Global Optimization of Compressor/inter-cooler Sequences for Constant Compressibility Factor and Variable Heat Capacity Gases

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2015

Peer reviewed|Thesis/dissertation
Global Optimization of Compressor/inter-cooler Sequences
for Constant Compressibility Factor and Variable Heat Capacity Gases

A thesis submitted in partial satisfaction
of the requirements for the degree Master of Science
in Chemical Engineering

by

Wenbo Sun

2015
ABSTRACT OF THE THESIS

Global Optimization of Compressor/inter-cooler Sequences

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by

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Master of Science in Chemical Engineering

University of California, Los Angeles, 2015

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This work identifies the global optimum for an infinite number of instances of the Total Annualized Cost (TAC) problem for compressor-intercooler sequences of constant compressibility factor gases with variable heat capacity. The prior state of the art involved the identification of the global optimum for an infinite number of instances of the TAC problem for compressor-intercooler sequences of constant compressibility factor gases with constant heat capacity. The resulting mathematical formulation of the TAC problem involves functions whose evaluation requires integral calculations. The necessary conditions of optimality for the TAC problem are then used to establish a number of mathematical properties, which are then used to establish the foundation of the proposed global optimization procedure. An illustrative case study
on methane compression is then presented. Its results suggest that the use of an average (constant) heat capacity over the allowable operating temperature range is a reasonable approximation that yields a global TAC minimum that is within 1% of the true global TAC minimum.
The thesis of Wenbo Sun is approved.

Panagiotis D. Christofides
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University of California, Los Angeles
2015
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LIST OF SYMBOLS

Letters

\( a \)  
Power law exponent of compressor capital cost

\( A \)  
Operating cost coefficient of compressor-intercooler series, \( \left( \frac{\$}{s \cdot K} \right) \)

\[ A \triangleq \dot{n} \cdot R \left( \frac{C_{\text{oper.}, \text{compr.}}}{C_{p,c}} \cdot \frac{C_{\text{oper.}, \text{cooler}}}{T_{c,out} - T_{c,in}} \right) \]

\( B \)  
Capital cost coefficient of compression, \( \left( \frac{\$}{s \cdot K^a} \right) \)

\[ B \triangleq F C_{\text{cap.}, \text{compr.}} \cdot \dot{n}^a \cdot (R)^a \geq 0 \]

\( c \)  
Coefficient of variable heat capacity polynomial function with respect to temperature

\( C \)  
Pressure ratio to \( Z \) exponent,

\[ C \triangleq \left( \frac{P_n}{P_0} \right)^Z \]

\( C_{p} \)  
Constant-pressure molar heat capacity of gas, \( \left( \frac{J}{mol \cdot K} \right) \)

\( C_{p,c} \)  
Constant-pressure molar heat capacity of coolant, \( \left( \frac{J}{mol \cdot K} \right) \)

\( C_{\text{cap.}, \text{compr.}} \)  
Capital cost coefficient of compression, \( \left( \frac{\$}{s^a} \right) \)

\( C_{\text{oper.}, \text{compr.}} \)  
Operating cost coefficient of compression, \( \left( \frac{\$}{J} \right) \)

\( C_{\text{oper.}, \text{cooler}} \)  
Operating cost coefficient of cooling, \( \left( \frac{\$}{mol} \right) \)
\[ D \text{ Maximum modified work of compressor } i, \ (K) \]

\[
D \triangleq f \left( T_{max}^{i} \right) = \int_{T_{0}^{i}}^{T_{max}^{i}} \frac{C_{p} \left( T' \right)}{R} dT'
\]

\[ E \text{ Maximum modified pressure ratio of compressor } i, \]

\[
E \triangleq g \left( T_{max}^{i} \right) \triangleq \exp \left( \int_{T_{0}^{i}}^{T_{max}^{i}} \frac{C_{p} \left( T' \right)}{RT'} dT' \right)
\]

\[ f' \text{ Modified work function of compressor } i, \]

\[
f : \left[ T_{0}, T_{max} \right] \rightarrow [0, D], \quad f : T_{i}^{*} \rightarrow W_{i} = f \left( T_{i}^{*} \right) \triangleq \int_{T_{0}^{i}}^{T_{i}^{*}} \frac{C_{p} \left( T' \right)}{R} dT' \quad \forall i = 1, n
\]

\[ F \text{ Annualization factor, } \left( \frac{1}{s} \right) \]

\[ F_{k} \text{ Objective function of optimal conditions to be analyzed} \]

\[
F_{k} \left( W_{k} \right) \triangleq \left( \frac{1}{\eta_{k}} \right)^{a} A \left( W_{k} \right)^{a-1} + \frac{1}{\eta_{k}} \right] f^{-1} \left( W_{k} \right) \ \forall k \in S_{i}^{w}
\]

\[ F_{i} \text{ General objective function of optimal conditions} \]

\[ g \text{ Modified pressure ratio function of compressor } i \text{ with respect to } T_{i}^{*}, \]

