Lawrence Berkeley National Laboratory
Recent Work

Title
THEORETICAL STUDY OF A REACTION-DIFFUSION MODEL FOR FLAME PROPAGATION IN A GAS

Permalink
https://escholarship.org/uc/item/5kr664tn

Author
Lin, Shao-Shiung.

Publication Date
1979-07-01
THEORETICAL STUDY OF A REACTION-DIFFUSION MODEL FOR FLAME PROPAGATION IN A GAS

Shao-Shiung Lin
(Ph. D. thesis)

July 1979

TWO-WEEK LOAN COPY

This is a Library Circulating Copy which may be borrowed for two weeks. For a personal retention copy, call Tech. Info. Division, Ext. 6782.

Prepared for the U. S. Department of Energy under Contract W-7405-ENG-48
DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.
THEORETICAL STUDY OF A REACTION-DIFFUSIÓN
MODEL FOR FLAME PROPAGATION IN A GAS

Shao-Shiung Lin
(Ph.D. Dissertation)

Department of Mathematics and Lawrence Berkeley Laboratory†
University of California, Berkeley, California  94720

July 1979

†Partially supported by the Engineering, Mathematical, and Geosciences Division of the U.S. Department of Energy under contract W-7405-ENG-48.
THEORETICAL STUDY OF A REACTION-DIFFUSION
MODEL FOR FLAME PROPAGATION IN A GAS

Shao-Shiung Lin

Department of Mathematics and Lawrence Berkeley Laboratory
University of California, Berkeley, California 94720

Ph.D. Dissertation

July 1979

Abstract

We consider a system of two equations of a reaction-diffusion model for flame propagation in a gas. The system of two equations contains a parameter, the Lewis number, which is the ratio of the heat conduction coefficient to the coefficient of chemical species diffusion. Flame fronts are traveling wave solutions connecting an unburned gas state and the corresponding adiabatic burned gas state. If we assume the existence of a positive ignition temperature, then we prove that, for arbitrary Lewis number,

(a) there exists a unique flame front associated with each unburned gas state.

This result is new. We use an iteration scheme along with an estimate on the total enthalpy to prove the result.

When the Lewis number is one, the model equation can be reduced to a scalar reaction-diffusion equation. Under the assumption of positive ignition temperature, we prove further that

(b) if the initial gas states are close to a flame front, then the gas mixture evolves into the flame front at an exponential rate. In particular, each flame front is stable under finite amplitude perturbation.
(c) for general initial gas states, the gas mixture evolves asymptotically into a suitable combination of flame fronts. These results are generalizations of the corresponding assertions proved by Kanel' (for (b)) and Fife and McLeod (for (b) and (c)). The tools used in the proof include the phase-plane technique for autonomous systems of ordinary differential equations, the spectral analysis for operators of Sturm-Liouville type and an a priori estimate for solutions of parabolic equations.

In the case of Lewis number equal to one, we also discuss the dependence of the flame speed on the ignition temperature. We find that
(d) the flame speed increases monotonically to a finite limit as the ignition temperature tends to zero.

Moreover, we prove that
(e) without the assumption of a positive ignition temperature, there exists a family of traveling wave solutions connecting the same unburned and burned gas states.

The limit flame speed of (d) is the minimal wave speed of (e). The results of (d) and (e) give the first proof that it is not essential to assume the existence of a positive ignition temperature in order for a flame front to be formed. However, we give evidence showing that, without the assumption of a positive ignition temperature, the flame front may fail to be stable under finite amplitude perturbations.

We also include a discussion of the relationship between our model and physical reality.
Acknowledgments

I would like to thank Professor Alexander J. Chorin for his constant encouragement and support, and, most importantly, for his confidence in my ability, although the results presented in this thesis are not always what he originally expected me to finish.

Thanks are also due to my friends, Mr. Albert Erkip and Mr. Ai-Nung Wang, for some helpful discussions.

My friend Mr. Chuck Livingston, has taken the painful task of correcting my English.

Finally, I would also like to thank Dr. Paul Concus and the Lawrence Berkeley Laboratory for their support and provision of research facilities during this work.
# Table of Contents

1. Introduction .................................................. 1  

2. The Reaction-Diffusion Model Equations for Flame Propagation 6  

3. The Case of Lewis Number Equal to One .......................... 15  
   (A) Comparison Principles and A Priori Estimates .................. 17  
   (B) A Continuity Lemma ........................................ 19  
   (C) The Traveling Wave Solutions .................................. 21  
   (D) The Stability of the Flame Fronts ............................... 47  
   (E) Asymptotic Behavior ......................................... 56  

4. The Case of Arbitrary Lewis Numbers ................................. 75  

5. Conclusions ................................................................ 98  

Bibliography .......................................................... 100  

Figures ...................................................................... 103
1. Introduction

We consider a reaction-diffusion model for one-dimensional flame propagation in a combustible gas mixture. The model equations are consequences of the conservation of energy and the continuity of chemical species. We assume that the chemical reaction involved in the combustion process is one-step and first-order (for a definition, see [38]). Then the model equations are

\[
\begin{align*}
\frac{dT}{dt} &= T_{xx} + \frac{Q}{\rho c_p} CA(T), \\
\frac{dC}{dt} &= \frac{1}{L} C_{xx} - \frac{1}{\rho} CA(T),
\end{align*}
\]

(1.1)

where \(T(x,t)\) and \(C(x,t)\) are the temperature and the concentration of the reactant for the gas mixture, respectively, and

\[
A(T) = \begin{cases} 
\exp\left(-\frac{E}{RT}\right), & T > T_i, \\
0, & T \leq T_i.
\end{cases}
\]

\(Q, \rho,\) and \(c_p\) are positive constants (for a definition, see Section 2). \(T_i\) and \(L\) are nonnegative constants (while \(L = \infty\) is permitted). \(T_i\) is the ignition temperature. \(L\) is the Lewis number which is the ratio of the heat conduction coefficient to the coefficient of chemical species diffusion. \(A(T)\) is assumed to be smoothed around \(T_i\). The flame fronts are visualized as traveling wave solutions of (1.1), that is, solutions of the form

\[
T(x,t) = T(\xi), \quad C(x,t) = C(\xi), \quad \xi = x + vt, \quad v \geq 0
\]

with

\[
T(-\infty) = T_a, \quad C(-\infty) = C_u, \quad 0 \leq T_u \leq T_i, \\
T(\infty) = T_b, \quad C(\infty) = 0, \quad T_i < T_b.
\]
\((T_u, C_u)\) is an unburned state of the gas mixture such that the corresponding adiabatic burned state \((T_b, 0)\) is given by

\[
(1.2) \quad T_b = \frac{Q}{c_p} C_u + T_u.
\]

We are concerned with the existence and the stability of the flame fronts, along with the asymptotic behavior of the model equations at large time. The basic tools we use are the phase plane technique for systems of autonomous ordinary differential equations, the spectral analysis for operators of Sturm-Liouville type, the comparison principles and an a priori estimate for equations of parabolic type. In the proof of the existence of the flame fronts, a physical quantity, the total enthalpy, also plays an important role.

We state explicitly in Section 2 the assumptions for the gas mixture which lead to the system of equations (1.1). A salient feature of the assumptions is the constancy of the density during the combustion process which enables us to decouple the gas dynamical effects from the reaction-diffusion effects. A discussion on the usefulness of this model can be found in [33], [39].

In Section 3, we discuss the case of \(L\) equal to one. In consistence with (1.2), the total enthalpy (defined to be \(QC + c_pT\)) is assumed to be a constant during the combustion process. Then equations (1.1) can be reduced to a scalar reaction-diffusion equation of the form

\[
(1.3) \quad T_t = T_{xx} + g(T), \quad g(0) = g(1) = 0, \quad g'(1) < 0,
\]

\[
g(T) = 0 \quad \text{for} \quad 0 < T < T_1, \quad g(T) > 0 \quad \text{for} \quad T_1 < T < 1.
\]

In subsection (C), we generalize Gel'fand's argument in [17] to show that
(1.4) (a) if \( T_1 > 0 \), then there exists a unique traveling wave solution for (1.3) connecting \( T = \alpha \) and \( T = 1 \) for each \( \alpha \) with \( 0 \leq \alpha < T_1 \).

(b) if \( T_1 = 0 \), then there exists a family of traveling wave solutions for (1.3) connecting \( T = 0 \) and \( T = 1 \).

(a) is also proved in [2], [12], [21], etc. In case of \( T_1 \) equal to 0, \( g^{(n)}(0) = 0 \) for every \( n \). The conclusion of (b) resembles the Kolmogorov type equations (where \( g(T) \) is concave on \( 0 < T < 1 \), see [23]). However, the conclusion of (b) is new and provides an example showing that flame fronts can still exist without the assumption of a positive ignition temperature. Furthermore, we show that the minimal speed wave in (b) can be obtained as the limit of the flame front in (a) as \( T_1 \rightarrow 0 \). The flame speed of (a) increases monotonically to the minimal wave speed of (b) as \( T_1 \rightarrow 0 \). Thus, we give some evidence showing that the minimal speed wave in (b) is physically realizable.

When \( T_1 > 0 \) in (1.3), the flame fronts in (a) are further shown to be the building blocks of the asymptotic gas states for general Cauchy problems of (1.3). Sattinger's stability criterion in [32] is applied in subsection (D) to study the stability of the flame fronts of (1.4)(a). In subsection (E), we strengthen the stability results in (D) to study the asymptotic behavior of general Cauchy problems for (1.3). The main results we obtain are

1) Suppose that

\[ T(x,0) = 0 \text{ for } x < a, \quad T(x,0) > T_1 \text{ for } x > b \]

where \( a < b \). Then \( T(x,t) \) approaches the flame front of
(1.4)(a) with \( \alpha = 0 \) in an exponential rate as \( t \to \infty \).

(2) Suppose that

\[
T(x,0) = \alpha \quad \text{for} \quad x \leq a, \quad T(x,0) = \beta \quad \text{for} \quad x > b, \\
T(x,0) > T_i \quad \text{for some subinterval of} \quad a \leq x \leq b,
\]

where \( a < b \) and \( 0 \leq \alpha, \beta < T_i \). Then \( T(x,t) \) is asymptotically equivalent to two flame fronts propagating in a diverging directions as \( t \to \infty \).

These results generalize the study of Kanel' [22] for (1.3) where assertion (1) is proved under more restrictive conditions and without an exponential decay rate. Fife and McLeod [12] obtained also similar results as (1) and (2) for different nonlinear source term (where \( g(T) < 0 \) for \( T \) close to 0). We apply the special result of Kanel' and the technique presented in [12], along with Sattinger's stability criterion, to prove these results.

We consider system (1.1) with arbitrary \( L \) in Section 4. We prove that

Given an unburned state \((T_u, C_u)\) such that \( T_b \), as computed in (1.2), is greater than \( T_i \), then there exists a unique flame front connecting \((T_u, C_u)\) and \((T_b, 0)\).

\( T_b \) is the adiabatic flame temperature corresponding to the unburned state \((T_u, C_u)\), as explained in [18]. This result is the first rigorous proof that such flame fronts exist. An interesting heuristic discussion of this fact was provided in [39]. Such existence is a natural consequence of the assumptions stated in Section 2 on which the model equations are based. Basically, we devise an iteration scheme which enables us to find a sequence of candidates for temperature
and concentration profiles. However, the important estimate which ensures that such sequence converges to a flame front is provided by an "almost" conservation of the total enthalpy. A study of how the total enthalpy changes during the whole process of flame propagation is also included.

We now give a brief history of our reaction-diffusion model. Flame fronts which occur as a result of energy balance between the heat release of the chemical reactions and other mechanisms was proposed in late nineteenth century by Le Chatelier et al.; see the historical account in [18], [26]. However, the influential work by Kolomogroff et al. [23] (1937) gave the first indication that heat conduction coupled with a nonlinear source term is able to sustain a steady wave form. After this work, a Russian school (notably, Zeldovich, Landau, Barenblatt, etc.) began to formulate this model for flame propagation during the decade 1938-1948. The conjecture that flame fronts actually exist in this model in the case of Lewis number equal to one was stated in [40] (1948). During the fifties, many chemical and biological phenomena, especially in so-called dissipative structures, were modelled on the interactions between chemical reactions and dissipative mechanism. Notable examples are Huxley's model for nerve pulse propagation (1952) and the Belousov reactions (1958) (see [24]). As for the combustion model, Spalding [36] used this model to compute laminar flame speed. Gel'fand [17] provided a rigorous proof showing the existence of flame fronts for a single equation model. Kanel' [21], [22] studied the asymptotic behavior for a single equation model. The recent studies of this model trying to keep a flavor of physical reality are provided by Sivashinsky [33], [34], [35].
2. The Reaction-Diffusion Model Equations for Flame Propagation

Consider an infinitely long tube filled with a combustible gas mixture (Figure 1). We assume that the gas mixture is motionless at some initial instant. It is ignited at some place so that, after a while, we can see the "flames" propagating through the tube. It will be shown in the following sections that these flame fronts are maintained by a balance of chemical energy release and the heat conduction along with chemical species diffusion. We now formulate the assumptions and the consequent equations which reflect these assumptions.

We use $x$ to denote the one-dimensional coordinate along the tube. The first assumption is

**Assumption (a):** We assume that the gas states are one-dimensional. Thus each state variable is a function of $(x,t)$ only where $t$ is the time parameter. The state variables relevant to us are

- the density of the gas mixture $\rho(x,t)$,
- the temperature $T(x,t)$,
- the mass concentration $C_i(x,t)$ of the $i^{th}$ chemical species participating as reactants in the chemical reactions, $i = 1, \ldots, n$, where $n$ is the total number of reactants.

As we shall show in the next section, assumption (a) will have a definite influence on the stability of the flame fronts (see Schivashinsky [33], [35] for a case of a two-dimensional model where instability of the flame fronts is possible).

As discussed in Glassman [18] or Lewis and Von Elbe [26], the flame can be considered hydrodynamically as a subsonic wave maintained by chemical energy release. In the case we consider, the flame speed
will be much smaller than the sound speed. Thus we can ignore the
dynamic compressibility of the gas mixture. Hence the density $\rho$ can
only depend on the temperature $T$. However, we introduce the main (and
most restrictive) assumption:

**Assumption (B):** We assume that the density $\rho$ remains constant
during the whole combustion process. That is, there is no thermal
expansion.

A consequence of this assumption is that the hydrodynamical effects can
be considered separately from the thermal effects. In particular,
since the gas mixture is initially motionless, it remains motionless
afterwards. Note that assumption (B) is rather artificial, and it
forces us to consider the case of an open tube behind the flame fronts;
that is, there is no boundary effect. See Glassman [12] for more
experimental descriptions. Thus our problem will be posed on the whole
real line.

We now give some preliminary assumptions on the chemical reaction
part. To simplify our presentation, we assume that the reactants in
the gas mixture only participate in an (overall) one-step chemical
reaction. General cases of multi-steps chemical reactions present no
new basic principles, and are just a matter of more complicated nota-
tions; see Williams [38] or Gavalas [16] for the details of general
chemical kinetics. Thus, if $A_i$ represents the reactant with concen-
tration $C_i$, the chemical reaction can be represented by

\[
\sum_{j=1}^{n} \alpha_j A_j \rightarrow \sum_{j=1}^{n} \beta_j A_j + \text{other products}
\]

where $\alpha_j$, $\beta_j$ are nonnegative integers called the stoichiometric
coefficients of the reaction. Note that $\beta_j = 0$ whenever $A_j$ does not reappear as a reaction product. The law of mass action gives the disappearance rate $R_i$ of the reactants $A_i$ as

$$R_i = (\beta_i - \alpha_i) k \prod_{j=1}^{n} C_j^{\alpha_j}, \quad i = 1, \ldots, n.$$  

The $k$ in (2.2) is called the specific reaction rate constant, and is assumed to be the same for each $R_i$. Actually, Arrhenius (1889), on the basis of arguments arising from statistical molecule collision theory, deduced that, for virtually all chemically reacting systems, $k$ depends mainly on the temperature. In fact, the Arrhenius relation

$$k = z \exp \left( \frac{E}{RT} \right)$$

holds. $R$ is the universal gas constant. $E$ is a constant called the activation energy which represents a potential barrier which the reactants must overcome to start the reaction (2.1). $z$ represents a statistical molecule collision frequency such that the actual collision energy overcomes the potential barrier $E$. We shall assume $z$ to be a constant. The Arrhenius relation (2.3) is widely verified by experimental evidence; see Williams [38] or Frank-Kamenetzky [14] for some experimental and theoretical discussions. We define

$$R(C_1, C_2, \ldots, C_n, T) = z \exp \left( -\frac{E}{RT} \right) \prod_{j=1}^{n} C_j^{\alpha_j}.$$  

Then

$$R_j = (\beta_j - \alpha_j) R, \quad j = 1, \ldots, n.$$  

For the purpose of describing the combustion process, we make the following assumption on chemical kinetics.
Assumption (γ): Reaction (2.1) is assumed to be exothermic with a "high" activation energy $E$ accompanied by a constant heat release of quantity $Q$. We also assume that the reactants $A_i$ of the gas mixture in the tube are in exact stoichiometric ratio.

The meaning of "exothermic" can be understood as $E > 0$ and $Q > 0$; see [38] or [14]. However, since we want to deal with gas systems capable of quick energy release, we require more of reaction (2.1), i.e. we require that the activation energy $E$ be high. Suppose that the gas system goes through reaction (2.1); its products have the adiabatic temperature $T_b$ [which can be computed solely from (2.1) and thermodynamic considerations, and is dependent only upon the concentrations of the reactants and the temperature of the gas before reaction; see Chapter I of Glassman [12]]. The requirement that $E$ be high can be expressed as

$$\frac{E}{RT_b} \gg 1. \tag{2.4}$$

A consequence of this assumption is that the gas system with low initial temperature has a negligible chemical reaction rate so that it may take a very long time to complete reaction (2.1). On the other hand, if the initial temperature of the gas system is high enough, reaction (2.1) is quickly completed due to (2.4). Thus it is reasonable to assume that, for the gas system under consideration, there exists a fixed positive temperature $T_i$, called the ignition temperature, such that the specific reaction rate constant of (2.3) satisfies

$$k = 0 \text{ for } 0 \leq T \leq T_i. \tag{2.5}$$

That is, the gas system can be ignited only if we raise the initial
temperature over the temperature $T_1$. Except for some special cases which will be mentioned explicitly, we always assume (2.5) in the following sections. We shall see that (2.5) is essential for the "quick" formation of flame fronts—a fact mentioned in [33] or [28]. However, in a particular case discussed in Section 3, (2.5) is not responsible for the existence of flame fronts. We should remark that the assumption of exact stoichiometricity of the gas mixture in $(\gamma)$ is not essential in the formulation of the equations. However, it has to do with the "inflammability" of the gas mixture, i.e. the ability to sustain a flame, and hence the stability of the flame fronts. Some experimental and theoretical discussions on the effect of non-stoichiometry of the gas system can be found in [26], [18], [27], [33], and [35].

Under the assumptions $(a)$, $(b)$, $(\gamma)$, we can now formulate the model equations. The basic connections among the gas states are provided by the conservation laws. Since we have no hydrodynamical effects, the conservation of energy and the continuity of the chemical species yield the following set of equations.

**Conservation of energy:**

\[
(2.6) \quad \rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}(\lambda \frac{\partial T}{\partial x}) + QR(C_1, \ldots, C_n, T).
\]

**Continuity of chemical species:**

\[
(2.7) \quad \rho \frac{\partial C_i}{\partial t} = \rho \frac{\partial}{\partial x}(\mu_i \frac{\partial C_i}{\partial x}) + R_i(C_1, \ldots, C_n, T), \quad i = 1, \ldots, n.
\]

The parameters in (2.6) and (2.7) are

- $\lambda = $ heat conductivity of the gas mixture,
- $\mu_i = $ diffusivity of the chemical species $A_i$, 
- $C_i = $ concentration of the chemical species $A_i$, 
- $R_i = $ reaction rate of the chemical species $A_i$, 
- $QR = $ heat release rate per unit volume.
\[ c_p = \text{specific heat capacity of the gas mixture under a given constant pressure } p. \]

Equations (2.6) and (2.7) are considered in the upper half plane: \(-\infty < x < \infty, \ t \geq 0\). For simplicity of analysis, we will assume that \(\lambda, \mu_1, \text{ and } c_p\) are all constants. We can justify this assumption on \(c_p\). As discussed in Courant and Friedriech [9] or Glassman [18], the pressure change across a deflagration wave (a flame) is approximately zero. We will assume that the pressure is exactly constant. It is at this pressure which the specific heat capacity \(c_p\) is computed.

Equations (2.6) and (2.7) are a system of parabolic equations with a nonlinear source term. It is a special case of a class of nonlinear partial differential equations called reaction-diffusion equations. Hence we will call our model a reaction-diffusion model for flame propagation. The resulting theory is called a thermal-diffusional theory of laminar flames in [4] or [33].

