = |T_{3n+3s+1}^{(3, \text{Set}(3\alpha))}|.\)
3. \((3n + 3t + 2)!\xi^{(3)}(h_{3\alpha} p_{3t+3}) = |T_{3n+3t+2}^{(3, \text{Set}(3\alpha))}|.\)

**Proof.** The proof is a variation of the proof of Theorem 4.1.1. For example, for
part (1), suppose that \( \alpha = (2, 3, 2) \). Then clearly \( h_{3\alpha} = h_\gamma \) where \( \gamma = (6, 6, 9) \). Thus

\[
(21)!\xi^{(3)}(h_{3\alpha}) = \sum_{\mu \vdash 7} (-1)^{21-\ell(\mu)} B_{3\mu, \gamma} \xi^{(3)}(e_{3\mu})
\]

\[
= \sum_{\mu \vdash 7} (-1)^{21-\ell(\mu)} \sum_{T = (b_1, \ldots, b_k) \in B_{3\mu, \gamma}} \left( \begin{array}{c} 21 \\ b_1, \ldots, b_k \end{array} \right) \prod_{i=1}^{k} (-1)^{3b_i + b_i/3}
\]

\[
= \sum_{\mu \vdash 7} \sum_{T = (b_1, \ldots, b_k) \in B_{3\mu, \gamma}} \left( \begin{array}{c} 21 \\ b_1, \ldots, b_k \end{array} \right) \prod_{i=1}^{k} (-1)^{(b_i/3)-1}.
\]

The main difference in this case is we start out with a brick tabloid \( T \) of shape \( \gamma \) filled with bricks whose size is a multiple of 3 instead of a brick tabloid \( T \) of shape \((3n)\) filled with bricks whose size is a multiple of 3. However, we can still use the binomial coefficient \( \left( \begin{array}{c} 21 \\ b_1, \ldots, b_k \end{array} \right) \) to fill the bricks with the numbers 1, \ldots, \( n \) so that numbers in each brick increase. For example, Figure 4.4 pictures such two filled brick tabloids. Again we use the factor \( \prod_{i=1}^{k} (-1)^{(b_i/3)-1} \) to place a \(-1\) on top of every third cell of a brick except the final cell and we define the weight \( W(F) \) of such a filled brick tabloid as the product of \(-1\)'s on top of the cells in \( F \).

In this way, we interpret \((3n)!\xi^{(3)}(h_{3\alpha})\) as the sum of the weights \( W(F) \) over the set \( \mathcal{FBT}_{3\alpha} \) of all filled brick tabloids \( F = (B_1, \ldots, B_k) \) of shape \( \gamma \) where \( \gamma \) is the partition induced by the parts of \( 3\alpha \) such that size of each brick is a multiple of 3.

We can then use the same involution \( I \) described in Theorem 4.1.1 except this time we read the bricks in rows from bottom to top and within rows from left
to right to determine which is the first cell which has a $-1$ on top or where two bricks can be combined. We then define $I(F)$ by either breaking a brick into two or combining two consecutive bricks that lie in a row just as in Theorem 4.1.1. For example, for the filled brick tabloid $F$ pictured at the left of Figure 4.4, $I(F)$ is pictured at the right of Figure 4.4 since the first cell with a $-1$ on top in our ordering of cells is the cell containing the number 12.

Then as before, $I$ shows that $(3n)!\xi^{(3)}(h_{3\alpha})$ is the sum over the weights of the fixed points of $I$. Once again, if $I(F) = F$, then $F$ must consist entirely of bricks of size 3 and there must be decreases between any two consecutive bricks that lie in the same row. For example, Figure 4.5 pictures such a fixed point of $I$. The only thing to do is interpret such a fixed point as a permutation. In our case, since $3\alpha = (6,9,6)$, we will read the cells in each row from left to right and read the rows so that we read the top row of size 6, followed by the row of size 9, followed by the bottom row of size 6 to obtain the permutation

$$3 \ 11 \ 14 \ 2 \ 5 \ 6 \ 13 \ 16 \ 15 \ 12 \ 18 \ 20 \ 8 \ 9 \ 21 \ 1 \ 4 \ 10 \ 7 \ 17 \ 19.$$

In general, we make the convention that the rows of the same size in $F$ reading
from top to bottom corresponds to the parts of $3\alpha$ of the same size, reading from left to right. Note that we are guaranteed that there is a descent at each position $i$ which is multiple of 3 unless $i$ corresponds to a cell which at the end of row. In such cases, $i$ is not forced to be the position of descent so that $i$ may or may not be in $Des(\sigma_F)$. In this way, we see that $(3n)!\xi^{(3)}(h_{3n}) = |T_{3n}^{(3,Set(3\alpha))}|$ as desired.

The proof of parts (2) and (3) is similar expect that the row corresponding to $3s + 3$ or $3t + 3$ will have a weight on the last brick which will make sure that one achieves the appropriate binomial coefficient to fill in all but the last one or two cells of the last brick in that row. Also, when we read the permutation off of a fixed point, we will assume that elements in that brick are read last.

Finally, we shall show how we can use ribbon Schur functions to get our desired generating function in the case where $k = 3$ and $A = \{3, 9\}$. We start out by finding the image of $Z_{(3,6,3n)}$ under $\xi^{(3)}$. Expanding $Z_{(3,6,3n)}$ in terms of the homogeneous symmetric functions as we did in section 2 and using Theorem 4.2.1, we can interpret $(9 + 3n)!\xi^{(3)}(Z_{(3,6,3n)})$ as sum of filled special rim hook tabloids of the form pictured in Figure 4.6. That is, it is easy to see that all the special rim hooks of shape $F_{(3,6,3n)}$ must all have sizes which are multiples of 3. Thus, if we read the rim hooks from top to bottom, then we will induce a composition of the form $3\alpha$ for some composition $\alpha$ of $3 + n$. For example, in Figure 4.6, $n = 2$ and reading the rim hooks of the filled special rim hook tabloid at the top, induces the composition $3\alpha = (9, 6)$. Now $(9 + 3n)!\xi^{(3)}(h_{(9,6)})$ equals the number of permutations $\sigma$ of $S_{15}$ such that $\{3, 6, 12\} \subseteq Des(\sigma) \subseteq \{3, 6, 9, 12\}$ so we have filled the special rim hook with such a permutation. On the other hand, reading the rim hooks of the filled special rim hook tabloid which is in the middle of Figure 4.6, induces the composition $(3, 6, 6)$. Now $(9 + 3n)!\xi^{(3)}(h_{(3,6,6)})$ equals the number of permutations $\sigma$ of $S_{15}$ such that $\{6, 12\} \subseteq Des(\sigma) \subseteq \{3, 6, 9, 12\}$ so we can use the same permutation to fill it. Now these two filled special rim hook tabloids only differ in the first vertical segment is part of the first filled special rim hook tabloid while the it is not part of the second filled special rim hook tabloid. Thus these

\[\square\]
two filled special rim hook tabloids will cancel as before.

Figure 4.6: Two canceling filled special rim hook tabloids, and a fixed point of \( I \).

However, if \( \sigma \) is a permutation which has a rise over the first vertical segment like the filled special rim hook tabloid at the bottom of Figure 4.6, then that filled special rim hook tabloid cannot have that vertical segment be part of that special rim hook tabloid. Thus our cancelation leaves us only with filled special rim hook tabloids whose first special rim hook is horizontal and whose permutation has a rise over the first vertical segment. Then we perform a similar cancelation for the remaining filled special rim hook tabloids relative to the second vertical segment to conclude that 

\[
(9 + 3n)!\xi^{(3)}(Z_{(3,6,3n)}) = |\{\sigma \in S_{9+3n} : Des(\sigma) = \{6\} \cup \{3k : 12 \leq k \leq n - 1\}||
\]

Continuing on this way, we can prove that

\[
(9 + 3n)!\xi^{(3)}(Z_{(3,6,3n)}) = |\{\sigma \in S_{9+3n} : Des(\sigma) = \{6\} \cup \{3k : 12 \leq k \leq n - 1\}||
\]
In general, we can prove

**Theorem 4.2.2.**

\[(3\alpha + 3n)! \xi^{(3)}(Z_{(3\alpha, 3n)}) = |\{\sigma \in S_{3|\alpha|+3n} : \text{Des}(\sigma) = \{3k : k = 1, \ldots, |\alpha| + n - 1\} - \text{Set}(3\alpha)\}|\]

Then we can apply \(\xi^{(3)}\) to both sides of (2.1.1) to get our desired generating function.

To obtain generating functions for lengths which are not divisible by 3, we have to use a variant of the ribbon Schur functions. That is, we will use the same collection of special rim hook tabloids that would appear in the expansion of \(Z_{(3\alpha, 3n+3)}\) in terms of homogeneous symmetric functions. However, we must weight each such special rim hook tabloid differently. That is, in the usual expansion of \(Z_{(3\alpha, 3n+3)}\) each special rim hook tabloid \(T\) whose hook lengths are \((a_1, \ldots, a_r)\) reading from top to bottom is weighted with \(\text{sgn}(T)h_{a_1} \cdots h_{a_r}\). If we are interested in lengths which are equivalent to 1 mod 3, then we will weight \(T\) with \(\text{sgn}(T)h_{a_1} \cdots h_{a_{r-1}}p_{a_r}^{\vec{v}}\) so that when we apply \(\xi^{(3)}\), we can pick up the proper weight for the last special rim hook which has the effect of allowing us to have the last two cells of the special rim hook empty. Similarly, if we are interested in lengths which are equivalent to 2 mod 3, then we will weight \(T\) with \(\text{sgn}(T)h_{a_1} \cdots h_{a_{r-1}}p_{a_r}^{\vec{u}}\) so that when we apply \(\xi^{(3)}\), we can pick up the proper weight for the last special rim hook which has the effect of allowing us to have the last cell of that special rim hook empty.

More formally, we will define a modified skew-Schur function

\[s_{\lambda/\mu}^{\vec{g}} = \sum_{(g_1, g_2, \ldots, g_k) \in SRT(\lambda/\mu)} \left( \prod_{i=1}^{k-1} \text{sgn}(g_i) h_{g_i} \right) \text{sgn}(g_k) p_{g_k}^{\vec{g}}\]

and

\[s_{\lambda/\mu}^{\vec{v}} = \sum_{(g_1, g_2, \ldots, g_k) \in SRT(\lambda/\mu)} \left( \prod_{i=1}^{k-1} \text{sgn}(g_i) h_{g_i} \right) \text{sgn}(g_k) p_{g_k}^{\vec{v}}\]

We then apply this definition to the specific case when \(s_{\lambda/\mu}\) is a ribbon.
From this definition, we would like to continue as before, and find a theorem analogous to Theorem 2.1.1, to which we then apply our homomorphisms. We will present our results on this matter in a forthcoming paper.
Chapter 5

Generating functions for permutations with repeating patterns

In this chapter, we will modify pieces of the machinery described in the chapters above. These modifications will give us a large number of new generating functions, which we group into 3 sections based on the modifications involved. In general, for each main type of modification, we will prove only one general example of the idea.

The three types of modifications are:

1. Nonconsecutive descents between the largest elements between repeating patterns,

2. Consecutive descents for patterns whose smallest element is at the end, and

3. Consecutive descents for patterns whose largest element is at the end.
5.1 Generating functions for consecutive descents between the largest elements between repeating patterns.

In this section we will draw on the ideas in Mendes’ work [30]. Our permutations will be of length $kn$, and as in [30], we will be working with blocks of length $k$ as our smallest unit. However, in the proof, instead of inserting natural numbers in decreasing order, we will insert them in a patterned order. The example we will demonstrate is inserting the numbers as $r − 1$ consecutive rises, followed by $s$ consecutive descents, where $r + s = k$. We will also be finding the generating function for a different notion of descents. Rather than counting descents between consecutive elements, we will count descents between the peaks of consecutive “blocks” of length $k$.

5.1.1 Our example pattern $A_r$, with $r − 1$ rises followed by $s$ descents.

Let $A_{r,s}$ be the set of all $\sigma \in S_{j(r+s)}$ such that

i) $\sigma_i < \sigma_{i+1}$ if $i \cong 1, \ldots , r − 1 \pmod (r + s),$

ii) $\sigma_i > \sigma_{i+1}$ if $i \cong r, \ldots , r + s − 1 \pmod (r + s)$, and

iii) $\sigma_{ir} > \sigma_{(i+1)r}, 1 \leq i \leq j − 1.$

We also define a block to be a sequence of the form $\sigma_{ki+1}\sigma_{ki+2}\ldots\sigma_{ki+k}$. We call the sequence $\sigma_{ki+1}\sigma_{ki+2}\ldots\sigma_{ki+k}$ the $i$th block. We define the peak of a block to be the $r$th element of the block.

To describe this in words, we would like permutations that have $r$ elements increasing, followed by $s$ elements decreasing. This forms one $r + s$ block, with a “peak” at position $r$. We then place $j$ of these patterns in a row, with the extra restriction that the “peak” of each block be larger than the peak that follows it. This is illustrated in Figures 5.1 and 5.2.

Define $A_j = |A_{r,j}|$. 
How many permutations of $S_{r+s}$ satisfy the restrictions shown in 5.1? Clearly the largest element is located at the peak, which is position $r$, so $\sigma_r = r + s$. Then we have $r+s-1$ elements left, and we need to choose $s$ of them to put in decreasing order for $\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_{r+s}$. The rest go in increasing order for $\sigma_1, \ldots, \sigma_{r-1}$. So

$$A_1 = \binom{r+s-1}{s}$$

and

$$\frac{A_1}{(r+s)!} = \frac{1}{(r+s)(r-1)!s!}$$

We can obtain a formula for $A_n$ by recursion. The largest element of the permutation is at position $r$, so we have $n(r+s) - 1$ numbers left to choose from. Then to choose the other $r+s-1$ elements, we choose $r-1$ for the left side ($\sigma_1, \ldots, \sigma_{r-1}$), and we choose $s$ for the right side ($\sigma_{r+1}, \ldots, \sigma_{r+s}$). Then we have $(n-1)(r+s)$ numbers left for the rest of the permutation. Thus
\[ A_n = \left( \frac{n(r+s)-1}{r-1,s,(n-1)(r+s)} \right) A_{n-1} \]

and

\[
\frac{A_n}{(n(r+s))!} = \frac{1}{(r-1)!s!(n(r+s))\left((r+s)(n-1)!\right)} \cdot \frac{A_{n-1}}{1}
= \frac{1}{(r+s)^n((r-1)!)^n(s!)^n n!}
\]

**Lemma 5.1.1.** Let \( r, s, k \) be positive integers with \( r+s = k \) and \( \mu \) be a partition of \( n \). Let \( T \) be a brick tabloid \((b_1, b_2, \ldots, b_{l(\mu)}) \in B_{\mu,n}\). Then

\[
\prod_{i=1}^{l(\mu)} A_{b_i} \left( \begin{array}{c} kn \\ kb_1, kb_2, \ldots, kb_{l(\mu)} \end{array} \right)
\]

is the number of ways to fill a brick tabloid \( kT = (kb_1, kb_2, \ldots, kb_{l(\mu)}) \) with permutations \( \sigma \) such that for each brick \( kb_i \), the elements of \( \sigma \) follow the pattern \( A_{b_i} \).

**Proof.** We first choose which numbers from \( \{1, \ldots, kn\} \) will be in each brick, which is counted by our multinomial coefficient. Then, given those numbers, the \( A_{b_i} \) counts the number of ways to arrange those numbers that follow our pattern \( A_{b_i} \). \( \square \)

We define a homomorphism

\[ \Theta^{r,s} : \Lambda \rightarrow \mathbb{Q}[x] \]

so that

\[
\Theta^{r,s}(e_{(r+s)n}) = \frac{(-1)^{(r+s)n-1}(x-1)^{n-1}}{((r+s)n)!} A_n
= \frac{(-1)^{(r+s)n-1}(x-1)^{n-1}}{(r+s)^n((r-1)!)^n(s!)^n n!}
\]

and \( \Theta^{r,s}(e_k) = 0 \) if \( k \not\equiv 0 \mod r+s \).
Define $S_{(r+s)n}^r$ be those $\sigma \in S_{(r+s)n}$ such that each block follows the pattern $A_1$.

Define $\text{des}_{r,s}(\sigma) = |\{\sigma_{i(r+s)+r} > \sigma_{(i+1)(r+s)+r}, 0 \leq i \leq n-1\}|$.