\[
g : \left[ T_{0}, T_{max} \right] \rightarrow [1, E], \quad g : T_{i}^{*} \rightarrow g \left( T_{i}^{*} \right) \triangleq \exp \left( \int_{T_{0}^{i}}^{T_{i}^{*}} \frac{C_{p} \left( T' \right)}{RT'} dT' \right) \quad \forall i = 1, n
\]

\[ h \text{ Modified pressure ratio function of compressor } i \text{ with respect to } W_{i} \]

\[
h \left( W_{i} \right) \triangleq g \left( f^{-1} \left( W_{i} \right) \right) \triangleq g \left( T_{i}^{*} \right) \triangleq \exp \left( \int_{T_{0}^{i}}^{T_{i}^{*}} \frac{C_{p} \left( T' \right)}{RT'} dT' \right) \quad \forall i = 1, n
\]

\[ H \text{ Molar enthalpy of fluid stream, } \left( \frac{J}{mol} \right) \]
\[ \dot{n} \] Molar flow rate of gas stream, \((\text{mol/s})\)

\[ \dot{n}_{c,i} \] Molar flow rate of coolant stream through cooler \(i\), \((\text{mol/s})\)

\[ N^w_c \] Cardinality of set \(S^w_c\)

\[ N^w_D \] Cardinality of set \(S^w_D\)

\[ N^w_i \] Cardinality of set \(S^w_i\)

\(p_k\) Local minimum of objective function of optimal conditions to be analyzed

\(p_i\) Local minimum of general objective function of optimal conditions

\(P\) Gas pressure, \((\text{Pa})\)

\(P_0\) Inlet gas pressure of the entire compressor-intercooler series, \((\text{atm})\)

\(P_i\) Outlet gas pressure of compressor \(i\), \((\text{atm})\)

\(P_n:\) Outlet gas pressure of the entire compressor-intercooler series, \((\text{atm})\)

\(q\) Number of discretized uneven intervals of \(W\)

\(r\) Discretization size of \(W\), \((K)\)

\(R\) Universal gas constant, \(\left(\frac{J}{\text{mol} \cdot \text{K}}\right)\)

\(S\) Molar entropy of gas, \(\left(\frac{J}{\text{mol} \cdot \text{K}}\right)\)

\(S^w_D\) The set of the compressor which is operated at the maximum workload,

\[ S^w_D \triangleq \{i = 1, n : W_i = \eta_i D\} \]

\(S^w_i\) The set of the compressor which is practically in use and operated below the
maximum workload,

\[ S_i^W \triangleq \{ i = 1, n : 0 < W_i < \eta_i D \} \]

\( S_i^W \) The set of the compressor which is not practically in use,

\[ S_i^W \triangleq \{ i = 1, n : W_i = \eta_i e \} \]

\( T \) Gas temperature, \( (K) \)

\( T_0 \) Inlet temperature of the entire compressor-intercooler series, \( (K) \)

\( T_{c,i,in} \) Inlet coolant temperature of cooler \( i \), \( (K) \)

\( T_{c,i,out} \) Outlet coolant temperature of cooler \( i \), \( (K) \)

\( T_i \) Outlet gas temperature of compressor \( i \), \( (K) \)

\( T_{i-1} \) Inlet gas temperature of compressor \( i \), \( (K) \)

\( T_i'' \) Hypothetical isentropic outlet gas temperature of compressor \( i \), \( (K) \)

\( T_{max} \) Maximum allowable operating temperature for all compressors, \( (K) \)

\( T_n \) Inlet temperature of the entire compressor-intercooler series, \( (K) \)

\( V \) Gas molar volume, \( \left( \frac{m^3}{mol} \right) \)

\( W_i \) Modified hypothetical isentropic work of compressor \( i \), \( (K) \)

\[ W_i \triangleq f(T_i'') = \int_{T_n}^{T_i''} \frac{C_p(T')}{R} dT' \quad \forall i = 1, n \]

\( W_{ideal} \) Hypothetical isentropic work of compressor, \( \left( \frac{J}{mol} \right) \)

\( W_{real} \) Actual work of compressor with considering compressor thermal efficiency, \( \left( \frac{J}{mol} \right) \)
\( Z \)  
Compressibility factor

Greek Letters

\( \eta_i \)  
Efficiency of compressor \( i \)

\( \nu \)  
Objective function value, \( \left( \frac{s_i}{s} \right) \)

Subscripts

\( in \)  
Inlet stream to process unit

\( out \)  
Outlet stream to process unit
ACKNOWLEDGEMENT

I am sincerely thankful to the UCLA Chemical and Biomolecular Engineering Department for providing me a teaching assistant position, and to my advisor, Vasilios Manousiouthakis for his generosity and education on me. I would also like to acknowledge the academic input of Jeremy Conner, and the spiritual support and encouragement from Flavio Da Cruz, Abdulrahman Albassam, Fernando Olmos, and Secgin Karagoz. This work will be submitted for publication to an archival journal with Jeremy Conner as a second co-author, and Vasilios Manousiouthakis as the corresponding author.
1. Introduction

In recent years, there has been growing interest in natural gas, as an alternative energy resource. According to [1], U.S. natural gas production is estimated to increase by about 40% from 2005 to 2050. Considering the issue of reducing carbon dioxide emissions, the Natural Gas Combined Cycle (NGCC) generates 8% less CO₂ emissions nationwide than Coal base power generation, [2]. Moreover, natural gas has an economic advantage over other low-carbon energy alternatives in electric power generation, [1]. The increased natural gas use in power generation applications, necessitates its compression, so it can be transported (in gaseous form) across the U.S.