The main question in this model is the existence of flame fronts, as shown in Fig. 1. Flame fronts are visualized as traveling wave solutions of system (2.6) and (2.7) connecting the burned and unburned states. Suppose that an unburned state of the gas mixture is given by

\[ T = T_u, \ C_j = C^u_j, \ j = 1, \ldots, n, \]

where \(T_u, C^u_j\) are constants so that \(T_u \leq T_i\) (whenever \(T_i\) is assumed), and the \(C^u_j\) are in exact stoichiometric ratio for (2.1). After burning, the temperature reaches its adiabatic temperature \(T_b\) and the concentrations have new states \(C^b_j\). Thus,

\[ T = T_b, \ C_j = C^b_j, \ j = 1, \ldots, n \]
can be viewed as a burned state. We seek a traveling wave solution for (2.6) and (2.7) of the form

\[(2.8) \quad T(x,t) = T(\xi), \quad C_j(x,t) = C_j(\xi), \quad \xi = x + Vt\]

such that

\[(2.9) \quad \lim_{\xi \to \infty} T(\xi) = T_b, \quad \lim_{\xi \to \infty} C_j(\xi) = C^b_j,\]

\[\lim_{\xi \to -\infty} T(\xi) = T_u, \quad \lim_{\xi \to -\infty} C_j(\xi) = C^u_j, \quad j = 1, \ldots, n.\]

\(V\) is an unknown constant to be determined, called the speed of the traveling wave. We will call \(V\) the flame speed determined by the given burned and unburned states. Note that the burned and unburned states all satisfy

\[R(C_1, \ldots, C_n, T) = 0,\]

since, after burning, some \(C_j^b = 0\) (i.e. \(\beta_j = 0\) in reaction (2.1)) due to the stoichiometric assumption \((\gamma)\). Thus the traveling wave (2.8) connects two stationary states of a dynamical system. Typically, it is a difficult problem to determine whether these traveling wave solutions, (2.8) and (2.9), exist or not. However, we will work theoretically with a simpler model of the chemical reaction to show the existence of flame fronts and leave the general problem to a numerical investigation. More generally, we are also interested in showing whether the model equations, (2.6) and (2.7), live up to our expectation. That is, if we consider the general Cauchy problem for (2.6) and (2.7), we are interested not only in showing the existence of the solution, but mainly in the properties of the solution and, in
particular, in determining the relationships between the general solutions and the particular solutions which are flame fronts. In some particular cases, we shall see in Section 3 that flame fronts play a dominant role in forming the general solutions of (2.6) and (2.7). They are the intermediate asymptotic solutions for (2.6) and (2.7) discussed in [3].

Through a change of variable \[ \bar{x} = \sqrt{\frac{\rho c_p}{\lambda}} x \], equations (2.6) and (2.7) can be written as

\[
\begin{align*}
\frac{\partial T}{\partial t} &= \frac{\partial^2 T}{\partial x^2} + \frac{Q}{\rho c_p} R(C_1, \ldots, C_n, T) \\
\frac{\partial C_i}{\partial t} &= \frac{1}{L_i} \frac{\partial^2 C_i}{\partial x^2} + \frac{1}{\rho} R_i(C_1, \ldots, C_n, T), \quad i = 1, \ldots, n
\end{align*}
\]

where the numbers

\[ L_i = \frac{\lambda}{\mu_i \rho c_p}, \quad i = 1, \ldots, n, \]

are the Lewis numbers of the gas mixture. The Lewis numbers measure the relative importance of heat conduction and chemical species diffusion. Suppose now that \( n = 1 \) and \( \alpha_1 = 1 \) in (2.1), i.e., the chemical reaction is of the form

\[ A + \text{diluents} \rightarrow \text{products} + \text{diluents} \]

[where we have suppressed the index], then equations (2.10) are the system

\[
\begin{align*}
\frac{\partial T}{\partial t} &= \frac{\partial^2 T}{\partial x^2} + \frac{Q}{\rho c_p} R(C, T) , \\
\frac{\partial C}{\partial t} &= \frac{1}{L} \frac{\partial^2 C}{\partial x^2} - \frac{1}{\rho} R(C, T) ,
\end{align*}
\]

where
\[ R(C,T) = \begin{cases} \frac{z C \exp(-\frac{E}{RT})}{R_{T_{i}}} , & \text{if } T > T_{i} \\ 0 , & \text{if } T < T_{i} \end{cases} \]

and we assume that \( R(C,T) \) has been smoothed around \( T = T_{i} \) so that \( \frac{\partial R(C,T_i)}{\partial T} > 0 \) for all \( C > 0 \). We will give a theoretical study of equations (2.11).

Some simple observations concerning the solution of general Cauchy problem for (2.11) can be readily made. Suppose that equations (2.11) are given the initial data

(2.12) \( T(x,0) = T_{0}(x) , \ C(x,0) = C_{0}(x) , \ -\infty < x < \infty , \)

where \( T_{0}(x) \) is positive and bounded, \( 0 \leq C_{0}(x) \leq 1 \). Then

(a) if \( T_{0}(x) \) and \( C_{0}(x) \) are piecewise continuous, there exists
\n a unique global solution \( T(x,t) , C(x,t) \) for the systems
\n(2.11) and (2.12);
(b) if \( T_{0}(x) = T_{0} \) and \( C_{0}(x) = C_{0} \) are both constants, this
unique solution is just the solution of the chemical reaction part, i.e. the solutions satisfy the ordinary differential equation

\[
\begin{cases}
\frac{dT}{dt} = \frac{Q}{\rho C_{p}} R(C,T) , & T(0) = T_{0} , \\
\frac{dC}{dt} = -\frac{1}{\rho} R(C,T) , & C(0) = C_{0} .
\end{cases}
\]

Basically, (a) is due to the fact that \( R(C,T) \) is Lipschitz continuous on every compact subset of the \((C,T)\) plane, and the region \( \{(C,T) \mid 0 < C \leq 1, T \geq 0\} \) is an invariant domain for (2.11) and (2.12). For a detailed proof, see Friedman [15] or Edel'man [10]. We remark that
(a) and (b) are also true for the more general system (2.10).
3. The Case of Lewis Number Equal to One

When \( L = 1 \), the system (2.11) can be reduced to a single equation. Define

\[
H = T + \frac{\varrho Q^C}{C_p}.
\]

H satisfies the heat equation

\[
H_t = H_{xx},
\]

where a subscript denotes differentiation with respect to the corresponding variable, e.g. \( H_t = \frac{\partial H}{\partial t} \). If \( H \) is constant initially, \( H \) remains constant. In this section (except in subsection (F)), we shall assume that

(3.1) \( H(x,t) = H_0 = \text{constant}, \quad |x| < \infty, \quad t \geq 0 \).

Then (2.11) is equivalent to the equation

(3.2) \( T_t = T_{xx} + h(T) \)

where

\[
h(T) = \begin{cases} \frac{Z}{\varrho}(H_0 - T)\exp(-\frac{E}{RT}), & \text{if } T > T_i \\ 0, & \text{if } T < T_i \end{cases}
\]

and \( h(T) \) is made continuous around \( T_i \) with \( h'(T_i^+) > 0 \). The assumption (3.1) implies that

\[
0 \leq T(x,t) \leq H_0, \quad \text{for all } (x,t).
\]

The change of variables

\[
\bar{T} = \frac{T}{H_0}, \quad \bar{x} = \frac{1}{\sqrt{H_0}} x, \quad \bar{T}_i = \frac{T_i}{H_0}
\]
transforms (3.2) into

\[(3.3) \quad T_t = T_{xx} + g(T)\]

(we omit the tildes ~), with

\[g(T) = \begin{cases} 
\frac{z}{\rho} (1-T) \exp \left( -\frac{E}{R \rho \Omega T} \right), & \text{if } T > T_1 \\
0, & \text{if } T < T_1. 
\end{cases}\]

Note that \(g(T)\) is smoothed around \(T_1\) and \(g'(T_1^+) > 0\). Note also that \(T\) in (3.3) is dimensionless. The main purpose of this section is to analyze (3.3) theoretically. The graph of \(g(T)\) has a shape as in Figure 2. The assumption on the chemical reaction (2.11) and (3.1) imply that \(T = 1\) is the only burned state of (3.3). However, any \(T\) with \(0 < T < T_1\) can be viewed as an unburned state.

For our convenience, we shall simultaneously consider the following class of reaction-diffusion equations:

\[(3.4) \quad u_t = u_{xx} + f(u), \quad |x| < \infty, \quad t > 0.\]

The function \(f(u)\) is assumed to be continuous and piecewise smooth on the real line, and satisfy either

\[(3.5) \quad \begin{aligned}
f(u) &\leq 0 \text{ for } 0 \leq u \leq u_1, \quad f(u) > 0 \text{ for } u_1 < u < 1; \\
f(0) = f(1) = 0, \quad f'(1) < 0, \quad f'(u_1^+) < \infty, \quad \text{and } \int_0^1 f(u)du > 0, 
\end{aligned}\]

where \(u_1\) is some fixed number with \(0 < u_1 < 1\), or

\[(3.6) \quad \begin{aligned}
f(u) &> 0 \text{ for } 0 < u < 1, \quad f(0) = f(1) = 0, \quad \text{and} \\
f'(0) < \infty, \quad f'(1) < 0. 
\end{aligned}\]

Note that equation (3.3) is a special case of (3.4) with \(g(T)\)
satisfying (3.5) with \( u_i = T_i \). If equation (3.4) is given an initial condition

\[
u(x,0) = \phi(x), \quad |x| < \infty
\]

which satisfies \( 0 \leq \phi(x) \leq 1 \) and is continuous except for some discrete jumps, then a consequence of (3.5) or (3.6) is that (3.4) possesses a unique global solution \( u(x,t) \) which is smooth for \( t > 0 \) and \( 0 \leq u(x,t) \leq 1 \) for all \((x,t)\). We have remarked on this point in Section 2. Thus we will call \( u(x,t) \) "the solution determined by \( \phi(x) \). Our main concern will be the asymptotic behavior of \( u(x,t) \) as \( t \) becomes large.

The remainder of this section is divided into several subsections. We begin by proving the existence of traveling wave solutions. Then we clarify what we mean by flame fronts. Eventually we examine the stability and asymptotic behavior of the flame fronts. Our ultimate objective will be to clarify the relationships between the general solutions of (3.3) and the flame fronts.

(A) Comparison Principles and A Priori Estimates

The basic tools we will use are the comparison principle and a priori estimates of Schauder type for equations of parabolic type.

Comparison Principle: Let \( u(x,t) \) be a solution of (3.4) and \( v(x,t) \) be a function defined for all \( |x| < \infty, \ t > 0 \) such that \( v, v_x, v_{xx}, v_t \) are continuous for \( t > 0 \) and uniformly bounded for \( t > 0 \). Suppose that \( v \) satisfies the inequalities:
\[ v_t \leq v_x + f(v), \]
\[ v(x,0) \leq u(x,0), \quad |x| < \infty, \quad t > 0. \]

Then

\[ v(x,t) \leq u(x,t) \text{ for all } (x,t). \]

The proof of this comparison principle can be found in [29] or [15]. As \( f(u) \) in (3.4) satisfies either (3.5) or (3.6), 0 and 1 are two solutions of (3.4). Thus if an initial condition for (3.4) lies between 0 and 1, the solution lies between 0 and 1 also. These kinds of bounds for solutions of (3.4) can be strengthened to include bounds for their derivatives via the use of the following a priori estimates.

**A Priori Estimates:** Suppose that \( u(x,t) \) is a classical solution of (3.4). Then, for each \( t_0 > 0 \), there exists a constant \( C(t_0) \) [independent of \( u(x,t) \)] such that

\[
\sup_{|x| < \infty, \ t \geq t_0} \{ u_x(x,t), u_{xx}(x,t), u_t(x,t) \} \\
\leq C \left( \sup_{|x| < \infty, \ t \geq 0} |f(u(x,t))| + \sup_{|x| < \infty, \ t \geq 0} |u(x,t)| \right). 
\]

Thus, if initially \( 0 \leq u(x,0) \leq 1 \), then, for each \( t_0 > 0 \), \( u_x, u_{xx} \) and \( u_t \) are uniformly bounded for \( t \geq t_0 \). This bound depends only on \( t_0, \max_{0 \leq u \leq 1} |f(u)| \text{ and } \max_{0 \leq u \leq 1} |f'(u)|. \) This form of a priori estimate is stated in [12]. It is a simplified version of the general interior Schauder estimates for parabolic equations, as can be found in [15] or [10].
(B) **A Continuity Lemma**

Suppose that the functions \( f_n(u) \), \( n = 0, 1, 2, \ldots \), satisfy (3.5) or (3.6). Define \( u^n(x,t) \) as the solution of

\[
\begin{align*}
\begin{cases}
  u_t &= u_{xx} + f_n(u) \\
  u(x,0) &= \phi_n(x), \\n  |x| &< \infty, \ t \geq 0,
\end{cases}
\end{align*}
\]

where \( 0 \leq \phi_n(x) \leq 1 \) and \( \phi_n(x) \) is piecewise continuous for each \( n = 0, 1, 2, \ldots \).

**Lemma 1:** For each \( (x,t) \), the estimate

\[
\left| u^n(x,t) - u^0(x,t) \right| \leq \frac{\epsilon_n}{M} (e^{Mt} - 1) + \epsilon_n e^{Mt}, \quad n = 1, 2, \ldots,
\]

holds, where \( M = \max_{0 \leq u \leq 1} |f'_0(u)| \) and

\[
\epsilon_n = \sup_{0 \leq u \leq 1} |f_n(u) - f_0(u)|, \quad \delta_n = \sup_{|x| \leq \infty} |\phi_n(x) - \phi_0(x)|.
\]

Furthermore, if \( \lim_{n \to \infty} \epsilon_n = \lim_{n \to \infty} \delta_n = 0 \), then

\[
(3.8) \quad u^n(x,t) \to u^0(x,t), \quad u^n_x(x,t) \to u^0_x(x,t), \quad \text{as } n \to \infty
\]

uniformly for all \( x \) and for all \( t \) in any compact subinterval of \( 0 < t < \infty \).

**Proof.** Pick any \( n \) and fix it. Define \( \omega(x,t) = u^n(x,t) - u^0(x,t) \). Then

\[
\begin{align*}
\omega_t &= \omega_{xx} + f_n(u^n) - f_0(u^0) \\
&= \omega_{xx} + (f_n(u^n) - f_0(u^n)) + (f_0(u^n) - f_0(u^0)).
\end{align*}
\]
Thus \( \omega \) satisfies the inequality:

\[
\omega_{xx} - \epsilon_n - M|\omega| \leq \omega_t \leq \omega_{xx} + \epsilon_n + M|\omega|,
\]

\[-\delta_n \leq \omega(x,0) \leq \delta_n, \quad |x| < \infty, \quad t \geq 0.
\]

Since the right hand side of (3.7) is the solution for the initial value problem

\[
\begin{align*}
\tilde{\omega}_t &= \tilde{\omega}_{xx} + \epsilon_n + M|\tilde{\omega}|, \\
\tilde{\omega}(x,0) &= \delta_n,
\end{align*}
\]

the comparison principle yields the estimate (3.7).

In particular, if \( \lim \epsilon_n = \lim \delta_n = 0 \), it follows directly from (3.7) that \( u^n(x,t) \rightarrow u^0(x,t) \) uniformly for all \( x \) and for all \( t \) in any compact subset of \( 0 < t < \infty \). In order to show the remaining part of (3.8), define

\[
A = \{(x,t) \mid |x| < \infty, \quad t_1 \leq t \leq t_2\}
\]

where \( t_1 \) and \( t_2 \) are two fixed constants satisfying \( 0 < t_1 < t_2 < \infty \).

The a priori estimates in subsection (A) imply that \( u^n_{xx}, u^n_x, u^n_t \) are uniformly bounded on \( A \). Hence, by the mean value theorem, the family \( \{u^n_x\} \) must be equicontinuous on \( A \). Therefore, by the Arzela-Ascoli theorem (see [30]), any subsequence of \( \{u^n_x\} \) must contain a sub-subsequence which converges uniformly on \( A \) to a limit function. This limit function must be \( u^0_x \), since \( u^n \rightarrow u^0 \) uniformly on \( A \). That is, each subsequence of \( \{u^n_x\} \) has a further subsequence which
converges uniformly on $A$ to $u_x^0$. Hence, $u_x^n$ converges uniformly on $A$ to $u_x^0$. This completes the proof of (3.8).

This lemma shows the continuity of solutions of (3.4) under perturbations of the nonlinear source term $f(u)$ and the initial data. We will use this fact repeatedly in the following subsections.

(C) The Traveling Wave Solutions

In this subsection, we will discuss the existence of traveling wave solutions of the combustion equation (3.3), and, more generally, of equation (3.4) whose nonlinear source term satisfies either (3.5) or (3.6). By definition, a traveling wave solution is a solution of the form

$$(3.9) \quad u(x,t) = U(\xi), \quad \xi = x + Vt,$$

where $V$ is some real number. In order for $U(\xi)$ to be a solution for (3.4), $U$ must satisfy a second order ordinary differential equation containing $V$ as a parameter. At $\xi = \pm \infty$, $U$ equals one of the stationary states of $f(u)$. $U$ satisfies an ordinary differential equation which can be transformed into an autonomous system of two first order ordinary differential equations. Such a transformation enables us to apply the phase plane technique discussed in standard textbooks of ordinary differential equations such as Hartman [19] or Coddington and Levinson [7]. Thus, we are able to apply this technique to show the existence of traveling wave solutions for the equation (3.4) [Theorem 1 and Theorem 2 below]. All those results depend on a fundamental lemma [Lemma 2] concerning the variation of possible candidates for $U(\xi)$ as the parameter $V$ changes. This lemma was first
proved by Gel'fand [17]. Our strategy in this subsection will be first to discuss the trajectories (or phase portraits) of the associated autonomous system of ordinary differential equations. This will lead us to the fundamental lemma. Then, after proving Theorem 1 and Theorem 2, we will specialize to the combustion equation (3.3). A family of flame fronts is found. Finally, we apply the fundamental lemma again to discuss the continuity properties of those flame fronts with respect to the unburned states or the ignition temperature. This subsection will be the basis of the following subsections.

A Fundamental Lemma

We seek a traveling wave solution of the form (3.9) for the equation (3.4). Then $U(\xi)$ satisfies the ordinary differential equation

$$VU' = U'' + f(U),$$

where $U' = \frac{dU}{d\xi}$, etc. Since the equation (3.4) is invariant under the reflection $x \rightarrow -x$, $V$ will be assumed to be nonnegative throughout the whole subsection. Introducing the variable

$$W = U',$$

(3.10) is then transformed into the autonomous system

$$\begin{cases}
U' = W, \\
W' = VW - f(U).
\end{cases}$$

(3.11)

The singular points of (3.11) are those points $(U,0)$ where $f(U) = 0$. In particular, $(0,0)$ and $(1,0)$ are two singular points of (3.11). If (3.11) is linearized around a singular point $(U,0)$, the linearized
system has the coefficient matrix

\[
\begin{bmatrix}
0 & 1 \\
-f'(U) & V
\end{bmatrix}
\]

At (1,0), the eigenvalues of this matrix are

\[
\lambda_1(V) = \frac{1}{2}(V + \sqrt{V^2 - 4f'(1)}) , \quad \lambda_2(V) = \frac{1}{2}(V - \sqrt{V^2 - 4f'(1)})
\]

Since \( f(u) \) satisfies (3.5) or (3.6), \( f'(1) < 0 \). Hence,

\[
\lambda_2(V) < 0 < \lambda_1(V) \text{ for all } V \geq 0.
\]

That is, (1,0) is a saddle point of (3.11) [for a definition, see [19]]. This fact will be used to prove Lemma 2. However, we will work mostly with the equivalent form of (3.11)

(3.12)

\[
\frac{dW}{dU} = V - \frac{f(U)}{W},
\]

and consider \( V \) as a parameter.

**Lemma 2:** Suppose that \( f(u) \) satisfies (3.5) or (3.6). Then, for each \( V \geq 0 \), there exists a unique solution \( W(U,V) \) of (3.12) defined on \( 0 \leq U \leq 1 \) such that

\[
W(1,V) = 0 , \quad \frac{\partial W}{\partial U}(1,V) = \lambda_2(V).
\]

The solution \( W(U,V) \) is a differentiable function of the parameter \( V \), and

(3.13)

\[
\frac{\partial W}{\partial V}(W,V) < 0 \text{ whenever } 0 < U < 1 \text{ and } V \geq 0,
\]

i.e., for each fixed \( U \) with \( 0 < U < 1 \), \( W(U,V) \) is a strictly decreasing function in \( V \).
Gel'fand [17] proved this lemma for the case of the combustion equation (3.3), i.e. when \( f(u) = 0 \) for \( 0 < u < u_1 \) in (3.5). His proof carries over to our case, in which \( f(u) \) satisfies (3.5) or (3.6). Typical phase portraits for trajectories of (3.11) are given in Figure 3 and Figure 4, where the solution \( W(U,V) \) is easily identified.

Proof [Gel'fand]: We show first that such \( W(U,V) \) exists for each \( V > 0 \). Let \( V \) be fixed in the following existence proof. Since \( (1,0) \) is a saddle point for (3.11), there exists a unique trajectory \( (U(\xi),W(\xi)) \) defined for \( \xi > 0 \) which satisfies

\[
\lim_{\xi \to \infty} (U(\xi),W(\xi)) = (1,0), \quad \lim_{\xi \to \infty} \frac{W'(\xi)}{U'(\xi)} = \lambda_2(V),
\]

and \( W(\xi) > 0 \) for \( \xi \) sufficiently large. That such a trajectory exists follows from the existence of an invariant manifold for a saddle point of an autonomous system (see, for example, Chapter 9 of Hartman [19]). It is easy to see that

\[
\frac{f'(1)}{V} < \lambda_2(V) < 0.
\]

Therefore this trajectory must lie in the domain

\[
0 < W < \frac{f(U)}{V}, \quad U < 1
\]

for \( \xi \) large enough; see Fig. 3 or Fig. 4. However, the direction of the vector field defined by the right hand side of (3.11) shows that we can continue this trajectory backward (i.e., as \( \xi \) decreases) until it touches either the \( U \)-axis or the \( W \)-axis. In any case, we can find \( \xi_0 \) such that
Then, since \( W = U'(\xi) \), \( U(\xi) \) is strictly increasing in \( \xi_0 < \xi < \infty \). \( U = U(\xi) \) can be inverted so that \( \xi = \xi(U) \) becomes a function defined on \( U_0 < U < 1 \). Define

\[
W(U,V) = W(\xi(U)), \quad U_0 < U < 1.
\]

Then, \( W(U,V) \) satisfies (3.12) and

\[
W(1,V) = 0, \quad \frac{\partial}{\partial U} W(1,V) = \frac{W'(\infty)}{U'(\infty)} = \lambda_2(V).
\]

If \( 0 < U_0 < 1 \) (i.e., \( W(\xi_0) = 0 \)), \( W(U,V) \) can be continued backwards again (i.e., \( 0 < U < U_0 \)) as a solution of (3.12) until it reaches the \( W \)-axis. Note that, in such a case, \( W(U,V) < 0 \) for \( 0 < U < U_0 \). This solution \( W(U,V) \) is what we seek. The uniqueness of \( W(U,V) \) follows from the uniqueness of the invariant manifold of the saddle point \((1,0)\).

