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Wave-particle interactions in space and laboratory plasmas

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Wave-particle Interactions in Space and Laboratory Plasmas

A dissertation submitted in partial satisfaction
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Doctor of Philosophy in Atmospheric and Oceanic Sciences

by

Xin An

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This dissertation presents a study of wave-particle interactions in space and in the laboratory. To be concrete, the excitation of whistler-mode chorus waves in space and in the laboratory is studied in the first part. The relaxation of whistler anisotropy instability relevant to whistler-mode chorus waves in space is examined. Using a linear growth rate analysis and kinetic particle-in-cell simulations, the electron distributions are demonstrated to be well-constrained by the whistler anisotropy instability to a marginal-stability state, consistent with measurements by Van Allen Probes. The electron parallel beta $\beta_{\parallel e}$ separates the excited whistler waves into two groups: (i) quasi-parallel whistler waves for $\beta_{\parallel e} \gtrsim 0.02$ and (ii) oblique whistler waves close to the resonance cone for $\beta_{\parallel e} \lesssim 0.02$. The saturated magnetic field energy of whistler waves roughly scales with the square of the electron beta $\beta_{\parallel e}^2$, as shown in both satellite observations and particle-in-cell simulations. Motivated by the puzzles of chorus waves in space and by their recognized importance, the excitation of whistler-mode chorus waves is studied in the Large Plasma Device by the injection of a helical electron beam into a cold plasma. Incoherent broadband whistler waves similar to magnetospheric hiss are observed in the laboratory plasma. Their mode structures are identified by the phase-correlation technique. It is demonstrated that the waves are excited through a combination of Landau resonance, cyclotron resonance and anomalous cyclotron resonance. To account for the finite size effect of the electron beam, linear unstable eigenmodes of whistler waves are calculated by matching the eigenmode solution at the boundary. It is shown that the perpendicular wave number inside the beam is quantized due to the constraint imposed
by the boundary condition. Darwin particle-in-cell simulations are carried out to study the simultaneous excitation of Langmuir and whistler waves in a beam-plasma system. The electron beam is first slowed down and relaxed by the rapidly growing Langmuir wave parallel to the background magnetic field. The tail of the core electrons are trapped by the large amplitude Langmuir wave and are accelerated in the parallel direction. The excitation of whistler waves through Landau resonance is limited by the saturation of Langmuir waves, due to a faster depletion rate of the beam free energy from $\partial f_b/\partial v_\parallel > 0$ by the latter compared to the former. The second part of the thesis considers the interaction between electromagnetic ion cyclotron (EMIC) waves and relativistic electrons. Nonlinear interactions between them are investigated in a two-wave oscillator model. Three interaction regimes are identified depending on the separation of the two wave numbers. Both the decoupled and degenerate regimes are characterized by phase bunching, in which the resonant electrons are scattered preferentially to one direction rather than diffusively. In the coupled regime, resonant electrons experience alternate trapping and de-trapping in the two overlapped resonant islands, from which stochastic motion of electrons arises. For a continuous spectrum of EMIC waves, test particle simulations are compared against quasi-linear diffusion theory (QLT) description of the wave-particle interactions. QLT gives similar results as test particle simulations for the small amplitude and broadband waves, whereas it fails for large amplitude and narrowband waves. By varying the wave spectral width and wave intensity systematically, a regime map is constructed to indicate the applicability of QLT in the wave parameter space.
The dissertation of Xin An is approved.

Bart Van Compernolle

George Morales

Larry Lyons

Richard M. Thorne

Jacob Bortnik, Committee Chair

University of California, Los Angeles

2017
To my family,

Zhibin An and Aifang Wang, Huan An and Fangzhou Sun
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4.1 A comparison between plasma parameters in the laboratory and in the magnetosphere: plasma density [Sheeley et al., 2001]; magnetic field strength; ratio of plasma frequency to cyclotron frequency [Li et al., 2012]; ratio of beam density to plasma density [Gao et al., 2014]; ratio of whistler wave amplitude to background magnetic field strength [Gao et al., 2014; Li et al., 2012]; ratio of energetic electron velocity to speed of light; ratio of electron thermal pressure to magnetic pressure. ......................................................... 50
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CHAPTER 1

Introduction

In the essentially collisionless magnetosphere, wave-particle interactions are one of the most important processes that controls the dynamic variability of energetic particles. This thesis, in particular, studies the interaction of energetic electrons with two of the plasma waves relevant to the inner magnetosphere, i.e., whistler-mode chorus waves and electromagnetic ion cyclotron (EMIC) waves.

Whistler-mode chorus waves play a critical role in accelerating energetic electrons (hundreds of keV) to highly relativistic energies (∼MeV) in the heart of the outer radiation belt, as well as in precipitating energetic electrons (tens of keV) to the upper atmosphere. Despite its recognized importance, there are a number of questions remaining to be answered: what parameters control the saturation amplitude of chorus waves? What determines the wave normal angle distribution of chorus waves? What triggers the frequency chirping of chorus waves? Very broadly, the characteristics of chorus waves must be dependent on the level of magnetic storm activity, since the plasma properties, such as the electron beta and electron anisotropy, may change vastly from quiet times to active times. One approach to quantify the dependence of the characteristics of chorus waves on the plasma parameters is the linear kinetic theory, which is used to determine the unstable wave modes and their corresponding wave normal angles. The whistler anisotropy instability is saturated by the reduction of electron anisotropy in velocity space, i.e., the scattering of electrons in velocity space. Such wave saturation process of whistler anisotropy instability is studied using a self-consistent particle-in-cell method. One should note, however, that for the growth of chorus waves, the saturation of whistler anisotropy instability is followed by a bursty nonlinear growth of wave energy [Omura and Nunn, 2011; Tao, 2014], during which the wave frequency shifts
substantially with time. Such nonlinear wave growth is attributed to the inhomogeneity of the background magnetic field [e.g., Nunn, 1974; Matsumoto, 1979; Vomvoridis et al., 1982; Omura et al., 2008]. Though such conjecture seems to be plausible, the formulated theory is still unsatisfactory in predicting the key features of chorus waves, e.g., the wave saturation amplitude, the frequency sweep rate and the subpacket structure. There is a major gap to be filled in between theory and satellite observations.

EMIC waves play a major role in precipitating the relativistic electrons (a few MeV) into the Earth’s upper atmosphere. Because their phase velocity is much smaller than the speed of relativistic electrons, these electrons are mainly scattered in pitch angle. There is a negligible amount of energy exchange between EMIC waves and relativistic electrons. Traditionally, quasi-linear diffusion theory is used to describe the pitch angle scattering of relativistic electrons by EMIC waves. It assumes that EMIC waves are small amplitude and broadband. However, spacecraft observations show that the amplitude of EMIC waves can reach a few percent of the background magnetic field and they are even narrowband sometimes. These observations pose a challenge to the validity of use of the quasi-linear theory in the observed wave environment. The applicability of quasi-linear theory in different regimes of wave parameter space needs to be tested.

1.1 The structure of the inner magnetosphere: plasmasphere, ring current and radiation belts

We begin by examining the structure of the inner magnetosphere, since this region is the home to various plasma waves of interest and sets up the context for this dissertation. The Earth’s magnetic field is sufficient to dominate the pressure balance with the solar wind flow upstream of the Earth, forming a cavity region, known as the magnetosphere. As a result, the geomagnetic field lines are compressed in the dayside and are stretched into an extended tail behind the Earth. A schematic picture of the Earth’s magnetosphere with its distorted geomagnetic field lines is shown in upper left panel of Figure 1.1. The region of thermalized solar wind plasma behind the bow shock is termed the magnetosheath. The boundary
between the magnetosphere and the thermalized solar wind is called the magnetopause. The continual stream of solar wind imposes a large-scale convection electric field across the magnetosphere from dawn to dusk. Particles in the Earth’s plasma sheet can be transported to the inner magnetosphere by such a dawn-to-dusk convection electric field. With another particle source from the ionosphere, the inner magnetosphere is populated by particles of distinct energies. Three groups of particle population are identified by their energy and spatial extent, namely, plasmasphere, ring current and radiation belts. The configuration of these regions is shown in the lower right panel of Figure 1.1.

![Figure 1.1: A schematic picture of the magnetosphere. Lower right: An expanded picture of the inner magnetosphere, showing the configuration of the plasmasphere, ring current and radiation belts with distinct particle energies. The gray-white sphere with an extended plume is the plasmasphere. The orange-red torus is the ring current. Two green-yellow-red tori with a slot region in between are the inner and outer radiation belts. [Collado-Vega, 2015].](image)

The plasmasphere is a region of cold (a few eV in temperature), dense ($\sim 10^3$ cm$^{-3}$) plasma located from just above the ionosphere and extending to about $7 \, R_E$ (Earth radius).
It was discovered by Gringauz [1963] using measurements from the LUNIK spacecraft and by Carpenter [1963] using ground-based measurements of whistler waves. It originates from the ionospheric outflows along the magnetic field lines. The plasmasphere terminates in a few tenths of an Earth radius, across which the plasma density drops by a factor of $10^2$ or more. This boundary is known as the plasmapause. A separatrix between open and closed streamlines exists, which is a zero order estimate of the plasmapause but not exactly the same [Grebowsky, 1970]. Inside the separatrix, the geomagnetic field lines are “frozen” into the ionosphere and co-rotate with the Earth because of the highly conducting ionosphere.

The cold bulk plasma of the plasmasphere co-rotates with Earth. Outside the separatrix, the large-scale dawn-to-dusk convection electric field dominates over the co-rotation electric field and the plasma is transported from the tail to the dayside magnetopause. The basic shape of the plasmasphere is asymmetric and has a bulge in the dusk extending to the dayside magnetopause [Carpenter, 1966; Grebowsky, 1970; Chappell et al., 1971; Chen et al., 1975], which is determined by the competition between the solar wind-driven convection electric field and the co-rotating electric field. The plasmasphere erodes during geomagnetic active times due to a stronger convection electric field. As a consequence, the plasma inside the pre-storm separatrix but outside the separatrix in active times erodes away. Extreme ultraviolet (EUV) images taken by EUV instrument [Sandel et al., 2000] on the IMAGE spacecraft [Burch, 2000] provide a global picture of the evolution of plasmaspheric structure and advance our understanding of the physics of the plasmasphere. EUV light from the sun is absorbed and reemitted by helium ions, making the plasmasphere luminous in 30.4 nm light. A series of images of the plasmasphere taken by the EUV instrument is shown in Figure 1.2, which begins near the main phase of a geomagnetic storm and endures over the next 3 hours. The global EUV images clearly show the predicted plasmaspheric plume or tail formed by the enhanced solar wind-driven convection electric field. This plume connects to the main body of plasmasphere in the duskside and extends toward the sun.

The ring current consists of energetic electrons and ions with energies of $\sim 10 - 200$ keV trapped on magnetic field lines from 2 to $7 R_E$. Due to the gradient and curvature of the magnetic field lines, electrons drift eastward while ions drift westward. The resulting current
Figure 1.2: Extreme ultraviolet emissions from the plasmasphere as imaged by the EUV instrument on the IMAGE spacecraft from above the north pole during a geomagnetic storm on 24 May 2000. The yellow circle represents the Earth. The sun is to the lower right corner of each image, opposite the dark shadow region. Note that a plasmaspheric plume develops in the duskside and extends to the dayside. Each image pixel is an integral of the EUV volume emission rate along the corresponding line of sight. The volume emission rate is proportional to the local He\textsuperscript{+} density. [Burch et al., 2001]
is thus westward, producing a southward magnetic field opposite to the Earth’s intrinsic
dipole field at low latitudes on Earth’s surface. The magnitude of this southward magnetic
field is proportional to the total energy content of the ring current particles, which is mainly
contributed by the ions. During the main phase of a storm, the ring current particles are
simply convected inward by the enhanced convection electric field and gain energy in this
process. The Dst index (or SYM-H index), which measures the perturbation of magnetic field
at low latitudes on the Earth’s surface, shows a significant reduction in the storm main phase.
As revealed by a recent observation from Van Allen Probes shown in Figure 1.3, the magnetic
field depressions (indicated by SYM-H index in Figure 1.3a) during the storm main phase are
mainly contributed by the $7 - 80$ keV protons (the blue trace in Figure 1.3b) [Gkioulidou et al.,
2016], which are driven by the convection electric field and mesoscale dynamic injections.
In contrast, protons with energy $> 100$ keV (the orange trace in Figure 1.3b) contribute to
the energy content during the storm recovery phase. A correlation analysis in Figures 1.3c
and 1.3d clearly indicates that the minimum of the SYM-H index is correlated with the
increase of the energy content of low energy protons ($7 - 80$ keV). While the energy content
contributed by protons is dominant during quiet times, the relative energy content from $O^+$
ions increases significantly during a storm. This implies that the Earth’s ionosphere is an
important source of $O^+$ ions in the ring current, which can be easily identified since oxygen
ions only exists in a higher ionized state in the solar wind. During the storm recovery phase,
the Dst index gradually recovers to the pre-storm value in two or three days, due to the
slowing down of the particle transport to the ring current and the removal of the trapped
particles from the ring current. Both collisional processes (charge exchange of energetic ions
with hydrogen atoms in the exosphere) as well as pitch-angle scattering (precipitation into
the atmosphere) by intense plasma waves contribute the overall loss of ring current particles.
The energetic neutral atoms (ENA) emissions produced in the collisions between energetic
ions and hydrogen atoms have been used to image the global ring current on NASA’s IMAGE
spacecraft [Burch, 2000].

The radiation belts consists of electrons and protons trapped in the geomagnetic field
from $1.2 R_E$ up to $8 R_E$. Protons with energies $> 100$ keV have a one-zone structure, for which
Figure 1.3: (a) SYM-H index from 18 March to 31 December 2013. SYM-H index is similar to the $Dst$ index but has a higher time resolution. (b) The energy content calculated for two group of protons, (i) $7 - 80$ keV in blue and $100 - 600$ keV in orange. The SYM-H index is overplotted in grey for reference. (c) The correlation coefficient between the SYM-H index and the partial energy content between $L$-shells $L = 3$ and $L = 7$, for parallel pressures (red) and perpendicular pressures (cyan). (d) The correlation coefficient between SYM-H index and the partial energy content for perpendicular pressures for three ranges of $L$-shells. [Gkioulidou et al., 2016]
the source mechanism is inward radial diffusion [Nakada and Mead, 1965]. The electrons in
the radiation belts have energies from a few hundred keV up to 10 MeV. They are found in
two distinct regions or belts (Figure 1.1): an inner belt between 1.2 and 2.5\(R_E\) and an outer
belt from 4 to 8\(R_E\). The gap in electron fluxes between the inner and outer belts is referred
to as the slot region. The inner belt electrons are relatively stable and are formed by a
balance between the slow inward radial diffusion and loss processes (Coulomb collisions and
pitch angle scattering by whistler waves) [Lyons and Thorne, 1973; Abel and Thorne, 1998].
The electron slot region during quiet times is an equilibrium structure also due to inward
radial diffusion and pitch angle scattering by plasmaspheric hiss [Lyons and Thorne, 1973;
Abel and Thorne, 1998]. The electron flux of the outer radiation belt is very dynamic and is
constantly subject to dramatic changes, which results from the competition between source
and loss processes. Wave-particle interactions are the major mechanism of these source
and loss processes in such a collisionless plasma environment. For instance, whistler-mode
chorus waves can play a dual role in stochastic local acceleration of electrons to relativistic
energies [e.g., Thorne et al., 2013a] as well as in precipitation of electrons to the upper
atmosphere [e.g., Thorne et al., 2010]. Ultra-low-frequency waves can transport electrons to
the outer radiation belt by inward radial diffusion from high \(L\)-shells, which accelerates the
electrons by the conservation of the first adiabatic invariant [e.g., Su et al., 2015; Mann et al.,
2016]. The dynamics of the outer radiation belt is dependent on geomagnetic storm activity
since various waves get enhanced during storms. This dissertation studies the excitation of
one such wave, namely the whistler-mode chorus wave, that is important for the electron
dynamics of the outer radiation belt.

1.2 Whistler-mode waves in the inner magnetosphere

Whistler-mode waves are an important type of magnetospheric wave, occurring in a large
region in the inner magnetosphere. They can have cyclotron resonant interactions with
energetic electrons, leading to the violation of the first adiabatic invariant. Consequently,
whistler waves can cause pitch angle scattering and energy diffusion of electrons. A survey
plot in Figure 1.4 shows two typical types of whistler-mode emissions in the inner magnetosphere, i.e., chorus waves outside the plasmasphere and plasmaspheric hiss. Another type of whistler-mode wave exhibiting discrete chirps is the set of triggered emissions by very-low-frequency (VLF) transmitters. Here we introduce the types of whistler-mode emissions relevant to this dissertation.

Figure 1.4: A survey plot of the wave electric spectral intensity observed on the Combined Release and Radiation Effects Satellite (CRRES) during orbit 119. The solid white line is the local electron cyclotron frequency $f_{ce}$. Three dashed lines, from bottom to top, denote the local lower hybrid frequency, $0.1f_{ce}$ and $0.5f_{ce}$. The first four harmonics of $f_{ce}$ are represented by the dotted lines. The local upper hybrid frequency is shown using the red solid line, indicating the location of plasmapause. Whistler-mode waves below $f_{ce}$, and electron cyclotron harmonic (ECH) waves between harmonics of $f_{ce}$ are labeled. [Meredith et al., 2004]

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Chorus emissions are coherent electromagnetic waves occurring in the whistler branch
of the dispersion relation, which occur in two distinct bands separated by the half electron cyclotron frequency [Tsurutani and Smith, 1974]. They exhibit various forms, such as broadband hiss-like emissions and discrete rising or falling tones as shown in Figure 1.5 [Pope, 1963; Burtis and Helliwell, 1969; Cornilleau-Wehrlin et al., 1978; Koons, 1981; Santolík et al., 2009; Li et al., 2012]. Spacecraft observations show that the source location of chorus waves is near the magnetic equator (determined due to the divergence of Poynting fluxes) [e.g., LeDocq et al., 1998; Santolík et al., 2003; Li et al., 2013a]. The wave normal angles of lower band chorus waves are observed to be quasi-parallel under disturbed geomagnetic conditions, while they tend to be oblique and close to the resonance cone under relatively quiet geomagnetic conditions [Agapitov et al., 2015; Li et al., 2016a]. The wave normal angles of upper band chorus waves, on the other hand, are found to have a wide distribution from 0° up to the resonance cone [Li et al., 2013b, 2016a; Taubenschuss et al., 2014]. Here the wave normal angle (WNA) denotes the angle between the wave vector and the background magnetic field. The source of free energy for whistler-mode chorus waves is the unstable population of electrons that is injected from Earth’s plasma sheet and hence it is dependent on the level of geomagnetic activity [Kennel and Petschek, 1966; Anderson and Maeda, 1977; Meredith et al., 2001]. Chorus waves play a dual role in both the loss and local acceleration of radiation belt electrons [Bortnik and Thorne, 2007]. For instance, the diffuse auroral precipitation at \( L < 8 \) is mainly attributed to chorus waves [Thorne et al., 2010]. Recent observations by Van Allen Probes as well as modeling show that chorus plays a major role in accelerating the relativistic electrons in the heart of the outer radiation belt [e.g., Reeves et al., 2013; Thorne et al., 2013a].

Plasmaspheric hiss is a broadband incoherent whistler-mode emission confined within the plasmasphere and dayside plasmaspheric plumes [Thorne et al., 1973; Tsurutani et al., 2015]. Their frequency range is from \( \lesssim 100 \text{ Hz} \) to serveral \( \text{kHz} \) [Li et al., 2015]. The origin of plasmaspheric hiss has been controversial [Green et al., 2005, 2006; Thorne et al., 2006; Meredith et al., 2006], although the leading theory obtained using the ray tracing method shows that hiss originates from a subset of chorus waves that avoid Landau damping when propagating from the equatorial region to higher latitude [Bortnik et al., 2008b, 2009a]. These
chorus waves propagate to lower $L$-shells, get trapped in the plasmasphere and eventually merge together to form plasmaspheric hiss. This theory has been confirmed by a simultaneous observation of chorus and hiss by two THEMIS spacecrafts [Bortnik et al., 2009b]. It is noted that this theory does not exclude other possible sources of origination [Bortnik et al., 2016]. Recently observed unusually low frequency hiss is demonstrated to be locally excited by injected energetic electrons in the outer plasmasphere [Li et al., 2013c; Chen et al., 2014]. Pitch angle scattering of relativistic electrons by hiss is mainly responsible for creating the quiet time slot region between the inner and outer radiation belts [Lyons and Thorne, 1973; Abel and Thorne, 1998]. An elegant observation of a pure eigenmode of pitch angle diffusion by hiss was recently done by Van Allen Probes, in which the decay of electron flux is synchronized at all pitch angles [O’Brien et al., 2014].

“Triggered” whistler emissions, as suggested by the name, are stimulated by VLF signals injected from the ground and are observed near the magnetically conjugate point of the injection location. It was first discovered in the course of experimental studies of whistler
waves and related ionospheric effects at Stanford University, in which the injected VLF signals were from stations operated by the U.S. Navy [Helliwell et al., 1964; Helliwell, 1965]. In later controlled experiments begun in 1973, coherent VLF signals are injected to the magnetosphere from Siple Station ($L = 4$) in Antarctica (see Helliwell [1983] for a review). The triggered whistler emissions had a variety of emission forms: rising tones, falling tones, rising tone followed by falling tone (known as hooks), falling tone followed by rising tone (known as inverted hooks) and simultaneous rising and falling tones, as shown in Figure 1.6. Nonlinear coherent wave-particle interactions have long been associated with many of the physical features of these emissions [e.g., Helliwell, 1967; Dysthe, 1971; Nunn, 1971, 1974; Karpman et al., 1974; Matsumoto, 1979; Vomvoridis et al., 1982; Omura and Matsumoto, 1982; Sagdeev et al., 1987; Molvig et al., 1988; Omura et al., 1991]. These features of triggered emissions in the active experiments in Siple were listed by Matsumoto [1979] as they constrain related theoretical work. Here it is worthy to repeat this list, with minor revisions and original citations.

- Emission forms have (a) a narrow bandwidth ($\sim$ 100 Hz) and (b) a sizable frequency variation including risers, fallers, hooks, inverted hooks and simultaneous risers and fallers [Helliwell et al., 1964].

- The dash-dot anomaly indicates a threshold behavior as a function of pulse length in triggering the stimulated emissions [Helliwell et al., 1964; Helliwell, 1965]. The triggered emissions change from fallers to risers as the pulse length increases [Helliwell and Katsufrakis, 1974]. Emissions can be triggered by both high- and low-power transmitters, while the low-power transmitter have a threshold with longer pulse length [Kimura, 1967, 1968].

- All triggered emissions start at the triggering frequency [Stiles and Helliwell, 1975] and initially rise on leaving of the triggering frequency regardless of their final slopes [Kimura, 1967].

- These features are repeatable.
The active experiments of VLF triggered emissions can shed light on the mechanisms that are responsible for the excitation of chorus waves. In fact, the leading theory in the excitation of chorus waves [Omura et al., 2008, 2009; Omura and Nunn, 2011] is very similar to the theoretical interpretation of triggered emissions [e.g., Matsumoto, 1979], in which phase space dynamics of electrons interacting with a quasi-monochromatic whistler wave play an important role. However, chorus waves are thought to be excited from a band of waves instead of a single triggering frequency, which should be treated properly in the new theory.

1.3 Previous laboratory experiments on whistler waves

Although the space environment readily provides a natural laboratory to study plasma physics, the laboratory environment is advantageous in isolating different physical processes and performing comprehensive diagnostics on one specific process of interest. Laboratory experiments on whistler waves generally fall into one of two categories: (i) the propagation properties of whistler waves, such as dispersion relation, resonance cones and ducting phenomena; (ii) the instabilities of whistler waves, including the instabilities in velocity space and parametric instabilities. A description of these experiments can be found in an early review paper by Stenzel [1999]. Here we briefly review previous experiments closely related to this dissertation and to possible future work in our experiment.

Whistler waves in the laboratory are often launched by electric dipole antennas or magnetic loop antennas where the propagation pattern of whistler waves depends on the shape of the antennas. For a “point antenna” \( (k_{\parallel} L_{\text{antenna}} \ll \pi) \), both theory and observations show that the radiation pattern is characterized by resonance cones [Fisher and Gould, 1969, 1971]. For a “finite sized antenna” \( (k_{\parallel} L_{\text{antenna}} \gtrsim \pi) \), the radiation pattern is a resonance cone pattern in the near field region \( (k_{\parallel} r < \pi) \) whereas it is a field aligned narrow lobe in the far field region \( (k_{\parallel} r > \pi) \) [Boswell and Gonfalone, 1975; Stenzel, 1976c]. Interferometry with movable probes can be done in the laboratory, which allows direct measurements of wave lengths. Thus the dispersion relation of whistler waves can be determined experimentally.
Figure 1.6: Typical examples of triggered emissions. [Kimura, 1967]
The ducting of small amplitude whistler waves in density troughs has been demonstrated in a large magnetized plasma [Stenzel, 1976a; Sugai and Takeda, 1980; Gekelman et al., 2011]. A large amplitude ($\delta B/B_0 \approx 1\%$) whistler wave was observed to create a field aligned density trough in which the wave became ducted [Stenzel, 1976b; Sugai et al., 1978]. The width of the density trough is on the order of one wave length of the launched whistler wave. The density depression was found to be created by electron heating by the antenna near field rather than the radiation pressure, since the density trough was observed not only in the wave propagation regime ($\omega/\Omega_e < 1$) but also in the wave evanescence regime ($\omega/\Omega_e > 1$) [Sugai et al., 1978].

An electron beam is relatively straightforward to produce in the laboratory, which is used as a free energy source for whistler waves. Whistler waves can be excited by the electron beam through Landau resonance and cyclotron resonance. The former has been demonstrated in a large magnetized plasma [Stenzel, 1977], in which broadband whistler waves are generated and are observed to propagate near the resonance cone. The parallel phase velocity of these oblique whistler waves matches the beam velocity. A density-modulated beam, acting as an antenna, has also been used to generate whistler waves involving both cyclotron resonance [Starodubtsev and Krafft, 1999] and Landau resonance [Krafft et al., 1994]. The electron temperature anisotropy driven whistler instability has been observed in a mirror confined, electron-cyclotron-resonance heated plasma [Garner et al., 1987]. Parametric instabilities are observed in the laboratory, in which a pump wave in the very oblique whistler branch is scattered into a daughter whistler wave with a slightly downshifted frequency [Tejero et al., 2015, 2016]. The beat wave between this daughter wave and the pump wave is in Landau resonance with the thermal electrons, i.e., $\Delta \omega \approx v_{te} \Delta k_\parallel$. The scattered wave can be observed as long as the nonlinear growth rate of the scattered wave exceeds its linear damping rate, which yields a threshold of the pumping wave as $\delta B/B_0 = 5 \times 10^{-6}$. A three-wave interaction process is occasionally observed, in which a pump wave in the oblique whistler branch decays into an ion Bernstein mode wave and a whistler wave [Tejero et al., 2015]. However, momentum conservation (matching of wave numbers) for a three-wave interaction is difficult to satisfy in a finite laboratory device. Thus the three-wave interaction is only
occasionally observed.

1.4 Electromagnetic ion cyclotron waves in the inner magnetosphere

Electromagnetic ion cyclotron (EMIC) waves are natural plasma emissions in the magnetosphere, excited by anisotropic hot ring current protons with energies of a few tens of keV. One favoured region for the wave growth is near the storm-time plasmapause [Thorne and Kennel, 1971; Thorne and Horne, 1997] and within plasmaspheric plume with density structures [Chen et al., 2009], because the density gradients can guide the waves along the magnetic field line and enhance the convective gain of the waves. These waves can be separated into oxygen band, helium band and hydrogen band due to the presence of heavier O$^+$ and He$^+$ ions [Gomberoff and Neira, 1983; Young et al., 1981; Horne and Thorne, 1994]. The observed propagation direction of EMIC waves is generally along the magnetic field line and directed from the equatorial region to higher latitudes [Fraser et al., 1996; Loto’aniu et al., 2005], although the wave vector does not necessarily need to be field-aligned.

Electrons can experience cyclotron resonance with EMIC waves when they co-stream with the waves and overtake the wave phase velocity to reverse the polarization of the waves [Thorne and Kennel, 1971; Thorne et al., 2013b]. Thus the resonant electrons must reach relativistic energies and essentially see a stationary magnetic field helix. The resonance condition is \( \omega - kv_\parallel = -\Omega_e/\gamma \), where \( k \) is the wave number, \( v_\parallel \) the electron resonant velocity, \( \Omega_e \) the unsigned electron cyclotron frequency and \( \gamma \) the relativistic factor. Since the EMIC wave frequency \( \omega \) is much smaller than electron gyrofrequency, \( \omega \) can usually be neglected and the resonance condition can be rewritten as

\[
kv_\parallel = \frac{\Omega_e}{\gamma}
\]  

(1.1)

There exists a minimum energy of electrons that can be resonant with a specific wave number \( k \), when the pitch angle is zero. It is evident from the resonance condition that the minimum resonant energy is \( E_{\text{min}} = \left( \frac{\sqrt{1 + \left( \frac{\Omega_e}{kc} \right)^2} - 1}{1 + \left( \frac{\Omega_e}{kc} \right)^2} \right) m_e c^2 \), where \( m_e \) is the electron mass and \( c \) the speed of light.
the speed of light. To be resonant with observed helium band EMIC waves in the inner magnetosphere, the electron energy was estimated on the order of 2 MeV [Meredith et al., 2003]. Since the electron resonant velocity is much larger the wave phase velocity, the energy of the resonant electrons is roughly a constant and scattering of the electrons occurs only in pitch angle [Kennel and Engelmann, 1966]. The scattering rates are shown to be close to the strong diffusion limit for typically observed EMIC wave amplitudes (1 − 10 nT), resulting in relativistic electron lifetimes of several hours to one day in the outer radiation belt [Summers and Thorne, 2003; Albert, 2003; Li et al., 2007; Su et al., 2011].