In addition, due to the large oil-gas price ratio, Compressed Natural Gas (CNG) light duty vehicles have significantly lower operating costs than gasoline vehicles. In fact, gasoline to CNG replacement is being increasingly practiced, since the replacement cost can be recovered in a relatively short time period, from the fuel cost savings, [2]. Again, the use of natural gas as fuel, in CNG form, necessitates its compression to high pressures (e.g. around 250 bar).

Another application of compression is in Compressed Air Energy Storage (CAES) systems. These are used to stabilize the intermittent and unstable nature of renewable energy sources, such as solar and wind energy, by storing excess energy, in the form of compressed air, during periods of over-supply, and release it, in the form of electricity generated by turbines running on released compressed air, during periods of high-demand, [3], [4].

Compression is also a critical component of fossil fuel based power plants with CO₂ capture processes. Indeed, following the removal of CO₂ from the power plant’s gaseous emissions, it must first be compressed, so it can be transported via pipeline in supercritical form, [5], to locations where it can be sequestered. Candidate locations often feature mile-deep
underground porous rock layers that can hold the CO₂, and are overlaid by impermeable, non-porous layers of rock that prevent its upward migration. Once on location, CO₂ must be further compressed, so it can be injected into these underground porous rock enclosures, [6].

The multitude of gas compression applications outlined above, combined with the typically high energetic and capital costs of compression, [7], makes apparent the need to identify the optimal manner in which compression should be carried out, and to minimize its adverse energetic, and economic impact. To this end, the optimization of a commonly employed compression configuration will be undertaken, namely of the multistage compressor/inter-cooler sequence.

1.1. Multistage compressor/inter-cooler sequence application

In most industrial applications, the gas compression process is often carried out in a series of compressors with intermediate coolers. The reason is that gas compression inevitably leads to higher temperatures, which in turn lead to significant increases in power consumption during the compression process. To reduce power consumption, the compression process is broken into stages, in between of which the compressed gas is cooled before further compression. As both the capital cost and operating cost of compression are closely related to the compressor power consumption, it is then apparent that compression economics can be significantly affected by the design of the compressor/inter-cooler sequences.

Multistage compressor/inter-cooler sequences are widely utilized for CH₄ transport, via pipeline, from production to consumption sites; for CO₂ transport, via pipeline, from fossil fueled power plants with CO₂ capture processes to CO₂ sequestration locations, [5]; and for CO₂ injection into underground porous rock layers, at CO₂ sequestration sites. Multistage compressor/inter-cooler sequences are also frequently employed in Compressed Air Energy
Storage (CAES) systems, [3], [4], [8]. In these applications, the heat removed from the air during its compression for storage purposes, is added to the air during its expansion for electricity generation purposes.

The multitude of industrial applications for multistage compressor-intercooler sequences, necessitates that their techno-economic optimization be carried out. First, prior literature efforts in this direction are described.

1.2. Review of multistage compressor/inter-cooler sequence optimization methods

The focus of the first optimization studies, for multistage compressor/inter-cooler sequences, was the minimization of compressor work. [9] minimized the work (power) for a sequence of two compressors, [10] for a sequence of three compressors, and [11] for a multi-compressor sequence. They all considered the compressors to be isentropic, and the gas to be ideal, have constant heat capacity, and be compressed from an initial temperature and pressure to a desired final pressure and a final temperature equal to the initial temperature. The optimal works of the compressors were shown to be equal at the optimum. These results have found themselves in textbooks by [12], [13]. More recently, [14] minimized compression work in the presence of pressure losses associated with cooling, and established that the optimum number of compressors and exchangers is finite under these conditions.

The Total Annualized Cost (TAC) minimization problem for chemical process flowsheets is highly non-convex, and methods for its global solution are scarce. [15] proposed a hybrid algorithm, which included branch and bound underestimation with imbedded interval analysis, to identify the global optimum of a TAC optimization problem for heat/mass exchange networks. Similarly, [16] identified the TAC global minimum for an isothermal reactor network within the
infinite dimensional state-space (IDEAS) framework, by solving globally a minimization problem with a concave, separable objective function, and a feasible region defined by linear constraints.

TAC minimization for multistage compressor/inter-cooler sequences has been largely pursued through locally optimal methods. [17] carried out TAC minimization for the staged compression of a CO₂-H₂O non-ideal gas mixture, using simulation and local minimization methods. In [18], aside from energy and exergy requirements, total (capital, operating, and maintenance) annualized cost (TAC) was also locally minimized for a multistage, CO₂ compressor/inter-cooler sequence, heat integrated with the steam cycle of the CO₂ capturing fossil fuel based power plant.