**Theorem 5.1.2.**

$$((r+s)n)! \Theta^{r,s}(h_{(r+s)n}) = \sum_{\sigma \in S_{(r+s)n}^r} x^{\text{des}_{r,s}(\sigma)}.$$

**Proof.**

$$((r+s)n)! \Theta^{r,s}(h_{(r+s)n}) = ((r+s)n)! \sum_{\mu \vdash (r+s)n} (-1)^{(r+s)n-l(\mu)} B_{\mu,(r+s)n} \Theta^{r,s}(e_\mu).$$

But $\Theta^{r,s}(e_\mu) = 0$ unless each part of $\mu$ is a multiple of $r+s$, so we can restrict ourselves to those $\mu = (r+s)\lambda$ for some $\lambda \vdash n$.

$$= ((r+s)n)! \sum_{\mu \vdash n} (-1)^{(r+s)n-l(\mu)} B_{(r+s)\mu,(r+s)n} \Theta^{r,s}(e_{(r+s)\mu})$$

$$= ((r+s)n)! \sum_{\mu \vdash n} (-1)^{(r+s)n-l(\mu)} \sum_{T=(b_1, \ldots, b_l(\mu)) \in B_{\mu,n}} \prod_{i=1}^{l(\mu)} \frac{(-1)^{b_{i-1}(x-1)b_{i-1}}}{(r+s)b_i \cdot ((r-1)!)^{b_i} b_i!}$$

$$= ((r+s)n)! \sum_{\mu \vdash n} (-1)^{rn+sn-l(\mu)} \times$$

$$\sum_{T=(b_1, \ldots, b_l(\mu)) \in B_{\mu,n}} (-1)^{sn-l(\mu)}(x-1)^{n-l(\mu)} \prod_{i=1}^{l(\mu)} \frac{A_{b_i}}{(r+s)b_i!}$$

$$= \sum_{\mu \vdash n} \sum_{T=(b_1, \ldots, b_l(\mu)) \in B_{\mu,n}} (x-1)^{n-l(\mu)}(r+s)^n \prod_{i=1}^{l(\mu)} \frac{A_{b_i}}{(r+s)b_i \cdot \ldots \cdot (r+s)b_{l(\mu)}}.$$

Next we interpret this as a weighted sum of objects, and we will call the set of objects $\emptyset$. By Lemma 5.1.1, we can think of this sum as all allowable fillings of all possible brick tabloids following our pattern, which we then weight with the $(x-1)^{n-l(\mu)}$ term. This terms allows us to place either an $x$ or a $-1$ on every $k^{th}$ (recall $k = r+s$) cell in each brick, except for the final cell in each brick.
We then define the weight of this labeled brick tabloid to be the product of the 
−1s and xs on the cells. Hence the brick tabloid in 5.3 has weight −x. Thus 

\((r + s)n!\Theta^n(r_{(r+s)n})\) is the sum of the weights of objects in \(\mathcal{O}\).

![Figure 5.3: An example of a labelled \(A\)-filled tabloid.](image)

As in previous chapters, we define an involution on \(\mathcal{O}\) as follows. We scan the labeled brick tabloid from left to right until either:

i) we find a cell which has a −1 on top in which case we split the brick into two pieces immediately after that cell and change the −1 to a 1, or

ii) we find two consecutive bricks \(B_i, B_{i+1}\) such that the final peak of \(B_i\) is greater than the first peak of \(B_{i+1}\), in which case we combine these two bricks into a single brick and replace the 1 on top of the last cell of \(B_i\) with a −1.

If neither case i) or ii) happens, we define \(I(o) = o\). For example, if \(o\) is the labeled tabloid in 5.3, then \(I(o)\) is pictured in 5.4. Looking at 5.4, we see that since \(\sigma_6 = 15\) which is larger than \(\sigma_{10} = 12\), then \(I(I(o)) = o\) as expected.

![Figure 5.4: The image of 5.3 under the involution \(I\).](image)

As with our previous proofs, we must now examine the fixed points of \(I\). None of the fixed points have a label of −1, and for every brick, the final peak is less than the first peak in the following brick. An example of such a fixed point is given in 5.5.
It is clear that the \( x \) labels are over exactly those blocks for which the next peak is filled by a smaller number. Thus

\[
((r + s)n)!\Theta^{r,s}(h_{(r+s)n}) = \sum_{\sigma \in S_{(r+s)n}^{r,s}} x^{\text{des},r,s}(\sigma).
\]

**Theorem 5.1.3.**

\[
1 + \sum_{n \geq 1} \frac{t^{(r+s)n}}{((r + s)n)!} \sum_{\sigma \in S_{(r+s)n}^{r,s}} x^{\text{des},r,s}(\sigma) = \frac{1 - x}{-x + e^{\exp\left(\frac{(r+s)(x-1)}{(r+s)(r-1)!s!}\right)}}.
\]

**Proof.**

\[
1 + \sum_{n \geq 1} \frac{t^{(r+s)n}}{((r + s)n)!} \sum_{\sigma \in S_{(r+s)n}^{r,s}} x^{\text{des},r,s}(\sigma) = 1 + \sum_{n \geq 1} t^{(r+s)n} \Theta^{r,s}(h_{(r+s)n})
\]

\[
= \Theta^{r,s}(H(t)) = \frac{1}{\Theta^{r,s}(E(-t))}
\]

Examining the denominator of this expression:

\[
\Theta^{r,s}(E(-t)) = 1 + \sum_{n \geq 1} \frac{(-t)^{(r+s)n} \Theta^{r,s}(e_{(r+s)n})}{(r+s)^n((r-1)!s!)^nn!}
\]

\[
= 1 + \sum_{n \geq 1} \frac{(-t)^{(r+s)n}(-1)^{(r+s)n-1}(x - 1)^{n-1}}{(r + s)^n((r-1)!)^nn!}
\]

\[
= 1 + \sum_{n \geq 1} \frac{-t^{(r+s)n}(x - 1)^{n-1}}{((r + s)(r-1)!s!)^nn!}
\]
\[
= \frac{1}{1-x} \left( 1 - x + \sum_{n \geq 1} \frac{t(r+s)n(x-1)^n}{((r+s)(r-1)!s!)^n n!} \right)
= \frac{1}{1-x} \left( -x + e^{\frac{t(r+s)(x-1)}{(r+s)(r-1)!s!}} \right)
\]

So

\[
1 + \sum_{n \geq 1} \frac{t(r+s)n}{((r+s)n)!} \sum_{\sigma \in S_{(r+s)n}^r} x^{\text{des}_{r,s}(\sigma)} = \frac{1 - x}{-x + exp\left(\frac{(-t)^r s(x-1)}{(r+s)(r-1)!s!}\right)}.
\]

We will now examine an example of this when \( r = 2 \) and \( s = 1 \). Our generating function becomes:

\[
\frac{1 - x}{-x + exp\left(\frac{t}{3(x-1)}\right)}
\]

The \( n \)th term of the exponential has coefficient of \( \frac{1}{3^n n!} \). But \( 3^n n! \) can be rewritten as \((3n)(3n-3) \cdots (3)\) which can be thought of as a “3 analogue” of a factorial. However, if we let \( r = 3 \) and \( s = 1 \), we do not get a “4 analogue”, because of the \( r - s! \) term. In Subsection 5.1.3, we show a variation of our example that has generating functions with these k-factorials in them.

5.1.2 \( q \)-Analogue of pattern \( A \) example.

In this section, we give an outline of the proof from the previous subsection extended to \( q \)-analogue. Of course we could also extend it to \( p, q \)-analogue, just as we did in Chapter 2; however, here we only present the \( q \)-analogue.

Define \( A_k(q) = \sum_{\sigma \in S_k} q^{\text{inv}(\sigma)} \).

We will proceed as before, finding a closed form for \( A_1(q) \), and then finding a recursive formula for \( A_k(q) \), and then solving the recursion.

Recall that \( \text{inv}(\sigma) = \sum_{i<j} \chi(\sigma_i > \sigma_j) \). As before, the largest element of our permutation is \( \sigma_r \), so we only need to place the other \( r + s - 1 \) elements and count
the inversions. Then for \( A_1(q) \), the last \( s + 1 \) elements form a decreasing sequence, so the inversions among the last \( s + 1 \) elements are given by \( q^{\binom{s+1}{2}} \). The first \( r \) elements have no inversions among themselves because they form an increasing sequence. Therefore we only need to concern ourselves with the inversions between the first \( r - 1 \) and the last \( s + 1 \). Carlitz [11] showed that this is counted by a \( q \)-multinomial coefficient. Thus

\[
A_1(q) = \binom{r + s - 1}{r - 1, s}_q q^{\binom{s+1}{2}}
\]

and

\[
\frac{A_1(q)}{[r + s]_q!} = \frac{q^{\binom{s+1}{2}}}{[r + s]_q[r - 1]_q!![s]_q!}.
\]

Then for our recursion for \( A_n(q) \), we have three parts:

- the inversions between the first \( r + s \) elements and the other \((n - 1)(r + s)\) elements,
- the inversions among elements in the first \( r + s \) elements,
- and the inversions among the last \((n - 1)(r + s)\) elements.

We will first consider inversions between the first \( r + s \) and the other elements. We must first \( q \)-choose which \( r + s \) elements we will be choosing \( \sigma_1, \ldots, \sigma_{r+s} \) from. As before, \( \sigma_r = n(r + s) \), so we are \( q \)-choosing \( r + s - 1 \) from \( n(r + s) - 1 \), and the \( q \)-multinomial coefficient counts this [11]. We still need to count the inversions between \( \sigma_r \) and \( \sigma_{r+s+1}, \ldots, \sigma_{n(r+s)} \), which equals \( q^{\binom{n-1}{r+s}} \). Next, inversions among the first \( r + s \) elements are counted by \( \binom{r+s-1}{r+s}_q q^{\binom{s+1}{2}} \), as in our calculation of \( A_1(q) \). Finally, inversions among the last \((n - 1)(r + s)\) elements are given by \( A_{n-1}(q) \). Thus,

\[
A_n(q) = \binom{n(r + s) - 1}{r + s - 1}_q q^{\binom{n-1}{r+s}}\binom{r + s - 1}{s}_q q^{\binom{s+1}{2}}(A_{n-1}(q)).
\]
Then

\[
\frac{A_n(q)}{[n(r + s)]_q!} = \frac{q^{\binom{n+1}{2}}q^{(n-1)(r+s)}}{[n(r + s)]_q!} \frac{[r + s - 1]_q!}{[s]_q! [r - 1]_q!} \frac{[n(r + s) - 1]_q!}{(n-1)(r + s)]_q!} A_{n-1}(q)
\]

\[
= \frac{q^{\binom{n+1}{2}}q^{(n-1)(r+s)}}{[n(r + s)]_q[s]_q! [r - 1]_q!} \frac{[n(r + s)]_q!(n-1)(r + s)]_q!}{[(n-1)(r + s)]_q!}
\]

\[
= \left(\frac{q^{\binom{n+1}{2}}}{[s]_q! [r - 1]_q!}\right)^n q^{(r+s)\binom{n}{2}} \frac{1}{[n(r + s)]_q!(n-1)(r + s)]_q!} \cdots [1(r + s)]_q!
\]

where the final equality holds by a simple manipulation of the \(q\)-series factoring differently.

Then we proceed as before, defining a homomorphism based on this formula for \(A_n(q)\).

\[
\Theta_{r,s}^e(r+s)_n = \frac{(-1)^{(r+s)n-1}(x - 1)^{n-1}A_n(q)}{[(r + s)n]_q!}
\]

and \(\Theta_{r,s}^e(k) = 0\) if \(k \not\equiv 0 \mod r + s\).

Note that the involution is unchanged, because the only thing that changes in our \(\text{AFT}\) are the \(-1\) and 1 labels. The underlying permutations do not change, so the inversion count never changes between \(o\) and \(I(o)\).

We are then able to obtain generating functions for \(x^{\text{des}_{r,s}(\sigma)} q^{\text{inv}(\sigma)}\). Below we give the resulting generating function. Define

\[
\text{exp}_q^{r,s}(u) = \sum_{n \geq 0} \frac{u^n}{[n]_{q^{r+s}}!} q^{(r+s)\binom{n}{2}}.
\]

Then

\[
1 + \sum_{n \geq 1} \frac{t^{(r+s)n}}{[(r + s)n]_q!} \sum_{\sigma \in S_{(r+s)n}^{r,s}} x^{\text{des}_{r,s}(\sigma)} q^{\text{inv}(\sigma)} = \frac{1 - x}{-x + \text{exp}_q^{r,s}(t^{(r+s)(x-1)q^{\binom{s+1}{2}}})}.
\]
We will present the details for obtaining these generating functions and the associated \( p, q \)-analogues in a future paper.

### 5.1.3 Other patterns

In the previous discussion, we chose to describe the machinery modification using a particular example, pattern \( A \). However, there are a huge number of other patterns we could have chosen. The main requirement for this method to work is that we have a pattern where the position of the largest element is obvious. Also we require a simple recursion that yields a formula for the number of patterns of length \( kn \) for \( n \geq 1 \).

Below, we present figures depicting some of the variety of patterns whose generating functions can be obtained using this method and outline the changes they require.

Let \( A_j^{(1)} \) be the pattern pictured below with \( j \) blocks of length 5, where the first element in each block is the largest, and peaks in each block form a decreasing sequence.

![Figure 5.6: 15 8 3 11 12 14 2 4 7 10 9 1 5 6 ∈ A_3^{(1)}.](image)

We next show the closed form solution to the recursion for this pattern. Define \( A_j^{(1)} = A_j^{(1)} \). Then \( A_1^{(1)} = 3 \). For \( A_n^{(1)} \), we build our recursion by examining the 5 left-most elements. Clearly \( \sigma_1 \) is the largest element of the entire permutation, so \( \sigma_1 = 5n \). For the other 4 elements, we choose which numbers we will use with \( \binom{5n-1}{4} \). Then for those chosen 4 elements, we have 3 ways we can arrange them. Thus
\[ A_n^1 = \binom{5n - 1}{4} \cdot 3A_{n-1}^1 \]
\[ = \frac{(5n - 1)(5n - 2)(5n - 3)(5n - 4)}{4 \cdot 3 \cdot 2 \cdot 1} A_{n-1}^1 \]
\[ = \frac{(5n)(5n - 1) \cdots (5n - 4)}{2^3 \cdot 5n} A_{n-1}^1 \]
\[ = \frac{5n!}{2^{3n}5^n n!}. \]

Now we alter our homomorphism to reflect the pattern \( A^{(1)} \). Define

\[ \Theta A^1(e_{5m}) = \frac{(-1)^{5m-1}}{5m!}(x - 1)^m A_m^{(1)}. \]

Then \((5m)!\Theta A^1(h_{5m}) = \sum_{\sigma \in S_{5m, A^{(1)}}} x^{d e s_5(\sigma)}\), where \( d e s_5 \) is the descents between peaks for each block of size 5 in this example.

The resulting generating function:

\[ 1 + \sum_{n \geq 1} \frac{t^{5n}}{5n!} \sum_{\sigma \in S_{5n, A^{(1)}}} x^{d e s_5(\sigma)} = \frac{1}{1 + \sum_{n \geq 1} \frac{(-t)^{5n}(1)^{n-1}}{40^n n!} (x - 1)^{n-1}} \]
\[ = \frac{1 - x}{-x + \exp\left(\frac{t(x-1)}{40}\right)}. \]

These cases all had the elements inserted into the bricks with restrictions on the relative sizes of consecutive elements in our blocks. Here we show that we also could have loosened that restriction and instead filled a brick of length \( mk \) with subsequences of length \( k \) with largest element first, and the next \( k - 1 \) without any decreasing/increasing restrictions. Then for our pattern \( \Lambda_m^{(2)} \), we must have decreases between the first elements of each block.

Let \( A_m^{(2)} = |\Lambda_m^{(2)}| \). Then it is easy to see that \( A_1^{(2)} = (k - 1)! \) so that \( \frac{A_m^{(2)}}{k!} = \frac{1}{k} \). Then
\[
A_n^{(2)} = \binom{kn - 1}{k - 1}(k - 1)!A_{n-1}^{(2)} \text{ and }
\]
\[
\frac{A_n^{(2)}}{(kn)!} = \frac{A_{n-1}^{(2)} kn}{kn} = \frac{1}{k^n n!}
\]

Figure 5.7: An example of a more unrestricted pattern. Note that the blocks do not have a repeated pattern except for the first element largest.

In our first example, we accounted for the increasing and decreasing sequences with our multinomial coefficients and \(a_k\)'s. We need to modify that for this case, which we do by altering our homomorphism. We also modify our notion of descent to compare the first elements of each \(k\)-subsequence, which we will call \(des_k\).

\[
\Theta^{(2)}(e_{kn}) = \frac{(-1)^{kn-1}(x-1)^{n-1}}{k^n n!}.
\]

Then the resulting generating function is:

\[
1 + \sum_{n \geq 1} \frac{t^{kn}}{kn!} \sum_{\sigma \in S_{k, kn}} x^{des_k(\sigma)} = \frac{1 - x}{-x + exp(\frac{t^{k(x-1)}}{k})}
\]

where \(S_{k, kn} \subseteq S_{kn} : \forall i \in \{0, \ldots, n - 1\} \forall j \in \{2, \ldots, k\}, \sigma_{ki+1} > \sigma_{ki+j}\).

Thus we have found the pattern we were looking for at the end of Section 5.1, with the "\(k\)-analogue" of factorial in the denominator.

In a future paper, we hope to classify all patterns whose generating functions can be obtained in this fashion.
5.2 Generating functions for consecutive descents where the smallest element is at the end of the pattern

We will be examining a pattern that is similar to pattern \( A \) from the previous section. However, we will add the restriction that \( \sigma_1 \) be larger than \( \sigma_{r+s} \). This is illustrated in 5.8. This small change will allow us to find generating functions for \( x^{\text{des}(\sigma)} \) instead of \( x^{\text{des},r,s(\sigma)} \).