That \( W(U,V) \) is differentiable in \( V \) is an obvious consequence of the continuous dependence of solutions of (3.11) on the parameter \( V \).

To prove (3.13), define

\[
P(U,V) = \omega_W(U,V), \quad 0 < U < 1, \quad V > 0.
\]

Then

\[
(3.14) \quad \frac{\partial P(U,V)}{\partial U} = 1 + \frac{f(U)}{W^2}, \quad 0 < U < 1, \quad P(1,V) = 0.
\]

But

\[
\frac{\partial P(U,V)}{\partial U} = \frac{\partial}{\partial V} \left( \frac{\partial P(U,V)}{\partial U} \right) = \frac{\partial}{\partial V} \lambda_2(V)
\]

\[
= \frac{\lambda_1(V)}{\sqrt{V^2 - 4f'(1)}} > 0.
\]
Hence, there exists an $\epsilon > 0$ such that
\[
\frac{dP}{dU}(U,V) > 0 \text{ for } 1-\epsilon \leq U \leq 1.
\]
Thus
\[
P(U,V) < P(1,V) = 0 \text{ for } 1-\epsilon < U < 1.
\]
If $P(U,V) > 0$ for some $U$, we can find a point $0 < \bar{U} < 1-\epsilon < 1$ such that
\[
P(\bar{U},V) = 0 \text{ and } P(U,V) < 0 \text{ for } 0 < U < 1.
\]
At $\bar{U}$, $\frac{dP}{dU}(\bar{U},V) \leq 0$. However, (3.14) implies that
\[
\frac{dP}{dU}(\bar{U},V) = 1 > 0.
\]
This contradiction establishes (3.13).

We have completed the proof of the lemma.

After this lemma, our objective becomes obvious. The traveling wave, $U(\xi)$, which we seek must be equal to the stationary states of $f(u)$ at $\xi = \pm\infty$. Due to the assumptions, (3.5) or (3.6), on $f(u)$, $U(\infty)$ must be equal to the state 1. If we can pick $V$ so that $W(0,V) = 0$, then the corresponding $U(\xi)$ must satisfy $U(-\infty) = 0$ (since $(0,0)$ is a singular point of (3.11)). This is the traveling wave connecting states 1 and 0.

The Case of the Nonlinear Source Term Satisfying (3.5)

When $f(u)$ satisfies condition (3.5), we will prove the existence of traveling fronts. The basic idea is to see how the phase portrait in Fig. 4 changes as $V$ changes. In particular, since $W(U,V)$ is the possible candidate for the trajectory of the traveling wave, (3.13) is
As \( P(O,V) = \frac{\partial W}{\partial V}(O,V) \), the inequality

\[
 W(O,V) \leq -u_i V + W(0,0), \quad V \geq 0
\]

holds. Since \( u_i > 0 \), this proves the claim (3.15).

Now, obviously,

\[
 W(0,0) = (2\int_0^1 f(s) ds)^{1/2}.
\]

Since \( \int_0^1 f(s) ds > 0 \), \( W(0,0) > 0 \). Lemma 2 shows that \( W(0,V) \) decreases as \( V \) increases. Thus, combining this with the claim (3.15), there exists a unique \( V > 0 \) such that \( W(0,V) = 0 \). The corresponding \( U(\xi) \) is the traveling wave we seek.

**Step 2 (uniqueness proof).** Suppose that \( \tilde{U}(\xi) \), \( \xi = x + \tilde{V}t \), is a second traveling wave solution satisfying \( \tilde{V} > 0 \), \( \tilde{U}(0) = 0 \), \( \tilde{U}(\infty) = 0 \).

On the phase plane, this wave, \( \tilde{U}(\xi) \), introduces a solution \( \tilde{W}(U) \) of (3.12) defined on \( 0 < U < 1 \) such that

\[
 \tilde{W}(0) = \tilde{W}(1) = 0.
\]

\( \tilde{W}(U) \) cannot be \( W(U,\tilde{V}) \), since traveling waves of the form \( W(U,V) \) are shown to be unique in Step 1. Hence \( \tilde{W}'(1) = \lambda_1(\tilde{V}) > 0 \). Thus,

\[
 \tilde{W}(U) < 0 \quad \text{for } U \text{ sufficiently close to } 1.
\]

We claim that

(3.16) \( \tilde{W}(U) < 0 \) for all \( 0 < U < 1 \).

Suppose that, on the contrary, (3.16) is false. Then we can find a
The basic tool we need.

**Theorem 1:** Suppose that \( f(u) \) satisfies condition (3.5). Then the reaction-diffusion equation

\[
  u_t = u_{xx} + f(u)
\]

has a unique traveling wave solution of the form

\[
  u(x, t) = U(\xi), \quad \xi = x + Vt
\]

such that \( V > 0, \ U(-\infty) = 0, \ U(\infty) = 1. \)

\( U(\xi) \) has the following properties:

1. \( 0 \leq U(\xi) \leq 1; \quad \frac{dU}{d\xi} > 0; \quad \lim_{\xi \to \pm \infty} \frac{dU}{d\xi} = 0. \)

2. \( U(\xi) = O(e^{V\xi}) \) as \( \xi \to -\infty; \quad U(\xi) = 1 + O(e^{-\mu\xi}) \) as \( \xi \to \infty, \)

where \( 0 < \mu < |\lambda_2(V)|. \)

**Proof.** We prove the theorem in several steps.

**Step 1** (existence proof). We use the notation of Lemma 2. We claim that

\[
  \lim_{V \to \infty} W(0, V) = -\infty.
\]

To see this, note that \( f(U)P > 0 \) for \( 0 \leq U \leq u_i. \) Hence, from (3.14),

\[
  \frac{\partial P}{\partial U}(U, V) \geq 1, \quad 0 \leq U \leq u_i, \quad V > 0.
\]

Integrating this inequality from 0 to \( u_i, \) we get

\[
  0 > P(u_i, V) \geq u_i + P(0, V),
\]

i.e.,

\[
  P(0, V) < -u_i \quad \text{for} \quad V > 0.
\]
with $0 < U_1 < 1$ such that

$$\tilde{W}(U_1) = \max_{0 < U < 1} \tilde{W}(U) > 0.$$ 

At $U = U_1$, $\tilde{W}'(U_1) = 0$, hence from (3.12),

$$\tilde{W}(U_1) = \frac{f(U_1)}{\bar{V}} > 0.$$ 

Hence $0 < U_1 < 1$. We can then find a $U_2$ with $U_1 < U_2 < 1$ such that

$$\tilde{W}(U_2) = 0, \quad \tilde{W}(U) < 0 \quad \text{for} \quad U_2 < U < 1.$$ 

At $U = U_2$, $\tilde{W}'(U_2) \leq 0$. However, from (3.12),

$$\tilde{W}'(U) = \bar{V} - \frac{f(U)}{\tilde{W}(U)} \geq \bar{V} \quad \text{for} \quad U_2 < U < 1.$$ 

Let $U + U_2$, the inequality

$$\tilde{W}'(U_2) \geq \bar{V} > 0$$

follows. This is a contradiction to the choice of $U_2$. We have established the claim (3.16).

$\tilde{W}(\xi)$ satisfies (3.12) with $V = \bar{V}$. Hence

$$\frac{d}{dU}(1_{\tilde{W}}^2) = \tilde{W}W - f(U), \quad 0 \leq U \leq 1.$$ 

Integrating this identity from 0 to 1, we find

$$\bar{V} \int_0^1 \tilde{W}(s)ds = \int_0^1 f(s)ds > 0.$$ 

Since $\bar{V} > 0$, this is a contradiction to (3.16). Thus the traveling wave solution in this theorem is unique.
Step 3. It remains to establish (1) and (2). Let \( U(\xi) \) be the traveling wave solution whose speed is \( V \). Then \( U(\xi) \) satisfies

\[ \frac{dU}{d\xi} = W(U(\xi), V), \quad |\xi| < \infty. \]

It is easy to see that \( W(U,V) > 0 \) for \( 0 < U < 1 \) [see Fig. 4 for reference]. Thus property (1) is obvious.

To prove property (2), let \( \tilde{\xi} \) be chosen so that \( U(\tilde{\xi}) = u_i \). This is possible by property (1). Then

\[ U(\xi) \leq u_i \text{ for } \xi \leq \tilde{\xi}. \]

Hence

\[ VU' = U'' + f(U) \leq U'', \quad \text{for } \xi \leq \tilde{\xi}. \]

Integrating this inequality from \( \xi = -\infty \) to \( \xi \leq \tilde{\xi} \) and using property (1), we get

\[ VU(\xi) \leq U'(\xi) \text{ for } \xi \leq \tilde{\xi}. \]

Integrating again from \( \xi \) to \( \tilde{\xi} \), we find

\[ 0 \leq U(\xi) \leq e^{V\xi}(e^{-V\tilde{\xi}}U(\xi)), \quad \xi \leq \tilde{\xi}. \]

Thus \( U(\xi) = O(e^{V\xi}) \) for \( \xi < \tilde{\xi} \).

The second inequality on \( \xi \rightarrow \infty \) is proved in the same way.

The basic idea of the proof of this theorem is essentially due to Gel'fand [17]. We reproduced a detailed proof since we will need the properties stated in this theorem in the following subsections. This theorem can also be found in Fife and McLeod [12] or Aronson and Weinberger [2] where different proofs are presented.
The Case of Nonlinear Source Term Satisfying (3.6)

If the nonlinear source term \( f(u) \) satisfies (3.6), the argument in Step 1 and Step 2 of the proof of Theorem 1 can not be applied. However, there still exist traveling wave solutions. The basic tool we use is still Lemma 2. The idea is still to find a \( V \) so that \( W(0,V) = 0 \). Hence, the difficulty is how to control the behavior of \( W(U,V) \) in a neighborhood of \( U = 0 \). In this case, we compare (3.12) with a linear equation to get an estimate of \( W(U,V) \) near 0.

Theorem 2: Suppose that \( f(u) \) satisfies the condition (3.6). Then there exists a \( V_* > 2\sqrt{f'(0)} \), \( V_*>0 \) such that the reaction-diffusion equation

\[
\frac{u_t}{u_{xx}} + f(u)
\]

(3.4)

has a traveling wave solution of the form

\[
\begin{align*}
    u(x,t) &= U(\xi), \quad \xi = x + Vt, \quad V > 0, \\
    U(-\infty) &= 0, \quad U(+\infty) = 1
\end{align*}
\]

if and only if \( V \geq V_* \).

Each traveling wave solution \( U(\xi) \) has the following properties:

(1) \( 0 \leq U(\xi) \leq 1; \frac{dU}{d\xi} > 0; \lim_{|\xi|\to\infty} \frac{dU}{d\xi} = 0. \)

(2) If \( f'(0) > 0 \), then \( U(\xi) = O(e^{\xi/V}) \) as \( \xi \to -\infty \), for any \( 0 < \mu < \frac{1}{2} (V - \sqrt{V^2 - 4f'(0)}) \).

(3) \( U(\xi) = 1 + O(e^{-\xi/V}) \) as \( \xi \to \infty \), for any \( 0 < \mu < |\lambda_2(V)| \).

When \( f(u) \) is concave on \( 0 \leq u \leq 1 \), Theorem 2 has been well-studied from the pioneering work of Kolomogoroff et al. [23] on the KPP equation to, for example, [2] or Stokes [37]. However, our theorem
contains the important case where \( f'(0) = 0 \). The Arrhenius nonlinear source

\[
(1-T) \exp\left( -\frac{E}{RT} \right)
\]

fits condition (3.5) in Theorem 2. Thus it is possible to have flame fronts even if we do not truncate this expression at some ignition temperature. Our theorem can also handle nonlinear source terms of the form

\[ u^n(1-u), \quad n \text{ a positive integer}, \]

which were studied extensively in Spalding [36] and Adler [1] as an approximation to the Arrhenius reaction term. Actually, the assertion made by Cohen [8] that, when \( f(u) = u^2(1-u) \), (3.4) has no traveling wave solutions is incorrect.

**Proof.** We will use the notation and assertions of Lemma 2.

**Step 1:** Obviously, \( W(U,0) = \left( 2 \int_0^1 f(s) ds \right)^{1/2} > 0 \) for \( 0 < U < 1 \).

**Step 2:** We claim that \( W(U,V) > 0 \) whenever \( 0 < U < 1 \) and \( V > 0 \).

For \( V = 0 \), this claim is just Step 1. Let \( V > 0 \) be fixed. Since

\[
\frac{d}{dU} W(1,V) = \lambda_2(V) < 0, \quad W(U,V) > 0 \text{ for } U \text{ sufficiently close to 1 and less than 1.}
\]

See Figure 5 as reference. If \( W(U,V) \leq 0 \) for some \( U, 0 < U < 1 \), we can choose \( U_1, 0 < U_1 < 1 \), such that

\[
W(U_1,V) = 0 \text{ and } W(U,V) > 0 \text{ for } U_1 < U < 1.
\]

Thus \( \max_{U_1 < U < 1} W(U,V) \) occurs at some point \( U_2 \), \( U_1 < U_2 < 1 \). At \( U_2 \),

\[
\frac{dW}{dU}(U_2,V) = 0, \text{ i.e., the graph of } W(U,V) \text{ intersects the curve } W = \frac{f(U)}{V} \text{ at } U = U_2.
\]

Since \( W(U_2,V) > 0 \) and \( W(U_1,V) = 0 \), \( \frac{dW}{dU} > 0 \) somewhere in \( U_1 < U < U_2 \).
From (3.12) and $W > 0$ in $U_1 < U < U_2$, it follows that the graph of $W(U,V)$ lies above the curve $VW = f(U)$ somewhere in the interval $U_1 < U < U_2$. But $W(U_1,V) = 0$ lies below the curve $VW = f(U)$; hence the graph of $W(U,V)$ will intersect the curve $VW = f(U)$ again in $U_1 < U < U_2$. This is impossible due to the vector field direction defined by the right hand side of (3.11). This step is proved.

**Step 3:** If $W(0,V) > 0$ for all $V > 0$, then

$$\lim_{V \to \infty} W(U,V) = 0 \ \text{uniformly on} \ 0 < U < 1.$$  

To prove this assertion, we estimate $\max_{0 < U < 1} W(U,V)$. Let $\tilde{U}$ be the point where this maximum occurs. Then $0 < \tilde{U} < 1$, and hence $\frac{dW}{dU}(\tilde{U},V) = 0$. That is,

$$\max_{0 < U < 1} W(U,V) = W(\tilde{U},V) = \frac{f(\tilde{U})}{V} \leq \frac{1}{V} \left( \max_{0 < U < 1} f(U) \right).$$

Since $W(U,V) > 0$ on $0 < U < 1$, we prove Step 3 by letting $V \to \infty$ in the above inequality.

**Step 4:** There exists $V > 0$ such that $W(0,V) = 0$. That is, there exists at least one traveling wave solution. It is in this step that we compare (3.12) with some linear equation near $U = 0$. By assumption, $0 < f'(0) < \infty$. We can find $\delta > 0$ and $\varepsilon > 0$ such that

$$0 < f(U) < \delta U, \ \text{for} \ 0 < U < \varepsilon.$$  

For any $V$ with $V^2 > 4\delta$, define

$$\Delta(U,V) = W(U,V) - \mu(V)U, \ 0 < U < 1,$$
where \( \mu(V) = \frac{1}{2}(V + \sqrt{V^2 - 4\delta}) \). Then, for \( 0 \leq U \leq \varepsilon \),

\[
\frac{\partial \Delta}{\partial U} = V - f(U) - \mu(V) \geq -\frac{\delta U}{W} + \frac{\delta}{\mu(V)} = \frac{\delta}{\mu(V)W^\Delta}.
\]

Integrating this differential inequality on \( 0 \leq U \leq \varepsilon \), we find

\[
\exp\left(-\int_0^U \frac{\delta}{\mu(V)W(x,V)} \, dx\right) \Delta(U,V) \geq \Delta(0,V) = W(0,V) \geq 0.
\]

Thus, \( \Delta(U,V) \geq 0 \) for \( 0 \leq U \leq \varepsilon \). In particular,

\[
W(\varepsilon,V) \geq \mu(V)\varepsilon \geq \mu(2\sqrt{\delta})\varepsilon > 0, \text{ for } V > 2\sqrt{\delta}.
\]

Therefore,

\[
\lim_{V \to \infty} W(\varepsilon,V) > 0.
\]

Hence Step 3 implies Step 4.

**Step 5:** Define

\[
V_* = \inf\{V > 0 | W(0,V) = 0\}.
\]

Then \( W(0,V_*) = 0 \) and \( V_* > 0 \) (due to Step 1). For any \( V > V_* \), Lemma 2 and Step 2 imply that

\[
0 \leq W(0,V) \leq W(0,V_*) = 0.
\]

Hence \( W(0,V) = 0 \). We have proved the "if" part of the theorem.

**Step 6:** The "only if" part of the theorem follows from the same argument as Step 2 in Theorem 1.

**Step 7:** The traveling wave \( U(\xi) \) is the solution of the ordinary differential equation

\[
\frac{du}{d\xi} = W(U(\xi),V), \quad U(-\infty) = 0, \quad U(\infty) = 1.
\]
Property (1) is obvious from Step 2.

Now if \( f'(0) > 0 \), it is easy to see from (3.12) that

\[
\frac{dW}{du}(0, V) \geq \frac{1}{2}(V - \sqrt{V^2 - 4f'(0)}) > 0.
\]

Thus, for any \( 0 < \mu < \frac{1}{2}(V - \sqrt{V^2 - 4f'(0)}) \), there exists an \( \varepsilon > 0 \)

such that

\[
\frac{dW}{du}(u, V) \geq \mu > 0 , \text{ for } 0 \leq u \leq \varepsilon.
\]

Integrating from 0 to \( u \), we find

\[
W(u, V) \geq \mu u , \text{ for } 0 \leq u \leq \varepsilon.
\]

Choose \( \tilde{\xi} \) so that \( U(\xi) = \varepsilon \). Then \( U(\xi) \leq \varepsilon \) for \( \xi < \tilde{\xi} \). Thus

\[
\frac{dU}{d\xi} \geq \mu U(\xi) \text{ for } \xi < \tilde{\xi}.
\]

Integrating this inequality, we obtain the inequality

\[
0 \leq U(\xi) \leq (e^{-\mu\tilde{\xi}}U(\tilde{\xi}))e^{\mu\xi} \text{ for } \xi < \tilde{\xi},
\]

which proves property (2).

The proof of property (3) is entirely similar.

Thus, in contrast to Theorem 1, there is a family of traveling wave solutions for (3.4) in the case of Theorem 2. We shall see in subsection (E) that the minimal speed wave can be physically realized as an asymptotic state arising from some (physical) initial conditions for equation (3.4). We shall see in subsection (D) that the minimal speed wave is in general more stable than the higher speed waves. Especially when we apply Theorem 2 to the combustion equation (3.3), we shall see that the minimal speed wave can also arise as a limiting
case of flame fronts. In other words, we claim that the minimal speed wave in Theorem 2 plays a more important role than any higher speed waves. The justification of this claim will be more evident in the following subsections. The qualitative differences between the minimal speed wave and higher speed waves in the case where \( f'(0) = 0 \) in Theorem 2 can be seen in the following statement.

(3.17) If \( f'(0) = 0 \) in Theorem 2, then

\[
\frac{\partial^2 W(0,V)}{\partial U^2} = 0 \quad \text{for} \quad V > V_*,
\]

and either \( \frac{\partial^2 W(0,V_*)}{\partial U^2} = 0 \) or \( \frac{\partial^2 W(0,V_*)}{\partial U^2} = V_* \).

Thus, if \( V > V_* \), \( W(U,V) \) has the same order of singularity as \( f(U) \) near \( U = 0 \).

If \( \frac{\partial^2 W(0,V_*)}{\partial U^2} = V_* \), the minimal speed wave still has the asymptotic behavior as property (2) in Theorem 2 at \( -\infty \) [however, \( 0 < \mu < V_* \)]. Any higher speed wave does not exponentially decay to zero at \( -\infty \).

To prove (3.17), we note from Lemma 2 that

\[
0 \leq W(U,V_1) \leq W(U,V_2), \quad \text{if} \quad V_1 > V_2.
\]

Hence

\[
0 \leq \frac{\partial W}{\partial U}(0,V_1) \leq \frac{\partial W}{\partial U}(0,V_2), \quad \text{if} \quad V_1 > V_2.
\]

If \( \frac{\partial W}{\partial U}(0,V_*) \neq 0 \), then \( \frac{\partial W}{\partial U}(0,V_*) = V_* \) from (3.12). If \( \frac{\partial W}{\partial U}(0,V) \neq 0 \) for some \( V > V_* \), \( \frac{\partial W}{\partial U}(0,V) = V \) also. However, the above inequality implies that \( V \leq V_* \). This contradicts \( V > V_* \). Thus (3.17) holds.

Roughly speaking, (3.17) shows that the transition zone [from state 0 to state 1] of any higher speed wave in Theorem 2 is in general very
"wide" in the case that $f'(0) = 0.$

We will apply (3.17) in subsections (D) and (E) to determine the stability and asymptotic behavior of those waves.

**Traveling Fronts Arising from the Combustion Equation**

We will now concentrate on the combustion equation

\[
T_t = T_{xx} + g(T),
\]

or

\[
T_t = T_{xx} + R(T),
\]

where

\[
R(T) = \frac{Z(1-T)\exp(-\frac{E}{\rho_h T})}{\rho}, \quad 0 \leq T \leq 1.
\]

As usual, we will look for a traveling wave solution of the form

\[
T(x,t) = T(\xi), \quad \xi = x + Vt, \quad V > 0.
\]

For later use, we use $(T,S)$ as phase variables for (3.3) or (3.18)

where $S = \frac{dT}{d\xi}.$ Thus, the traveling wave solutions satisfy

\[
\begin{align*}
\frac{dT}{d\xi} &= S, \\
\frac{dS}{d\xi} &= VS - f(T),
\end{align*}
\]

and the equivalent form

\[
\frac{dS}{dT} = V\frac{f(T)}{S},
\]

where $f(T) = g(T)$ or $R(T).$ For both equations, we use $S(T,V)$ to denote the solutions defined in Lemma 2. Thus
\[
\frac{\partial}{\partial T}S(T,V) = V - \frac{f(T)}{S(T,V)} , \quad S(1,V) = 0 , \quad \frac{\partial}{\partial T}S(1,V) = \lambda_{2}(V) .
\]

In the case of \( f(T) = g(T) \) [i.e. equation (3.3)], \( S(T,V) \) also depend on the choice of the ignition temperature \( T_{i} \). Whenever it is necessary, we shall denote this dependence by writing \( S(T,V) = S_{T}(T,V) \).