1.3. Conceptual Approach

More recently, [19], [20] performed two TAC optimization studies of a multistage compressor/inter-cooler sequence for a constant compressibility factor gas. [19] first presented the mathematical problem formulation, established its optimality properties, and solved analytically the operating cost problem. Then, [20] proposed a methodology that can identify the global TAC minimum, for the infinite number of instances of the TAC problem generated as the overall compression ratio is parametrically varied over a range of values. Figure 1, [20], is the schematic diagram of a multistage compressor/inter-cooler sequence. In [20], it was assumed that all compressors are non-isentropic with possibly different efficiencies, all intercoolers are isobaric, process operation is steady-state, and the gas being compressed has a constant compressibility factor \( Z \), and a constant heat capacity \( C_p \). Since the heat capacity of a gas can vary significantly over a large temperature range, it would thus be of interest to develop a global
optimization method for the case of a constant compressibility factor gas with temperature-dependent heat capacity. Thus, a global TAC optimization method is developed next, for a multistage compressor-intercooler sequence processing a constant compressibility factor $Z$ gas, with a temperature-dependent heat capacity $C_v$. Moreover, the comparison of the constant heat capacity and temperature-dependent heat capacity cases is carried out.

![Schematic Diagram of the investigated system](image-url)

**Figure 1. Schematic Diagram of the investigated system. Source: [20] (Figure 1, p. 1862)**

1.4. Mathematical Formulation

The optimization formulation considered in this work relies on the thermodynamic description of the gas as a constant compressibility factor gas, as was the case for [19], [20]:

Constant compressibility factor $Z$:

$$ PV = ZRT $$

During isentropic (ideal) compression, the relations between gas properties at the inlet and outlet of an isentropic compressor are:
\[ W_{\text{ideal}} = H_{\text{out}}' - H_{\text{in}} = \int_{T_{\text{in}}}^{T_{\text{out}}} C_p(T') \, dT' \]

\[ S_{\text{out}}' - S_{\text{in}} = \int_{T_{\text{in}}}^{T_{\text{out}}} \frac{C_p(T')}{T'} \, dT' - ZR \ln \left( \frac{P_{\text{out}}}{P_{\text{in}}} \right) = 0 \]

For non-isentropic, adiabatic (real) compression, the relations between gas properties at the inlet and outlet of a real compressor are related to those at the inlet and outlet of an ideal compressor, via the efficiency of the real compressor:

\[ W_{\text{real}} = H_{\text{out}}' - H_{\text{in}} = \int_{T_{\text{in}}}^{T_{\text{out}}} C_p(T') \, dT' \]

\[ \eta = \frac{W_{\text{ideal}}}{W_{\text{real}}} = \frac{H_{\text{out}}' - H_{\text{in}}}{H_{\text{out}}' - H_{\text{in}}} = \frac{\int_{T_{\text{in}}}^{T_{\text{out}}} C_p(T') \, dT'}{\int_{T_{\text{in}}}^{T_{\text{out}}} C_p(T') \, dT'} \]

Applying these fundamental thermodynamic relations, the mathematical formulation of the optimization problem is formed, [19], [20]. In Figure 2 [19], the TAC objective function of the formulation includes compressor capital costs, compressor operating costs, and cooler operating costs. Cooler capital costs are not considered, because they are significantly lower than the compressor capital costs. The optimization problem equality constraints consist of relations between compressor efficiencies and gas temperature conditions across the ideal and real compressors, and a relation between overall sequence compression ratio and gas temperature conditions across the sequence’s ideal compressors. The optimization problem inequality constraints ensure thermodynamically feasible, and safe operation.
In Figure 2, [19], the optimization formulation only has two sets of optimization variables, real compressor outlet temperatures $T_i$, and ideal compressor outlet temperatures $T_i''$. This formulation is the result of the proof provided in [19], that at the TAC optimum, the compressor inlet temperatures must be equal to the sequence inlet temperature, i.e. that $T_{i-1} = T_0$ $\forall i = 1, n$. 

In Figure 2, [19], the optimization formulation only has two sets of optimization variables, real compressor outlet temperatures $T_i$, and ideal compressor outlet temperatures $T_i''$. This formulation is the result of the proof provided in [19], that at the TAC optimum, the compressor inlet temperatures must be equal to the sequence inlet temperature, i.e. that $T_{i-1} = T_0$ $\forall i = 1, n$. 