1.5 Outline and objective of the thesis

The two parts of the work presented in this dissertation are unified under the general theme of wave-particle interactions in space and in the laboratory. The first part consists of chapters 3 - 6 that study the excitation of whistler waves in space and in the laboratory. The second part consists of chapters 7 - 8 that study the nonlinear interaction between EMIC waves and relativistic electrons. To start with, chapter 2 elaborates on the theoretical background on the coherent and incoherent wave-particle interactions required to understand the thesis. It also presents the technical aspects of the Darwin particle-in-cell method, which is used extensively in the thesis. Chapter 3 discusses the relaxation process of whistler anisotropy instability responsible for the excitation of chorus waves and its dependence on electron beta. It is targeted at demonstrating that the observed electron distributions in the inner magnetosphere are in a marginally stable state constrained by the whistler anisotropy instability. Chapter 4 presents the excitation of incoherent broadband whistler waves in a laboratory plasma, in which a correlation analysis is used to identify the wave mode structure and the corresponding resonance. Chapter 5 presents an analytical treatment of the linear unstable eigenmodes of whistler waves in a finite electron beam. The perpendicular wave number inside the electron beam is shown to be quantized due to the boundary condition and the associated mode structure is constructed. Chapter 6 presents kinetic simulations in a beam-plasma system. It is aimed at understanding the evolution of beam electron dis-
tributions relevant to the laboratory experiment. The beam whistler instability is shown to be suppressed by the saturation of Langmuir wave. Chapter 7 studies the nonlinear interaction between EMIC waves and relativistic electron using a two-wave model. It is intended to investigate the electron dynamics in EMIC waves including an inhomogeneous background magnetic field. A transition from coherent interaction to incoherent interaction where stochastic motion of electrons arises is shown by varying the separation of two wave numbers. For a continuous spectrum of EMIC waves, chapter 8 compares test particle simulations against quasi-linear diffusion theory. This chapter is intended to construct a regime map for the applicability of quasi-linear theory in the wave parameter space composed of the wave spectral width and wave intensity. Each chapter comes with its own introduction to motivate the work and a summary to recap the key results. Finally, an outlook of the future is given in chapter 9. In addition, an appendix is given to analytically calculate the susceptibility tensor for a gyrating beam. This result is used in chapter 5.
CHAPTER 2

Theoretical and numerical background

2.1 Incoherent and coherent resonant interactions

We start with the general resonance condition. The electromagnetic fields of plasma waves exert a Lorentz force on charged particles. The changes of particle motion induced by the wave force are permanent or non-adiabatic when the wave and the particle are in resonance. The resonant condition between particles and waves can be written as

\[ \omega - k_\parallel v_\parallel = n\Omega_s/\gamma \]  \hspace{1cm} (2.1)

where \( n = 0, \pm 1, \pm 2, \ldots \) \( \omega \) is the wave frequency, \( k_\parallel \) is the wave number parallel to the background magnetic field, \( v_\parallel \) is the parallel velocity of the particles and \( \gamma \) is the relativistic factor. \( \Omega_s = q_s B_0/m_s c \) is the signed cyclotron frequency of species \( s \). \( n = 0 \) corresponds to the Landau resonance, for which the wave phase velocity is equal to the particle parallel velocity. In Landau resonance, the wave parallel electric field \( E_\parallel \) remains in phase with particle parallel velocity, allowing efficient energy transfer between waves and particles. For electromagnetic waves, Landau resonance only occurs for obliquely propagating waves since \( E_\parallel \) is needed. Other resonances with \( n \neq 0 \) corresponds to cyclotron resonant interactions. In contrast to Landau resonance, cyclotron resonance occurs in gyro-phase of particle trajectories. For parallel propagating waves with \( k_\perp = 0 \), only first order cyclotron resonance \( (n = \pm 1) \) contributes to the wave growth/damping. As the wave vector becomes more oblique with \( k_\perp \neq 0 \), higher order cyclotron resonance comes into play, since the elliptically polarized wave can exchange energy with particles over several cyclotron periods. In the case of whistler-electron interactions, the cyclotron resonance \( (n = -1) \) corresponds to counter-streaming electrons and whistler waves, whereas Landau resonance \( (n = 0) \) corresponds to
co-streaming electrons and whistler waves.

Whistler waves exhibit both broadband, hiss-like emissions and discrete, chirping emissions. The dynamics of electrons in these two types of emissions is very different. For an incoherent broadband whistler wave, the electrons in the velocity space (or action space in terms of Hamiltonian dynamics) execute a random walk. At every time step, the resonant electrons get a random kick by the wave since the gyro-phase of resonant electrons with respect to the wave phase is randomized. For a coherent narrow band whistler wave, the scattering experienced by the electrons is not random. Some individual electrons can be phase locked with the coherent wave for many cyclotron periods and undergo large changes in both pitch angle and energy in a single encounter with the wave.

The diffusion of electrons in velocity space by a broadband wave spectrum was formulated as a Fokker-Planck equation \[Kennel and Engelmann, 1966\]. A useful concept is the diffusion surface in velocity space along which the particles diffuse \[Kennel and Petschek, 1966; Gendrin, 1981; Summers et al., 1998; Walker, 2013\]. When interacting with a particular wave with frequency $\omega$ and parallel wave number $k_\parallel$, the particle energy is conserved in the wave frame moving with wave phase velocity $\omega/k_\parallel$. This can be written as

$$v_\perp dv_\perp + \left( v_\parallel - \frac{\omega}{k_\parallel} \right) dv_\parallel = 0 \quad (2.2)$$

In the limit of $v \gg \omega/k_\parallel$, resonant particles are scattered primarily in pitch angle and the particle energy is roughly a constant. For a single wave, the diffusion surface is the surface of a sphere centered at $(v_\perp = 0, v_\parallel = \omega/k_\parallel)$. For a band of waves, the particle can only resonate with a particular wave mode $(\omega, k_\parallel)$ at one point in velocity space $(v_\perp, v_\parallel)$. By integrating Equation (2.2) across the resonant surfaces (defined by the resonant conditions), the diffusion surface can be readily obtained for this band of waves. Figure 2.1 shows such resonant diffusion surfaces (red solid lines) of electrons during first order cyclotron resonance with a broadband of chorus waves between $0.2 \Omega_e$ and $0.5 \Omega_e$. Due to a positive gradient of phase space density in the vicinity of the loss cone, $10 – 100$ keV electrons diffuse toward the loss cone along the red curves and transfer energy to the chorus waves. In the mean time, waves can be absorbed by quasi-isotropic $\sim$ MeV electrons at large pitch angles. Thus the
energy is transferred from low to high energy electrons, mediated by the whistler waves.

The discovery of large amplitude (>100 mV/m) chorus waves in the Earth’s outer radiation belt [Cattell et al., 2008; Cully et al., 2008; Li et al., 2011a] raised question about the validity of the quasi-linear description of wave-particle interactions, which assumes a small amplitude and broadband wave spectrum. The interactions between electrons and a monochromatic whistler wave in the Earth’s dipole field were studied extensively using both a test particle approach [e.g., Inan et al., 1978; Matsumoto and Omura, 1981; Omura and Summers, 2006; Bortnik et al., 2008a] and a Hamiltonian formulation [e.g., Albert, 1993, 2002]. The motion of electrons strongly depends on a dimensionless parameter $R$ which is proportional to the ratio of the wave amplitude to the magnetic field gradient. The effect of a large amplitude, monochromatic wave on the electron motion is demonstrated in Figure 2.2. The electrons are scattered diffusively in both pitch angle and energy during resonant interactions with a small amplitude monochromatic whistler wave (Figure 2.2a, 2.2b), since the gyro-phase of electrons with respect to the wave phase is randomly distributed between 0 and $2\pi$. However, the electrons are scattered deterministically to smaller pitch angles and energies during resonant interactions with a large amplitude monochromatic whistler wave at low latitude (Figure 2.2c, 2.2d), since the gyro-phase of electrons with respect to the wave phase is bunched which allows advection of the pitch angle and energy in only one direction. In some situation, the gyro-phase of electrons with respect to the wave phase can also be trapped which leads to rapid electron energization and scattering to higher pitch angle (Figure 2.2e, 2.2f). The resonant scattering of electrons in both pitch angle and energy by a large amplitude monochromatic whistler wave is large in magnitude and advective in direction due to phase bunching and phase trapping. In contrast, the resonant scattering by a small amplitude monochromatic wave is small and diffusive due to the random phase. While a monochromatic wave model can capture the coherent nature of constant-amplitude chorus waves, recent satellite observations show the subpacket structure of chorus waves with amplitude modulation [Santolák et al., 2004; Santolák et al., 2014]. Using a full wave spectrum model for interactions between chorus wave and electrons, Tao et al. [2012a] demonstrated that the coherent interactions, such as phase trapping and phase bunching, are destroyed
Figure 2.1: Cyclotron resonant ellipse (solid black) and resonant diffusion surface (solid red) for a broadband of whistler waves ($0.2 < \omega/\Omega_e < 0.5$) at the equator at $L = 4.5$. Also shown are constant energy surfaces (dotted) at 10, 29, 63, 188 and 603 keV. Blue surfaces indicate the contours of phase space density (darker blue means higher values of phase space density). The boundary of loss cone at $L = 4.5$ is shown as a straight line. [Horne and Thorne, 2003]
due to the overlap of two resonant islands [Tao et al., 2013]. Instead, the resulted scattering of electrons is neither diffusive like nor advective, but something in between.

Figure 2.2: Scattering of electrons in both pitch angle and energy by a counter-streaming monochromatic whistler wave. Case A (panels a and b) corresponds to small amplitude waves at low latitude; Case B (panels c and d) corresponds to large amplitude waves at low latitude; Case C (panels e and f) corresponds to large amplitude, oblique waves at high latitude. Panels (a), (c) and (e) are equatorial pitch angles. Panels (b), (d) and (f) are total electron energy. [Bortnik et al., 2008a]

2.2 The Darwin particle-in-cell method

Progress in understanding the excitation of whistler-mode chorus waves relies heavily on particle-in-cell (PIC) simulations, due to the kinetic, non-equilibrium and nonlinear nature of the problem. For example, PIC methods have been used to demonstrate the damping of a large amplitude whistler wave [Ossakow et al., 1972a], to study the whistler sideband instability [Denavit and Sudan, 1975], to study the triggered whistler emission [Vomvoridis and Denavit, 1980; Omura and Matsumoto, 1982] and to study the whistler anisotropy instabilities [Ossakow et al., 1972b; Pritchett et al., 1991; Devine et al., 1995]. In the past
ten years, great progress in understanding the excitation of chorus waves has been made possible by PIC simulations [Katoh and Omura, 2007; Omura et al., 2008; Hikishima et al., 2009a,b; Omura et al., 2009; Hikishima et al., 2010; Katoh and Omura, 2011; Tao, 2014; Tao et al., 2014a]. In the PIC method, the plasma is modeled as a set of discrete particles. Electrons and ions are tracked in terms of position and velocity, while interacting with their self-generated electromagnetic fields and external fields. At each time step of the PIC method, charge and current densities are first accumulated on the grids from the particle positions and velocities according to some interpolation scheme. The electromagnetic fields are then calculated on the grids by a finite difference method or Fast Fourier transform based on the integration of Maxwell’s equation. The fields are then interpolated to obtain the Lorentz force at the particle’s position. The particles are advanced to the next time step with a new velocity and position by the finite difference method. This describes the procedure for one time step and the loop is repeated until the desired number of time steps is reached.

Compared to a conventional electromagnetic PIC method, the Darwin PIC method excludes the transverse component of the displacement current in Ampere’s law and hence excludes retardation effects and light waves, but leaves the physics of whistler waves unaffected [Busnardo-Neto et al., 1977; Geary et al., 1986; Hewett, 1985]. The elimination of light waves in the Darwin PIC models removes the constraint on the time step set by Courant’s condition

$$\Delta t \lesssim \frac{\delta}{c}$$

where $\Delta t$ is the time step, $\delta$ is the grid spacing and $c$ is the speed of light. The grid spacing is constrained by the requirement that the Debye length, $\lambda_D = \frac{v_{te}}{\omega_{pe}}$, must be resolved. Here $v_{te}$ is the electron thermal velocity. Thus the constraint on the time step can be written as

$$\Delta t \lesssim \frac{v_{te}}{c} \frac{1}{\omega_{pe}}$$

However, the Darwin particle-in-cell model does not have the restriction on the time step.
set by Equation (2.4), especially for small \(v_{te}/c\), and thus greatly improves computation efficiency. This is beneficial for our study, since it covers a wide range of \(v_{te}/c\).

Here we briefly review the main iteration loop in the Darwin PIC code as part of the UCLA particle-in-cell (UPIC) framework Decyk [2007]:

- Deposit charge density, current density and derivative of current density on a mesh from the particles:

\[
\rho(x) = \sum_i q_i S(x - x_i) \quad (2.5)
\]
\[
\mathbf{j}(x, t) = \sum_i q_i \mathbf{v}_i(t) S(x - x_i(t)) \quad (2.6)
\]
\[
\frac{\partial \mathbf{j}(x, t)}{\partial t} = \sum_i q_i \left[ \frac{d\mathbf{v}_i}{dt} S(x - x_i(t)) + \mathbf{v}_i \frac{\partial S(x - x_i(t))}{\partial t} \right]
= \sum_i q_i \left[ \frac{d\mathbf{v}_i}{dt} S(x - x_i(t)) + \mathbf{v}_i \frac{\partial S(x - x_i(t))}{\partial (x - x_i)} \cdot \frac{d(x - x_i)}{dt} \right]
= \sum_i q_i \left[ \frac{d\mathbf{v}_i}{dt} S(x - x_i(t)) - \mathbf{v}_i \mathbf{v}_i \cdot \nabla S(x - x_i(t)) \right] \quad (2.7)
\]

Here \(x_i\) and \(v_i\) are the position and velocity of the \(i^{th}\) particle, respectively. \(S(x)\) is the particle shape function. For point particles, this would be a Dirac Delta function. But in computer modeling, extended particle shapes are commonly used. In the code, two quantities are deposited separately for the derivative of current density, an acceleration density and a velocity flux,

\[
a(x) = \sum_i q_i \frac{d\mathbf{v}_i}{dt} S(x - x_i) \quad (2.8)
\]
\[
\mathbf{M}(x) = \sum_i q_i \mathbf{v}_i \mathbf{v}_i S(x - x_i) \quad (2.9)
\]

Thus the derivative of current density is calculated as

\[
\frac{\partial \mathbf{j}(x, t)}{\partial t} = \mathbf{a} - \nabla \cdot \mathbf{M} \quad (2.10)
\]

Finally, note that conservation of charges automatically satisfies

\[
\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \quad (2.11)
\]
using the identity
\[
\frac{\partial S(x - x_i(t))}{\partial t} = -\mathbf{v}_i \cdot \nabla S(x - x_i(t))
\]
as in Equation (2.7).

- Solve Maxwell’s equation: The electric field \( \mathbf{E} \) is separated into longitudinal and transverse parts, \( \mathbf{E} = \mathbf{E}_L + \mathbf{E}_T \), which satisfies the condition

\[
\begin{align*}
\nabla \times \mathbf{E}_L & = 0 \\
\nabla \cdot \mathbf{E}_T & = 0
\end{align*}
\]  

(2.12)  
(2.13)

Maxwell’s equations are then written as

\[
\begin{align*}
\nabla \times \mathbf{B} & = \frac{4\pi}{c} \mathbf{j}_T = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}_L}{\partial t} \\
\nabla^2 \mathbf{E}_T & = \frac{1}{c} \nabla \times \frac{\partial \mathbf{B}}{\partial t} = \frac{4\pi}{c^2} \frac{\partial \mathbf{j}_T}{\partial t} \\
\nabla \cdot \mathbf{E}_L & = 4\pi \rho \\
\n\nabla \cdot \mathbf{B} & = 0
\end{align*}
\]  

(2.14)  
(2.15)  
(2.16)  
(2.17)

Note that in Equation (2.14) the longitudinal component of the current density cancels the longitudinal component of the displacement current due to conservation of charges. This can be demonstrated as follows. Substitute the charge density \( \rho \) in Equation (2.11) by Poisson’s equation and note that the divergence of the transverse part of a vector field vanishes. We obtain

\[
\nabla \cdot \left( \mathbf{j}_L + \frac{1}{4\pi} \frac{\partial \mathbf{E}_L}{\partial t} \right) = 0
\]  

(2.18)

Since the term in the parenthesis is curl free by definition, it can be written as the gradient of a scalar field

\[
\mathbf{j}_L + \frac{1}{4\pi} \frac{\partial \mathbf{E}_L}{\partial t} = \nabla \phi
\]  

(2.19)

Thus the scalar field \( \phi \) satisfies the Laplace equation

\[
\nabla^2 \phi = 0
\]  

(2.20)
For a strictly periodic system of interest here, the solution is
\[ \phi = \text{const.} \rightarrow \nabla \phi = 0 \quad (2.21) \]
and thus
\[ \mathbf{j}_L + \frac{1}{4\pi} \frac{\partial \mathbf{E}_L}{\partial t} = 0 \quad (2.22) \]

• Advance the particle co-ordinates using the Lorentz force:
\[ m_i \frac{dv_i}{dt} = q_i \int \left[ \mathbf{E}(x) + \mathbf{v}_i \times \mathbf{B}(x)/c \right] S(x_i - x) dx \quad (2.23) \]
\[ \frac{dx_i}{dt} = v_i \quad (2.24) \]

On a computer, these field equations are solved in discrete space and time co-ordinates. Discretizing time for the field equations in a Darwin code are more complex than that for those in the electromagnetic code. This is because the transverse electric field \( \mathbf{E}_T \) depends on the acceleration \( \frac{dv_j}{dt} \) of all particles, whereas the acceleration of a particle depends on \( \mathbf{E}_T \). So it is a system of coupled equations.

A simple iterative scheme that uses old values of \( \frac{dv_j}{dt} \) and hence old values of \( \frac{\partial \mathbf{j}_T}{\partial t} \) to find new values of \( \mathbf{E}_T \), i.e.,
\[ \mathbf{E}_T^n(x_j) = - \sum_{k=-\infty}^{\infty} \frac{4\pi}{k^2 c^2} \frac{\partial \mathbf{j}_T^o(t)}{\partial t} \exp(i \mathbf{k} \cdot \mathbf{x}_j) \quad (2.25) \]
is unstable when \( kc < \omega_{pe} \). The superscripts \( n \) and \( o \) refer to “new values” and “old values”, respectively. Numerical stability can be achieved by subtracting a scaled quantity from both sides, i.e.,
\[ \nabla^2 \mathbf{E}_T^n - \frac{\omega_{p0}^2}{c^2} \mathbf{E}_T^n = \frac{4\pi}{c^2} \frac{\partial \mathbf{j}_T}{\partial t} - \frac{\omega_{p0}^2}{4\pi} \mathbf{E}_T^o \quad (2.26) \]
where the average plasma frequency \( \omega_{p0} \) is defined as
\[ \omega_{p0}^2 = \frac{4\pi}{V} \sum_i \frac{q_i^2}{m_i} \quad (2.27) \]
The solution of the modified equation is
\[ \mathbf{E}_T^n(x_j) = - \sum_{k=-\infty}^{\infty} \frac{4\pi}{k^2 c^2 + \omega_{p0}^2} \left( \frac{\partial \mathbf{j}_T(t)}{\partial t} - \frac{\omega_{p0}^2}{4\pi} \mathbf{E}_T^o \right) \exp(i \mathbf{k} \cdot \mathbf{x}_j) \quad (2.28) \]
When the solution has converged, this equation reduces to the original one.

Solving for the fields requires knowledge of the positions, velocities and accelerations of particles at time $t$. Because of the leapfrog scheme, the positions are already known at time $t$, but the velocities are retarded by half the time step and the accelerations are yet to be determined. The time-centered velocities and accelerations are obtained by using

$$v_j(t) = \frac{v_j(t + \Delta t/2) + v_j(t - \Delta t/2)}{2} \quad (2.29)$$

$$\frac{dv_j(t)}{dt} = \frac{v_j(t + \Delta t/2) - v_j(t - \Delta t/2)}{\Delta t} \quad (2.30)$$

The iteration starts by first calculating $E_L(t)$ from $x(t)$. Next, estimate $B(t)$ from $x(t)$ and $v(t - \Delta t/2)$ and still use the previous value $E_T(t - \Delta t/2)$. Then advance the particles, calculate $\frac{dv_j(t)}{dt}$ and $v_j(t)$, and deposit $\frac{\partial j(t)}{\partial t}$ and $j(t)$. Here do not update the positions and velocities of particles in memory. Finally, solve for improved $B(t)$ and $E_T(t)$. Repeat this procedure as $B(t)$ and $E_T(t)$ converge. This iteration scheme works well and converges in one or two iterations if $\max(\omega_p^2(x)) < 1.5 \omega_{p0}^2$. 

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CHAPTER 3

The relaxation of the whistler anisotropy instability
and its dependence on electron beta

3.1 Introduction

It has long been appreciated that a sufficient temperature anisotropy ($T_{\perp e}/T_{\parallel e} > 1$) of electrons is needed to provide the free energy for the excitation of whistler waves in space [e.g., Tsurutani and Smith, 1974; Li et al., 2009; Santolík et al., 2010; Schriver et al., 2010; Liu et al., 2011; Omura and Nunn, 2011; Tao, 2014; Fu et al., 2014]. Hereafter $T_e$ is the electron temperature; and the subscripts $\perp$ and $\parallel$ denote the directions perpendicular and parallel to the background magnetic field, respectively. The excitation process starts with an exponential growth of an infinitesimally small seed wave in wave amplitude, known as the linear stage. The enhanced whistler waves cause electron scattering in velocity space leading to reduced electron anisotropy, which in turn saturates the wave growth. Eventually, the whistler waves relax the electron distribution toward a marginally stable state [e.g., Kennel and Petschek, 1966; Kennel and Engelmann, 1966]. Consequently, the whistler anisotropy instability imposes an upper bound on the relaxed electron anisotropy [Gary and Wang, 1996; Gary et al., 2000, 2011].

The source of free energy for whistler-mode chorus waves is the unstable population of electrons that is injected from Earth’s plasma sheet and hence it is dependent on the level of geomagnetic activity [Kennel and Petschek, 1966; Anderson and Maeda, 1977; Meredith et al., 2001]. Recent observations [Li et al., 2016a] indicate that the lower band chorus waves tend to be quasi-parallel to the magnetic field during disturbed geomagnetic periods at higher $L$-shells, whereas they become very oblique to the background magnetic field and
close to the resonance cone during quiet geomagnetic periods at lower $L$-shells. Spacecraft observations related to this study suggest that the wave normal angle of lower band chorus waves is also organized by the electron beta $\beta_{\parallel e}$ [Yue et al., 2016]. Here the wave normal angle (WNA) denotes the angle between the wave vector and the background magnetic field. $\beta_{\parallel e}$ is defined as

$$\beta_{\parallel e} = \frac{n_e T_{\parallel e}}{B_0^2/8\pi}$$

where $n_e$ is total plasma density, $T_{\parallel e}$ is the parallel electron temperature calculated as the second order velocity moment for the whole electron distribution and $B_0$ is the background magnetic field. Note a Gaussian unit system is used in this study. One would expect that the electron distribution under active geomagnetic conditions at higher $L$-shells corresponds to larger $\beta_{\parallel e}$ while the electron distribution under quiet geomagnetic conditions at lower $L$-shells corresponds to smaller $\beta_{\parallel e}$. Thus the dependence of the WNAs of lower band chorus waves on the electron $\beta_{\parallel e}$ is consistent with their dependence on geomagnetic activities observed by Li et al. [2016a]. The WNAs of upper band chorus waves, on the other hand, have a wide distribution between $0^\circ$ and the resonance cone angle, while showing a weak dependence on $\beta_{\parallel e}$ [Yue et al., 2016]. These observations raise the need to study the impact of electron $\beta_{\parallel e}$ on the excitation of whistler waves. Here, using an ensemble of particle-in-cell simulations, we study the whistler anisotropy instability for both large and small electron beta $\beta_{\parallel e}$ and compare them with satellite observations.

### 3.2 Linear growth rate analysis

Whistler-mode waves excited by an anisotropic electron distribution grow in amplitude exponentially in the initial stage of the growth process, known as the linear stage. These waves saturate subsequently because of the reduction in the electron anisotropy, caused by pitch angle scattering. During and after the wave saturation, the enhanced whistler waves force the electron distribution function toward a marginally stable state. To find the bounds of this marginally stable state and compare with satellite observations, we solve the hot plasma dispersion relation using the HOTRAY code [Horne, 1989]. This code implements the hot...
plasma dispersion relation for an arbitrary number of plasma species with Maxwellian distributions. In our case, the electrons are modeled as a single bi-Maxwellian distribution, i.e.,

\[
f_e(v_{\perp}, v_{\parallel}) = \frac{1}{(2\pi)^{3/2}v_{t, e}v_{t||e}} \exp \left( -\frac{v_{\parallel}^2}{2v_{t||e}^2} - \frac{v_{\perp}^2}{2v_{t, e}^2} \right)
\]  

(3.2)

Here \(v_{\perp}\) and \(v_{\parallel}\) are the perpendicular and parallel velocities, respectively, relative to the external magnetic field. The thermal velocities \(v_{t, e}\) and \(v_{t||e}\) are related to the electron temperatures by \(T_{\perp e} = m_e v_{t, e}^2\) and \(T_{|| e} = m_e v_{t||e}^2\), respectively. Note that \(\beta_{|| e}\) can be written as

\[
\beta_{|| e} = \frac{2T_{|| e}}{m_e c_{Ac}^2}
\]  

(3.3)

where \(c_{Ac} = c\Omega_e/\omega_{pe}\) is the electron Alfvén velocity and \(\Omega_e/\omega_{pe}\) is the ratio of electron cyclotron frequency and plasma frequency, which is set to 0.2, a typical value in the generation region of chorus waves. As \(\Omega_e/\omega_{pe}\) is fixed, the electron distribution can be uniquely described by \(T_{\perp e}/T_{|| e}\) and \(\beta_{|| e}\). Neutrality is ensured by adding cool protons with a Maxwellian distribution of 1 eV temperature. It is noted that a single bi-Maxwellian electron distribution captures the hot electron population but misses the cold electron component. The hot electrons can resonate with whistler waves and exchange energy with the waves. The addition of a cold electron component reduces the minimum cyclotron resonant energy and hence increases the number density of resonant electrons, but also increases the total electron density \([Li et al., 2011b,c]\). Such a competition between the number density of resonant electrons and total electron density can affect the fraction of resonant electrons in the total electron population \([Gary et al., 2012; Wu et al., 2013]\). But as long as the total electron density is unchanged or \(\omega_{pe}/\Omega_e\) is kept constant, the cold electron component only reduces the absolute values of linear growth rate but does not change the results of the marginal stability state \([Gary et al., 2011]\).

The results of the linear growth rate analysis are shown in Figure 3.1. For three representative parameters, \((\beta_{|| e}, T_{\perp e}/T_{|| e}) = (0.01, 4), (0.023, 3.5), (0.088, 2.5)\), the linear growth rates of whistler waves in wave number space \((k_{\perp}, k_{||})\) are shown in Figures 3.1a, 3.1b
and 3.1c, respectively. The linear growth rate maximizes near the resonance cone for \((\beta_{||e}, T_{\perp e}/T_{||e}) = (0.01, 4)\), and \((0.023, 3.5)\) as shown in Figures 3.1a and 3.1b. In contrast, it maximizes in the parallel direction for \((\beta_{||e}, T_{\perp e}/T_{||e}) = (0.088, 2.5)\) as shown in Figure 3.1c. The maximum linear growth rate in each panel is extracted from the wave number space for each state of plasma, and displayed as in Figure 3.1d. If some small dissipation rate \(\gamma_d\) is assumed (for example due to Landau damping), there exists a threshold \(\gamma = \gamma_d\) for whistler anisotropy instability. A contour of \(\gamma = \gamma_d = 5 \times 10^{-3} \Omega_e\) (solid white line) is drawn to indicate such a typical threshold. This threshold scales as \(\beta_{||e}^{-0.5}\) for \(\beta_{||e} > 0.025\) [Gary and Wang, 1996] and becomes independent of \(\beta_{||e}\) for \(\beta_{||e} < 0.025\) [Gary et al., 2011]. Comparison of the instability threshold with satellite observations (in Figure 3.5) clearly demonstrates that the observed plasma state is constrained to a marginally stable state by whistler anisotropy instability. The wave frequency corresponding to the maximum linear growth rate is shown in Figure 3.1e. The linear theory, however, cannot produce both lower and upper band of chorus waves with a single electron bi-Maxwellian distribution. Instead, the most unstable frequency predicted by linear theory continuously shifts from upper band to lower band with increasing \(\beta_{||e}\). The WNA at which the maximum linear growth rate occurs, on the other hand, shows an abrupt change around \(\beta_{||e} = 0.025\) as in Figure 3.1f. Below \(\beta_{||e} = 0.025\), unstable waves preferentially grow near the resonance cone, whereas above \(\beta_{||e} = 0.025\), unstable waves preferentially grow parallel to the background magnetic field. This transition is due to the ability of oblique waves to satisfy the cyclotron resonance condition for small \(\beta_{||e}\), i.e., the only way to satisfy the resonance condition for small values of \(\beta_{||e}\) is for the \(k_{||}\) to tend to very large values which can only occur near the resonance cone.

### 3.3 Computational setup

The Darwin particle-in-cell model used in this study is based on a two-dimensional spectral code developed as part of the UCLA particle-in-cell (UPIC) framework [Decyk, 2007], known as MDPIC2. The computational domain in this study is comprised of \(512 \times 512\) cells with 81 particles in each cell. The grid spacing is \(0.02c/\omega_{pe}\). The time step is \(0.1\omega_{pe}^{-1}\).
Figure 3.1: Examples of the linear growth rates in wave number space for three plasma states, $(\beta_{\parallel e}, T_{\perp e}/T_{\parallel e}) = (a) (0.01, 4), (b) (0.023, 3.5), (c) (0.088, 2.5)$. Note that the wave number is normalized by the characteristic electron gyro-radius in each case, $\rho_e = \alpha_{\perp} / \Omega_e$. (d) The maximum linear growth rate extracted from the wave number space, for each plasma state. The white solid line is a contour of $\gamma_{\text{max}} / \Omega_e = 5 \times 10^{-3}$ indicating the threshold of whistler anisotropy instability. (e) The wave frequency and (f) the wave normal angle corresponding to the maximum linear growth rate for each plasma state. The solid white lines in (e) and (f) are the same as that in (d).
Both electromagnetic fields and particles use periodic boundary conditions in two directions. Electrons are initialized to be uniformly distributed in the computation domain with a single bi-Maxwellian distribution in velocity space ($T_{\perp e} > T_{\parallel e}$). Ions are treated as a fixed, charge neutralizing background. A uniform external magnetic field is applied in the $x$ direction, which is set as $\Omega_e/\omega_{pe} = 0.2$, corresponding to the typical value in the generation region of whistler-mode chorus waves. The value of $\beta_{\parallel e}$ (equation (3.3)) is scanned by changing $v_{t\parallel e}/c$. The temperature anisotropy of electrons $T_{\perp e}/T_{\parallel e}$ is scanned by changing $v_{t\perp e}/v_{t\parallel e}$. The initial electron distribution can be uniquely determined by the electron $\beta_{\parallel e}$ and the temperature anisotropy of electrons $T_{\perp e}/T_{\parallel e}$.