Consider the equation (3.3) where we fix the ignition temperature \( T_{i} \) and fix a way of smoothing \( g(T) \) around \( T_{i} \). An obvious corollary of Theorem 1 is that, for each unburned state \( 0 < \alpha < T_{i} \), there exists a unique traveling front \( T_{\alpha}(\xi_{\alpha}) \), \( \xi_{\alpha} = x + V_{\alpha}t \), such that

\[
T_{\alpha}(-\infty) = \alpha , \quad T_{\alpha}(+\infty) = 1 , \quad \text{and} \quad \frac{dT_{\alpha}(\xi_{\alpha})}{d\xi_{\alpha}} > 0 .
\]

\( V_{\alpha} > 0 \) is the wave speed. Actually, in terms of phase plane variables, \( T_{\alpha}(\xi_{\alpha}) \) is the wave corresponding to \( S(T,V_{\alpha}) \). That is,

\[
\frac{dT_{\alpha}}{d\xi_{\alpha}} = S(T_{\alpha}(\xi),V_{\alpha}) , \quad T_{\alpha}(-\infty) = \alpha , \quad T_{\alpha}(+\infty) = 1
\]

and

\[
S(T,V_{\alpha}) = V_{\alpha}(T-\alpha) \quad \text{for} \quad 0 \leq T \leq T_{i} ,
\]

\[
S(T,V_{\alpha}) > 0 \quad \text{for} \quad T_{i} < T < 1 ,
\]

\[
S(1,V_{\alpha}) = 0 , \quad \text{and} \quad \frac{dS}{dT}(1,V_{\alpha}) = \lambda_{2}(V_{\alpha}) < 0 .
\]

A typical graph for \( S(T,V_{\alpha}) \) is depicted in Figure 5. As for the unburned state \( T_{i} \), \( T_{i} \) is related to the burned state \( 1 \) by a nonlinear source term of the type in Theorem 2. Hence, there exists a \( V^{*} \geq 2\sqrt{g'(T_{i}+)} \) such that (3.3) has a family of traveling wave solutions connecting \( T_{i} \) and \( 1 \) with \( V^{*} \) as the minimal wave speed. We will denote the minimal speed traveling front by \( T^{*}(\xi) \), \( \xi = x + V^{*}t \), and the corresponding phase plane solution of (3.19) by \( S^{*}(T) \). Thus,
\( S^*(T) \) satisfies (3.19) with

\[
\frac{dT^*}{d\xi} = S^*(T^*(\xi)) , \quad T^*(-\infty) = T_i , \quad T^*(+\infty) = 1 ;
\]
\[
S^*(T) = V^*(T-T_i) \quad \text{for} \quad 0 \leq T \leq T_i ,
\]
\[
S^*(T) > 0 \quad \text{for} \quad T_i < T < 1 ,
\]
\[
S^*(1) = 0 ,
\]
\[
\frac{dS^*}{dT}(1) = \lambda_2(V^*) < 0 ,
\]
\[
\frac{dS^*}{d\tau}(T_i) = \frac{1}{2}(V^* - \sqrt{V^*-4g^*(T_i+)}).
\]

The wave speed \( V_\alpha \) depends on the choice of ignition temperature. Whenever it is necessary, such dependence is denoted by \( V_\alpha = V_\alpha(T_i) \).

Consider now equation (3.18) with a non-truncated Arrhenius reaction term. \( R(T) \) is of the type described in Theorem 2. Thus there exists a \( V_* > 0 \) such that (3.18) has a family of traveling wave solutions with \( V_* \) as the minimal speed. The minimal speed wave will be denoted by \( T^*(\xi) , \xi = x + V_*t , \) and the corresponding phase plane solution of (3.19) will be denoted by \( S^*(T) \). Thus \( S^*(T) \) is defined on \( 0 \leq T \leq 1 \) and satisfies (3.19) with

\[
\frac{dT^*}{d\xi} = S^*(T^*(\xi)) , \quad T^*(-\infty) = 0 , \quad T^*(+\infty) = 1 ;
\]
\[
S^*(0) = S^*(1) = 0 , \quad \frac{dS^*}{dT}(0) = V^* > 0 , \quad \frac{dS^*}{dT}(1) = \lambda_2(V^*) < 0 ,
\]
\[
S^*(T) > 0 \quad \text{for} \quad 0 < T < 1 .
\]

This \( T^*(\xi) \) has properties (1), (2), (3) in Theorem 2. Since \( R'(0) = 0 , \) any higher speed wave for (3.18) has a qualitative property near \( T = 0 \) different from \( T^*(\xi) , \) as shown in (3.17). We shall have more justification on singling out the minimal speed wave \( T^*(\xi) \) in the following subsections. See Figure 6 for the graph of \( T^*(\xi) \).
In the remainder of this subsection, we will discuss how \( V_\alpha \) and \( T_\alpha \) behave as \( \alpha \) or \( T_i \) changes. In particular, we will consider the existence of the limits \( \lim_{\alpha \to T_i} V_\alpha \) or \( \lim_{T_i \to 0} V_0(T_i) \). We will use the notations developed here.

**The Limit \( \lim_{\alpha \to T_i} V_\alpha \)**

To discuss this limit, the basic tool will be Lemma 2. Fix the ignition temperature \( T_i \) and a way of smoothing \( g(T) \) around \( T_i \) in equation (3.3). We claim the following

**Proposition 1:** \( V_\alpha \) is a continuous increasing function on \( 0 \leq \alpha < T_i \), and

\[
\lim_{\alpha \to T_i} V_\alpha = V^* .
\]

The corresponding wave \( S(T,V_\alpha) \) converges to \( S^*(T) \) uniformly on \( T_i \leq T \leq 1 \).

**Proof.** From Lemma 2, \( S(T,V) \) is a decreasing function of \( V \). If \( 0 \leq \alpha < \beta < T_i \), then \( 0 = S(\beta,V_\beta) < S(\beta,V_\alpha) \) [see Fig. 6 for a clear picture or (3.20)]. Since \( S(\beta,V) \) is strictly decreasing in \( V \), this is possible only if \( V_\alpha < V_\beta \). Thus,

\[
V_\alpha < V_\beta \quad \text{if and only if} \quad 0 \leq \alpha < \beta < T_i .
\]

Property (3.21) for \( T^*(\xi) \) clearly implies that

\[
S^*(T) = S(T,V^*) , \quad 0 \leq T \leq 1 .
\]

The minimality of \( V^* \) implies that
$S(T_i, V) > 0$ for all $V < V^*$, \\
$S(T_i, V) = 0$ for all $V \geq V^*$. \\
Hence, by looking at $S(T_i, V)$, the inequality \\

$$V_\alpha < V^* \text{ for all } 0 \leq \alpha < T_i$$

follows. Hence $\lim_{\alpha \to T_i} V_\alpha = \bar{V}$ exists and $\bar{V} \leq V^*$. \\
On the other hand, $S(T_i, V_\alpha) = V_\alpha(T_i - \alpha)$. Thus \\

$$S(T_i, \bar{V}) = \lim_{\alpha \to T_i} S(T_i, V_\alpha) = \lim_{\alpha \to T_i} V_\alpha(T_i - \alpha) = 0.$$ \\
This is possible only if $\bar{V} \geq V^*$. Therefore we have proved that \\

$$\lim_{\alpha \to T_i} V_\alpha = V^*.$$ \\
Obviously, $S(T, V_\alpha) \to S(T, V^*) = S^*(T)$ uniformly by Dini's Theorem. \\
This proposition shows the continuous dependence of wave speed on the unburned states. We will explain more later.

The Limit $\lim_{T_i \to 0} V_0(T_i)$

Recall that $V_0(T_i)$ is the wave speed of the unique traveling wave solution of (3.3) connecting state 0 and state 1 when the ignition temperature $T_i$ is fixed. In order to analyze this limit, we set up the problem more precisely. If we truncate $R(T)$ at $T = \alpha > 0$ [i.e. $T_i$ is assumed to be $\alpha$], we will smooth the truncated $R(T)$, which is denoted by $g_\alpha(T)$, in such a way that
\( g_\alpha(T) = 0 \) for \( 0 \leq T \leq \alpha \);
\[
0 < g_\alpha(T) < R(T), \quad \text{for} \quad \alpha < T < \alpha + \epsilon(\alpha);
\]
\( g_\alpha(T) = R(T) \) for \( \alpha + \epsilon(\alpha) \leq T \leq 1 \),

so that \( g_\alpha(T) \) satisfies (3.5) with \( u_i = \alpha \). Here \( \epsilon(\alpha) \) is some small positive number which decreases to 0 as \( \alpha \) decreases to 0. See Figure 7. We also assume that \( g_\alpha(T) \) is smoothed in such a way that

\[
g_\alpha(T) \leq g_\beta(T) \quad \text{for} \quad 0 < \beta \leq \alpha < 1.
\]

Now, corresponding to \( g_\alpha(T) \), \( V_0(\alpha) \) is the unique wave speed for the wave connecting 0 and 1. We will show that \( \lim_{\alpha \to 0} V_0(\alpha) \) exists. The basic idea is that we are able to compare the solutions \( S_\alpha(T,V) \) as \( \alpha \) changes. Then Lemma 2 is applied to get the desired limit.

\( S_\alpha(T,V) \) denotes the solutions in Lemma 2 corresponding to the nonlinear source term \( g_\alpha(T) \). Let \( S(T,V) \) be the solutions in Lemma 2 using \( R(T) \) as the source term. We first prove a comparison lemma.

**Lemma 3:** For each fixed \( V, V \geq 0 \), \( S_\beta(T,V) \leq S_\alpha(T,V) \leq S(T,V) \) whenever \( 0 < \alpha < \beta < 1 \).

**Proof.** Since \( g_\alpha(T) = g_\beta(T) = R(T) \) for \( \beta + \epsilon(\beta) \leq T \leq 1 \), equation (3.3) with \( g_\alpha(T), g_\beta(T) \) as source terms are the same on \( \beta + \epsilon(\beta) \leq T \leq 1 \). Hence

\[
S_\alpha(T,V) = S_\beta(T,V) = S(T,V) \quad \text{for} \quad \beta + \epsilon(\beta) \leq T \leq 1.
\]

Define

\[
\Delta(T) = S_\alpha(T,V) - S_\beta(T,V), \quad 0 \leq T \leq 1.
\]
Then
\[ \frac{d\Delta}{dT} = (V - \frac{g_\alpha}{S_\alpha}) - (V - \frac{g_\beta}{S_\beta}) = -\frac{g_\alpha}{S_\alpha} + \frac{g_\beta}{S_\beta}. \]

However, \( g_\beta \leq g_\alpha \) and \( S_\beta(T,V) > 0 \) on \( 0 < T < 1 \). Hence
\[ \frac{d\Delta}{dT} \leq -\frac{g_\alpha}{S_\alpha} + \frac{g_\beta}{S_\beta} = \frac{g_\alpha}{S_\alpha} \Delta. \]

Integrating this inequality, we find
\[ \exp\left(-\int_T^{\beta+\epsilon(\beta)} \frac{g_\alpha}{S_\alpha} dx\right) \Delta(T) \geq \Delta(\beta + \epsilon(\beta)) = 0, \]
for \( 0 \leq T \leq \beta + \epsilon(\beta) \). Therefore,
\[ S_\beta(T,V) \leq S_\alpha(T,V) \text{ for } 0 \leq T \leq 1. \]

The proof that \( S_\alpha(T,V) \leq S(T,V) \) is similar.

Using this lemma, we now can prove

**Proposition 2:** \( V_0(T_i) \) is a decreasing function of \( T_i \), and
\[ \lim_{T_i \to 0} V_0(T_i) = V_* \]
where \( V_* \) is the minimal wave speed in Theorem 2 using \( R(T) = \frac{Z(1-T)}{\rho} \exp(-\frac{E}{R \rho H_0 T}) \) as the nonlinear source term.

**Proof.** Let \( 0 < \alpha < \beta < 1 \) be fixed. By Lemma 3,
\[ S_\beta(T,V_0(\beta)) \leq S_\alpha(T,V_0(\beta)) \leq S(T,V_0(\beta)), \quad 0 \leq T \leq 1. \]
In particular, \( 0 = S_\beta(0,V_0(\beta)) \leq S_\alpha(0,V_0(\beta)) \). But \( S_\alpha(0,V) \) is decreasing in \( V \), and \( V_0(\alpha) \) is the first \( V \) such that \( S_\alpha(0,V) = 0 \).
Hence
\[ V_0(\beta) \leq V_0(\alpha) \text{ for } 0 < \alpha < \beta < 1. \]

Similarly, using \( S_\alpha(T,V) \leq S(T,V) \), we have
\[ V_0(\alpha) \leq V_\star \text{ for } 0 < \alpha < 1. \]

Hence the limit
\[ \tilde{V} = \lim_{\alpha \to 0} V_0(\alpha) \leq V_\star \]
exists.

For convenience, define \( h_\alpha(T) = S_\alpha(T,V_0(\alpha)) \). Let \( 0 < T_0 \leq 1 \) be fixed; let \( \tilde{\alpha} \) be chosen so that \( \tilde{\alpha} + \varepsilon(\tilde{\alpha}) \leq T_0 \). Then, since \( \varepsilon(\beta) \to 0 \) and (3.23) holds,
\[ h_\beta(T) = S_\beta(T,V_0(\beta)) = S(T,V_0(\beta)) \]
for all \( 0 < \beta \leq \tilde{\alpha} \) and \( T_0 \leq T \leq 1 \). Since \( V_0(\beta) \to \tilde{V} \) as \( \beta \to 0 \),
\[ S(T,V_0(\beta)) \to S(T,\tilde{V}) \text{ uniformly on } 0 \leq T \leq 1. \]
Thus \( h_\alpha(T) \) converges to \( S(T,\tilde{V}) \) uniformly on \( T_0 \leq T \leq 1 \). But \( T_0 \) is arbitrary, therefore
(3.24) \( \lim_{\alpha \to 0} h_\alpha(T) = S(T,\tilde{V}) \) uniformly on every compact subset of \( 0 < T \leq 1 \).

However, \( h_\alpha(T) \) satisfies the identity
\[ h'_\alpha(T) = V_0(\alpha) - \frac{g_\alpha(T)}{h_\alpha(T)}, \quad 0 < T < 1. \]

Since \( \{V_0(\alpha)\} \) is bounded, \( \{h'_\alpha(T)\} \) is uniformly bounded on \( 0 \leq T \leq 1 \). Hence, \( \{h_\alpha(T)\} \) is equicontinuous on \( 0 \leq T \leq 1 \). This fact, along with (3.24), implies that
Thus \( S(T, \tilde{V}) \) is a traveling wave connecting 0 and 1 for equation (3.18). By Theorem 2, \( \tilde{V} \geq V_\star \). Therefore \( \tilde{V} = V_\star \).

This proposition is proved.

The process of truncating and smoothing \( R(T) \) at some ignition temperature which leads us to the limit \( V_\star \) in Proposition 2 is a natural way to do the job. Note that the limiting speed \( V_\star \) is independent of the way we smooth the source term around the ignition temperature. It is intrinsic to the physical system we considered, i.e. depends only on the physical parameters \( (\rho, z, E, H_0, \text{etc.}) \) defining the physical system and burned and unburned states chosen.

I suspect that, no matter how we smooth the source term around the ignition temperature, as long as we keep \( 0 < g_\alpha(T) < R(T) \), the speed \( V_0(\alpha) \) will tend to \( V_\star \) also. The assumption that

\[
g_\alpha(T) \leq g_\beta(T), \quad \text{if} \quad 0 < \beta < \alpha < 1
\]

is just a technical assumption which enables us to prove Lemma 2. As we shall see in subsections (D) and (E) below, the truncation on the Arrhenius expression \( R(T) \) not only stabilizes the traveling fronts, but also makes those fronts emerge very quickly from suitable initial states of the gas. However, if, without the assumption of positive ignition temperature, the gas mixture can sustain a "flame front", its speed should be computable from the truncated speeds. The existence of an infinite family of traveling fronts in the case of no positive ignition temperature poses the questions of which front is the physical one, whether they are stable, what initial states of the gas give rise
to the physical fronts, etc. These questions are not easy to answer. However, some observations are possible. The existence of those fronts partly explains why the formation of a flame front is harder to observe in actual numerical computation when we do not assume the existence of a positive ignition temperature. Proposition 2 suggests that the minimal speed front is the physically realizable front. Any higher speed front cannot be obtained by truncating the source term $R(T)$. The following subsections will provide more evidence of these suggestions.

The Flame Fronts

Equations (3.3) and (3.18) are translation invariant and also reflection invariant. That is, if $T(x,t)$ is a solution, then $T(\varepsilon x + c, t)$ is also a solution for any real $c$, and $\varepsilon = \pm 1$.

For the combustion equation (3.3), we visualize the flame fronts as the family of traveling wave solutions given by

$$T(x + \alpha t + c), \quad \varepsilon = \pm 1, \quad 0 \leq \alpha \leq T_1, \quad c \text{ any real,}$$

where, when $\alpha = T_1$, define $V_{T_1}$ and $T_{T_1}(\xi)$ to be the minimal speed $V^*$ and corresponding $T^*$ given in (3.21). We shall see in the following subsections that the fronts in (3.25) are the building blocks of many general initial-value problems for equation (3.3).

For the combustion equation (3.18), the flame fronts are visualized as

$$T_*(\varepsilon x + V_*(t + c), \quad \varepsilon = \pm 1, \quad c \text{ any real.}$$
(D) **The Stability of the Flame Fronts**

This subsection discusses the stability under finite amplitude perturbation of the flame fronts appearing in (3.25). The typical approach to attack such a problem is to carry out an infinitesimal analysis first. That is, we linearize equation (3.3) around a flame front and consider the corresponding linear operator (the derivative) associated with this flame front. Then the spectral properties of this linear operator can be used to determine the local behavior of the equation (3.3) around the flame front. If the spectrum of the linear part lies in the stability region (usually the left half plane of the complex plane) and we have a nice control on the (nonlinear) remainder operator, the finite amplitude stability is then established. Such an analysis for the stability of traveling wave solutions of scalar reaction-diffusion equations has been carried out in great detail in Sattinger [31], [32]. The typical linear operators encountered in such cases are (non-self adjoint) Sturm-Liouville operators whose coefficients depend on the wave under study. Thus the spectral analysis of the Sturm-Liouville operator such as presented in Chapter 8 of Coddington and Levinson [7] is the basic tool for the stability analysis. However, we will not repeat Sattinger's analysis here. We will quote a special result of Sattinger [Proposition 4 below], and apply it to the combustion equations either with truncation or without truncation. We will use both the $L^2$ norm and the $L^\infty$ norm (uniform norm). Thus a finite amplitude perturbation of the flame front $T_\alpha(\xi)$ with $0 \leq \alpha < T_1$ returns to $T_\alpha(\xi)$ in an exponential rate. However, if $g'(T_1^+) > 0$ in (3.3), the flame front $T_{T_1}(\xi)$ is not even stable under infinitesimal $L^\infty$ perturbation. The wave $T_\star(\xi)$ of (3.26) is
stable under infinitesimal perturbation, but not in an exponential rate. We will use the properties of these fronts listed in (3.20), (3.21), (3.22) to derive these results. Those stability properties will also be used in subsection (E) to determine the asymptotic behavior for the combustion equation.

**Stability Under Infinitesimal Perturbation**

We discuss the stability of the flame fronts appearing in (3.25) or (3.26) under infinitesimal perturbation. Let \( \phi(\xi), \xi = x + vt, \) be any wave appearing in (3.25) or (3.26) and let \( f(T) \) be either \( g(T) \) or \( R(T) \) in (3.3) or (3.18). We linearize the equation

\[
T_t = T_{xx} + f(T)
\]

around the solution \( \phi(\xi) \). The linear equation

\[
v_t = v_{\xi \xi} - vv_{\xi} + f'(\phi)v
\]

results where we have transformed to \( \xi \)-coordinate. Let

\[
L = \frac{d^2v}{d\xi^2} - V \frac{dv}{d\xi} + f'(\phi(\xi))v(\xi)
\]

be the corresponding linear operator defined on the space of functions \( v(\xi) \) such that

\[
(3.27) \quad \frac{d}{d\xi}v(\xi), \quad \frac{d^2v}{d\xi^2} \text{ exist in } L^2 \text{ sense, and } \lim_{|\xi| \to \infty} |v(\xi)| = 0.
\]

Note that such \( v(\xi) \) must be continuous. Since \( V > 0 \), \( L \) is not self-adjoint in \( L^2 \) inner product. If all the nonzero spectra of \( L \) lie in the open left half-plane of the complex plane, then \( \phi \) is stable under infinitesimal perturbation; we then call \( \phi \) an
infinitesimally stable front of (3.3) or (3.18). If we introduce

$$\omega(\xi) = \exp(-\frac{V}{2}\xi)v(\xi),$$

then

$$Lv = e^{\frac{V}{2}\tilde{\omega}}$$

where

$$\tilde{\omega} = \frac{d^2 \omega}{d\xi^2} + \left(f'(\phi(\xi)) - \frac{V^2}{4}\right) \omega.$$ 

$L$ is self-adjoint with respect to $L^2$ inner product if $\omega$ satisfies (3.27). However, if we require that eigenfunctions of $L$ and $\tilde{L}$ satisfy (3.27), $L$ and $\tilde{L}$ need not have the same spectrum, since the transformation from $v(\xi)$ to $\omega(\xi)$ involves $\exp(-\frac{V}{2}\xi)$ which grows exponentially at $\xi = -\infty$.