Figure 2. TAC Optimization formulation. Source: [19] (Formula (20), p. 4138)
2. Optimization of multistage compressor/inter-cooler sequence for a constant compressibility factor, variable heat capacity gas

In order to facilitate the statement of the necessary conditions of optimality for a variable heat capacity and a constant compressibility factor gas, the formulation in Figure 2, [19], is slightly modified as follows:

\[
\nu = \min \sum_{i=1}^{n} \left[ \left( \frac{1}{\eta_i} \right)^a FC_{\text{compr}}(\dot{n} \cdot R)^a \left( \int_{T_0}^{T_i} C_p(T') \frac{dT'}{R} \right)^a + \frac{1}{\eta_i} \dot{n} \cdot R \left( C_{\text{compr}} + C_{\text{cool}} \frac{1}{C_{p,c}} \left( T_{c,\text{out}} - T_{c,\text{in}} \right) \right) \left( \int_{T_0}^{T_i} C_p(T') \frac{dT'}{R} \right)^a \right] 
\]

s.t.

\[
\left( \frac{P_n}{P_0} \right)^z = \prod_{i=1}^{n} \exp \left( \int_{T_0}^{T_i} \frac{C_p(T')}{RT'} dT' \right) 
\]

\[
\eta_i = \frac{\int_{T_0}^{T_i} C_p(T') \frac{dT'}{R}}{\int_{T_0}^{T_i} \frac{C_p(T')}{R} dT'} 
\]

\[
0 \leq \int_{T_0}^{T_i} \frac{C_p(T')}{R} dT' \leq \frac{1}{\eta_i} \int_{T_0}^{T_i} \frac{C_p(T')}{R} dT' \leq \int_{T_0}^{T_{\text{max}}} \frac{C_p(T')}{R} dT' < \infty, 
\]

\[
0 < T_0 \leq T_i \leq T_{\text{max}} < \infty, \quad i = 1, n 
\]

Introducing a number of new parameters, simplifies the formulation above as follows:

\[
\nu = A \cdot \min \sum_{i=1}^{n} \left[ \left( \frac{1}{\eta_i} \right)^a B(W_i)^a + \frac{1}{\eta_i} W_i \right] 
\]

s.t. \[(a)\]

\[
\prod_{i=1}^{n} h(W_i) - C = 0 
\]

\[
0 \leq W_i \leq \eta_i D \quad i = 1, n, 
\]

Where,
$$A \triangleq \hat{n} \cdot R \left( \frac{C_{\text{compr.}}^{\text{oper.}}}{C_{p,c}^{\text{compr.}}} + \frac{C_{\text{cooler}}^{\text{oper.}}}{(T_{c,\text{out}} - T_{c,\text{in}})} \right) \geq 0 \text{, } B \triangleq F C_{\text{compr.}}^{\text{oper.}} \hat{n}^{a} \left( R \right)^{a} \geq 0 \text{, } C \triangleq \left( \frac{P_{n}}{P_{0}} \right)^{Z} > 1 \text{,}$$

$$D \triangleq f \left( T_{\text{max}} \right) = \int_{T_{0}}^{T_{\text{max}}} \frac{C_{p} \left( T' \right)}{R} dT' > 0 \text{, } E \triangleq g \left( T_{\text{max}} \right) \triangleq \exp \left( \int_{T_{0}}^{T_{\text{max}}} \frac{C_{p} \left( T' \right)}{RT'} dT' \right) \text{,}$$

$$W_{i} \triangleq f \left( T_{i}^{*} \right) = \int_{T_{0}}^{T_{i}^{*}} \frac{C_{p} \left( T' \right)}{R} dT' \forall i = 1, n \text{, so } T_{i}^{*} = f^{-1} \left( W_{i} \right) \forall i = 1, n \text{,}$$

$$h \left( W_{i} \right) \triangleq g \left( f^{-1} \left( W_{i} \right) \right) \triangleq g \left( T_{i}^{*} \right) \triangleq \exp \left( \int_{T_{0}}^{T_{i}^{*}} \frac{C_{p} \left( T' \right)}{RT'} dT' \right) \forall i = 1, n \text{,}$$

Though most of the theoretical results that follow are derived without having to specify a particular mathematical relation for the temperature dependence of heat capacity, in the case study presented later, a formula for temperature dependent heat capacity of gases in the ideal gas state is employed, [21]:

$$\frac{C_{p} \left( T' \right)}{R} = c_{0} + c_{1}T' + c_{2}T'^{2} + c_{3}T'^{3}, \forall T' \in \left[ T_{0}, T_{\text{max}} \right]$$

In formulation (a), A and B are cost coefficients, C is the total compression ratio raised to a power equal to the compressibility factor Z, and D, E are the images of $T_{\text{max}}$ by the functions, $f \left( \cdot \right), g \left( \cdot \right)$ respectively.