5.2.1 Our example pattern \( \mathbb{C} \)

![Figure 5.8: The shape of a permutation that follows pattern \( \mathbb{C} \).](image)

Define \( \mathbb{C}_k \) to be the set of all \( \sigma \in S_{k(r+s)} \) such that:

i) \( \sigma_i < \sigma_{i+1} \) if \( i \equiv 1, \ldots, r-1 \text{ mod } r+s \),

ii) \( \sigma_i > \sigma_{i+1} \) if \( i \equiv r, \ldots, r+s-1 \text{ mod } r+s \),

iii) \( \sigma_{i(r+s)+1} > \sigma_{(i+1)(r+s)}, 1 \leq i \leq k \), and

iv) \( \sigma_{i(r+s)} > \sigma_{i(r+s)+1}, 1 \leq i \leq k-1 \).

To describe this in words, we would like patterns that have \( r \) elements increasing, followed by \( s \) elements decreasing. This forms one \( r+s \) block, with a “peak” at position \( r \). We also restrict that the first element \( (\sigma_1) \) in each block be larger than the last element \( (\sigma_{r+s}) \). Then place \( k \) of these patterns in a row, with the extra restriction that the last element of each \( r+s \) group \( (\sigma_{i(r+s)}) \) be larger than the element that directly follows it \( (\sigma_{i(r+s)+1}) \). This is illustrated in 5.9. We define \( C_k = |\{\mathbb{C}_k\}| \).
As before, we will first calculate $C_1$, then find a recursion for $C_n$, and try to solve it in closed form. For $C_1$, we can see that $\sigma_r = r + s$, and $\sigma_{r+s} = 1$, so two of our elements are forced from the start. Then from the remaining $r + s - 2$ elements, we choose $s - 1$ for the decreasing sequence $\sigma_{r+1}, \ldots, \sigma_{r+s-1}$, and the rest from the increasing sequence $\sigma_1, \ldots, \sigma_{r-1}$. Thus,

$$C_1 = \binom{r + s - 2}{s - 1}$$

and

$$\frac{C_1}{(r + s)!} = \frac{1}{(r + s)(r + s - 1)(r - 1)!s!(s - 1)!}.$$ 

For our recursion, this time we will be recursing from right to left, instead of left to right as we did in the previous section. From the definition and Figure 5.9, we can see that $\sigma_{n(r+s)} = 1$. Let’s examine $a = \sigma_{(n-1)(r+s)+1}$ (the first element of the final block). We can see that $a$ has $(n - 1)(r + s) + (r - 1)$ elements that must be greater than it and 1 element that must be smaller than it, so that $a \in \{2, \ldots, s + 1\}$.
In Figure 5.10, we show the case where \( a = 2 \). Then we must first choose which other \( r + s - 2 \) elements will be in the final block of \( r + s \) elements. Then the largest of these must be at the peak \( \sigma_{(n-1)(r+s)+r} \), so we simply choose \( r - 2 \) of them to be in our increasing subsequence, and the others fill the decreasing subsequence. Thus when \( a = 2 \),

\[
C_n = \binom{n(r+s) - 2}{r+s-2} \binom{r+s-3}{r-2} C_{n-1},
\]

In Figure 5.11, we show the case where \( a = 3 \). Then we know that \( \sigma_{n(r+s)-1} = 2 \).

![Figure 5.11: \( C_n \), with \( \sigma_{n(r+s)} = 3 \).](image)

So we choose \( r + s - 2 \) from the remaining \( n(r+s) - 3 \) choices. The largest of those will have to be \( \sigma_{(n-1)(r+s)+r} \), so we choose \( r - 2 \) of the remaining \( r + s - 4 \) to be the increasing subsequence. Thus when \( a = 3 \),

\[
C_n = \binom{n(r+s) - 3}{r+s-3} \binom{r+s-4}{r-2} C_{n-1},
\]

More generally, when \( a = k \), we reason in the same manner.

![Figure 5.12: \( C_n \), with \( \sigma_{n(r+s)} = k + 1 \).](image)

When \( a = s + 1 \),
\[ C_n = \binom{n(r+s) - (s-1)}{r+s-(s+1)} \binom{r+s-(s+2)}{r-2} C_{n-1} \]

So for any value of \( a \),

\[ C_n = \left( \sum_{j=2}^{s+1} \binom{n(r+s) - j}{r+s-j} \binom{r+s-j-1}{r-2} \right) C_{n-1} \]

\[ \frac{C_n}{n(r+s)!} = \sum_{j=2}^{s+1} \frac{(n(r+s) - j)!}{(n(r+s))!(r+s-j)(r-2)!(s-j+1)! ((n-1)(r+s))!} C_{n-1} \]

We can see that even with a relatively minor change in the pattern, our formula for the number of these restricted permutations is much more complicated. In fact, the formula becomes quite messy when \( s > 3 \).

We will continue the proof with \( s = 1 \). Then

\[ \frac{C_n}{n(r+1)!} = \frac{(n(r+1) - 2)!}{(n(r+1))!(r+1-2)(r-2)!(1-2+1)! ((n-1)(r+1))!} \]

\[ = \frac{1}{(r-1)^n((r-2))!^n n!(r+1)^n} \times \]

\[ \frac{1}{(n(r+1) - 1)((n-1)(r+1) - 1) \cdots (r+1 - 1)} \]

\[ = \frac{1}{((r-1)!)^n(r+1)^n n! \prod_{i=1}^{n} \frac{1}{i(r+1) - 1}} \]

We define a homomorphism

\[ \hat{\Theta}^{r,1} : \Lambda \to \mathbb{Q}[x] \]

so that
\[ \hat{\Theta}^{r,1}(e_{(r+1)n}) = \frac{(-1)^{(r+1)n-1}(x-1)^{n-1}x^n}{((r+1)!)^n} C_n \]

and \( \hat{\Theta}^{r,1}(e_k) = 0 \) if \( k \not\equiv 0 \mod (r+1) \).

Define \( S_{r,s,n} \subset S_{(r+s)n} \) to be the set of permutations \( \sigma \) such that each block of size \( r+s \) follows the pattern \( C_1 \).

**Theorem 5.2.1.**

\[ ((r+1)n)! \hat{\Theta}^{r,1}(h_{(r+1)n}) = \sum_{\sigma \in S_{r,1,n}} x^\text{des}(\sigma). \]

**Proof.**

\[ ((r+1)n)! \hat{\Theta}^{r,1}(h_{(r+1)n}) = ((r+1)n)! \sum_{\mu \vdash (r+1)n} (-1)^{(r+1)n-l(\mu)} B_{\mu,(r+1)n} \hat{\Theta}^{r,1}(e_{(r+1)\mu}). \]

But \( \hat{\Theta}^{r,1}(e_\mu) = 0 \) unless each part of \( \mu \) is a multiple of \( r+1 \), so we can restrict ourselves to those \( \mu = (r+1)\lambda \) for some \( \lambda \vdash n \).

\[ = \sum_{\mu \vdash n} \sum_{\mu \vdash n} (-1)^{(r+1)n-l(\mu)} B_{\mu,(r+1)n} \hat{\Theta}^{r,1}(e_{(r+1)\mu}) \]

Next we interpret this as a weighted sum of objects, and we will call the set of objects \( \hat{O} \). By 5.1.1, we can think of this sum as all allowable fillings of all possible
brick tabloids following our pattern, which we then weight with the $x^n(x-1)^{n-l(\mu)}$ term. The $x^n$ allows us to label every “peak” cell with an $x$, i.e., every cell equivalent to $r \mod k$. The $(x-1)^{n-l(\mu)}$ term allows us to place either an $x$ or a $-1$ on every $k^{th}$ cell in each brick, except for the final cell. We then define the weight of this labeled brick tabloid to be the product of the $-1$s and $x$'s on the cells. Hence the brick tabloid in 5.13 has weight $x^5$. Thus $(((r+1)n)!\Theta^r.1(h_{(r+1)n})$ is the sum of the weights of objects in $\hat{O}$.

\[
\begin{array}{ccccccc}
\text{x} & 1 & \text{x} & \text{x} & \text{x} & 1 & 1 \\
7 & 11 & 5 & 9 & 10 & 6 & 4 & 12 & 3 & 2 & 8 & 1
\end{array}
\]

Figure 5.13: An example of a labelled $\mathbb{C}$ filled tabloid.

As in previous chapters, we define an involution on $\hat{O}$ as follows. We scan the labeled brick tabloid from left to right until either:

i) we find a cell which has a $-1$ on top in which case we split the brick into two bricks immediately after that cell and change the $-1$ to a 1, or

ii) we find two consecutive bricks $B_i, B_{i+1}$ such that the final cell of $B_i$ is greater than the first cell of $B_{i+1}$, in which case we combine these two bricks into a single brick and replace the 1 on top of the last cell of $B_i$ with a $-1$.

If neither case i) or ii) happens, we define $I(o) = o$. For example, if $o$ is the labeled tabloid in 5.13, then $I(o)$ is pictured in 5.14.

\[
\begin{array}{ccccccc}
\text{x} & 1 & \text{x} & \text{x} & \text{x} & -1 & 1 \\
7 & 11 & 5 & 9 & 10 & 6 & 4 & 12 & 3 & 2 & 8 & 1
\end{array}
\]

Figure 5.14: The image of 5.13 under the involution $I$.

As with our previous proofs, we must now examine the fixed points of $I$. None of the fixed points have a label of $-1$, and for every brick, the final cell is less than the first cell in the following brick. An example of such a fixed point is given in
5.15.

![Figure 5.15: A fixed point of the involution $I$.](image)

It is clear that the $x$ labels are over exactly those cells for which the next cell is filled by a smaller number. Thus

$$(r+1)n!\tilde{\Theta}^{r,1}(h_{(r+1)n}) = \sum_{\sigma \in S_{r,1,n}} x^{\text{des}(\sigma)}.$$  

Note that if we had not limited ourselves to $s = 1$, then we would have built a homomorphism that had a weight of $x^s n$ to distribute on each cell in our decreasing subsequences except the last cell, which we would again weight using the $(x - 1)^{n-l(\mu)}$ labels.

As before, we use our symmetric function identity to find the generating function, except in this section, we find it for descent in the standard sense. Recall that we have simplified this example by restricting to the case where $s = 1$.

**Theorem 5.2.2.**

$$1 + \sum_{n \geq 1} \frac{t^{r+1}n}{((r+1)n)!} \sum_{\sigma \in S_{r,1,n}} x^{\text{des}(\sigma)} = \frac{1 - x}{-x + \sum_{n \geq 0} \frac{(x-1)^{n-1} t^{(r+1)n} \prod_{i=1}^n i(r+1)-1}{((r-1)!(r+1))^n n!}}.$$  

**Proof.**

$$1 + \sum_{n \geq 1} \frac{t^{r+1}n}{((r+1)n)!} \sum_{\sigma \in C_n} x^{\text{des}(\sigma)}$$

$$= \tilde{\Theta}^{r,1}(H(t))$$

$$= \frac{1}{\tilde{\Theta}^{r,1}(E(-t))}$$
\[ \hat{\Theta}^{r,1}(E(-t)) = \sum_{n \geq 0} \hat{\Theta}^{r,1}(e_n)(-t)^n \]

\[ = 1 + \sum_{n \geq 1} \frac{(-1)^{(r+1)n-1}(x-1)^{n-1}C_n(-t)^{(r+1)n}}{((r+1)n)!} \]

\[ = 1 + \sum_{n \geq 1} \frac{-(x-1)^{n-1}(r+1)^n}{((r-1)!(r+1))n! \prod_{i=1}^{n} \frac{1}{i(r+1)-1}} \]

We then use the reciprocal of this expression. Multiplying numerator and denominator by \((1-x)\), we obtain our theorem.

5.2.2 Other patterns with minimal elements at the end

We end this section with a discussion of other patterns that can be counted using the same technique. We note that we could have used almost any pattern whose blocks of size \(k\) have the final cell filled with the smallest number. As an example, consider the pattern \(C^{(1)}\) picture in Figure 5.16.

![Figure 5.16: The shape of permutations in \(C^{(1)}_{3}\).](image)

Define \(C^{(1)}_m = |C^{(1)}_m|\). Then for \(C^{(1)}_1\), only two permutations satisfy our pattern, 5 3 1 2 4, and 5 2 1 3 4, so \(C^{(1)}_1 = 2\). For \(C^{(1)}_n\), we look at the left-most 5 elements. It is obvious that \(\sigma_1 = 5n\) and \(\sigma_5 = 5n - 1\). We choose the remaining 3 elements
\(\sigma_2, \sigma_3, \sigma_4\), from the remaining \(5n - 2\). There are 2 ways to arrange those three. The remaining elements are counted by \(C^{(1)}_{n-1}\).

\[
C^{(1)}_n = \binom{5n - 2}{3} \cdot 2 \cdot C^{(1)}_{n-1} = \frac{5n \cdot 5n - 1 \cdots 5n - 4}{5n \cdot 5n - 1 \cdot 3} C^{(1)}_{n-1} = \frac{5n!}{15^n n! (\prod_{i=1}^n 5n - 1)}
\]

Consequently, our generating function in this case does not simplify to have an exponential in the denominator.

As we saw in the two examples we worked out, sometimes the recursions we can develop for these patterns do not have simple closed forms, and so do not lead to an easy definition for our homomorphism or a compact generating function. In future work, we hope to categorize those patterns whose recursions lead to closed forms and compact generating functions.

Finally we note that by replacing each \(\sigma_i\) with \(kn - \sigma_i\), we obtain rises if and only if we had descents in our original permutation. Thus we have demonstrated a method for obtaining generating functions for similar types of patterns where we use the statistic \(x^{\text{ris(}\sigma)}\).

### 5.3 Generating functions for consecutive descents with largest elements at the end of the pattern

Our next modification is similar to Section 5.2 in that we will be finding generating functions for \(x^{\text{des(}\sigma)+1}\) in the standard sense of descents, rather than the \(r, s\) peak version of descents in Section 5.1. However, in this section, we will show how to deal with patterns which have the largest element at the end of each block.
5.3.1 Our example pattern \( \mathbb{D} \)

We will demonstrate our method via an example. The example pattern is pictured in Figure 5.17. We can see that this pattern forces:

\[ \sigma_1 < \sigma_2 \]
\[ \sigma_2 > \sigma_3 \]
\[ \sigma_3 < \sigma_4, \text{ and} \]
\[ \sigma_2 < \sigma_4. \]

We will call this pattern \( P \). Define \( \mathbb{D}_k \) to be the set of all \( \sigma \in S_{4k} \) such that:

\[ \sigma_i < \sigma_{i+1} \text{ if } i \equiv 0, 1, 3 \mod 4. \]
\[ \sigma_i > \sigma_{i+1} \text{ and } \sigma_i < \sigma_{i+2} \text{ if } i \equiv 2 \mod 4. \]

In 5.18, we show the two permutations in \( \mathbb{D}_1 \).

Define \( D_k = |\mathbb{D}_k| \). As an example, we give the shape of the permutations in \( \mathbb{D}_3 \) in Figure 5.19.
We find our recursion for $D_n$ in a straightforward way. We look at the final 4 elements of our permutation. For all $1 \leq i \leq n$, $\sigma_{4i}$ is larger than $\sigma_{4i-1}$, $\sigma_{4i-2}$, and $\sigma_{4i-3}$. We also know that $\sigma_{4i} < \sigma_{4i+1}$, so our permutation must have $\sigma_{4n} = 4n$. Likewise, $\sigma_{4n-2} = 4n - 1$, since it is larger than all elements except $\sigma_{4n}$. Next we examine by cases the two choices for $\sigma_{4n-3}$.

Case i: If $\sigma_{4n-3} = 4n - 2$, then we have $4n - 3$ choices left for $\sigma_{4n-1}$.

Case ii: If $\sigma_{4n-3} = 4n - 3$, then $\sigma_{4n-1}$ can only be $4n - 2$.

Altogether, we have $4n - 2$ choices for the last 4 elements of $D_n$, and $D_{n-1}$ choices for the rest. Thus

$$D_n = (4n - 2)D_{n-1} = \prod_{k=1}^{n} (4k - 2).$$

We define a homomorphism

$$\tilde{\Theta}^P : \Lambda \rightarrow \mathbb{Q}[x]$$

so that $\tilde{\Theta}^P(e_k) = 0$ if $k \not\equiv 0 \mod 4$, and
\[ \tilde{\Theta}^P(e_{4n}) = \frac{(-1)^{4n-1}x(1-x)^{n-1}x^n}{(4n)!} D_n \]
\[ = \frac{(-1)^{4n-1}(1-x)^n}{(4n)!} \prod_{k=1}^{n} (4k-2). \]

Define \( S_n^P \subset S_{4n} \) such that each block follows the pattern \( \mathbb{D}_1 \).