We are now in a position to prove

**Proposition 3:**

(a) Whenever $0 < \alpha < T_1$, the flame front $T_\alpha(\xi)$ in (3.25) is infinitesimally stable with respect to equation (3.3).

(b) $T_{T_1}(\xi)$ of (3.25) is unstable under infinitesimal perturbation in the $L^\infty$ norm, whenever $g'(T_1) > 0$.

(c) $T_*(\xi)$ of (3.26) is infinitesimally stable with respect to equation (3.18).

**Proof.** (a) Without loss of generality, we may assume $\alpha = 0$. Let $\phi(\xi) = T_0(\xi)$. Formally, $L$ and $\tilde{L}$ have the same spectrum, since the transformation between $v$ and $\omega$ formally transforms eigenfunctions of $L$ into those of $\tilde{L}$, and vice versa.

Since $\phi(\xi)$ satisfies the equation
\[
\frac{d^2 \phi}{d \xi^2} - V_0 \frac{d \phi}{d \xi} + g(\phi) = 0 ,
\]

\(\phi = \frac{d \phi}{d \xi}\) is an eigenfunction of \(L\) with eigenvalue 0, i.e. \(L(\frac{d \phi}{d \xi}) = 0\) and \(\frac{d \phi}{d \xi}\) satisfies (3.24), which follows from (3.20). Then

\[
\omega(\xi) = \exp\left(-\frac{V_0}{2} \xi \right) \frac{d \phi}{d \xi}
\]
satisfies (3.27) also, since \(\phi(\xi) = 0(e^0 \xi)\) as \(\xi + \infty\) [see property (2) of Theorem 1]. Thus \(\omega(\xi)\) is an eigenfunction of \(\tilde{L}\) with eigenvalue 0. But, by property (1) of Theorem 1, \(\omega(\xi)\) has no zero. Therefore, 0 must be the largest eigenvalue of \(\tilde{L}\), since \(\tilde{L}\) is self-adjoint and the largest eigenvalue of a self-adjoint operator is characterized by the property that its eigenfunction has no roots. Thus the nonzero spectrum of \(\tilde{L}\) lies on the open left half-plane of the complex plane. We have proved (a).

(b) Let \(\phi(\xi) = T_{T_1}(\xi)\). We will show that \(v_0(\xi)\) can be chosen so that \(\sup_{|\xi| < \infty} |v_0(\xi)|\) is arbitrarily small and the solution of the linear equation

\[
v_t = L v , \quad v(\xi,0) = v_0(\xi)
\]
grows without bound as \(t \to \infty\). To prove this assertion, pick any number \(\mu\), with \(0 < \mu < g'(T_1+)\). We claim that the system

(3.28) \[
\frac{d^2 y}{d \xi^2} - V_{T_1} \frac{d y}{d \xi} + g'(\phi(\xi)) y = \mu y , \quad \sup_{|\xi| < \infty} |y(\xi)| < \infty
\]

has a nontrivial solution \(y_0(\xi)\). Then if we set

\(v_0(\xi) = \epsilon y_0(\xi) , \quad \epsilon\) any fixed real number,

the function \(v(\xi,t) = \epsilon e^{\mu t} y_0(\xi)\) solves \(v_t = L v\) and grows without
bound as \( t \to \infty \), hence showing (3.28) holds would prove (b).

It remains to prove the claim (3.28). At \( \xi = \infty \), the solutions of (3.28) are asymptotically equal to the solutions of

\[
\frac{d^2y}{d\xi^2} - V_T \frac{dy}{d\xi} + g'(1)y = \mu y.
\]

[See Chapter 8 of Coddington and Levinson [7].] Thus (3.28) has at least one solution \( y_0(\xi) \) such that

\[
y_0(\xi) \sim \exp(\lambda \xi) \text{ as } \xi \to \infty,
\]

where

\[
\lambda = \frac{1}{2} \left( V_{T_1} - (V_{T_1}^2 - 4g'(1) + 4\mu)^{1/2} \right) < 0.
\]

Since (3.28) is linear, \( y_0(\xi) \) is defined for all \( \xi \). But, at \( \xi = -\infty \), the solutions of (3.28) are asymptotically equal to the solutions of

\[
\frac{d^2y}{d\xi^2} - V_T \frac{dy}{d\xi} + g'(T_1) y = \mu y.
\]

Hence there exist constants \( d_1 \) and \( d_2 \) such that

\[
y_0(\xi) \sim d_1 \exp(\sigma_+ \xi) + d_2 \exp(\sigma_- \xi) \text{ as } \xi \to -\infty,
\]

where

\[
\sigma_\pm = \frac{1}{2} \left( V_{T_1} \pm (V_{T_1}^2 - 4g'(T_1) + 4\mu)^{1/2} \right).
\]

By Theorem 2, \( V_{T_1}^2 \geq 4g'(T_1) \). Hence \( \sigma_+ > 0 \) due to the choice that \( 0 < \mu < g'(T_1) \). Thus \( \sup_{|\xi| < \infty} |y_0(\xi)| < \infty \). This proves the claim (3.28).

(c) \( R(T) \) in (3.18) has derivative \( R'(0) = 0 \). The proof of (c) is the same as that of (a), using the properties of \( T_*(\xi) \) listed in (3.22).
If \( g'(T_1^+) = 0 \) in case (b), the front \( T_1^+(\xi) \) is still infinitesimally stable, similar to the case (c). It should be remarked that any higher speed traveling wave connecting the states 1 and \( T_1 \) in (3.3) or the states 0 and 1 in (3.18) is unstable under infinitesimal perturbation. The proof of (a) had been indicated in Gel'fand [17] where he failed to point out that the transformation from the non-self-adjoint operator \( L \) to the self-adjoint operator \( \tilde{L} \) is necessary. The proof also appears in Sattinger [31], [32] where a more detailed study of \( L \) and \( \tilde{L} \) is given. We remark that the stability of fronts in cases (a) and (c) is with respect to infinitesimal perturbation which satisfies (3.27). Their stability with respect to infinitesimal perturbation under the \( L^\infty \) norm is undetermined. The general case of stability under finite amplitude perturbation is a much more delicate problem. We shall rely on Sattinger's investigation.

**Sattinger's Proposition**

**Proposition 4** (Sattinger [32]): Suppose that the reaction-diffusion equation

\[
\begin{align*}
    u_t &= u_{xx} + f(u), \\
    f(0) &= f(1) = 0,
\end{align*}
\]

has a traveling wave solution \( U(\xi), \xi = x + Vt, V > 0, \) which satisfies the conditions

(i) \( 0 \leq U(\xi) \leq 1, \quad U(-\infty) = 0, \quad U(\infty) = 1, \)

(ii) \( \frac{dU}{d\xi} > 0 \) for all \( \xi, \quad \frac{dU}{d\xi}(\pm\infty) = 0, \)

(iii) \( \int_{-\infty}^{\infty} |f'(U(\xi)) - f'(L)| d\xi < \infty, \quad \int_{-\infty}^{0} |f'(U(\xi)) - f'(0)| d\xi < \infty. \)

If \( u(x,t) \) is a solution such that the norm

(3.29) \[
    \sup_{|x|<\infty} |(u(x,0) - U(x))(1 + \exp(-\frac{V}{2}x))|
\]
is sufficiently small, then \( u(x,t) \) satisfies the inequality
\[
\sup_{|\xi|<\infty} |u(\xi,t)-U(\xi+\xi_0)| \leq Ke^{-\omega t}, \quad t \geq 0
\]
where \( \omega > 0, \ K > 0, \ \xi_0 \) are some real constants depending on the magnitude of (3.29).

This proposition exhibits the exponential decay to the traveling wave front when the initial condition is close to the front in a suitable way. Note that, since \( V > 0 \), (3.29) puts a very restrictive condition on \( u(x,0) \) at \( x = -\infty \). It was shown in [32] that we may fail to get the exponential decay if we do not insert the weight function \( V \left( 1+e^{-\frac{V}{2} \xi} \right) \) in (3.29). Especially when \( f'(0) > 0 \), it is essential to measure the perturbed norm (3.29) in terms of the weight function \( 1+e^{-\frac{V}{2} \xi} \) if we want a result of exponential decay. In such cases, this traveling front may fail to be a \( C_0 \) stable asymptotic state for the reaction-diffusion equation considered as discussed in the survey article by Fife [13]. We shall see that in the case of the combustion equation (3.3), we may get rid of the weight \( 1+\exp(-\frac{V}{2} \xi) \) by putting some less restrictive condition on \( u(x,0) \).

**Stability Under Finite Amplitude Perturbation**

Direct application of Proposition 4 gives the following

**Corollary 1:** The flame fronts in (3.25) and (3.26) have the stability property stated in Proposition 4.

**Proof.** We only have to check the conditions (i), (ii), (iii). From Theorems 1 and 2, these flame fronts satisfy (i) and (ii). For
(iii), we will use properties listed in (3.20), (3.21), (3.22).

Consider first the front \( T_\alpha(\xi), \xi = x + V_\alpha t, 0 \leq \alpha < T_i \). Since, for \( \xi \) sufficiently small, \( T_\alpha(\xi) < T_i \). The condition

\[
\int_{-\infty}^{0} |g'(T_\alpha(\xi)) - g'(\alpha)| d\xi < \infty
\]

is satisfied. On the other hand,

\[
\int_{0}^{\infty} |g'(T_\alpha(\xi)) - g'(1)| d\xi = \int_{T_\alpha(0)}^{1} \frac{|g'(T) - g'(1)|}{S(T,V_\alpha)} dT,
\]

where \( S(T,V_\alpha) \) is the phase plane representation of the front \( T_\alpha(\xi) \), see (3.20). The above integral has a singular point at \( T = 1 \). But

\[
\frac{d}{dT}S(1,V_\alpha) = \lambda_2(V_\alpha) \neq 0,
\]

by l'Hospital's rule,

\[
\lim_{T \to 1} \frac{|g'(T) - g'(1)|}{S(T,V_\alpha)} = \frac{|g''(1)|}{|\lambda_2(V_\alpha)|} \neq 0.
\]

Thus,

\[
\int_{T_\alpha(0)}^{1} \frac{|g'(T) - g'(1)|}{S(T,V_\alpha)} dT < \infty.
\]

Condition (iii) is verified.

The verification for the other fronts is similar.

Note that the norm computed in (3.29) is dependent upon the flame speed. As we have remarked, it is in general impossible to obtain exponential decay under a small uniform perturbation (i.e., without the weight function \( 1 + \exp(-\frac{V_\xi}{2}) \)). However, a special case is of interest to us. This is
Corollary 2: Consider the combustion equation (3.3)

\[ T_t = T_{xx} + g(T) \]

with an initial condition

\[ T(x,0) = \phi(x), \quad 0 \leq \phi(x) \leq 1, \quad |x| < \infty. \]

Suppose that, for some \( 0 \leq \alpha < T_i \), \( \phi(x) = \alpha \) for all \( x \leq x_0 \) with \( x_0 \) some fixed number. Then, whenever \( \sup_{|x|<\infty} \sup_{0 \leq \alpha < T_i} |T_\alpha(x) - \phi(x)| \) is sufficiently small, there exists \( \xi_0 \) such that the solution \( T(x,t) \) has the asymptotic behavior

\[ \sup_{|x|<\infty} |T(x,t) - T_\alpha(x+V_\alpha t + \xi_0)| \leq Ke^{-\omega t}, \quad t \geq 0, \]

for some \( K > 0, \omega > 0 \).

Proof. From Theorem 1, there exists \( \bar{x} \leq x_0 \) sufficiently small such that

\[ T_\alpha(x) = \alpha + O(e^{-\alpha}) \quad \text{for} \quad x \leq \bar{x}. \]

This fact yields the estimate

\[ \sup_{|x|<\infty} \left |(T_\alpha(x) - \phi(x))(1 + \exp(-\frac{\alpha}{2} x)) \right | \leq \sup_{|x|<\infty} |T_\alpha(x) - \phi(x)|. \]

Thus, if \( \sup_{|x|<\infty} |T_\alpha(x) - \phi(x)| \) is sufficiently small, (3.29) is satisfied, and hence the corollary follows.

In the next subsections, we shall obtain similar asymptotic behavior for a more general initial condition \( \phi(x) \).

The assumption that \( 0 \leq \alpha < T_i \) is essential in Corollary 2. When \( \alpha = T_i \) in Corollary 2, the estimate (3.29) cannot be obtained.
Indeed, if $g'(T_{i+}) > 0$, Theorem 2 gives the estimate

$$T_{T_i}(x) = T_i + O(e^{ux}) \text{ as } x \to -\infty,$$

where $0 < u < \frac{1}{2}\left[\sqrt{V_{T_i}^2 - (V_{T_{i+}}^2 - 4g'(T_{i+}))}\right]$. Thus,

$$\left|\phi(x) - T_{T_i}(x)\right|(1 + \exp(-\frac{T_i}{2}x)) \geq 0\left(\exp(-\frac{1}{2}V_{T_i}^2 - 4g'(T_{i+})x)\right) \text{ as } x \to -\infty.$$

Since $V_{T_i}^2 > 4g'(T_{i+})$, the estimate (3.29) cannot be obtained. However, in the next subsection, we will show that $T_{T_i}(\zeta)$ can be obtained as a final asymptotic state from some special initial conditions. The rate of approach to this flame front is very slow. This fact, along with Proposition 3(b), partially accounts for the fact that, without the assumption of a positive ignition temperature, the formation of the flame front is harder to observe in actual numerical computation.

(E) Asymptotic Behavior

The basic problem in this subsection is to consider the general Cauchy problem for the combustion equation (3.3) or (3.18). The combustion equation

$$T_t = T_{xx} + g(T), \quad t \geq 0, \quad |x| < \infty,$$

is given a Cauchy initial condition

$$T(x,0) = \phi(x), \quad 0 \leq \phi(x) \leq 1, \quad |x| < \infty.$$

In this subsection, $\phi(x)$ is assumed to be a piecewise continuous function whose discontinuities are discrete jumps. The general problem is what $\lim_{t \to \infty} T(x,t)$ looks like, when $T(x,t)$ is the solution.
determined by (3.25). It is the purpose of this subsection to show that the flame fronts in (3.25) or (3.26) are "intrinsic" to the gas mixture combustion, i.e. the flame fronts act as building blocks in the sense that the \( \lim_{t \to \infty} T(x,t) \) consists of staked combinations of the flame fronts.

Such considerations are not new. Kanel' [21], [22] proved that if \( \phi(x) \) in (3.30) has the properties

\[
\phi(x) = 0 \text{ for } x < 0 \text{ and sufficiently large, } \frac{d\phi}{dx} > 0 \text{ for all } x, \text{ and } \phi(x) = 1 \text{ for } x > 0 \text{ and sufficiently large,}
\]

then \( T(x,t) \) approaches the flame front \( T_0(x+V_0 t) \). He did not determine the rate of approach. One of our main results (Theorem 3) is to show that the rate of approach is exponential under more general initial condition \( \phi(x) \) of the form

\[
\phi(x) = 0 \text{ for } x < 0 \text{ and sufficiently large,} \quad \phi(x) > T_1 \text{ for } x > 0 \text{ and sufficiently large.}
\]

(3.31)

The strategy to obtain the result is simple-minded. We apply Kanel's theorem to the special case of initial condition

\[
\phi(x) = \begin{cases} 
1, & \text{if } x > 0 \\
0, & \text{if } x < 0.
\end{cases}
\]

Then the assumption that a positive ignition temperature exists enables us to obtain an estimate of the form (3.29) in Proposition 4 of subsection (D). Thus the exponential decay follows from Sattinger's stability criterion. This is our Proposition 5. The general case of \( \phi(x) \) satisfying (3.31) is a consequence of the comparison principle in
subsection (A) and a series of lemmas proved by Fife and McLeod [12] (Lemma 5.1 to Lemma 4.5 on pp. 347-351) which we summarize as follows:

**Lemma 4** (Fife & McLeod [12]): Suppose that $f(u)$ satisfies condition (3.5). Consider the Cauchy problem

\[
\begin{aligned}
&u_t = u_{xx} + f(u), \quad t > 0 \\
&u(x,0) = \phi(x), \quad |x| < \infty
\end{aligned}
\]

with $0 \leq \phi(x) \leq 1$. Let $U(\xi), \xi = x + Vt$, be a traveling wave solution. If there exists $K_0 > 0, \xi_1, \xi_2$ real constants such that the solution $u(x,t)$ satisfies the inequality

\[
U(\xi+\xi_1) - K_0 e^{-\omega t} \leq u(x,t) \leq U(\xi+\xi_2) + K_0 e^{-\omega t}, \quad t > 0, \quad |x| < \infty
\]

for some $\omega > 0$, then there exists $\xi_0$ such that

\[
\limsup_{t \to \infty} \sup_{|\xi| < \infty} |u(\xi,t) - U(\xi+\xi_0)| = 0.
\]

(3.31) states that initial states vaguely similar to flame fronts adjust themselves and evolve into the flame fronts in an exponential rate. This is a global stability result of the flame fronts as compared to Corollary 2 of Proposition 4.

A similar strategy is applied to prove results concerning more general initial conditions $\phi(x)$. Thus we are able to prove that, if the gas mixture is ignited at some region of the tube, two flame fronts propagating in diverging directions are generated. However, if two flame fronts propagate in converging directions, eventually the "hotter" flame dominates the evolution process.
Finally, we quote a result of Stokes [37] to show that the flame front in (3.26) for the equation (3.18) is an asymptotic state of the special initial condition

\[
\phi(x) = \begin{cases} 
1, & \text{if } x > 0 \\
0, & \text{if } x \leq 0.
\end{cases}
\]

(See Proposition 6 below.) In this case, there does not exist an ignition temperature. Sattinger's stability criterion cannot be applied to obtain a rate of exponential decay. I am not able to generalize Stokes' theorem to more general initial conditions, say, initial conditions of Kanel's type. My general feeling is that such a result should be true without exponential decay.

**Preliminary Observations**

If, in (3.30), \(0 < \phi(x) \leq T_i\) for all \(x\), then equations (3.3) with initial condition (3.30) is just the heat equation with \(\phi(x)\) as the initial condition. Thus:

(3.32) If \(0 < \phi(x) \leq T_i\) for all \(x\), and the average

\[
\bar{\phi} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \phi(x) dx
\]

exists, then

\[
\lim_{t \to \infty} \sup_{|x| < \infty} |T(x, t) - \bar{\phi}| = 0.
\]

This fact is just a consequence of the decay of heat conduction.
On the other hand, if $ \phi(x) $ exceeds $ T_i $ everywhere the chemical reaction proceeds everywhere in the tube. We expect the gas in the whole tube to be burned very quickly. Hence

(3.33) If there exists $ T_i < \beta < 1 $ so that $ \beta \leq \phi(x) \leq 1 $ for all $ x $, then

$$ |T(x,t)-1| \leq Ke^{-\omega t}, \quad |x| < \infty, \quad t > 0, $$

for some $ K > 0, \omega > 0 $.

The proof of (3.33) is as follows. Let $ \beta(t) $ be the solution of the ordinary differential equation

$$ \frac{d\beta}{dt} = g(\beta(t)), \quad \beta(0) = \beta. $$

The comparison principle in subsection (A) implies that

$$ \beta(t) \leq T(x,t) \leq 1, \quad |x| < \infty, \quad t > 0. $$

We claim that there exists $ K > 0, \omega > 0 $ such that

$$ 0 \leq 1 - \beta(t) \leq Ke^{-\omega t}, \quad t > 0. $$

This fact would prove (3.27). The claim is true because $ g'(1) < 0 $, so that 1 is an attractive stationary point for the equation $ \frac{d\beta}{dt} = g(\beta) $. Standard books on ordinary differential equations (say [7]) will give the asymptotic behavior. We note that this argument also shows that

$$ |T(x,t)-\beta(t)| \leq Ke^{-\omega t} \text{ for } |x| < \infty, \quad t > 0. $$

The rate $ \omega $ of decay is determined by the quantities $ |g'(1)|, \ 1-\beta $ and $ \beta-T_i $. 
The Flame Fronts as Asymptotic States

Suppose that the initial data \( \phi(x) \) is close to the burned state \( 1 \) on one end of the tube, while it is in an unburned state on the other end of the tube. We would expect that a flame front appears and propagates along the tube. Thus we claim

**Theorem 3:** Let \( \phi(x) \) in (3.30) satisfy the conditions that

\[
\phi(x) = \alpha \text{ for all } x < 0 \text{ sufficiently large, } \alpha < \phi(x) \leq 1 \text{ for all } x
\]

\[
\beta < \phi(x) \text{ for all } x > 0 \text{ sufficiently large,}
\]

where \( 0 < \alpha < T_i < \beta < 1 \). Then there exist constants \( c, \tau_0 > 0 \) such that

\[
|T(x,t) - T_\alpha(x+V_\alpha t+c)| \leq Ke^{-\omega t}, \quad |x| < \infty, \quad t \geq \tau_0
\]

for some \( K > 0, \omega > 0 \). \( V_\alpha \) is the flame speed of the flame front \( T_\alpha \) [see (3.25)].

We shall prove this theorem by first proving a special case of it, namely, the case of a single jump initial condition. Since this special case is of special interest to us, we list it as

**Proposition 5:** Suppose that \( \phi(x) \) in (3.30) is of the form

\[
\phi(x) = \begin{cases} 
\alpha, & \text{if } x < 0, \\
\beta, & \text{if } x > 0,
\end{cases}
\]

where \( 0 < \alpha < T_i < \beta < 1 \). Then the conclusion of Theorem 3 holds.
Proof. Without loss of generality, we shall assume that \( \alpha = 0 \). Let \( T(x,t) \) be the solution of (3.3) with (3.35) as initial condition. The proof is divided into several steps.

**Step 1:** Let

\[
\tilde{\phi}(x) = \begin{cases} 
1, & \text{if } x \geq 0 \\
0, & \text{if } x < 0
\end{cases}
\]

and \( \tilde{T}(x,t) \) be the solution of (3.3) with \( \tilde{\phi}(x) \) as initial condition.

