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A Probabilistic Approach to the Air Traffic Management in The Next Generation Air Transportation System: Optimal Routing Decision With Geometric Recourse Model

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A Probabilistic Approach to the Air Traffic Management in The Next Generation Air Transportation System: Optimal Routing Decision With Geometric Recourse Model

By
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A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Engineering – Civil and Environmental Engineering in the Graduate Division of the University of California, Berkeley

Committee in charge:
Professor Mark Hansen, Chair
Professor Samer Madanat
Professor Andrew Lim
Professor Zou-Jun Shen

Fall 2010
To my mother

Now in heaven

Always in my heart
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ABSTRACT

A Probabilistic Approach to Air Traffic Management in The Next Generation Air Transportation System: Optimal Routing Decision With Geometric Recourse Model

by

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Doctor of Philosophy in Civil and Environmental Engineering
University of California, Berkeley

Professor Mark Hansen, Chair

There has been growing interest in air transportation community to develop a routing decision model based on probabilistic characterization of severe weather. In the probabilistic air traffic management (PATM), decisions are made based on the stochastic weather information in the expected total cost sense. Probabilistic approach aims to enhance routing flexibility and reduce the risks associated with uncertainty of the future weather.

In this research, a geometric model is adopted to generate optimal route choice when the future weather is stochastic. The geometric recourse model (GRM) is a strategic PATM model that incorporates route hedging and en-route recourse options to respond to weather change. Hedged routes are routes other than the nominal or detour route, and aircraft is re-routed to fly direct to the destination, which is called recourse, when the weather restricted airspace become flyable. Aircraft takes either the first recourse or the second recourse: The first recourse occurs when weather clears before aircraft reaches it flying on the initial route. The second recourse occurs when the aircraft is at the weather region.

There are two variations of GRM: Single Recourse Model (SRM) with first recourse only and Dual Recourse Model (DRM) with both the first and second recourse options. When the weather clearance time follows a uniform distribution, SRM becomes convex with optimal route being either the detour or a hedged route. The DRM has a special property when the maximum storm duration time is less than the flight time to the tip of the storm on the detour route: it is always optimal to take the nominal route. The performance study is conducted by measuring the cost saving from either SRM or DRM. The result shows that there are cases with substantial cost saving, reaching nearly 30% with DRM.

The ground-airborne hybrid model is an extension of the GRM, where both ground holding as well as route hedging are considered. The optimal combination of ground delay and route choice is determined by weather characteristics as well as the ground-airborne cost ratio. The numerical analysis reveals that whenever ground delay is required, the optimal route choice is the nominal one, while a non-nominal route is optimal when the ground delay is zero. There exists a unique critical cost ratio associated with given weather condition, which determines whether ground holding is optimal or not.
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I know that my father, whose love and support is simply unfathomable to me, will be happier and prouder than myself as he reads my dissertation. My mother, who passed away after battling a terminal cancer for three years in my fourth year in Berkeley, made me want to, and persevere to finish up this chapter of my life. I dedicate my work to my mother, who never stopped loving her little girl.
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I. INTRODUCTION

There is a growing interest in air traffic management (ATM) strategies that incorporate uncertainty in the national airspace system (NAS). Research in “probabilistic air traffic management” (PATM) seeks to guide decisions on ground-holding or otherwise modifying aircraft four-dimensional trajectories (4DTs) in order to minimize the expected cost, or to hedge against “worst case” scenarios in the Next Generation Air Transportation System (NextGen).

This research studies the problem of developing a minimum-cost aircraft routing strategy when some weather condition inhibits the use of nominal route for an indefinite period. In conventional Air Traffic Management (ATM), two options are commonly considered in such situation; the flight is either held at the origin airports until the nominal route becomes flyable or rerouted to avoid the weather region. The choice between these options is based upon a deterministic and conservative characterization of future weather, often resulting in underutilized airspace and unnecessary delay if the weather clears early.

This research proposes a geometric model to find an optimal route when the weather clearance time is stochastic. The route decision takes into account the probability distribution of storm clearance time, the possibility of route hedging, and recourse opportunities. When facing uncertain weather, there are two potential risks to hedge against: persistence risk and clearance risk. Persistence risk is the risk when we take an “optimistic” route and weather persists, resulting in unplanned re-routing and delay. Clearance risk is the risk when we take a “pessimistic” route and weather clears sooner, resulting unnecessary flight time. To mitigate these risks, we need to consider intermediate routing options that may not be optimal under either persistence or clearance, but hedge against either possibility. In doing so, we must consider how the route might be adjusted if the storm clears during the course of flight. We assume that the flight plan can be amended in such an event so that the plane can go direct to the destination.

In our model, the routing decision is made based on four parameters; nominal route between origin and destination airport, storm location, storm size, and maximum storm duration time. The optimistic route is the nominal one while pessimistic route goes around the storm. A hedged route is one that is between the optimistic and the pessimistic ones. We use the term recourse for a change in a routing that results from the storm clearing. We consider two recourse possibilities. First, the storm may clear before the aircraft reaches it, so that it can be rerouted directly to its destination. This is called first recourse. The storm may instead persist beyond the time when the aircraft reaches it—so that the plane must turn and begin to fly around it-- but clear before the tip of the storm is reached. The aircraft may then be rerouted direct to the destination; we refer to this as second recourse.

In our model, which we term the geometric recourse model (GRM), a triangle is drawn in which the base is the nominal path between the origin and destination airport, and the vertex is the tip of the storm, which we assume to be a straight line perpendicular to the nominal path. We seek routes that minimize expected total flight cost, which in some cases are hedged routes. We consider two variations of the geometric recourse model: the single recourse model (SRM) and dual recourse model (DRM). The SRM allows first
recourse only, while the DRM allows both first and second recourse. The SRM is more conservative, since it entails a higher penalty in the case when the aircraft reaches the storm region before it has cleared. The DRM assumes greater responsiveness to changing conditions and consequently results in reduced cost.

This research introduces the concept of the geometric recourse model and formulates nonlinear stochastic optimization models for the SRM and DRM. We assume that the storm clearance time follows a uniform distribution. With this assumption, we show that the SRM becomes convex, and find optimality conditions and the approximate analytic solution in closed form. We also find a condition that guarantees the nominal route to be optimal in the DRM. Through numerical study, we compare the total expected flight cost and cost saving for optimal routes obtained from the SRM and DRM under a wide range of parameter values.
II. BACKGROUND

While traffic in the national airspace system has temporarily abated, its pre-recession level was approaching the capacity limit, with air travelers frequently experiencing flight delays and cancellations. Out of all causes of such delays, weather has been the most dominant one. According to the US Department of Transportation, air travelers experienced the worst flight delay in 2007 since year 2000, and weather accounted for more than 75% of these delays, as shown in Figure 1.

In the event of adverse weather, one of the most widely used delay mitigation processes is the ground delay program (GDP). In a GDP, flights are held on the ground at the origin airport and assigned to new departure times based on available capacity at the destination airport. While the GDP is well-suited for airport arrival capacity restrictions, it is not for airspace capacity restrictions. Consequently, the Federal Aviation Administration (FAA) implemented Airspace Flow Programs (AFP) in June 2006. The purpose of AFPs is to control the en-route traffic demand in regions of airspace that are capacity-constrained, most commonly as the result of severe weather.

Neither GDPs nor AFPs explicitly recognize that future weather is uncertain. As a result, when weather changes unexpectedly, a significant amount of reactive and tactical control is required, often resulting in inefficient system utilization. The motivation of this research is to integrate probabilistic weather information into strategic planning to provide flexible and effective decision support in order to reduce losses from imperfect information about future weather.
There have been numerous efforts to address weather-related disruptions in the air traffic management. Earlier traffic flow management models such as Bertsimas [1] and Goodhart [2], often have a deterministic setting. More recently, Nilim et al. [4] proposed a dynamic aircraft routing model with robust control. This paper adopted shortest-path algorithms in a grid structure, by discretizing time into stages when the routing decisions are made, and airspace as a two-dimensional grid. The weather condition in each potential storm region is assumed and modeled as a Markovian process with two states: 0 (No storm) and 1 (Storm). The transition matrix is estimated based on the historical weather forecasts. Optimization results show a promising improvement compared to flying around the storm without recourse. The method in their paper has a robust control algorithm that with a wide range of applications.

In the air transportation system, however, the frequent routing adjustments entailed by this approach may place undue workload on controllers and pilots. Moreover, the Markovian assumption is of doubtful validity in the context of convective weather. Two of the goals in our study are to set up a model that has the flexibility to adopt a variety of probability distributions of storm clearance times, and to limit re-routing decisions to a reasonable number.

Bertsimas et al. [3] proposed a two-stage optimization model based on a dynamic network flow approach. The authors set up a multi-aircraft optimization model minimizing the weather delay cost, based on a deterministic weather scenario. One important aspect of their study is that the cost function covers all components of aircraft operation costs, such as fixed cost, ground holding cost, and airborne cost. From the air traffic management perspective, it would be ideal to utilize both Ground Delay Program (GDP) and airborne rerouting to mitigate weather related disruptions, especially since ground delay is less costly than extra flight time. Here, we do not explicitly consider the ground delay option, but instead focus on the choice of routing for a given time of departure. The extension of the model to support choice among alternate departure times is discussed at the conclusion of this paper.
IV. GEOMETRIC RECOURSE MODEL (GRM)

A. Geometric Recourse Model Concept

Consider the problem of routing a single flight in the presence of a single storm. Given an origin and destination pair, assume there is a linear storm of known size blocking the direct route at a certain location. Based on those five parameters—origin (O), destination (D), storm-route intersection ($S_L$), and storm tip $S_T$—construct a triangle $\triangle ODS_T$, where the nominal route is the base OD and storm size is the altitude $S_LS_T$, as illustrated in Figure 2.

Note that while the storm has two tips, we choose the one nearer to $S_L$, since this is the one that the aircraft would be routed around. Defining the unit of distance such that the aircraft cruises at a constant speed of 1, we refer to the base OD as the nominal route, the altitude $S_LS_T$ as the front of the storm and the vertex $S_T$ as the tip of the storm. The route $OS_TD$, which goes around the storm, is called the detour route. Upon departure, the aircraft may set a course along the nominal route, the detour route, or one in between.

During the course of the flight, aircraft may be re-routed to fly direct to the destination when the storm clears; we refer to such route changes as recourse. Depending on the timing of storm clearance, there are three recourse possibilities as illustrated in Figure 3: (a) recourse if the storm clears before the aircraft reaches it; (b) recourse at the storm front if the storm persists until after the aircraft reaches it, but clears as the aircraft flies along the storm front toward the tip; or (c) no recourse because the storm persists until after the aircraft reaches the tip of the storm.

We define the case (a) as the first recourse, the case (b) as the second recourse, and the case (c) as no recourse. Given the geometric setup, the objective is to find the route that minimizes expected total flight cost, where choosing a route is equivalent to choosing an angle between zero and the base angle $\angle S_TOS_L$. Although such a decision variable is
intuitive, the resulting objective function involves complex trigonometric terms that make it difficult to analyze. Instead, we propose a ratio-based model in which complexity is reduced without loss of generality.

![Figure 3 Recourse Options in Geometric Model](image)

In the ratio-based model, the nominal route and weather parameters are expressed as ratios to the nominal route as illustrated in Figure 4. In other words, we define the unit of distance as the distance of the nominal route, and the unit of time as the time required to fly that route. Now we introduce a new decision variable \( x \), which is the distance from the origin to the storm front along the course set from the origin. The ratio-based model is then formulated as follows.

![Figure 4 Ratio-based Geometric Model](image)

1: flight time (equivalent to distance) of nominal route between origin and destination
\( \alpha \): ratio of storm distance from origin in units of nominal route flight time: \( 0 < \alpha < 1 \)
\( \beta \): storm size in units of nominal route nominal route flight time: \( \beta > 0 \)
\( \mu \): random variable representing the storm clearance time with probability density function \( p(\mu) \)
\( x \): distance to the storm along course set from origin in units of nominal route flight time:
\[
\alpha \leq x \leq \sqrt{\alpha^2 + \beta^2}
\]
We consider two variations of geometric recourse model – Single Recourse (SRM) and Dual Recourse (DRM). The Single Recourse Model (SRM) allows the first recourse only, while the Dual Recourse Model (DRM) allows both the first and second recourse. The DRM, because it allows for immediate rerouting of flights moving along the storm region when the storm burns off, is more responsive to changing conditions. The SRM, because it assumes a large penalty for flights that reach the storm region prior to storm clearance, is more conservative. In addition, SRM is an upper bound to DRM.

B. Formulation of Single Recourse Model (SRM) and Dual Recourse Model (DRM)

The SRM is formulated as follows.

\[
\begin{align*}
\min & \int_0^\alpha (\mu + \sqrt{1 + \mu^2 - 2\mu\frac{\alpha}{x}}) p(\mu) d\mu + \\
& \int_0^\alpha (x + \beta - \sqrt{x^2 - \alpha^2} + \sqrt{(1-\alpha)^2 + \beta^2}) p(\mu) d\mu \\
\text{s.t.} & \quad \alpha \leq x \leq \sqrt{\alpha^2 + \beta^2}, \text{ where } 0 < \alpha < 1, \beta > 0
\end{align*}
\]

(1)

In the objective function, the first integral is the expected total flight cost when the first recourse is taken, and the second integral is the case when no recourse is possible. In other words, the first integrand is the flight time when the flight sets a course corresponding to \(x\) but turns to take a direct course to the destination when the storm clears at time \(\mu\). The second integrand is the time to fly \(x\) distance to the storm region, then along that region to the tip, and then direct to the destination. In our analytic study, we show that SRM becomes convex and analytically identify optimality conditions when the weather clearance time follows a uniform distribution. We also find approximate analytic solution using Taylor series approximation.