**Theorem 5.3.1.**

\[ (4n)! \tilde{\Theta}^P(h_{4n}) = \sum_{\sigma \in S_n^P} x^{\text{des}(\sigma)+1}. \]

**Proof.**

\[ (4n)! \tilde{\Theta}^P(h_k) = (4n)! \sum_{\mu^4=4n} (-1)^{4n-l(\mu)} B_{\mu,4n} \tilde{\Theta}^P(e_{\mu}). \]

But \( \tilde{\Theta}^P(e_{\mu}) = 0 \) unless each part of \( \mu \) is a multiple of 4, so we can restrict ourselves to those \( \mu = 4\lambda \) for some \( \lambda \vdash n \).

\[ \begin{align*}
(4n)! \sum_{\mu^4=4n} (-1)^{4n-l(\mu)} B_{\mu,4n} \tilde{\Theta}^P(e_{\mu}) &= (4n)! \sum_{\mu^4=4n} (-1)^{4n-l(\mu)} B_{\mu,4n} \prod_{i=1}^l \tilde{\Theta}^P(e_{4\mu_i}) \\
&= (4n)! \sum_{\mu^4=4n} (-1)^{4n-l(\mu)} \sum_{T=(b_1, \ldots, b_l(\mu)) \in B_{\mu,n}} \prod_{i=1}^l \frac{(-1)^{4b_i-1}}{(4b_i)!} D_{b_i} x^{b_i+1}(1-x)^{b_i-1} \\
&= \sum_{\mu^4=4n} \sum_{T=(b_1, \ldots, b_l(\mu)) \in B_{\mu,n}} \left( 4n \prod_{i=1}^l (4b_i, 4b_2, \ldots, 4b_l(\mu)) \right) x^{n+l(\mu)}(1-x)^{n-l(\mu)} \prod_{i=1}^l D_{b_i} 
\end{align*} \]

Next we interpret this as a weighted sum of objects, and we will call the set of objects \( \tilde{\Theta} \). By an argument identical to 5.1.1, we can think of this sum as all allowable permutations filling all possible brick tabloids following our pattern, which we then weight with the \( x^{n+l(\mu)}(1-x)^{n-l(\mu)} \) term. First we look at the \( x^{n+l(\mu)} \) term. \( x^n \) allows us to label every \( 4i+2 \)th cell with an \( x \), i.e. every cell equivalent
to $2 \mod 4$. Note that this labels each internal descent (within each block) with an $x$. The $x^{|\mu|}$ term allows to place an $x$ over the last cell of each brick. The $(1 - x)^{n - |\mu|}$ term allows us to place either an 1 or a $-x$ on every 4th cell in each brick, except for the final cell. We then define the weight of this labeled brick tabloid to be the product of the 1s and $-x$’s on the cells. Hence the brick tabloid in 5.20 has weight $-x^7$. Thus $(4n)!\tilde{\Theta}^P(h_k)$ is the sum of the weights of objects in $\tilde{\mathcal{O}}$.

As in previous chapters, we define an involution on $\tilde{\mathcal{O}}$ as follows. We scan the labeled brick tabloid from left to right until either:

i) we find a cell which has a $-x$ on top in which case we split the brick into two pieces immediately after that cell and change the $-x$ to an $x$, or

ii) we find two consecutive bricks $B_i, B_{i+1}$ such that the final cell of $B_i$ is less than the first cell of $B_{i+1}$, in which case we combine these two bricks into a single brick and replace the $x$ on top of the last cell of $B_i$ with a $-x$.

If neither case i) or ii) happens, we define $I(o) = o$. For example, if $o$ is the labeled tabloid in 5.20, then $I(o)$ is pictured in 5.21. Note that though we have altered our standard involution used in previous sections, this involution is still well-defined, sign-reversing and the weight of $I(o)$ is unchanged except for the sign.

![Figure 5.20: An example of a $\mathbb{D}$ filled tabloid in $\tilde{\mathcal{O}}$.](image)

![Figure 5.21: The image of 5.20 under the involution $I$.](image)
As with our previous proofs, we must now examine the fixed points of $I$. None of the fixed points have a label of $-x$, and for every brick, the final cell of the brick is greater than the first cell in the following brick. An example of such a fixed point is given in 5.22.

$$
\begin{array}{ccccccccc}
3 & 5 & 2 & 6 & 7 & 10 & 4 & 11 & 13 & 14 & 16 & 8 & 12 & 9 & 15
\end{array}
$$

Figure 5.22: A fixed point of the involution $I$.

It is clear that for all except the final $x$, the $x$ labels are over exactly those cells for which the next cell is filled by a smaller number. Thus

$$(4n)!\Theta_P(h_{4n}) = \sum_{\sigma \in S_{4n}^P} x^{\text{des}(\sigma)+1}.$$

From here, we continue as usual to obtain our generating function. Because of the product in our recursive calculation of $D_n$, we do not obtain a nice closed form for our generating function, and the proof is identical to the previous proof. However we include the result as a comparison to our other generating functions.

**Theorem 5.3.2.**

$$
1 + \sum_{n \geq 1} \frac{t^{4n}}{(4n)!} \sum_{\sigma \in D_n} x^{\text{des}(\sigma)+1} = \frac{1}{1 - \sum_{n \geq 1} \frac{t^{4n}x^{n+1}(1-x)^{n-1} \prod_{k=1}^{n} 4k-2}{(4n)!}}
$$
Chapter 6

Some Expansions of the Dual Basis of $Z_{\lambda}$

6.1 Introduction

Zigzag (or ribbon) Schur functions are the skew Schur functions with a ribbon shape and are indexed by compositions. A composition $\beta = (\beta_1, \ldots, \beta_k)$ of $n$, denoted $\beta \models n$, is a sequence of positive integers such that $\beta_1 + \beta_2 + \ldots + \beta_k = n$. We define a zigzag shape to be a connected skew shape that contains no $2 \times 2$ array of boxes. Given a composition $\beta = (\beta_1, \ldots, \beta_k)$, we let $Z_{\beta}$ denote the skew Schur function corresponding to the zigzag shape whose row lengths are $\beta_1, \ldots, \beta_k$ reading from top to bottom. For example Figure 6.1 shows the zigzag shape corresponding to the composition $(2, 3, 1, 4)$. As pointed out in [23], zigzag Schur functions arise

![Zigzag Shape](image)

Figure 6.1: The ribbon shape corresponding to the composition $(2, 3, 1, 4)$, so that $s_{(7,4,4,2)}/(3,3,1) = Z_{(2,3,1,4)}$. 

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in many contexts. For example, the scalar product of any two zigzags gives the number of permutations \( \sigma \) such that \( \sigma \) and \( \sigma^{-1} \) have the associated pair of descent sets [39]. Zigzags can also be used to compute the number of permutations with a given descent set and cycle structure [20]. MacMahon [29] showed their coefficients in terms of the monomial symmetric functions count descents in permutations with repeated elements. They also show up as \( sl_n \)-characters of the irreducible components of the Yangian representation in level 1 modules of \( \hat{sl}_n \)[22].

The zigzag Schur functions corresponding to partitions of \( n \) form a basis of \( \Lambda_n \), the space of homogeneous symmetric functions of degree \( n \), and therefore they have a dual basis relative to the Hall inner product which we denote by \( \{DZ_{\lambda}\}_{\lambda \vdash n} \). We shall call \( DZ_{\lambda} \) the dual zigzag symmetric function corresponding to \( \lambda \). The basis \( \{DZ_{\lambda}\}_{\lambda \vdash n} \) has not been extensively studied. Let \( \{m_{\lambda}\}_{\lambda \vdash n} \) denote the set of monomial symmetric functions, \( \{h_{\lambda}\}_{\lambda \vdash n} \) denote the set of homogeneous symmetric functions, and \( \{s_{\lambda}\}_{\lambda \vdash n} \) denote the set of Schur functions. The main result of this chapter is to give a combinatorial interpretation to coefficients that arise in the expansion of \( DZ_{\lambda} \) in terms of the monomial symmetric functions. That is, we shall give a combinatorial interpretation to \( a_{\mu,\lambda} \) where

\[
DZ_{\lambda} = \sum_{\mu} a_{\mu,\lambda} m_{\mu}. \tag{6.1.1}
\]

Our main result will show that \( a_{\mu,\lambda} \) is a signed sum over the weights of certain paths in the lattice of partitions under refinement. In general such a signed sum is complicated, but we will show that in many special cases, we can explicitly evaluate this sum. For example, we will show that \( a_{\mu,(n)} = 1 \) for all \( \mu \) so that

\[
DZ_{(n)} = \sum_{\mu} m_{\mu} = s_{(n)}
\]

where \( s_{(n)} \) is the Schur function associated to the partition with only one part.

Once we have found our combinatorial interpretation for \( a_{\mu,\lambda} \), we can obtain combinatorial interpretations for the expansion of \( DZ_{\lambda} \) in terms of any other basis by using the combinatorial interpretations of the transition matrices between bases
of symmetric functions found in [5]. In particular, we shall use this method to find explicit values for $b_{\mu,\lambda}$ where

$$DZ_\lambda = \sum_\mu b_{\mu,\lambda} s_\mu$$  \hspace{1cm} (6.1.2)$$

for certain special cases.

We now give brief explanations of the concepts to state our main result. There is a natural correspondence between a composition $\beta$ of $n$ and subsets of $[n-1]$. That is, given a composition $\beta = (\beta_1, \ldots, \beta_k)$ of $n$, we define a subset of $[n-1]$ by

$$\text{Set}(\beta) = \{\beta_1, \beta_1 + \beta_2, \beta_1 + \beta_2 + \beta_3, \ldots, \beta_1 + \beta_2 + \ldots + \beta_{k-1}\}. \hspace{1cm} (6.1.3)$$

We also let $\lambda(\beta)$ denote the partition that arises from $\beta$ by arranging its parts in decreasing order and $\ell(\beta)$ denote the number of parts of $\beta$. For example, if $\beta = (2, 3, 1, 2)$, then $\text{Set}(\beta) = \{2, 5, 6\}$ and $\lambda(\beta) = (3, 2, 2, 1)$. Given a subset $S = \{a_1 < a_2 < \ldots < a_r\} \subseteq [n-1]$, we define a composition of $n$ by

$$\beta_n(S) = (a_1, a_2 - a_1, \ldots, a_r - a_{r-1}, n - a_r). \hspace{1cm} (6.1.4)$$

For example, if $S = \{2, 4, 8\}$, then $\beta_{10}(S) = (2, 2, 4, 2)$. We also define $\text{shape}_n(S) = \lambda(\beta_n(S))$. Given two compositions $\beta$ and $\gamma$, we say that $\beta$ is a refinement of $\gamma$, denoted $\beta \leq_r \gamma$, if by adding together adjacent components of $\beta$, we can obtain $\gamma$. For two partitions $\mu$ and $\lambda$ with $\mu \leq_r \lambda$, we define $\text{Path}(\mu, \lambda)$ to be the set of all $P = (\mu_0, \mu_1, \ldots, \mu_k)$, such that $\mu = \mu_0 <_r \mu_1 <_r \ldots <_r \mu_k = \lambda$. If $P = (\mu_0, \mu_1, \ldots, \mu_k)$ is such a path, we let $\ell(P) = k$ denote the length of $P$. Finally, $\mu$ and $\lambda$ are partitions of $n$, then we define

$$[\mu \to \lambda] = |\{S \subseteq \text{Set}(\mu) : \text{shape}_n(S) = \lambda\}|$$

For example, if $\mu = (2, 2, 2, 1)$ and $\lambda = (4, 2, 1)$, then $[\mu \to \lambda] = 2$, since $\text{Set}(\mu) = \{2, 4, 6\}$ and $\lambda(\beta_7(\{2, 6\})) = \lambda(\beta_7(\{4, 6\})) = (4, 2, 1)$.

This given, our main result is to give a combinatorial interpretation of for the coefficients $a_{\mu,\lambda}$ that arise in (6.1.1).
Theorem 6.1.1. If \( \lambda \) and \( \mu \) are partitions of \( n \), then

\[
a_{\mu,\lambda} = (-1)^{l(\mu) - l(\lambda)} \sum_{P \in \text{Path}(\mu,\lambda)} [P] (-1)^{l(P)}
\]

where \( P = (\mu_0, \mu_1, \ldots, \mu_k) \), \( \mu = \mu_0 <_r \mu_1 <_r \ldots <_r \mu_k = \lambda \) and \( [P] = [\mu_0 \to \mu_1 \to \mu_2 \ldots \to \mu_{k-1} \to \mu_k] \).

As one application of our main result, we can give a combinatorial interpretation of the expansion of \( Z_\alpha \) in terms of \( Z_\lambda \)'s, where \( \alpha \) is a composition of \( n \), and \( \lambda \) is a partition of \( n \). It is known, see [18], that

\[
Z_\alpha = \sum_{T \subseteq \text{Set}(\alpha)} (-1)^{|\text{Set}(\alpha) - T|} h_{\lambda(\beta(T))}.
\]

Thus if \( Z_\alpha = \sum_{\mu \vdash n} f_{\mu,\alpha} Z_\mu \), then

\[
f_{\mu,\alpha} = \langle Z_\alpha, DZ_\mu \rangle = \sum_{T \subseteq \text{Set}(\alpha)} (-1)^{|\text{Set}(\alpha) - T|} a_{\lambda(\beta(T)),\mu}.
\tag{6.1.5}
\]

In principle, (6.1.5) gives rise to a combinatorial algorithm to compute the coefficients \( f_{\mu,\alpha} \). However, such an algorithm is not necessarily the most efficient way to compute these coefficients.

The outline of this chapter is as follows. In Section 6.2, we shall review the necessary background for symmetric functions and the combinatorial interpretation of the entries of the transition matrices between various bases of symmetric functions that we shall need. In particular, we shall use the Jacobi-Trudi identity to give a combinatorial interpretation of the coefficients \( Z_\lambda |_{h_\mu} \). In Section 6.3, give some examples of the computations involved in computing the coefficients \( a_{\mu,\lambda} \). In Section 6.4, we present the proof of our main theorem. In Section 6.5, we give closed forms for several of the coefficients, independent of the size of the composition. In Subsection 6.5.1, we give the expansion of several dual zigzags in terms of monomial functions which are independent of the size of the partition, while in 6.5.2 we do the same for the Schur functions. In Section 6.6, we give a brief explanation of two applications of our main result. In Subsection 6.6.1, we
show how our result can express a ribbon Schur function in terms of ribbon Schur functions indexed by partitions. In Subsection 6.6.2, we show how our result gives a coefficient-free decomposition of Schur functions indexed by hooks.

### 6.2 Background Information

In this section, we review some of the background information presented in 1.2

We say that \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_k) \) is a partition of \( n \), written \( \lambda \vdash n \) if \( \lambda_1 + \cdots + \lambda_k = n = |\lambda| \). We \( \ell(\lambda) \) denote the number of parts of \( \lambda \). We let \( F_\lambda \) denote the Ferrers diagram of \( \lambda \). If \( \mu = (\mu_1, \ldots, \mu_m) \) is a partition where \( m \leq k \) and \( \lambda_i \geq \mu_i \) for all \( i \leq m \), we let \( F'_\lambda/\mu \) denote the skew shape that results by removing the cells of \( F_\mu \) from \( F_\lambda \).

![Figure 6.2: The skew Ferrers diagram of (3,3,2,1)/(2,1).](image)

A column-strict tableau \( T \) of shape \( \lambda \) is any filling of \( F_\lambda \) with natural numbers such that entries in each row are weakly increasing from left to right, and entries in each column are strictly increasing from bottom to top. We define the content of \( T \) to be \( c(T) = (\alpha_1, \alpha_2, \ldots) \) where \( \alpha_i \) is the number of times that \( i \) occurs in \( T \). If

![Figure 6.3: A column strict tableau of shape (3,2^2,1) and content (2^2,1,2,1).](image)

\( \lambda \) is a partition denoted by \( \lambda = (\lambda_1, \ldots, \lambda_l) = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n}) \), where \( m_i \) is the
number of parts of $\lambda$ equal to $i$, then we define $z_\lambda = 1^{m_1}2^{m_2} \cdots n^{m_n}m_1!m_2! \cdots m_n!$.

There are six standard bases of the space of homogeneous symmetric functions of degree $n$, $\Lambda_n(x)$, which are generally notated as: $\{m_\lambda\}_{\lambda \vdash n}$ (the monomial symmetric functions), $\{h_\lambda\}_{\lambda \vdash n}$ (the complete homogeneous symmetric function), $\{e_\lambda\}_{\lambda \vdash n}$ (the elementary symmetric functions), $\{p_\lambda\}_{\lambda \vdash n}$ (the power sum symmetric functions), $\{f_\lambda\}_{\lambda \vdash n}$ (the forgotten symmetric functions) and $\{s_\lambda\}_{\lambda \vdash n}$ (the Schur functions), where $\lambda$ is a partition of $n$.

The Hall inner product is a standard scalar product on the space of homogeneous symmetric functions $\Lambda_n(x)$, which is defined by:

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$$

where

$$\delta_{\lambda,\mu} = \begin{cases} 
1 & \text{if } \lambda = \mu, \\
0 & \text{otherwise}. 
\end{cases}$$

Under this scalar product, $\{s_\lambda\}_{\lambda \vdash n}$ and $\{p_\lambda/\sqrt{z_\lambda}\}_{\lambda \vdash n}$ are known to be self-dual, and $\{e_\lambda\}_{\lambda \vdash n}$ and $\{f_\lambda\}_{\lambda \vdash n}$ are dual [5].