Kanel' [2] proved the following facts about \( \tilde{T}(x,t) \):

(a) there exists \( \xi_0 \) such that

\[
\lim_{t \to \infty} \sup_{|x| < \infty} |\tilde{T}(x,t) - T_0(x + V_t \xi + \xi_0)| = 0.
\]

(b) for each \( t \), \( \lim_{x \to \infty} \tilde{T}(x,t) = 0 \) and \( \lim_{x \to -\infty} \tilde{T}(x,t) = 1 \).

Define a new variable \( \xi = x + V_t \xi + \xi_0 \); the equation (3.3) is then transformed into

\[
T_t = T_{\xi \xi} - V_t T_\xi + g(T).
\]

Let \( T(\xi,t) \) and \( \tilde{T}(\xi,t) \) be the solution \( T(x,t) \) and \( \tilde{T}(x,t) \) in the new variable \( (\xi,t) \). Fact (a) is then translated into

(a) \( \lim_{t \to \infty} \sup_{|\xi| < \infty} |\tilde{T}(\xi,t) - T_0(\xi)| = 0 \).

We claim that (a) and (b) imply the following fact:

(c) there exists \( K > 0, \omega > 0, \tau_0 > 0 \) such that

\[
\sup_{|\xi| < \infty} |\tilde{T}(\xi,t) - T_0(\xi)| \leq K e^{-\omega t}, \text{ for all } t \geq \tau_0.
\]

That is, this proposition is true for initial condition \( \tilde{\phi}(\xi) \). Indeed,
choose any $0 < \varepsilon < T_i$, let $\tau_1$ be such that

$$\sup_{|\xi| < \infty} |\tilde{T}(\xi, t) - T_0(\xi)| < \varepsilon, \text{ for all } t > \tau_1.$$ 

Let $\tilde{\xi}$ be a fixed number such that $T_0(\tilde{\xi}) < T_i - \varepsilon$. Then

$$\tilde{T}(\xi, t) \leq T_i \text{ for all } \xi \leq \tilde{\xi}, \ t > \tau_1.$$ 

By property (b), we may assume that $\tilde{\xi}$ is chosen in such a way that the above inequality is true for all $t > 0$. Then $\tilde{T}(\xi, t)$ satisfies the equation

$$T_t = T_{\xi \xi} - V_0 T_\xi, \ \xi \leq \tilde{\xi}, \ t > 0$$

with initial-boundary conditions

$$T(\xi, 0) = 0 \text{ for } \xi \leq \tilde{\xi}, \ T(-\infty, t) = 0, \ T(\tilde{\xi}, t) \leq T_i, \text{ for all } t.$$ 

Consider $v(\xi, t) = T(\xi, t) - T_i e^{V_0(\xi - \tilde{\xi})}$. $v$ satisfies the properties

$$v_t = v_{\xi \xi} - V_0 v_\xi, \ \text{for } \xi \leq \tilde{\xi}, \ t > 0$$

$$v(\xi, 0) = -T_i e^{V_0(\xi - \tilde{\xi})} \leq 0; \ v(-\infty, t) = 0, \ v(\tilde{\xi}, t) = T(\tilde{\xi}, t) - T_i \leq 0$$

for all $t$.

Thus, by the maximum principle, we find $v(\xi, t) \leq 0$ for all $\xi \leq \tilde{\xi}$ and $t > 0$, i.e.

$$0 \leq T(\xi, t) \leq T_i e^{V_0(\xi - \tilde{\xi})}, \ \text{for } \xi \geq \tilde{\xi} \text{ and } t > 0.$$ 

Hence

$$\sup_{|\xi| < \infty} |(T(\xi, t) - T_0(\xi))(1 + \exp(-\frac{V_0}{2} \xi))| \leq O(\sup_{|\xi| < \infty} |T(\xi, t) - T_0(\xi)|)$$

for all $t > 0$ (the bound in $O(\cdots)$ is independent of $t$). Since,
by property (a), \( \tau_0 \) can be chosen so that \( \sup_{|\xi|<\infty} |T(\xi,\tau_0)-T_0(\xi)| \) is sufficiently small. Sattinger's Proposition 4 implies fact (c).

**Step 2:** Since \( \beta < 1 \), \( T(\xi,0) \leq \tilde{T}(\xi,0) \) by definition of \( \phi \) and \( \tilde{\phi} \). 

Hence, by the comparison principle in subsection (A),

\[ 0 \leq T(\xi,t) \leq \tilde{T}(\xi,t) \text{ for all } \xi \text{ and } t \geq 0. \]

Thus, from Step 1,

\[ T(\xi,t) = O(e^{\mu \xi}) \text{ as } \xi \to -\infty, \ t \geq 0, \]

uniformly in \( t \).

**Step 3:** Since \( \beta > T_1 \), the state with \( \xi \) sufficiently large goes to the burned state 1 exponentially [see (3.3)]. Thus, it is easy to see that there exists \( \xi_1 \) such that

\[ \lim_{t \to \infty} \sup_{|\xi|<\infty} |T(\xi,t)-\tilde{T}(\xi+\xi_1,t)| = 0. \]

Then, from property (a) in Step 1,

\[ \lim_{t \to \infty} \sup_{|\xi|<\infty} |T(\xi,t)-T_0(\xi+\xi_1)| = 0. \]

That is, \( T(\xi,t) \) has property (a) in Step 1.

**Step 4:** Step 2 shows that property (b) in Step 1 holds for \( T(\xi,t) \).

Thus, from Step 3, \( T(\xi,t) \) has also properties (a) and (b) in Step 1. The same argument proves that (c) in Step 1 holds also for \( T(\xi,t) \).

This proves the proposition.

The assertion in Proposition 5, without exponential decay, was proved in Kanel' [21]. We use his result along with Sattinger's
stability criterion to get the exponential decay. Similarly, Kanellakopoulos [22] had similar results as in Theorem 3 without the assertion of exponential decay. His assumption on $\phi(x)$ was more restrictive than what we assume in (3.34). We will rely on a series of lemmas proved in Fife and McLeod [12] to get the exponential decay in Theorem 3.

Proof (of Theorem 3). We may assume that $\alpha = 0$ in (3.34). Define two jump functions as follows:

$$
\phi_x(x) = \begin{cases} 
\beta, & \text{if } x > 0 \\
0, & \text{if } x \leq 0 
\end{cases} \quad \phi_y(x) = \begin{cases} 
1, & \text{if } x > 0 \\
0, & \text{if } x \leq 0 
\end{cases}
$$

Let $T_\xi(x,t)$ and $T_\gamma(x,t)$ be the solution of (3.3) with initial condition $\phi_x(x)$ and $\phi_y(x)$ respectively. Define $\xi = x + V_0 t$ and change to $(\xi,t)$ variables. The equation (3.3) is transformed into

$$
(3.36) \quad T_t = T_{\xi_x} - V_0 T_{\xi_y} + g(T).
$$

Choose $\xi_\xi$ and $\xi_\gamma$ such that

$$
\phi(\xi) = 0 \text{ for } \xi \leq \xi_\xi, \quad \phi(\xi) > \beta \text{ for } \xi \geq \xi_\gamma.
$$

This is possible by assumption (3.34). Then, obviously,

$$
\phi_x(\xi - \xi_\gamma) \leq \phi(\xi) \leq \phi_y(\xi - \xi_\xi), \quad |\xi| < \infty.
$$

Applying the comparison principle in subsection (A) to the transformed equation (3.36), the inequality

$$
T_\xi(\xi - \xi_\gamma,t) \leq T(\xi,t) \leq T_\gamma(\xi - \xi_\xi,t), \quad |\xi| < \infty, \ t \geq 0
$$

follows. But Proposition 5 implies that there exists $\mu > 0$ such
that, for some real constants $c_1$ and $c_2$,

$$T_2(\xi-\xi, t) \geq T_0(\xi-\xi_1) - k_1 e^{-\mu t},$$

$$T_2(\xi-\xi_2, t) \leq T_0(\xi-\xi_1) + k_1 e^{-\mu t}$$

for all $t$ and $\xi$. Thus, we obtain the important estimate

(3.37) \quad \quad T_0(\xi-\xi_1) - k_1 e^{-\mu t} \leq T(\xi, t) \leq T_0(\xi-\xi_2) + k_1 e^{-\mu t}

where $\xi_1 = \xi - c_1$, $\xi_2 = \xi - c_2$.

(3.37) is the kind of estimate stated in Lemma 4 of Fife and McLeod. Thus, there exists $\xi_0$ such that

(3.38) \quad \quad \lim_{t \to \infty} \sup_{|\xi| < \infty} |T(\xi, t) - T_0(\xi + \xi_0)| = 0.

However, since $T(\xi, 0) = 0$ for sufficiently large $\xi < 0$, (3.37) can be used to prove that

$$T(\xi, t) = O(e^{\mu \xi}) \quad \text{as} \quad \xi \to -\infty, \ t > 0,$$

uniformly in $t$. The process of this proof is exactly the same as in Step 1 of Proposition 5. This estimate, along with (3.38), gives the desired result in Theorem 3, since we can then apply Sattinger's Proposition 4. This completes the proof.

The assumption that $\alpha < T_1$ in Theorem 3 or Proposition 5 is essential in order to get the exponential decay. This is best illustrated in the proof of Corollary 2 in subsection (D). The assumption that $\phi(x)$ be of constant $\alpha$ for sufficiently large $x < 0$ also seems to be essential. Allowing the unburned states of $\phi(x)$ to vary infinitely often for sufficiently small $x$ results in the difficulty
of which flame front should be approached. In other words, in order to get a clear-cut asymptotic flame front, assumption (3.34) on the initial condition seems to be best possible. Theorem 3 gives some sense of global stability as compared to finite stability in Corollary 2 of subsection (D). In a later part of this subsection, we shall consider the case where the unburned state of the initial condition is close to \( T_i \).

It should be remarked that such a strong stability property of flame fronts may not be possible in "reality". Especially when gas dynamical effects are taken into consideration, the flame fronts are mostly unstable; see Markerstein [28]. The assumptions that \( L = 1 \) and that the total enthalpy be preserved stabilize the flame fronts in this reaction-diffusion model. However, in multi-dimensional reaction-diffusion model, "self-turbulence" may occur intrinsically without gas dynamical effects; see Schivaskinsky [35].

**Generation and Interaction of Flame Fronts**

If the gas mixture is ignited somewhere in the tube there should be generated two flame fronts propagating in diverging directions. We justify this intuition by showing

**Theorem 4:** Given three gas states \( 0 \leq \alpha_1, \alpha_2 < T_i < \beta \leq 1 \). If the initial condition \( \phi(x) \) in (3.30) has the properties:

\[
\phi(x) = \alpha_1 \quad \text{for} \quad x \leq x_1; \quad \alpha_1 \leq \phi(x) \leq 1 \quad \text{for} \quad x_1 \leq x < x_2;
\]
\[
\beta \leq \phi(x) \quad \text{for} \quad x_2 \leq x \leq x_3; \quad \alpha_2 \leq \phi(x) \leq 1 \quad \text{for} \quad x_3 < x \leq x_4;
\]
\[
\phi(x) = \alpha_2 \quad \text{for} \quad x \geq x_4,
\]

(3.39)
where \( x_1 \leq x_2 < x_3 \leq x_4 \) are some fixed points. Then if \( x_3 - x_2 \) is large enough, the solution \( T(x,t) \) has the asymptotic behavior

\[
|T(x,t) - T_{\alpha_2}(x + \alpha_2 t + c_2)| \leq K e^{-\omega t} \text{ for } x > 0, \ t \geq T_0,
\]
\[
|T(x,t) - T_{\alpha_1}(x + \alpha_1 t + c_1)| \leq K e^{-\omega t} \text{ for } x < 0, \ t \geq T_0,
\]

where \( c_1, c_2, K > 0, \ \omega > 0, \ T_0 > 0 \) are some real constants.

Proof. Without loss of generality, we may assume \( \alpha_1 = 0 \) and then suppress the index of \( \alpha_2 \), i.e., let \( \alpha = \alpha_2 \).

Step 1: Suppose that \( \phi(x) \) is of the particular form

\[
\tilde{\phi}(x) = \begin{cases} 
0 & \text{if } x < y_1 \\
\beta & \text{if } y_1 \leq \beta \leq y_2 \\
\alpha & \text{if } x > y_2 
\end{cases}
\]

We shall show that this theorem, with \( \tilde{\phi}(x) \) as the initial condition, is true.

Define two jump functions \( \phi_1 \) and \( \phi_2 \) as follows:

\[
\phi_1(x) = \begin{cases} 
\beta & \text{if } x > 0 \\
0 & \text{if } x \leq 0 
\end{cases}
\]
\[
\phi_2(x) = \begin{cases} 
\beta & \text{if } x > 0 \\
\alpha & \text{if } x \leq 0 
\end{cases}
\]

Then \( \phi(x) = \min\{\phi_1(x-y_1), \phi_2(-x+y_2)\} \). Let \( \tilde{T}(x,t), T_1(x,t), T_2(x,t) \) be the solution of (3.3) with initial condition \( \phi(x), \phi_1(x), \phi_2(x) \) respectively. By the comparison principle in subsection (A),

\[
\tilde{T}(x,t) \leq \min\{T_1(x-y_1,t), T_2(-x+y_2,t)\}
\]
for all \((x,t)\) in the upper half plane. Now, for any sufficiently small \(\epsilon > 0\), we can always find \(d_1\) and \(d_2\) such that

\[
\min\{T_0(x+d_1), T_\alpha(-x+d_2)\} \leq \tilde{\phi}(x) + \epsilon, \quad -\infty < x < \infty
\]

and

\[
\beta \leq \min\{T_0(x+d_1), T_\alpha(-x+d_2)\}
\]

for \(x\) in some smaller subinterval of \(y_1 < x < y_2\). The latter is possible whenever \(y_2 - y_1\) is large enough. Moreover, since \(\epsilon\) is small and \(y_2 - y_1\) is large, \(d_1\) and \(d_2\) may be chosen to be bounded as \(\epsilon\) changes. However, \(T_0(x+V_0t+d_1)\) and \(T_\alpha(-x+V_\alpha t+d_2)\) are solutions of the equation (3.3). Therefore the comparison principle in subsection (A) again implies that

\[
\min\{T_0(x+V_0t+d_1), T_\alpha(-x+V_\alpha t+d_2)\} \leq T_\star(x,t)
\]

where \(T_\star(x,t)\) is the solution of (3.3) with \(\min\{\tilde{\phi}(x) + \epsilon, 1\}\) as initial condition. But Lemma 2 in subsection (B) gives the inequality

\[
T_\star(x,t) \leq \tilde{T}(x,t) + \epsilon e^{Mt}, \quad |x| < \infty, \quad t \geq 0
\]

where \(M = \sup_{0 \leq T \leq 1} |g'(T)|\) is independent of \(\epsilon\). Hence, from (3.41)

\[
\min\{T_0(x+V_0t+d_1), T_\alpha(-x+V_\alpha t+d_2)\} \leq \tilde{T}(x,t) + \epsilon e^{Mt}
\]

As \(\epsilon \to 0\), \(d_1\) and \(d_2\) may assume to tend to a definite limit which we will still denote by \(d_1\) and \(d_2\) (note that \(d_1\) and \(d_2\) depend only on \(\epsilon\), and are uniformly bounded). Then the above inequality implies that

\[
T_0(x+V_0t+d_1) + T_\alpha(-x+V_\alpha t+d_2) - 1 \leq \tilde{T}(x,t)
\]
On the other hand, by Proposition 5, there exists $K > 0$, $\omega > 0$, $c_1$, $c_2$ such that

$$|T_1(x-y_1, t) - T_0(x + V_0 t - y_1 + c_1)| \leq \frac{K}{2} e^{-\omega t},$$

$$|T_2(-x+y_2, t) - T_\alpha(-x + V_\alpha t + y_2 + c_2)| \leq \frac{K}{2} e^{-\omega t}.$$

Thus, from (3.40) and (3.42), the inequality

(3.43) \quad \tilde{T}_0(x + V_0 t + d_1) + T_\alpha(-x + V_\alpha t + d_2) - 1 - K e^{-\omega t}

$$\leq \tilde{T}(x, t) \leq T_0(x + V_0 t - y_1 + c_1) + T_\alpha(-x + V_\alpha t + y_2 + c_2) - 1 + K e^{-\omega t}$$

follows. Here we have used the obvious facts that

$$T_2(-x+y_2, t) \leq T_1(x-y_1, t) \quad \text{for } x \text{ large, } t > 0,$$

$$T_1(x-y_1, t) \leq T_2(-x+y_2, t) \quad \text{for } x < 0 \text{ small, } t > 0,$$

to deduce (3.43) from (3.40).

Once we have an inequality of the type (3.43), we may apply the same arguments as Fife and McLeod [12], pp. 354-357, to prove the conclusion. This finishes Step 1.

**Step 2:** We define two step functions as follows:

$$\tilde{\phi}_1(x) = \begin{cases} 0, & \text{if } x < x_1 \\ 1, & \text{if } x_1 \leq x \leq x_4 \\ \alpha, & \text{if } x_4 < x \end{cases}$$

$$\tilde{\phi}_2(x) = \begin{cases} 0, & \text{if } x < x_2 \\ \beta, & \text{if } x_2 \leq x \leq x_3 \\ \alpha, & \text{if } x_3 < x \end{cases}$$

Then, obviously, $\tilde{\phi}_2(x) \leq \phi(x) \leq \tilde{\phi}_1(x)$ for all $x$. Let $\tilde{T}_1(x, t)$ and
\( \tilde{T}_2(x,t) \) be the solution of (3.3) with initial conditions \( \tilde{\phi}_1 \) and \( \tilde{\phi}_2 \) respectively. The comparison principle yields the inequality

\[ \tilde{T}_2(x,t) \leq T(x,t) \leq \tilde{T}_1(x,t), \quad |x| < \infty, \quad t \geq 0. \]

Thus, by Step 1, we can obtain a similar estimate as (3.43) for \( T(x,t) \). This implies that \( T(x,t) \) has the asymptotic property described in the theorem. The reason is as in Step 1.

Again, the assumption that \( \alpha_1, \alpha_2 < T_i \) plays an essential role in Theorem 4.

A similar technique as used in the proof of Theorem 4 can be applied to formulate and prove the following:

**Theorem 5:** Suppose that \( \phi(x) \) in (3.25) has the following properties:

\[ \phi(x) \geq \beta \quad \text{for} \quad |x| \text{ sufficiently large}, \]
\[ \phi(x) = \alpha_1 \quad \text{for} \quad x_1 \leq x \leq x_2, \quad \alpha_1 \leq \phi(x) \leq 1 \quad \text{for} \quad x \leq x_2, \]
\[ \alpha_2 \leq \phi(x) \leq \alpha_1 \quad \text{for} \quad x_2 \leq x \leq x_3, \quad \phi(x) = \alpha_2 \quad \text{for} \quad x_3 \leq x \leq x_4, \]
\[ \alpha_2 \leq \phi(x) \leq 1 \quad \text{for} \quad x \geq x_4, \]

where \( x_1 < x_2 < x_3 < x_4 \) are four real constants and \( 0 \leq \alpha_1, \alpha_2 < T_i < \beta \). Then, if \( x_2-x_1 \) and \( x_4-x_3 \) are sufficiently large, the solution \( T(x,t) \) has the asymptotic formula

\[ (3.44) \quad T(x,t) = \max\{T(\alpha_1(-x+V_{\alpha_1}t+c_1), T(\alpha_2(x+V_{\alpha_2}t+c_2)) \} + O(e^{-\omega t}) \]

as \( t \rightarrow \infty \), for some \( c_1, c_2 \) and \( \omega > 0 \).

We will not give the complete proof as in Theorem 4. The steps of the proof are the same. First, we prove (3.44) for the case of initial step functions, and then use comparison principle to obtain
estimates for solutions of general $\phi(x)$. Note that, when $0 < \alpha_1 < \alpha_2 < T_1$, $V_{\alpha_1} < V_{\alpha_2}$. The two waves in (3.44) move toward each other. Their main transition parts will sooner or later interact. If we assume $0 < \alpha_1 < \alpha_2$ in (3.44), the wave $T_{\alpha_2}$ moves faster than $T_{\alpha_1}$. (3.44) shows that, as soon as $T_{\alpha_2}$ dominates $T_{\alpha_1}$, $T_{\alpha_2}$ "absorbs" $T_{\alpha_1}$ as time goes to $\infty$. This kind of flame interaction is very similar to phenomena like the Belousov reaction, which is also modeled on a reaction-diffusion type equation; see Howard and Koppell [24].

For more general initial conditions, we can expect that a lot of traveling waves arise. During the evolution process, either the gas states in the whole tube evolve into burned state $I$ or eventually at most two flame fronts dominate the evolution process. We are content with Theorems 4 or 5 as an example of such an evolution process.

We remark that (3.33) can be improved via the use of Theorem 3 or Theorem 4 as follows:

(3.45) Suppose that $\phi(x)$ in (3.30) exceeds the ignition temperature $T_1$ on a sufficiently large interval of the line. Then the solution $T(x,t)$ has the asymptotic behavior

$$\lim_{t \to \infty} |T(x,t) - 1| = 0$$

uniformly on every compact subset of the line.

Similar results were proved, for example, in Aronson and Weinberger [2] or Kopell and Howard [24] for different nonlinear sources.
The Case of No Positive Ignition Temperature

When a positive ignition temperature is not assumed, the asymptotic behavior for the general Cauchy problem is more difficult. The discussion on stability of wave fronts in subsection (D) suggests that similar results like Theorem 3 or 4 may fail to be true for the wave \( T_1(\xi) \) in (3.25) or the minimal speed wave of the non-truncated combustion equation (3.18). It is more difficult to determine which class of initial conditions gives rise to these waves. We quote a result which was proved by Kolomogoroff et al. [23] for their KPP equation and later extended by Stokes [37] to rather general nonlinear sources. This is similar to Proposition 5:

**Proposition 6 (Stokes [37]):** Suppose that the function \( f(u) \) satisfies the conditions in (3.6). Let \( u(x,t) \) be the solution to the reaction-diffusion equation

(3.46) \[ u_t = u_{xx} + f(u) \]

with the initial condition

(3.47) \[ u(x,0) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \]

Then, if \( U(\xi), \xi = x + Vt \), denotes the minimal speed traveling wave solution of (3.46) as exhibited in Theorem 2, there exists some constant \( c \) such that

(3.48) \[ \limsup_{t \to \infty} |u(x,t) - U(x + Vt + c)| = 0. \]

We will not reproduce the proof of Stokes in [37]. This proposition
exhibits the minimal speed wave as an asymptotic state of some physically reasonable initial states. The approach to the traveling front in (3.48) is not likely to be exponential; see the corresponding discussion in Sattinger [32]. It is also an unsettled question what class of initial conditions should be put in (3.47) in order to give rise to the minimal speed wave, or higher speed waves.