3.4 Whistler anisotropy instabilities for large and small $\beta_{\parallel e}$

The whistler anisotropy instability undergoes a transition from exciting predominantly quasi-parallel waves at $\beta_{\parallel e} \gtrsim 0.02$ to oblique waves at $\beta_{\parallel e} \lesssim 0.02$, which is described by the linear kinetic dispersion relation [Gary et al., 2011; Yue et al., 2016]. Here the evolution of both the electron distribution and field pattern are shown for these two distinct regimes from MDPIC2.

Figure 3.2 corresponds to $\beta_{\parallel e} = 0.56$ and $T_{\perp e}/T_{\parallel e} = 2$. The time history of the wave magnetic energy and the electron anisotropy are shown in Figure 3.2a. The energy of the wave magnetic field has an exponential growth as a function of time before $t = 100\Omega_e^{-1}$. The saturation of the wave amplitude occurs around $t = 100\Omega_e^{-1}$ and is accompanied with a reduction of $T_{\perp e}/T_{\parallel e}$ from 2 to 1.4. Damping of the wave magnetic field is seen after the saturation, during which the electron anisotropy further relaxes. The reduced electron distribution function parallel to the background magnetic field is shown in Figure 3.2b, where four colored lines denote the initial state at $t = 0$ (dark blue), the linear stage $t = 40\Omega_e^{-1}$ (light blue), the saturation state $t = 100\Omega_e^{-1}$ (yellow) and the relaxed state $t = 300\Omega_e^{-1}$ (red). The wave power in wave number space at $t = 40, 100, 300\Omega_e^{-1}$ is displayed in Figures 3.2c, 3.2d and 3.2e, with the associated wave magnetic field pattern displayed in Figures 3.2f, 3.2g and 3.2h, respectively. Quasi-parallel propagating whistler waves are excited during
the linear stage, with a maximum power at the parallel wave number \( k_x = 0.7\omega_{pe} \cdot c^{-1} \), corresponding to the wave frequency \( \omega = 0.3\Omega_e \) (not shown), as expected from linear theory. The enhanced whistler waves lead to cyclotron heating parallel to the background magnetic field as seen in Figure 3.2b. The wave field, in turn, gets damped as seen from Figure 3.2d to 3.2e. Also note that some wave power slightly oblique to the background magnetic field gets damped after wave saturation, making the wave normal direction more field aligned in the relaxed state.

Figure 3.3 illustrates the whistler anisotropy instability in the small \( \beta_{||e} \) regime for \( \beta_{||e} = 0.01 \) and \( T_{\perp e}/T_{||e} = 5 \), in the same format as Figure 3.2. A calculation of the maximum linear growth rate of all wave modes gives \( 0.03\Omega_e^{-1} \), which is significantly smaller than that in Figure 3.2a \( (0.1\Omega_e^{-1}) \), and consistent with the time history of wave magnetic field energy in Figure 3.3a. The electron anisotropy \( T_{\perp e}/T_{||e} \) drops significantly from 5 to 3 during the saturation phase and then decreases gradually during the relaxation stage. The reduced electron distribution functions at four selected snapshots are shown in Figure 3.3b: \( t = 0 \) (dark blue), \( t = 100\Omega_e^{-1} \) (light blue), \( t = 250\Omega_e^{-1} \) (yellow) and \( t = 300\Omega_e^{-1} \) (red). Cyclotron resonance contributes to the heating of electrons in the parallel direction, while Landau resonance makes the plateau near the wave phase velocity in the parallel direction \( v_{ph,||} = \omega/k_x \) and hence leads to damping of whistler waves. It is noticeable that the plateau of the reduced distribution function moves to larger \( |v_{||}| \) from the snapshot of \( t = 250\Omega_e^{-1} \) to \( t = 300\Omega_e^{-1} \). This is consistent with the down-shift of the parallel wave number shown in Figures 3.3d and 3.3e, where the Landau resonance velocity increases from \( \omega/k_x = 0.7\Omega_e/(3\omega_{pe} \cdot c^{-1}) = 0.046c \) to \( \omega/k_x = 0.7\Omega_e/(2\omega_{pe} \cdot c^{-1}) = 0.07c \). Here a frequency of \( \omega = 0.7\Omega_e \) corresponding to these wave numbers is read from the frequency spectrum (not shown). The distinct feature in this small electron beta regime is that whistler waves propagate obliquely (WNA \( \approx 45^\circ \)) close to the resonance cone (resonance cone angle \( \psi_{res} = 45.6^\circ \) for \( \omega/\Omega_e = 0.7 \)), as seen both in the wave number space (Figures 3.3c, 3.3d, 3.3e) and in the configuration space (Figures 3.3f, 3.3g, 3.3h). The oblique nature of whistler waves in this regime makes Landau damping significant during the saturation and relaxation stage as seen in Figure 3.3b.
Figure 3.2: The evolution of the whistler anisotropy instability for $\beta_{\parallel e} = 0.56$ and $T_{\perp e}/T_{\parallel e} = 2$ from MDPIC2. (a) The time history of the square of wave magnetic amplitude (black solid line) and the temperature anisotropy $T_{\perp e}/T_{\parallel e}$ (blue solid line). The amplitude of wave magnetic field $\delta B$ is normalized by the amplitude of the background magnetic field $B_0$. (b) The reduced electron distribution parallel to the background magnetic field at four selected snapshots $t = 0, 40, 100, 300 \Omega_e^{-1}$. Each colored line corresponds to the time indicated by the dashed line with the same color in panel (a). The line of dark blue is the initial distribution function (overlap with the line of light blue). The wave power in the wave number space at three selected snapshots: (c) $t = 40 \Omega_e^{-1}$, (d) $t = 100 \Omega_e^{-1}$ and (e) $t = 300 \Omega_e^{-1}$. The pattern of the wave magnetic field corresponding to the same snapshots: (f) $t = 40 \Omega_e^{-1}$, (g) $t = 100 \Omega_e^{-1}$ and (h) $t = 300 \Omega_e^{-1}$. Length is normalized by electron inertial length $d_e = c/\omega_{pe}$. 
Figure 3.3: The evolution of the whistler anisotropy instability for $\beta_{\parallel e} = 0.01$ and $T_{\perp e}/T_{\parallel e} = 5$ from MDPIC2. The format is the same as that of Figure 3.2.
3.5 An observed upper bound of electron anisotropy

To find the bounds of the relaxed electron anisotropy constrained by the whistler anisotropy instability, we scan the parameter space comprised of $\beta_{\parallel e}$ and $T_{\perp e}/T_{\parallel e}$. For each $T_{\perp e}/T_{\parallel e}$, 10 values of $\beta_{\parallel e}$ are scanned from 0.005 to 1 with a uniform step size on a logarithmic scale. For each $\beta_{\parallel e}$, 8 values of $T_{\perp e}/T_{\parallel e}$ are scanned from 1.5 to 5 with a uniform step size on a linear scale. A total number of 80 runs are conducted, which are distributed to 80 nodes with 16 cores on each node. Using our current computational facilities, it takes about 10 hours to complete the computation. The results are shown in Figure 3.4. The evolution paths of the electron distributions for 80 runs are shown in the plane of $\beta_{\parallel e}$ and $T_{\perp e}/T_{\parallel e}$ in Figure 3.4a, color-coded by the time elapsed from the start of the simulation. The envelope of the final relaxed states of the evolution gives the threshold of whistler anisotropy instability, for which the temperature anisotropies are greatly reduced, linear growth rates are zero and the marginal stability condition is reached. The time to approach the marginal stability condition decreases as $\beta_{\parallel e}$ increases, which can be understood by the increase of linear growth rates with increasing $\beta_{\parallel e}$ for fixed electron temperature anisotropy [Yue et al., 2016]. The cooling of the electrons in the perpendicular direction and the accompanied heating in the parallel direction are due to the constraint on the resonant diffusion path in velocity space of resonant electrons [Brice, 1964; Kennel and Petschek, 1966], i.e.,

$$\frac{dE_{\perp}}{dE_{\parallel}} = -\frac{1}{1 - \frac{\omega}{\Omega_e}} < 0$$

(3.4)

where $dE_{\perp}$ and $dE_{\parallel}$ are the change in perpendicular and parallel energies of resonant electrons, respectively, interacting with a wave of frequency $\omega$. Surprisingly, the evolution of whistler anisotropy instability follows a linear path with the same slope on a log-log scale of $\beta_{\parallel e}$ and $T_{\perp e}/T_{\parallel e}$. This path can be characterized as $T_{\perp e}/T_{\parallel e} \cdot \beta_{\parallel e}^{p} = C_I$ or $T_{\perp e} \cdot T_{\parallel e}^{p-1} = C'_I$, where $p = 1.24$ and $C_I$ and $C'_I$ are constants determined by the initial condition. It seems plausible to obtain the wave amplitude if one knows the initial state of the electron distribution function and its evolution path. However, obtaining the wave amplitude by energy conservation is not practical since the ratio between the wave energy density and particle kinetic energy density is on the order of $10^{-4} - 10^{-3}$. The subtraction of two large numbers of
particle kinetic energy energy gives a large uncertainty on the resulting wave energy density. The evolution path in Figure 3.4b is in the same format as in Figure 3.4a, but color-coded by the wave vector anisotropy angle $\theta_B$ [Shebalin et al., 1983], which is defined as

$$\tan^2 \theta_B = \frac{\sum_k k_{\perp}^2 |\delta B_k|^2}{\sum_k k_{\parallel}^2 |\delta B_k|^2}$$

(3.5)

Here $k_{\perp}$ and $k_{\parallel}$ are wave numbers perpendicular and parallel to the background magnetic field, respectively. $\delta B_k$ is the magnetic spectral density in wave number space. $\theta_B$ is equal to the WNA for a monochromatic wave and becomes 45° for isotropic turbulent fluctuations. The wave vector anisotropy angle is output with a time step of about half the cyclotron period. Above the instability threshold, it is seen that small wave vector anisotropy angles for quasi-parallel propagating waves transition to more oblique wave vector anisotropy angle as $\beta_{\parallel,e}$ decreases. This corresponds well to the correlation between the WNAs and the electron beta for lower band whistler-mode chorus waves observed by Van Allen Probes [Yue et al., 2016]. Below the instability threshold, there are only isotropic thermal fluctuations and $\theta_B$ is about 45°.

To compare the obtained instability threshold from MDPIC2 with satellite observations, three years of whistler-mode chorus wave events and the associated particle measurements from the NASA’s twin satellites, Van Allen Probes [Mauk et al., 2014], are used. An unstable anisotropic electron distribution takes tens of electron cyclotron periods to approach the marginally stable state, which is much shorter than the time resolution of the particle instrument. Thus the electron distributions associated with whistler-mode chorus waves are usually observed in a marginally stable state. These wave events are binned by $\beta_{\parallel,e}$ and $T_{\perp,e}/T_{\parallel,e}$. The methodology of processing the dataset is described in a companion study [Yue et al., 2016]. Note that in spacecraft observations, the electron beta is calculated by excluding the cold electron component ($< 15$ eV), which is below the measurable limit of the instrument. One caveat in the comparison is that the same $\beta_{\parallel,e}$ may refer to different electron distributions, especially in the small $\beta_{\parallel,e}$ regime. The simulations use one single Maxwellian distribution as a simplified representation while the real distribution may have a mixture of electron components with different temperatures. Figure 3.5 shows the electron beta $\beta_{\parallel,e}$
Figure 3.4: (a) The evolution paths of the whistler anisotropy instability in the plane of $\beta_{\parallel e}$ and $T_{\perp e}/T_{\parallel e}$ for 80 runs from MDPIC2, color-coded by the time in terms of electron cyclotron period in the simulations. (b) The evolution path of the whistler anisotropy instability in the same format as that of panel (a), but color-coded by the wave vector anisotropy angle $\theta_B$. 

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with respect to the electron temperature anisotropy $T_{\perp e}/T_{\parallel e}$, color-coded by the number of whistler-mode chorus events in each bin. The final state of relaxed electron distribution from MDPIC2 as in Figure 3.4 (except those runs with low signal to noise ratio) is over-plotted as solid black diamonds in Figure 3.5. It is clear that the observed chorus wave events are constrained by the instability threshold, indicated by the envelope of the relaxed state from MDPIC2. This instability threshold of the temperature anisotropy $T_{\perp e}/T_{\parallel e}$ roughly scales as $\beta_{\parallel e}^{-0.5}$ for $\beta_{\parallel e} \gtrsim 0.1$, indicated by the theoretical upper bound of electron anisotropy (the light blue light in Figure 3.5) given by *Gary and Wang* [1996]. For $\beta_{\parallel e} \lesssim 0.1$, this upper bound of electron anisotropy becomes weakly dependent on $\beta_{\parallel e}$ and approaches a constant [Gary et al., 2011; Yue et al., 2016].

3.6 The scaling of the saturated wave magnetic field energy with $\beta_{\parallel e}$

Here we explore the dependence of the saturated wave magnetic field energy on the electron beta. Figure 3.6a shows the saturated wave magnetic field energy with respect to the electron beta $\beta_{\parallel e}$ for all the runs as in Figure 3.4 except those with the saturated wave energy near the noise level. Each run is color-coded by the initial electron temperature anisotropy $T_{\perp e}/T_{\parallel e}$. The excited frequency spectra transition from the lower band (marked by circles) to the upper band (marked by squares) at about $\beta_{\parallel e} = 0.2$, consistent with linear kinetic theory (Figure 4e in *Yue et al.* [2016]). The noise level of wave magnetic energy scales with $\beta_{\parallel e}^2$, which is indicated by the gray crosses in Figure 3.6a for each run. Above the noise level, the saturated wave energy increases with electron anisotropy rapidly and finally reaches some limit where the saturated wave energy $(\delta B/B_0)^2$ roughly scales with $\beta_{\parallel e}^{\alpha} (\alpha = 1.2 - 2)$. A gray line indicating the scaling of $(\delta B/B_0)^2 \propto \beta_{\parallel e}^2$ is plotted as a reference. Note that the saturated wave energy in the small $\beta_{\parallel e}$ regime shown has not reached this limit, since the performed runs in this regime do not exceed the anisotropy threshold enough. On the spacecraft observation side, the normalized magnetic field energy of whistler-mode chorus waves with respect to the electron beta $\beta_{\parallel e}$ from Van Allen Probes is shown in Figure 3.6b, using the
Figure 3.5: The distribution of the electron temperature anisotropy measurements with respect to the electron beta $\beta_{\parallel e}$, color-coded by the number of whistler-mode chorus wave events in each bin. The final states of relaxed electron distributions from all the runs of MDPIC2 as in Figure 3.4 (except those runs of low signal to noise ratio) are shown as solid black diamonds. The theoretical upper bound of electron anisotropy for $0.1 \leq \beta_{\parallel e} \leq 10$ calculated by Gary and Wang [1996], $\frac{T_{\perp e}}{T_{\parallel e}} = 1 + \frac{S_e}{\beta_{\parallel e}}$ with $S_e = 0.42$ and $\alpha_e = 0.5$, is shown as the light blue line.
same dataset as in Figure 3.5. The summation of the total magnetic field energy across the whole band is marked with black dots. For upper band chorus (red dots), its magnetic field energy \((\delta B/B_0)^2\) scales with \(\beta_{||e}^2\) for \(\beta_{||e} \lesssim 0.05\) and drops rapidly for \(\beta_{||e} \gtrsim 0.05\). This can be understood from two aspects. First, linear kinetic theory predicts that the most unstable frequency band continuously shifts from upper band to lower band with increasing \(\beta_{||e}\) (Figure 4e of Yue et al. [2016]). The rapid drop of upper band chorus wave energy \((\delta B/B_0)^2\) for \(\beta_{||e} \gtrsim 0.05\) is a natural consequence of linear kinetic theory, since the resonant excitation of upper band chorus in the large \(\beta_{||e}\) regime is unfavorable. Second, particle-in-cell simulations suggest the scaling of wave magnetic field energy with \(\beta_{||e}^2\) in the small \(\beta_{||e}\) regime, which is consistent with spacecraft observations. For lower band chorus, it is seen that the wave magnetic field energy scales with \(\beta_{||e}^2\) for all the observed electron beta. This is consistent with the results of the particle-in-cell simulations in the large \(\beta_{||e}\) regime. However, with a single Maxwellian distribution, the lower band chorus waves do not occur in the small \(\beta_{||e}\) regime. Recent spacecraft observations suggest that lower band chorus in the small \(\beta_{||e}\) regime may be related to the plateau distribution or beam distribution in 100–500 eV energy range [Li et al., 2016b]. It is noted that the saturated magnetic field energy of whistler waves in MDPIC2, though also scales to \(\beta_{||e}^2\), is about four orders of magnitude larger than that observed by Van Allen Probes. The main reason for this discrepancy is that only one hot electron component (100 eV-10 keV in temperature) is assumed in the simulation. In reality, a thermal electron component (a few eV in temperature) is usually present, which lowers the fraction of resonant electrons in the total electron population dramatically and hence reduces the wave amplitude. But a thermal electron component has a smaller Debye length and therefore requires more computation grids if the system size is fixed, which is more computationally expensive. It can be reasonably expected that the magnetic field energy of whistler waves in MDPIC2 with an additional component of thermal electrons would drop substantially but still scale to the electron beta \(\beta_{||e}\) with a similar power index as in satellite observations.
Figure 3.6: (a) The normalized saturated wave magnetic field energy as a function of the electron beta $\beta_{||e}$ for all the runs shown in Figure 3.4 except those with the saturated wave energy near the noise level. Each run is color-coded by the initial electron temperature anisotropy $T_{\perp e}/T_{|| e}$ shown in the color bar. The excited frequency spectra in the lower band are marked by circles and those in upper band are marked by squares. A gray dashed line with the scaling $(\delta B/B_0)^2 \propto \beta_{|| e}^2$ is plotted as a reference. The gray cross markers indicate the noise level of $(\delta B/B_0)^2$ at each $\beta_{|| e}$. Note that the gray cross markers are clustered together and appear as thick short lines. (b) The normalized magnetic field energy of whistler-mode chorus waves with respect to the electron beta $\beta_{|| e}$ observed by Van Allen Probes. Blue and red dots indicate lower and upper band chorus waves, respectively. Black dots denote the summation over lower and upper band chorus waves. A gray dashed line with the scaling $(\delta B/B_0)^2 \propto \beta_{|| e}^2$ is plotted as a reference.
3.7 Summary and discussion

In this study, an ensemble of particle-in-cell simulations is performed to study the dependence of the whistler anisotropy instability on the electron beta $\beta_{||e}$, relevant to the excitation of whistler-mode chorus waves in the near-Earth space environment. The final relaxed electron distributions are constrained to marginal stability states where the linear growth rates are zero. The bounds of this marginal stability state computed from MDPIC2 show remarkable consistency with Van Allen Probe observations. Quasi-parallel propagating whistler waves at $\beta_{||e} \gtrsim 0.02$ transition to very oblique whistler waves propagating close to the resonance cone at $\beta_{||e} \lesssim 0.02$. The electron distribution in both small and large $\beta_{||e}$ regime gets heated in the parallel direction due to cyclotron resonance. While in the small $\beta_{||e}$ regime, the wave field is greatly damped after wave saturation due to significant Landau damping. Correspondingly, a plateau of the reduced electron distribution in the parallel direction is formed due to Landau damping. The saturation magnetic field energy of whistler waves scales with $\beta_{||e}^{\alpha} (\alpha = 1.2 - 2)$ in MDPIC2 and scales with a similar power law of $\beta_{||e}^2$ in Van Allen Probe observations, though larger amplitudes of whistler waves are obtained in simulations due to a missing thermal electron component. These results suggest the importance of the electron beta $\beta_{||e}$ in determining the properties of whistler waves, such as the WNAs and wave amplitudes.

By assuming a single electron component with a bi-Maxwellian distribution, we can readily explore the role of different electron temperatures in the excitation of whistler waves. However, in spacecraft observations, the plasma has multiple electron components with different temperatures. For example, if two electron components exist and have distinct temperatures, two bands of whistlers can be excited independently due to the linear instability of these two electron components [Liu et al., 2011; Fu et al., 2014]. The characteristics of the banded whistlers can be determined separately by the corresponding electron component that excites them. This study will be helpful in interpreting the whistler anisotropy instability of multiple electron components with distinct temperatures. On the other hand, if the temperatures of two electron components are not well separated, the colder compo-
nent would get cyclotron heating in the parallel direction and develop a high energy tail, which can reduce the anisotropy of the warmer component and suppress the corresponding anisotropy instability of the warmer component. Such interplay between different electron components is both important and interesting, which can be addressed in future studies.
CHAPTER 4

Excitation of incoherent broadband whistler waves in a laboratory plasma

In this chapter we discuss broadband whistler waves that are excited on the Large Plasma Device by the injection of a helical electron beam into a cold plasma [An et al., 2016]. The mode structure of the excited whistler wave is identified using a phase-correlation technique showing that the waves are excited through a combination of Landau resonance, cyclotron resonance and anomalous cyclotron resonance. The dominant wave mode excited through cyclotron resonance is quasi-parallel propagating, whereas wave modes excited through Landau resonance and anomalous cyclotron resonance propagate at oblique angles that are close to the resonance cone. An analysis of the linear wave growth rates captures the major observations in the experiment.

4.1 Introduction

Several mechanisms have been proposed to account for the banded structure of chorus [Omura et al., 2009; Liu et al., 2011; Fu et al., 2014; Mourenas et al., 2015; Fu et al., 2015]. Among the proposed mechanisms, Mourenas et al. [2015] suggested that the less frequently occurring but statistically significant very oblique lower band chorus waves can be generated through a combination of cyclotron resonance and Landau resonance with low energy electron beams having energies of a few keV. But these theoretical ideas remain to be tested by satellite observations and laboratory experiments.

In order to study the excitation of chorus-like whistler waves in a laboratory plasma, an electron beam is used as the free energy source, injected into a cold background plasma. In
a beam-plasma system, various instability processes can excite a variety of waves including whistler-mode waves \cite{Bell1964}, Bernstein-mode waves \cite{Kusse1970, Mizuno1971} and Langmuir waves \cite{ONeill1971, Gentle1973}. Whistler-mode emissions by beam-plasma interaction have been studied extensively in the past, such as in the generation of auroral hiss \cite{Maggs1976, Gurnett1983, Sazhin1993}, in active experiments in the space environment \cite{Lavergnat1979, Tokar1984, Gurnett1986, Farrell1988, Neubert1992} and in controlled laboratory settings \cite{Stenzel1977, Krafft1994, Starodubtsev1999, Starodubtsev2009, VanCompernolle2015}. During active experiments of the Spacelab 2 mission, for instance, beam-generated whistler-mode emissions were observed to propagate near the resonance cone and were attributed to Landau resonance \cite{Gurnett1986, Farrell1988}. In controlled laboratory experiments, whistler-mode emissions were also generated through both Landau resonance \cite{Krafft1994} and cyclotron resonance \cite{Starodubtsev1999} using a density-modulated electron beam. However, if the electron beam is not modulated, broadband whistler waves are produced instead of a single wave with predetermined frequency. The experiment reported in this chapter is unique in that there is no imposed frequency on the electron beam. Whistler waves are spontaneously excited by different resonance modes simultaneously, which results in frequency spectra having clear upper and lower bands.

4.2 Experimental setup

The experiment is performed on the upgraded Large Plasma Device (LAPD) \cite{Gekelman2016, Leneman2006} at the Basic Plasma Science Facility (BaPSF) at University of California, Los Angeles. The LAPD is a long cylindrical device, with an axial magnetic field and an 18 m long, 60 cm diameter quiescent plasma column. A schematic diagram of the experimental setup is shown in Figure 4.1. The background plasma is pulsed at 1 Hz. A 10 cm diameter electron beam source \( (0.5 \text{ kV} \leq V_{\text{beam}} \leq 4 \text{ kV}) \) \cite{VanCompernolle2014, VanCompernolle2015} is introduced into the machine 15 m from the LAPD source at the opposite end of the
The beam source is angled at $30^\circ$ with respect to the background magnetic field in order to provide sufficient free energy in the electron distribution for the cyclotron growth of the whistler waves. The magnetic field near the beam source is uniform at 60 G for 7 meters, and then transitions to 350 G near the LAPD source for reliable operation of the source. The Helium plasma in the experiment has a Helium fill pressure of $3 \times 10^{-5}$ Torr. The experiment is performed after the active phase of the LAPD discharge, in the afterglow plasma, when the electrons are relatively cold, i.e., $T_e \leq 0.5$ eV. Typical absolute plasma parameters in the laboratory are quite different from those found in the magnetosphere but the dominant scaled dimensionless quantities are set to be very similar (see Table 4.1 below for the range of plasma parameters). Measurements of plasma parameters and wave activity were taken with Langmuir probes and high frequency magnetic loop probes (which is positioned 1 meter away from the beam source). Volumetric data is obtained by moving computer controlled probes through the plasma over the course of thousands of identical plasma shots. The start of the electron beam pulse is taken as $t = 0$ and the streaming direction of the electron beam as positive direction along $z$.

![Figure 4.1: A schematic picture of the experimental setup (not to scale). A 10 cm diameter electron beam source launches energetic electrons with energies up to 4 keV. Probes measure plasma parameters and detect wave activity. A fixed B-dot probe is used as a reference for the moving B-dot probe to measure the phase difference between the two locations.](image-url)
Table 4.1: A comparison between plasma parameters in the laboratory and in the magnetosphere: plasma density [Sheeley et al., 2001]; magnetic field strength; ratio of plasma frequency to cyclotron frequency [Li et al., 2012]; ratio of beam density to plasma density [Gao et al., 2014]; ratio of whistler wave amplitude to background magnetic field strength [Gao et al., 2014; Li et al., 2012]; ratio of energetic electron velocity to speed of light; ratio of electron thermal pressure to magnetic pressure.

### 4.3 An example of broadband whistler waves in the laboratory

Figure 4.2 shows an overview of the beam-generated whistler waves. The beam voltage and beam current are shown in panel (a). The time series of the perpendicular component of the magnetic field perturbation is shown in panel (b). The wave activity is seen to turn on when the beam is turned on, which demonstrates that these waves are spontaneously excited by the electron beam. Panel (c) shows the dynamic spectrogram in the frequency range of the whistler waves. Whistler waves are present in a broad frequency range with $0.1 < \omega/\Omega_e < 0.9$, where $\omega$ is the wave frequency. Four selected snapshots of the wave spectra are shown in panel (d) for $t = 25\mu s$ (blue), $t = 40\mu s$ (green), $t = 60\mu s$ (yellow) and $t = 80\mu s$ (red). The time-averaged spectrum between $t = 25\mu s$ and $t = 80\mu s$ is shown as a thick black line. Two spectral peaks clearly arise, similar to the two-band structure of whistler mode chorus waves in space [Burtis and Helliwell, 1976] which will be demonstrated below to be due to different resonance modes.
Figure 4.2: (a) Time series of beam source voltage (blue) and total beam current (red) flowing into the LaB$_6$ disk. (b) Time series of magnetic fluctuations transverse to the background magnetic field. (c) Dynamic spectrogram of the time series. (d) Spectra at four selected time instants: $t = 25\mu s$ (blue), $t = 40\mu s$ (green), $t = 60\mu s$ (yellow) and $t = 80\mu s$ (red). Spectrum averaged between $t = 25\mu s$ and $t = 80\mu s$ is shown as a thick black line, showing the typical two-band structure and a minimum wave power at $\omega = 0.5\Omega_e$. 

Spectrum averaged between $t = 25\mu s$ and $t = 80\mu s$ is shown as a thick black line, showing the typical two-band structure and a minimum wave power at $\omega = 0.5\Omega_e$. 

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4.4 The mode structure of broadband whistler waves

Using a moving magnetic field probe together with a similar fixed probe as a reference, the phase-delay $\Delta \phi(x, \omega)$ between these two probes can be measured accurately, given that the step size of the moving probe is less than one wavelength. This is the so-called cross-correlation technique. Assume that the time series measured by the moving probe and by the reference probe are $B(x, t)$ and $B(x_r = 0, t)$, respectively. By taking a Fourier transform of these two time series, multiplying one’s Fourier transform with the conjugate of the other one’s and extracting the phase of the complex product, one can obtain the phase-delay between these two probes as $\Delta \phi(x, \omega) = \text{arg}\left(\tilde{B}(x, \omega)\tilde{B}^*(x_r = 0, \omega)\right)$. The mode structure can be constructed as

$$|\tilde{B}(x, \omega)| \exp\left(i\Delta \phi(x, \omega) - i\omega t\right)$$

with volumetric data obtained for many identical plasma shots. Figure 4.3 shows the mode structure and corresponding refractive index surfaces of broadband whistlers at 4 representative frequencies. From the mode structure, one can extract both parallel and perpendicular wave numbers. The phase-delay between the moving probe and the reference probe in the cylindrical geometry is assumed to be $\Delta \phi(x, \omega) = k_z(\omega)z + k_\perp(\omega)\rho$, which has been obtained in the process of constructing the mode structure. $k_z$ and $k_\perp$ are the parallel and perpendicular wave numbers, respectively. $z$ and $\rho$ are the axial and radial coordinates in the cylindrical geometry, respectively. To extract the parallel wave number, for a given frequency, one can linearly fit $\Delta \phi(\rho, z, \omega)$ in the $x-z$ plane as a function of $z$ at each $\rho$ position. The perpendicular wave number is obtained, in a similar fashion, by fitting $\Delta \phi(\rho, z, \omega)$ as a linear function of $\rho$ at each $z$ position. The parallel wave number and perpendicular wave number for each mode structure is marked using the red asterisk in the wave number space in the third row in Figure 4.3. The term resonance cone is used to describe a region in the wave number space, where whistler wave propagation is prohibited ($n^2 < 0$), approximately where $\psi > \arccos(\omega/\Omega_e)$. Here $n$ is the refractive index. Wave normal angle (WNA) $\psi$ is used to denote the angle between the wave vector and the background magnetic field. More generally, whistler waves are strongly damped in the resonance cone region when thermal effects of the
background plasma are taken into account [Horne and Sazhin, 1990]. The mode structure at $\omega = 0.15\Omega_e$ is displayed in column (a). The wave fronts propagate towards the center, where the beam is located, in the $x - y$ plane (top panel). These features can be understood by locating the $\mathbf{k}$-vector in wave number space. The group velocity $\partial \omega / \partial \mathbf{k}$ is normal to the refractive index surface. In this case the perpendicular group velocity is directed in an opposite sense to the perpendicular phase velocity, which means energy is flowing out and away from the beam while phase front propagates into the beam. Consistently, these waves propagate close to the resonance cone with WNA $\approx 80^\circ$ in the $x - z$ plane. The co-streaming of these waves with the electron beam in the parallel direction indicates that the excitation is likely due to Landau resonance.