When given two bases of $\Lambda_n(x)$, $\{a_\lambda\}_{\lambda \vdash n}$ and $\{b_\lambda\}_{\lambda \vdash n}$, we first fix some ordering of the partitions of $n$, e.g. the lexicographic order, and then we may think of the bases as row vectors, $\langle a_\lambda \rangle_{\lambda \vdash n}$ and $\langle b_\lambda \rangle_{\lambda \vdash n}$. We can define the transition matrix $M(a, b)$ that transforms the basis $\langle a_\lambda \rangle_{\lambda \vdash n}$ into the basis $\langle b_\lambda \rangle_{\lambda \vdash n}$ by

$$\langle b_\lambda \rangle_{\lambda \vdash n} = \langle a_\lambda \rangle_{\lambda \vdash n} M(a, b).$$

The $(\lambda, \mu)$ entry of $M(a, b)$ is given by the equation

$$b_\lambda = \sum_{\mu \vdash n} a_\mu M(a, b)_{\mu,\lambda}.$$
In addition, we shall also be interested in finding a combinatorial interpretation for the entries of $M(s, DZ)$. That is, we want to find a combinatorial interpretation for $b_{\mu, \lambda}$ where 
\[ DZ_{\lambda} = \sum_{\mu} b_{\mu, \lambda} s_{\mu}. \]

Next we shall describe the combinatorial interpretation of the coefficients that arise in expanding a skew Schur function in terms of the homogeneous symmetric functions. In particular, we will need to use the expansion of skew-Schur functions in terms of $h_{\lambda}$. To do so, we introduce rim hooks, special rim hooks and special rim hook tabloids. More detail is given in [13] where they are used to give a combinatorial interpretation of the inverse Kostka matrix.

For a partition $\lambda$, consider the Ferrers diagram $F_{\lambda}$. A rim hook of $\lambda$ is a sequence of cells, $h$, along the northeast boundary of $F_{\lambda}$ such that any two consecutive cells in $h$ share an edge and if we remove $h$ from $F_{\lambda}$, we are left with the Ferrers diagram of another partition. More generally, $h$ is a rim hook of a skew shape $\lambda/\mu$ if $h$ is a rim hook of $\lambda$ which does not intersect $\mu$.

A rim hook tableau of shape $\lambda/\nu$ and type $\mu$, $T$, is a sequence of partitions 
\[ T = (\nu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \lambda^{(k)} = \lambda), \]
such that for each $1 \leq i \leq k$, $\lambda^{(i)}/\lambda^{(i-1)}$ is a rim hook of $\lambda^{(i)}$ of size $\mu_i$. We define the sign of a rim hook $h_i = \lambda^{(i)}/\lambda^{(i-1)}$ to be 
\[ sgn(h_i) = (-1)^{r(h_i)-1}, \]
where $r(h_i)$ is the number of rows that $h_i$ occupies. The sign of a rim hook tableau $T$ is 
\[ sgn(T) = \prod_{i=1}^{k} sgn(h_i). \]

Given two partitions $\lambda^{(i-1)} \subset \lambda^{(i)}$, we say that $\lambda^{(i)}/\lambda^{(i-1)}$ is a special rim hook if $\lambda^{(i)}/\lambda^{(i-1)}$ is a rim hook of $\lambda^{(i)}$ and $\lambda^{(i)}/\lambda^{(i-1)}$ contains a cell from the first column of $\lambda$. A special rim hook tabloid (SRHT) $T$ of shape $\lambda/\mu$ is a sequence of partitions 
\[ T = (\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \lambda^{(k)} = \lambda), \]
such that for each $1 \leq i \leq k$, $\lambda^{(i)}/\lambda^{(i-1)}$ is a special rim hook of $\lambda^{(i)}$. We have a partition determined by the integers $|\lambda^{(i)}/\lambda^{(i-1)}|$ which is the type of the special rim hook tabloid $T$. Notice that we have used the word *tabloid* instead of *tableau* in order to highlight there is no implicit order in the size of each successive special rim hook, unlike in a rim hook tableau.

The sign of a special rim hook, $h_i = \lambda^{(i)}/\lambda^{(i-1)}$, and the sign of a special rim hook tabloid $T$, are defined as we did for rim hooks and rim hook tableaux. We show an example of a special rim hook tabloid of type $(6,5,4,2)$ and shape $(5,4,4,3,1)$ in Figure 6.4. For $|\lambda/\nu| = |\mu|$, Eğecioğlu and Remmel [13] show that

![Figure 6.4: A special rim hook tabloid of shape (5,4,4,3,1) and type (6,5,4,2).](image)

$$s_{\lambda/\nu} = \sum_{\mu} K_{\mu,\lambda/\nu}^{-1} h_{\mu}$$

(6.2.1)

where

$$K_{\mu,\lambda/\nu}^{-1} = \sum_{T \text{ is a SRHT of shape } \lambda/\nu \text{ and type } \mu} \text{sgn}(T).$$

Hence we obtain a combinatorial description of

$$M(s,m)_{\lambda,\mu} = K_{\mu,\lambda}^{-1}.$$

We now give examples of the expansion of $\{DZ_{\lambda}\}_{\lambda \vdash n}$ when $n = 6$. We first give the expansion of $DZ_{\lambda}$ in terms of the monomial symmetric functions, when $\lambda \vdash 6$. 

where $|K^{-1}_{\mu,\lambda}|$ is the inverse Kostka matrix which will be described below. Thus

$$DZ_\lambda = \sum_{\mu \leq \lambda} a_{\mu,\lambda} \sum_{\gamma} s_\gamma K^{-1}_{\mu,\gamma} = \sum_{\gamma} s_\gamma \sum_{\mu \leq \lambda} a_{\mu,\lambda} K^{-1}_{\mu,\gamma}. \quad (6.2.2)$$

Hence

$$b_{\mu,\gamma} = \sum_{\mu \leq \lambda} a_{\mu,\lambda} K^{-1}_{\mu,\gamma}. \quad (6.2.3)$$

The expansion of $DZ_\lambda$ in terms of the Schur functions, when $\lambda \vdash 6$, is given below.
\[
\begin{align*}
DZ_{(6)} &= s_6 & DZ_{(3,1,1,1)} &= s_{3,1,1,1} - s_{2,2,1,1} \\
DZ_{(5,1)} &= s_{5,1} - s_{4,2} + s_{3,2,1} - s_{2,2,2} - s_{2,2,1,1} & DZ_{(2,2,2)} &= s_{2,2,2} \\
DZ_{(4,2)} &= s_{4,2} - s_{3,3} - s_{3,2,1} + 2s_{2,2,2} + s_{2,2,1,1} & DZ_{(2,2,1,1)} &= s_{2,2,1,1} \\
DZ_{(4,1,1)} &= s_{4,1,1} - s_{3,2,1} + s_{2,2,2} + s_{2,2,1,1} & DZ_{(2,1,1,1,1)} &= s_{2,1,1,1,1,1} \\
DZ_{(3,3)} &= s_{3,3} - s_{2,2,2} & DZ_{(1,1,1,1,1,1)} &= s_{1,1,1,1,1,1} \\
DZ_{(3,2,1)} &= s_{3,2,1} - 2s_{2,2,2} - s_{2,2,1,1}.
\end{align*}
\]

Recall that we defined a composition \( \beta \) of \( n \), denoted \( \beta \vdash n \), as a list of positive integers \((\beta_1, \beta_2, \ldots, \beta_k)\) such that \( \beta_1 + \beta_2 + \ldots + \beta_k = n \). We call \( \beta_i \) a component of \( \beta \), and we say that \( \beta \) has length \( l(\beta) = k \) and size \( |\beta| = n \). From this definition, we can see that \( \beta \) is a partition if each of its components are weakly decreasing. For any composition \( \beta \), we define the partition determined by \( \beta \), \( \lambda(\beta) \), which we obtain by reordering the components of \( \beta \) in weakly decreasing order, e.g. \( \lambda(2, 8, 9, 4) = (9, 8, 4, 2) \). Notice that two compositions \( \beta, \gamma \) can determine the same partition, e.g. if \( \beta = (2, 8, 9, 4) \) and \( \gamma = (2, 9, 8, 4) \), then \( \lambda(2, 8, 9, 4) = (9, 8, 4, 2) = \lambda(2, 9, 8, 4) \).

There is a natural correspondence between a composition \( \beta \vdash n \) and a subset \( \text{Set}(\beta) \subseteq [n - 1] = \{1, 2, \ldots, n - 1\} \) where

\[
\text{Set}(\beta) = \{\beta_1, \beta_1 + \beta_2, \beta_1 + \beta_2 + \beta_3, \ldots, \beta_1 + \beta_2 + \ldots + \beta_{k-1}\}.
\]

We can also reverse this process so that for any subset \( S = \{j_1, j_2, \ldots, j_{k-1}\} \subseteq [n - 1] \), we can find the composition \( \beta_n(S) \vdash n \) where

\[
\beta_n(S) = (j_1, j_2 - j_1, \ldots, n - j_{k-1}).
\]

For example, the composition \( \beta = (2, 9, 8, 4) \) has \( \text{Set}(\beta) = \{2, 11, 19\} \subseteq [22] \). We also define \( \text{shape}_n(S) = \lambda(\beta_n(S)) \). For example if \( S = \{2, 5, 6, 10\} \) and \( n = 11 \), then \( \beta_{11}(S) = (2, 3, 1, 4, 1) \), and \( \text{shape}_{11}(S) = (4, 3, 2, 1, 1) \).

Given two partitions \( \lambda \) and \( \mu \) of \( n \), we say that \( \lambda \) is a refinement of \( \mu \), written \( \lambda \leq_r \mu \), if \( \lambda \) can be created from \( \mu \) by splitting some of the parts of \( \mu \) into pieces.
For example, \((4, 2, 1, 1, 1) \leq_r (5, 3, 2)\) since we can split 5 into 4 + 1 and 3 into 1 + 1 + 1 to obtain \(\lambda\). The cover relations in the lattice of partitions of \(n\) under refinement arise by starting with a partition \(\lambda\) and combining two of the parts of \(\lambda\) to get \(\mu\). Similarly, given two compositions \(\beta\) and \(\gamma\), we say that \(\beta\) is a refinement of \(\gamma\), denoted \(\beta \leq_r \gamma\), if by adding together adjacent components of \(\beta\), we can obtain \(\gamma\). For example, \(421131 \leq_r 4314\), meaning \(\gamma = 421131\) is a refinement of \(\beta = 4314\). If we only add together a single pair of adjacent components of a partition \(\beta\) to get \(\gamma\), then we will say that \(\gamma\) covers \(\beta\).

The refinement ordering restricted to the set of partitions forms a lattice which we call the lattice of partitions under refinement, or more briefly, the refinement lattice. For two partitions \(\mu\) and \(\lambda\), with \(\mu \leq_r \lambda\) we define \(\text{Path}(\mu, \lambda)\) to be the set of all \(P = (\mu_0, \mu_1, \ldots, \mu_k)\), such that \(\mu = \mu_0 <_r \mu_1 <_r \ldots <_r \mu_k = \lambda\). We define the length of \(P\), \(l(P) = k\).

Given two partitions of \(\lambda\) and \(\mu\) of \(n\) such that \(\mu \leq_r \lambda\), we define

\[ [\mu \rightarrow \lambda] = |\{S \subseteq \text{Set}(\mu) : \text{shape}_n(S) = \lambda\}|. \]

As an example, let’s calculate \([2, 1^4] \rightarrow (4, 2)\]. Note that \(\text{Set}(2, 1^4) = \{2, 3, 4, 5\}\). We want to find \(|\{S \subseteq \{2, 3, 4, 5\} : \text{shape}_6(S) = (4, 2)\}|. The only two subsets of \(\{2, 3, 4, 5\}\) that have the appropriate shape are \(\{2\}\) and \(\{4\}\), so \([2, 1^4] \rightarrow (4, 2)\] = 2.

### 6.3 An example of the calculations

Before proceeding with the proof of Theorem 6.1.1, we shall demonstrate how it can be used to calculate \(a_{\mu, \lambda}\) in the case where \(\mu = (1^6)\) and \(\lambda = (3, 2, 1)\). Since our theorem says we sum over all paths in the refinement lattice, we give the relevant portion of the refinement lattice in Figure 6.5. First we give several examples of how to calculate \([\alpha \rightarrow \beta]\). Recall that \(\text{Set}(\lambda) = \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \cdots + \lambda_{k-1}\}\). We first calculate \([(1^6) \rightarrow (2, 1^4)]\), which is equal to \(|\{S \subseteq \text{Set}(1^6) : \text{shape}_6(S) = (2, 1^4)\}|. \text{Set}(1^6) = \{1, 2, 3, 4, 5\}\), and the subsets
Figure 6.5: The refinement lattice from (1,1,1,1,1,1) to (3,2,1).

Table 6.1: Values for \([\alpha \rightarrow \beta]\) for pairs in the refinement lattice from \((1^6)\) to \((3,2,1)\).

<table>
<thead>
<tr>
<th>([1^6 \rightarrow 2, 1^4]) = 5</th>
<th>([2, 1^4 \rightarrow 3, 1^3]) = 1</th>
<th>([3, 1^3 \rightarrow 3, 2, 1]) = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>([1^6 \rightarrow 3, 1^3]) = 4</td>
<td>([2, 1^4 \rightarrow 2^2, 1^1]) = 3</td>
<td>([2^2, 1^2 \rightarrow 3, 2, 1]) = 1</td>
</tr>
<tr>
<td>([1^6 \rightarrow 2^2, 1^2]) = 6</td>
<td>([2, 1^4 \rightarrow 3, 2, 1]) = 4</td>
<td>([3, 1^3 \rightarrow 3, 2, 1]) = 2</td>
</tr>
<tr>
<td>([1^6 \rightarrow 3, 2, 1]) = 6</td>
<td>([2^2, 1^2 \rightarrow 3, 2, 1]) = 1</td>
<td></td>
</tr>
</tbody>
</table>

\(\{2,3,4,5\}, \{1,3,4,5\}, \{1,2,4,5\}, \{1,2,3,5\}\), and \(\{1,2,3,4\}\) all have shape equal to \((2,1^4)\). Therefore \([(1^6) \rightarrow (2,1^4)] = 5\). Similarly \([(1^6) \rightarrow (3,2,1)] = 6\) since \(\{3, 4\}, \{3, 5\}, \{2, 5\}, \{2, 3\}, \{1, 3\}, \{1, 4\}\) are the only subsets \(T\) of \(Set(1^6) = \{1, 2, 3, 4, 5\}\) such that \(shape_6(T) = (3, 2, 1)\). Finally we calculate \([(2,1^4) \rightarrow (3,1^3)]\). In this case, \(Set(2,1^4) = \{2, 3, 4, 5\}\) and the only subset \(T\) of \(Set(2,1^4)\) such that \(shape_6(T) = (3,1^3)\) is \(\{3, 4, 5\}\). Thus \([(2,1^4) \rightarrow (3,1^3)] = 1\).

From these three examples we see that a considerable amount of work goes into calculating \([\alpha \rightarrow \beta]\) for every possibility in our refinement lattice. In Table 6.1, we give the values needed to calculate \([\alpha \rightarrow \beta]\) for all pairs in the refinement lattice from \((1^6)\) to \((3,2,1)\).

Once we have calculated those values, we can easily calculate the weights of each possible path in our refinement lattice. These paths and weights are listed in
Table 6.2: The weight of each possible path in the refinement lattice from \((1^6)\) to 
\((3, 2, 1)\).

<table>
<thead>
<tr>
<th>Possible Paths (P)</th>
<th>Length((P))</th>
<th>Weight((P))</th>
</tr>
</thead>
<tbody>
<tr>
<td>([(1^6) \rightarrow (3, 2, 1)])</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>([(1^6) \rightarrow (3, 1^3)][(3, 1^3) \rightarrow (3, 2, 1)])</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>([(1^6) \rightarrow (2^2, 1^2)][(2^2, 1^2) \rightarrow (3, 2, 1)])</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>([(1^6) \rightarrow (2, 1^4)][(2, 1^4) \rightarrow (3, 2, 1)])</td>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td>([(1^6) \rightarrow (2, 1^4)][(2, 1^4) \rightarrow (3, 1^3)][(3, 1^3) \rightarrow (3, 2, 1)])</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>([(1^6) \rightarrow (2, 1^4)][(2, 1^4) \rightarrow (2^2, 1^2)][(2^2, 1^2) \rightarrow (3, 2, 1)])</td>
<td>3</td>
<td>15</td>
</tr>
</tbody>
</table>

Table 6.2, which will be used in our calculation of \(a_{\mu,\lambda}\).

Finally, we combine this information:

\[
a_{(1^6), (3, 2, 1)} = (-1)^{6-3} \sum_{P \in Path((1^6), (3, 2, 1))} (-1)^{l(P)}[P]
\]

\[
= -1^3(-1^1(6) + 1^2(8 + 6 + 20) + 1^3(10 + 15))
\]

\[
= -(-6 + 34 - 25)
\]

\[
= -3.
\]

We should note that although this first example required many calculations, we 
have now done almost all of the work for several other coefficients for \(n = 6\) since 
our the set of paths that we considered also arise in the computation of \(a_{\alpha,\beta}\) for 
other pairs of partitions. In addition, we will see later that the same calculations 
allow us to evaluate an infinite number of coefficients \(a_{\alpha,\beta}\) where \(\alpha\) and \(\beta\) are 
partitions of \(n > 6\).