When we apply Proposition 6 to the wave $T_{T_i}(\xi)$ or the combustion equation with a non-truncated Arrhenius nonlinear source, one is easily convinced that these waves are physically realizable flame fronts. However, their possible unstability and the existence of higher speed waves present the difficulty of actually computing them.
4. **The Case of Arbitrary Lewis Numbers**

We present in this section a proof of the existence of flame fronts in the case of arbitrary Lewis numbers. As usual, we assume that the flame fronts are traveling wave solutions to (2.11) of the form

\[ T(x,t) = T(\xi), \quad C(x,t) = C(\xi), \quad \xi = x + Vt. \]

When substituted into (2.11), \( T(\xi) \) and \( C(\xi) \) satisfy a system of two second order (singular) boundary value problems which are related through the Arrhenius nonlinear source term. The speed \( V \) appears as a parameter. If the concentration profile \( C(\xi) \) is known, the temperature profile equation for \( T(\xi) \) looks very much like equation (3.3) (where \( L = 1 \) is assumed), and hence the fundamental lemma, Lemma 2, in subsection 3(C) can be applied again. On the other hand, if the temperature profile \( T(\xi) \) is known, the concentration profile equation for \( C(\xi) \) is a singular linear boundary value problem which can be transformed into a type of equation discussed in Hille [20]. The solution for \( C(\xi) \) is then secured. Thus, our general strategy will be an iteration scheme based on these observations. The main difficulty in such an iteration process is to obtain correct estimates which ensure convergence. When a Lewis number is fixed, the salient feature of the combustion process is the almost conservation of the total enthalpy which provides the desired estimate. The existence of flame fronts then follows under a natural physical condition which is a consequence of the conservation of energy, that is, the temperature of the burned gas is the unique temperature reached from the temperature of the unburned gas through an adiabatic burning process (see Chapter I of Glassman [18]). We shall present a discussion of the stability of
these flame fronts under infinitesimal perturbation at the end of this section. We should remark that a positive ignition temperature is assumed throughout the whole section.

The Main Theorem

Recall from (2.11) that the model equations are

\[
\begin{align*}
T_t &= T_{xx} + \frac{Q}{C_p} \tilde{A}(T), \\
C_t &= \frac{1}{L} C_{xx} - \tilde{A}(T),
\end{align*}
\]

where

\[
\tilde{A}(T) = \begin{cases} 
\frac{Z}{\rho} \exp(-\frac{E}{RT}) & \text{for } T > T_i, \\
0 & \text{for } T < T_i,
\end{cases}
\]

with the usual convention around \( T_i \). Assume that an unburned state for the gas is given by

\[
C = C_u, \ T = T_u, \ T_u < T_i.
\]

In view of the assumptions about the type of chemical reactions considered in Section 2, after burning, the gas reaches a burned state of the form

\[
C = 0, \ T = T_b, \ T_b > T_i.
\]

The main theorem in this section is then

**Theorem 6:** There exists a flame front (i.e., traveling wave solution of (4.1)) connecting the unburned state (4.2) and the burned state (4.3) if and only if

\[
QC_u = C_p (T_b - T_u).
\]
Whenever the flame front exists, the flame speed and the profiles of
the temperature and the concentration are unique.

Condition (4.4) in this theorem is a natural consequence of the
conservation of energy under the assumptions stated in Section 2 (e.g.,
no gas dynamical effects, etc.). All the heat released by the chemical
reaction is transformed into the heat content of the burned gas and
manifests as the burned gas temperature. That $T_b$ relates to $T_u$
in the form of (4.4) is clearly stated in Chapter I of Glassman [18] in
the computation of the adiabatic flame temperature of the burned gas.

Most of this section is devoted to a proof of this theorem. We
begin with some a priori estimates for the temperature and concentration
profiles, then define an iteration scheme for the candidates for
these profiles, and eventually establish the existence of the flame
front.

We should remark that the only properties of $\tilde{A}(T)$ in (4.1)
which we shall need in the proof are

\[
\tilde{A}(T) > 0 , \quad \frac{d\tilde{A}}{dT}(T) > 0 \quad \text{for} \quad T > T_i .
\]

The exponential form of $\tilde{A}(T)$ is of minor importance.

Reduction of Equations (4.1)

We normalize equations (4.1) with respect to states (4.2) and
(4.3). By the change of variables

\[
\tilde{T} = \frac{T-T_u}{T_b-T_u} , \quad \tilde{C} = \frac{C}{C_u} , \quad \tilde{T}_i = \frac{T_i-T_u}{T_b-T_u} ,
\]
equations (4.1) are transformed into
\[
\begin{align*}
T_t &= T_{xx} + \frac{Q_C u}{c_p(T_b - T_u)} C(T), \\
C_t &= \frac{1}{L} C_{xx} - C(A(T)),
\end{align*}
\]

where we have omitted the tildes, and

\[
A(T) = \begin{cases} 
\frac{Z}{\rho} \exp\left(-\frac{E}{R((T_b - T_u)T + T_u)}\right) & \text{for } T > T_i \\
0 & \text{for } T < T_i
\end{cases}
\]

In the new variables, the unburned and burned states of (4.2) and (4.3) now become

\[(C = 1, T = 0) \text{ and } (C = 0, T = 1)\]

respectively. The expected flame front

\[T(x,t) = T(\xi), \; C(x,t) = C(\xi), \; \xi = x + Vt, \; V > 0\]

should satisfy (4.5) with boundary conditions

\[
\begin{align*}
T(-\infty) &= 0, \; T(+\infty) = 1, \\
C(-\infty) &= 1, \; C(+\infty) = 0.
\end{align*}
\]

Thus \(T(\xi)\) and \(C(\xi)\) satisfy the system of ordinary differential equations

\[
\begin{align*}
T'' - VT' + \frac{Q_C u}{c_p(T_b - T_u)} C(A(T)) &= 0, \\
C'' - LVC' - LCA(T) &= 0,
\end{align*}
\]

where \(T' = \frac{dT}{d\xi}\), etc. The problem is to find a \(V\) such that a solution of (4.7) exists and satisfies (4.6).
Monotonicity of Profiles

If (4.7) has a solution \((T(\xi), C(\xi))\) satisfying boundary conditions (4.6), then

(a) \(0 < C(\xi) < 1, \ 0 < T(\xi) < 1\) for all \(|\xi| < \infty\),

\[
(4.8) \quad (b) \ C'(\xi) < 0, \ T'(\xi) > 0 \text{ for all } |\xi| < \infty,
\]

(c) \(\lim_{|\xi| \to \infty} C'(\xi) = \lim_{|\xi| \to \infty} T'(\xi) = 0\).

Proof. (1) We claim that \(C(\xi) > 0\) for all \(|\xi| < \infty\). Otherwise, \(C(\xi) < 0\) for some \(\tilde{\xi}\). Then, since \(C(-\infty) = 1 > 0\) and \(C(\infty) = 0\), there exists \(\tilde{\xi}\) such that

\[
C(\tilde{\xi}) < 0, \ C'(\tilde{\xi}) = 0, \ C''(\tilde{\xi}) > 0
\]

(i.e., \(C(\tilde{\xi})\) is the minimum). Thus, from \(C\)-equation of (4.7),

\[
C''(\tilde{\xi}) = LC(\tilde{\xi})A(T(\tilde{\xi})) \leq 0.
\]

This contradiction establishes claim (1).

Basically, this claim is a maximum principle for a second-order boundary value problem.

(2) We claim that \(T'(\xi) \geq 0\) and \(C'(\xi) \leq 0\) for all \(|\xi| < \infty\).

Hence \(0 \leq T(\xi) \leq 1, \ 0 \leq C(\xi) \leq 1\) for all \(|\xi| < \infty\).

Suppose that the claim is false, say \(T'(\tilde{\xi}) < 0\) for some \(|\tilde{\xi}| < \infty\). The \(T\)-equation of (4.7) implies that

\[
T'' - VT' = -\frac{QC_u}{C_p(T_b-T_u)}CA(T) \leq 0.
\]

Thus, integrating the inequality,

\[
T'(\xi) \leq T'(\tilde{\xi})\exp(V(\xi-\tilde{\xi})) \text{ for } \xi \geq \tilde{\xi}.
\]
A further integration yields

$$T(\xi) \leq \frac{1}{V} T'(\tilde{\xi}) \exp(V(\xi-\tilde{\xi})) + T(\tilde{\xi}) \quad \text{for} \quad \xi \geq \tilde{\xi}.$$ 

Since $T'(\tilde{\xi}) < 0$, letting $\xi \to \infty$, we get

$$\lim_{\xi \to \infty} T(\xi) = -\infty.$$ 

This limit contradicts $T(\infty) = 1$.

(3) We shall prove (a), and then (b) follows.

If $C(\tilde{\xi}) = 0$ for some $|\tilde{\xi}| < \infty$, then $C(\xi) = 0$ for all $\xi \geq \tilde{\xi}$ due to the fact that $C'(\xi) \leq 0$. Hence,

$$T'' - VT' = 0 \quad \text{for all} \quad \xi \geq \tilde{\xi}.$$ 

This is incompatible with $T(\infty) = 1$. Therefore (a) follows.

(4) (c) follows from (a) and (b).

Thus, if a flame front exists as in Theorem 6, the qualitative temperature and concentration profiles look as in Figure 8, as expected. We shall justify a more precise asymptotic behavior of flame fronts as $|\xi| \to \infty$ later.

Proof of the Necessity of (4.4)

Assume that a flame front of Theorem 6 exists. Then (4.7) has a solution $(T(\xi), C(\xi))$ satisfying (4.6). Define

$$\alpha = \frac{QC_u}{c_l(T_b - T_u)}$$

for convenience. Then, by cancelling the nonlinear source in (4.7),
Integrating (4.9) from \(-\infty\) to \(\xi\), and using (4.8), we obtain
\[
LT'' + \alpha C'' = LV(T' + \alpha C') \quad \text{for all } \xi.
\]
Similarly, integrating (4.9) from \(\xi\) to \(\infty\), the following identity follows:
\[
LT' + \alpha C' = LV(T + \alpha(C - 1)) \quad \text{for all } \xi.
\]
Thus, comparing these two identities, \(\alpha = 1\). By definition of \(\alpha\), we have proved (4.4).

From now on, we assume that (4.4) holds. Then (4.7) becomes
\[
\begin{align*}
T'' - VT' + CA(T) &= 0, \\
C'' - LVC' - LCA(T) &= 0,
\end{align*}
\]
with (4.6) as boundary conditions.

The Total Enthalpy and Its A Priori Estimate

The total enthalpy of the physical system (4.1) is \(c_pT + QC\). In terms of the transformed variables in (4.10), this total enthalpy is a constant multiple of \(T + C\) if condition (4.4) holds. Thus, we call
\[
H(\xi) = T(\xi) + C(\xi), \quad |\xi| < \infty,
\]
the total enthalpy for (4.10), and call
\[
E(\xi) = LT(\xi) + C(\xi), \quad |\xi| < \infty,
\]
the associated enthalpy for (4.10). We now claim that
(4.14) If \((T(\xi), C(\xi))\) is a solution of (4.10) satisfying boundary conditions (4.6), then

(a) \(E'(\xi) = LV(H(\xi) - 1)\) for all \(\xi\),

(b) \(E(\xi) = VL(L-1)exp(LV\xi)\int_{\xi}^{\infty}T(\eta)exp(-LV\eta)d\eta + 1\) for all \(\xi\),

(c) when \(L \geq 1\), \(1 \leq H(\xi) \leq E(\xi) \leq L\) for all \(\xi\);

when \(0 \leq L < 1\), \(L \leq E(\xi) \leq H(\xi) \leq 1\) for all \(\xi\),

(d) when \(L = 1\), \(H(\xi) = E(\xi) = 1\) for all \(\xi\);

when \(L > 1\), \(H(\xi)\) has a unique maximum point;

when \(L < 1\), \(H(\xi)\) has a unique minimum point.

**Proof.** (a) Since (4.4) holds, \(\alpha = 1\) in (4.9). Thus,

\[E'' = LT'' + C'' = LV(T' + C') = LVH'.\]

Integrating this identity from \(-\infty\) to \(\xi\), and using boundary conditions (4.6) and (4.8), (a) is proved.

(b) From (4.11) and (4.13), \(H = E + (1-L)T\). Thus,

\[E'(\xi) - LVE(\xi) = LV(1-L)T(\xi) - LV\]

which follows from (a). Hence,

\[\frac{d}{d\xi}(exp(-LV\xi)E(\xi)) = LV(1-L)exp(-LV\xi)T(\xi) + \frac{d}{d\xi}(exp(-LV\xi)).\]

Integrating from \(\xi\) to \(\infty\), and using (4.8), we get

\[- exp(-LV\xi)E(\xi) = VL(1-L)\int_{\xi}^{\infty}exp(-LV\eta)T(\eta)d\eta - exp(-LV\xi).\]

Thus (b) is proved. Note that, from (4.8), the integral

\[\int_{\xi}^{\infty}exp(-LV\eta)T(\eta)d\eta\]

exists for all \(\xi\).
(c) From (4.8), \( T(\eta) \) is a strictly increasing function of \( \eta \).

Hence,

\[
T(\xi) \leq T(\eta) \leq 1 \quad \text{for} \quad \xi \leq \eta \leq \infty .
\]

Then,

\[
\frac{1}{V_L} \exp(-LV\xi) T(\xi) \leq \int_{\xi}^{\infty} \exp(-LV\eta) T(\eta) d\eta \leq \frac{1}{V_L} \exp(-LV\xi) .
\]

Therefore, if \( L \geq 1 \),

\[
(L-1)T(\xi) + 1 \leq E(\xi) \leq (L-1) + 1 = L .
\]

This proves (c).

(d) The case of \( L \) equal to 1 follows directly from (c).

Assume now that \( L > 1 \). We note that \( H(-\infty) = H(\infty) = 1 \), and

\[ 1 \leq H(\xi) \leq 2 . \]

Hence, \( \max_{|\xi|<\infty} H(\xi) \) exists. We claim that there exists only one extreme point \( \xi \) of \( H(\xi) \) such that

\[
H(\xi) = \max_{|\xi|<\infty} H(\xi) .
\]

To show this fact, define

\[
u(\xi) = V_L \exp(LV\xi) \int_{\xi}^{\infty} \exp(-LV\eta) T(\eta) d\eta .
\]

Then \( E = (L-1)u + 1 \), and

\[
(L-1) \frac{du}{d\xi} = \frac{dE}{d\xi} = LV(H(\xi)-1) \geq 0
\]

due to (c). Therefore, \( u(\xi) \) is an increasing function of \( \xi \). But it is easy to see that

\[
u(-\infty) = T(-\infty) = 0 , \quad \nu(\infty) = T(\infty) = 1 .
\]
Then, since $T(\xi)$ is strictly increasing, $u(\xi)$ must be strictly increasing. Thus, it is legitimate to change coordinate $\xi$ to $u$ via the use of $u = u(\xi)$. Define

$$
\omega(u) = \frac{du}{d\xi}(\xi(u)), \quad 0 \leq u \leq 1.
$$

From the definition of $u$, it follows that $\omega(u)$ satisfies the ordinary differential equation

$$
\begin{cases}
\omega \frac{d\omega}{du} = V(u - S(u)), \\
\omega(0) = \omega(1) = 0, \quad \frac{d\omega}{du}(1) < 0,
\end{cases}
$$

(4.15)

where $S(u) = \frac{dT}{d\xi}(\xi(u))$. But, from (4.8),

$$
S(0) = S(1) = 0, \quad \frac{dS}{du}(1) < 0, \quad S(u) > 0,
$$

equation (4.15) is of the form appearing in Theorem 2 of Section 3. Thus $\omega(u)$ has a unique extreme point (see the phase portrait in Fig. 4). Correspondingly, $\frac{du}{d\xi}$ has a unique extreme point. But,

$$
(L-1)\frac{du}{d\xi} = LV(H(\xi)-1) \text{ for all } \xi,
$$

$H(\xi)$ has a unique extreme point $\bar{\xi}$. $H(\bar{\xi})$ must then be $\max_{|\xi|<\infty} H(\xi)$. This proves (d).

Note that, from the proof of (d), $\frac{du}{d\xi}(\xi) < 0$ for all $|\xi| < \infty$. Hence, if $L > 1$,

$$
H(\xi) > 1 \text{ for all } |\xi| < \infty.
$$

Similarly, if $L < 1$, $H(\xi) < 1$ for all $|\xi| < \infty$. The case of $L$ equal to 1 corresponds to Section 3 where $C = 1-T$. Thus, if $L \neq 1$, ...
the total enthalpy of (4.10) is not conserved. The qualitative comparison is as in Figure 9. However, (4.14)(c) shows that the total enthalpy is almost conserved for each fixed $L$.

The main idea of the existence proof for Theorem 6 is to use (4.14)(b) to define an iteration scheme, and then use the a priori estimates in (c) to ensure convergence.

An Iteration Scheme

We fix now a Lewis number $L > 1$. The iteration scheme is defined as follows.

(a) Choose any $\tilde{C}_0(T)$, $0 \leq T \leq 1$, such that

\begin{align}
(4.16) \quad (a) \quad & \tilde{C}_0(T) \text{ is of class } C^2 \text{ on } 0 \leq T \leq 1, \\
& (b) \quad \tilde{C}_0(0) = 1, \quad \tilde{C}_0(1) = 0, \\
& (c) \quad -L \leq \frac{d\tilde{C}_0}{dT}(T) < 0 \text{ for } 0 < T < 1, \\
& (d) \quad 1 \leq \tilde{H}_0(T) \leq \tilde{E}_0(T) \leq L \text{ for } 0 \leq T \leq 1,
\end{align}

where

\[ \tilde{H}_0(T) = \tilde{C}_0(T) + T, \quad \tilde{E}_0(T) = \tilde{C}_0(T) + LT. \]

(b) Suppose that $\tilde{C}_n(T)$ (and, correspondingly, $\tilde{H}_n(T)$ and $\tilde{E}_n(T)$) has been defined so that $\tilde{C}_n(T)$ satisfies properties listed in (4.16). Let $\tilde{S}_n(T)$ and $V_n$, $V_n > 0$, be the unique solution to the ordinary differential equation

\begin{align}
(4.17) \quad & \begin{cases}
\tilde{S} \frac{d\tilde{S}}{dT} = V\tilde{S} - \tilde{C}_n(T)A(T), & 0 \leq T \leq 1, \\
\tilde{S}(0) = \tilde{S}(1) = 0, \quad \frac{d\tilde{S}}{dT}(1) < 0.
\end{cases}
\end{align}

$\tilde{S}_n(T)$ and $V_n$ exist, and are unique, due to Theorem 1 in Section 3.
\( S_n(T) \) is of class \( C^1 \) on \( 0 \leq T \leq 1 \).

(c) Let \( T_n(\xi) \) be the unique solution such that

\[
\frac{dT_n}{d\xi} = S_n(T_n(\xi)), \quad T_n(0) = T_i.
\]

Due to properties of \( S_n(T) \) listed in Theorem 1 of Section 3, \( T_n(\xi) \) is defined for all \( \xi \), and is strictly increasing with \( T_n(-\infty) = 0 \), \( T_n(\infty) = 1 \). Thus its inverse function exists. This inverse function is denoted by \( \xi_n(T) \), \( 0 \leq T \leq 1 \). Both \( T_n(\xi) \) and \( \xi_n(T) \) are of class \( C^2 \) on their respective defining domain.

(d) Define, for all \( \xi \),

\[
E_{n+1}(\xi) = V_6(L-1)\exp(LV_n\xi)\int_0^\infty \exp(-LV_nT_n(\eta))d\eta + 1,
\]

\[
C_{n+1}(\xi) = E_{n+1}(\xi) - LT_n(\xi),
\]

\[
H_{n+1}(\xi) = C_{n+1}(\xi) + T_n(\xi).
\]

(e) Define, for all \( 0 \leq T \leq 1 \),

\[
\tilde{C}_{n+1}(T) = C_{n+1}(\xi_n(T)),
\]

\[
\tilde{H}_{n+1}(T) = H_{n+1}(\xi_n(T)),
\]

\[
\tilde{E}_{n+1}(T) = E_{n+1}(\xi_n(T)).
\]

Then \( \tilde{C}_{n+1}, \tilde{H}_{n+1} \) and \( \tilde{E}_{n+1} \) have properties listed in (4.16). Thus we can iterate the scheme further.

This completes the iteration scheme.

**Verification of the Iteration Scheme**

We will show that \( \tilde{C}_{n+1}(T) \) defined in (e) above actually satisfies all the properties listed in (4.16). A direct substitution from (4.19) shows that \( E_{n+1}(\xi) \) has the following properties:
\[ E_{n+1}(\infty) = 1, \quad E_{n+1}(\infty) = L, \quad \frac{dE_{n+1}(\infty)}{d\xi} = 0, \]

(4.21) \[ \frac{dE_{n+1}}{d\xi} = V_n L[\eta_{n+1}]^{-1}, \text{ for all } \xi. \]

Compare with the formulas in (4.14).

**Proof (of (4.16) for \( \tilde{C}_{n+1}(T) \)):** (a) By definition of \( C_n(\xi) \), it is of class \( C^2 \) on \( |\xi| < \infty \). But \( \xi_n(T) \) is of class \( C^2 \), hence \( \tilde{C}_{n+1}(T) \) is of class \( C^2 \).

(b) follows from (4.21).

(o) follows from the defining equation (4.19) for \( E_{n+1}(\xi) \).