The Dual Recourse Model (DRM) allows recourse both before and at the storm region. The optimization model is formulated as follows.

\[
\begin{align*}
\min & \int_0^\alpha (\mu + \sqrt{1 + \mu^2 - 2\mu\frac{\alpha}{x}}) p(\mu) d\mu \\
& + \int_x^{x+\beta-\sqrt{x^2-\alpha^2}} (\mu + \sqrt{(\mu - x + \sqrt{x^2 - \alpha^2})^2 + (1-\alpha)^2}) p(\mu) d\mu \\
& + \int_{x+\beta-\sqrt{x^2-\alpha^2}}^{\infty} (x + \beta - \sqrt{x^2 - \alpha^2} + \sqrt{(1-\alpha)^2 + \beta^2}) p(\mu) d\mu \\
\text{s.t.} & \quad \alpha \leq x \leq \sqrt{\alpha^2 + \beta^2}, \text{ where } 0 < \alpha < 1, \beta > 0
\end{align*}
\]

(2)

In the objective function, the first integral is the expected total flight cost when the first recourse is taken, the second integral is the flight cost when the second recourse is taken, and the third integral is for the case when no recourse is possible. It is obvious that the SRM is an upper bound to the DRM with the additional second recourse option.
C. Uniform Weather Distribution and Numerical Solutions

We assume that the weather (storm) clearance time follows a uniform distribution ranging between 0 and T, or \( \mu \sim \text{Uniform}[0, T] \). The uniform distribution assumption not only makes the models analytically tractable, but it is a reasonable choice from a practical point of view as well. A forecast of convective weather is included in several weather forecast products published by National Oceanic and Atmospheric Administration (NOAA)’s Storm Prediction Center (SPC). One of the forecasts that is widely used in both practice and research is the convective outlook watch. According to the SPC, roughly 1,000 watches are published each year to address possible severe weather conditions in the subsequent few hours, and each convective activity is associated with a probability. The probability value provided in the forecast can be associated with the weather clearance probability we assume.

It should also be noted that while we maintain the assumption of a uniform distribution for clearance time, our results apply equally to the case where the clearance time distribution is uniform only up to the latest time when a flight can reach the tip of the storm. If the storm persists beyond that time, the details of its clearance time distribution are no longer important, since the flight cost is independent from the timing of the weather clearance.

With the uniform distribution assumption, we now have an additional parameter \( T \), which is the latest possible time that storm will remain, or maximum storm duration time. With the introduction of \( T \), it is clear that when \( T \leq \alpha \), both the SRM and the DRM have the optimal solution \( x^* = \alpha \) with expected total cost of one. In other words, if the storm will definitely clear before the aircraft reaches it by flying on the nominal route, then the nominal route is optimal without a weather interruption to the destination.

In addition to analytic research on the model properties, we utilize numerical solutions to enhance our knowledge of model behaviors and to obtain further insights. For numerical analysis, we adopt a sampling-based method to solve the geometric recourse models for a large set of parameter combinations. A 3-D grid of spacing 0.05—i.e. 5% of the nominal route distance— is created in \( \alpha - \beta - T \) space. Then, the upper limits are set on storm size and the maximum storm duration time at 2 and 4 respectively. In other words, the tip of the largest storm is twice the nominal route distance, and the longest storm duration is four times the flight time on the nominal route. Therefore, we have \( \alpha \in [0.05, 0.95], \beta \in [0.05, 2], T \in [0.05, 4] \) resulting in a set of 60,800 parameter combinations. Note that when \( \alpha = 0 \) or \( \alpha = 1 \), the storm is located either at the origin or the destination airport, in which case the ground delay program works best.

The numerical solution sets provide a solid foundation to further explore critical model properties such as projected improvement from adopting DRM or SRM. Since SRM is an upper bound to DRM, DRM guarantees lower flight time than SRM. Likewise, SRM guarantees less cost than taking the detour route. In the following chapters, individual models are discussed first, and the performance review is followed including optimal cost sensitivity and cost savings.
V. Single Recourse Model (SRM)

A. Convex Optimization and Optimality Conditions

When the weather clearance time follows a uniform distribution, we can show that the SRM becomes convex with negative gradient at the lower bound.

Let \( f_s(x) \) be the expected total cost function of the SRM in \( x \in I, \) where \( I = [\alpha, \sqrt{\alpha^2 + \beta^2}] \). Given the weather clearance time \( \mu \sim \text{Uniform}[0, T] \), we have

\[
f_s(x) = \int_0^x \left( \mu + \sqrt{1 + \mu^2 - 2\mu \frac{\alpha}{x}} \right) \frac{1}{T} d\mu + \int_x^T (x + \beta - \sqrt{x^2 - \alpha^2} + \sqrt{(1 - \alpha)^2 + \beta^2}) \frac{1}{T} d\mu \tag{3}
\]

Define \( f_{s1}(x, \mu) \) and \( f_{s2}(x, \mu) \) as follows.

\[
f_{s1}(x, \mu) = \mu + \sqrt{1 + \mu^2 - 2\mu \frac{\alpha}{x}} \tag{4}
\]

\[
f_{s2}(x, \mu) = x + \beta - \sqrt{x^2 - \alpha^2} + \sqrt{(1 - \alpha)^2 + \beta^2} \tag{5}
\]

Then, we have

\[
f_s'(\alpha) = \frac{1}{T} \left( f_{s1}(\alpha, \alpha) + \int_0^\alpha \frac{d}{dx} f_{s1}(x, \mu) d\mu \right)_{x=\alpha} - f_{s2}(\alpha, \alpha) + \int_\alpha^T \frac{d}{dx} f_{s2}(x, \mu) d\mu \right)_{x=\alpha} \tag{6}
\]

Since \( \lim_{x \to \alpha} \int_0^\alpha \frac{d}{dx} f_{s1}(x, \mu) d\mu \right)_{x=\alpha} + \int_\alpha^T \frac{d}{dx} f_{s2}(x, \mu) d\mu \right)_{x=\alpha} = -\infty \), we have \( f_s'(\alpha) < 0 \). Therefore, the gradient at the lower bound is always negative.

We also show that \( f_s(x) \) is a convex function based on \( f_s''(x) > 0 \). The convexity and negative gradient yields the optimality condition of SRM as follows.

\[
x^* \begin{cases} 
\in (\alpha, \sqrt{\alpha^2 + \beta^2}), & f_s'(\sqrt{\alpha^2 + \beta^2}) > 0 \\
\sqrt{\alpha^2 + \beta^2}, & f_s'(\sqrt{\alpha^2 + \beta^2}) \leq 0 
\end{cases} \tag{7}
\]

The optimality condition states that if the gradient at the upper bound is positive, then there exists an interior solution. Otherwise, the upper bound of \( x \) is the optimal solution. Since the SRM has the optimal solution either at the interior or the upper bound, the nominal route is never optimal.

An interior solution is equivalent to taking an intermediate route inside the triangle, and therefore represents the case when a hedged route provides the minimum cost flight path. By rearranging parameters in (3), the first condition becomes \( T < g_s(\alpha, \beta) \), where \( g_s(\alpha, \beta) \) is a function of \( \alpha, \beta \). One of the major insights we’re seeking with the geometric recourse model, whether it is the SRM or the DRM, is whether or how much hedging works under a certain weather situation. Since \( T < g_s(\alpha, \beta) \) is the condition to have an interior

---

1. Note that \( f_s(x) \) a continuous differentiable function over \( I \).
2. We obtain the minimum of \( f_s(x, \alpha, \beta, T) \) by solving the following problem
   \[
   \min f_s'(x, \alpha, \beta, T) \text{ s.t. } 0 < \alpha < 1, \alpha \leq x \leq \sqrt{\alpha^2 + \beta^2}, \beta > 0, T > \alpha, \text{ which yields } f_s''(x) \geq 9.70482 \times 10^{-23}.
   \]
   Numerical solution is found from 250 randomly selected starting point in the domain.
3. See Appendix A.1. for the complete formula.
solution, the function \( g_s(\alpha, \beta) \) is in fact the boundary between the interior solution region and the upper bound solution region: if \( T \) is below \( g_s(\alpha, \beta) \), then an intermediate route is optimal. Otherwise, the detour route is optimal.

The function \( g_s(\alpha, \beta) \) however, is a complex analytic function. Instead, it is shown in a contour plot in the \( \alpha-\beta \) plane in Figure 5. In the contour map, each contour line corresponds to a value of \( g_s(\alpha, \beta) \), which is shown in white squares. Using this map, one can determine whether there is an interior solution or not once the weather parameters are known. For example, if \( \alpha = 0.6 \) and \( \beta = 0.4 \), there is an interior solution when \( T = 1 \). However, the upper bound or taking the detour path is optimal when \( T = 2 \). As \( T \) gets larger, the interior solution region diminishes, which follows our intuition since as \( T \) approaches infinity, the interior solution is never optimal. The contour map provides a convenient decision map to determine whether hedging is worth considering or not.

![Figure 5 Interior Solution Condition of SRM](image)

**B. Analytic Solution Approximation using Taylor Series**

The SRM doesn’t have a closed-form interior solution. Instead, Taylor series is applied to our objective function to find a polynomial of degree 2 in \( x \) around the middle point of its domain \( \frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2} \). The Taylor series approximation, which we call \( f_{ts}(x) \) is quite complex, but we’re only interested in whether the minimizing \( x \) falls inside the \( x \) domain or not. Therefore, we can summarize optimal solution as follows.
\[ x^* = \begin{cases} 
\alpha & T \leq \alpha \\
\approx \frac{\text{Coefficient } (f_{ts,1})}{2\text{Coefficient } (f_{ts,2})} & \alpha < T < \theta_s(\alpha, \beta) \\
\sqrt{\alpha^2 + \beta^2} & T \geq \theta_s(\alpha, \beta) 
\end{cases} \]  \tag{8}

In (4), \text{Coefficient}(f, n) denotes the coefficient of \(x^n\) of polynomial function \(f\). The approximation error was well contained in less than 1\% in most cases we've tested. Selected results are shown in Table I.

**Table I. Analytic Solution Approximation and Errors**

<table>
<thead>
<tr>
<th>Geometry</th>
<th>E(Total Cost)</th>
<th>Int. sol</th>
<th>(x^*)</th>
<th>(x^*_{ts})</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha=0.2) &lt;br&gt;(\beta=0.5) &lt;br&gt;(T=2.5)</td>
<td>(f(x))</td>
<td>N</td>
<td>0.53</td>
<td>0.53</td>
<td>0%</td>
</tr>
<tr>
<td>(\alpha=0.5) &lt;br&gt;(\beta=0.5) &lt;br&gt;(T=1.1)</td>
<td>(f(x))</td>
<td>Y</td>
<td>0.66</td>
<td>0.64</td>
<td>2%</td>
</tr>
<tr>
<td>(\alpha=0.8) &lt;br&gt;(\beta=0.5) &lt;br&gt;(T=1.1)</td>
<td>(f(x))</td>
<td>Y</td>
<td>0.85</td>
<td>0.85</td>
<td>0.3%</td>
</tr>
</tbody>
</table>
VI. DUAL RECURSE MODEL (DRM)

A. Nominal Route Theorem

The Dual Recourse Model (DRM) is neither always convex nor concave and its properties are best addressed in our numerical analysis. However, DRM has a unique property that guarantees the nominal route to be optimal when the maximum storm duration time is below a specific threshold value.

Nominal Route Theorem. \( x^* = \alpha \) if \( 0 < T \leq \sqrt{\alpha^2 + \beta^2} \) given the second recourse option.

Proof.

It is trivial that \( x^* = \alpha \) if \( T \leq \alpha \).

If \( \alpha < T \leq \sqrt{\alpha^2 + \beta^2} \), the objective function \( f_d(x) \) is as follows.

\[
f_d(x) = \int_0^x \left( \mu + \sqrt{\mu^2 + 1 - 2\mu \frac{\alpha}{x}} \right) p(\mu) \, d\mu + \int_x^T \left( \mu + \sqrt{(1 - \alpha)^2 + (\mu - x + \sqrt{x^2 - \alpha^2})^2} \right) p(\mu) \, d\mu
\]

Then, we have

\[
f_d(x) - f_d(\alpha) = \int_0^x \left( \mu + \sqrt{\mu^2 + 1 - 2\mu \frac{\alpha}{x}} - \sqrt{\mu^2 + 1 - 2\mu} \right) p(\mu) \, d\mu + \int_x^T \left( \mu + \sqrt{(1 - \alpha)^2 + (\mu - x + \sqrt{x^2 - \alpha^2})^2} - \sqrt{(1 - \alpha)^2 + (\mu - \alpha)^2} \right) p(\mu) \, d\mu
\]

To show (11) is positive for \( \forall x \in (\alpha, T) \), we show that each integrand in (11) is positive. It is trivial that

\[
\sqrt{\mu^2 + 1 - 2\mu \frac{\alpha}{x}} - \sqrt{\mu^2 + 1 - 2\mu}.
\]

Since \( \left( \mu^2 + 1 - 2\mu \frac{\alpha}{x} \right) - ((1 - \alpha)^2 + (\mu - \alpha)^2) > 0 \) when \( \alpha < \mu \leq x \), we have

\[
\sqrt{\mu^2 + 1 - 2\mu \frac{\alpha}{x}} - \sqrt{(1 - \alpha)^2 + (\mu - \alpha)^2} > 0
\]

Similarly,

\[
\sqrt{(1 - \alpha)^2 + (\mu - x + \sqrt{x^2 - \alpha^2})^2} - \sqrt{(1 - \alpha)^2 + (\mu - \alpha)^2} > 0, \text{ where } x < \mu \leq T.
\]

From (12), (13) and (14), we have \( f_d(x) - f_d(\alpha) > 0, \forall x \in (\alpha, T) \). Therefore,

\[
x^* = \alpha, \text{ if } 0 < T \leq \sqrt{\alpha^2 + \beta^2}. \quad (Q.E.D)
\]

The nominal route theorem states that it is always optimal to fly on the nominal route when the maximum storm duration time is less than the time to fly to the tip of the
storm on the detour path. With the second recourse option added, the DRM enables a flight to utilize the nominal route, which is never optimal under the SRM. Moreover, the theorem holds regardless of the weather probability distribution.