### 6.4 Proof of Theorem 6.1.1:

We start by expanding the zigzag Schur functions in terms of the homoge-
neous symmetric functions \(\{h_\lambda\}_{\lambda \vdash n}\) derived from the Jacobi-Trudi by Eğecioğlu 
and Remmel [13],

\[
s_{\lambda/\mu} = \det(h_{\lambda_i-\mu_j-i+j}) = \sum_{\nu} K_{\nu, \lambda/\mu}^{-1} h_\mu
\]
where \( h_0 = 1 \) and \( h_k = 0 \) if \( k < 0 \). Applying it specifically to zigzag Schur functions and using compositions as subscripts, we can show that for any \( \alpha \models n \),

\[
Z_{\alpha} = (-1)^{l(\alpha)} \sum_{\beta \leq \alpha} (-1)^{l(\beta)} h_{\lambda(\beta)}.
\]

Alternatively,

\[
Z_{\alpha} = h_{\lambda(\beta(\alpha))} + \sum_{T \subseteq \text{Set}(\alpha)} (-1)^{|\text{Set}(\alpha) - T|} h_{\lambda(\beta(T))}. \tag{6.4.1}
\]

The result in 6.4.1 is well-known and can be proved by inclusion-exclusion [18]. Recall that \([\mu \rightarrow \lambda] = |\{S \subseteq \text{Set}(\mu) : \text{shape}_{n}(S) = \lambda\}|. So

\[
Z_{\lambda} = h_{\lambda} + \sum_{\lambda \leq \alpha} (-1)^{l(\lambda) - l(\alpha)} [\lambda \rightarrow \alpha] h_{\alpha}.
\]

In order to prove our theorem, we will take the result for \( \{DZ_{\lambda}\}_{\lambda \models n} \) in 6.1.1 as a definition, and show it is dual under the Hall inner product to \( \{Z_{\lambda}\}_{\lambda \models n} \). As a result of the Cauchy identity [28], we need only show that

\[
\sum_{\gamma} Z_{\gamma}(x) DZ_{\gamma}(y) = \sum_{\gamma} h_{\gamma}(x) m_{\gamma}(y)
\]

or, equivalently,

\[
\sum_{\gamma} Z_{\gamma}(x) DZ_{\gamma}(y)|_{h_{\lambda}(x)m_{\mu}(y)} = \delta_{\lambda,\mu}.
\]

Given our expansion of \( Z_{\lambda}(x) \) in terms of \( h_{\lambda}(x) \)'s and the fact that \( \langle h_{\lambda}(x), m_{\mu}(x) \rangle = \delta_{\lambda,\mu} \), we need to show that

\[
\sum_{\gamma} Z_{\gamma}(x) DZ_{\gamma}(y)|_{h_{\lambda}(x)} = \sum_{\alpha \leq r, \lambda} (-1)^{l(\alpha) - l(\lambda)} [\alpha \rightarrow \lambda] m_{\alpha}(y)
\]

or

\[
\sum_{\gamma} Z_{\gamma}(x) DZ_{\gamma}(y)|_{h_{\lambda}(x)m_{\mu}(y)} = \sum_{\mu \leq r, \alpha \leq r, \lambda} (-1)^{l(\alpha) - l(\lambda)} [\alpha \rightarrow \lambda] a_{\mu,\alpha}
\]

Thus we need only show that \( \sum_{\mu \leq r, \alpha \leq r} (-1)^{l(\alpha) - l(\lambda)} [\alpha \rightarrow \lambda] a_{\mu,\alpha} = \delta_{\lambda,\mu} \). We proceed in two cases:
Case 1. \( \mu = \lambda \).

It is easy to see from our definitions that \( a_{\lambda, \lambda} = 1 \), so that
\[
\sum_{\mu \leq r, \alpha \leq \lambda} (-1)^{l(\alpha)-l(\lambda)}[\alpha \rightarrow \lambda] a_{\mu, \alpha} = (-1)^0[\lambda \rightarrow \lambda] a_{\lambda, \lambda} = 1,
\]
which is equivalent to \( \delta_{\lambda, \lambda} \).

Case 2. \( \mu \neq \lambda \).

Then we must show that
\[
\sum_{\mu \leq r, \alpha < r} (-1)^{l(\alpha)-l(\lambda)}[\alpha \rightarrow \lambda] a_{\mu, \alpha} =
\sum_{\mu \leq r, \alpha < r} \sum_{P \in \text{Path}(\mu, \alpha)} (-1)^{l(\lambda)-l(\mu)}[\alpha \rightarrow \lambda][P](-1)^{\ell(P)} = 0
\]

This appears to be adding another step to our path \( P \). But we must be careful, because if \( \alpha = \lambda \), then the length of our path hasn’t changed. So we divide it into two terms: \( \alpha = \lambda \), and \( \alpha \neq \lambda \). That is, let
\[
A = \sum_{P \in \text{Path}(\mu, \lambda)} (-1)^{\ell(\lambda)-\ell(\mu)}[P](-1)^{\ell(P)}
\]
\[
B = \sum_{\mu \leq r, \alpha < r} \sum_{P \in \text{Path}(\mu, \alpha)} (-1)^{\ell(\lambda)-\ell(\mu)}[\alpha \rightarrow \lambda][P](-1)^{\ell(P)}
\]

Now suppose that \( P = (\mu = \mu_0 \leq_r \ldots \leq_r \mu_s = \lambda) \) is a path in \( \text{Path}(\mu, \lambda) \). Note \( s \geq 1 \) since we are assuming \( \mu \neq \lambda \). Let \( Q = (\mu_0 \leq_r \ldots \leq_r \mu_{s-1} = \alpha) \). Then \( P \) contributes
\[
(-1)^{\ell(P)}[P] = (-1)^s[\mu_0 \rightarrow \mu_1][\mu_1 \rightarrow \mu_2] \cdots [\mu_{s-1} \rightarrow \mu_s]
\]
to \( A \) and contributes
\[
(-1)^{\ell(Q)}[Q][\mu_{s-1} \rightarrow \mu_s] = (-1)^{s-1}[\mu_0 \rightarrow \mu_1][\mu_1 \rightarrow \mu_2] \cdots [\mu_{s-1} \rightarrow \mu_s]
\]
to \( B \). Thus \( A + B = 0 \) as desired.
6.5 Examples of these Coefficients

6.5.1 Special Cases of the $a_{\mu,\lambda}$’s

We saw in our example calculating $a_{(1^6),(3,2,1)}$ how difficult and time-consuming it can be to find these coefficients. However, in a number of special cases, we can actually compute a closed form for the sum

$$a_{\mu,\lambda} = (-1)^{l(\mu)-l(\lambda)} \sum_{P \in \text{Path}(\mu,\lambda)} [P] (-1)^{l(P)}.$$ 

For example, if $\mu <_r \lambda$ is a cover relation in the refinement lattice, then there is only one path and the formula for the coefficient $a_{\mu,\lambda}$ consists of a single term. In fact, we can prove the following.

Observations:

1. As previously noted, it is clear from our interpretation that $a_{\mu,\mu} = 1$ for all $\mu$.

2. If $\lambda$ and $\mu$ are a cover relation in the refinement lattice, then $a_{\mu,\lambda} = [\mu \rightarrow \lambda]$.

3. If $l(\mu) - l(\lambda) \geq 2$, then

$$a_{\mu,\lambda} = (-1)^{l(\mu)-l(\lambda)-1}([\mu \rightarrow \lambda] + \sum_{\mu <_r \alpha <_r \lambda} (-1)^{l(\mu)-l(\lambda)}[\mu \rightarrow \alpha] a_{\alpha,\lambda}).$$

This observation will be used in the proofs of several theorems later, and in calculations, allows us to build tables of coefficients recursively.

4. For any $\mu$ such that $\mu \vdash n$, $a_{\mu,(n)} = 1$, so that we find $DZ_{(n)} = \sum_{\mu} m_{\mu} = s(n)$.

The proofs of 1 and 2 are obvious from our main theorem. We outline a proof of 3 by induction on the length of the refinement.

$$a_{\mu,(n)} = (-1)^{l(\mu)-1} \sum_{P \in \text{Path}(\mu,(n))} (-1)^{l(P)}[P]$$

$$= (-1)^{l(\mu)-1} \sum_{\mu <_r \alpha <_r (n)} (-1)[\mu \rightarrow \alpha] \sum_{P \in \text{Path}(\alpha,(n))} (-1)^{l(P)}[P]$$

$$+ (-1)^{l(\mu)-1}(-1)[\mu \rightarrow (n)].$$
Our inductive assumption that $a_{\alpha, (n)} = 1$ gives that $\sum_{P \in \text{Path}(\alpha, (n))} (-1)^{(P)}[P] = (-1)^{(\alpha) - 1}$. Thus note that

$$a_{\mu, (n)} = (-1)^{l(\mu) - 1} \sum_{\mu < \alpha < \epsilon(n)} (-1)[\mu \to \alpha](-1)^{(\alpha) - 1} + (-1)^{l(\mu) - 1}(-1)[\mu \to (n)].$$

But if we think about the definition of $[\mu \to \alpha]$, now we are summing over all possibilities of ways to remove at least one element from $\text{Set}(\mu)$ so

$$a_{\mu, (n)} = (-1)^{l(\mu) - 1} \sum_{\emptyset \subseteq S \subseteq \text{Set}(\mu)} (-1)^{|\text{Set}(\mu)| - |S|}$$

$$= (-1)^{l(\mu) - 1}((-1)^{|\text{Set}(\mu)| - |S|} - (-1)^{|\text{Set}(\mu)|}).$$

But $\sum_{S \subseteq \text{Set}(\mu)} (-1)^{|S|} = 0$. So

$$a_{\mu, (n)} = (-1)^{l(\mu)}(0 - (-1)^{|\text{Set}(\mu)|}) = (-1)^{l(\mu)}((-1)^{|\text{Set}(\mu)| + 1})$$

But $|\text{Set}(\mu)| + 1 = l(\mu)$, so $a_{\mu, (n)} = 1$.

Other results can be found using careful examination of the lattice of refinement.

**Theorem 6.5.1.** We have the following results with $\mu = (1^k)$ and $\lambda = (b, 1^{k-b})$ for $b = 1, 2, \ldots, 7$:

1. $a_{(1^k), (2, 1^{k-2})} = k - 1$
2. $a_{(1^k), (3, 1^{k-3})} = 1$
3. $a_{(1^k), (4, 1^{k-4})} = \binom{k-1}{2} - 2$
4. $a_{(1^k), (5, 1^{k-5})} = -\frac{1}{2}(k - 1)(k - 4) + 3$
5. $a_{(1^k), (6, 1^{k-6})} = \frac{1}{6}(k^3 - 3k^2 - 16k - 6)$
6. $a_{(1^k), (7, 1^{k-7})} = -\frac{1}{3}(k)(k + 1)(k - 7) + 1$
Proof. The proofs of some of the above items are very straightforward. For example, the proof of item 1 is plain because the relevant portion of the refinement lattice contains only two shapes. Moreover, \( \text{Set}(1^k) = \{1, 2, \ldots, k-1\} \) and when we remove any element from \( \text{Set}(1^k) \), one ends up with a set that has shape \((2, 1^{k-2})\). Since there are \( k - 1 \) ways to remove one element from \( \text{Set}(1^k) \), it follows that \( a_{(1^k),(2,1^{k-2})} = k - 1 \). In other words,

\[
a_{(1^k),(2,1^{k-2})} = (-1)^1 \sum_{P \in \text{Path}((1^k),(2,1^{k-2}))} |P| (-1)^{|P|}
\]

\[
= (-1)(1^k) \to (2,1^{k-2})(-1) = k - 1.
\]

We now include a table of values for \([\alpha \to \beta]\) that will aid in our proofs. Each of these entries is an elementary counting problem in itself, but they are all easy calculations so that we do not include explanations of any of them here.

For the proof of 2, we refer to Table 6.3.

\[
a_{(1^k)\to(3,1^{k-3})} = (-1)^2([1^k] \to (2,1^{k-2})[2,1^{k-2}] \to (3,1^{k-3})]
\]

\[
+([1^k] \to (3,1^{k-3})(-1))
\]

\[
= (k - 1)(1) + (k - 2)(-1)
\]

\[
= 1.
\]

For the proof of 3, we will use Table 6.3. We don’t list the individual paths,
but rather calculate the weights of the paths directly.

\[
a_{(1^k), (4,1^{k-3})} = (-1)^3 \left[ - (k-1)(k-3) + - (k-1)(1)(1) + (k-1) \right] \\
+ \left( \frac{k-1}{2} \right) - (k-2) + (k-2) - (k-3) \\
= \frac{k^2}{2} - \frac{3k}{2} - 1 = \left( \frac{k-1}{2} \right) - 2.
\]

We omit the proofs of 4, 5, and 6, but they are similar to 1, 2, and 3 in that they are just careful evaluations of the lattice of partitions under refinement.

\[\square\]

Here are some other results which are useful for the computation of the coefficients \(b_{\mu,\lambda}\) of (6.1.2):

**Theorem 6.5.2.** For any positive integer \(k\),

7. \(a_{(1^k), (3^2, 1^{k-6})} = 0\)

8. \(a_{(1^k), (3,2,1^{k-5})} = -\frac{1}{2}k(k-5)\)

9. \(a_{(2^{k-2}), (4,1^{k-4})} = k - 3\)

10. \(a_{(2^{k-2}), (3,2,1^{k-5})} = 1\)

**Proof.** Here we will prove only 8, but the others are similar. Again, we refer to Table 6.3. We list the paths by increasing length.

\[
a_{(1^k), (3^2, 1^{k-5})} = (-1)^{l(1^k) - l(3^2,1^{k-5})} \sum_{P \in \text{Path}(1^k, (3,2,1^{k-5}))} (-1)^l(P)[P]
\]
\[
\begin{align*}
&= -1^3 \left( [(1^k) \to (3, 2, 1^{k-5})](-1) \\
&\quad + [(1^k) \to (3, 1^{k-3})][((3, 1^{k-3}) \to (3, 2, 1^{k-5})](-1)^2 \\
&\quad + [(1^k) \to (2, 2, 1^{k-4})][((2, 2, 1^{k-4}) \to (3, 2, 1^{k-5})](-1)^2 \\
&\quad + [(1^k) \to (2, 1^{k-2})][((2, 1^{k-2}) \to (3, 2, 1^{k-5})](-1)^2 \\
&\quad + [(1^k) \to (2, 1^{k-2})][((2, 1^{k-2}) \to (2, 2, 1^{k-4}) \times \\
&\quad [(2, 2, 1^{k-4}) \to (3, 2, 1^{k-5})](-1)^3 \\
&\quad + [(1^k) \to (2, 1^{k-2})][((2, 1^{k-2}) \to (3, 1^{k-3}) \times \\
&\quad [(3, 1^{k-3}) \to (3, 2, 1^{k-5})](-1)^3 \\
&\quad + (k-2)(k-5) + 2)(-1) + (k-2)(k-4) \\
&\quad + \binom{k-2}{2} + (k-1)(2k-8) + (k-1)(k-3)(-1) + \\
&\quad (k-1)(k-4)(-1) \\
&\quad + \frac{1}{2} k(k-5)
\end{align*}
\]

Before we begin our next theorem, we give two lemmas.