In addition, the domains and ranges for the above defined functions are:

$$f : \left[ T_{0}, T_{\text{max}} \right] \to \left[ 0, D \right], \text{ } f : T_{i}^{*} \to W_{i} = f \left( T_{i}^{*} \right) \triangleq \int_{T_{0}}^{T_{i}^{*}} \frac{C_{p} \left( T' \right)}{R} dT' \forall i = 1, n$$

$$f^{-1} : \left[ 0, D \right] \to \left[ T_{0}, T_{\text{max}} \right], \text{ } f : W_{i} \to T_{i}^{*} \triangleq f^{-1} \left( W_{i} \right) : W_{i} = \int_{T_{0}}^{T_{i}^{*}} \frac{C_{p} \left( T' \right)}{R} dT' \forall i = 1, n$$
The monotonicity of the functions above plays an important role in establishing the global optimality of the obtained solution. It is thus important to point out that:

\[
\begin{align*}
\left\{ \frac{C_p(T')}{R} > 0 \quad &\forall T' \in \left[T_0, T_{\max}\right] \text{ for all gas species} \\
W_i = f(T_i) &\triangleq \int_{T_0}^{T_i} \frac{C_p(T')}{R} dT' \quad &\forall i = 1, n \\
f : [T_0, T_{\max}] \to [0, D], \quad f : T_i \to W_i \quad &\forall i = 1, n \text{ monotonically increases}
\end{align*}
\]

\[
\left\{ f : [T_0, T_{\max}] \to [0, D], \quad f : T_i \to W_i \quad &\forall i = 1, n \text{ monotonically increases} \right\} \Leftrightarrow \left\{ f^{-1} : [0, D] \to [T_0, T_{\max}], \quad f : W_i \to T_i' \triangleq f^{-1}(W_i) \quad &\forall i = 1, n \text{ monotonically increases} \right\}
\]

\[
\begin{align*}
\left\{ \frac{C_p(T')}{RT'} dT' > 0 \quad &\forall T' \in \left[T_0, T_{\max}\right] \right\} \Rightarrow \\
\left\{ \int_{T_0}^{T_i} \frac{C_p(T')}{RT'} dT' \quad &\forall T' \in \left[T_0, T_{\max}\right], \quad \forall i = 1, n \text{ monotonically increases} \right\} \Leftrightarrow \\
g(T_i) &\triangleq \exp\left(\int_{T_0}^{T_i} \frac{C_p(T')}{RT'} dT'\right) \quad &\forall i = 1 \\
g : [T_0, T_{\max}] \to [1, E], \quad g : T_i \to g(T_i) \quad &\forall i = 1, n \text{ monotonically increases}
\end{align*}
\]
\[ W_i > W_j \iff \int_{T_0}^{T_i'} \frac{C_p(T')}{R} dT' > \int_{T_0}^{T_j'} \frac{C_p(T')}{R} dT' \iff f^{-1}(W_i) = T_i^* > T_j^* = f^{-1}(W_j) \iff \int_{T_0}^{T_i'} \frac{C_p(T')}{RT'} dT' > \int_{T_0}^{T_j'} \frac{C_p(T')}{RT'} dT' \iff \exp \left( \int_{T_0}^{T_i'} \frac{C_p(T')}{RT'} dT' \right) > \exp \left( \int_{T_0}^{T_j'} \frac{C_p(T')}{RT'} dT' \right) \iff h(W_i) > h(W_j) \iff \]

\[ h : [0, D] \rightarrow [1, E] , \ h : W_i \rightarrow h(W_i) \ \forall i = 1, n \text{ monotonically increases} \]

Similar to the previous study, [19], the feasibility of the optimization problem needs to be verified, so the corresponding theorem is proved.

**Theorem 1:**

Let \( D > 0, \ C > 1, \ \eta_i \in (0,1) \ \forall i = 1, n \). Then the optimization problem \((a)\) is feasible iff

\[ C \leq \prod_{i=1}^{n} h(\eta_i D) . \]

See Appendix for the proof.

Since the feasibility of \((a)\) is verified by *Theorem 1*, the optimization problem \((a)\) is slightly modified so as to facilitate the derivation of the necessary conditions for optimality.

\[ \nu = A \cdot \min_{\{h_i \}_{i=1}^{n}} \sum_{i=1}^{n} \left[ \left( \frac{1}{\eta_i} \right)^a \frac{B}{A} (W_i)^a + \frac{1}{\eta_i} W_i \right] \]

\[ \text{s.t.} \]

\[ \prod_{i=1}^{n} h(W_i) - C = 0, \ \eta_i \varepsilon \leq W_i \leq \eta_i D \ \ i = 1, n, \]

\[ \varepsilon \triangleq f(T) = \frac{T}{\tau_0} \frac{C_p(T')}{R} dT' = 0, \ \varepsilon \in \left( 0, \frac{\min_{i=1}^{n} \eta_i}{\max_{i=1}^{n} \eta_i} D \right) ; \ 1 = \frac{h(\eta \varepsilon)}{h(D \eta)} < \frac{h(\eta D)}{h(\eta_j D)} \ \forall i = 1, n | \eta_i > \eta_j \]
The derivation of the optimality conditions of the above problem necessitates the introduction of the following sets representing the workload status of each compressor:

\[ S_D^w \triangleq \{ i = 1, n : W_i = \eta_i D \}, \quad S_D^w \] contains all compressors which are operated at the maximum workload.

\[ S_I^w \triangleq \{ i = 1, n : \eta_i \varepsilon < W_i < \eta_i D \}, \quad S_I^w \] contains all compressors which are in use and operated below the maximum workload.

\[ S_e^w \triangleq \{ i = 1, n : W_i = \eta_i \varepsilon \}, \quad S_e^w \] contains all compressors which are not practically in use.