**B. Optimal Solution Approximation**

One of our primary interests is to understand the model behavior with respect to each parameter. Such understanding enhances our capability to prepare for future changes in the weather condition in general terms, and strengthens strategic value of the geometric recourse model. Since there is no analytic solution to the DRM, the model behaviors are extracted from a large numerical solution set based 60,800 parameter combinations.

To define the optimal solution in terms of weather parameters, we adopt a statistical inference approach, in which each parameter-solution combination is treated as a data point in a four-dimensional space that comprises of four axes - three weather parameters $\alpha, \beta, T$ and the optimal solution $x^*$. Each data point then is categorized into three solution classes as shown in Table II. Parameter combinations in the solution class A choose the nominal route, and those in the class B chooses an intermediate route. Likewise, the solution class C contains parameters that choose the detour route.

<table>
<thead>
<tr>
<th>Class A</th>
<th>Class B</th>
<th>Class C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^* = x_{\text{min}} = \alpha$</td>
<td>$x^* = x_{\text{int}}, \alpha &lt; x_{\text{int}} &lt; \sqrt{\alpha^2 + \beta^2}$</td>
<td>$x^* = x_{\text{max}} = \sqrt{\alpha^2 + \beta^2}$</td>
</tr>
</tbody>
</table>

On the left side of Figure 6, parameter combination $(\alpha, \beta, T)$’s in each solution class are shown in a color-coded point plot in $\alpha - \beta - T$ space. Class A solutions are shown in yellow and blue, and class B and C solutions are shown in red and green respectively. Note that class A solutions are shown in two different colors, yellow and blue. The yellow point region contains both the cases satisfying the trivial solution and the nominal route theorem. In other words, $(\alpha, \beta, T)$’s in yellow color satisfy $T \leq \sqrt{\alpha^2 + \beta^2}$ and $x^* = \alpha$. the blue point region contains parameters that choose the nominal route when $T > \sqrt{\alpha^2 + \beta^2}$. Therefore, blue points satisfy $T > \sqrt{\alpha^2 + \beta^2}$ and $x^* = \alpha$. We notice that that there exists a large set of $(\alpha, \beta, T)$’s for which the nominal route is optimal when $T > \sqrt{\alpha^2 + \beta^2}$, especially when $\alpha$ and $\beta$ is large.

The boundary surfaces between categories are shown on the right side of Figure 6.\textsuperscript{4} The yellow surface is $T = \alpha$ which borders between the trivial and non-trivial case, and the

\textsuperscript{4} Each surface is plotted by taking the minimum $T$ for each $(\alpha, \beta)$’s in each solution category.
blue surface is $T = \sqrt{\alpha^2 + \beta^2}$. The red and green surface borders between the solution class A and B, and the class B and C, respectively.

![Figure 6. Point Plot of Weather Parameters Based On Their DRM Solution Class](image)

Transition from one category to the next is gradual, and the optimality conditions are captured in the boundary surfaces: one can determine the solution class of given $\alpha$, $\beta$, and $T$ from the location of that parameter combination in the surface plot. For example, if the current $(\alpha, \beta, T)$ falls between the red and green surface, then its optimal route choice is an intermediate one. To find a formula of boundary surfaces, we adopt a standard curve fitting technique as follows.

Define the boundary surface between the solution class A and B as $T_{AB}$ and the surface between class B and C as $T_{BC}$. Using standard curve fitting technique, we obtain the following models.5

$$T_{AB} = -0.37 + 1.48\alpha + 1.82\beta$$
$$T_{BC} = -0.27 + 0.91\alpha + 2.11\beta + 0.41\frac{\beta}{\alpha}$$

The following model is selected for the interior solution.6

$$x_{int}^* (\alpha, \beta, T) = 0.16 + 0.97\alpha + 1.11T\alpha - 1.05\alpha^2 - 2.08\alpha\beta$$

5 Find the ANOVA table and residual plots in Appendix B.1.
6 Find the ANOVA table and residual plots in Appendix B.2.
Now, the optimal solution of DRM is summarized as follows.

\[
x^* = \begin{cases} 
\alpha & 0 < T \leq T_{AB}(\alpha, \beta) \\
\frac{x_{int}(\alpha, \beta, T)}{\sqrt{\alpha^2 + \beta^2}} & T_{AB}(\alpha, \beta) < T \leq T_{BC}(\alpha, \beta) \\
1 & T > T_{BC}(\alpha, \beta)
\end{cases}
\]  

(18)

C. Sensitivity Analysis

Although an estimation, the optimal solution in (18) provides a reasonable framework for our sensitivity analysis. In the sensitivity analysis, we take the partial derivatives of optimal solution at a local point. However, direct use of \(x^*\) has certain limitations. Recall that the decision variable \(x\) is the distance from the origin to the storm front and lies between \(\alpha\) and \(\sqrt{\alpha^2 + \beta^2}\). Now, consider the following two cases where the storm location is different while the size and the maximum storm duration time are identical: \(\alpha = 0.9, \beta = 0.1, T = 0.9\) and \(\alpha = 0.1, \beta = 0.1, T = 0.9\). The first case yields \(x^* = \alpha = 0.9\), while the second case yields \(x^* = \sqrt{\alpha^2 + \beta^2} = 0.14\). Simple comparison of the optimal solution values seems to suggest that for the same \(\beta\) and \(T\), DRM gives larger solution for larger \(\alpha\). At a closer look however, one can easily find that the optimal solution is the lower bound of \(x\) when \(\alpha = 0.9\), and the upper bound when \(\alpha = 0.1\).

The previous example brings out the point that a direct analysis on \(x^*\) does not measure the degree of divergence of the optimal route from either the nominal or the detour one. There are several alternative measures that we can adopt to correctly address such shortcomings. For instance, we can analyze \(\frac{x^*}{\alpha}\) instead, which is the secant of the base angle formed by the nominal and optimal route. We may also analyze the base angle itself which is equivalent to \(ArcSec\left(\frac{x^*}{\alpha}\right)\). Among various alternatives, we select the divergence ratio \(\rho\), which represent the degree of divergence of the optimal route from the nominal one as defined below.

\[
\rho = \frac{x^* - \alpha}{\sqrt{\alpha^2 + \beta^2} - \alpha}
\]  

(19)

The divergence ratio \(\rho\) measures optimal route distance on a scale in which the nominal route distance is 0 and the detour route distance is 1. Any intermediate route therefore has \(0 < \rho < 1\). For example \(\rho = 0.3\) means that additional distance of the optimal route compared to the nominal route is 30% of the difference between the detour and nominal routes. From (18) and (19), we have the formula for \(\rho\) as follows.

\[
\rho = \begin{cases} 
0 & 0 < T \leq T_{AB}(\alpha, \beta) \\
\frac{x_{int}(\alpha, \beta, T) - \alpha}{\sqrt{\alpha^2 + \beta^2} - \alpha} & T_{AB}(\alpha, \beta) < T \leq T_{BC}(\alpha, \beta) \\
1 & T > T_{BC}(\alpha, \beta)
\end{cases}
\]  

(20)
We begin our sensitivity analysis with the discussion of the sensitivity of \( \rho \) with respect to the maximum weather clearance time \( T \). The first and second partial derivatives of \( \rho \) with respect to \( T \) are shown in (21) and (22).

\[
\frac{\partial \rho}{\partial T} = \begin{cases} 
0 & 0 < T \leq T_{AB}(\alpha, \beta) \\
\frac{1.11 \alpha}{\sqrt{\alpha^2 + \beta^2 - \alpha}} & T_{AB}(\alpha, \beta) < T \leq T_{BC}(\alpha, \beta) \\
0 & T > T_{BC}(\alpha, \beta) 
\end{cases}
\]  

(21)

\[
\frac{\partial^2 \rho}{\partial T^2} = 0
\]  

(22)

From (21) and (22), we find that the divergence ratio \( \rho \) is linear to \( T \). In other words, for storms of the same size \( \beta_0 \) and located at the same distance \( \alpha_0 \) from the origin, the optimal route is the nominal when \( T \leq T_{AB}(\alpha_0, \beta_0) \), then linearly increases when \( T_{AB}(\alpha_0, \beta_0) < T \leq T_{BC}(\alpha_0, \beta_0) \), until it becomes the detour one when \( T > T_{BC}(\alpha_0, \beta_0) \).\(^7\)

In Figure 7, selected \( T - \rho \) plots are shown. The series “1” is when \( \alpha = 0.25, \beta = 0.15 \), series “2” is when \( \alpha = 0.45, \beta = 1 \), and series “3” is when \( \alpha = 0.65, \beta = 0.75 \). In the series “1”, the optimal route is either the nominal or the detour one, and the nominal route is optimal only when \( T \) is fairly small. This follows our intuition that the benefit of the hedged route is likely to be negligible compared to the detour route for small storms located near the origin. In the series “2” and “3”, there exists a range of \( T \) that hedged routes are optimal, and such range is wider for the series “2”.

\[1: \alpha = 0.25, \beta = 0.15 \quad 2: \alpha = 0.45, \beta = 1 \quad 3: \alpha = 0.65, \beta = 0.75\]

Figure 7. Divergence Ratio With Respect To T

\(^7\) Note that not all three solution categories are possible for all weather conditions. For example, when \( \alpha_0 = 0.25, \beta_0 = 0.15 \), an intermediate route is never optimal since \( \{T | T_{AB}(\alpha_0, \beta_0) < T \leq T_{BC}(\alpha_0, \beta_0)\} = \emptyset \).
We now consider the effect of the storm location. The optimality conditions in (18) are rearranged to be expressed with respect to $\alpha$ as follows.

\[
\begin{align*}
T_{AB}(\alpha, \beta) &\leq T \\
T_{AB}(\alpha, \beta) &< T \leq T_{BC}(\alpha, \beta) \\
T &< T_{BC}(\alpha, \beta)
\end{align*}
\]

\[
\begin{align*}
\alpha &\geq \alpha_{AB} \\
\alpha < \alpha_{AB} &\cap (\alpha \geq \alpha_{BC2} \cup \alpha \leq \alpha_{BC1}) \\
\alpha_{BC1} < \alpha &< \alpha_{BC2}
\end{align*}
\]

(23)

where

\[
\begin{align*}
\alpha_{AB} &= 0 \\
\alpha_{BC1} &= 0.25 + 0.67T - 1.23\beta \\
\alpha_{BC2} &= 0.15 + 0.55T - 1.15\beta - 0.20\sqrt{0.52 + 3.89T + 7.29T^2 - 19.16\beta - 30.71T\beta + 32.35\beta^2} \\
\alpha_{BC1} < \alpha &< \alpha_{BC2}
\end{align*}
\]

(24)

Now, the first and second derivatives of $\rho$ with respect to $\alpha$ are shown below.

\[
\frac{\partial \rho}{\partial \alpha} = \begin{cases} 
\frac{\partial x^*_{\int}}{\partial \alpha} \div \left(1 - \frac{\alpha}{\sqrt{a^2 + \beta^2}}\right) & \alpha \geq \alpha_{AB} \\
0 & \alpha_{BC1} < \alpha < \alpha_{BC2}
\end{cases}
\]

(25)

\[
\frac{\partial^2 \rho}{\partial \alpha^2} = \begin{cases} 
\frac{\partial^2 x^*_{\int}}{\partial \alpha^2} \div \left(1 - \frac{\alpha}{\sqrt{a^2 + \beta^2}}\right) & \alpha \geq \alpha_{AB} \cup \alpha_{BC1} < \alpha < \alpha_{BC2} \\
0 & \alpha_{BC1} < \alpha < \alpha_{BC2}
\end{cases}
\]

(26)

Overall, it is optimal to choose the nominal route for storms located near the destination. Intermediate routes provide optimal cost operation for storms located near the origin and near the midway point. For certain combinations of storm size and maximum duration, the optimal route for small $\alpha$ is an interior one, with the divergence ratio gradually increasing with $\alpha$ until the detour route becomes optimal, and then decreasing until the nominal route becomes optimal. As one can see in Figure 6, not all three solution classes occur for some combinations of $\beta$ and $T$. From (23), for a combination of values of these parameters for which the detour route is never optimal, the optimal route is interior until $\alpha = \alpha_{AB}$, after which it becomes the nominal route. Likewise, in cases where the nominal route is never optimal, then the optimal route is an interior one when $\alpha_{BC1} < \alpha < \alpha_{BC2}$, and the detour one otherwise. The second partial derivative $\frac{\partial^2 \rho}{\partial \alpha^2}$ of the interior solution as shown in (26) is negative in its domain\(^8\), and the divergence ratio $\rho$ is thus concave with respect to $\alpha$.