Lemma 6.5.3. If \(0 \leq f \leq d\),

\[
[(2^{c+f}, 1^{b-2f}) \to (2^{c+d}, 1^{b-2d})] = \binom{b-d-f}{d-f}.
\]

Proof. For \([(2^{c+f}, 1^{b-2f}) \to (2^{c+d}, 1^{b-2d})]\], we need to count the sets \(\beta\) that results by removing \(d-f\) elements from \(\text{Set}((2^{c+f}, 1^{b-2f}))\) so that \(\text{shape}_n(\beta) = (2^{c+d}, 1^{b-2d})\). All the spaces we create by removing elements need to be spaces of size 2, so we can only remove those elements corresponding to 1’s in the partition, and no two consecutive. There are \(b-2f-1\) elements of the set corresponding to 1’s in the partition, and need to choose \(d-f\) to remove with no two consecutive, which is \(\binom{b-d-f}{d-f}\). \(\square\)
Lemma 6.5.4. For $1 \leq j < d$

$$(-1)^{d-1}inom{b-d}{d} + \sum_{f=1}^{j} (-1)^{d-f-1}inom{b-d-f}{d-f} \frac{b(b-1) \cdots (b-f+2)}{f!} (b-2f+1)$$

$$= (-1)^{d+j-1} \frac{b(b-1) \cdots (b-j+1)}{d!} (b-d-j) \cdots (b-2d+1) \binom{d-1}{j}$$

Proof. By induction. The base case where $j = 1$:

$$(-1)^{d-1}inom{b-d}{d} + (-1)^{d-2}inom{b-d-1}{d-1} (b-1)$$

$$= (-1)^{d-1} \frac{(b-d-1)!}{(b-2d)!d!} ((b-d) - d(b-1))$$

$$= (-1)^{d-1} \frac{(b-d-1)!}{(b-2d)!d!} (b(1-d))$$

$$= (-1)^{d} \frac{b}{d!} (b-d-1) \cdots (b-2d+1) \binom{d-1}{1}$$

We assume the statement is true for $j = i$. For $j = i + 1$ (where $i + 1 < d$), the left hand side of 6.5.4 is:

$$= \sum_{f=1}^{i+1} (-1)^{d-f-1} \binom{b-d-f}{d-f} \frac{b(b-1) \cdots (b-f+2)}{f!} (b-2f+1)$$

$$+ (-1)^{d-1} \frac{b}{d!}$$

$$= \sum_{f=1}^{i} (-1)^{d-f-1} \binom{b-d-f}{d-f} \frac{b(b-1) \cdots (b-f+2)}{f!} (b-2f+1)$$

$$+ (-1)^{d-(i+1)} \frac{b(b-1) \cdots (b-i+1)}{d-(i+1)!} (b-2i+1)$$

$$+ (-1)^{d-1} \frac{b}{d!}$$

$$= (-1)^{d-1+i} \frac{b(b-1) \cdots (b-1+i)}{d!} (b-d-i) \cdots (b-2d+1) \binom{d-1}{i} +$$

$$(-1)^{d-(i+1)} \frac{b(b-1) \cdots (b-i+1)}{d-(i+1)!} (b-2i-1)$$
Theorem 6.5.5. If \( d = 1 \),

\[
\alpha_{(2^c, 1^b), (2^{c+1}, 1^{b-2})} = b - 1.
\]

If \( d > 1 \),

\[
\alpha_{(2^c, 1^b), (2^{c+d}, 1^{b-2d})} = \frac{b(b - 1) \cdots (b - d + 2)}{d!} (b - 2d + 1)
\]

Proof. Note that the case where \( d = 1 \) means that \( \mu \) and \( \lambda \) are a cover relation in the lattice of partitions under refinement, so according to result 1,

\[
\alpha_{(2^c, 1^b), (2^{c+1}, 1^{b-2})} = [(2^c, 1^b) \rightarrow (2^{c+1}, 1^{b-2})] = b - 1.
\]

When \( d = 2 \),
\[ a_{\mu,\lambda} = (-1)^{l(\lambda) - l(\mu)} \sum_{P \in \text{Paths}(\mu,\lambda)} [P](-1)^{l(P)} \]
\[ = (-1)^2(\left( (2^c, 1^b) \rightarrow (2^{c+2}, 1^{b-4}) \right) - \left( (2^c, 1^b) \rightarrow (2^{c+1}, 1^{b-2}) \right) \times \left( (2^{c+1}, 1^{b-2}) \rightarrow (2^{c+2}, 1^{b-4}) \right)). \]

By 6.5.3, that is
\[ \left( \begin{array}{c} b - 2 \\ 2 \end{array} \right) - \left( \begin{array}{c} b - 1 \\ 1 \end{array} \right) \left( \begin{array}{c} b - 3 \\ 1 \end{array} \right) = -1(b)(b - 3) \]

For \( d > 2 \), we will proceed by induction on \( d \), and factor out the last step in each path. Assume that the theorem is true for \( d = k \). Now let \( d = k + 1 \). To simplify notation, let \( \mu = (2^c, 1^b) \), \( \lambda = (2^{c+k+1}, 1^{b-2(k+1)}) \), and \( \beta_f = (2^{c+f}, 1^{b-2f}) \). We will use Lemmas 6.5.3 and 6.5.4.

\[ a_{\mu,\lambda} = (-1)^{l(\lambda) - l(\mu)} \sum_{P \in \text{Paths}(\mu,\lambda)} [P](-1)^{l(P)} \]
\[ = (-1)^2\left( \left[ \mu \rightarrow \lambda \right](-1) + \sum_{f=1}^{\beta} \sum_{P \in \text{Paths}(\mu,\beta_f)} [P](-1)^{l(P)+1}[\beta_f \rightarrow \lambda] \right) \]
\[ = (-1)^2 + (-1)^2\sum_{f=1}^{\beta} [\beta_f \rightarrow \lambda](-1)^{l(\beta_f) - l(\mu)}(-1)^{l(\beta_f) - l(\mu)} \times \]
\[ \sum_{P \in \text{Paths}(\mu,\beta_f)} [P](-1)^{l(P)} \]
\[ = (-1)^2\left( \frac{b - j - 1}{j + 1} \right) + (-1)^2\sum_{f=1}^{\beta} (-1)^{j} \left( \frac{b - (j + 1) - f}{j + 1 - f} \right) \times \]
\[ \frac{b(b-1) \cdots (b-f+2)}{f!} (b - 2f + 1) \]

Using Lemma 6.5.4, with \( j = d - 1 \), we obtain
\[ = (-1)^{j+1} b(b - 1) \cdots (b - j + 1) (b - 2j - 1) \cdots (b - 2(j + 1) - 1) \]
\[ = \frac{b(b - 1) \cdots (b - (j + 1) + 2)}{(j + 1)!} (b - 2(j + 1) + 1). \]

Finding the value of one coefficient also tells us the value of an infinite number of other coefficients. Let \( \mu = (\mu_1, \ldots, \mu_j) \). That is, define \( k\mu \) to be the partition obtained when each part of \( \mu \) is multiplied by \( k \) so that \( k\mu = (k\mu_1, \ldots, k\mu_j) \). Then we can prove the following result.

**Theorem 6.5.6.** For all \( k \in \mathbb{N} \),

\[ a_{\mu, \lambda} = a_{k\mu, k\lambda}. \]

**Proof.** The key point is that if \( k\mu < r, \beta < r, k\lambda \), then all the parts of the partition \( \beta \) must have sizes which are multiples of \( k \). Thus \( \beta \) must be of the form \( k\alpha \) for some \( \mu < r \alpha < r \lambda \). It follows that the refinement lattice between \( \mu \) and \( \lambda \) is isomorphic to the refinement lattice between \( k\mu \) and \( k\lambda \). Moreover it is easy to see \( [k\mu \to k\lambda] = [\mu \to \lambda] \). The proof now follows by our recursions.

\[ \square \]

In particular, if we apply 6.5.6 to 6.5.5, we obtain infinite number of cases where we have explicit formulas for \( a_{\mu, \lambda} \).

Here is another result of the same sort.

**Theorem 6.5.7.** Let \( \mu = (\mu_1, \ldots, \mu_s) \) and \( \lambda = (\lambda_1, \ldots, \lambda_t) \). Then for any \( j \) such that \( 1 \leq j < \min(\mu_s, \lambda_t) \),

\[ a_{\mu, \lambda} = a_{(\mu_1, \ldots, \mu_s, j), (\lambda_1, \ldots, \lambda_t, j)}. \]

The proof of 6.5.7 follows from examining the compositions and noticing that we must always have the last element of the composition in our subsets \( S \) in order
for shapeₙ(S) to match (λ₁, ..., λₜ, k). This theorem works in "both directions", so to speak. Knowledge of the coefficients aₜ,λ where µ ⊨ n and λ ⊨ n both with smallest part larger than 1 allows us to compute values of aₐ,β for certain partitions α and β of size larger than n. Conversely, knowledge of coefficients a₉,λ where µ and λ have identical unique smallest part allows us to compute values of aₐ,β where α and β are partitions of size smaller than n by removing that smallest part from both µ and λ.

Thus the combination of 6.5.6 and 6.5.7 enables us to calculate the value aₐ,β for infinitely many α and β starting with a single value of a₉,λ. That is, starting with a₉,λ, we can first multiply each part by k, then add smaller parts on the end, and so on.

6.5.2 Special Cases of the b₉,λ's

Our method of expansion in terms of Schur functions is useful not only in calculating particular expansions, but can also be used to make general statements independent of the size of λ.

We can use the fact that b₉,λ can be expressed as a₉,λ to prove further results. In particular we have the following results.

Theorem 6.5.8. 1. DZₙ(n) = sₙ(n)

2. DZ₁(n) = s₁ₙ

3. DZ₂ₙ = s₂ₙ,₁ₙ⁻₂ₖ ∀ k

4. DZ₃ₙ = s₃ₙ,₁ₙ⁻₃ₖ ∀ k

5. DZ₄ₙ = s₄ₙ,₁ₙ⁻₄ₖ

6. DZ₅ₙ = s₅ₙ,₁ₙ⁻₅ₖ

Proof. The proof of 1 was given above. The proofs of the others involve using the combinatorial interpretation of the coefficients that arise in (6.2.2). We include here only the proof of 5.
We will show 5 by showing that if we expand both sides in terms of the monomial symmetric functions, the coefficients of $m_\mu$ are equal for any $\mu$.

$$DZ_{(3,2,1k-5)}m_\mu = a_{(\mu),(3,2,1k-5)}$$

Now if $\mu$ is not a refinement of $(3, 2, 1k-5)$, then $a_{\mu,(3,2,1k-5)} = 0$. Recall that when we write a partition, $(i^j)$ is the partition with $j$ copies of $i$.

\[
a_{(3,2,1k-5),(3,2,1k-5)} = 1
\]
\[
a_{(2,2,1k-4),(3,2,1k-5)} = 1
\]
\[
a_{(3,1k-3),(3,2,1k-5)} = k - 4
\]
\[
a_{(2,1k-2),(3,2,1k-5)} = 1
\]
\[
a_{(1k),(3,2,1k-5)} = \frac{-1}{2}k(k-5)
\]

We let $\alpha = (3, 2, 1k-5)$, $\beta = (2, 2, 1k-4)$.

Table 6.4: Necessary values for $S_\lambda|_{m_\mu}$.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$(2^21k-4)$</th>
<th>$(3, 1k-3)$</th>
<th>$(2, 1k-2)$</th>
<th>$(1k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{(3,2,1k-5)}$</td>
<td>2</td>
<td>1</td>
<td>2($k-4$)</td>
<td>$k-4$</td>
<td>$(k-4)(k-2)$</td>
</tr>
<tr>
<td>$s_{(2^3,1k-6)}$</td>
<td>1</td>
<td>0</td>
<td>$k-5$</td>
<td>0</td>
<td>$\frac{1}{2}(k-5)(k-2)$</td>
</tr>
<tr>
<td>$s_{(2^2,1k-4)}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$k-3$</td>
</tr>
<tr>
<td>$s_{(3,2,1k-5)}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$k-4$</td>
<td>1</td>
</tr>
<tr>
<td>$-2s_{(2^3,1k-6)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-s_{(2^2,1k-4)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The entries of this table are obtained by recalling that $s_{\lambda|m_\mu}$ is the number of column-strict fillings of a Ferrers diagram of shape $\lambda$ with $\mu_1 1s$, $\mu_2 2s$, $\ldots$, etc. Again, each of the entries in this table requires is minor exercise in counting, so we do not include the details. Comparing our results, we can see 5 holds. $\square$
6.6 Applications of Our Main Result

6.6.1 Ribbon Schur Functions in terms of Ribbon Schur Functions indexed by partitions

As noted in the introduction, one application of our main result is to give a combinatorial interpretation of the expansion of $Z_\alpha$ in terms of $Z_\lambda$’s, where $\alpha$ is a composition of $n$ and $\lambda$ is a partition of $n$. Note that if 

$$Z_\alpha = \sum_{\mu \vdash n} f_{\mu,\alpha} Z_\mu,$$

then

$$f_{\mu,\alpha} = \langle Z_\alpha, DZ_\mu \rangle = \langle \sum_{T \subseteq \text{Set}(\alpha)} (-1)^{|\text{Set}(\alpha) - T|} h_{\lambda(\beta(T))}, \sum_{\lambda \vdash n} a_{\mu,\lambda} m_\lambda \rangle$$

$$= \sum_{T \subseteq \text{Set}(\alpha)} (-1)^{|\text{Set}(\alpha) - T|} a_{\lambda(\beta(T)),\mu}$$

since $\{h_\lambda\}_{\lambda \vdash n}$ and $\{m_\lambda\}_{\lambda \vdash n}$ and dual bases. We now present an example of this fact; we will expand $Z_{(2,2,4,2)}$ as a sum of $Z_\lambda$’s indexed by partitions of 10.

Table 6.5: Values for $(-1)^{|\text{Set}(\alpha) - T|}$ and $\lambda(\beta(T))$ for all $T \subseteq \text{Set}(2,2,4,2)$.

| $T$   | $(-1)^{|\text{Set}(\alpha) - T|}$ | $\lambda(\beta(T))$ |
|-------|-----------------------------------|---------------------|
| $\emptyset$ | $-1$                             | $(10)$              |
| $\{2\}$   | $1$                              | $(8,2)$             |
| $\{4\}$   | $1$                              | $(6,4)$             |
| $\{8\}$   | $1$                              | $(8,2)$             |
| $\{2,4\}$ | $-1$                             | $(6,2,2)$           |
| $\{2,8\}$ | $-1$                             | $(6,2,2)$           |
| $\{4,8\}$ | $-1$                             | $(4,4,2)$           |
| $\{2,4,8\}$ | $1$                          | $(4,2,2,2)$         |

Table 6.5 tells us that

$$f_{\mu,(2,2,4,2)} = a_{(4,2,2,2),\mu} - a_{(4,4,2),\mu} - 2a_{(6,2,2),\mu} + a_{(6,4),\mu} + 2a_{(8,2),\mu} - a_{(10),\mu}.$$ 

Then Table 6.6 gives that $Z_{(2,2,4,2)} = Z_{(4,2,2,2)} + Z_{(4,4,2)} - Z_{(6,2,2)} - Z_{(6,4)} + Z_{(8,2)}$. 

Table 6.6: Values for $a_{\gamma,\mu}$ used to compute $Z_{(2,2,4,2)} = \sum_{\mu \vdash n} f_{\mu,(2,2,4,2)}Z_{\mu}$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>(4, 2, 2)</th>
<th>(4, 4, 2)</th>
<th>(6, 2, 2)</th>
<th>(6, 4)</th>
<th>(8, 2)</th>
<th>(10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{(4,2,2),\mu}$</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$-a_{(4,4,2),\mu}$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$-2a_{(6,2,2),\mu}$</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>$a_{(6,4),\mu}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$2a_{(8,2),\mu}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$-a_{(10),\mu}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>Sum for each $\mu$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

As another application of our results is that we can give a combinatorial interpretation of the coefficients that arise in the expansion of a Schur function $s_{\gamma}$ in terms of the $Z_{\lambda}$’s where $\gamma, \lambda \vdash n$. That is, we can give a combinatorial interpretation of $e_{\mu,\gamma}$ where $s_{\gamma} = \sum_{\mu \vdash n} e_{\mu,\gamma}Z_{\mu}$.

Note that by 6.2.1, $s_{\gamma} = \sum_{\mu} K_{\mu,\gamma}^{-1}h_{\mu}$, so that

$$e_{\lambda,\gamma} = \langle s_{\gamma}, DZ_{\lambda} \rangle$$
$$= \langle \sum_{\mu} K_{\mu,\gamma}^{-1}h_{\mu}, \sum_{\beta \leq \lambda} a_{\beta,\lambda} m_{\beta} \rangle$$
$$= \sum_{\beta \leq \lambda} K_{\beta,\gamma}^{-1}a_{\beta,\lambda}.$$  

We now present an example by expanding $s_{(3,2,1)}$ as a sum of ribbon Schur functions indexed by partitions. We can easily see that $s_{(3,2,1)} = h_{1}h_{2}h_{3} - h_{1}h_{1}h_{4} - h_{3}h_{3} + h_{1}h_{5}$ by writing down all the special rim hook tabloids of shape $(3,2,1)$. Then

$$\langle s_{(3,2,1)}, DZ_{\lambda} \rangle = a_{(3,2,1),\lambda} - a_{(4,1,1),\lambda} - a_{(3,3),\lambda} + a_{(5,1),\lambda}.$$  

In Table 6.7, we present the relevant values of $a_{\mu,\lambda}$.

Thus

$$s_{(3,2,1)} = Z_{(3,2,1)} - Z_{(4,1,1)} - Z_{(4,2)} + Z_{(5,1)}.$$  

This may not be the most efficient algorithm in all cases, for example another approach is to use a result of Lascoux and Pragacz [26] which gives the expansion
Table 6.7: Values for $a_{\gamma, \lambda}$ used to compute $s_{(3,2,1)} = \sum_{\mu \vdash n} e_{\mu,(3,2,1)} Z_{\mu}$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>(3, 2, 1)</th>
<th>(4, 1, 1)</th>
<th>(3, 3)</th>
<th>(4, 2)</th>
<th>(5, 1)</th>
<th>(6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{(3,2,1),\lambda}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$-a_{(4,1,1),\lambda}$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$-a_{(3,3),\lambda}$</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$a_{(5,1),\lambda}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Sum for each $\lambda$</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

of a Schur function as a product of ribbon Schur functions using a determinantal formula. Any product ribbon Schur functions can be simplified to a sum of ribbon Schur functions. However the ribbon Schur functions that result from such an expansion are just arbitrary $Z_\alpha$ where $\alpha$ is a composition. Thus one would need to expand $Z_\alpha = \sum_{\lambda \vdash n} f_{\lambda,\alpha} Z_{\lambda}$, where $\alpha$ is a composition of $n$ and $\lambda$ is a partition of $n$, as we did above. In special cases, such as when $\gamma$ is a double hook, this method may be more efficient. However this method does not give a combinatorial interpretation of the coefficients of the $Z_\lambda$’s that arise in the expansion.

### 6.6.2 A decomposition of Schur functions indexed by hooks into $DZ_\lambda$s.

A surprising result of our work in this chapter is that we discovered a coefficient-free decomposition of hook Schur functions. We would like to show the following results that was suggested by Francois Bergeron [7].