Using the property that \( T_n(\xi) \leq T_n(n) \leq 1 \), the proof is exactly the same as (4.14)(c).

(γ) Since \( H_{n+1} \geq 1 \) for all \( \xi \), \( \frac{dE_{n+1}}{d\xi} \geq 0 \) for all \( \xi \) (from equation (4.21)). Hence,

\[ \frac{d\tilde{E}_{n+1}}{dT} = \frac{dE_{n+1}}{d\xi} \frac{d\xi}{dT} = \frac{1}{\xi_n(T)} \frac{dE_{n+1}}{d\xi} \geq 0. \]

On the other hand,

\[ \frac{d\tilde{E}_{n+1}}{dT} = \frac{d\tilde{C}_{n+1}}{dT} + L \]

This proves part of (γ). The other part follows from the observation that \( C_{n+1}(\xi) \) satisfies the differential equation

\[ C_{n+1}'' - LV_n C_{n+1}' - LC_n A(T_n) = 0. \]

Since \( \tilde{C}_{n+1} + T \geq 1 \) (from (δ)), \( \frac{d\tilde{C}_{n+1}}{dT}(1) \leq -1 < 0. \) But

\[ \frac{d\tilde{C}_{n+1}}{dT} = \frac{dC_{n+1}}{d\xi} \frac{1}{\xi_n(T)}. \] Hence, \( \frac{dC_{n+1}}{d\xi}(\xi) < 0 \) for \( \xi \) sufficiently close
to $\infty$. If $\frac{dC_{n+1}(\xi)}{d\xi}(\xi) > 0$ for some $\xi$, let $\bar{\xi}$ be the largest such point. Then,
\[
\frac{dC_{n+1}(\bar{\xi})}{d\xi} = 0, \quad \frac{d^2C_{n+1}(\bar{\xi})}{d\xi^2} \leq 0 .
\]

However,
\[
C''_{n+1}(\bar{\xi}) = LC_n(\bar{\xi})A(T_n(\bar{\xi})) > 0 .
\]

This contradiction establishes $(\gamma)$.

Thus the iteration scheme gives rise to a sequence of functions

(4.22) \quad \check{c}_n(T), \check{s}_n(T), V_n, \quad n = 1,2,3,\ldots \]

Note that we can start from, say, $\check{c}_0(T) = 1 - T$. We shall show that functions in (4.22) are uniformly bounded and equicontinuous on $0 < T < 1$. Hence we may apply Arzela-Ascoli's theorem to (4.22).

A Comparison Lemma

We shall develop a comparison lemma to show that $\{V_n\}$ in (4.22) is bounded. This lemma is essentially a generalization of Lemma 3 in subsection (c) of Section 3.

Lemma 5: Let $g_i(T), i = 1,2$, be two functions satisfying condition (3.5), and $g_1(T) \leq g_2(T)$ for all $0 \leq T \leq 1$. Let $S_i(T,V), i = 1,2$, be the unique solution in Lemma 2 such that
\[
\begin{cases} 
\frac{dS_i}{dT} = VS_i - g_i(T), & 0 \leq T \leq 1, \\
S_i(1,V) = 0, & \frac{d}{dT}S_i(1,V) < 0, \quad i = 1,2, 
\end{cases}
\]

where $V > 0$ is fixed. Then, whenever $S_1(T,V) > 0$,
Proof. Define

$$\Delta(T) = S_1(T,V) - S_2(T,V) .$$

Then,

$$\frac{d\Delta}{dT} = (V - \frac{g_1}{s_1}) - (V - \frac{g_2}{s_2})$$

$$= \frac{g_2}{s_2} - \frac{g_1}{s_1} > \frac{g_2}{s_2} - \frac{g_2}{s_1} = \frac{g_2}{s_1 s_2} \Delta .$$

Thus,

$$\frac{d}{dT} \left( \exp \left( \int_{T}^{1} \frac{g_2}{s_1 s_2} dx \right) \Delta \right) > 0 .$$

Integrating from $T$ to 1, we get

$$\Delta(1) > \Delta(T) .$$

But $\Delta(1) = 0$. This proves the lemma.

Note that, from Lemma 2, $S_1(T,V) > 0$ for $T_i < T < 1$. Thus, the conclusion of Lemma 5 holds at least for $T_i < T < 1$.

Estimates for $S_n(T)$ and $V_n$

Let

$$g_{\ell}(T) = (1-T)A(T), \quad g_r(T) = L(1-T)A(T) .$$

Then $g_{\ell}(T)$ and $g_r(T)$ satisfy (3.5). Let $S_{\ell}(T,V)$ be the unique solution of

$$\begin{cases}
S \frac{dS}{dT} = VS - g_{\ell}(T), & 0 \leq T \leq 1 \\
S(1) = 0, & \frac{dS}{dT}(1) < 0 .
\end{cases}$$

(4.23)
$S_r(T,V)$ is similarly defined. Define $V_i^*$, $i = r, \ell$, to be the unique $V$ such that

\begin{equation}
S_i(0,V_i^*) = 0, \quad i = r, \ell.
\end{equation}

Such $V_i^*$ exists due to Theorem 1 in Section 3. Note that $V_i^* > 0$ for $i = r, \ell$.

Now, let $\tilde{S}_n(T,V)$ be the unique solution to

\begin{equation}
\begin{cases}
\frac{d\tilde{S}}{dT} = V\tilde{S} - \tilde{C}_n(T)A(T), & 0 \leq T \leq 1 \\
\tilde{S}(1) = 0, & \frac{d\tilde{S}}{dT}(1) < 0.
\end{cases}
\end{equation}

Thus, $\tilde{S}_n(T,V_n) = \tilde{S}_n(T)$. But, from property (4.16)(6) for $\tilde{C}_n(T)$,

\[ g_\ell(T) \leq \tilde{C}_n(T)A(T) \leq g_r(T), \quad 0 \leq T \leq 1. \]

Hence, from Lemma 5,

\[ S_\ell(T,V) \leq \tilde{S}_n(T,V) \leq S_r(T,V) \]

whenever $S_\ell(T,V) \geq 0$. Applying this inequality to $V_\ell^*$ at $T = 0$ (note that $S_\ell(T,V_\ell^*) > 0$ for $0 < T < 1$), then

\[ V_\ell^* \leq V_n, \]

where we have used Theorem 1 to deduce this inequality. Similarly,

\[ V_n \leq V_r^*. \]

In particular, for all $0 \leq T \leq 1$,

\[ S_\ell(T,V_\ell^*) \leq S_\ell(T,V_n) \leq \tilde{S}_n(T,V_n) = \tilde{S}_n(T). \]

Since $g_r(T)$ and $g_\ell(T)$ are independent of $n$, we get the following
estimates of $\tilde{S}_n(T)$ and $V_n$:

**Lemma 6**: Let $V_i^*, i = r, \xi, \lambda$, and $S_\lambda(T, V)$ be defined as in (4.23) and (4.24). Then, for $n = 1, 2, 3, \ldots$,

$$0 < V_i^* \leq V_n \leq V_r^*,$$

$$S_\lambda(T, V_r^*) \leq \tilde{S}_n(T) \leq \frac{\delta}{V_r^*} \text{ for all } 0 \leq T \leq 1,$$

where $\delta = A(1)$.

**Proof.** It remains to show that $\tilde{S}_n(T) \leq \frac{\delta}{V_r^*}$ for $0 \leq T \leq 1$. Let $\bar{T}$ be the point where $\tilde{S}_n(\bar{T}) = \max_{0 < T < 1} \tilde{S}_n(T)$ holds. Then $0 < \bar{T} < 1$, and $\frac{d\tilde{S}_n}{dT}(\bar{T}) = 0$. Thus, form (4.17),

$$\tilde{S}_n(\bar{T}) = \frac{\tilde{C}_n(\bar{T})A(\bar{T})}{V_n} \leq \frac{\delta}{V_r^*}.$$

Since $0 \leq \tilde{S}_n(T)$, we have proved the lemma.

Lemma 6 gives a uniform estimate for $\{\tilde{S}_n(T)\}$ and $\{V_n\}$. It will also help us to establish a derivative estimate.

We know from (4.16)(γ) for $\tilde{C}_n(T)$ that $\frac{d\tilde{C}_n}{dT}$ is uniformly bounded on $0 \leq T \leq 1$. As for $\{\frac{d\tilde{S}_n}{dT}\}$, we will use Lemma 6. Lemma 2 of Section 3 implies that

$$S_\lambda(T, V_r^*) > 0 \text{ for } T_1 < T < 1.$$

Hence, from Lemma 6, $\tilde{S}_n(T)$ has a positive uniform lower bound on every compact subset of $T_1 < T < 1$. But, from (4.17),

$$\frac{d\tilde{S}_n}{dT} = V_n - \frac{\tilde{C}_n(T)A(T)}{\tilde{S}_n(T)} \text{ for } 0 \leq T \leq 1.$$
Hence,

\[ \frac{d\tilde{S}_n}{dT} \] is uniformly bounded on every compact subset of \( T_1 < T < 1 \).

Thus we have proved

**Lemma 7:** The sequence of functions in (4.22) has the following properties:

1. They are uniformly bounded.
2. \( \{\tilde{C}_n(T)\} \) is equicontinuous on \( 0 < T < 1 \).
3. \( \{\tilde{S}_n(T)\} \) is equicontinuous on every compact subinterval of \( T_1 < T < 1 \).

Property (3) in Lemma 7 is a consequence of (4.25).

We are now in a position to complete the proof of Theorem 6.

**Proof of the Sufficiency of (4.4) in Theorem 6**

**Proof (of Theorem 6):** Suppose that (4.4) holds in Theorem 6 and \( L \geq 1 \). Then Lemma 7 holds for (4.22). Thus, by Arzela-Ascoli's theorem, we can find a common subsequence of (4.22), say,

\[ \tilde{C}_{n_k}(T), \tilde{S}_{n_k}(T), V_{n_k}, k = 1, 2, 3, \ldots \]

such that

\[
\lim_{k \to \infty} V_{n_k} = V, \\
\lim_{k \to \infty} \tilde{C}_{n_k}(T) = \tilde{C}(T) \text{ uniformly on } 0 < T < 1, \\
\lim_{k \to \infty} \tilde{S}_{n_k}(T) = \tilde{S}(T) \text{ uniformly on every compact subinterval of } T_1 < T < 1.
\]
From Lemma 6, \( V \) and \( \tilde{S}(T) \) satisfy

\[
0 < V_2^* \leq V \leq V_r^*, \quad S_2(T,V_r^*) \leq \tilde{S}(T) \leq \frac{\delta}{V_2^*} \quad \text{for } T_i < T < 1.
\]

In particular, \( \tilde{S}_{n_k}(T) \), \( k = 1,2,3,... \), and \( \tilde{S}(T) \) have common positive lower bound on every compact subinterval of \( T_i < T < 1 \). Hence,

\[
\lim_{k \to \infty} \frac{\tilde{C}_{n_k}(T)A(T)}{\tilde{S}_{n_k}(T)} = \frac{\tilde{C}(T)A(T)}{\tilde{S}(T)}
\]

uniformly on every compact subset of \( T_i < T < 1 \). Thus,

\[
\lim_{k \to \infty} \int_{T_1}^{T_2} \frac{\tilde{C}_{n_k}(T)A(T)}{\tilde{S}_{n_k}(T)}dT = \int_{T_1}^{T_2} \frac{\tilde{C}(T)A(T)}{\tilde{S}(T)}dT,
\]

for any \( T_1, T_2 \) such that \( T_i < T_1 \leq T_2 < 1 \).

However, from (4.17),

\[
\tilde{S}_{n_k}(T_2) - \tilde{S}_{n_k}(T_1) = V_n(T_2 - T_1) - \int_{T_1}^{T_2} \frac{\tilde{C}_{n_k}(T)A(T)}{\tilde{S}_{n_k}(T)}dT,
\]

for any \( T_i < T_1 \leq T_2 < 1 \). Letting \( k \to \infty \), we obtain

\[
\tilde{S}(T_2) - \tilde{S}(T_1) = V(T_2 - T_1) - \int_{T_1}^{T_2} \frac{\tilde{C}(T)A(T)}{\tilde{S}(T)}dT,
\]

for any \( T_i < T_1 \leq T_2 < 1 \). This implies that \( \tilde{S}(T) \) is differentiable on \( T_i < T < 1 \) and satisfies

\[
(4.26) \quad \tilde{S} \frac{d\tilde{S}}{dT} = VS - \tilde{C}(T)A(T), \quad T_i < T < 1.
\]

Since \( \tilde{C}(1) = 0 \) [which is from \( \tilde{C}_{n_k}(1) = 0 \)], (4.26) is also satisfied at \( T = 1 \). On the other hand, if \( 0 \leq T \leq T_i \), \( A(T) = 0 \). Hence,
\[ \tilde{S}_{n_k}(T) = V_{n_k}T, \quad 0 \leq T \leq T_i. \]

Therefore \( \lim_{k \to \infty} \tilde{S}_{n_k}(T) = VT \) for \( 0 \leq T \leq T_i \). This means that, if we define \( \tilde{S}(T) = VT \) on \( 0 \leq T \leq T_i \), \( \tilde{S}(T) \) satisfies (4.26) on \( 0 \leq T \leq 1 \). Then \( \lim_{k \to \infty} \tilde{S}_{n_k} = \tilde{S} \) uniformly on \( 0 \leq T \leq 1 \).

\( \tilde{S}(T) \) is the solution of (4.26) with \( \tilde{S}(0) = \tilde{S}(1) = 0 \), and \( \tilde{S}(T) > 0 \) for \( 0 < T < 1 \). Define \( T(\xi) \) to be the solution of

\[ \frac{dT}{d\xi} = \tilde{S}(T(\xi)), \quad T(0) = T_i. \]

Then \( T(\xi) \) is of class \( \mathcal{C}^2 \), defined for all \( \xi \), and

\[ \frac{d^2T}{d\xi^2} - V\frac{dT}{d\xi} + \tilde{c}(T(\xi))A(T) = 0 \quad \text{for all} \quad \xi, \]

(4.26)

\[ T(-\infty) = 0, \quad T(\infty) = 1. \]

Since \( \tilde{S}_{n_k}(T) \to \tilde{S}(T) \) uniformly, the definition of \( T_{n_k}(\xi) \) in (4.18) shows that

\[ \lim_{k \to \infty} T_{n_k}(\xi) = T(\xi), \quad \lim_{k \to \infty} T'_{n_k}(\xi) = T'(\xi) \]

uniformly for \( |\xi| \leq \infty \). This fact follows from the continuous dependence of solutions of ordinary differential equations on a parameter (see Hartman [19]). Thus, from (4.19), \( W(\xi) = \lim_{k \to \infty} C_{n_{k+1}}(\xi) \) exists and

(4.27) \[ W(\xi) = VL(L-1)\exp(LV\xi) \int_{\xi}^{\infty} \exp(-LV\eta)T(\eta)d\eta + 1 - LT(\xi). \]

Thus \( W(\xi) \) is of class \( \mathcal{C}^2 \), and \( W(\xi) \) satisfies

(4.28) \[ \frac{d^2W}{d\xi^2} - LV\frac{dW}{d\xi} - LW(\xi)A(T(\xi)) = 0, \quad \text{for all} \quad \xi. \]
On the other hand, from (4.19),

\[ \frac{d^2 C_{n+1}}{d\xi^2} - LV_n C_{n+1} + L\tilde{C}_{n+1}(T_n C_{n+1})A(T_n C_{n+1}) = 0. \]

Since \(C_{n+1}(\xi) = W(\xi), C'_{n+1}(\xi) = W'(\xi)\) and \(C''_{n+1}(\xi) = W''(\xi)\), it follows from the integral formulas (4.19) and (4.27). Therefore,

\[ \frac{d^2 W}{d\xi^2} - LV \frac{dW}{d\xi} - L\tilde{C}(T(\xi))A(T(\xi)) = 0. \]

Comparing with (4.28), we get

\[ W(\xi) = \tilde{C}(T(\xi)), \quad |\xi| < \infty. \]

Thus, from (4.26) and (4.28),

\[ \begin{cases} \frac{dT}{d\xi} - V \frac{dT}{d\xi} + W(\xi)A(T) = 0, & T(-\infty) = 0, \quad T(\infty) = 1, \\ \frac{d^2 W}{d\xi^2} - LV \frac{dW}{d\xi} - LW(\xi)A(T) = 0, & W(-\infty) = 1, \quad W(\infty) = 0. \end{cases} \]

i.e. \((T(\xi), W(\xi))\) satisfies (4.7) with boundary conditions (4.6). We have proved the existence of a flame front in Theorem 6.

The proof for \(L \leq 1\) is similar.

The uniqueness proof follows the same line of arguments as the uniqueness proof in Theorem 1.

Note that the proof of Theorem 6 essentially uses the assumption of a positive ignition temperature. The definition of the iteration scheme uses the uniqueness of \(\tilde{S}_n(T)\) which needs this assumption.
Spatial Asymptotic Behavior of Flame Fronts

If \((T(\xi), C(\xi))\) is the unique solution of (4.7) with boundary conditions (4.6), then

\[ T(\xi) = O(e^{V\xi}) \text{ as } \xi \to -\infty, \quad T(\xi) = 1 + O(e^{-\lambda \xi}) \text{ as } \xi \to \infty; \]
\[ C(\xi) = 1 + O(e^{LV\xi}) \text{ as } \xi \to -\infty, \quad C(\xi) = 0(e^{-\mu \xi}) \text{ as } \xi \to \infty, \]

where \(\lambda, \mu > 0\) are some constants depending on \(V\) and \(L\).

**Proof.** Change coordinate \(\xi\) to \(T\) by \(T = T(\xi)\) and define

\[ S(T) = \frac{dT}{d\xi}(\xi(T)), \quad 0 \leq T \leq 1. \]

Then,

\[ S \frac{dS}{dT} = VS - C(\xi(T))A(T), \]
\[ \frac{dT}{d\xi} = S(T(\xi)). \]

But (4.14)(c) implies that \(\frac{d}{dT}C(1) < 0\). Hence,

\[ \frac{dS}{dT}(1) = \frac{1}{2}(V - \sqrt{V^2 - 4C'(1)\delta}) < 0, \]

where \(\delta = A(1) > 0\). Thus the proof of this asymptotic behavior follows the same line of arguments as the corresponding results of Theorem 1.

Stability Under Infinitesimal Perturbation

Since equations (4.1) are translation invariant, the linearized equations of (4.1) around the flame front of Theorem 6 have \(C'(\xi)\) and \(T'(\xi)\) as eigenfunctions with eigenvalue 0. Since the linearized equation is similar to a system of self-adjoint operators, 0 is the largest eigenvalue of the linearized equations due to the fact that
$C'(\xi) < 0$ and $T'(\xi) > 0$ for all $|\xi| < \infty$. Thus the flame front of Theorem 6 is stable under infinitesimal perturbation.

The whole argument is the same as the case of $L$ equal to 1 in subsection (D) of Section 3. We will not carry out the detailed proof. The assumption of positive ignition temperature is still essential.
5. **Conclusions**

The flame propagation in a combustible gas mixture is a product of complicated interactions among various physical and chemical mechanisms. We isolate in this thesis the effects of the chemical reactions and heat conduction or chemical species diffusion. In particular, we show that a laminar flame is a result of energy balance between the heat release from the chemical reactions and the heat conduction or the chemical species diffusion.

The model equations as formulated in Section 2 are of reaction-diffusion type. The nonlinear source term is the Arrhenius expression for the chemical reaction rate which is exponentially dependent on the temperature. The flame fronts are visualized as traveling wave solutions to the model equations connecting an unburned state of the gas mixture with the corresponding adiabatic burned state. If we assume the existence of a positive ignition temperature, we show that a unique flame front exists for each unburned gas state. However, we give evidence to the effect that a positive ignition temperature is not necessary for the flame fronts to exist. But, without an ignition temperature, the flame fronts may become unstable, and difficult to compute numerically.

If a positive ignition temperature is assumed, these flame fronts fulfill a lot of intuitive expectations. Specifically, we show that the formation of a flame front proceeds at an exponential rate, and, if we ignite the gas mixture simultaneously at several places in the tube, several flame fronts arise and eventually there remain at most two flame fronts. We should remark that these conclusions are shown only in a special case of the general model equations.
A realistic case where this model typically applies is the Bunsen flame; see [26], [18] or [33]. In this case, the experimental flame speed and the computed flame speed are well-confirmed in [18] or [26].

We ask now what is the position of this model with respect to general flame propagation where gas dynamical effects are taken into account. The most restrictive condition of this model is the constancy of the gas density during the combustion process. Presumably, if a boundary is present behind the flame front, the burned gas typically expands and the gas dynamics comes into play to propagate the flame front. However, this model still acts as a local source of flame speed in addition to the gas velocity. Thus, we conclude that this model in general occupies a "local" position even in two or three dimensional flame propagation. In higher dimensions, this (one-dimensional) model always provides a flame speed in the normal direction of the flame surface. We believe that the theoretical study in this thesis still provides a useful understanding of local flame propagation even in higher dimensions.

It is still to be explored how to fit this reaction-diffusion model into the gas-dynamical model of von Neumann. The approach, in the author's opinion, should be numerical for every practical purpose. In this direction, Chorin's advection and propagation algorithm in [5] should be useful.
Bibliography


Figures

1. Typical flame front in the tube.
2. The Arrhenius nonlinear source term in the case of $L$ equal to 1
3. Typical phase portrait for (3.11) with $f(u)$ satisfying (3.5)
4. Typical phase portrait for (3.11) with $f(u)$ satisfying (3.6)
5. The wave $T_\alpha(\xi)$ on the phase plane
6. The minimal speed wave of (3.18)
7. Truncated $g_\alpha(T)$ and $R(T)$
8. Temperature and concentration profiles in a flame front with arbitrary Lewis number
9. Total enthalpy of a flame front for different Lewis numbers
Figure 1.

Tube

burned state II I unburned state

I : induction zone } Flame front.
II: reaction zone

Figure 2.
Figure 3.
Figure 4.
Figure 5.

Figure 6.

Figure 7.
Figure 8.

Temperature profile

Concentration profile
Figure 9.

H = total enthalpy
This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.