The mode structure at $\omega = 0.25\Omega_e$ is displayed in column (b). In contrast to the waves at $\omega = 0.15\Omega_e$, these waves are counter-streaming with the beam, having WNA $\approx 105^\circ$, which is also close to resonance cone. It will be shown below that the excited waves at this frequency are due to first-order cyclotron resonance. The mode structure at $\omega = 0.35\Omega_e$ is displayed in column (c). This mode has slightly oblique wave fronts whereas the group velocity is parallel to the background magnetic field, in a configuration known as the Gendrin mode. This is the location of the spectral peak shown earlier in Figure 4.2d, and hence constitutes the most intense waves in this experiment. Waves travelling in the opposite direction to the electron beam can efficiently gain energy from the beam through cyclotron resonance, resulting in wave amplitudes that are larger than that of other modes. The mode structure at $\omega = 0.65\Omega_e$ is displayed in column (d). These waves are co-streaming with the electron beam with WNA $\approx 49^\circ$. Animation of $B_x - B_y$ vector fields show the co-existence of both a strong right-hand polarized component and a weak left-hand polarized component. This mode is more complicated than the previous ones in the sense that it is excited through both Landau resonance and anomalous cyclotron resonance, as shown below.
Figure 4.3: Mode structure of whistler waves at 4 representative frequencies corresponding to each column. The first two rows show $B_y$ in the $x - y$ and the $x - z$ planes, respectively. Wave amplitudes are normalized to the maximum wave amplitude in each panel. Arrows in the second row represent the wave vector direction. The third row shows the refractive index surface (which is a curve in 2D projection) for each frequency in the wave number space. The wave number corresponding to each mode structure is marked by the red asterisk. Note that $k_\perp < 0$ represents radially inward propagating waves.
4.5  $\omega - k_z$ diagram

Using the method of extracting the wave numbers described above and shown in Figure 4.3, we repeat the procedure for all frequencies in the range $0.1 < \omega/\Omega_e < 0.9$, and show the corresponding results in panels (a) and (b) of Figure 4.4. The data is organized in the $k_z - \omega$ plane in panel (a), color coded by the power spectral density. Panel (b) is organized in the same way as panel (a), but color coded by WNA computed as $\psi = \arctan(k_{\perp}/k_z)$. Note that at each wave frequency, the parallel wave numbers obtained at each radial location are plotted in Figure 4.4. These data points beautifully aggregate into three resonance modes, corresponding to $\omega - k_z u = n\Omega_e$ with $n = 1, 0, -1$ ($u$ is the initial beam velocity in the parallel direction). Cyclotron resonance occurs in the frequency range below $0.4\Omega_e$ as shown by the cluster of orange-red points falling on the $n = 1$ line. Even for normal, first-order cyclotron resonance, the WNA is seen to be close to the resonance cone in the frequency range $0.2 < \omega/\Omega_e < 0.3$, which becomes less oblique with increasing frequency and eventually reaches about $160^\circ$ in the most intense range between $0.35 < \omega/\Omega_e < 0.4$. The cyclotron resonance mode results in the most efficient energy transfer and hence the largest wave amplitudes compared to the other two resonance modes. For Landau resonance below $0.25\Omega_e$, the measured data is seen to fall slightly below $\omega = k_z u$, which may be evidence that the beam electrons are slowed down from their initial velocity $u$, by the time they reach the primary wave excitation region in the experiment. The Landau resonance mode below $0.25\Omega_e$ has a WNA of $80^\circ$ or so. Such oblique WNA implies a finite wave electric field (though small) in the parallel direction, which allows energy transfer between beam electrons and whistler waves through Landau resonance. The WNA for Landau resonance in the frequency range $0.4 < \omega/\Omega_e < 0.9$ is near the edge of the resonance cone. Anomalous cyclotron resonance co-exists with Landau resonance in the frequency range $0.4 < \omega/\Omega_e < 0.9$. The WNA for anomalous cyclotron resonance is in the range from $20^\circ$ to $50^\circ$.

To compare the experimental results with the predictions of linear theory, the hot plasma dispersion relation is solved to obtain linear growth rates and the accompanying WNA using the HOTRAY code [Horne, 1989]. This code implements the hot plasma dispersion relation.
relation in a magnetized plasma for an arbitrary number of summed Maxwellian distributions including an optional drift in the parallel direction and also a loss cone. The beam electrons are modeled as a beam ring distribution, implemented in HOTRAY as

\[
f(v_{\perp}, v_z) = \frac{1}{\pi^{\frac{3}{2}} \alpha_{\perp}^2 \alpha_{\parallel}(1 - \beta)} \exp \left( -\frac{(v_z - v_d)^2}{\alpha_{\parallel}^2} \right) \times \left( \exp \left( -\frac{v_{\perp}^2}{\alpha_{\perp}^2} \right) - \exp \left( -\frac{v_{\perp}^2}{\beta \alpha_{\perp}^2} \right) \right)
\]

(4.1)

Here \( v_{\perp} \) and \( v_z \) are perpendicular and parallel velocities, respectively, relative to the background magnetic field. \( \alpha_{\parallel}, \alpha_{\perp}, \beta \) and \( v_d \) are free parameters that control the shape of the distribution function. Since direct measurements of the distribution function are not available at this stage, the distribution function is roughly inferred based on physical arguments. As the beam enters the plasma, the fastest growing Langmuir waves slow down and relax the beam electrons in the parallel direction \([O’Neil et al., 1971; Gentle and Lohr, 1973]\). The beam electrons move locally over a single Langmuir wave with a relative velocity of \( \Delta v = 2^{-\frac{1}{2}}(n_b/n_0)^{\frac{3}{2}}u \) \([O’Neil et al., 1971]\). The Langmuir wave eventually reaches an amplitude \( \phi \approx m_e(\Delta v)^2/e \) which is enough to trap the beam electrons, and also causes nonlinear saturation of wave growth. Thus the beam electrons are modeled to be a Maxwellian centered at the phase velocity of the Langmuir wave \( v_d = u - \Delta v \) with a thermal spread \( \alpha_{\parallel} = \sqrt{2}\Delta v \). In the perpendicular direction, we set \( \alpha_{\perp} = v_{\perp,0} \) and \( \beta = 0.8 \). Here \( v_{\perp,0} \) is the initial beam velocity in perpendicular direction. As such, the corresponding distribution in the perpendicular direction peaks at \( (v_{\perp})_{\text{max}} = v_{\perp,0} \) and the full width at half Maximum is \( \sim v_{\perp,0} \), which is likely much broader than that in the experiment, but is the lower limit that can be reached by equation (4.1). The results of solving the hot plasma dispersion relation for the distribution described above are shown in panels (c) and (d) of Figure 4.4, in the same format as the first two panels of Figure 4.4. The cyclotron resonance mode, Landau resonance mode and anomalous cyclotron resonance mode show up in approximately the right \( k_z - \omega \) locations, with appreciable growth rates, and having consistent WNA with the experimental results. The dominant wave mode is excited through cyclotron resonance when the resonant velocity is at the negative gradient side \( \partial f/\partial v_z < 0 \) \([Kennel and Petschek, 1966; Kennel, 1966]\). Thus the calculated data points for cyclotron resonance have a slope larger...
than \( v_d \) but near the initial beam velocity \( u \). The Landau resonance mode, on the other hand, is excited by the positive gradient \( \partial f / \partial v_z > 0 \). Therefore the calculated data points for Landau resonance have a slope below \( v_d \). A broad spectrum between \( 0.4 < \omega / \Omega_e < 0.85 \) is excited through anomalous cyclotron resonance with WNA close to resonance cone. We note that waves in Landau resonance with beam electrons in the frequency range \( \omega / \Omega_e > 0.4 \) are not well captured by the linear growth rate calculations. However, one should also note that in reality spatial wave growth takes place. Waves with \( k_z < 0 \) (\( k_z > 0 \)) are growing toward (away from) the injection point of the beam electrons, whereas we make a simplifying assumption and construct one distribution function to account for all waves with different profiles of spatial growth. This very likely results in some inconsistency between the linear analysis and experimental results, although the general trends are remarkably similar.

4.6 Parameter scans

To investigate whether the linear growth rates scale in a similar way as the observed waves under a variety of plasma conditions, we perform a series of parameter scan. The plasma parameters used in the example of Figure 4.2 were chosen to serve as the control case, and the beam source was tilted to 45°. For clarification, the parameters for the control case are \( \omega_{pe} / \Omega_e = 9.6 \), \( E_b = 3 \text{ keV} \), \( n_b / n_0 = 1.6 \times 10^{-3} \) and pitch angle \( \alpha = 45^\circ \). For a given frequency, a calculation of linear growth rates is performed over all possible WNA and the largest linear growth rate is extracted. Repeating this procedure for many frequencies in the whistler wave range gives a spectrum of linear growth rate. Thus a comparison between power spectral densities and linear growth rates can be made for different plasma parameters.

The first parameter scan was performed by varying the cold plasma density at a fixed beam energy \( E_b = 3 \text{ keV} \) and fixed beam density \( n_b = 5 \times 10^7 \text{ cm}^{-3} \). This changes both \( \omega_{pe} / \Omega_e \) and \( n_b / n_0 \). Figure 4.5a shows power spectral densities for different \( \omega_{pe} / \Omega_e \) from the experiment. The corresponding linear growth rates from HOTRAY are shown in Figure 4.5b. The power spectral peaks clearly shift to higher frequencies as the plasma density decreases, which is well captured by the spectra of the linear growth rate. These spectral
Figure 4.4: Wave properties plotted on a $k_z - \omega$ diagram from the LAPD experiment (a, b) and corresponding HOTRAY calculations (c, d), respectively, showing multiple resonance modes, color-coded by (a) power spectral density, (b) wave normal angle $\psi$ from the experiment, (c) linear growth rates and (d) wave normal angle $\psi$ from the HOTRAY code.
peaks are parallel propagating wave modes and are dominantly excited through normal cyclotron resonance, described by $\omega - k_z u = \Omega_e$. An up-shift in the wave frequency is required to satisfy the resonance condition as the plasma density decreases. The very oblique wave modes excited through Landau resonance also show an up-shift in frequency as shown in Figure 4.5b in yellow-green bands. The wave modes excited through anomalous cyclotron resonance and higher order resonance are relatively insensitive to plasma density changes, ranging between $0.7 - 0.9 \Omega_e$. A second scan, displayed in Figure 4.5c and 4.5d, was preformed with $\omega_{pe}/\Omega_e = 9.6$ and $n_b/n_0 = 1.6 \times 10^{-3}$ by varying electron beam energy. An up-shift of the power spectral peak is observed as the beam energy decreases (Figure 4.5c), which agrees well with linear analysis (Figure 4.5d). It can be understood by observing that, to satisfy $\omega - k_z u = \Omega_e$, the wave frequency has to increase to compensate for the reduction of the Doppler shift term. The secondary peaks in the upper band $\omega/\Omega_e > 0.5$ in Figure 4.5c, primarily due to Landau resonance, are not well reproduced by the linear analysis. A third scan was performed by varying beam density at $\omega_{pe}/\Omega_e = 9.6$ and $E_b = 3$ keV as shown in Figure 4.5e and 4.5f. Both the power spectral densities and linear growth rates increase as $n_b/n_0$ increases, which is expected. It is noted that there is frequency spectrum broadening in Figure 4.5e (also in Figure 4.5a) as $n_b/n_0$ increases. Preliminary investigation shows that wave-wave interactions get enhanced as $n_b/n_0$ increases. This possibly leads to the broadening of power spectrum. Another possibility is the broadening of wave-particle resonance resulting from strong turbulence [Dupree, 1966].

Finally, the peak of each wave power spectrum from the experiment and that of each linear growth rate spectrum are extracted for all parameter scans. This serves to compare the spectral peak locations from both experiment and linear theory (Figure 4.6a, 4.6c, 4.6e), and also to compare the maximal saturated wave power with the maximal linear growth rate (Figure 4.6b, 4.6d, 4.6f). Three rows in Figure 4.6, from top to bottom, correspond to plasma density scan, beam energy scan and beam density scan, respectively. Comparisons of the spectral peak frequencies show good agreement between experiment and linear theory. The shift of the spectral peak frequencies with respect to plasma density and beam energy can be understood in terms of linear excitation through cyclotron resonance as explained
Figure 4.5: A comparison of observed wave properties (panels a, c, and e) and corresponding maximum linear growth rates (panels b, d, and f) obtained from three parameter scans: variation of the plasma density (a, b), beam energy (c, d) and beam density (e, f). $\delta B_n$ is the spectral density of magnetic noise level as a function of frequency. $(\delta B/\delta B_n)^2$ measures the amplification of the magnetic field from the noise level.
above. The shift of the spectral peak frequencies with respect to beam density is due to the modification of plasma dispersion relation by hot electrons. One may expect some positive correlation between saturated wave power and linear growth rate, though the wave saturation process is governed by nonlinear processes. This expectation is consistent with the beam energy scan and the beam density scan. However, the saturated wave power and the linear growth rate show an inverse relation in the low plasma density regime. This indicates that nonlinear processes become a stronger factor in mediating wave-particle interactions as $n_b/n_0$ gets larger.

4.7 Summary

In summary, we have used a novel experimental setup to reveal the complex mode structure and excitation mechanisms of whistler-mode waves in a laboratory plasma, designed to closely resemble the plasma characteristics in the near-Earth space environment. Our results show that the whistler waves are excited primarily due to three basic resonance regimes simultaneously: the normal cyclotron resonance mode, Landau resonance, and first order anomalous cyclotron resonance mode. Linear wave growth calculations show consistent behavior both in intensity (or growth rate) and wave normal angle, shedding new light on the excitation process of whistler waves in space. However, a bi-Maxwellian distribution is believed to be responsible for the excitation of whistler-mode chorus waves in space based on satellite observations, whereas our experiment essentially used a beam ring distribution in velocity space. The difference in the distribution function of hot electrons leads to some of the different wave characteristics between the laboratory experiment and space observations. Diagnostics on the electron distribution function are also desired. These issues constitute the directions of future studies.
Figure 4.6: A comparison of the spectral peak frequencies from experiment and linear theory (a, c, e), and also comparisons of maximal saturated wave power with maximal linear growth rate (b, d, f) for three parameter scans. Comparisons for the plasma density scan are displayed in (a, b), beam energy scan in (c, d), and beam density scan in (e, f), for which the corresponding parameters are plotted as color-bars on the right.
5.1 Introduction and problem setup

In the experiment conducted at the Large Plasma Device, a circular electron beam is injected into the background plasma to excite whistler waves. The linear instability of an electron beam of a finite size differs from that of an infinite homogeneous beam, since the unstable waves spend a limited amount of time inside the beam for amplification and eventually propagate out of the beam region. Such a finite electron beam may lead to a decrease of the linear wave growth rate compared to an infinite electron beam. Note that the size of the electron beam is only a few times the gyro-radius of the beam electrons and is comparable to the wavelength of whistler waves. This ordering of the length scale violates the assumption behind the ray tracing method and hence it cannot be applied here. Corresponding to each complex wave frequency (eigenvalue), the wave has a certain mode structure (eigenmode) that may either leak out of the beam region or be locked in the beam region.

Here we consider a finite electron beam in a slab geometry which nevertheless allows us to capture the essential ingredients of the experiment but does not complicate the solution too much mathematically. The background magnetic field $B_0$ is along $z$ axis. The density of the electron beam is assumed to have a simple top-hat profile in the $x$ direction, namely

$$ n_b(x) = \begin{cases} 
  n_b, & |x| < a \\
  0, & otherwise 
\end{cases} \quad (5.1) $$

Where $a$ is the half width of the beam. The background electrons are cold and uniform, with
a density $n_0$. The background ions are cold and are distributed to maintain the neutrality of the system. The frequency range considered here is much higher than the lower hybrid frequency so that ions are treated as a fixed background. The unperturbed electron beam distribution function is

$$ f_{0b}(v_\parallel, v_\perp) = \frac{n_b}{2\pi v_\perp} \delta(v_\perp - v_{\perp 0}) \delta(v_z - u) \quad (5.2) $$

The wave dispersion relation inside and outside the beam, respectively, can be treated as a uniform medium. The key step in the analysis is to match the eigenmode solutions inside and outside the beam at the boundary. In section 5.2, a linear instability analysis is performed for an infinite electron beam. In section 5.3, the eigenmode solutions are matched at the boundary and the final results are presented.

### 5.2 Linear instability analysis for an infinite electron beam

To begin with, the linearized Vlasov equation is

$$ \frac{df_b}{dt} - \frac{e}{m} \left( \hat{E} + \frac{v \times \hat{B}}{c} \right) \cdot \frac{\partial f_{0b}}{\partial v} = 0 \quad (5.3) $$

where $\hat{f}_b$ is the perturbed distribution function and $\hat{E}$, $\hat{B}$ are the perturbed fields. The total time derivative $\frac{d}{dt}$ is along the unperturbed electron orbit. We consider perturbations of the form

$$ \hat{E} = \tilde{E} \exp(-i\omega t + ik_x x + ik_z z) \quad (5.4) $$

$$ \hat{B} = \tilde{B} \exp(-i\omega t + ik_x x + ik_z z) \quad (5.5) $$

Equation (5.3) can be integrated along its characteristics, i.e., the unperturbed helical orbits of electrons. This integral can be calculated as

$$ \hat{f}_b = \frac{e}{m} \int_{-\infty}^{t} dt' \exp(-i\omega t' + ik_x x' + ik_z z') \tilde{S} \quad (5.6) $$

Here the integral kernel $\tilde{S}$ is

$$ \tilde{S} = \left( \frac{\hat{E} + v' \times \hat{B}}{c} \right) \cdot \frac{\partial f_{0b}(v')}{\partial v'} \quad (5.7) $$
The perturbed beam current can be calculated from the velocity moments of the perturbed beam distribution \( \hat{n}_b \) as

\[
\hat{j}_b = -e \int_0^{\infty} 2\pi v_\perp dv_\perp \int_{-\infty}^{\infty} dv_z \langle \mathbf{v} \hat{n}_b \phi \rangle
\]  

(5.8)

Note that \( \hat{j}_b \) has the form \( \hat{j}_b = \tilde{j}_b \exp(-i\omega t + ik_x x + ik_z z) \). In fact, \( \tilde{j}_b \) can be expressed as a linear superposition of \( \tilde{E}_x(x) \), \( \tilde{E}_y(x) \) and \( \tilde{E}_z(x) \), with the coefficients being integrals of gradients over velocity space, namely

\[
\frac{4\pi i}{\omega} \tilde{j}_b = \chi_b \cdot \mathbf{E}
\]  

(5.9)

Here \( \chi_b \) is the susceptibility tensor for the electron beam, which is derived in detail in appendix A. The susceptibility tensor is summarized here as the following

\[
\chi_{xx} = -\frac{\omega_{pb}^2}{\omega^2} - \frac{\omega_{pb}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{2\pi}{\lambda} J_n^2 \right) \frac{n\Omega_e}{\omega - k_z u - n\Omega_e} + (J_n^2 \cot^2 \theta) \frac{n^2\Omega_e^2}{(\omega - k_z u - n\Omega_e)^2} \right]
\]  

(5.10)

\[
\chi_{yy} = -\frac{\omega_{pb}^2}{\omega^2} - \frac{\omega_{pb}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{\pi}{\lambda} (\lambda J_n' )^2 \right) \frac{n\Omega_e}{\omega - k_z u - n\Omega_e} + (\lambda^2 (J_n')^2 \cot^2 \theta) \frac{\Omega_e^2}{(\omega - k_z u - n\Omega_e)^2} \right]
\]  

(5.11)

\[
\chi_{zz} = -\frac{\omega_{pb}^2}{\omega^2} \tan^2 \theta - \frac{\omega_{pb}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{2\pi}{\lambda} J_n J_n' \tan^2 \theta \right) \frac{(\omega - n\Omega_e)^2}{\Omega_e (\omega - k_z u - n\Omega_e)} + J_n^2 \frac{(\omega - n\Omega_e)^2}{(\omega - k_z u - n\Omega_e)^2} \right]
\]  

(5.12)

\[
\chi_{xy} = \frac{i\omega_{pb}}{\omega} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{\pi}{\lambda} (\lambda J_n J_n' )' \right) \frac{n\Omega_e}{\omega - k_z u - n\Omega_e} + (n\lambda J_n J_n' \cot^2 \theta) \frac{\Omega_e^2}{(\omega - k_z u - n\Omega_e)^2} \right]
\]  

(5.13)

\[
\chi_{yx} = -\chi_{xy}
\]  

(5.14)

\[
\chi_{xz} = \frac{\omega_{pb}}{\omega} \tan \theta - \frac{\omega_{pb}}{\omega} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{2\pi}{\lambda} J_n J_n' \tan \theta \right) \frac{\omega - n\Omega_e}{\omega - k_z u - n\Omega_e} + (n J_n^2 \cot \theta) \frac{\Omega_e (\omega - n\Omega_e)}{(\omega - k_z u - n\Omega_e)^2} \right]
\]  

(5.15)

\[
\chi_{zx} = \chi_{xz}
\]  

(5.16)

\[
\chi_{yz} = -\frac{i\omega_{pb}}{\omega} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{\pi}{\lambda} (\lambda J_n J_n' )' \tan \theta \right) \frac{\omega - n\Omega_e}{\omega - k_z u - n\Omega_e} + (\lambda J_n J_n' \cot \theta) \frac{\Omega_e (\omega - n\Omega_e)}{(\omega - k_z u - n\Omega_e)^2} \right]
\]  

(5.17)

\[
\chi_{zy} = -\chi_{yz}
\]  

(5.18)

Here \( \omega_{pb}^2 = 4\pi n_b e^2/m_e \) and \( \omega \) is the wave frequency. \( \theta \) is the angle between the background magnetic field \( \mathbf{B}_0 \) and the wave vector \( \mathbf{k} \). \( J_n \) is the Bessel function of the first kind of \( n \)-th order and its argument is essentially the wave number normalized by the gyro-radius of the beam electron, given by \( \lambda = \frac{k_z u/\Omega_e}{\Omega_e} \). Note that the susceptibility tensor \( \chi_b \) is Hermitian for any real \( \omega \). \( \chi_b \) is also proportional to the beam density, i.e., \( \chi_b \propto n_b \). Since the beam is
tenuous, $\chi_b$ is significant only when the residual of $\omega - k_z u - n \Omega_e$ is small. The singularity at the resonance is avoided by navigating in the complex plane of wave frequency $\omega$.

Combining Faraday’s and Ampere’s Law, one obtains

$$\frac{c^2}{\omega^2} k \times k \times \vec{E} = -\left( \frac{4\pi i}{\omega} \vec{j}_b + \frac{4\pi i}{\omega} \vec{j}_c + \vec{\tilde{E}} \right)$$

(5.19)

where $\vec{j}_b$ and $\vec{j}_c$ are perturbed plasma currents generated by beam electrons and cold background electrons, respectively. Making use of the cold plasma dielectric tensor, one can combine the currents generated by background electrons $\hat{\vec{j}}_c$ and the displacement current as

$$\frac{4\pi i}{\omega} \hat{\vec{j}}_c + \vec{\tilde{E}} = \vec{\epsilon}_c \cdot \vec{E}$$

(5.20)

where $\vec{\epsilon}_c$ is the well-known cold plasma dielectric tensor [Stix, 1962], written as

$$\vec{\epsilon}_c = \begin{pmatrix} \epsilon_\perp & -i \epsilon_H & 0 \\ i \epsilon_H & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix}$$

(5.21)

Combining equation (5.19) - (5.21) leads to $\vec{M}_b \cdot \vec{\tilde{E}} = 0$ with the dispersion matrix having the form

$$\vec{M}_b = \frac{c^2}{\omega^2} \vec{k} \times \vec{k} \times \vec{E} - \frac{k^2 c^2}{\omega^2} \vec{I}_3 + \vec{\epsilon}_c + \chi_b$$

$$= \begin{pmatrix} \epsilon_\perp + \chi_{xx} - \frac{k^2 c^2}{\omega^2} & -i \epsilon_H + \chi_{xy} & \chi_{xz} + \frac{k_x k_z c^2}{\omega^2} \\ i \epsilon_H + \chi_{yx} & \epsilon_\perp + \chi_{yy} - \frac{(k_x^2 + k_z^2) c^2}{\omega^2} & \chi_{yz} \\ \chi_{zx} + \frac{k_x k_z c^2}{\omega^2} & \chi_{zy} & \epsilon_\parallel + \chi_{zz} - \frac{k^2 c^2}{\omega^2} \end{pmatrix}$$

(5.22)

For nontrivial solution, we write $\det(\vec{M}_b) = 0$, which gives the dispersion relation of the beam-plasma system. For a given point in wave number space $(k_x, k_z)$, a root finding procedure is implemented to solve the dispersion relation for a complex wave frequency. The input background parameters for the numerical calculation are $\omega_{pe}/\Omega_e = 5.0$, $n_b/n_0 = 0.01$. The input beam parameters for a 3 keV beam with 45 degree pitch angle are $u/c = 0.0766$ and $v_{\perp 0}/c = 0.0766$. Figure 5.1 shows the numerical result. The dispersion relation in the wave number space is displayed in Figure 5.1a, color coded by the real wave frequency. The corresponding wave growth rate, i.e., the imaginary part of the frequency, is also shown in
the wave number space in Figure 5.1b. Three resonance modes (i.e., cyclotron resonance, Landau resonance and anomalous cyclotron resonance) show up prominently. Each of the resonance modes has a wide distribution in wave normal angle. It is seen that the real wave frequency near resonance has a small down-shift compared to that of the cold plasma dispersion relation. Away from the resonance, the beam has a negligible effect on the dispersion relation.

5.3 Matching the eigenmodes at the boundary

Here we match the eigenmodes inside and outside the beam at the boundary. The electric field outside the beam region has the form

\[
\hat{E}^{\text{out}} = \begin{cases} 
  \begin{pmatrix} 
  \tilde{E}^{\text{out}}_x \\
  \tilde{E}^{\text{out}}_y \\
  \tilde{E}^{\text{out}}_z 
  \end{pmatrix} \exp(-i\omega t + ik_z z + i k_x^{\text{out}} x), & x > a \\
  \begin{pmatrix} 
  -\tilde{E}^{\text{out}}_x \\
  -\tilde{E}^{\text{out}}_y \\
  \tilde{E}^{\text{out}}_z 
  \end{pmatrix} \exp(-i\omega t + ik_z z - i k_x^{\text{out}} x), & x < -a 
  \end{cases}
\]

(5.23)

Here the flip of sign for \( E_x \) and \( E_y \) component is due to that \( k_x \) is opposite in \( x > a \) and \( x < -a \). Note that \( k_x^{\text{out}} > 0 \) indicates outgoing wave fronts away from the beam and \( k_x^{\text{out}} < 0 \) indicates incoming wave fronts toward the beam. The sign of \( k_x^{\text{out}} \) should be chosen properly so that the Poynting flux is directed away from the beam. The phase difference of \( \pi \) in \( x \) and \( y \) direction between \( x > a \) and \( x < -a \) is due to the change of wave vector direction.