The corresponding cardinalities in each set are

\[ N_D^w, N_I^w, N_e^w, \quad N_D^w + N_I^w + N_e^w = n. \]

The possible combinations of the workload status of each compressor generate a variety of cases to be considered, some of which can be readily shown to be impossible at the global optimum.

To this end, Theorem 2 below is established.

**Theorem 2:**

Let \( A > 0, B > 0, D > 0, 1 < C \leq \prod_{i=1}^{n} h(\eta_i D), \eta_i \in (0,1] \) \( \forall i = 1, n \) and consider the problem \( \nu_{\varepsilon} \),

where \( \varepsilon = 0; \quad \varepsilon \in \left( \min_{i=1,n} \frac{\eta_i}{D}, \max_{i=1,n} \eta_i \right) \)

\[ 1 = \frac{h(\eta_i \varepsilon)}{h(\eta_i D)} < \frac{h(\eta_j D)}{h(\eta_j \varepsilon)} \quad \forall i = 1, n \]

\[ \eta_i > \eta_j. \] Then, for all gas species with a non-decreasing \( \frac{C_p(T')}{R} \) with respect to \( T' \),

a. \( 1 \leq N_D^w < n \) implies \( \min_{i \in S_D^w} \eta_i \geq \max_{i \in S_I^w \cup S_e^w} \eta_i \)
b. $1 \leq N_{\varepsilon}^{w^*}$ implies $\max_{i \in S_i^{w^*}} \eta_i \leq \min_{i \in S_i^{w^*} \cup S_j^{w^*}} \eta_i$

See Appendix for the proof.

2.1. The necessary conditions of optimality

The Lagrangian for the optimization problem under consideration is:

$$L(W_i, \lambda, \mu, \omega_i) = \sum_{i=1}^{n} \left[ \left( \frac{1}{\eta_i} \right)^a \frac{B}{A} (W_i)^{a-1} + \frac{1}{\eta_i} W_i \right] + \lambda \left( \prod_{i=1}^{n} h(W_i) - C \right) + \sum_{i=1}^{n} \mu_i (\eta_i - W_i) + \sum_{i=1}^{n} \omega_i (W_i - \eta_i D)$$

Then the optimal conditions for $\nu_\varepsilon$ are:

$$\frac{\partial L(W_i, \lambda, \mu, \nu_i)}{\partial W_k} \triangleq \frac{1}{\eta_k} + \frac{aB}{A} \left( \frac{1}{\eta_k} \right)^{(a-1)} + \lambda \frac{\partial h(W_k)}{\partial W_k} \prod_{i=1 \neq k}^{n} h(W_i) - \mu_k + \omega_k = 0 \ \forall k = 1, n$$

$$\prod_{i=1}^{n} h(W_i) - C = 0, \ \eta_i, \epsilon \leq W_i \leq \eta_i D \ \forall i = 1, n$$

$$\mu_i \geq 0, \mu_i (\eta_i, \epsilon - W_i) = 0, \omega_i \geq 0, \omega_i (W_i - \eta_i D) = 0 \ \forall i = 1, n$$

$$\begin{align*}
\left\{ \begin{array}{l}
\frac{1}{\eta_k} aB \left( \frac{1}{\eta_k} \right)^{(a-1)} + \lambda \frac{\partial h(W_k)}{\partial W_k} \prod_{i=1 \neq k}^{n} h(W_i) = 0 \ \forall k \in S_i^{w^*} \\
\mu_i = \frac{1}{\eta_i} aB \left( \frac{1}{\eta_i} \right)^{(a-1)} + \lambda \frac{\partial h(W_i)}{\partial W_i} \prod_{i=1 \neq i}^{n} h(W_i) \geq 0 \ \forall i \in S_i^{w^*} \\
\omega_m = -\frac{1}{\eta_m} aB \left( \frac{1}{\eta_m} \right)^{(a-1)} - \lambda \frac{\partial h(W_m)}{\partial W_m} \prod_{i=1 \neq m}^{n} h(W_i) \geq 0 \ \forall m \in S_m^{w^*} \\
\prod_{i \in S_i^{w^*}} h(\eta_i, \epsilon) \prod_{i \in S_i^{w^*}} h(W_i) \prod_{i \in S_i^{w^*}} h(\eta_i, \epsilon) - C = 0,
\end{array} \right.
\end{align*}$$

$$0 < \eta_k, \epsilon < W_k < \eta_k D \ \forall k \in S_i^{w^*},$$

$$\mu_i \geq 0 \ W_i = \eta_i, \epsilon \ \forall i \in S_i^{w^*}, \ \omega_m \geq 0 \ W_m = W_m D \ \forall m \in S_m^{w^*}$$
Since $W_i \triangleq f(T_i') = \int_{T_{t_0}}^{T_i'} \frac{C_p(T')}{R} dT' \quad \forall i = 1, n$, $h(W_i) \triangleq \exp\left(\int_{T_{t_0}}^{T_i'} \frac{C_p(T')}{RT'} dT'\right) \quad \forall i = 1, n$, so