---

\(^8\) One can show that $\frac{\partial^2 \rho}{\partial \alpha^2} < 0$ when $\alpha < \alpha_{AB} \cap (\alpha \geq \alpha_{BC2} \cup \alpha \leq \alpha_{BC1})$, by solving the following optimization model when decision variables are $\alpha$, $\beta$ and $T$.\footnote{\(\frac{\partial x^*}{\partial \alpha^2} = 0.16 + 0.97\alpha + 1.117\alpha - 1.05\alpha^2 - 2.08\alpha\beta, \alpha < \alpha_{AB}, \alpha \geq \alpha_{BC2} \cup \alpha \leq \alpha_{BC1}, 0 < \alpha < 1, \beta > 0, T \geq 0\)}
In Figure 8, selected $\alpha - \rho$ plots are shown. The series “1” where $\beta = 0.35, T = 1.35$ starts at with the intermediate route until $\alpha = 0.15$, after which it stays at the detour route until $\alpha = 0.8$. When $0.8 < \alpha < 1$, the nominal route is optimal. The series “2” and “3” are when $\beta = 0.55, T = 1.55$ and $\beta = 0.85, T = 1.85$ respectively. These two series share a similar trend, starting with an interior solution for small $\alpha$, with the deviation first increasing and then decreasing with $\alpha$, ultimately reaching the nominal route. For both of these cases, the detour route is never optimal.

\[
\begin{array}{c}
1: \beta = 0.35, T = 1.35 \\
2: \beta = 0.55, T = 1.55 \\
3: \beta = 0.85, T = 1.85
\end{array}
\]

**Figure 8. Divergence Ratio and With Respect To $\alpha$**

We now discuss the effect of the storm size. The optimality condition is expressed with respect to $\beta$ as follows.\(^9\)

\[
\begin{align*}
T_{AB}(\alpha, \beta) &\leq T \\
T_{AB}(\alpha, \beta) &< T \leq T_{BC}(\alpha, \beta) \\
T &< T_{BC}(\alpha, \beta)
\end{align*}
\]

\[
\beta \geq \beta_{BA}
\]

\[
\beta_{CB} < \beta < \beta_{BA}, \text{ where } \beta_{BA} = \left\{ \begin{array}{ll}
0.20 + 0.55 T - 0.82 \alpha & \text{if } 0 < \beta \leq \beta_{CB} \\
\frac{(0.26 + 0.987 - 0.89 \alpha) \alpha}{0.40 + 2.05 \alpha} & \text{if } 0 < \beta \leq \beta_{CB}
\end{array} \right.
\]

(27)

Consider the first and second derivatives of $\rho$ with respect to $\beta$.

\[
\frac{\partial \rho}{\partial \beta} = \begin{cases}
0 & \beta \geq \beta_{BA} \\
\frac{1}{\sqrt{\alpha^2 + \beta^2 - \alpha}} \left( \frac{\partial x^*_\text{int}}{\partial \beta} - (x^* - \alpha) \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right) & \beta_{CB} < \beta < \beta_{BA} \\
0 & 0 < \beta \leq \beta_{CB}
\end{cases}
\]

(28)

\(^9\) As discussed before, note that some of the conditions in (17) will result in an empty set for certain $\alpha$ and $T$.
$\frac{\partial^2 \rho}{\partial \beta^2} = \begin{cases} 
0 & 0 < \beta \leq \beta_{CB} \cup \beta \geq \beta_{BA} \\
-2 \frac{\partial x_{int}}{\partial \beta} \frac{\beta}{\sqrt{\alpha^2 + \beta^2(\sqrt{\alpha^2 + \beta^2 - \alpha})}} & \beta_{CB} < \beta < \beta_{BA} 
\end{cases}$ (29)

Overall, the optimal route is the detour one for storms of smaller size, and the nominal one for storm of larger size. In between, the optimal route transitions from the detour to the nominal route more or less gradually. The first partial derivative of the interior solution is negative while the second derivative is positive.\textsuperscript{10} Therefore, $\rho$ is a monotonic, decreasing and convex function in $\beta$ for the solution class B. Selected $\beta - \rho$ cases as shown in Figure 9.

\begin{tabular}{|c|c|c|}
\hline
1: $\alpha = 0.6, T = 0.9$ & 2: $\alpha = 0.7, T = 1.25$ & 3: $\alpha = 0.9, T = 2.05$ \\
\hline
\end{tabular}

Figure 9. Divergence Ratio With Respect To $\beta$

To conclude, in this section, we measure the degree of deviation of the optimal solution from the nominal route, when the nominal route represents the minimum deviation and the detour route represents the maximum deviation. We summarize our findings in the list below.

- When $\alpha$ and $\beta$ is fixed, the deviation increases from the minimum to the maximum with increasing $T$. In other words, the optimal route transitions from the nominal to the intermediate then to the detour as $T$ increases. We also find that the rate of increase is constant.

\textsuperscript{10} $\frac{\partial x_{int}^*}{\partial \beta} = -2.08\alpha < 0$ and $x^* > \alpha$
- When $\beta$ and $T$ is fixed, the nominal route is preferred for storms near the destination, while a non-nominal route is preferred for storms near the origin. The interior solutions exhibits concavity with increasing $\alpha$, thus the solution deviates from the nominal route at a decreasing rate as $\alpha$ increases before it converges back toward the nominal at a decreasing rate. We also find that the optimal route is almost always the nominal one, when $\alpha$ is close to 1.

- When $\alpha$ and $T$ are fixed, deviation from the nominal route decreases from the maximum to the minimum as $\beta$ increases. In other words, the optimal route is the detour one for small $\beta$, and the intermediate one for sufficiently large $\beta$. Once $\beta$ becomes large enough, the optimal route is always the nominal one. We also find that for the interior solutions, the divergence ratio decreases at a decreasing rate with $\beta$. 
VII. PERFORMANCE REVIEW OF GEOMETRIC RECOURSE MODELS

In addition to the properties of individual geometric recourse model, we are also interested in measure the performance improvement of two models. We first define three performance scenarios and find the solutions of each of them based on 60,800 parameter combinations as described in chapter IV. Three performance scenarios are as follows.

- **Baseline**: take the detour route, and fly direct to the destination whenever storm clears.
- **SRM**: use SRM with the first recourse option only
- **DRM**: use DRM with both the first and second recourse options.

Available operational options in each scenario are summarized in Table III. Since the baseline is an upper bound of the SRM, which in turn is an upper bound of the DRM, it is natural to measure the improvement of one model to the next in that order. Improvement of SRM compared to the baseline shows the value of hedging and the first recourse. Likewise improvement of DRM compared to SRM highlights the value of the second recourse.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Hedging</th>
<th>First Recourse</th>
<th>Second Recourse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>N</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>SRM</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>DRM</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
</tbody>
</table>

A. Minimum Expected Total Cost (ETC*)

The first performance metric we measure is the total flight cost. The average and the maximum of all ETC* with respect to each parameter are obtained from the set of ETC* with one of the parameters fixed a specific value. For example, to obtain the average and maximum of ETC* with respect to the storm location $\alpha$, set $\alpha=0.05$ and collect ETC* for all possible combinations of $\beta$ and $T$ to find the average and the maximum of the collection. Repeat the process for all $\alpha$ values $0.1, 0.15, ..., 0.95$ to generate the $\alpha - \text{Mean}(ETC^*)$ and $\alpha - \text{Max}(ETC^*)$ plots. $\beta$ and $T$ plots are obtained in the same manner, and the results are summarized in Table IV. In the table, the second column contains $\text{Mean}[ETC^*]$, which shows the average performance of each model. The third column contains $\text{Max}[ETC^*]$, showing the changes in the maximum, or the worst-case performance.

The $\alpha - \text{Mean}[ETC^*]$ plot shows how the average optimal cost changes with respect to $\alpha$. We can see that on average, the $ETC^*$ of the baseline and that of the SRM are nearly the same, which implies that, on average, there is little cost saving from hedging without the second recourse option. On the other hand, the DRM performs considerably better than the other models except when the storm is very close to the origin since its average $ETC^*$
gradually decreases as \( \alpha \) increases until \( \alpha \) is almost 1. In the baseline and SRM, the optimal flight cost exhibits convexity reaching the minimum when \( \alpha \) is near 0.6—i.e. the storm located just over halfway between the origin and destination. However, the DRM exhibits steady improvement in performance as the storm gets closer to the destination.

The \( \alpha - \text{Max}[ETC^*] \) plot shows the worst-case performance with respect to \( \alpha \). There is little difference among three models when the storm is very near the origin. In the case of baseline and SRM, the cost decreases at a decreasing rate as the storm location approaches the destination, while the DRM shows a steady of decrease in the worst-case flight cost. This plot again confirms the value of the second recourse allowed under DRM.

The \( \beta \) plot shows that \( \text{Mean}[ETC^*] \) increases as \( \beta \) increases overall. Increasing gap between the DRM and the other models shows the advantage of DRM increases with storm size. The need to fly all the way to the tip of the storm under the persistence scenario in the SRM becomes an increasingly great disadvantage as the storm size \( \beta \) increases. Performance of the models is identical in the worst case, however, when the long expected storm duration and closeness of the storm region to the origin dictate use of the detour route under all three models.

In the \( T \) plots, we find, as expected, that \( ETC^* \) increases as \( T \) increases, while average performance of DRM is better than the baseline and SRM, especially when \( T \) is between 1 and 2. It is intuitive that when a storm is expected to last for a very long time, the detour route will be chosen under all three models. It is when the maximum storm duration is roughly between the flight time on the nominal route and twice of it, that the route hedging and recourse options provide the greatest cost saving potential.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean[ETC*]</td>
<td>Max[ETC*]</td>
</tr>
<tr>
<td>![Graph for ( \alpha )</td>
<td>![Graph for ( \beta )</td>
</tr>
</tbody>
</table>

**Table IV. Mean and Maximum of ETC* with respect to weather parameters**
In summary, there is little performance difference between the SRM and baseline models in terms of the flight cost. On the other hand, there is a range of weather parameter values when DRM performs substantially better than either. On average, expected total cost increases with decreasing rate as the storm size and the maximum duration time increases. On the other hand, expected total cost is convex with respect to the storm location, as it decreases up to a minimum point then increases. We observe that DRM reduces the weather risk further with a storm located near the destination airport. This is also true for storms of larger size. There is a range of T values where DRM shows substantially less expected total cost, although this advantage disappears as T becomes very large.

**B. Cost Savings**

The second performance metric we measure is the reduction in the total flight cost, or the cost savings. Define cost saving of the SRM and DRM as follows.

\[
S(\text{SRM}) = 1 - \frac{ETC^*[\text{SRM}]}{ETC^*[\text{Baseline}]} \quad (30)
\]
\[
S(\text{DRM}) = 1 - \frac{ETC^*[\text{DRM}]}{ETC^*[\text{SRM}]} \quad (31)
\]

where \(ETC^*[\cdot]\) is the minimum expected total cost of the selected model.

The cumulative distribution functions of \(S(\text{SRM})\) and \(S(\text{DRM})\) are shown in Figure 10. The SRM yields cost savings of less than 1% in 90% of cases, and savings of less than 5% in nearly 99% of cases, with the largest cost saving just under 6%. On the other hand, more than 32% of cases have savings larger than 5% with the DRM compared to the SRM, with the largest saving close to 30%. In addition, nearly 20% of cases show savings of larger than 10%.
In Table V, the average, minimum and maximum cost savings are plotted with respect to each parameter. We observe that the maximum and average values of both $S(SRM)$ and $S(DRM)$ increase monotonically with $\alpha$. From $\alpha$-$S(SRM)$ plot, we find that the SRM works best with storms very near the destination, while the DRM works well for a wider range of storm locations according to $\alpha$-$S(DRM)$ plot. We observe that while the overall cost saving of SRM is marginal, it does provide meaningful cost saving opportunities in certain cases.

Comparison of the $\beta$-$S(SRM)$ and $\beta$-$S(DRM)$ plot, one can observe that DRM saves substantial cost for a wider range of $\beta$’s than SRM, which coincides with our finding in the optimal flight cost study. It is intuitive that without the second recourse option, SRM is limited in its capability to hedge against the persistence risk. In $\beta$-$S(DRM)$ plot, we observe that the average cost saving maintains an increasing trend even after the maximum saving plateaus, suggesting even after the upper end of the distribution stabilizes, its mass continues to shift toward the higher values.

In the $T$-$S(SRM)$ and $T$-$S(DRM)$ plots, we observe non-monotonic relationships featuring an interior maximum. For the SRM, both the maximum and average cost saving peaks, at about 6% and 1.5% respectively, when $T$ is between 1 and 1.1. In a fairly narrow range around these values, the SRM can yield modest cost savings for certain geometries. In the case of the DRM, the peaks for maximum and average saving do not coincide. The former peaks near 30% when $T$ is between 1.9 and 2.1, while the latter peaks at about 10% when $T$ is around 1.3. The sensitivity of delay savings to maximum storm duration revealed in these plots point to the need to consider ground delay—which in effect reduces the maximum storm duration time—in combination with route hedging. In addition, the large gap between the maximum and average cost saving demonstrate the importance of considering all three weather parameters when assessing whether route hedging can yield a significant payoff.
Table V Minimum, Average and Maximum Saving By Weather Parameters

<table>
<thead>
<tr>
<th></th>
<th>S(SRM) = 1 - ( \frac{ETC^{SRM}}{ETC^{Baseline}} )</th>
<th>S(DRM) = 1 - ( \frac{ETC^{DRM}}{ETC^{SRM}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>![Graph of S(SRM) for ( \alpha )]</td>
<td>![Graph of S(DRM) for ( \alpha )]</td>
</tr>
<tr>
<td>( \beta )</td>
<td>![Graph of S(SRM) for ( \beta )]</td>
<td>![Graph of S(DRM) for ( \beta )]</td>
</tr>
<tr>
<td>( T )</td>
<td>![Graph of S(SRM) for ( T )]</td>
<td>![Graph of S(DRM) for ( T )]</td>
</tr>
</tbody>
</table>

Minimum (Dotted) | --- |
Average (Dashed) | --- |
Maximum (Solid)  | --- |

C. Sensitivity Analysis on Cost Saving

Although the cost saving metrics defined in section B provide us an overall understanding on how the cost saving changes with respect to one of the weather parameters, the interaction between parameters are lost in the aggregated statistics. In this section, we take a detailed look at the cost saving sensitivity from a set of contour plots.
1) **Single Recourse Model (SRM) vs Baseline**

Consider the cost saving of the SRM over the baseline case. Based on the same performance metric as defined in (20), a set of contour plots are generated for a range of parameter values, where each contour line corresponds to the cost saving. For example, the contour plot when $\alpha = \alpha_0$ is generated in the $\beta - T$ plane, where each contour line corresponds to the cost saving when $\alpha = \alpha_0$, or $S(SRM)|_{\alpha=\alpha_0} = 1 - \frac{ETC^{*}(\text{DRM})|_{\alpha=\alpha_0}}{Baseline |_{\alpha=\alpha_0}}$. Contour plots of $\beta$ and $T$ likewise.

Selected cost saving contour plots with respect to $\alpha$ are shown in Figure 11, when $\alpha=0.2, 0.5, 0.9$. First of all, the overall cost saving improves as $\alpha$ increases, which agrees with our finding in the previous section. We make several other observations as well as described below.

- Level of the cost saving is much more sensitive to the maximum storm duration time $T$ than the storm size $\beta$. In fact, $T$ alone completely determines the cost saving when the storm size is larger than a certain value as noted in the parallel contours for large $\beta$.
- In the optimal cost study, we found that the largest saving of SRM reaches near 6%. In $\alpha=0.9$ plot, we have a region that identifies those region with savings larger than 0.05. One can expect best performance with SRM for storms located very near the destination with the maximum storm duration time near 1.