The electric field inside the beam region has the form

\[
\hat{E}^{\text{in}} = \begin{pmatrix} 
  \tilde{E}^+_x \\
  \tilde{E}^+_y \\
  \tilde{E}^+_z 
  \end{pmatrix} \exp(ik_x^{\text{in}} x) + \begin{pmatrix} 
  \tilde{E}^-_x \\
  \tilde{E}^-_y \\
  \tilde{E}^-_z 
  \end{pmatrix} \exp(-ik_x^{\text{in}} x) \exp(-i\omega t + ik_z z) \]

(5.24)

The boundary condition considered is the continuity of tangential electric field across the boundary, i.e., \( E_y^{\text{in}}|_{x=\pm a} = E_y^{\text{out}}|_{x=\pm a} \) and \( E_z^{\text{in}}|_{x=\pm a} = E_z^{\text{out}}|_{x=\pm a} \). Note that the continuity of
Figure 5.1: (a) The real wave frequency as a function of wave number $k_x$ and $k_z$. (b) The corresponding wave growth rate in wave number space. All the wave numbers are normalized to the electron inertial length $c/\omega_{pe}$. All the frequencies are normalized to the electron cyclotron frequency $\Omega_e$. 
$B_x$ across the boundary is equivalent to the continuity of $E_y$ because of $B_x = -\frac{k_x}{\omega} E_y$. This boundary condition can be written explicitly as

\[
\begin{align*}
E_y^+ \exp(i k_x a) + E_y^- \exp(-i k_x a) &= E_y^{\text{out}} \exp(i k_x a) \\
E_y^+ \exp(-i k_x a) + E_y^- \exp(i k_x a) &= -E_y^{\text{out}} \exp(i k_x a)
\end{align*}
\tag{5.25}
\]

\[
\begin{align*}
E_z^+ \exp(i k_x a) + E_z^- \exp(-i k_x a) &= E_z^{\text{out}} \exp(i k_x a) \\
E_z^+ \exp(-i k_x a) + E_z^- \exp(i k_x a) &= -E_z^{\text{out}} \exp(i k_x a)
\end{align*}
\tag{5.26}
\]

Adding the conditions at $x = \pm a$ in equation (5.25) and subtracting the conditions at $x = \pm a$ in equation (5.26), we obtain

\[
\begin{align*}
(E_y^+ + E_y^-) \cos(k_x a) &= 0 \\
(E_z^+ - E_z^-) \sin(k_x a) &= 0
\end{align*}
\]

The only possible solution is

\[
\begin{align*}
E_y^+ &= -E_y^- \\
E_z^+ &= E_z^-
\end{align*}
\tag{5.29, 5.30}
\]

This pair of equations means that the mode structure of $E_y$ is odd and that of $E_z$ is even. In fact, the mode structure of $E_x$ is also odd. Equations (5.25) and (5.26) can be rewritten to relate the field inside and outside the beam through

\[
\begin{align*}
2i E_y^+ \sin(k_x a) &= E_y^{\text{out}} \exp(i k_x a) \\
2 E_z^+ \cos(k_x a) &= E_z^{\text{out}} \exp(i k_x a)
\end{align*}
\tag{5.31, 5.32}
\]

In principle, one can write two of the electric field components in terms of the third component, e.g.,

\[
\begin{align*}
\frac{E_x^+}{E_y^+} &= r^+ \\
\frac{E_z^+}{E_y^+} &= s^+
\end{align*}
\tag{5.33, 5.34}
\]
The symmetry requires that

\[
\frac{E_x^-}{E_y^-} = r^+ 
\]

\[
\frac{E_z^-}{E_y^-} = -s^+ 
\]  

(5.35) \hspace{1cm} (5.36)

Outside the beam, one has

\[
\frac{E_{x}^{\text{out}}}{E_y^{\text{out}}} = r_{\text{out}} 
\]

\[
\frac{E_{z}^{\text{out}}}{E_y^{\text{out}}} = s_{\text{out}} 
\]  

(5.37) \hspace{1cm} (5.38)

Using equations (5.34) and (5.38), we can write equations (5.31) and (5.32) as a linear system about \(E_y^+\) and \(E_y^{\text{out}}\). The determinant of this linear system must vanish, i.e,

\[
is_{\text{out}} \tan(k_{x}^{\text{in}}a) = s^+ 
\]  

(5.39)

The dispersion relation inside the beam is

\[
\det(M^{\text{in}}) = 0 
\]  

(5.40)

where

\[
M^{\text{in}} = \frac{c^2}{\omega^2} k^{\text{in}} k^{\text{in}} - \frac{(k^{\text{in}})^2 c^2}{\omega^2} I_3 + \epsilon_e + \chi_b 
\]

(5.41)

with \(k^{\text{in}} = (k_{x}^{\text{in}}, 0, k_z)\). The dispersion relation outside the beam is

\[
\det(M^{\text{out}}) = 0 
\]  

(5.42)

where

\[
M^{\text{out}} = \frac{c^2}{\omega^2} k^{\text{out}} k^{\text{out}} - \frac{(k^{\text{out}})^2 c^2}{\omega^2} I_3 + \epsilon_e 
\]

(5.43)

with \(k^{\text{out}} = (k_{x}^{\text{out}}, 0, k_z)\). Translation symmetry in the \(z\) direction implies that the spectrum of \(k_z\) is continuous. Therefore, for a given \(k_z\), equations (5.39), (5.40) and (5.42) are solved simultaneously for values of \(\omega\), \(k_{x}^{\text{in}}\) and \(k_{x}^{\text{out}}\) in the complex domain. To be able to make a comparison with the result of an infinite beam in section 5.2, the parameters of the
background plasma and the beam are kept the same. The half beam width is 10 times of the beam electron gyro-radius, i.e., \(a = 10v_{\perp0}/\Omega_e\). An overview of the numerical results is shown in Figure 5.2. The solutions are plotted in Figure 5.2a with respect to the parallel wave number \(k_z\) and the real part of the perpendicular wave number \(k_{x\perp}^{\text{in}}\) inside the beam, color coded by the growth rate. It is seen that the perpendicular wave number \(k_{x\perp}^{\text{in}}\) inside the beam is quantized, resulting from the constraint imposed by the boundary condition (5.39). The vertical dashed blue lines \(k_{x\perp}^{\text{in}}a = n\pi\) (\(n\) is an integer) are plotted as a reference. These data are re-organized into the plane of \(k_z - \omega\) in Figure 5.2b, also color coded by the growth rate. Clearly, the solutions with appreciable growth are clustered around three resonance lines \(\omega - k_z u = n\Omega_e\) with \(n = 1, 0, -1\) from left to right. Three representative mode structures of \(E_y\), taken from the cyclotron resonance mode, Landau resonance mode and anomalous resonance mode, respectively, are shown in Figure 5.3, 5.4 and 5.5. In all three plots, \(E_y\) with respect to \(x\) is displayed in the upper panel and an image of \(E_y\) in the \(x - z\) plane is shown in the lower panel. As shown in treating the boundary condition, \(E_y\) is an odd mode. The amplitude of \(E_y\) peaks in the vicinity of the boundary and decays to zero as \(x \rightarrow \pm \infty\). For the chosen cyclotron resonance mode shown in Figure 5.3, the wave has a frequency of 0.52\(\Omega_e\) and a temporal growth rate of 0.03\(\Omega_e\). The beam width is less than the perpendicular wave length inside the beam. The wave front outside the beam is oblique and converges toward the beam while the Poynting flux flows out of the beam. For the Landau resonance mode shown in Figure 5.4, the beam width is roughly 2 perpendicular wave lengths inside the beam. The wave frequency is 0.47\(\Omega_e\) and the wave growth rate is 0.04\(\Omega_e\). The wave front is oblique. In contrast to the cyclotron resonance, this mode is co-streaming with the beam in the positive \(z\) direction. For the anomalous cyclotron resonance mode shown in Figure 5.5, the beam width is roughly 5.5 perpendicular wavelength inside the beam. This mode has a frequency of 0.7\(\Omega_e\) and a growth rate of 0.04\(\Omega_e\). It is also co-streaming with the beam.
Figure 5.2: (a) The solutions in the space of the parallel wave number $k_z$ and the real part of the perpendicular wave number $k_{x}^{in}$ inside the beam. Each solution is color coded by the growth rate, with the color bar shown on the right. The vertical dashed blue lines indicate $k_{x}^{in}a = n\pi$ ($n$ is an integer) to manifest the quantization of $k_{x}^{in}$. (b) The solutions as a function of the parallel wave number $k_z$ and the wave frequency $\omega$, color coded by the growth rate. Three solid black lines, from left to right, represents $\omega - k_z u = n\Omega_e$ with $n = 1, 0, -1$. The length scale is normalized to the beam half width.
Figure 5.3: A chosen mode structure excited by cyclotron resonance. The corresponding parallel wave number $k_z$, wave frequency $\omega$ and wave growth rate $\gamma$ is displayed on the top. (a) The $y$ component of the electric field $E_y$ as a function of position $x$. The shaded region indicates where the beam is located. (b) The mode structure of $E_y$ in the $x - z$ plane.
Figure 5.4: A chosen mode structure excited by Landau resonance. The format is similar to that of Figure 5.3.
Figure 5.5: A chosen mode structure excited by anomalous cyclotron resonance. The format is similar to that of Figure 5.3.
5.4 Summary and Discussion

The linear unstable eigenmodes of whistler waves excited by a gyrating electron beam in a slab geometry are studied here. A linear instability analysis is first performed to find the resonance modes and the associated wave growth rate for an infinite beam. By matching the eigenmodes of whistler waves at the boundary for a finite beam, a complex wave frequency is solved for each wave mode and the corresponding mode structure is constructed. It is shown that the perpendicular wave number inside the beam with an appreciable growth is quantized. The parity of \( x \) and \( y \) components of the wave electric field is odd whereas that of \( z \) component of the wave electric field is even. The wave electric field peaks in the vicinity of the boundary and can leak out of the electron beam, decaying to zero at infinity.

It should be pointed out that the calculation in this study only works for a tenuous beam, i.e., \( n_b \ll n_0 \). On one hand, the validity of linear instability analysis relies on the wave growth rate being much smaller than the wave frequency, which requires \( n_b \ll n_0 \). In addition, the continuity of the displacement electric field at the boundary is ensured only when \( n_b \ll n_0 \). This means the beam does not change the dispersion properties of the background plasma significantly but only provides a small wave growth rate. The enforcement of the continuity of the displacement electric field into the computation leads to an over-determined system\(^1\).

The problem setup is self-consistent only when the beam is tenuous. Such a dilemma can be reconciled if we consider a finite beam with a smooth density profile. Under the setup of a smooth beam, the only boundary condition is that all the wave fields vanish at infinity. By expanding the wave fields and the electron distribution around the gyro-orbit of beam electrons, the linearized Vlasov equation can be integrated properly. A set of differential equations can be obtained, which defines the eigenvalue problem. This type of geometry, which is both more realistic and more complex than the one considered in this chapter, will be studied in the future.

\(^1\)For a given parallel wave number, this system has four equations but only three unknowns. Four equations are the dispersion relations inside and outside the beam, one boundary condition from the continuity of tangential electric field and the other boundary condition from the continuity of the displacement electric field. Three unknowns are the wave frequency, the perpendicular wave numbers inside and outside the beam.
CHAPTER 6

Kinetic simulations of the beam-plasma instability

6.1 Introduction

The excitation of incoherent broadband whistler waves was studied by the injection of a gyrating electron beam into a cold plasma on the Large Plasma Device. In the experiment, a measurement of the electron distribution function is needed to study the self-consistent wave-particle interactions. But such a diagnostic of the electron distribution is not available at the current stage. On the other hand, linear kinetic theory can predict the growth rate of Langmuir waves and whistler waves for a given beam distribution. But such a linear theory cannot resolve how the unstable waves modify the electron distribution and therefore cannot resolve how the fast-growing Langmuir instabilities affect the slow-growing whistler instabilities. Here, using the self-consistent Darwin particle-in-cell method described in section 2.2, we study the excitation Langmuir waves and whistler waves in a beam-plasma system and the associated evolution of the electron distribution.

6.2 Computational setup

A two dimensional beam-plasma system periodic in both directions is explored using the Darwin particle-in-cell method. The computation domain consists of \( L_x = 4096 \) grids in \( x \) direction and \( L_y = 1024 \) grids in \( y \) direction with a grid spacing of \( 0.02d_e \). Here \( d_e = c/\omega_{pe} \) is the electron inertial length. \( c \) is the speed of light and \( \omega_{pe} \) is the plasma frequency. The time step is \( 0.1\omega_{pe}^{-1} \). A uniform external magnetic field \( B_0 \) is applied in the \( x \) direction with a magnitude \( \Omega_e/\omega_{pe} = 0.2 \). In this study, the ions are immobile and form a charge neutralizing
background. A beam ring distribution is initialized in the system, which is localized in the center of the domain in $y$ direction ($|y - \frac{L_y}{2}| < \frac{L_y}{8}$) and uniform in the $x$ direction. It has a streaming velocity $U_{\parallel b}/c = 0.0766$ parallel to the magnetic field and a velocity ring concentrated at $V_{\perp b}/c = 0.0766$ in the perpendicular direction, which corresponds to an electron beam of 3 keV in kinetic energy and 45 degree in pitch angle, typical in the experiment. The beam width $L_y/4$ is about 13 times of the gyro-radius of the beam electrons, which is comparable to that in the experiment. In the beam region, the ratio between the beam density $n_b$ and the total plasma density is $n_b/(n_b + n_0) = 1/8$, where $n_0$ is background plasma density in the beam region. The background electrons form a return current that cancels the beam current in the parallel direction, i.e., $n_bU_{\parallel b} + n_0U_{\parallel 0} = 0$. Here $U_{\parallel 0} = -U_{\parallel b}/7$ is the streaming velocity of background electrons in the beam region. Aside from this small streaming velocity in the beam region, the background electrons have an isotropic Maxwellian distribution with a thermal velocity of $0.01c$ (about 50 eV in thermal temperature). Outside the beam region, the density of background electrons is $n_b + n_0$ so that the total plasma density is uniform.

6.3 The excitation of Langmuir and whistler waves and the associated evolution of the electron distribution

Figure 6.1 shows the field pattern of Langmuir waves and whistler waves of two snapshots. In Figure 6.1a, the longitudinal electric field, $\delta E_L = -\nabla \phi$, along the $x$ direction is displayed at $t = 30\omega_{pe}^{-1}$ when Langmuir waves saturate. This field pattern is dominantly due to Langmuir waves, since the electrostatic electric field energy of Langmuir waves is much larger than that of whistler waves. A Fourier analysis of the Langmuir wave field shows that substantial wave energy ranges in the parallel wave number $k_x$ of 10 - 15$\omega_{pe}/c$, corresponding to 0.42 - 0.63$d_e$ in wavelength. The perpendicular wave number $k_y$ of Langmuir waves ranges between 0 - 4$\omega_{pe}/c$ at the time of wave saturation, which is much smaller than the parallel wave number $k_x$. It is worthy to note that the Langmuir waves only exist in the beam region. In contrast, whistler waves can propagate out of the beam region, as shown by the wave magnetic field
\( \delta B_x \) in \( x \) direction in Figure 6.1b. This snapshot is taken at \( t = 100\omega_{pe}^{-1} \) when whistler waves saturate. The beam generated whistler waves have very oblique wave fronts with \( k_x = 1 - 2\omega_{pe}/c \) and \( k_y = 1 - 4\omega_{pe}/c \) based on a Fourier analysis of the wave field, corresponding to a wavelength on the order of several electron inertial lengths. To demonstrate that the energy is flowing out of the beam, the Poynting flux is integrated for all the wave modes along the \( x \) direction through the system. The \( y \) component of the integrated Poynting flux is shown in Figure 6.2. Inside the beam, the Poynting flux can be in both the \( +y \) and \( -y \) directions; while outside the beam, it is directed away from the beam indicating the energy is flowing out of the beam. The region outside of the beam in Figure 6.2a is expanded in Figure 6.2b. It is seen that the front of the Poynting flux propagates out as time is advanced.

![Figure 6.1:](image)

Figure 6.1: (a) The field pattern of longitudinal electric field along the \( x \)-direction at \( t = 30\omega_{pe}^{-1} \). (b) The field pattern of magnetic field along the \( x \)-direction at \( t = 100\omega_{pe}^{-1} \).

Now we are in a position to explore the excitation of Langmuir and whistler waves and the associated evolution of the electron distribution. The time series data of the electromagnetic fields is sampled at 32 locations centered in the \( x \) direction and equally spaced in the \( y \) direction inside the electron beam. A wavelet analysis of the parallel electric field \( \delta E_x \) and the \( y \) component of the magnetic field \( \delta B_y \) are shown in Figures 6.3a and 6.3b, respectively.
Figure 6.2: (a) The $y$ component of the integrated Poynting flux as a function $y$ position. It is color coded by different time instants indicated by the legend on the right. The beam region is between the two dashed lines. (b) An expanded display of the integrated Poynting flux for one side out of the beam.
Note that the power spectrum is averaged over 32 sampling locations to minimize its variance. The Langmuir waves at $\omega/\Omega_e = 3 - 5$ dominate over other wave modes in the power spectrum of $\delta E_x$ as shown in Figure 6.3a. They saturate in five plasma oscillations (around $t = 30\omega_{pe}^{-1}$) and gradually damp out. Whistler waves show up prominently below the electron cyclotron frequency in the power spectrum of $\delta B_y$. Around $t = 100\omega_{pe}^{-1}$ ($\sim 3$ cyclotron periods), whistler waves saturate with a primary peak at $\omega/\Omega_e = 0.6$ and a secondary peak at $\omega/\Omega_e = 0.25$. After saturation, the magnitude of these oblique whistler waves further decreases through Landau damping. To contrast the very different growth rate between Langmuir waves and whistler waves, two line cuts are taken from the wavelet spectral peaks, one at $\omega/\Omega_e = 3.5$ for Langmuir waves and the other at $\omega/\Omega_e = 0.6$ for whistler waves. The result is shown in a linear-log plot of Figure 6.4. The linear growth rate is indicated by $1/2$ of the slope in the linear part of the wave energy evolution. This linear growth rate is calculated as $0.15\omega_{pe}$ for Langmuir waves at $\omega/\Omega_e = 3.5$, and is equal to $0.015\omega_{pe}$ ($= 0.075\Omega_e$) for whistler waves $\omega/\Omega_e = 0.6$. This calculation characterizes the fastest growing Langmuir waves and relatively slow-growing whistler waves. Note that before the Langmuir wave saturates, whistler waves can also extract free energy from the inverted slope region (i.e., $\partial f_b/\partial v_{\parallel} > 0$) of the beam through Landau resonance, although the rate of such energy transfer is slower than that for the Langmuir wave as shown in Figure 6.4. After the Langmuir wave saturates, whistler waves can only be excited through cyclotron resonance since the free energy from $\partial f_b/\partial v_{\parallel} > 0$ has been exhausted by the Langmuir instability. Correspondingly, the electron distribution responds to the Langmuir and whistler instabilities on two different time scales. Figure 6.5 shows the electron histogram in the velocity space of $v_{\parallel} - v_{\perp}$ at four representative snapshots. Note that the electrons are counted over the entire computation domain. To begin, the distribution is initialized as the core electrons and the beam ring electrons (Figure 6.5a). Shortly before the Langmuir wave saturation at $t = 28\omega_{pe}^{-1}$, the beam electrons are trapped and relaxed by the Langmuir waves in the parallel direction (Figure 6.5b). As the the magnitude of the Langmuir wave grows, the width of its resonant island broadens in $v_{||}$ due to $\Delta v_{\parallel} \propto \sqrt{\delta E}$, where $\Delta v_{\parallel}$ is the width of the resonant island and $\delta E$ is the Langmuir wave amplitude. This large amplitude Langmuir wave becomes
resonant with, and traps the tail of the core electrons and subsequently gets the tail of the core electrons accelerated to the beam energy level, as shown in Figure 6.5c at $t = 35\omega_{pe}^{-1}$. At a later time, the relaxed beam electrons are scattered to lower pitch angles and lose energy, through which whistler waves gain energy further and grow in magnitude. This is shown in Figure 6.5d taken at $t = 100\omega_{pe}^{-1}$ when the whistler waves saturate.

Figure 6.3: (a) The power spectrum of $\delta E_x$ evolving as a function of time. (b) The power spectrum of $\delta B_y$ evolving as a function of time.

6.4 The suppression of beam whistler instabilities by Langmuir wave

The growth of whistler-mode waves through Landau resonance is limited by the growth of Langmuir waves. The fast growing Langmuir waves saturate rapidly in a few plasma oscilla-
Figure 6.4: The evolution of the power spectral density as a function of time. The power spectral density of $\delta E_x$ at $\omega/\Omega_e = 3.5$ is shown as the red line with the $y$ axis on the left. The power spectral density of $\delta B_y$ at $\omega/\Omega_e = 0.6$ is shown as the blue line with the $y$ axis on the right.

Figure 6.5: The electron number histogram in the velocity space of $v_\parallel - v_\perp$ at four selected time instants: (a) $t = 0$; (b) $t = 28\omega_{pe}^{-1}$; (c) $t = 35\omega_{pe}^{-1}$; (d) $t = 100\omega_{pe}^{-1}$.
tions and deplete the beam free energy in the parallel direction through Landau resonance. Whistler waves saturate soon after the saturation of Langmuir waves since there is little free energy left for the Landau resonant excitation of whistler waves. Such a competition between Langmuir and whistler instabilities depends on $\omega_{pe}/\Omega_e$, which characterizes the ratio between the linear growth rate of Langmuir instabilities and that of whistler instabilities. To minimize the effect of cyclotron resonance, a field-aligned electron beam is used here while the rest of the setup is kept the same. Figure 6.6a shows the magnetic field energy of whistler waves with respect to time for a set of $\omega_{pe}/\Omega_e$ values. Each of the color-coded lines corresponds to the colored spot in Figure 6.6b, in which the ratio of the saturated magnetic field energy to initial magnetic field energy is shown as a function of $\omega_{pe}/\Omega_e$. Under the special scenario of $\omega_{pe}/\Omega_e = 1$, whistler waves and Langmuir waves saturate over the same time scale and whistler waves saturate at a substantially larger amplitude compared to other cases. As $\omega_{pe}/\Omega_e$ increases, the saturated whistler wave energy decreases and eventually is immersed in the noise level beyond $\omega_{pe}/\Omega_e = 7$. Linear theory predicts that Landau resonance between whistler waves and the electron beam does not occur beyond a critical value of $(\omega_{pe}/\Omega_e)_{\text{critical}} = 6.5$ for our parameter regime. This inhibits the energy transfer between the beam electrons and whistler waves and results in a low signal to noise ratio in the high $\omega_{pe}/\Omega_e$ regime. Below the critical value of $\omega_{pe}/\Omega_e$, Langmuir instabilities limit the saturation energy level of whistler instabilities by extracting the free energy of the beam at a faster rate than the whistler instabilities as long as $\omega_{pe}/\Omega_e > 1$.

6.5 Summary and discussion

Using the self-consistent Darwin particle-in-cell method, we study the excitation of Langmuir and whistler waves in a beam-plasma system. The Langmuir wave grows in magnitude rapidly and saturates in a few plasma oscillations, while the electron beam is slowed down and relaxed in the parallel direction. As the amplitude of Langmuir waves approaches saturation, resonance with the tail of the background core electrons occurs and accelerates them parallel to the background magnetic field. Whistler waves grow in magnitude and
Figure 6.6: (a) the evolution of magnetic field energy of whistler waves as a function of time. From black-blue line to orange-red line, the corresponding values of $\omega_{pe}/\Omega_e$ are 1, 2, 3, 4, 5, 7, 9, 10. (b) Corresponding to each run in (a), the ratio of saturated energy to initial energy is shown as a function of $\omega_{pe}/\Omega_e$. Each colored spot corresponds to the line of the same color in panel (a).
saturate over the time scale of a few cyclotron periods. They are excited through Landau resonance and cyclotron resonance. In terms of wave propagation, the Langmuir waves are localized to the beam region, whereas whistler waves can leak out of the beam and transport energy away from the beam. Finally, the competition between Langmuir and whistler instabilities are tested for a field-aligned beam. Due to a faster depletion of the beam free energy by Langmuir waves with increasing $\omega_{pe}/\Omega_e$, the saturation amplitude of whistler waves decreases. Beyond a critical $\omega_{pe}/\Omega_e$, Landau resonance does not occur for whistler waves and the saturation amplitude of whistler waves is immersed in the noise.
CHAPTER 7

An oscillator model representative of electron interactions with electromagnetic ion cyclotron waves

Nonlinear interactions of relativistic electrons with a monochromatic electromagnetic ion cyclotron wave were investigated in previous studies by solving the Lorentz equation for test particles. In the present study, we simplify the Lorentz equation to a set of oscillator equations, which capture the key features of the wave-particle interaction. These reduced oscillator equations confirm that the inhomogeneity ratio plays an important role in determining the nonlinear nature of the interaction with a monochromatic wave. When considering the case of two discrete wave elements, three general regimes were identified in the nonlinear interactions, depending on the separation of the two wave numbers: There were 1) a decoupled regime in which electrons pass only one of the two distant resonance points, 2) a coupled regime in which electrons pass both of the two moderately separated resonance points and 3) a degenerate regime in which electrons pass two closely situated resonance points. The decoupled and degenerate regimes in the nonlinear interaction are characterized by phase bunching. However, the electrons experience alternate trapping and de-trapping near the separatrix in the coupled regime, leading to chaotic motion. The linear interaction region is also considered, where the linear superposition principle is tested. The linear approximation results generally agree with the two-wave simulation except in the coupled regime.

7.1 Introduction

Quasi-linear theory assumes that waves are broadband and the wave amplitudes are small [Kennel and Engelmann, 1966]. However, Albert and Bortnik [2009] showed, using an inho-
mogeneity ratio to map out the nonlinear region, that the electrons can respond nonlinearly to a monochromatic EMIC wave with a typical amplitude of 2 nT. Nonlinear phase bunching leads to advection away from the loss cone, while nonlinear phase trapping induces advection toward the loss cone. The nonlinear interactions are further complicated due to the broadband spectrum of EMIC waves [Liu et al., 2012] or alternatively a coherent EMIC wave with a variable frequency [Omura and Zhao, 2012]. The quasi-linear diffusion theory and test particle simulations were compared in a recent study to determine the amplitude threshold when quasi-linear diffusion theory breaks down for EMIC waves [Su et al., 2012]. Su et al. [2013] also introduced the latitudinal dependence of nonlinear interactions between EMIC and relativistic electrons.

A realistic EMIC wave band contains multiple frequency components, which should be considered in the nonlinear wave particle interaction. Tao et al. [2013] established a two-wave model to investigate the effects of amplitude modulation on nonlinear interactions between electrons and large amplitude whistler-mode waves. As the first step of an investigation of the interaction between electrons and a realistic EMIC wave packet, in the present work we follow the procedure of Tao et al. [2013] and investigate how the electron behavior changes for a two-wave EMIC model, compared with a one-wave model.

### 7.2 The oscillator dynamical system

We consider multiple EMIC waves with finite amplitudes propagating along the dipole field line towards the northern hemisphere. The relativistic electrons move in the same direction as the EMIC waves in order to undergo cyclotron resonance. For realistic magnetospheric conditions, the following relations hold for relativistic electrons traversing the EMIC wave field:

\[
\frac{\omega}{k} \ll \frac{p_\parallel}{\gamma m}, \quad \frac{|B_w|}{B} \ll 1, \quad \text{and} \quad \left| \frac{\omega}{\Omega_e} \right| \ll 1
\]  

(7.1)

\(\omega\) and \(k\) are the wave frequency and wave number, respectively, \(p_\parallel = \gamma m v_\parallel\) is the parallel momentum of the electrons, \(\gamma\) is the relativistic factor and \(m\) is the mass of the electrons. \(B_w\) and \(B\) are the magnitudes of the wave magnetic field and background magnetic field,
respectively. $\Omega_e = \frac{eB_{\text{max}}}{mc}$ is the unsigned electron gyro-frequency, where $e$ is the magnitude of the elemental charge and $c$ is the light speed. By using these relations, we obtain the simplified Lorentz equation [Matsumoto and Omura, 1981; Chang and Inan, 1983]:

$$\frac{dp_\|}{dt} = \sum_i e (B_w)_i \frac{p_\perp}{c} \frac{\sin \zeta_i}{\gamma m} - \frac{p_\perp^2}{2\gamma mB} \frac{\partial B}{\partial z}$$

$$\frac{dp_\perp}{dt} = \sum_i -e (B_w)_i \frac{p_\|}{c} \frac{\sin \zeta_i}{\gamma m} + \frac{p_\perp p_\|}{2\gamma mB} \frac{\partial B}{\partial z}$$

$$\frac{d\zeta_i}{dt} = \frac{\Omega_e}{\gamma} - k_i \frac{p_\|}{\gamma m}$$

(7.2)

where $p_\perp$ is the momentum of electrons perpendicular to the background magnetic field. The subscript $i$ represents the quantity corresponding to the $i^{th}$ wave. Thus $(B_w)_i$ is the magnitude of $i^{th}$ wave magnetic field. The phase angle $\zeta_i$ is the angle from the $i^{th}$ wave magnetic field vector $(B_w)_i$ to the perpendicular momentum vector $p_\perp$. $k_i$ is the wave number of the $i^{th}$ wave. The total momentum $p = \sqrt{p_\|^2 + p_\perp^2}$ is roughly constant since the wave electric force is much smaller than the wave magnetic force. With the definition of $\theta_i = \frac{\Omega_e}{\gamma} - k_i \frac{p_\perp}{\gamma m}$, the Lorentz equation can be written as a set of oscillator equations [Vomvoridis and Denavit, 1979; Matsumoto and Omura, 1981]

$$\frac{d\zeta_i}{dt} = \theta_i$$

$$\frac{d\theta_i}{dt} = \omega_{ti}^2 \left( R_i - \sum_{j=1}^N \epsilon_j \sin \zeta_j \right)$$

(7.3)

where the trapping frequency $\omega_{ti}^2 = k_i \frac{\Omega_e}{\gamma} \frac{p_\perp}{\gamma m}$ and the inhomogeneity ratio $R_i = \frac{1}{\omega_{ti}^2} \left[ \frac{1}{\Omega_e} \frac{\partial \Omega_e}{\partial z} \left( \Omega_e \frac{p_\|}{\gamma m} + k_i \frac{p_\perp^2}{2\gamma^2 m^2} \right) \right]$. The maximum magnitude of all the discrete waves is $B_{w}^{\text{max}} = \max_{1 \leq j \leq N} (B_w)_j$. $\epsilon_j = \frac{(B_w)_j}{B_{w}^{\text{max}}}$ is the relative magnitude of the $j^{th}$ wave to the maximum magnitude of all the discrete waves. $\Omega_e$ is defined as $\frac{eB_{w}^{\text{max}}}{mc}$. To make the problem tractable, the following assumptions are made: (1) the magnitudes of the multiple waves are the same, i.e., $\epsilon_j = 1$ for $j = 1, \cdots, N$. (2) both the inhomogeneity ratio and the trapping frequency are constants. (3) the inhomogeneity ratio is the same for each wave, i.e. $R_1 = R_2 = \cdots = R_N = R$.

An assumption made in the analysis is that the inhomogeneity ratio $R$ is a constant for any specific wave and that is essentially the same for each wave. With this assumption, the
interaction of multiple waves with electrons reduces to a two-degree-of-freedom Hamiltonian
system (as shown in equation (7.5)), which has been analyzed in a number of related fields
of physics [equation 5.9 of Chirikov, 1979; Escande and Doveil, 1981; Month and Herrera,
1979; Laval and Gresillon, 1979] and stands to benefit from insights gained from those areas.
The inhomogeneity ratio $R$ only applies a “mirror force” to the electrons. For the interaction
of one-wave with electrons, the dynamics comes out of the competition between the wave
term and $R$. While in the interaction of two-waves with electrons, the dynamics come out of
the coupling between the two wave terms as shown in section 7.4. The simplified oscillator
model is not intended to simulate the wave-particle interaction in every detail, but to capture
the fundamental modes of the wave-particle interaction in the case of multiple resonances.

We normalize time in equation (7.3) by the $N^{th}$ wave trapping frequency $\omega_{tN} \mapsto t$
and hence $\theta_i/\omega_{tN} \mapsto \theta_i$. The phase and the rate of change of the phase
$(\zeta_1, \theta_1; \zeta_2, \theta_2; \cdots; \zeta_N, \theta_N)$
are not a set of independent variables. Only two independent variables exist, which are
chosen to be $\zeta_N$ and $\theta_N$. $\zeta_i$ and $\theta_i$ can be represented in terms of the two independent
variables,

$$\begin{align*}
\theta_i &= \kappa_i \theta_N + \nu_i \\
\zeta_i &= \kappa_i \zeta_N + \nu_i t + \phi_i
\end{align*}$$

(7.4)

where $\nu_i = (1 - \kappa_i) \left( \frac{V_R}{V_t} \right)_N$, $\phi_i = (\zeta_i)_0 - \kappa_i (\zeta_N)_0$, $\kappa_i = \frac{k_i}{k_N}$. The subscripts $i$ and $N$ indicate
the quantity of the $i^{th}$ and $N^{th}$ waves, respectively. $(\zeta_i)_0$ and $(\zeta_N)_0$ are values of $\zeta_i$ and $\zeta_N$
at $t = 0$. $V_R$ is the resonant velocity given by $V_R = \frac{\Omega}{\gamma k}$. $V_t$ is the trapping velocity given
by $V_t = \frac{\omega}{k}$ [Matsumoto and Omura, 1981], which is equivalent to the width of the resonant
island in the phase portrait. $\frac{V_R}{V_t}$ is on the order of 10 for typical magnetospheric conditions.
For example, for electrons with kinetic energy 2 MeV ($\gamma = 5$) and helium band EMIC
waves with magnitude 2 nT and wave number $10^{-6}$ cm$^{-1}$ at $L=4$, the resonant velocity and
trapping velocity are 0.56$c$ and 0.047$c$, respectively.