**Theorem 6.6.1.**

\[
\sum_{\lambda \vdash n} DZ_\lambda = \sum_{\lambda \vdash n, \lambda = \text{a hook}} S_{\lambda}.
\]

**Proof.**

\[
\sum_{\lambda \vdash n, \lambda = \text{a hook}} S_{\lambda} = \sum_{k=0}^{n-1} S_{(1^k, n-k)}.
\]

Let’s examine the coefficient of $m_\mu$ when we write expand this sum into the monomial basis elements, where $\mu = (\mu_1, \ldots, \mu_l)$. $S_{(1^k, n-k)} | m_\mu$ counts the number
of ways to fill a \((1^k, n-k)\) shape with \(\mu_i\) is. We know that all \(u_1\) 1s have to be in the bottom row of the shape, and we can have only at most 1 of each other number in any other row, so we at least have \(\mu_1\) 1’s, \(\mu_2 - 1\) 2’s, \(\ldots\), \(\mu_l(\mu) - 1\) \(l(\mu)\)’s in the first row. The other numbers we have left to place are \(2, 3, \ldots, l\), and we need to choose \(k - 1\) of them to be in the other rows. Thus,

\[
\sum_{k=0}^{n-1} S_{1^k, n-k}^{(1^k, n-k)} m_\mu = \sum_{k=0}^{n-1} \binom{l(\mu) - 1}{k - 1} = 2^{l(\mu) - 1}
\]

We will now show that \(\sum_{\lambda \vdash n} DZ_{\lambda} |_{m_\mu} = 2^{l(\mu) - 1}\) By definition, \(\sum_{\lambda \vdash n} DZ_{\lambda} |_{m_\mu} = \sum_{\lambda \vdash n} a_{\mu, \lambda}\).

We proceed by induction on the length of \(\mu\). Base cases: If \(l(\mu) = 1\), then

\[
\sum_{\lambda \vdash n} a_{\mu, \lambda} = a_{(n), (n)} = 1 = 2^{1 - 1}
\]

If \(l(\mu) = 2\), then

\[
\sum_{\lambda \vdash n} a_{\mu, \lambda} = a_{(k, n-k), (k, n-k)} + a_{(k, n-k), (n)} = 1 + 1 = 2^{2 - 1}
\]

Assume the theorem is true for \(l(\mu) = n\), and let \(l(\mu) = n + 1\)

\[
\sum_{\lambda} a_{\mu, \lambda} = \sum_{\lambda \geq \mu} (-1)^{l(\mu) - l(\lambda)} \sum_{P \in Path(\mu, \lambda)} [P](-1)^{l(P)}
\]

\[
= (\sum_{\lambda \geq \mu} (-1)^{l(\mu) - l(\lambda)} \sum_{P \in Path(\mu, \lambda)} [P](-1)^{l(P)}) + [\mu \rightarrow \mu]
\]

\[
= (\sum_{\lambda \geq r \mu} (-1)^{l(\mu) - l(\lambda)} \sum_{\mu < r \tau \leq r \lambda} [\mu \rightarrow \tau](-1) \sum_{P' \in Path(\tau, \lambda)} [P'](-1)^{l(P')})
\]

\[
+ [\mu \rightarrow \mu]
\]

\[
= (\sum_{\lambda \geq r \mu} \sum_{\mu < r \tau \leq r \lambda} (-1)^{l(\mu) - l(\tau)} [\mu \rightarrow \tau](-1)\left(\sum_{P' \in Path(\tau, \lambda)} (-1)^{l(P')}\right)) + [\mu \rightarrow \mu]
\]

(6.6.1)
\[
\sum_{\lambda > r} \sum_{\mu \leq r \leq \lambda} (-1)^{l(\mu)} \mu \rightarrow \tau a_{\tau, \lambda} + \mu \rightarrow \mu
\]

\[
= (-1)^{l(\mu)} \sum_{\lambda > r} \sum_{\mu < r} (-1)^{l(\tau)+1} \mu \rightarrow \tau a_{\tau, \lambda} + 1
\]

\[
= (-1)^{l(\mu)} \sum_{\tau > r} \mu \rightarrow \tau (-1)^{l(\tau)+1} 2^{l(\tau)-1} + 1
\]

\[
= (-1)^{n+1} \sum_{\tau > r} \mu \rightarrow \tau (-1)^{l(\tau)+1} 2^{l(\tau)-1} + 1
\]

Note that to sum over \([\mu \rightarrow \tau]\) over all \(\tau\) of length \(k\) means we count all the subsets of \(\beta(\mu)\) of size \(k - 1\).

\[
= (-1)^{n+1} \sum_{k=1}^{n} (-1)^{k} 2^{k-1} \binom{n}{k-1} + 1
\]

\[
= (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^{k} 2^{k} \binom{n}{k} + 1
\]

\[
= \sum_{k=0}^{n-1} (-1)^{n-k} 2^{k} \binom{n}{k} + 1
\]

\[
= 2^n - 1 + 1
\]

\[
= 2^n = 2^{l(\mu)-1}
\]

where the antepenultimate equality holds after using the binomial theorem.

We will need the following binomial coefficient result in order to prove our next theorem.

**Lemma 6.6.2.** For \(m \geq k, j \geq k\),

\[
\binom{m-1}{k-1} = \binom{m-1}{k-1} (-1)^{m-k-1} + \sum_{i=k+1}^{m-1} (-1)^{i+k-1} \binom{i-1}{k-1} \binom{m-1}{i-1}.
\]
Proof. By the binomial theorem,

\[ 0 = (1 + (-1)^n = \sum_{s=0}^{n} (-1)^s \binom{n}{s} = \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} \]

Then

\[ 1 = \sum_{s=0}^{n-1} (-1)^{s-1} \binom{n}{s} \]

Substitute \( n = m - k, \ s = i - k. \)

\[ 1 = \sum_{i-k=0}^{m-k-1} (-1)^{i-k-1} \binom{m-k}{i-k} \]

\[ = \sum_{i=k}^{m-1} (-1)^{i-k-1} \binom{m-k}{i-k} \]

Multiplying both sides by a binomial coefficient,

\[ \binom{m-1}{k-1} = \sum_{i=k}^{m-1} (-1)^{i-k-1} \binom{m-1}{k-1, \ i-k, m-i} \]

\[ = \sum_{i=k+1}^{m-1} (-1)^{i-k-1} \binom{m-1}{k-1, \ i-k, m-i} + (-1)^{m-k-1} \binom{m-1}{k-1} \]

\[ = (-1)^{m-k-1} \binom{m-1}{k-1} + \sum_{i=k+1}^{m-1} (-1)^{i+k-1} \binom{m-1}{i-1} \binom{i-1}{k-1}. \]

\[ \square \]

Our next theorem is a stronger result than Theorem 6.6.1.

**Theorem 6.6.3.**

\[ s_{(n-k+1,1^{k-1})} = \sum_{\lambda \vdash n, l(\lambda) = k} DZ_\lambda. \]
Proof. We would like to show that for any \( \mu \),

\[
\sum_{\lambda \vdash n, l(\lambda) = k} DZ_{\lambda}|_{m(\mu)} = \sum_{\lambda \vdash n, l(\lambda) = k} DZ_{\lambda}|_{m(\mu)}.
\]

The left-hand side of this equation is the number of column-strict tableaux for the Ferrers diagram of \((n - k + 1, 1^{k-1})\) with \( \mu_1 \) 1s, \( \mu_2 \) 2s, \ldots, \( \mu_{l(\mu)} \) \( l(\mu) \) s. But since this shape is a hook, we know all \( \mu_1 \) 1s must go in the bottom row. For the other rows, we must fill with \( k - 1 \) of \( 2, \ldots, l(\mu) \), which is \( \binom{l(\mu) - 1}{k - 1} \). The rest of the filling is completely determined.

So we would like to show

\[
\sum_{\lambda \vdash n, l(\lambda) = k} DZ_{\lambda}|_{m(\mu)} = \sum_{\lambda \vdash n, l(\lambda) = k} a_{\mu, \lambda} = \binom{l(\mu) - 1}{k - 1}.
\]

We will proceed by induction on \( l(\mu) - l(\lambda) \). For \( l(\mu) - l(\lambda) = 0 \), we have \( \mu = \lambda \). Then

\[
\sum_{\lambda \vdash n, l(\lambda) = k} DZ_{\lambda}|_{m(\mu)} = \sum_{\lambda \vdash n, l(\lambda) = k} a_{\mu, \lambda} = a_{\mu, \mu} = 1,
\]

which equals \( \binom{l(\mu) - 1}{l(\mu) - 1} \).

Assume the theorem is true for \( l(\mu) - l(\lambda) \leq j \). We will prove it for \( l(\mu) - l(\lambda) = j + 1 \).

\[
\sum_{\lambda \vdash n, l(\lambda) = k} a_{\mu, \lambda} = \sum_{\lambda \vdash n, l(\lambda) = k} (-1)^{l(\mu) - l(\lambda)} \sum_{P \in \text{Path}(\mu, \lambda)} [P](-1)^{l(P)} - \sum_{P \in \text{Path}(\mu, \lambda)} [P](-1)^{l(P)} - [\mu \rightarrow \lambda])
\]

\[
= (-1)^{l(\mu) - l(\lambda) - 1} \sum_{\lambda \vdash n, l(\lambda) = k} [\mu \rightarrow \lambda]
\]

\[
+ (-1)^{l(\mu) - l(\lambda)} \sum_{\lambda \vdash n, l(\lambda) = k} \sum_{P \in \text{Path}(\mu, \lambda), l(P) \geq 2} [P](-1)^{l(P)}
\]

(6.6.2)
\[
\sum_{\lambda \vdash n, \ell(\lambda) = k} [\mu \rightarrow \lambda] \text{ is exactly the number of subsets of } \text{Set}(\mu) \text{ with size } k - 1, \text{ which is equal to } \binom{l(\mu) - 1}{k - 1}.
\]

Next we will examine \[\sum_{\lambda \vdash n, \ell(\lambda) = k} \sum_{P \in \text{Path}(\mu, \lambda), \ell(P) \geq 2} [P](-1)^{l(P)}\]. Recall that partitions of \(n\) under the refinement order form a lattice. We can grade this lattice by length of the partition. Then every path in this lattice from \(\mu\) to \(\lambda\) with two steps must have final step whose grade is strictly in between \(l(\lambda)\) and \(l(\mu)\). We will sum over all such possible penultimate steps \(\nu\).

\[
\sum_{\lambda \vdash n, \ell(\lambda) = k} \sum_{P \in \text{Path}(\mu, \lambda), \ell(P) \geq 2} [P](-1)^{l(P)} = \sum_{\lambda \vdash n, \ell(\lambda) = k} \sum_{i = l(\mu) + 1}^{l(\mu) - 1} \sum_{\nu \vdash n, \ell(\nu) = i} [\nu \rightarrow \lambda](-1)^{l(\nu) - l(\mu)} a_{\mu, \nu}
\]

\[
= \sum_{\lambda \vdash n, \ell(\lambda) = k} \sum_{i = l(\mu) + 1}^{l(\mu) - 1} \sum_{\nu \vdash n, \ell(\nu) = i} [\nu \rightarrow \lambda](-1)^{l(\nu) - l(\mu)} a_{\mu, \nu}
\]

\[
= \sum_{i = l(\mu) + 1}^{l(\mu) - 1} \sum_{\nu \vdash n, \ell(\nu) = i} a_{\mu, \nu}(-1)^{l(\nu) - l(\mu) + 1} \sum_{\lambda \vdash n, \ell(\lambda) = k} [\nu \rightarrow \lambda]
\]

\[
= \sum_{i = l(\mu) + 1}^{l(\mu) - 1} a_{\mu, \nu}(-1)^{l(\nu) - l(\mu) + 1} \left( \binom{i - 1}{k - 1} \right)
\]

\[
= \sum_{i = l(\mu) + 1}^{l(\mu) - 1} \left( \binom{i - 1}{k - 1} \right) (-1)^{i - l(\mu) + 1} \sum_{\nu \vdash n, \ell(\nu) = i} a_{\mu, \nu}
\]

\[
= \sum_{i = l(\mu) + 1}^{l(\mu) - 1} (-1)^{i - l(\mu) + 1} \binom{i - 1}{k - 1} \binom{l(\mu) - 1}{i - 1}
\]

We now combine these pieces back to continue simplifying equation 6.6.2.
\begin{align*}
\sum_{\lambda \in n} a_{\mu,\lambda} &= (-1)^{l(\mu) - l(\lambda) - 1} \sum_{\lambda \in n} [\mu \rightarrow \lambda] \\
&\quad + (-1)^{l(\mu) - l(\lambda)} \sum_{\lambda \in n} \sum_{P \in \text{Path}(\mu, \lambda)} [P] \left( -1 \right)^{l(P)} \\
&= \binom{l(\mu) - 1}{k - 1} \left( -1 \right)^{l(\mu) - l(\lambda) - 1} \\
&\quad + (-1)^{l(\mu) - l(\lambda)} \sum_{i = l(\lambda) + 1}^{l(\mu) - 1} (-1)^{i + l(\mu) - 1} \binom{i}{k - 1} \left( \binom{l(\mu) - 1}{i - 1} \right)
\end{align*}

Recall that \( l(\lambda) = k \), and letting \( l(\mu) = m \),

\begin{align*}
&= \binom{m - 1}{k - 1} \left( -1 \right)^{m - k - 1} + \sum_{i = k + 1}^{m - 1} (-1)^{i + k - 1} \binom{i - 1}{k - 1} \left( \binom{m - 1}{i - 1} \right) \\
&= \binom{m - 1}{k - 1}
\end{align*}

by our Lemma 6.6.2.

\[\Box\]

We note here that our previous theorem 6.6.1 is actually a corollary to 6.6.3. However, the theorem 6.6.1 came several months earlier than 6.6.3, so we include it for a sense of chronology.

Here we would also like to point out that after seeing our proof, Garsia developed an alternate proof of 6.6.3[16]. Garsia’s proof relies on the relationship between ribbon Schur functions and quasisymmetric functions. Garsia used this
relationship to show that the coefficients in the expansion of $s_{(1^k,n-k)}$ are 1 exactly when the partition has $k + 1$ parts, and 0 otherwise. Garsia’s proof is shorter, given background in quasisymmetric functions. However, the theorem was, as far as the authors know, unknown before our proof, and arises quite naturally from our combinatorial interpretation (simply by classifying paths by their penultimate step).

6.7 Conclusions and Further Research

In this chapter we have given combinatorial interpretations of the coefficients in the expansion of $DZ_{\lambda}$ in terms of the monomial symmetric functions. We also found more indirect combinatorial interpretations of the expansion $DZ_{\lambda}$ in terms of the Schur functions by using the inverse Kostka matrix. Moreover, we have given explicit formulas for such coefficients in many special cases.

There are many unanswered questions in this area. Of particular interest is what happens when we apply the $\omega$ transformation to $DZ_{\lambda}$. That is, recall the $\omega : \Lambda_n \rightarrow \Lambda_n$ is defined by the fact for all $\lambda \vdash n$, $\omega(h_{\lambda}) = e_{\lambda}$. Then the question is: can we give a combinatorial interpretation of $\omega(DZ_{\lambda})$ in terms of $\{Z_{\lambda}\}_{\lambda \vdash n}$ or $\{DZ_{\lambda}\}_{\lambda \vdash n}$? We can clearly give a combinatorial interpretations of $\omega(DZ_{\lambda})$ in terms of $\{f_{\lambda}\}_{\lambda \vdash n}$, since we can already expand $DZ$ in terms of $\{m_{\lambda}\}_{\lambda \vdash n}$ and $\omega(m_{\lambda}) = f_{\lambda}$.

There are several other areas of interest, in particular we would like to find a natural extension of the $DZ_{\lambda}$ in the nonsymmetric case or noncommutative case. It is possible that our interpretation as sums of weighted paths in the lattice of partitions under refinement could be extended to unweighted paths in the lattice of compositions under refinement.

We examined the expansion of products $DZ_{\lambda} \cdot DZ_{\mu}$ in terms of the Schur basis and the $\{DZ_{\lambda}\}$ basis. As is standard, we shall say that such a product is Schur positive if the coefficients in the expansion in terms of Schur functions are all positive. We shall say that such a product is dual ribbon positive if the coefficients in the expansion in terms of $DZ_{\lambda}$’s are all positive. We have several results, whose
• $DZ_\lambda \cdot DZ_\mu$ is Schur positive and dual ribbon positive when $\lambda = (j)$ and $\mu = (1^{n-j})$.

• $DZ_\lambda \cdot DZ_\mu$ is Schur positive and dual ribbon positive when $\lambda = (1^j)$ and $\mu = (1^{n-j})$.

Our conjecture in this area is that $DZ_\lambda \cdot DZ_\mu$ is dual ribbon positive whenever $\mu$ and $\lambda$ are hook shapes. Another area for research would be to find a combinatorial interpretation to the coefficients of products of dual ribbons in terms of dual ribbons.

We also examined the coefficients of $DZ_\lambda$ in their expansion in terms of the power and elementary symmetric functions. Again the coefficients that arise in such expansions are not all positive. Thus another unanswered question is to find good combinatorial interpretations for the coefficients in the expansion of $DZ_\lambda$ in terms of the other standard bases for the space of symmetric functions.
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