![Figure 11. Cost Saving of SRM when $\alpha=0.2$, $\alpha=0.5$ and $\alpha=0.9$](image)

Selected cost saving plots with respect to $\beta$ are shown in Figure 12, when $\beta=0.5, 1.0$ and 2.0 in $\alpha$-$T$ plane. The overall trend agrees with our previous finding that cost saving increases as $\beta$ increases. We also observe that a larger saving occurs with storms located near the destination when $T$ is near 1. Comparing $\beta=1.0$ and 2.0 plot side by side, we observe that they are almost identical, which confirms the diminishing return of $\beta$. Note that $T$ needs to be smaller than 2 to have any saving.
Figure 12. Cost Saving of SRM when $\beta=0.5$, $\beta=1.0$ and $\beta=2.0$

Figure 13 shows selected cost saving plots when $T=0.5, 0.8, 1.0$ and 1.5 on $\alpha$-$\beta$ plane. Overall, the cost saving increases to reach its maximum when $T=1.0$, then decreases thereafter. In fact, the saving is always less than 2% when $T > 1.5$, and less than 1% when $T \geq 1.9$. The vertical contour lines demonstrate the higher sensitivity to $\alpha$ than $\beta$.

Figure 13. Cost Saving of SRM When $T=0.5$, $T=0.8$, $T=1.0$ and $T=1.5$
2) **Single Recourse Model (SRM) vs Double Recourse Model (DRM)**

Now, we discuss the cost savings of the DRM compared to the SRM. In section B, we learned that the DRM shows a considerable improvement to the SRM, and the cost saving reaches nearly 30% in some cases. A set of contour plots are generated based on the formula in (21). For example, the contour plot when \( \alpha = \alpha_0 \) is generated in the \( \beta - T \) plane, where each contour line corresponds to the cost saving when \( \alpha = \alpha_0 \), or \( S(DRM)\vert_{\alpha=\alpha_0} = 1 - \frac{ETC^{\alpha}(DRM)\vert_{\alpha=\alpha_0}}{ETC^{\alpha}(SRM)\vert_{\alpha=\alpha_0}} \cdot \beta \) and \( T \) plots are obtained in the same manner.

In Figure 14, selected DRM-SRM cost saving contour plots are shown when \( \alpha=0.2, 0.5, 0.9 \) in \( \beta - T \) plane. Overall, performance of the DRM improves as \( \alpha \) increases. When \( T \leq \sqrt{\alpha^2 + \beta^2} \), the DRM shows substantial improvement over the SRM due to the nominal route theorem. One of the biggest differences of the contour plots from those of the SRM is the widening shape of the contour lines. Such shape confirms our previous finding that the second recourse option of the DRM provides a unique capability to hedge against the persistence risk of larger storms. It also suggests that the DRM cost saving does depend on both \( \beta \) and \( T \), with the largest cost saving occurring when \( 1.4 \leq T \leq 2.1 \) and \( \beta \geq 1.1 \). From the increasing positive saving region, we also conclude that the DRM reduces costs for a wide range storm size and the maximum duration time when the storm is located closer the destination.

![Figure 14. DRM Cost Saving When \( \alpha=0.2, 0.5 \) and \( \alpha=0.9 \)](image)

In Figure 15, selected cost savings are shown when \( \beta=0.5, 1.0, \) and \( 2.0 \) in \( \alpha - T \) plane. Overall, the positive saving region grows as \( \beta \) increases, which confirms the strength of the DRM to handle the persistence risk of larger storms. For a small storms such as \( \beta=0.5 \), cost saving from the DRM over the SRM is marginal. On the contrary, for a large storm such as \( \beta=2.0 \), most combinations of \( \alpha \) and \( \beta \) incurs meaningful savings. Such observation suggests that the effect of the second recourse option is more evident when facing a large storm.
In Figure 16, selected cost saving contour plots are shown when T=0.5, 1.0, 1.5, 2.0 and 3.0 on α-β plane. As discussed in section B, the average cost saving peaks when T is near 1.5, while the maximum cost saving peaks when T is near 2. Such trend is evident with the larger saving regions in T=1.0, T=1.5 and T=2.0 plot, and significantly reduced regions when T=0.5 and T=3.0. It also suggests that there is not only a specific range of T values that the DRM generates considerable savings, but also the cost savings are negligible otherwise.
VIII. GROUND-AIRBORNE HYBRID GEOMETRIC RE COURSE MODEL

The ground-airborne hybrid geometric recourse model concerns hedging against weather risk through ground delay in addition to route hedging. In this model, aircraft have the option to wait on the ground at the origin airport before taking off on the route with minimum expected flight cost. Optimal ground delay decisions are affected not only by the storm parameters included in the previous models but also by the ground-airborne cost ratio. It is conventional wisdom that the unit cost of ground delay is less than that of extra airborne time, but their ratio varies from flight to flight. The effect of the cost ratio is critical in determining the optimal ground delay: if the ground delay costs significantly less than airborne, then it will be best to wait on the ground for en route weather to clear. On the other hand, if the ratio is close to 1, an early take-off will be preferred.

The formulation of the hybrid geometric recourse model involves two decision variables -- the ground delay amount and the route choice -- and four parameters: storm location, storm size, maximum storm duration time, and ground-airborne cost ratio. The objective is to find the combination of ground delay and route choice that minimizes the expected total cost, which now consists of a ground delay and a flight time component. Since we found that the DRM performs considerably better than the SRM for choosing routes, we consider a hybrid model which combines ground holding with the DRM. In the following sections, we discuss the formulation and solutions of the hybrid DRM. We investigate how the optimal solutions and associated expected costs vary with model parameters. As in the geometric recourse model study, the weather clearance time is assumed to follow a uniform distribution between 0 and T.

A. Hybrid Dual Recourse Model (hDRM) Formulation

Adopting the ratio-based framework where variables and parameters are defined in units of nominal route flight distance, the hybrid DRM is formulated as follows.

\( \alpha \): storm location from origin in units of nominal route: \( 0 < \alpha < 1 \)
\( \beta \): storm size in units of nominal route: \( \beta > 0 \)
\( c \): ground-airborne cost ratio: \( 0 < c < 1 \)
\( x \): distance to the storm front along the course set from the origin in units of nominal route: \( \alpha \leq x \leq \sqrt{\alpha^2 + \beta^2} \)
\( g \): ground delay amount: \( 0 \leq g \leq T - \alpha \)
\( \mu \): random variable representing the storm clearance time with probability density function \( p(\mu) \).

\[
\min \int_0^g (1 + c\mu) p(\mu) d\mu + \int_g^{g+x} \left( cg + \mu + \sqrt{1 + \mu^2 - 2\mu \frac{\alpha}{x}} \right) p(\mu) d\mu + \int_{g+x}^{g+x+\beta-x^2-\alpha^2} \left( cg + x + \sqrt{(\mu - x + \sqrt{x^2 - \alpha^2})^2 + (1 - \alpha)^2} \right) p(\mu) d\mu
\]
\[
+ \int_{g+x+\beta-\sqrt{x^2-\alpha^2}}^{\infty} \left( cg + x + \sqrt{(x + \beta - \sqrt{x^2 - \alpha^2})^2 + \sqrt{(1 - \alpha^2 + \beta^2)} \right) p(\mu) d\mu
\]
\[
\text{s.t. } \alpha \leq x \leq \sqrt{\alpha^2 + \beta^2}, \ g \geq 0
\]

In the formulation, we introduce an additional parameter \( c \), which is the ground-airborne cost ratio ranging between 0 and 1. We introduce an additional decision variable \( g \) as well, which is the optimal ground delay time. In the objective function, the first integral represents the case when the storm clears while the aircraft waits on the ground, after which the aircraft departs and flies along the nominal route with a total flight cost of 1. The second integral is the case when the aircraft is able to take the first recourse after spending \( g \) time units on the ground and departing on a course specified by \( x \). Likewise, the third and forth integral are the cases when the aircraft takes the second recourse, or no recourse is possible, respectively, after spending \( g \) time units on the ground respectively.

With the uniform distribution assumption on the weather clearance time, the objective function in (32) becomes (33), where \( f_h(x, g|c, \alpha, \beta, T) \) is the expected total cost of the hybrid DRM when the weather clearance time follows a uniform distribution between 0 and \( T \).

\[
f_h(x, g|c, \alpha, \beta, T) = \int_0^g \frac{1}{T} (1 + c\mu) \ d\mu + \frac{T-g}{T} f_d(x|\alpha, \beta, T-g)
\]

The first term in \( f_h \) represents the case when the weather clears before the assigned ground delay amount \( g \) and the aircraft depart on the nominal route without interruption. The second term is the case when the aircraft waits on the ground for the amount \( g \) to fly on the route determined by the DRM. Note that the effect of the ground delay is equivalent to reduction in the maximum storm duration time, and the optimal route choice is determined based on \( \alpha, \beta \) and \( T - g \). Since the weather clearance time follows a uniform distribution between zero and the maximum duration time, \( f_d(x|\alpha, \beta, T-g) \) is the expected total cost of the DRM when the storm clearance probability is \( \frac{1}{T-g} \). In other words, \( f_d(x|\alpha, \beta, T-g) \) will yield the optimal route choice based on the conditional probability that the storm didn’t clear in the first \( g \) time units. Since we seek an optimal route choice based on the unconditional probability, the correctional term \( \frac{T-g}{T} \) is needed.

Based on (33), it is reasonable to investigate a systematic way to determine the route choice portion of the hybrid DRM from the solutions of the DRM. Such an approach is particularly useful since \( f_h \) is neither convex nor concave with multiple local minima in some cases, and we already have the DRM solution set for a wide range of weather values. Before embarking on the discussion of the solution algorithms, we first assert the relationship between the DRM and the hybrid DRM in the following theorem.

**Hybrid DRM Theorem.** If \( x^* \) and \( g^* \) is the optimal solution of \( f_h(x, g|c, \alpha, \beta, T) \), then \( x^* \) is the optimal solution of \( f_d(x|\alpha, \beta, T-g^*) \).

**Proof.**
Suppose that there exists \( x_0 \) and \( g^* \) such that they are the optimal solutions of \( f_h(x, g|c, \alpha, \beta, T) \), but \( x_0 \) is not an optimal solution of \( f_d(x|\alpha, \beta, T - g^*) \). Then, there exists \( x \) satisfying

\[
\begin{align*}
 f_d(x|\alpha, \beta, T - g^*) &> f_d(x_0|\alpha, \beta, T - g^*) \\
 f_h(x, g^*|c, \alpha, \beta, T) &< f_h(x_0, g^*|c, \alpha, \beta, T)
\end{align*}
\]

(34) (35)

From \( f_h(x, g|c, \alpha, \beta, T) = \int_0^1 \frac{1}{T}(1 + c\mu) \ d\mu + \frac{T - g}{T} f_d(x|\alpha, \beta, T - g) \) (33), we have the following.

\[
\begin{align*}
 f_h(x, g^*|c, \alpha, \beta, T) - f_h(x_0, g^*|c, \alpha, \beta, T) &= \frac{T - g^*}{T} f_d(x_0|\alpha, \beta, T - g^*) - \frac{T - g}{T} f_d(x_0|\alpha, \beta, T - g^*) > 0
\end{align*}
\]

(36)

which contradicts \( 35f_h(x, g^*|c, \alpha, \beta, T) < f_h(x_0, g^*|c, \alpha, \beta, T) \).

Therefore, \( x_0 \) is an optimal solution of \( f_d(x|\alpha, \beta, T - g^*) \) and \( f_h(x, g^*|c, \alpha, \beta, T) \).

\( Q.E.D. \)

The theorem states that the optimal route choice of the hybrid DRM is determined by the DRM with the maximum storm duration time \( T \) reduced by the ground delay amount \( g \). In the following section, we introduce solution algorithms for the hybrid DRM based on this theorem.

**B. Solving the Hybrid Dual Recourse Model (Hybrid DRM)**

To solve the hybrid model, we propose an algorithm to find the range of cost ratio \( c \) that makes a specific combination of \( x \) and \( g \) optimal. The idea is as follows: given the weather parameters, find the DRM solution of them first. Continue solving the DRM for the same storm location and size, but with the maximum storm duration time varying from the current value to \( \alpha \) in decrements of \( \delta \). The difference between the original and reduced value of the maximum storm duration time is thus the ground delay amount \( g \). This process generates a set of expected total cost values of the hybrid DRM as functions of \( c \) for given \( \alpha, \beta, T \). For each member in the set, find the range of \( c \) that makes that member the minimum among all members in the set. If such a range exists, then the \( x \) and \( g \) associated with the current cost are the optimal solution of the hybrid DRM when the cost ratio is in that range. If no such range exists, then the selected \( x \) and \( g \) are never optimal. The pseudo code for the above is presented below.

Given \((\alpha_0, \beta_0, T_0)\)
Select \( \delta \) and generate the index set \( I \)

\[
I = \left\{ 0, 1, \ldots, \frac{T_0 - \alpha_0}{\delta} \right\}
\]

Create the parameter set \( P \) and the ground delay set \( G \)

\[
P = \{ p_i | p_i = (\alpha_0, \beta_0, T_0 - i\delta), i \in I \}, G = \{ g_i | g_i = i\delta, i \in I \}
\]

Find the DRM solution set \( S_\beta \)
\[ S_D = \left\{ x_i \bigg| x_i = \arg\min_x f_d(x)p_i, p_i \in P, i \in I \right\} \]

Generate the set of expected total cost of the hybrid DRM \( F_H \)
\[ F_H = \{ f_h(x_i,g_i|c,p_i) \big| x_i \in S_D, p_i \in P, g_i \in G, i \in I \} \]

For each \( i \in \{0,1,\ldots,\frac{T_0-\alpha}{\delta}\} \)
- Find the range of \( c \) satisfying \( f_h(x_i,g_i|c,p_i) < f_h(x_j,g_j|c,p_j) \) for \( j \neq i \)
  where \( i,j \in I \), where \( 0 < c < 1 \).
- If such range of \( c \) exists, which we call \( C_i \), then store \( (\alpha,\beta,T_0,x_i,g_i,C_i) \) in the hybrid DRM solution set.
  Else, repeat with the next \( i \).