The final oscillator dynamical system describing a relativistic electron in multiple EMIC
waves can be described as

\[ \frac{d\zeta}{dt} = \theta \]
\[ \frac{d\theta}{dt} = R - \sum_{j=1}^{N} \sin (\kappa_j \zeta + \nu_j t + \phi_j) \] (7.5)

where we suppress the subscript \( N \) for \((\zeta, \theta)\). The first equation gives the rate of change of the phase. The second equation is the dynamical equation, where the first term is the inhomogeneity ratio that represents the spatial variations in the system, including the external adiabatic force, and the second term is the force of multiple waves, which includes the main dynamical properties of the system.

The resonance condition for the \( i^{th} \) wave is \( \theta_i = 0 \), which corresponds to

\[ \hat{\theta} = -\frac{\nu_i}{\kappa_i} = - \left( \frac{V_R}{V_t} \right)_N \frac{1 - \kappa_i}{\kappa_i} \] (7.6)

\( \hat{\theta} \) represents the resonance point of the electron with the \( i^{th} \) wave, where we attach a hat to \( \theta \) to distinguish the resonance point from the ordinary phase velocity notation. The \( i^{th} \) resonance point is located at the phase velocity of the \( i^{th} \) wave potential, which is stationary as seen by the particles at resonance with the \( i^{th} \) wave. Specifically, the resonance point is located at \( \hat{\theta} = 0 \) for the \( N^{th} \) wave. When \( \kappa_i \to 0 \), the \( i^{th} \) wave drops out of resonance, i.e., \( \hat{\theta} \to -\infty \). When \( \kappa_i \to 1 \), the \( i^{th} \) resonance point becomes degenerate with the \( N^{th} \) resonance point.

Equation (7.5) includes two key parameters, \( R \) and \( \kappa \). The inhomogeneity ratio \( R \) determines the region of nonlinear interaction. The wave number spacing \( \kappa \) controls the distances of different resonance points as shown in equation (7.6) and determines the regime in which the system is located. We will demonstrate the important role of these parameters in the following sections.
7.3 Single wave oscillator dynamical system

For the single wave-particle interactions, equation (7.5) becomes

\[
\begin{align*}
\frac{d\zeta}{dt} &= \theta = \frac{\partial H}{\partial \theta} \\
\frac{d\theta}{dt} &= R - \sin \zeta = -\frac{\partial H}{\partial \zeta}
\end{align*}
\]  

(7.7)

The Hamiltonian \( H = \frac{\theta^2}{2} + (-R\zeta - \cos \zeta) \) of the system is a conserved quantity [Vomvoridis and Denavit, 1979], i.e. \( \frac{dH}{dt} = 0 \). The first term may be seen as the kinetic energy while the second term, \( V(\zeta) = -R\zeta - \cos \zeta \), plays the role of potential energy. The value of the inhomogeneity ratio \( R \) compared to unity determines the shape of the potential and hence the motion of the electron.

The utility of \( R \) in mapping out the nonlinear interaction region is shown in Figure 7.1. We specify \( R \) to solve the ordinary differential equation (ODE) set (7) numerically. \( R \) is chosen as 0.1, 1 and 5, respectively. In each run, 24 electrons, with initial phase \( \zeta_0 \) uniformly distributed between 0 and \( 2\pi \), are released with the same initial rate of change of phase \( \theta_0 = -10 \).

For \( R = 0.1 \) as shown Figure 7.1b, the potential has a weak gradient with relatively deep periodic wells. The adiabatic trajectory is obtained by only considering a constant acceleration represented by the constant inhomogeneity ratio \( R \), which is the dashed black line in Figure 7.1a. The wave induced change is most pronounced in the vicinity of the resonance. While the electrons are away from the resonance point, the wave induced motion could be seen as small perturbations over the adiabatic motion. Dramatic deviation from the adiabatic trajectory occurs in the vicinity of the resonance point, \( \hat{\theta} = 0 \), where the potential matches the constant Hamiltonian. During resonance, the direction of the rate of change of \( \zeta \) reverses, in analogy to the familiar situation where particles bounce back in the potential well when the total energy matches the potential. Due to the shape of the potential, the resonance only takes place in a limited range of phase angles where \( -\frac{\partial V}{\partial \zeta} > R \) (the square covered range in Figure 7.1b). Thus all the electrons become bunched over this range of phase angles at resonance and experience advection to higher \( \theta \) (\( \frac{d\theta}{dt} > R \) according equation...
(7.7)) compared to the adiabatic trajectory.

For $R = 1$ as shown in Figure 7.1d, the potential has periodic plateaus which cause pseudo-trapping. We call this effect pseudo-trapping because there is no potential well that really traps the particle. The pseudo-trapped particles with the Hamiltonian being near the value of potentials at plateaus spends an extended time period just above the shallow plateau in the vicinity of resonance point (Figure 7.1c). Pseudo-trapping leads to negative changes from the adiabatic trajectory, which is opposite to that of phase bunching.

For $R = 5$ as shown in Figure 7.1f, the potential is dominated by the adiabatic term with little deviations caused by the wave term. The phase angle at the resonance point is distributed almost uniformly a function of the initial phase angles. This leads to essentially stochastic diffusion in $\theta$ after resonance compared to an adiabatic trajectory.

Albert and Bortnik [2009] use the inhomogeneity ratio $R$ to indicate the nonlinearity of the wave particle interaction. $R \ll 1$ represents strong nonlinear motion, i.e., phase bunching, while $R \gg 1$ gives linear stochastic motion. We confirm this by using the single wave oscillator analysis. No phase trapping is observed for the parameters adopted in current simulation. Phase trapping can occur only when the electrons are initially trapped in the potential well, i.e., when the resonance condition is initially satisfied or nearly satisfied. In terms of the phase portrait, the electrons cannot cross over the separatrix that defines regions of trapped vs. untrapped electrons when $R \ll 1$. Phase trapping appears in the two-wave simulation considered in the next section.

7.4 Two-wave oscillator dynamical system

Considering the two-wave oscillator system, equation (7.5) becomes

$$\frac{d\zeta}{dt} = \theta$$
$$\frac{d\theta}{dt} = R - \sin \zeta - \sin (\kappa \zeta + \nu t + \phi)$$  (7.8)

with $\kappa = \frac{k_1}{k_2}$, $\nu = (1 - \kappa) \frac{V_R}{V_i}$, $\phi = (1 - \kappa) \zeta_0$, where $\kappa$ is the ratio of two wave numbers and $\zeta_0$ is the initial phase angle. In the following, we call the wave with small wave number the
Figure 7.1: Particle trajectories (a, c, e) and corresponding potentials (b, d, f) in three typical interaction regions of a single wave oscillator dynamical system. a) Nonlinear interaction region with $R = 0.1$; b) Transition region with $R = 1$; c) Linear interaction region with $R = 5$. Each left panel contains color-coded 24 electron trajectories with initial phase uniformly distributed between 0 and $2\pi$. The dashed black lines in (a) are the adiabatic trajectory. The adiabatic trajectories in (c) and (e) are the same as (a). The thin black lines in the left panels are resonance points located at $\hat{\theta} = 0$. Time in the left panels is a dimensionless quantity which is normalized by the inverse of trapping frequency $\omega_t^{-1}$. The squared area in (b) are the limited range where phase angles are possibly bunched.
first wave, and the wave with the larger wave number the second wave.

We solve this set of dynamical equations numerically both in the nonlinear \((R = 0.1)\) and linear \((R = 5)\) regions, with initial conditions \(\theta_0 = -10\) and \(\zeta_0\) uniformly distributed between 0 and \(2\pi\). We choose \(\frac{V_R}{V_t} = 10\), which is typical for magnetospheric conditions.

### 7.4.1 Nonlinear interaction region \((R = 0.1)\)

In the nonlinear interaction region, three general regimes can be identified: decoupled, coupled and degenerate. The representative trajectories of three regimes are shown in Figure 7.2, for \(\kappa = 0.01\) (Figure 7.2a), \(\kappa = 0.8\) (Figure 7.2b) and \(\kappa = 0.99\) (Figure 7.2c), respectively. In the left panel, \(\kappa\) equals 0.01, which indicates the second wave number is much larger than the first wave number. The first wave is out of resonance and the resonance point of the second wave is located at \(\hat{\theta} = 0\). The behavior of the electrons is characterized by phase bunching. The electron trajectories are advected to higher \(\theta\) compared to the adiabatic trajectory (the thick blue line) around the resonance point of the second wave. Only the second wave is responsible for the scattering of the electrons. In the middle panel, the two resonance points are moderately separated (indicated by two horizontal lines representing two resonance conditions respectively). The electrons pass through both resonance points. A portion of the electrons still undergo phase bunching, while other electrons are phase trapped by either of the two waves around the resonance points and become de-trapped subsequently. The de-trapping of the electrons is essentially random since small perturbations can drastically change the trajectories of electrons trapped near the separatrix of the resonant island in the phase portrait. Thus trapping and de-trapping characterize the particle behavior. The random de-trapping makes the scattering result more stochastic in nature, compared to the deterministic advection of the \(\kappa = 0.01\) case. Therefore, in the nonlinear coupled regime, the two-wave scattering results have a distinct nature from that of single wave. In the right panel, two wave numbers are almost the same. The scattering result is similar to a single wave. The advection is enhanced compared to the left panel because the resonant wave amplitude is larger. We identify this as the degenerate case.
Figure 7.2: Representative particle trajectories of three typical interaction regimes in the nonlinear interaction region of the two-wave oscillator dynamical system with $R = 0.1$. a) Decoupled regime; b) Coupled regime; c) Degenerate regime. Each panel contains 24 electron trajectories with initial phase uniformly distributed between 0 and $2\pi$, which are coded with 24 colors. The thick blue lines in each panel are the adiabatic trajectories. The resonance point of the second wave is indicated by the horizontal black line located at $\hat{\theta} = 0$ in each panel. The resonance point of the first wave moves from $\hat{\theta} = -\infty$ to $\hat{\theta} = 0$ as $\kappa$ goes from 0 to 1. The resonance point of the first wave is out of view in panel a. The resonance point of the first wave is below $\hat{\theta} = 0$ in panel b, which is indicated by the thin black line. The resonance point of the first wave merges with that of the second wave at $\hat{\theta} = 0$ in panel c. Time is a dimensionless quantity which is normalized by the inverse of the second wave trapping frequency $\omega_{r2}^{-1}$. 

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To show the dependence of $\theta$ change on the wave number spacing $\kappa$, we vary $\kappa$ from 0 to 1 in steps of 0.01 for fixed $R = 0.1$. Each run contains 24 electrons with uniform initial phase angles between 0 and $2\pi$. We obtain $\Delta \theta$ by subtracting the final value of adiabatic $\theta$ from the final scattering results of $\theta$ for each electron. In Figure 7.3, three general regimes are clearly shown: 1) the decoupled regime with $\Delta \theta > 0$, for the $\kappa$ range from about 0 to 0.6; 2) the coupled regime with both $\Delta \theta > 0$ and $\Delta \theta < 0$, $\kappa$ ranging from about 0.6 to 0.98; 3) the degenerate regime with $\Delta \theta > 0$, for $\kappa$ close to 1. Note that the transition between the two regimes is gradual, also confirmed in a related study that involves a superposition of two whistler waves [Tao et al., 2013]. Consequently, it is difficult to define a specific range for each regime. There is a portion of electrons clustered around $\kappa = 0.5$ with $\Delta \theta < -20$ due to phase trapping by the first wave. The particle velocity oscillates around the resonance point of the first wave (not shown here), which is the character of phase trapping.

7.4.2 Linear interaction region ($R = 5$)

The electrons experience a stochastic scattering if the adiabatic force dominates over the wave forces, i.e., the linear limit. In principle, we can approximate the effect of multiple waves by linearly superimposing the effect of each individual wave. We attempt to apply this principle to the two-wave oscillator dynamical system in the linear interaction region.

The inhomogeneity ratio is set to 5 in the present simulation, which is located in the linear interaction region. All other parameters are kept the same as that in the nonlinear interaction region simulation. The distribution of the scattered electrons is shown in Figure 7.4. The left panel shows the simulation for the two waves while the right panel shows the result of a linear approximation. In the linear approximation, we consider the scattering effect of one wave at a time and add the final scattering results of the two waves together for each electron. Again three general regimes show up in Figure 7.4a. The transition from one regime to another is gradual. We can identify the coupled regime by comparing the deviation of the simulation results from the linear superposition results. However, the specific range of the coupled regime is not clearly defined here. The decoupled regime occurs for $0 < \kappa < 0.5$ where only
Figure 7.3: The distribution of the scattering result of $\Delta \theta$ with respect to wave number spacing $\kappa$ in the nonlinear interaction region of the two-wave oscillator dynamical system with $R = 0.1$. For each $\kappa$, 24-color coded squares represent electrons with 24 uniform initial phases distributed between 0 and $2\pi$. The decoupled regime is located at about $0 < \kappa < 0.6$, coupled regime about $0.6 < \kappa < 0.98$ and degenerate regime close to $\kappa = 1$. 
the second resonant wave is responsible for the magnitude of diffusion and the first wave is out of resonance with electrons. The coupled regime is located around $\kappa = 0.5$ where the linear approximation deviates from the two-wave simulation. The range of coupled regime shrinks compared to that in the nonlinear interaction. In the degenerate regime $0.5 < \kappa < 1$, diffusion shows magnitude modulation. The total amplitude of two resonant waves at the resonance point determines the magnitude of the diffusion. Since the total amplitude is modulated by the difference of two closely spacing wave numbers at the resonance point, the magnitude of diffusion shows similar modulation pattern in the degenerate regime. The maximum diffusion magnitude is about double that in the decoupled regime, because the wave amplitude is about twice larger. Overall, the linear approximation is similar to the two-wave simulation except in the coupled regime.

### 7.5 Summary

We introduced an oscillator dynamical system to represent the key elements of multiple wave-particle interactions, and analyzed a special case of a single wave system and a more general case of a two-wave system. In the single wave oscillator dynamical system, the inhomogeneity ratio determines the potential shape in the Hamiltonian formulation. The interaction region is categorized as follows: 1) $R \ll 1$, phase bunching causes deterministic motion of the electrons; 2) $R \sim 1$, pseudo-phase-trapping produces scattering results that deviate significantly from phase bunching; 3) $R \gg 1$, the electrons undergo a stochastic process at the resonance point. For the two-wave oscillator dynamical system, we confirm the importance of the inhomogeneity ratio and identify three general regimes in the nonlinear interaction region:

1) The decoupled regime. In this regime the two resonance points are far apart. They affect the particle trajectory independently.

2) The coupled regime. In this regime the separation of two resonance points is moderate. The two-wave oscillator enables the trapping and the de-trapping of the particles near the separatix. Whether the particles are trapped or de-trapped is random. This makes the
Figure 7.4: The comparison between the distribution of the scattering result obtained from a) two-wave simulation b) linear superposition, with respect to wave number spacing $\kappa$ in the linear interaction region of the two-wave oscillator dynamical system with $R = 5$. 24-color coded lines represent electrons with 24 uniform initial phases distributed between 0 and $2\pi$. 
scattering results appear more stochastic rather than as a deterministic advection. It is
quite different from that of the single wave nonlinear interactions.

3) The degenerate regime. In this regime two resonance points are nearly overlapped. Both
waves are responsible for the nonlinear phase bunching at the resonance point. This makes
the advection larger than that in the decoupled regime.

Furthermore, we test the linear superposition principle in the linear interaction region
using the two-wave oscillator dynamical system. The linear approximation result generally
agrees with the two-wave simulation except in the coupled regime.

In the current work, we reduce the Lorentz equation to an oscillator dynamical system,
which captures the essential physical picture of a particle interacting with multiple-waves.
This is the first step toward understanding the interactions between electrons and a realistic
EMIC wave packet. A realistic EMIC wave spectrum can be modeled as a finite band
including multiple discrete frequencies. For the case of a finite number of monochromatic
waves but with sufficient large amplitude for each wave, the neighboring resonant island
is largely overlapped, leading to the coupled regime as we discussed in the two-wave case.
Although chaotic motion also arise in the coupled regime, this motion is nonlinear and
cannot be described in terms of the quasilinear diffusion, because of the dependence of
neighboring resonances on each other. For another limit of sufficiently large number of
discrete monochromatic waves with much smaller amplitude for each wave, the motion can
be well described by quasilinear theory [Kennel and Engelmann, 1966]. The threshold of the
transition between these two limits was investigated for both EMIC wave [Su et al., 2012]
and whistler mode waves [Tao et al., 2012b]. Reality should be somewhere between these
two limits, where global stochastic motion arises.
CHAPTER 8

A regime map for electron interactions with 
electromagnetic ion cyclotron waves

In this chapter, we compare quasi-linear theory (QLT) with test particle simulation results for electron interactions with electromagnetic ion cyclotron waves, and describe the particle dynamics for different wave spectra. In a parameter space comprised of wave amplitude and wave bandwidth, the particle dynamics can be divided into 4 regimes: (1) small amplitude and broadband spectrum, (2) small amplitude and narrowband spectrum, (3) large amplitude and broadband spectrum, (4) large amplitude and narrowband spectrum. The small amplitude and broadband wave spectrum regime satisfies the weak turbulence assumption and can hence be described by QLT. The small amplitude and narrowband wave spectrum in the Earth’s dipole field leads to “point-resonant” type diffusion, which is equivalent to the narrowband limit of stochastic type diffusion. The large amplitude and broadband wave spectrum gives rise to incomplete phase randomization of electrons, which cannot be described by QLT. But QLT correctly predicts the particle distribution in this regime due to strong scattering by large amplitude waves. Finally, QLT fails in the large amplitude and narrowband regime because of nonlinear phase trapping and phase bunching. A regime map is constructed to indicate where QLT can apply in the wave parameter domain.

8.1 Introduction

The interactions between electrons and broadband EMIC waves in geospace are traditionally treated by quasilinear theory (QLT) [Thorne and Kennel, 1971; Lyons and Thorne, 1972]. However, nonlinear effects due to phase-bunching and phase-trapping by a coherent
narrowband wave spectrum [Albert and Bortnik, 2009; Omura and Zhao, 2012, 2013; Liu et al., 2012; Su et al., 2012] are of particular interest since the report of narrowband EMIC rising tone emissions [Pickett et al., 2010; Nakamura et al., 2014]. The electrons undergo stochastic motion in velocity space driven by a broadband wave spectrum [Tao et al., 2011; Liu et al., 2010], whereas they execute regular motion driven by coherent wave packets or by a monochromatic infinite wave train. Two important parameters in controlling the dynamics of wave-particle interactions are found to be wave bandwidth and wave amplitude [Tao et al., 2012b,a, 2013; Liu et al., 2012]. This study is an extension of Tao et al. [2012b] and An et al. [2014], aimed at exploring the dynamics of wave-particle interactions in a broader parameter space comprised of wave bandwidth and wave amplitude. Particularly, we are interested in which region of parameter space is valid for the use of QLT.

8.2 Simulation model

8.2.1 Background plasma and wave spectrum

The background magnetic field is a simplified dipole field in the Cartesian coordinate system [Tao et al., 2012b]. In this coordinate system, $z$ is the distance along the field line from the equatorial plane to the observation point. The $z$-component of the field is $B_{0z}(\lambda) = B_{0z}(\lambda = 0) \sqrt{1 + 3 \sin^2 \lambda} / \cos^6 \lambda$ with $dz = LR_E \sqrt{1 + 3 \sin^2 \lambda} \cos \lambda d\lambda$ giving the conversion between latitude $\lambda$ and the $z$ coordinate. Here $R_E$ is the Earth’s radius. The L-shell number is 6 in this study and $B_{0z}(\lambda = 0) \approx 139\text{nT}$ correspondingly. The $x$ and $y$-components are constructed as $B_{0x} = -x(dB_{0z}/dz)/2$ and $B_{0y} = -y(dB_{0z}/dz)/2$ to ensure $\nabla \cdot \mathbf{B}_0 = 0$. We adopt a typical storm-time ion composition with $70\%$ H$^+$, $20\%$ He$^+$ and $10\%$ O$^+$ [Meredith et al., 2003], that are assumed to be cold. The cold electron density varies as $n_e = n_{e0} \cos^{-4} \lambda$ [Denton et al., 2002] with equatorial electron density $n_{e0} = 400\text{cm}^{-3}$ [Meredith et al., 2003]. We consider an EMIC wave in the helium band, due to its high occurrence rate [Meredith et al., 2014]. EMIC waves propagate from the equatorial plane to both hemispheres along the field line [Meredith et al., 2003]. The wave field is confined within $10^\circ$ from the equatorial plane.
The method of the discretization of a continuous wave spectrum is given by Tao et al. [2012b]. Here we only recap a few key steps. A discretized spectrum is used to represent a continuous wave spectrum in computer simulations. For a parallel propagating left-hand polarized EMIC wave, the wave field is described as

\[
B_W = \sum_{j=1}^{N_W} -B_{jx} W \sin \Phi_j e_x + B_{jy} W \cos \Phi_j e_y
\]

\[
E_W = \sum_{j=1}^{N_W} E_{jx} W \cos \Phi_j e_x + E_{jy} W \sin \Phi_j e_y
\]  

(8.1)

Here \(N_W\) is the total number of discrete wave frequencies. The spacing of \(\omega_j\) is discussed in the next subsection of test particle simulation. The wave phase of the \(j^{th}\) component is defined as \(\Phi_j = \int_0^z k_j dz' - \omega_j t + \Phi_{0j}\) with \(\Phi_{0j}\) randomly generated between 0 and 2\(\pi\). For a given \(y\)-component of wave magnetic field \(B_{jy} W\), we specify the other three amplitudes by using Faraday’s Law. They are given as \(B_{jx} W = B_{jy} W\) and \(E_{jx} W = E_{jy} W = B_{jy} W / n_j\), where the refractive index \(n_j\) is equal to \(k_j c / \omega_j\). The wave number \(k_j\) is determined from the wave frequency \(\omega_j\) using the cold plasma dispersion relation for L-mode waves [Stix, 1992]. It is worth noting that \(E_{jx} W\) and \(E_{jy} W\) will change sign, i.e., \(E_{jx} W \rightarrow -E_{jx} W\) and \(E_{jy} W \rightarrow -E_{jy} W\), if the wave number changes direction, i.e., \(k_j \rightarrow -k_j\), but the wave is still left-hand polarized with respect to the background magnetic field.

In this study, the one-sided wave power spectral density [Press et al., 2002] has the form

\[
P(\omega) \propto \begin{cases} 
\exp \left[ -\frac{(\omega - \omega_m)^2}{2\delta\omega^2} \right] & \omega_{LC} < \omega < \omega_{UC} \\
0 & \text{otherwise}
\end{cases}
\]  

(8.2)

and root mean square (RMS) wave amplitude is defined as

\[
B_{RMS}^W = \left[ \int_0^{\infty} P(\omega) d\omega \right]^{\frac{1}{2}}
\]  

(8.3)

Here \(\omega_{LC}\) and \(\omega_{UC}\) are the lower cut-off and the upper cut-off frequencies, respectively. \(\omega_m\) is the mean frequency. \(\delta\omega\) represents the spread of the wave spectrum. In the simulation, we fix
\[ \omega_{LC} = 0.6\Omega_{He^+}, \quad \omega_{UC} = 0.96\Omega_{He^+} \] and \( \omega_m = 0.78\Omega_{He^+} \) but vary \( \delta \omega \) to change the wave power spread. The continuous spectrum \( P(\omega) \) is discretized into \( I(\omega_j) \) where \( j = 1, \cdots, N_W \), and the power of the \( j \)th component is \( \frac{B_{Wj}^2}{\delta \omega^2} = B_{yj}^W \). The constraints on \( I(\omega_j) \) are

\[ I(\omega_j) \propto P(\omega_j) \]

\[ \sum_{j=1}^{N_W} I(\omega_j) = \int_0^\infty P(\omega) \, d\omega \quad (8.4) \]

Thus the \( y \)-component of wave magnetic field can be written as

\[ B_{yj}^W = B_{RMS}^W \left[ \frac{\exp \left( \frac{-(\omega_j - \omega_m)^2}{\delta \omega^2} \right)}{\sum_{i=1}^{N_W} \exp \left( \frac{-(\omega_i - \omega_m)^2}{\delta \omega^2} \right)} \right]^{1/2} \quad (8.5) \]

We will vary the wave RMS amplitude \( B_{RMS}^W \) and wave bandwidth \( \delta \omega \) to demonstrate their controlling effect on particle dynamics.

### 8.2.2 Test particle simulation

We simulate the motion of test particles by directly solving the Lorentz equation. An ensemble of 2000 electrons is used to obtain the evolution of a specific distribution over the time scale of a bounce period. The electrons are initialized with identical initial kinetic energies and identical initial pitch angles, and are uniformly distributed over latitude between two mirror points. The direction of electron momentum is randomly chosen to be either parallel to or anti-parallel to \( B_0 \). The electron gyrophase is uniformly distributed between 0 and 2\( \pi \).

The pitch angle diffusion coefficients can be obtained by

\[ D_{TP}^{\alpha_0\alpha_0} = \frac{\langle (\alpha_0 - \langle \alpha_0 \rangle)^2 \rangle}{2\tau} \quad (8.6) \]

from the test particle simulation [Tao et al., 2012b], where \( \alpha_0 \) is the equatorial pitch angle and \( \langle \cdot \rangle \) denotes averaging over all the electrons. \( \tau \) is the simulation time which is long enough to capture several wave-particle interactions. To compare \( D_{TP}^{\alpha_0\alpha_0} \) with theoretical results, we calculate the bounce-averaged diffusion coefficient from quasilinear theory as [Glauert and Horne, 2005]

\[ D_{QL}^{\alpha_0\alpha_0} = \frac{1}{T} \int_0^{\lambda_m} D_{\alpha\alpha} \frac{\cos \alpha}{\cos^2 \alpha_0} \cos^7 \lambda \, d\lambda \quad (8.7) \]
where $\lambda_m$ is the mirror latitude and $T(\alpha_0) \simeq 1.30 - 0.56 \sin \alpha_0$. The local resonant diffusion coefficient [Liu et al., 2010] is

$$D_{\alpha\alpha} (\lambda) = \frac{\pi \Omega_e}{4 \gamma} \epsilon^2 k \tilde{P} (k)$$

(8.8)

with $\epsilon = B_{RMS}^W / B_0 (\lambda)$. The one-sided power spectral density in wave number domain is given by $\tilde{P} (k) = \hat{P} (\omega) \frac{d\omega}{dk}$ and has the normalization $\int_0^\infty \tilde{P} (k) dk = \int_0^\infty \hat{P} (\omega) d\omega = 1$. Note there is an extra factor of $1/2$ in equation (8.8) compared to Summers [2005] due to the use of one-sided power spectrum [Tao et al., 2011].

The key driver of electron scattering is the wave spectrum. Interesting physics arise with different specification of the wave spectrum. First we investigate the diffusion regime driven by a small amplitude and incoherent broadband wave spectrum using a test-particle simulation. The discrete frequencies $\omega_j$'s are equally spaced between the lower and upper cutoff frequency. The total number of waves $N_W$ is 100. This set up is chosen to satisfy the resonant island overlap criterion [Chirikov, 1979] and will be used in all runs. We specify the wave spectrum as $B_{RMS}^W = 10 pT$ and $\delta \omega = 0.6 \Omega_{He^+}$, which is chosen to satisfy the weak turbulence assumption [Kennel and Engelmann, 1966]. The ensemble of 2000 test particles start with initial pitch angle $\alpha_0 = 55^\circ$ and initial kinetic energy $E_0 = 2$ MeV. The initial parameters are chosen so that electrons initially resonate with the center frequency of the spectrum at the equator. The simulation lasts for 5 unperturbed bounce periods. The simulation results are shown in Figure 8.1. Panel (a) shows the linear dependence of the variance of the equatorial pitch angle $\text{Var}[\alpha_0]$ on time. $\text{Var}[\alpha_0]$ is defined as $\langle (\alpha_0 - \langle \alpha_0 \rangle)^2 \rangle$. The slope of this line is double the diffusion coefficient at $\alpha_0 = 55^\circ$ as indicated by equation (8.6). The relation $\text{Var}[\alpha_0] \propto t$ illustrates the diffusive behaviour of electrons and provides validity for the calculation of $D_{\alpha\alpha \alpha \alpha}^{TP}$. Panel (b) shows the comparison between $D_{\alpha\alpha \alpha \alpha}^{TP}$ (solid blue line) and $D_{\alpha\alpha \alpha \alpha}^{QL}$ (red bubbles). $D_{\alpha\alpha \alpha \alpha}^{TP}$ is obtained from equation (8.6) by simulating the ensemble of electrons for one unperturbed bounce period, i.e., $\tau = \tau_B$ in equation (8.6). $D_{\alpha\alpha \alpha \alpha}^{QL}$ is obtained from equation (8.7) by directly integrating the weighted local diffusion coefficients over all resonant latitudes. The test particle simulation agrees with QLT for all
the resonant pitch angles and captures the cutoff pitch angle around $\alpha_0 = 80^\circ$, beyond which electrons of 2 MeV are out of resonance. In a small amplitude and incoherent broadband wave spectrum, the resonant region of phase space is globally stochastic in which adiabatic islands do not exist or occupy negligible phase space volume [Lichtenberg and Lieberman, 1992]. Phase randomization quickly takes place and allows diffusion over a 2-D velocity space (gyrophase is averaged out due to phase randomization) [Kennel and Engelmann, 1966]. These ingredients illustrate the characteristics of stochastic diffusion and form the foundation of quasilinear diffusion theory.

Figure 8.1: (a) The linear dependence of the variance of the equatorial pitch angle $\text{Var}[\alpha_0]$ on time. Time is normalized to bounce period $\tau_B$. The initial equatorial pitch angle is $\alpha_0 = 55^\circ$. (b) The comparison between $D^{TP}_{\alpha_0\alpha_0}$ (solid blue line) and $D^{QL}_{\alpha_0\alpha_0}$ (red bubbles). Both panels (a) and (b) corresponds to $B_{\text{RMS}}^{W} = 10pT$ and $\delta\omega = 0.6\Omega_{He^+}$.

8.2.3 Stochastic diffusion simulation

The interactions between EMIC waves and electrons can be treated as a stochastic diffusion process in pitch angle which can be described by quasilinear theory. The bounce-averaged
pitch angle diffusion equation is \cite{Albert2004}

\[
\frac{\partial f_0}{\partial t} = \frac{1}{G} \frac{\partial}{\partial \alpha_0} \left( GD_{\alpha_0\alpha_0} \frac{\partial f_0}{\partial \alpha_0} \right)
\]  

(8.9)

with Jacobian factor \( G(\alpha_0) = T(\alpha_0) \sin \alpha_0 \cos \alpha_0 \) and normalized bounce period \( T(\alpha_0) = 1.30 - 0.56 \sin \alpha_0 \). The bounce-averaged diffusion coefficient \( D^{QL}_{\alpha_0\alpha_0} \) is also obtained from equation (8.7). The bounce-averaged diffusion equation (8.9) can be written in a standard form

\[
\frac{\partial f_0}{\partial t} = D_{\alpha_0\alpha_0} \frac{\partial^2 f_0}{\partial \alpha_0^2} + b_{\alpha_0} \frac{\partial f_0}{\partial \alpha_0}
\]

(8.10)

where \( b_{\alpha_0} = \frac{1}{G} \frac{\partial}{\partial \alpha_0} (GD_{\alpha_0\alpha_0}) \). The second order derivative with respect to \( \alpha_0 \) is a diffusion term and the first order derivative with respect to \( \alpha_0 \) is an advection term. The standard form of diffusion equation can be linked to the stochastic differential equation \cite{Freidlin2016,Gardiner1985,Tao2008,Tao2014b}

\[
dA_0 = b_{\alpha_0} \, dt + \sigma \, dW
\]

(8.11)

Here \( A_0 \) is a random variable representative of \( \alpha_0 \) and the coefficient \( \sigma = \sqrt{2D_{\alpha_0\alpha_0}} \). \( W \) is a Wiener process and \( dW \sim \sqrt{dt} \mathcal{N}(0,1) \), where \( \mathcal{N}(0,1) \) denotes the normal distribution with expected value 0 and variance 1. We solve stochastic differential equation (8.11) numerically to obtain the electron distribution function. In order to compare the simulation results of stochastic diffusion equation (8.11) with that of the test particle method, we start an ensemble of 2000 electrons with identical kinetic energies \( E_0 = 2\text{MeV} \) and identical pitch angles \( \alpha_0 = 55^\circ \), and let them evolve for one unperturbed bounce period as in the test particle simulation. The wave spectrum satisfies the weak turbulence assumption with \( B^{W}_{\text{RMS}} = 10pT \) and \( \delta \omega = 0.6\Omega_{H_e^+} \). We run the test particle code and stochastic diffusion code to obtain electron distribution function \( g_{\text{TPS}}^{\alpha_0}(\alpha_0) \) and \( g_{\text{QL}}^{\alpha_0}(\alpha_0) \) respectively. The comparison between quasilinear theory (QLT) and the test particle simulation (TPS) is shown in Figure 8.2. Panel (a) shows the distribution function \( g_{\text{QL}}^{\alpha_0}(\alpha_0) \) from QLT (blue) and \( g_{\text{TPS}}^{\alpha_0}(\alpha_0) \) from TPS (red) respectively, where superscript ‘QL’ means quasilinear and ‘TPS’ means test particle simulation. Panel (b) shows the cumulative distribution function \( G(\alpha_0) \) corresponding to panel (a). The distribution functions \( g_{\text{QL}}^{\alpha_0}(\alpha_0) \) and \( g_{\text{TPS}}^{\alpha_0}(\alpha_0) \) show a good comparison as
expected. We quantify the difference between \( g^{QL}(\alpha_0) \) and \( g^{TPS}(\alpha_0) \) as

\[
d(g^{QL}, g^{TPS}) = \max_{0<\alpha_0<\pi/2} \left| G^{QL}(\alpha_0) - G^{TPS}(\alpha_0) \right|
\] (8.12)

where \( G(\alpha_0) \) is the cumulative distribution function of \( g(\alpha_0) \). The metric \( d(g^{QL}, g^{TPS}) \) ranges from 0 to 1, with “\( d = 0 \)” implying that the two distributions are identical. As the metric “\( d \)” increases, it indicates the difference between two distributions \( g^{QL} \) and \( g^{TPS} \) is larger. For the case shown in Figure 8.2, the difference is 0.0645. In the next section, we will use the metric defined in equation (8.12) to further compare the results of QLT and TPS using various wave spectra and obtain the regime map.

Figure 8.2: (a) The distribution function obtained from stochastic diffusion simulation (blue) and from test particle simulation (red) respectively. Simulation time is one unperturbed bounce period for initial pitch angle \( \alpha_0 = 55^\circ \). Wave RMS amplitude and wave bandwidth used in the simulation are \( B_{RMS}^W = 10pT \) and \( \delta \omega = 0.6\Omega_{He^+} \) respectively. (b) The corresponding cumulative distribution function (CDF) for the distribution function in (a). The maximum difference between CDF of stochastic diffusion simulation \( G^{QL}(\alpha_0) \) and CDF of test particle simulation \( G^{TP}(\alpha_0) \) is 0.0645.
8.3 The regime map

It is well known that nonlinear wave-particle interactions lead to drastically different particle behaviour in comparison with linear interactions. Here we investigate how nonlinear interactions alter the overall particle behaviour over a relatively long time scale and how wave-particle interactions over short time scale can be linked to that of long time scale. To achieve this goal, the comparison between TPS and QLT is made over a broad range of wave parameters over one bounce period. Specifically, 20 grid points of $B_{RMS}^W$ are distributed logarithmically between 0.01 and 10 nT. Another 20 grid points of $\delta \omega$ are distributed logarithmically between $0.005 \Omega_{He^+}$ and $\Omega_{He^+}$. Thus $20 \times 20$ grid points are used to construct the wave parameter space. In each grid point, we run the test particle code and stochastic diffusion code to obtain electron distribution function $g^{TP}(\alpha_0)$ and $g^{QL}(\alpha_0)$ respectively. In both simulations, an ensemble of 2000 electrons is released with kinetic energy $E_0 = 2\text{MeV}$ and pitch angle $\alpha_0 = 55^\circ$. The detailed description of initialization is given in subsection 8.2.2 and 8.2.3. Figure 8.3 shows the regime map obtained by the 400 runs. The color indicates the difference between $g^{TP}(\alpha_0)$ and $g^{QL}(\alpha_0)$ measured with the aid of equation (8.12). To facilitate interpretation of the regime map, we also show the electron pitch angle distribution for a range of wave bandwidths in Figure 8.4. The first and second row in Figure 8.4 is for 0.1 nT and 10 nT respectively, with left column obtained from TPS and right column from QLT. A sharp transition is found in the upper left part of the regime map (Figure 8.3), from the region roughly conforming with the predications of QLT to the region severely violating QLT. The critical wave amplitude of the transition tends to increase as the wave bandwidth widens. We will describe the particle dynamics in both linear and nonlinear regimes and explain how the short time scale resonant interactions relate to the electron behaviour in the regime map.

8.3.1 Linear regime

The linear regime refers to the small wave amplitude limit, including both broadband and narrowband spectra. The systematic behaviour of electrons in the linear regime is pitch
Figure 8.3: The regime map in the parameter space comprised by wave RMS amplitude $B_{RMS}^W$ and wave bandwidth $\delta \omega$. The coverage of wave RMS amplitude is $0.01 \sim 10 \text{ nT}$, and the coverage of wave bandwidth is $0.005 \sim 1\Omega_{He^+}$. The color indicates the difference between $g^{QL}(\alpha_0)$ and $g^{TP}(\alpha_0)$, which is measured by equation (8.12).
Figure 8.4: Pitch angle distribution of electrons obtained from TPS (left column) and QLT (right column) for a range of wave bandwidth. Wave RMS amplitude is 0.1 nT in panel (a) and (b), and is 10 nT in panel (c) and (d). The y-axis $\Delta \alpha_0$ in all panels denotes the difference between final equatorial pitch angle and initial equatorial pitch angle $\alpha_0 = 55^\circ$.

angle diffusion. However, diffusion is caused by different reasons for broadband spectrum and narrowband spectrum (panel (a) and (b) of Figure 8.4). In the following demonstration, 6 electrons are released from the magnetic equator towards the northern hemisphere, co-streaming with the northward propagating EMIC waves. These electrons have identical initial pitch angles $\alpha_0 = 55^\circ$ and energies $E_0 = 2\text{MeV}$, and are evenly distributed in initial gyro-phase. The results are shown in Figure 8.5. Panel (a) shows electron trajectories in a small amplitude and broadband wave spectrum with $B_{RMS}^W = 0.1 \text{nT}$ and $\delta \omega/\Omega_{He^+} = 0.6$. Panel (b) shows electron trajectories in a small amplitude and narrowband wave spectrum with $B_{RMS}^W = 0.1 \text{nT}$ and $\delta \omega/\Omega_{He^+} = 0.005$. In the broadband spectrum, strong phase randomization takes place due to overlapping resonant islands [Liu et al., 2012]. This results in stochastic trajectories as demonstrated in panel (a) of Figure 8.5, which is also evident in Figure 8.1. However, in the narrowband spectrum, all the resonant islands degenerate into one island [Zaslavsky, 1985]. Furthermore, the resonant island vanishes due to the Earth’s inhomogeneous magnetic field. The phase of electrons with respect to wave magnetic field...
varies regularly rather than randomly. The diffusion-like behaviour of electrons in a narrowband spectrum results from “point-resonant diffusion”. We coin the term “point-resonant diffusion” to indicate such short interaction time that phase coherence is quickly lost. In other words, the inhomogeneity of Earth’s dipole field effectively narrows the interaction time window. The electrons get one kick during the interaction and do not return. As a result, we get the trajectories as shown in panel (b) of Figure 8.5. The equivalence of point-resonant diffusion and stochastic diffusion in the narrowband limit is demonstrated by Albert [2010]. Namely, the narrowband limit of quasilinear diffusion coefficient is exactly the same as point-resonant diffusion coefficient obtained from a coherent interaction. The quasilinear diffusion coefficient for a broadband spectrum can also be constructed as a sum of point-resonant diffusion coefficient for a series of monochromatic waves. To confirm the equivalence of point-resonant diffusion and stochastic diffusion in narrowband limit, we repeat the simulation shown in Figure 8.1 but in the narrowband limit of the broadband wave spectrum, with $B_{RMS}^W = 0.01$ nT and $\delta \omega = 0.005\Omega_{He^+}$. The results are shown in Figure 8.6. Panel (a) shows the linear dependence of the variance of equatorial pitch angle on time for initial pitch angle $\alpha_0 = 55^\circ$, which illustrates the diffusive behaviour in the narrowband limit. Panel (b) shows the comparison of the diffusion coefficients from QLT (solid blue line) and TPS (red bubbles). The diffusion coefficients from QLT represents stochastic type of diffusion coefficients in the narrowband limit, whereas the diffusion coefficients from TPS represents point-resonant type of diffusion coefficients by a nearly monochromatic wave. Panel (b) gives strong support to the equivalence of point-resonant diffusion and stochastic diffusion in the narrowband limit. Therefore, the particle dynamics can be well described by QLT in the linear regime (including both broadband and narrowband spectra) as shown in the regime map (Figure 8.3).

We should also notice the moderate difference between QLT and TPS in the lower left part of the regime map (Figure 8.3) which is caused by moderate pitch angle advection, which is evident in panel (a) of Figure 8.4. This is reasonable because the resonant latitude in the narrowband limit is $2^\circ \sim 3^\circ$ from the equator. An indicator for the nonlinearity is the inhomogeneity ratio $R$, which is roughly the ratio of the background magnetic field.
Figure 8.5: The trajectories of 6 electrons with identical initial pitch angle $\alpha_0 = 55^\circ$ and evenly sampled initial gyro-phase in four different wave spectrum limit: (a) small amplitude and broadband spectrum, $B_{RMS}^W = 0.1$ nT and $\delta\omega/\Omega_{He^+} = 0.6$; (b) small amplitude and narrowband spectrum, $B_{RMS}^W = 0.1$ nT and $\delta\omega/\Omega_{He^+} = 0.005$; (c) large amplitude and broadband spectrum, $B_{RMS}^W = 10$ nT and $\delta\omega/\Omega_{He^+} = 0.6$; (d) large amplitude and narrowband spectrum, $B_{RMS}^W = 10$ nT and $\delta\omega/\Omega_{He^+} = 0.005$. 
Figure 8.6: Same as Figure 8.1 except for a small amplitude and narrowband spectrum, $B^W_{RMS} = 0.01$ nT and $\delta \omega = 0.005 \Omega_{He^+}$.

gradient to the wave amplitude $R \approx \frac{1}{kB^W_{RMS}} \frac{\partial B_0}{\partial z}$. The inhomogeneity ratio is about $R = 0.5$ (moderately nonlinear) for $B^W_{RMS} = 0.1$ nT and is about $R = 5$ (linear) for $B^W_{RMS} = 0.01$ nT. Thus the moderate difference here in this limit is caused by moderate nonlinearity.

### 8.3.2 Nonlinear regime

The characteristics of electron motion are subject to a drastic change from the linear regime to the nonlinear regime. We show the typical electron trajectories of the nonlinear regime in panels (c) and (d) of Figure 8.5. The initialization is the same as that for panels (a) and (b). Panel (c) shows electron trajectories in a large amplitude and broadband wave spectrum with $B^W_{RMS} = 10$ nT and $\delta \omega / \Omega_{He^+} = 0.6$. Panel (d) shows electron trajectories in a large amplitude and narrowband wave spectrum with $B^W_{RMS} = 10$ nT and $\delta \omega / \Omega_{He^+} = 0.005$. We explain the link between these sample trajectories and the regime map as follows.

In the nonlinear regime, electron motion driven by a broadband spectrum is not fully stochastic, but experiences alternate random walk and phase trapping, which means phase
mixing is not complete in this regime [Liu et al., 2012]. This may result from some adiabatic islands embedded in the stochastic region of the phase portrait [Lichtenberg and Lieberman, 1992]. So the dynamics of the electrons cannot be described by QLT in a large amplitude and broadband wave spectrum. However, the electron distribution produced by QLT after one bounce period agrees well with that obtained from TPS as indicated in the regime map. The strong diffusion limit is reached once the condition $D_{QL}^{α0} ≥ \frac{α^2}{τ_B/4}$ is satisfied [Kennel, 1969]. Using the scaling $D_{QL}^{α0} \propto (B_{RMS}^W/B_0)^2$ and the numerical computations in Figure 8.1, we estimate the wave RMS amplitude threshold for strong diffusion limit as $(B_{RMS}^W)_{critical} = 1.2$ nT. The electrons rapidly evolve toward equilibrium (i.e., $∂f_0/∂t ≈ 0$ in equation (8.9)) above this amplitude threshold in the quasilinear diffusion description (shown in panel (d) of Figure 8.4). The large amplitude and broadband waves in reality also make the electrons spread all over the resonant pitch angles (shown in panel (c) of Figure 8.4), even though the dynamics cannot be fully described by quasilinear diffusion.

We also examine the dynamics of particles in a large amplitude and narrowband limit wave spectrum. It is known that electrons undergo phase trapping and phase bunching in a large amplitude monochromatic wave in the Earth’s dipole field. Phase trapping and phase bunching violate the basic assumption (i.e., uniform phase or random phase) of QLT and hence cannot be described by QLT. Although nonlinear phase trapping and phase bunching is conceptually different from QLT, the consequence of both tend to spread the electron distribution over the time scale of one bounce period, which is evident in panel (c) of Figure 8.4. The main discrepancy actually arises from the different resonant pitch angle ranges given by the two descriptions. On one hand, the range of resonant pitch angle in QLT is determined by the resonance condition, i.e., equation (1.1). For this study in the narrowband limit $δω = 0.005Ω_{He^+}$, the range of resonant pitch angles is approximately between 30° and 60°. This can be shown as a similar curve to that in panel (b) of Figure 8.6, except for a larger wave amplitude $B_{RMS}^W = 10$ nT. On the other hand, from the point view of nonlinear wave-particle interactions, the range of resonant pitch angles can be interpreted as the width of the resonant island in the phase portrait. For a large amplitude monochromatic wave
(i.e., $B^W_{\text{RMS}} = 10 \, \text{nT}$ and $\omega = \omega_m = 0.78 \Omega_{He^+}$) corresponding to the narrowband limit in the current study, we plot the phase portrait in Figure 8.7. The phase portrait is comprised of the phase angle $\zeta$ (the angle between perpendicular momentum and wave magnetic field) and equatorial pitch angle $\alpha_0$. The pitch angle range of the resonant island is between $\alpha_0 = 28^\circ$ and $\alpha_0 = 80^\circ$, which roughly agrees with test particle simulation in panel (c) of Figure 8.4. So the main discrepancy in the regime map can be attributed to the difference of the resonant pitch angle range between QLT and TPS, which originates from the finite wave amplitude of a coherent wave.

### 8.4 Summary and discussion

In this study, we examined the effects of wave amplitude and wave bandwidth on the long term evolution of the electron distribution. Comparison is made between quasilinear diffusion theory and test particle approach over one bounce period. We also give a description of the particle dynamics in four regimes of the wave spectrum.

1. In a small amplitude and broadband wave spectrum, electrons undergo stochastic motion due to complete phase randomization. Quasilinear diffusion theory gives an accurate description in this limit.

2. In a small amplitude and narrowband wave spectrum, electrons undergo point-resonant diffusion caused by the inhomogeneous dipole field of the Earth. The point-resonant type diffusion is equivalent to the stochastic type diffusion in the narrowband limit. Thus quasilinear theory can also apply to the narrowband limit of a small amplitude spectrum.

3. In a large amplitude and broadband wave spectrum, electrons experience alternate random walk and phase trapping due to incomplete phase randomization. Quasilinear theory cannot describe the particle dynamics in this limit. But such large wave amplitudes make electrons spread all over the resonant pitch angle and hence quasilinear diffusion produces similar electron distributions as test particle simulation.
Figure 8.7: The phase portrait of electrons at the magnetic equator in a large amplitude monochromatic wave, i.e., $B_{RMS}^W = 10 \text{ nT}$ and $B_0 = 140 \text{ nT}$. The resonant pitch angle is chosen to be $55^\circ$. The red line is the separatrix, which separates the resonant island from the free unbounded oscillation. The $x$-axis and $y$-axis are phase angle $\zeta$ and equatorial pitch angle $\alpha_0$ respectively.
4. In a large amplitude and narrowband wave spectrum, nonlinear phase trapping and phase bunching are the characteristics of the electron motion. Quasilinear diffusion theory is invalid in such a nonlinear coherent wave spectrum. The resonant pitch angle range depends on wave amplitude.

Quoting the statistical results of EMIC waves from the CRRES satellite (see figure (9) of Meredith et al. [2014]), the range of intensity of helium band waves during active conditions is $0.01 - 100 \text{ nT}^2$ (i.e., $B_{RMS}^W = 0.1 - 10 \text{ nT}$), and the range of bandwidth of the moderate and strong helium band wave events during active conditions is $0.003 - 0.1 f_{cp}$ (i.e., $\delta \omega = 0.01 - 0.4 \Omega_{He^+}$). Therefore we expect that the typical scenario to be the transition of large amplitude narrowband wave regime to the large amplitude broadband wave regime, and is mostly located in the latter, where quasilinear theory can produce the right results of overall evolution of distribution function but cannot describe the dynamics of individual particles.

There exists a critical wave amplitude beyond which the deviation of electron scattering from the prediction of quasilinear theory starts to occur in the narrowband limit. This threshold value increases as wave bandwidth $\delta \omega$ increases as shown in the regime map. It should be noted that the threshold value also varies with various parameters, such as pitch angle, energy, background magnetic field and cold electron density. Analytical determination of the threshold wave amplitude would be valuable for the valid use of QLT and is left for future study. Though quasilinear theory gives correct results under a large amplitude and broadband wave spectrum, the particle dynamics are yet to be studied in this regime, under which phase randomization is incomplete. These questions are beyond the scope of current study and will be examined in future work.
CHAPTER 9

Outlook

Here, I attempt to give an outlook of future work.

**Excitation of whistler-mode chorus waves** - With the availability of massively parallel particle-in-cell simulations and a laboratory experiment, we are in a position to study the physics in the excitation of chorus waves. Obviously, the excitation of chorus waves is a nonlinear process. Whistler anisotropy instability provides the free energy for the initial growth of a banded whistler wave. Upon saturation of the whistler anisotropy instability, the wave frequency starts to chirp and the wave energy grows explosively. In a uniform background magnetic field, the whistler anisotropy instability would damp after saturation. While in a nonuniform background magnetic field, the chirping in frequency leads to the change in resonant velocity and allows to maintain the wave growth even after the saturation of whistler anisotropy instability. The key question to answer is what phase space structure initiates the frequency chirping, how the frequency chirping maintains a bursty wave growth, and how the chirping tone terminates. With a two-dimensional or even three-dimensional particle-in-cell code from the UPIC framework, a nonuniform dipole background magnetic field can be considered. Of particular importance, advanced particle diagnostics need to be developed to resolve the evolution of the phase space portrait, in which the cause of the initiation of frequency chirping would be identified. Laboratory experiments conducted at the Large Plasma Device are helpful in mapping out the parameter space favoring the excitation of chorus waves. What is more important, when the hot electron distribution can be measured in the laboratory, a new window will be opened to identify the exact mechanism responsible for the chorus excitation.

**The unstable eigenmodes excited by a finite electron beam** - The unstable
eigenmodes of whistler waves excited by an electron beam with a top-hat density profile was worked out in chapter 5. We assumed that the beam is so tenuous that it does not modify the cold plasma dispersion relation significantly but only provides a small growth rate for the wave. Thus the continuity of displacement electric field can be ensured automatically. On the other hand, by setting up a smooth beam profile, the only boundary condition is that the wave field vanishes at infinity. Consequently, a system of ordinary differential equations can be obtained by expanding the wave field and the density distribution around the gyro-orbit. This defines an eigenvalue problem, which is interesting to study in the future.

Electron scattering by a large amplitude and monochromatic EMIC wave in a dipole geometry - The stochastic motion of electrons in a small amplitude and broadband wave is described by quasi-linear diffusion theory, since the phase between waves and electrons is randomized. In contrast, it was shown in chapter 7 and 8 that the electron dynamics in a large amplitude and monochromatic EMIC wave in a dipole geometry is characterized by phase trapping and phase bunching. Therefore it is needed to develop a formulation to incorporate the electron scattering by a large amplitude and monochromatic EMIC wave into a global simulation of electron transport. Such a formulation must be based on the probability of electrons to undergo phase bunching and phase trapping after one encounter with the wave. An accurate estimation of the finite interaction time between waves and particles is crucial since it tells how large in pitch angle the electrons are scattered by multiplying the scattering rate. This finite interaction time originates from the finite wave amplitude.
APPENDIX A

The susceptibility tensor for a gyrating electron beam

Here the susceptibility tensor for a gyrating electron beam is calculated analytically, which is utilized in chapter 5. To begin with, the linearized Vlasov equation is

$$\frac{d\hat{f}_b}{dt} - \frac{e}{m} \left( \hat{E} + \frac{v \times \hat{B}}{c} \right) \cdot \frac{\partial f_{0b}}{\partial v} = 0 \quad (A.1)$$

where $\hat{f}_b$ is the perturbed distribution function and $\hat{E}$, $\hat{B}$ are the perturbed fields. We consider perturbations of the form

$$\hat{E} = \tilde{E} \exp(-i\omega t + ik_x x + ik_y y + ik_z z) \quad (A.2)$$
$$\hat{B} = \tilde{B} \exp(-i\omega t + ik_x x + ik_y y + ik_z z) \quad (A.3)$$

Equation (A.1) can be integrated along its characteristics, i.e., the unperturbed helical orbits of electrons. This integral can be calculated as

$$\hat{f}_b = \frac{e}{m} \int_{-\infty}^{t} dt' \exp(-i\omega t' + ik_x x' + ik_y y' + ik_z z') \tilde{S} \quad (A.4)$$

Here the integral kernel $\tilde{S}$ is

$$\tilde{S} = \left( \frac{\hat{E} + \frac{v' \times \hat{B}}{c}}{c} \right) \cdot \frac{\partial f_{0b}(v')}{\partial v'}$$

$$= \left( v'_x \tilde{E}_x + v'_y \tilde{E}_y \right) f_{0b\perp} + \tilde{E}_z f_{0bz}$$

$$+ \left[ \frac{v'_x}{c} (v'_y \tilde{B}_z - v'_z \tilde{B}_y) + \frac{v'_y}{c} (v'_x \tilde{B}_z - v'_z \tilde{B}_x) \right] f_{0b\perp} + \frac{1}{c} (v'_x \tilde{B}_y - v'_y \tilde{B}_x) f_{0bz}$$

$$= \tilde{E}_x v'_x f_{0b\perp} + \tilde{E}_y v'_y f_{0b\perp} + \tilde{E}_z f_{0bz}$$

$$+ \tilde{B}_x \left[ \frac{v'_y}{c} (v_z f_{0b\perp} - f_{0bz}) \right] + \tilde{B}_y \left[ -\frac{v'_x}{c} (v_z f_{0b\perp} - f_{0bz}) \right] \quad (A.5)$$

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We define

\[ f_{0\perp} = \frac{1}{v_{\perp}} \frac{\partial f_{0b}}{\partial v_{\perp}} \]  \hfill (A.6)

\[ f_{0z} = \frac{\partial f_{0b}}{\partial v_z} \]  \hfill (A.7)

Note that \( v_{\perp} \) and \( v_z \) are two constants along the particle orbit. Using Faraday’s equation, we can write \( \vec{B}_x \) and \( \vec{B}_y \) as

\[ \vec{B}_x = \frac{k_y c}{\omega} \vec{E}_z - \frac{k_z c}{\omega} \vec{E}_y \]  \hfill (A.8)

\[ \vec{B}_y = \frac{k_z c}{\omega} \vec{E}_x - \frac{k_x c}{\omega} \vec{E}_z \]  \hfill (A.9)

Replacing \( \vec{B}_x \) and \( \vec{B}_y \) with the electric field, one obtains

\[ \tilde{S} = \vec{E}_x v'_x \left[ f_{0\perp} - \frac{k_z}{\omega} (v_z f_{0\perp} - f_{0z}) \right] \]

\[ + \vec{E}_y v'_y \left[ f_{0\perp} - \frac{k_z}{\omega} (v_z f_{0\perp} - f_{0z}) \right] \]

\[ + \vec{E}_z \left[ f_{0z} + \frac{k_y v'_y}{\omega} (v_z f_{0\perp} - f_{0z}) + \frac{k_x v'_x}{\omega} (v_z f_{0\perp} - f_{0z}) \right] \]  \hfill (A.10)

The unperturbed particle orbit can be solved as [Stix, 1962]

\[ v'_x = v_x \cos(\Omega_e \tau) + v_y \sin(\Omega_e \tau) \]  \hfill (A.11)

\[ v'_y = -v_x \sin(\Omega_e \tau) + v_y \cos(\Omega_e \tau) \]  \hfill (A.12)

\[ v'_z = v_z \]  \hfill (A.13)

\[ x' = -\frac{v_x}{\Omega_e} \sin(\Omega_e \tau) - \frac{v_y}{\Omega_e} [1 - \cos(\Omega_e \tau)] + x \]  \hfill (A.14)

\[ y' = \frac{v_x}{\Omega_e} [1 - \cos(\Omega_e \tau)] - \frac{v_y}{\Omega_e} \sin(\Omega_e \tau) + y \]  \hfill (A.15)

\[ z' = -v_z \tau + z \]  \hfill (A.16)

where \( \tau = t - t' \). Note that \( \Omega_e = eB_0/mc \) is the unsigned electron cyclotron frequency. The particle orbit reaches \( \mathbf{v}' = \mathbf{v} \) and \( \mathbf{x}' = \mathbf{x} \) at \( t' = t \). Using the particle orbit, we further write
\[ S(\tau) = \bar{E}_x(x) (v_x \cos(\Omega_e \tau) + v_y \sin(\Omega_e \tau)) \left[ f_{0b,\perp} - \frac{k_x}{\omega} (v_z f_{0b,\perp} - f_{0b}) \right] + \bar{E}_y(x) (-v_x \sin(\Omega_e \tau) + v_y \cos(\Omega_e \tau)) \left[ f_{0b,\perp} - \frac{k_x}{\omega} (v_z f_{0b,\perp} - f_{0b}) \right] + \bar{E}_z(x) \left[ f_{0b} + (-v_x \sin(\Omega_e \tau) + v_y \cos(\Omega_e \tau)) \frac{k_y}{\omega} (v_z f_{0b,\perp} - f_{0b}) \right] + (v_x \cos(\Omega_e \tau) + v_y \sin(\Omega_e \tau)) \frac{k_z}{\omega} (v_z f_{0b,\perp} - f_{0b}) \] (A.17)

To write \( \bar{S} \) in a more compact form, we define

\[ v_x = v_{\perp} \cos \phi \] (A.18)
\[ v_y = v_{\perp} \sin \phi \] (A.19)
\[ k_x = k_{\perp} \cos \theta \] (A.20)
\[ k_y = k_{\perp} \sin \theta \] (A.21)
\[ g_{\perp} = \frac{\partial f_{0b}}{\partial v_{\perp}} = v_{\perp} f_{0b,\perp} \] (A.22)
\[ g_z = \frac{\partial f_{0b}}{\partial v_z} = f_{0b} \] (A.23)

\( \bar{S} \) becomes

\[ \bar{S}(\tau) = \bar{E}_x(x) \cos(\phi - \Omega_e \tau) \left[ g_{\perp} + \frac{k_x}{\omega} (v_{\perp} g_z - v_z g_{\perp}) \right] + \bar{E}_y(x) \sin(\phi - \Omega_e \tau) \left[ g_{\perp} + \frac{k_x}{\omega} (v_{\perp} g_z - v_z g_{\perp}) \right] + \bar{E}_z(x) \left[ g_z - \cos(\phi - \theta - \Omega_e \tau) \frac{k_{\perp}}{\omega} (v_{\perp} g_z - v_z g_{\perp}) \right] \] (A.24)