End

The pseudo code finds the solution set for one combination of weather parameters. We repeat the code to find the solution set for a dense grid of parameter combinations. A parameter grid is created in the \( \alpha - \beta - T \) space, where \( 0 < \alpha < 1, 0 < \beta \leq 3, 0 < T \leq 4 \) and each grid is spaced by 0.05. We increment the maximum storm duration time by 0.05; thus \( \delta = 0.05 \). Now, define the cost matrix \( H \) as follows.

\[
H = \begin{pmatrix}
0 & \arg\min_x f_d(x|\alpha_0,\beta_0,T_0) & \delta & \ldots & T_0 - \alpha_0 \\
\arg\min_x f_d(x|\alpha_0,\beta_0,T_0 - \delta) & \ldots & \arg\min_x f_d(x|\alpha_0,\beta_0,\alpha_0) \\
f_d(x|\alpha_0,\beta_0,T_0) & \ldots & f_d(x|\alpha_0,\beta_0,\alpha_0) \\
f_h(x^*,0|c,\alpha_0,\beta_0,T_0) & \ldots & f_h(x^*,\delta|c,\alpha_0,\beta_0,T_0 - \delta) & \ldots & f_h(x^*,T_0 - \alpha_0|c,\alpha_0,\beta_0,\alpha_0)
\end{pmatrix}
\]

The cost matrix \( H \) is a 4 by \( \frac{T_0-\alpha_0}{\delta} \) matrix, where each column contains the cost information when the ground delay amount specified in the first row is taken: the first row is the ground delay amount, the second row is the optimal route choice \( x^* \) based on the reduced maximum storm duration time. The third row is the expected total cost associated with the route using the DRM. The fourth row is the expected total cost of the hybrid DRM \( f_h^* \), when the ground delay in the first row and the route in the second are chosen.

In essence, the matrix \( H \) is the search space to find our optimal solution when \( \alpha = \alpha_0, \beta = \beta_0 \) and \( T = T_0 \). For each element in the fourth row, we find the range of the cost ratio \( c \) that makes the selected element minimum among all elements in the fourth row. If such a range exists, the ground delay and the route choice in the column of the selected element is the optimal solution when \( c \) is in that range. By repeating this process for all elements in the fourth row, we obtain the solution set of the hybrid DRM.

In Figure 17, three examples of hybrid model solution are presented. In the first case when \( \alpha=0.8, \beta=1.65, T=1.8 \), the optimal route is always the nominal, while the optimal ground hold time decreases as the cost ratio increases. In the second case when \( \alpha=0.05, \beta=0.5, T=2.55 \), the optimal choice is to take some positive ground delay to fly the nominal route until \( c=0.4 \). Once the cost ratio exceeds 0.4 however, the ground delay option is
discarded, and it is optimal to take an intermediate route immediately. The third case is similar to the second one, with the detour route being optimal instead of an intermediate route, once the ground delay cost is over the threshold value.

Case I) $\alpha=0.8, \beta=1.65, T=1.8$

Case II) $\alpha=0.05, \beta=0.5, T=2.55$

Case III) $\alpha=0.95, \beta=1, T=3$
In Table VI, we check the existence of the optimal solutions out of nine possible categories: the optimal route choice are categorized into three groups: the nominal, intermediate, and detour route. Likewise, the optimal ground delay is categorized into three groups: the minimum (zero), intermediate, and maximum delay. The optimal solution of the hybrid model therefore is grouped into nine categories. Out of nine categories, there are four categories that are never optimal: those which involve taking positive ground delay and then flying an intermediate or the detour route.

**Table VI. Existence of Optimal Solutions of the Hybrid DRM**

<table>
<thead>
<tr>
<th>$g_{min} = 0$</th>
<th>$g \in (0, T - \alpha)$</th>
<th>$g_{max} = T - \alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>$x_{min} = \alpha$</strong></td>
<td><strong>$x \in (\alpha, \sqrt{\alpha^2 + \beta^2})$</strong></td>
<td><strong>$x_{max} = \sqrt{\alpha^2 + \beta^2}$</strong></td>
</tr>
<tr>
<td>$Y$</td>
<td>$Y$</td>
<td>$Y$</td>
</tr>
<tr>
<td>$Y$</td>
<td>$N$</td>
<td>$N$</td>
</tr>
<tr>
<td>$Y$</td>
<td>$N$</td>
<td>$N$</td>
</tr>
</tbody>
</table>

The results in Figure 17 and Table VI indicate that a solution that involves taking ground delay is optimal only when it is combined with choosing the nominal route. Otherwise, the optimal choice is to depart without delay and fly the optimal route—which may be nominal, intermediate or detour—based on the DRM.\(^{11}\) In addition, there exists a unique threshold value of the ground-airborne cost ratio that triggers elimination of the ground delay option.

**C. Optimal Solution of the hybrid DRM (hDRM)**

We seek to further characterize the relationship between the solution to the hybrid DRM and its input parameters. As we concluded in the previous section, if the cost ratio is below a certain threshold value, the optimal solution is the combination of a positive ground delay with the nominal route. On the other hand, if the cost ratio is over the threshold value, an immediate take-off without ground delay is optimal and the optimal route is determined by the DRM.

\(^{11}\) When $T \leq \alpha$, the combination of no ground delay and the nominal route is always optimal regardless of the ground-airborne cost ratio and those cases are excluded from analysis.
We call the unique cost ratio which, when exceeded, yields an optimal solution with no ground delay the Critical Cost Ratio (CCR). The CCR defines the set of cases in which ground delay is too expensive to be part of the optimal strategy for avoiding en route weather. The higher the CCR, the more promising are weather avoidance strategies that involve ground delay.

The solution of the hybrid DRM solutions are summarized as follows.

\[
(g^*, x^*) = \begin{cases} 
(0, \alpha) & 0 < c \leq CCR(\alpha, \beta, T) \\
(\text{argmin}_x f_d(x|\alpha, \beta, T)) & CCR(\alpha, \beta, T) < c < 1
\end{cases}
\]  

(37)

To find the optimal ground delay \(g^*\), we utilize the fact that a positive ground delay is always associated with the nominal route. In other words, the optimal ground delay \(g^*\) is the one that minimizes expected total cost of the hybrid DRM when \(x = \alpha\), and we can find \(g^*\) by solving the following.

\[\min f_h(\alpha, g|\alpha, \beta, T) \quad \text{s.t.} \quad 0 < g \leq T - \alpha, \text{where} \quad 0 < c < 1, \quad 0 < \alpha < 1, \quad \beta > 0, \quad T > \alpha \quad (38)\]

We obtain \(f_h(\alpha, g|\alpha, \beta, T)\) from (23) when \(x = \alpha\), which results in a piecewise function as shown below.

\[f_h(\alpha, g|\alpha, \beta, T) = \begin{cases} 
h_1(g, \alpha, \beta, T) & g = T - \alpha \\
h_2(g, \alpha, \beta, T) & T - \alpha - \beta \leq g < T - \alpha \\
h_3(g, \alpha, \beta, T) & 0 < g < T - \alpha - \beta
\end{cases}
\]

, where

\[h_1(g, \alpha, \beta, T) = 1 + c\left(g - \frac{g^2}{2T}\right), \quad (40)\]

\[h_2(g, \alpha, \beta, T) = \frac{g\left(1-c\right)(g-2T)+2g+2\left(2-\alpha\right)\alpha-(g-T+\alpha)\sqrt{(g-T+\alpha)^2+(1-\alpha)^2-(1-\alpha)^2}}{2T} \frac{1-\alpha}{\left(-1+c\right)^2+(g-T+\alpha)^2} \quad (41)\]

\[h_3(g, \alpha, \beta, T) = \frac{\left(\alpha+\beta+\sqrt{(1-\alpha)^2+\beta^2}\right)\left(2\alpha-2T+2g+\beta\right)+cg\left(g-2T\right)-a\beta-2T+\beta+(1-\alpha)^2\ln\frac{1-\alpha}{\beta+(1-\alpha)^2+\beta^2}}{2T} \quad (42)\]

\(f_h(\alpha, g|\alpha, \beta, T)\) is a continuous piecewise function with respect to \(g\) and not always convex nor concave. To solve the optimization problem in (38), we first study each piece in (39). The first piece \(h_1\) is the expected total cost when the maximum ground delay \(T - \alpha\) is taken, which is \(1 + c\left(\frac{T-\alpha}{2}\right)^2\). \(h_2\) is the expected total cost when \(T - \alpha - \beta \leq g < T - \alpha\) and is a convex function. In fact, \(h_2\) is not only convex but non-monotonic with a unique interior optimal solution, which yields an optimal solution as follows.\(^{12}\)

\[g^*_2 = \frac{-1+c-2cT+c^2T+\alpha+\sqrt{1-2c+c^2-2\alpha+2c\alpha-c^2\alpha^2-4\alpha^2+2c^2\alpha^2}}{-2c+c^2}, T - \alpha - \beta \leq g^* < T - \alpha \quad (43)\]

\(^{12}\) See Appendix A.2. for proof
$h_3$ is the expected total cost when $0 \leq g < T - \alpha - \beta$ and is a concave function.\(^{13}\)

The optimal solution occurs either at the lower or the upper bound in this case.

$$ g_3^* = \begin{cases} 0 & f_h(\alpha, 0|c, \alpha, \beta, T) \leq f_h(\alpha, T - \alpha - \beta|c, \alpha, \beta, T) \\ (T - \alpha - \beta) & f_h(\alpha, 0|c, \alpha, \beta, T) > f_h(\alpha, T - \alpha - \beta|c, \alpha, \beta, T) \end{cases} \quad (44) $$

The zero ground delay condition $f_h(\alpha, 0|c, \alpha, \beta, T) \leq f_h(\alpha, T - \alpha - \beta|c, \alpha, \beta, T)$ in (44) is reduced as follows.

$$ 0 < \alpha < 1, \beta > 0, T > \alpha + \beta, c \geq \frac{2(-1+\alpha+\beta)}{T+\alpha+\beta} + 2 \sqrt{\frac{1-2\alpha+\alpha^2+\beta^2}{(T+\alpha+\beta)^2}} \quad (45) $$

Likewise, the non-zero ground delay condition $f_h(\alpha, 0|c, \alpha, \beta, T) > f_h(\alpha, T - \alpha - \beta|c, \alpha, \beta, T)$ in (44) is reduced as follows.

$$ 0 < \alpha < 1, \beta > 0, T > \alpha + \beta, 0 < c < \frac{2(-1+\alpha+\beta)}{T+\alpha+\beta} + 2 \sqrt{\frac{1-2\alpha+\alpha^2+\beta^2}{(T+\alpha+\beta)^2}} \quad (46) $$

From (45) and (46), we can rewrite (44) as follows.

$$ g_3^* = \begin{cases} 0 & c \geq \frac{2(-1+\alpha+\beta)}{T+\alpha+\beta} + 2 \sqrt{\frac{1-2\alpha+\alpha^2+\beta^2}{(T+\alpha+\beta)^2}}, 0 < \alpha < 1, \beta > 0, T > \alpha + \beta \\ T - \alpha - \beta & 0 < c < \frac{2(-1+\alpha+\beta)}{T+\alpha+\beta} + 2 \sqrt{\frac{1-2\alpha+\alpha^2+\beta^2}{(T+\alpha+\beta)^2}} \end{cases} \quad (47) $$

Given $T \geq \alpha + \beta$, the cost function $f_h(\alpha, g|c, \alpha, \beta, T)$ is concave when $0 \leq g < T - \alpha - \beta$, and convex when $T - \alpha - \beta \leq g \leq T - \alpha$. The fact that the cost function is a continuous function gives the optimal solution either at the lower bound of the concave piece $h_3$ or in the interior of the convex piece $h_2$. In other words, the optimal solution is as follows when $T \geq \alpha + \beta$.

$$ g^* = \begin{cases} 0, f_h(\alpha, 0|c, \alpha, \beta, T) \leq f_h(\alpha, g_2^*|c, \alpha, \beta, T) \\ g_2^*, f_h(\alpha, 0|c, \alpha, \beta, T) > f_h(\alpha, g_2^*|c, \alpha, \beta, T) \end{cases} \quad (48) $$

Neither of the conditions in (48) can be reduced analytically. However, we already found those conditions in (46): when $h_3$ has the minimum at the upper bound, $f_h(\alpha, g|c, \alpha, \beta, T)$ has the optimal solution in $h_2$. In other words, the condition (46) is the condition for $g_2^*$ to be optimal. In summary, the optimal ground delay is as follows where $g_{\text{int}}^* = g_2^*$.

$$ g^* = \begin{cases} 0 & c \geq \frac{2(-1+\alpha+\beta)}{T+\alpha+\beta} + 2 \sqrt{\frac{1-2\alpha+\alpha^2+\beta^2}{(T+\alpha+\beta)^2}}, T > \alpha + \beta \\ (g_{\text{int}}^*) & 0 < c \leq \frac{2(-1+\alpha+\beta)}{T+\alpha+\beta} + 2 \sqrt{\frac{1-2\alpha+\alpha^2+\beta^2}{(T+\alpha+\beta)^2}}, T > \alpha + \beta \\ g_{\text{int}}^* & 0 < c < 1, \alpha < T \leq \alpha + \beta \end{cases} \quad (49) $$

\(^{13}\) See Appendix A.3. for proof
When \( \alpha < T \leq \alpha + \beta \), it is always optimal to take ground delay at all cost levels. Recall that the DRM in general has the optimal solution of \( \alpha \) when \( 0 < T \leq \sqrt{\alpha^2 + \beta^2} \) according to the nominal route theorem. With the ground delay option added, we have a larger set of weather parameters that guarantees the nominal route to be optimal when weather clearance time follows a uniform distribution. Moreover, the ground delay option always reduces expected flight cost when \( \alpha < T \leq \alpha + \beta \).

### D. Critical Cost Ratio (CCR)

Now we introduce the critical cost ratio theorem.

**Critical Cost Ratio Theorem.** The critical cost ratio \( c_0 \) is the following.

\[
c_0 = \begin{cases} 
\frac{2(\alpha + \beta - 1 + \sqrt{(1-\alpha)^2 + \beta^2})}{T + \alpha + \beta} & T > \alpha + \beta \\
1 & \alpha < T \leq \alpha + \beta 
\end{cases}
\]

**Proof.**

When \( \alpha < T \leq \alpha + \beta \), it is always optimal to take positive ground delay according to (49), which means that \( c_0 = 1 \).

Given \( T > \alpha + \beta \), let \( c_1 = \frac{2(\alpha+\beta-1+\sqrt{(1-\alpha)^2+\beta^2})}{T+\alpha+\beta} \). Suppose that \( 0 < c_0 < c_1 < 1 \). Then, there must exist \( c_0 < c < c_1 \) satisfying \( g^* = 0 \) from (37) as well as \( g^* > 0 \) from (49), which is a contradiction. Therefore,

\[
c_0 \geq c_1. \tag{50}
\]

Now, suppose that \( 0 < c_1 < c_0 < 1 \). Then, there must exist \( c_1 < c < c_0 \) satisfying \( g^* > 0 \) from (37). From (49) we also have that \( g^* = 0 \), which establishes a contradiction. Therefore,

\[
c_0 \leq c_1. \tag{51}
\]

From (50) and (51), we have \( c_0 = c_1 = \frac{2(\alpha+\beta-1+\sqrt{(1-\alpha)^2+\beta^2})}{T+\alpha+\beta} \). \( \square \)

Based on the critical ratio theorem, the solution to the hybrid DRM is summarized as follows.

\[
(g^*, x^*) = \begin{cases} 
(0, \arg\min_x f_d(x|\alpha, \beta, T)) & c > \frac{2(-1+\alpha+\beta)}{T+\alpha+\beta} + \sqrt{\frac{1-2\alpha+a^2+\beta^2}{(T+\alpha+\beta)^2}}, T > \alpha + \beta \\
(g^*_{\text{int}}, \alpha) & 0 < c \leq \frac{2(-1+\alpha+\beta)}{T+\alpha+\beta} + \sqrt{\frac{1-2\alpha+a^2+\beta^2}{(T+\alpha+\beta)^2}}, T > \alpha + \beta \tag{52} \\
(g^*_{\text{int}}, \alpha) & 0 < c < 1, ~ \alpha < T \leq \alpha + \beta 
\end{cases}
\]
\[ g_{\text{int}}^* = \frac{-1+c-2cT+c^2T+\alpha+\sqrt{1-2c+2a+6ca-2c^2a+a^2-4ca^2+2c^2a^2}}{-2c+c^2}. \]

Once the weather parameters and cost ratio are known, one can determine if the optimal solution requires ground hold or not from the CCR formula. If a positive ground delay is required, \( g_{\text{int}}^* \) gives the required ground delay and the optimal route is the nominal one. If no ground delay is required, one can determine the optimal route based on the DRM.

Since the CCR is always 1 when \( T \leq \alpha + \beta \), consider the case when \( T > \alpha + \beta \). As stated in the critical cost ratio theorem, the CCR is \( c_0 = \frac{2\left(\alpha + \beta - 1 + \sqrt{(1-\alpha)^2 + \beta^2}\right)}{T+\alpha+\beta} \), which is always positive and can be larger than 1. Since the ground-airborne cost ratio is never larger than 1, the weather parameters satisfying \( c_0 > 1 \) have the CCR of 1, in which case, it is always optimal to take ground holding to fly on the nominal route. By rearranging terms, we obtain those set of weather parameters as shown below.

\[ 0 < \alpha < 1, \beta > \sqrt{2\alpha - \alpha^2}, \alpha + \beta < T < -2 + \alpha + \beta + 2\sqrt{1-2\alpha + \alpha^2 + \beta^2} \quad (53) \]

Before discussing the sensitivity of the CCR, we first consider limiting cases with respect to each weather parameter. When \( \alpha \to 0 \), or when the storm is very near the origin, we have \( c_0 = \frac{2\left((-1+\beta+\sqrt{1+\beta^2})\right)}{T+\beta} \). In Figure 18, we show the contour plot of those CCR’s in the \( \beta - T \) plane. We observe that the benefit of ground delay is limited when storms are small unless the cost ratio is small. The large region with CCR over 1 indicates that there are many combinations of storm size and maximum duration time that lead to benefit from ground delay even when it is fairly costly.\(^{14}\)

When \( \alpha \to 1 \), or when the storm is very near the destination, we have \( c_0 = \frac{4\beta}{1+T+\beta} \). In Figure 19, we show the contour plot of those CCR’s in the \( \beta - T \) plane. We observe trends similar to those in the previous plot.\(^{15}\) From the limiting cases of storm location, we conclude that a storm near the origin or the destination is more likely to take ground delay even if the cost ratio is high. There exist, however, weather conditions when the immediate take-off is optimal at a fairly low cost ratio, especially when the storm is small.

\(^{14}\) The contour line associated with CCR value 1 is \( T = -2 + \beta + 2\sqrt{1+\beta^2} \).

\(^{15}\) The contour line associated with CCR value 1 is \( T \to -1 + 3\beta \).
When $\beta \to 0$, the CCR is zero, while it is larger than 1 when $\beta \to \infty$. The result follows our intuition that ground delay is not appropriate for very small storms, while it is always justified for very large ones.

When $T = \alpha + \beta$, we have $c_0 = \frac{-1 + \alpha + \beta + \sqrt{(1-\alpha)^2 + \beta^2}}{\alpha + \beta}$, Since $\frac{-1 + \alpha + \beta + \sqrt{(1-\alpha)^2 + \beta^2}}{\alpha + \beta} > \frac{2(\alpha + \beta - 1 + \sqrt{(1-\alpha)^2 + \beta^2})}{T + \alpha + \beta}$, the current CCR for this maximum storm duration value is an upper bound to the CCR in general. In other words, the cost ratio to eliminate ground delay is the highest among all CCR’s associated with given $\alpha$ and $\beta$, when $T$ is close to $\alpha + \beta$. In Figure 20, we show the contour plot of those CCR’s in the $\alpha - \beta$ plane. We observe that CCR is always larger than 1 when $\beta$ is larger than 1, which indicates that ground delay is always appropriate for storms larger than 1 if the maximum storm duration time is close to the flight time to the tip of the storm on the nominal route without recourse. We also observe that the CCR is more sensitive to the storm size than to storm location, except when the location is very near the origin airport. When $T \to \infty$, we have $c_0 \to 0$, which follows our
intuition; for storms expected to last for an extended time, it is optimal not to take any ground delay to fly on the route determined by the DRM. In the next section, we discuss the sensitivity of the CCR as well as the positive ground delay.

![Figure 20. Contour Plot of CCR when $T \rightarrow \alpha + \beta$](image)

**E. Sensitivity Analysis**

We first discuss the sensitivity of the critical cost ratio $c_0 = \frac{2(\alpha + \beta - 1 + \sqrt{(1 - \alpha)^2 + \beta^2})}{T + \alpha + \beta}$.

Since $\frac{\partial c_0}{\partial T} < 0$, ground delay becomes less attractive for storms expected to last longer. Likewise, $\frac{\partial c_0}{\partial \beta} > 0$ suggests that ground delay becomes more valuable with a larger storm even at a higher cost. The sensitivity of the storm location, however, needs a closer look. $c_0$ is not monotonic with respect to $\alpha$, yet the partial second derivative $\frac{\partial^2 c_0}{\partial \alpha^2} > 0$. Therefore, $c_0$ is convex with respect to $\alpha$, which means for storms located near the origin or the destination, the optimal solution is more likely to include ground delay. It also means that an intermediate and the detour route are more likely to be optimal for storms in the vicinity of the route midpoint, although the nominal may still be preferable if ground delay cost is low.

If the optimal solution includes ground delay, optimal amount of such delay is $g^{*\text{int}} = \frac{-1 + c - 2cT + c^2T + a + \sqrt{1 - 2c + c^2 - 2a + 6ca - 2c^2a + a^2 - 4c^2a^2 + 2c^2a^2}}{-2c + c^2}$. We notice that $g^{*\text{int}}$ does not depend on the storm size $\beta$, which seems counter-intuitive. In fact, the effect of $\beta$ is captured in the critical cost ratio $c_0$.

$g^{*\text{int}}$ is a concave monotonic decreasing function with respect to $c$ since $\frac{\partial g^{*\text{int}}}{\partial c} > 0$ and $\frac{\partial^2 g^{*\text{int}}}{\partial c^2} < 0$. The decreasing trend is consistent with the observations in the numerical analysis. The concavity suggests that the amount of ground delay decreases at an increasing rate as the CCR increases.
As for the storm location $\alpha$, we have $\frac{\partial g^*}{\partial \alpha} < 0$ and $\frac{\partial^2 g^*}{\partial \alpha^2} > 0$, so $g_{\text{int}}^*$ is a convex monotonic decreasing function with respect to $\alpha$. In other words, the optimal ground delay decreases with the distance of the storm from the origin, and the rate of decrease is more rapid for storms located near the origin. It also suggests that for storms located near the destination, ground delay has limited value.

As for the maximum storm duration time, we have $\frac{\partial g_{\text{int}}^*}{\partial T} = 1$, and $g_{\text{int}}^*$ is linearly increasing function with respect to $T$. In other words, the amount of ground delay increases at exactly the same rate as $T$. Such result follows out intuition that the ground delay amount essentially reduces the maximum storm duration time.

### F. Performance Analysis of DRM and hybrid DRM (hDRM)

The performance analysis of hybrid DRM (hDRM) is based on the cost reduction from using the hybrid DRM compared to the DRM. Since the DRM is an upper bound of the hDRM, the solution of the hybrid DRM is never worse than that of the DRM.\(^{16}\) Now define the performance metric as the percentage cost saving as follows.

$$S(\text{hDRM}) = 1 - \frac{\text{Optimal Expected Total Cost of hDRM}}{\text{Optimal Expected Total Cost of DRM}}$$

In\(^\text{21}\), the cumulative distribution function plot of $S(\text{hDRM})$ is shown for $c = 0.1, 0.2, \ldots, 0.9$. When $c = 0.1$, almost 80% of all cases we tested have a cost saving of greater than 5% when ground delay is allowed. This percentage decreases as the cost ratio increases, reaching about 22% when $c = 0.9$. Likewise, we observe that nearly 50% of cases realize more than 25% saving when $c = 0.1$, while none of tested cases show this level of saving when $c = 0.9$. The CDF’s are also seen to be quite linear and nearly parallel when they approach their limiting value of 1.

\(^{16}\) In theory, the optimal expected total cost of DRM can be obtained from hDRM by assuming $c \to \infty$. 

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The average and maximum cost savings are plotted against the storm parameters in Figure 22. The average and maximum saving is obtained for each parameter by aggregating the other parameters. For example, the $\alpha - S(hDRM)$ plot is generated by finding the average and maximum cost saving for all values of $\beta$, $T$ and $c$ for each $\alpha$.

In the $\alpha - S(hDRM)$ plot, we notice that both the average and maximum cost savings reduce as $\alpha$ increases. Such trend suggests that the addition of ground delay is beneficial for storms located closer to the origin. The overall shape however, is nearly flat, which indicates that the effect of ground delay is more dependent on other factors than the location of the storm. The $\beta - S(hDRM)$ plot shows that both the average and maximum cost saving increases as $\beta$ increases. Recall that during our analysis of the DRM, we confirm that the second recourse option produces substantial saving especially for a larger storm. From this plot, it seems that a larger storm also benefits from the ground delay option as well. Note that when the storm is smaller than 0.4, the average saving below 3% even if the maximum saving can be high. This suggests that there are few cases involving small storms in which taking ground delay can significantly reduce costs. The $T - S(hDRM)$ plot shows trends similar to those observed in the $\beta - S(hDRM)$ plot, suggesting that the ground delay option is more useful for storms with longer duration time. The insensitivity of savings in the region where $T$ is larger than 2 indicates that a longer duration time doesn’t always result in greater savings from employing ground delay and is subject to other factors such as the cost ratio.
Figure 22. Average and Maximum $S(hDRM)$ With Respect To Weather Parameter

$G$. Value of Hedging

We have found that the optimal combination of the ground delay and route choice is either a non-negative ground delay with the nominal route or zero ground delay with a non-nominal route. Since a positive ground delay is always accompanied with the nominal route, the second recourse, which we found in the section B of this chapter, makes the nominal route optimal in many cases plays a crucial role in the hybrid-DRM. When an intermediate route is optimal, it means that the ground delay is useless and our hedged routing strategy combined with immediate departure works best.
In practice, the nominal and detour routes must always be available. Intermediate or hedged routes, on the other hand, may or may not be allowed. In this section, we assess the value of hedging by identifying the weather and cost combinations in which intermediate routes are optimal. Since the detour route implies the exclusive use of the first recourse while the nominal route implies the exclusive use of the second recourse, we can obtain insights on the value of making both recourse options available from the cases when the hedged route is optimal as well.

From (18) and (49), the condition for an intermediate route to be optimal is as follows.

\[ T_{AB} < T \leq T_{BC}, c_0 < c < 1 \]  

(55)

In Figure 23, \( \alpha, \beta, T \) combinations that satisfy (55) are shown in the point plot. The critical cost ratio (CCR) of corresponding \( \alpha, \beta, T \) is shown in gray scales.\(^{17}\) Among the weather parameters, those near the boundary surface \( T_{AB} \) require a high CCR for an intermediate route to be optimal, while those near the boundary surface \( T_{BC} \) require a lower CCR. One interpretation of this pattern is that for storms of the same size and at the same location, the longer the storm is expected to last, the less attractive the ground delay becomes even at a fairly low cost. Similarly, for storms with the same location and maximum duration time, the ground delay becomes less effective when its size is small. The range of CCR values shown in Figure 23 is between 13.3% and 97.2%.