Now we express the phase factor in equation (A.4) as a function of \( \tau \)

\[ -i \omega t' + i k_x x' + i k_y y' + i k_z z' = -i \omega (t - \tau) + i k_x \left[ x + \frac{v_{\perp}}{\Omega_e} (-\sin \phi + \sin(\phi - \Omega_e \tau)) \right] + i k_y \left[ y + \frac{v_{\perp}}{\Omega_e} (\cos \phi - \cos(\phi - \Omega_e \tau)) \right] + i k_z \left[ z - v_{\perp} \tau \right] \]

\[ = -i \omega t + i k_x x + i k_y y + i k_z z + i (\omega - k_z v_z) \tau \]

\[ + i \frac{k_x v_{\perp}}{\Omega_e} (-\sin \phi + \sin(\phi - \Omega_e \tau)) + i \frac{k_y v_{\perp}}{\Omega_e} (\cos \phi - \cos(\phi - \Omega_e \tau)) \]

\[ = -i \omega t + i k_x x + i k_y y + i k_z z + i (\omega - k_z v_z) \tau \]

\[ + i \frac{k_z v_{\perp}}{\Omega_e} (-\sin(\phi - \theta) + \sin(\phi - \theta - \Omega_e \tau)) \] (A.25)
Therefore the perturbed distribution can be integrated over $\tau$ as

$$\tilde{f}_b = \frac{e}{m} \exp(-i\omega t + ik_x x + ik_y y + ik_z z)$$

$$\times \int_0^\infty d\tau \exp \left[ i(\omega - k_z v_z)\tau + i \frac{k_y v_y}{\Omega_e} (-\sin(\phi - \theta) + \sin(\phi - \theta - \Omega_e \tau)) \right] \tilde{S}(\tau)$$

(A.26)

Using the Jacobi-Anger expansion $e^{iz \sin \theta} = \sum_{m=-\infty}^{\infty} J_m(z) e^{im\theta}$, we have

$$\exp\left(-i \frac{k_y v_y}{\Omega_e} \sin(\phi - \theta)\right) = \sum_{m=-\infty}^{\infty} (-1)^m J_m\left(\frac{k_y v_y}{\Omega_e}\right) \exp(im(\phi - \theta))$$

(A.27)

$$\exp\left(i \frac{k_y v_y}{\Omega_e} \sin(\phi - \theta - \Omega_e \tau)\right) = \sum_{n=-\infty}^{\infty} J_n\left(\frac{k_y v_y}{\Omega_e}\right) \exp(in(\phi - \theta - \Omega_e \tau))$$

(A.28)

Hereafter $k_y = 0$ or $\theta = 0$ is assumed using the symmetry perpendicular to the background magnetic field. Another simplification is averaging $\tilde{f}_b$ and its velocity moments over $\phi$ in velocity space when calculating the charge density and current density. The following identities
are useful in this procedure

\[
\frac{1}{2\pi} \int_0^{2\pi} d\phi \exp \left[ i \frac{k_{\perp} v_{\perp}}{\Omega_e} ( -\sin \phi + \sin(\phi - \Omega_e \tau) ) \right] \times \begin{pmatrix}
1 \\
\cos \phi \\
\sin \phi \\
\cos(\phi - \Omega_e \tau) \\
\sin(\phi - \Omega_e \tau) \\
\cos \phi \cos(\phi - \Omega_e \tau) \\
\cos \phi \sin(\phi - \Omega_e \tau) \\
\sin \phi \cos(\phi - \Omega_e \tau) \\
\sin \phi \sin(\phi - \Omega_e \tau)
\end{pmatrix}
\begin{pmatrix}
J_n^2 \\
\frac{\Omega_e}{k_{\perp} v_{\perp}} J_n^2 \\
i J_n J_n' \\
\frac{\Omega_e}{k_{\perp} v_{\perp}} J_n^2 \\
i \frac{\Omega_e}{k_{\perp} v_{\perp}} J_n J_n' \\
i \frac{\Omega_e}{k_{\perp} v_{\perp}} J_n J_n' \\
(\Omega_e)^2 \\
(\Omega_e)^2
\end{pmatrix}
\]  
(A.29)

\[
= \sum_{n=-\infty}^{\infty} \exp(-i n \Omega_e \tau) \times \begin{pmatrix}
1 \\
\cos \phi \\
\sin \phi \\
\cos(\phi - \Omega_e \tau) \\
\sin(\phi - \Omega_e \tau) \\
\cos \phi \cos(\phi - \Omega_e \tau) \\
\cos \phi \sin(\phi - \Omega_e \tau) \\
\sin \phi \cos(\phi - \Omega_e \tau) \\
\sin \phi \sin(\phi - \Omega_e \tau)
\end{pmatrix}
\begin{pmatrix}
J_n^2 \\
\frac{\Omega_e}{k_{\perp} v_{\perp}} J_n^2 \\
i J_n J_n' \\
\frac{\Omega_e}{k_{\perp} v_{\perp}} J_n^2 \\
i \frac{\Omega_e}{k_{\perp} v_{\perp}} J_n J_n' \\
i \frac{\Omega_e}{k_{\perp} v_{\perp}} J_n J_n' \\
(\Omega_e)^2 \\
(\Omega_e)^2
\end{pmatrix}
\]

Here the argument of both Bessel function and its derivative is \( \frac{k_{\perp} v_{\perp}}{\Omega_e} \). The integration over \( \tau \) in equation (A.26) can be performed as

\[
\int_0^\infty d\tau \exp [i(\omega - k_z v_z - n \Omega_e) \tau] = \frac{i}{\omega - k_z v_z - n \Omega_e}
\]
(A.30)
given that \( \Re(\omega) > 0 \). We denote the averaging procedure by \( \langle \rangle_\phi \). The zeroth- and first-order moment of \( \hat{f}_b \) becomes

\[
\langle \hat{f}_b \rangle_\phi = \frac{e}{m} \exp(-i\omega t + ik_x x + ik_z z) \sum_{n=-\infty}^{\infty} \frac{1}{\omega - k_z v_z - n \Omega_e}
\times \left\{ \tilde{E}_x(x) \left( \frac{\Omega_e}{k_{\perp} v_{\perp}} i J_n^2 \right) \left[ g_\perp + \frac{k_z}{\omega} (v_\perp g_z - v_z g_\perp) \right] \\
+ \tilde{E}_y(x) (J_n J_n') \left[ g_\perp + \frac{k_z}{\omega} (v_\perp g_z - v_z g_\perp) \right] \\
+ \tilde{E}_z(x) (i J_n^2) \left[ \left( 1 - \frac{n \Omega_e}{\omega} \right) g_z + \left( \frac{n \Omega_e v_z}{\omega v_{\perp}} \right) g_\perp \right] \right\}
\]
(A.31)
\[ \langle v_x \hat{f}_b \rangle_\phi = \frac{e v_{\perp}}{m} \exp(-i\omega t + i k_x x + i k_z z) \sum_{n=-\infty}^{\infty} \frac{1}{\omega - k_z v_z - n\Omega_e} \]
\[ \times \left\{ \tilde{E}_x(x) \left( \frac{n^2\Omega_e^2}{k_z^2 v_{\perp}^2} j_n^2 \right) \left[ g_\perp + \frac{k_z}{\omega} (v_{\perp} g_z - v_z g_\perp) \right] 
\times \tilde{E}_y(x) \left( \frac{n\Omega_e}{k_z v_{\perp}} J_n J'_n \right) \left[ g_\perp + \frac{k_z}{\omega} (v_{\perp} g_z - v_z g_\perp) \right] 
\times \tilde{E}_z(x) \left( \frac{n\Omega_e}{k_z v_{\perp}} i J_n^2 \right) \left[ (1 - n\Omega_e/\omega) g_z + (n\Omega_e v_z/\omega v_{\perp}) g_\perp \right] \right\} \] (A.32)

\[ \langle v_y \hat{f}_b \rangle_\phi = \frac{e v_{\perp}}{m} \exp(-i\omega t + i k_x x + i k_z z) \sum_{n=-\infty}^{\infty} \frac{1}{\omega - k_z v_z - n\Omega_e} \]
\[ \times \left\{ \tilde{E}_x(x) \left( -\frac{n\Omega_e}{k_z v_{\perp}} J_n J'_n \right) \left[ g_\perp + \frac{k_z}{\omega} (v_{\perp} g_z - v_z g_\perp) \right] 
\times \tilde{E}_y(x) (i(j_n')^2) \left[ g_\perp + \frac{k_z}{\omega} (v_{\perp} g_z - v_z g_\perp) \right] 
\times \tilde{E}_z(x) (-J_n J'_n) \left[ (1 - n\Omega_e/\omega) g_z + (n\Omega_e v_z/\omega v_{\perp}) g_\perp \right] \right\} \] (A.33)

Knowledge of velocity moments now leads to the calculation of first-order plasma currents generated by the perturbed electron beam distribution, i.e.,
\[ \hat{\mathbf{j}}_b = -e \int_0^\infty 2\pi v_{\perp} dv_{\perp} \int_{-\infty}^{\infty} dv_z \langle \mathbf{v} \hat{f}_b \rangle_\phi \] (A.34)

Note that \( \hat{\mathbf{j}}_b \) has the form \( \hat{\mathbf{j}}_b = \hat{\mathbf{j}}_b \exp(-i\omega t + i k_x x + i k_z z) \). From equation (A.31)-(A.34), one observes that \( \hat{\mathbf{j}}_b \) can be expressed as a linear superposition of \( \tilde{E}_x(x) \), \( \tilde{E}_y(x) \) and \( \tilde{E}_z(x) \), with the coefficients being integrals of gradients over velocity space, namely
\[ \frac{4\pi i}{\omega} \hat{\mathbf{j}}_b = \chi_b \cdot \mathbf{E} \] (A.35)
where
\[
\chi_b = \begin{pmatrix}
\chi_{xx} & \chi_{xy} & \chi_{xz} \\
\chi_{yx} & \chi_{yy} & \chi_{yz} \\
\chi_{zx} & \chi_{zy} & \chi_{zz}
\end{pmatrix}
\]

\[
= \frac{\omega^2}{\omega} \sum_{n=-\infty}^{\infty} \left( 2\pi v_\perp dv_\perp \int_{-\infty}^{\infty} dv_z \frac{1}{\omega - k_z v_z - n\Omega_e} \right)
\]

(A.36)

\[
\times \left( \begin{array}{ccc}
\frac{n^2\Omega^2}{k^2 v_\perp^2} & J_n^2 v_\perp U & -i\frac{n\Omega_e}{k_z v_\perp} J_n J_n' v_\perp U \\
-i\frac{n\Omega_e}{k_z v_\perp} J_n J_n' v_\perp U & (J_n')^2 v_\perp U & iJ_n J_n' v_\perp W \\
\frac{n\Omega_e}{k^2 v_\perp} J_n^2 v_\perp U & -iJ_n J_n' v_\perp U & J_n^2 v_z W
\end{array} \right)
\]

We define
\[
U = g_\perp + \frac{k_z}{\omega} (v_\perp g_z - v_z g_\perp) \quad \text{(A.37)}
\]
\[
W = \left( 1 - \frac{n\Omega_e}{\omega} \right) g_z + \frac{n\Omega_e v_z}{\omega v_\perp} g_\perp \quad \text{(A.38)}
\]

where we have re-scale $g_\perp$ and $g_z$ to be $g_\perp = \frac{1}{n_b} \frac{\partial f_0}{\partial v_\perp}$ and $g_z = \frac{1}{n_b} \frac{\partial f_0}{\partial v_z}$, respectively. $n_b$ is beam density and $\omega^2_{pb} = \frac{4\pi n_b e^2}{m}$. Now we can calculate the specific susceptibility tensor for a beam ring distribution. We write the beam electron distribution as the following
\[
f_{0b}(v_z, v_\perp) = n_b P(v_z)Q(v_\perp)
\]
\[
P(v_z) = \delta(v_z - u) \quad \text{(A.39)}
\]
\[
Q(v_\perp) = \frac{1}{2\pi v_\perp} \delta(v_\perp - v_{\perp0})
\]

Thus $U$ and $W$ can be rewritten as
\[
U = \left( 1 - \frac{k_z v_z}{\omega} \right) PQ_\perp + \frac{k_z v_\perp}{\omega} P_z Q \quad \text{(A.40)}
\]
\[
W = \frac{n\Omega_e v_z}{\omega v_\perp} PQ_\perp + \left( 1 - \frac{n\Omega_e}{\omega} \right) P_z Q \quad \text{(A.41)}
\]
where $Q_\perp = \frac{dQ}{dv_\perp}$ and $P_z = \frac{dP}{dv_z}$. To calculate the susceptibility tensor $\chi_b$, two types of integration by parts are useful as the following

$$
\int_{0}^{\infty} dv_\perp \int_{-\infty}^{\infty} dv_z h(v_\perp) l(v_z) PQ_\perp
= l(u) \int_{0}^{\infty} dv_\perp h(v_\perp) Q_\perp
= l(u) [h(v_\perp) Q(v_\perp)]_0^\infty - l(u) \int_{0}^{\infty} dv_\perp Q \frac{dh}{dv_\perp}
= - l(u) \frac{1}{2\pi v_{\perp 0}} \frac{dh}{dv_\perp}(v_{\perp 0})
$$

(A.42)

$$
\int_{0}^{\infty} dv_\perp \int_{-\infty}^{\infty} dv_z h(v_\perp) l(v_z) PzQ
= \frac{1}{2\pi v_{\perp 0}} h(v_{\perp 0}) \int_{0}^{\infty} dv_z l(v_z) P_z
= \frac{1}{2\pi v_{\perp 0}} h(v_{\perp 0}) [l(v_z) P(v_z)]_0^\infty - \frac{1}{2\pi v_{\perp 0}} h(v_{\perp 0}) \int_{0}^{\infty} dv_z P \frac{dl}{dv_z}
= - \frac{1}{2\pi v_{\perp 0}} h(v_{\perp 0}) \frac{dl}{dv_z}(u)
$$

(A.43)

Taking advantage of equation (A.42) and (A.43), we can start to calculate each component of the susceptibility tensor. $\chi_{xx}$ can be calculated as

$$
\chi_{xx} = - \frac{\omega^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \frac{2n^2}{\lambda} J_n J_n' + \frac{2n^2}{\lambda} J_n J_n' \frac{n\Omega_e}{\omega - k_z u - n\Omega_e} + J_n^2 \cot^2 \theta \frac{n^2 \Omega_e^2}{(\omega - k_z u - n\Omega_e)^2} \right]
$$

(A.44)

Here $\lambda = k_{\perp} v_{\perp 0}/\Omega_e$. The argument of Bessel function and its derivatives is $\lambda$. The first term can be further calculated as

$$
\sum_{n=-\infty}^{\infty} \frac{2n^2}{\lambda} J_n J_n' = \sum_{n=-\infty}^{\infty} \frac{n}{2} (J_{n-1}^2 - J_{n+1}^2)
= \sum_{n=-\infty}^{\infty} \left( \frac{n+1}{2} J_n^2 - \frac{n-1}{2} J_n^2 \right)
= \sum_{n=-\infty}^{\infty} J_n^2
= 1
$$

(A.45)

Here the recurrence equation of Bessel function and its derivative are used. Thus

$$
\chi_{xx} = - \frac{\omega^2}{\omega^2} - \frac{\omega^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{2n^2}{\lambda} J_n J_n' \frac{n\Omega_e}{\omega - k_z u - n\Omega_e} + (J_n^2 \cot^2 \theta) \frac{n^2 \Omega_e^2}{(\omega - k_z u - n\Omega_e)^2} \right) \right]
$$

(A.46)
\( \chi_{yy} \) can be calculated as

\[
\chi_{yy} = -\frac{\omega_0^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \frac{1}{\lambda} (\lambda^2 (J_n')^2)' + \frac{1}{\lambda} (\lambda^2 (J_n')^2)' \frac{n\Omega_e}{\omega-k_u-n\Omega_e} + \lambda^2 (J_n')^2 \cot^2 \theta \frac{\Omega_e^2}{(\omega-k_u-n\Omega_e)^2} \right] \tag{A.47}
\]

The first term can be further calculated as

\[
\sum_{n=-\infty}^{\infty} \frac{1}{\lambda} (\lambda^2 (J_n')^2)' = \sum_{n=-\infty}^{\infty} \left[ 2(J_n')^2 + 2\lambda J_n'' \right] = \sum_{n=-\infty}^{\infty} 2J_n' \left[ -\frac{1}{\lambda} (\lambda^2 - n^2) J_n \right] = -2\lambda \sum_{n=-\infty}^{\infty} J_n J_n' + \sum_{n=-\infty}^{\infty} \frac{2n^2}{\lambda} J_n J_n'
\]

\[
= 1
\]

Here we have used several identities, including the definition of the Bessel differential equation \( \lambda^2 J_n'' + \lambda J_n' + (\lambda^2 - n^2) J_n = 0 \), the identity \( \sum_{n=-\infty}^{\infty} J_n J_n' = 0 \) and the identity equation (A.45). Thus

\[
\chi_{yy} = -\frac{\omega_0^2}{\omega^2} - \frac{\omega_0^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{1}{\lambda} (\lambda^2 (J_n')^2)' \right) \frac{n\Omega_e}{\omega-k_u-n\Omega_e} + (\lambda^2 (J_n')^2 \cot^2 \theta \frac{\Omega_e^2}{(\omega-k_u-n\Omega_e)^2} \right] \tag{A.49}
\]

\( \chi_{zz} \) can be calculated as

\[
\chi_{zz} = -\frac{\omega_0^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{2n}{\lambda} J_n J_n' \tan^2 \theta \right) \frac{k_z u}{\Omega_e} \frac{k_z u}{\omega-k_z u-n\Omega_e} + J_n^2 \frac{(\omega-n\Omega_e)^2}{(\omega-k_z u-n\Omega_e)^2} \right] \tag{A.50}
\]

The first term in the summation can be further calculated as

\[
\sum_{n=-\infty}^{\infty} \left( \frac{2n}{\lambda} J_n J_n' \right) \frac{k_z u}{\Omega_e} \frac{k_z u}{\omega-k_z u-n\Omega_e} = \sum_{n=-\infty}^{\infty} \left( \frac{2n}{\lambda} J_n J_n' \right) \frac{k_z u}{\Omega_e} \left( -1 + \frac{\omega-n\Omega_e}{\omega-k_z u-n\Omega_e} \right) = \sum_{n=-\infty}^{\infty} \left( \frac{2n}{\lambda} J_n J_n' \right) \frac{k_z u}{\Omega_e} \frac{\omega-n\Omega_e}{\omega-k_z u-n\Omega_e} \frac{1}{(\omega-k_z u-n\Omega_e)} \tag{A.51}
\]

\[
= 1 + \sum_{n=-\infty}^{\infty} \left( \frac{2n}{\lambda} J_n J_n' \right) \frac{(\omega-n\Omega_e)^2}{\Omega_e(\omega-k_z u-n\Omega_e)}
\]
The second and fourth ‘=’ sign in this summation use equation (A.45) and the following identity

\[
\sum_{n=-\infty}^{\infty} \frac{2n}{\lambda} J_n J'_n = \sum_{n=-\infty}^{\infty} (J_{n-1} + J_{n+1}) \cdot \frac{1}{2} (J_{n-1} - J_{n+1}) \\
= \frac{1}{2} \sum_{n=-\infty}^{\infty} (J_n^2 - J_{n+1}^2)
\]  
(A.52)

Thus

\[
\chi_{zz} = -\frac{\omega_p^2}{\omega^2} \tan^2 \theta - \frac{\omega_p^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \frac{2n}{\lambda} (\lambda J_n J'_n) + (\omega-n\Omega_e)^2 \right] \left( \frac{\omega-n\Omega_e}{\omega-k_z u-n\Omega_e} \right)
\]  
(A.53)

\[
\chi_{xy} \text{ can be calculated as}
\]

\[
\chi_{xy} = i\frac{\omega_p^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \frac{n}{\lambda} (\lambda J_n J'_n) + \frac{n\Omega_e}{\omega-k_z u-n\Omega_e} + n\lambda J_n J'_n \cot^2 \theta \frac{\Omega_e^2}{(\omega-k_z u-n\Omega_e)^2} \right]
\]  
(A.54)

The first term in the summation can be further calculated as

\[
\sum_{n=-\infty}^{\infty} \frac{n}{\lambda} (\lambda J_n J'_n) = \sum_{n=-\infty}^{\infty} n [ (J_n^2 + J_n J'_n) ] \\
= \sum_{n=-\infty}^{\infty} \frac{n}{4} [(J_{n-1} - J_{n+1})^2 + J_n (J_{n-2} - 2J_n + J_{n+2})] \\
= \frac{1}{4} \sum_{n=-\infty}^{\infty} (n+1)J_n^2 + (n+1)J_n J_{n+2} + (n-1)J_n^2 + (n-1)J_n J_{n-2} + 2nJ_n J_{n+2} + nJ_n J_{n+2}
\]  
(A.55)

Here the first ‘=’ sign uses equation (A.52), the second ‘=’ sign uses the recurrence relation and the third ‘=’ sign changes the indexing of some terms. Thus

\[
\chi_{xy} = \frac{i}{\omega_p^2} \sum_{n=-\infty}^{\infty} \left[ \frac{n}{\lambda} (\lambda J_n J'_n) \sum_{n=\infty}^{\infty} \left( \frac{n\Omega_e}{\omega-k_z u-n\Omega_e} + n\lambda J_n J'_n \cot^2 \theta \frac{\Omega_e^2}{(\omega-k_z u-n\Omega_e)^2} \right) \right]
\]  
(A.56)

\[
\chi_{yx} \text{ can then be calculated as}
\]

\[
\chi_{yx} = -\chi_{xy}
\]  
(A.57)

\[
\chi_{xz} \text{ can be calculated as}
\]

\[
\chi_{xz} = -\frac{\omega_p^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \frac{2n^2}{\lambda} J_n J'_n \tan \theta + \frac{2n^2}{\lambda} J_n J'_n \tan \theta \frac{\omega-n\Omega_e}{\omega-k_z u-n\Omega_e} + nJ_n^2 \cot \theta \frac{\Omega_e(\omega-n\Omega_e)}{(\omega-k_z u-n\Omega_e)^2} \right]
\]  
(A.58)
Using equation (A.45), the first term in $\chi_{xz}$ can be simplified. Thus

$$\chi_{xz} = \frac{\omega^2}{\omega^2} \tan \theta - \frac{\omega}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{2n^2}{\lambda} J_n J'_n \tan \theta \right) \frac{\omega-n\Omega_e}{\omega-k_z u-n\Omega_e} + (n J_n^2 \cot \theta) \frac{\Omega_e(\omega-n\Omega_e)}{(\omega-k_z u-n\Omega_e)^2} \right] \quad (A.59)$$

$\chi_{zx}$ can be calculated as

$$\chi_{zx} = -\frac{\omega^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{2n^2}{\lambda} J_n J'_n \tan \theta \right) \frac{k_z u}{\Omega_e} \frac{\omega-k_z u}{\omega-k_z u-n\Omega_e} + (n J_n^2 \cot \theta) \frac{\Omega_e(\omega-n\Omega_e)}{(\omega-k_z u-n\Omega_e)^2} \right] \quad (A.60)$$

The first term in this summation can be simplified using equation (A.51). Thus

$$\chi_{zx} = -\frac{\omega^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{2n^2}{\lambda} J_n J'_n \tan \theta \right) \frac{\omega-n\Omega_e}{\omega-k_z u-n\Omega_e} + (n J_n^2 \cot \theta) \frac{\Omega_e(\omega-n\Omega_e)}{(\omega-k_z u-n\Omega_e)^2} \right] \quad (A.61)$$

It is seen that

$$\chi_{xz} = \chi_{zx} \quad (A.62)$$

$\chi_{yz}$ can be calculated as

$$\chi_{yz} = -i \frac{\omega^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ -\frac{n}{\lambda} (\lambda J_n J'_n) \tan \theta + \frac{n}{\lambda} (\lambda J_n J'_n) \tan \theta \frac{\omega-n\Omega_e}{\omega-k_z u-n\Omega_e} + \lambda J_n J'_n \cot \theta \frac{\Omega_e(\omega-n\Omega_e)}{(\omega-k_z u-n\Omega_e)^2} \right] \quad (A.63)$$

The first term in this summation vanishes by using equation (A.55). Thus

$$\chi_{yz} = -i \frac{\omega^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{n}{\lambda} (\lambda J_n J'_n) \tan \theta \right) \frac{\omega-n\Omega_e}{\omega-k_z u-n\Omega_e} + (\lambda J_n J'_n \cot \theta) \frac{\Omega_e(\omega-n\Omega_e)}{(\omega-k_z u-n\Omega_e)^2} \right] \quad (A.64)$$

$\chi_{zy}$ can be calculated as

$$\chi_{zy} = i \frac{\omega^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{n}{\lambda} (\lambda J_n J'_n) \tan \theta \right) \frac{\omega-k_z u}{\Omega_e} \frac{\omega-k_z u}{\omega-k_z u-n\Omega_e} + (\lambda J_n J'_n \cot \theta) \frac{\Omega_e(\omega-n\Omega_e)}{(\omega-k_z u-n\Omega_e)^2} \right] \quad (A.65)$$

The first term in this summation can be further simplified as the following

$$\sum_{n=-\infty}^{\infty} \left( \frac{n}{\lambda} (\lambda J_n J'_n) \right) \frac{\omega-k_z u}{\Omega_e} \frac{\omega-k_z u}{\omega-k_z u-n\Omega_e} = \sum_{n=-\infty}^{\infty} \left( \frac{n}{\lambda} (\lambda J_n J'_n) \right) \frac{\omega-k_z u}{\Omega_e} \frac{1+\omega-n\Omega_e}{\omega-k_z u-n\Omega_e}$$

$$= \sum_{n=-\infty}^{\infty} \left( \frac{1}{\lambda} (\lambda J_n J'_n) \right) \frac{\omega-k_z u}{\Omega_e} \frac{\omega-n\Omega_e}{\omega-k_z u-n\Omega_e}$$

$$= \sum_{n=-\infty}^{\infty} \left( \frac{1}{\lambda} (\lambda J_n J'_n) \right) \frac{\omega-n\Omega_e}{\Omega_e} \frac{1+\frac{n\Omega_e}{\omega-k_z u-n\Omega_e}}$$

$$= \sum_{n=-\infty}^{\infty} \left( \frac{n}{\lambda} (\lambda J_n J'_n) \right) \frac{\omega-n\Omega_e}{\omega-k_z u-n\Omega_e} \quad (A.66)$$
Here the second and fourth ‘=’ sign use equation (A.55) and the following identity

\[
\sum_{n=-\infty}^{\infty} \frac{1}{\lambda^n} (\lambda J_n J'_n)' = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda^n} (J_n J'_n + \lambda (J_n J'_n)')
\]

\[
= \sum_{n=-\infty}^{\infty} (J_n J'_n)'
\]

\[
= \sum_{n=-\infty}^{\infty} ((J'_n)^2 + J_n J''_n)
\]

\[
= \frac{1}{4} \sum_{n=-\infty}^{\infty} [((J_n-1 - J_{n+1})^2 + J_n (J_{n-2} - 2J_n + J_{n+2})]
\]

\[
= 0
\]

where we have used \( \sum_{n=-\infty}^{\infty} J_n J'_n = 0 \), the recurrence relation of Bessel function and the change of indexing technique. Thus \( \chi_{zy} \) can be rewritten as

\[
\chi_{zy} = i \frac{\omega_{pb}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{n}{\lambda} (\lambda J_n J'_n)' \tan \theta \right) \frac{\omega - n\Omega_c}{\omega - k z u - n\Omega_c} + (\lambda J_n J'_n \cot \theta) \frac{\Omega_c (\omega - n\Omega_c)}{(\omega - k z u - n\Omega_c)^2} \right]
\]

(A.68)

It is seen that

\[
\chi_{zy} = -\chi_{yz}
\]

(A.69)

To summarize the results, we collect all the components of the susceptibility tensor as the following

\[
\chi_{xx} = -\frac{\omega_{pb}^2}{\omega^2} - \frac{\omega_{pb}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{2n^2}{\lambda} J_n J'_n \right) \frac{-n\Omega_c}{\omega - k z u - n\Omega_c} + (J_n^2 \cot^2 \theta) \frac{n^2 \Omega_c^2}{(\omega - k z u - n\Omega_c)^2} \right]
\]

(A.70)

\[
\chi_{yy} = -\frac{\omega_{pb}^2}{\omega^2} - \frac{\omega_{pb}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{1}{\lambda} (\lambda J_n J'_n)' \right) \frac{-n\Omega_c}{\omega - k z u - n\Omega_c} + (\lambda J_n J'_n \cot^2 \theta) \frac{\Omega_c^2}{(\omega - k z u - n\Omega_c)^2} \right]
\]

(A.71)

\[
\chi_{zz} = -\frac{\omega_{pb}^2}{\omega^2} \tan^2 \theta - \frac{\omega_{pb}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{2n}{\lambda} J_n J'_n \tan^2 \theta \right) \frac{\Omega_c^2}{(\omega - n\Omega_c)^2} + J_n^2 \frac{(\omega - n\Omega_c)^2}{(\omega - k z u - n\Omega_c)^2} \right]
\]

(A.72)

\[
\chi_{xy} = i \frac{\omega_{pb}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{n}{\lambda} (\lambda J_n J'_n)' \tan \theta \right) \frac{-n\Omega_c}{\omega - k z u - n\Omega_c} + (n\lambda J_n J'_n \cot^2 \theta) \frac{\Omega_c^2}{(\omega - k z u - n\Omega_c)^2} \right]
\]

(A.73)

\[
\chi_{yx} = -\chi_{xy}
\]

(A.74)

\[
\chi_{xz} = \frac{\omega_{pb}^2}{\omega^2} \tan \theta - \frac{\omega_{pb}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{2n}{\lambda} J_n J'_n \tan \theta \right) \frac{-n\Omega_c}{\omega - k z u - n\Omega_c} + (nJ_n^2 \cot \theta) \frac{\Omega_c (\omega - n\Omega_c)}{(\omega - k z u - n\Omega_c)^2} \right]
\]

(A.75)

\[
\chi_{zx} = \chi_{xz}
\]

(A.76)

\[
\chi_{yz} = -i \frac{\omega_{pb}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \left[ \left( \frac{n}{\lambda} (\lambda J_n J'_n)' \tan \theta \right) \frac{-n\Omega_c}{\omega - k z v u - n\Omega_c} + (\lambda J_n J'_n \cot \theta) \frac{\Omega_c (\omega - n\Omega_c)}{(\omega - k z u - n\Omega_c)^2} \right]
\]

(A.77)

\[
\chi_{zy} = -\chi_{yz}
\]

(A.78)


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