![Figure 23. Range of CCR When Hedged Routes Are Optimal](image)

\(^{17}\) Note that every point in the figure belongs to the DRM solution category B.
In Table VII, a collection of contour maps show the regions in which it is optimal to depart immediately and choose an intermediate route for various cost ratio upper limits. Values in the white squares are the maximum storm duration time $T$. For example, in the first plot, the plotted weather combinations are those that yield an optimum with an intermediate route and no ground delay, if the cost ratio is 30%. Likewise, those in the 40% plot yield intermediate routes if the cost ratio is 40%. The fact that the storm size is well below 1 in most plots indicates that intermediate routes are most likely to be optimal when storms are small, especially when the CCR is also low.

Table VII. Contour Maps of the Weather Parameters When Hedged Routes are Optimal

<table>
<thead>
<tr>
<th>Cost Ratio</th>
<th>Contour Map</th>
</tr>
</thead>
<tbody>
<tr>
<td>30%</td>
<td><img src="image1.png" alt="Contour Map" /></td>
</tr>
<tr>
<td>40%</td>
<td><img src="image2.png" alt="Contour Map" /></td>
</tr>
</tbody>
</table>
In this chapter, we formulated the ground-airborne hybrid-DRM and find the solution set based on the relationship between the DRM and hybrid DRM. When the ground-airborne cost ratio is below the Critical Cost Ratio (CCR), the combination of the nominal route and positive ground delay is optimal. On the other hand, if the cost ratio is above the CCR, an immediate take-off without any ground delay is optimal. The route choice for an immediate take-off is determined by the DRM since no ground delay is required.

When the optimal cost under the hybrid-DRM is compared to that of the DRM, the maximum cost saving reaches nearly 60% when the ground hold cost is 10% of the airborne. Even when the ground-airborne ratio is 90%, there are cases that realize a cost saving close to 20%. Overall, there are opportunities for significant cost savings from
employing ground delay as a weather avoidance strategy. We also find that even if the ground delay is most effective when its relative cost is less than the additional airborne cost, there are cases when ground holding is still valuable when the cost associated with it is not too different from the airborne cost.

The cost saving decreases as $\alpha$ increases, while it increases for $\beta$ and $T$. In general, the addition of ground delay is most useful for storms of a larger size, located closer to the destination with a longer maximum storm duration time. In the next chapter, we incorporate such findings to a non-parametric model that approximates the optimality conditions, which provides the optimal solutions in a closed form and the opportunities for more complex analyses.

Unlike the nominal and the detour route, which must be available in practice, use of hedged routes may or may not be provided. We have found that there are many cases when the hedged route is optimal. We have also identified the specific weather and cost ratio combinations for which an immediate take-off on the hedged route is optimal. This occurs most often in situations featuring small storm sizes and low CCR values.
IX. CONCLUSIONS AND FUTURE RESEARCH

In this research, a probabilistic air traffic management strategy is proposed and studied. The main purpose of our approach is to reduce the risks associated with the weather uncertainty in the airspace at a minimum expected total cost, when a set of operational and control capabilities such as hedged routes and recourses are available. The geometric model is adopted to incorporate operational flexibility such as the first and the second recourses as well as hedged routes. The simple geometric setup enables us to conduct complex analyses and effective evaluation on the proposed model.

The geometric recourse model has two variations: Single Recourse Model (SRM) and Dual Recourse Model (DRM). The SRM allows the first recourse only and becomes convex with its optimal solution either being the detour or an intermediate route when the weather clearance time follows the uniform distribution. The cost improvement of the SRM compared to taking the detour route is marginal, however. The DRM allows both the first and second recourses and the cost improvement compared to the SRM is substantial, reaching over 35% in some cases. According to the nominal route theorem, regardless of the weather distribution, the nominal route is always optimal in the DRM if the flight time to the tip of the storm on the detour route is longer than the maximum storm duration time. Overall, the second recourse option enables to use the nominal route in many cases, which considerably increases the cost saving opportunity compared to the SRM.

While the SRM and DRM consider routing only, the ground-airborne hybrid model also allows weather avoidance through ground holding. Since the cost ratio between the ground and airborne delay is not fixed, we treat the cost ratio as an additional parameter to the model to find the optimal combination of the ground delay and route choice to minimize the total expected cost. Under the uniformly distributed weather clearance time, the optimal route of the hybrid model is determined by the geometric recourse model with the maximum storm duration time reduced by the ground delay amount.

We identify a unique threshold ground-airborne cost ratio, which we call the Critical Cost Ratio (CCR), that triggers elimination of the ground delay option. If the current cost ratio is below the CCR, it is optimal to take a positive ground delay to fly on the nominal route. On the other hand, if it is below the CCR, an immediate take-off without ground delay is optimal, and the optimal route is determined by the DRM. The formula of CCR is obtained in terms of the storm location, size and the maximum duration time. Through the sensitivity analysis, we find that storms located near the origin or the destination is more likely to benefit from ground delay then those in the midway. For storms about halfway between the origin and destination, non-nominal routes are especially useful unless the cost ratio is fairly low. It is also found that the optimal ground delay is larger in the following cases: (i) storms located near the origin, and (ii) storm with large maximum duration time. The optimal ground delay is invariant to the storm size, however, and the effect of the storm size is captured in the CCR.

Cost savings of the hybrid DRM over the DRM depends on the cost ratio. At 30% cost ratio, over 60% of all cases we tested show larger than 10% savings, with its maximum reaching nearly 45%. At 50% cost ratio, nearly 45% show larger than 10% savings, with its
maximum reaching 35%. The ground delay as it is available today proves its value under the stochastic setting as well.

In this research, we conclude that the capability to respond to weather change during the course of flight such as the recourses is most effective in reducing cost associated with uncertain weather. The first recourse enables an efficient use of the detour route, while the second recourse combined with the nominal route is one of the most effective ways to save cost when optimal. The ground delay option adds another cost saving opportunity from expanded use of the nominal route, but such saving is subject to the ground-airborne cost ratio. It is not rare when immediate take-off on the hedged routes are optimal, and those weather and cost ratio combinations are identified. The hedged routes are especially useful when the storms of sizes smaller than 1 are associated with low CCR.

There are several variations of the geometric recourse model that might be considered in future research. One can change the assumptions on the weather such as the clearance time probability distributions other than uniform. Other storm geometries, including storms that are not orthogonal to the nominal route, and moving storms, may also be considered. The choice of routes might also be extended to consider curvilinear ones that begin at a narrow angle relative to the nominal path but splay outward as the aircraft approaches an active storm. We can also introduce variable cost structure for operational capabilities. For example, the extensive use of the nominal route in the DRM and the hybrid DRM is due to the possibility of the second recourse at no additional cost. However, it is reasonable to suspect that the last minute change of the flight plan very near the restricted airspace might involve a higher cost than the first recourse or the ground holding. Similarly, we can impose a higher cost to the hedged routes as a way to compensate for a potential availability issue.

The development of future air traffic management strategies not only involves finding solution algorithms for complex optimization models but also requires extensive analyses to extract critical insights and policy implications for the air transportation system users and the air navigation service provider. Although the expected total cost functions of the geometric models we discussed are obtainable in analytic forms, this may not be the case for other variations, and different methods such as numeric simulation may be needed find solutions. We expect many of the qualitative insights gained from the research presented here to also hold in more general and complex settings in which analytical methods are no longer tractable.
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[19] “Roadmap for Performance Based Navigation”, Federal Aviation Administration

APPENDIX A

A.1. Boundary Condition of Interior Solution

\[ g_s(\alpha, \beta) = ((\alpha^2 + \beta^2)^2(\alpha^5 + \alpha^4(-2 + \beta) + \beta^3 + \beta^5 + \alpha^3(1 - 3\beta + 2\beta^2) \]
\[ + \alpha^2\beta(3 - 2\beta + 2\beta^2) + \alpha\beta(-1 + \beta - 3\beta^2 + \beta^3)) \]
\[ - \sqrt{\alpha^2 + \beta^2(\alpha^8 + \alpha^7(-2 + \beta) + 3\alpha^5(-1 + \beta)^2\beta + 3\alpha^3(-1 + \beta)^2\beta^3 + \beta^6} \]
\[ + \beta^8 + \alpha^6(1 - 3\beta + 4\beta^2) + \alpha^2\beta^3(1 + 3\beta - 3\beta^2 + 4\beta^3) \]
\[ + \alpha^4\beta(-1 + 3\beta - 6\beta^2 + 6\beta^3) + \alpha^3\beta^3(-1 - 2\beta^3 + \beta^4)) \]
\[ - \sqrt{1 - 2\alpha + \alpha^2 + \beta^2(\alpha^8 + \alpha^7(-1 + \beta) + 3\alpha^3(-1 + \beta)\beta^4 + \beta^8} \]
\[ + 2\alpha^6\beta(-1 + 2\beta) + 2\alpha^2\beta^5(-1 + 2\beta) + 2\alpha^4\beta^3(-2 + 3\beta) \]
\[ + \alpha\beta^5(-1 - \beta + \beta^2) + \alpha^5\beta(1 - 3\beta + 3\beta^2)) \]
\[ + \sqrt{-2\alpha^3 + \alpha^4 + \beta^2 - 2\alpha\beta^2 + \beta^4 + \alpha^2(1 + 2\beta^2)(\alpha^7 + \alpha^6(-1 + \beta) + \beta^7} \]
\[ + \alpha^3\beta^3(-4 + 3\beta) + \alpha^5\beta(-2 + 3\beta) + \alpha^4\beta(1 - 2\beta + 3\beta^2) \]
\[ + \alpha^2\beta^3(1 - \beta + 3\beta^2) + \alpha\beta^3(-1 - 2\beta^2 + \beta^3)) \]
\[ + \alpha^2\beta^2(\alpha^4 - \alpha^3(2 + \sqrt{\alpha^2 + \beta^2} + \sqrt{1 - 2\alpha + \alpha^2 + \beta^2})) \]
\[ + \beta^2(1 + \beta^2 + \sqrt{\alpha^2 - 2\alpha^3 + \alpha^4 + \beta^2 - 2\alpha\beta^2 + 2\alpha^2\beta^2 + \beta^4} \]
\[ + \alpha^2(1 + 2\beta^2 + 2\sqrt{\alpha^2 + \beta^2} + \sqrt{1 - 2\alpha + \alpha^2 + \beta^2} \]
\[ + \sqrt{\alpha^2 - 2\alpha^3 + \alpha^4 + \beta^2 - 2\alpha\beta^2 + 2\alpha^2\beta^2 + \beta^4}) \]
\[ - \alpha(\sqrt{\alpha^2 + \beta^2} + \sqrt{\alpha^2 - 2\alpha^3 + \alpha^4 + \beta^2 - 2\alpha\beta^2 + 2\alpha^2\beta^2 + \beta^4} \]
\[ + \beta^2(2 + \sqrt{\alpha^2 + \beta^2} + \sqrt{1 - 2\alpha + \alpha^2 + \beta^2})) \bigg) (\log [-\alpha + \sqrt{\alpha^2 + \beta^2} \]
\[ + \beta^2(1 - 2\alpha + \alpha^2 + \beta^2)(-\alpha + \alpha^2 + \beta^2 \]
\[ + \sqrt{(\alpha^2 + \beta^2)(1 - 2\alpha + \alpha^2 + \beta^2)(\alpha^3 + \alpha^2(\beta - \sqrt{\alpha^2 + \beta^2} + \alpha\beta(\beta \]
\[ - \sqrt{\alpha^2 + \beta^2}) + \beta^2(\beta - \sqrt{\alpha^2 + \beta^2})) \bigg) \]

A.2. Interior solution of \( h_2(g, c, \alpha, \beta, T) \)

First of all, we show that \( h_2(g, c, \alpha, \beta, T) \) is a non-monotonic function.

\[ \frac{\partial h_2}{\partial g} < 0 \] when \( \alpha = 0.35, \beta = 1, T = 0.8, c = 0.23, g = 0.33, \) and

\[ \frac{\partial h_2}{\partial g} > 0 \] when \( \alpha = 0.31, \beta = 1, T = 0.63, c = 0.62, g = 0.32. \) Therefore, \( h_2(g, c, \alpha, \beta, T) \) is not monotonic.
Now, we have \( \frac{\partial^2 h_2}{\partial g^2} = -\frac{-1+c+g-T+a}{\sqrt{1+g^2-2gT+T^2-2a+2ga-2Ta+2a^2}} > 0 \), which gives that \( h_2 \) is a convex function with respect to \( g \). Since \( h_2 \) is not monotonic, we have a unique minimum in the interior. We can find the minimum \( g \) by solving the following equation.

\[
\frac{\partial h_2}{\partial g} = - \frac{-1+(-1+c)g+T-cT+\sqrt{1+g^2-2gT+T^2-2a+2ga-2Ta+2a^2}}{g} = 0 , \text{ where } T - \alpha - \beta \leq g^* < T - \alpha
\]

And the solution is \( g^*_2 = \frac{-1+c-2cT+c^2T+\alpha+\sqrt{1-2c+2c^2T+T^2+2a+2a^2-4ca^2+2a^2}}{-2c+c^2} \).

A.3. Proof of non-monotonic and concave \( h_3 (g, c, \alpha, \beta, T) \)

We first show that \( h_3 (g, c, \alpha, \beta, T) \) is not monotonic.

\[
\frac{\partial h_3}{\partial g} = - \frac{-1+c(g-T)+\alpha+\beta+\sqrt{1+\alpha+\beta}\alpha}{T}
\]

\[
\frac{\partial h_3}{\partial g} < 0 \text{ when } \alpha = 0.625, \beta = 2, T = 3.125, c = 0.5, g = 0.25 , \text{ and } \frac{\partial h_3}{\partial g} > 0 \text{ when } \alpha = 0.56, \beta = 0.625, T = 3.1, c = 0.5, g = 0.375. \text{ Therefore } h_3 \text{ is not monotonic.}
\]

Now, \( \frac{\partial^2 h_3}{\partial g^2} = - \frac{c}{T} < 0 \), and \( h_3 \) is concave.