\[
\begin{align*}
\dot{h}(W_i) &= \frac{\partial h(W_i)}{\partial W_i} \frac{\partial W_i}{\partial T_i'} = \frac{\partial h(W_i)}{\partial T_i'} \frac{1}{\partial W_i} = \frac{\partial}{\partial T_i'} \left(\exp\left(\int_{T_{t_0}}^{T_i'} \frac{C_p(T')}{RT'} dT'\right)\right) \frac{1}{\partial T_i'} \left(\int_{T_{t_0}}^{T_i'} \frac{C_p(T')}{RT'} dT'\right) \\
&= \exp\left(\int_{T_{t_0}}^{T_i'} \frac{C_p(T')}{RT'} dT'\right) \frac{C_p(T_i')}{RT_i'} = \frac{1}{T_i'} \exp\left(\int_{T_{t_0}}^{T_i'} \frac{C_p(T')}{RT'} dT'\right) = h(W_i) = 0 \quad \forall i = 1, n,
\end{align*}
\]

\[
\begin{align*}
\left[\left(\frac{1}{\eta_k}\right)^a \frac{aB}{A} (W_k)^{a-1} + \frac{1}{\eta_k}\right] h(W_k) + \lambda C \frac{h(W_k)}{f^{-1}(W_k)} &= 0 \quad \forall k \in S^w_i \\
\mu_i &= \left[\left(\frac{1}{\eta_i}\right)^a \frac{aB}{A} (W_i)^{a-1} + \frac{1}{\eta_i}\right] h(W_i) + \lambda C \frac{h(W_i)}{f^{-1}(W_i)} \geq 0 \quad \forall l \in S^w_e \\
\omega_m &= -\left[\left(\frac{1}{\eta_m}\right)^a \frac{aB}{A} (W_m)^{a-1} + \frac{1}{\eta_m}\right] h(W_m) - \lambda C \frac{h(W_m)}{f^{-1}(W_m)} \geq 0 \quad \forall m \in S^w_D \\
\prod_{r \in S^w_e} h(\eta, \epsilon) \prod_{s \in S^w_l} h(W_s) \prod_{r \in S^w_l} h(\eta, \epsilon) - C &= 0, \\
0 < \eta_k < W_k < \eta_k D \quad \forall k \in S^w_i, \\
\mu_i \geq 0 \quad W_i = \eta_i \epsilon \quad \forall l \in S^w_e, \quad \omega_m \geq 0 \quad W_m = \eta_m D \quad \forall m \in S^w_D
\end{align*}
\]

\[
\begin{align*}
\left[\left(\frac{1}{\eta_k}\right)^a \frac{aB}{A} (W_k)^{a-1} + \frac{1}{\eta_k}\right] f^{-1}(W_k) + \lambda C &= 0 \quad \forall k \in S^w_i \\
\frac{\mu_i}{h(W_i)} &= \left[\left(\frac{1}{\eta_i}\right)^a \frac{aB}{A} (W_i)^{a-1} + \frac{1}{\eta_i}\right] f^{-1}(W_i) + \lambda C \geq 0 \quad \forall l \in S^w_e \\
\frac{\omega_m}{h(W_m)} &= -\left[\left(\frac{1}{\eta_m}\right)^a \frac{aB}{A} (W_m)^{a-1} + \frac{1}{\eta_m}\right] f^{-1}(W_m) - \lambda C \geq 0 \quad \forall m \in S^w_D \\
\prod_{r \in S^w_e} h(\eta, \epsilon) \prod_{s \in S^w_l} h(W_s) \prod_{r \in S^w_l} h(\eta, \epsilon) - C &= 0, \\
0 < \eta_k < W_k < \eta_k D \quad \forall k \in S^w_i, \\
\mu_i \geq 0 \quad W_i = \eta_i \epsilon \quad \forall l \in S^w_e, \quad \omega_m \geq 0 \quad W_m = \eta_m D \quad \forall m \in S^w_D
\end{align*}
\]
At the global optimum of $\nu_\epsilon$, the following conditions must be satisfied:

$$\left\{ \begin{array}{l}
\left( \frac{1}{\eta_k} \right)^a \frac{aB}{A} (W_k)^{a-1} + \frac{1}{\eta_k} f^{-1}(W_k) + \lambda C = 0 \quad \forall k \in S_i^W \\
\frac{\mu_i}{h(\eta_i)} = \left( \frac{aB}{A} \frac{1}{\eta_i} (\epsilon)^{a-1} + 1 \frac{1}{\eta_i} \right) f^{-1}(\eta_i \epsilon) + \lambda C \geq 0 \quad \forall l \in S_i^W \\
\frac{\omega_k}{h(\eta_m)} = -\left( \frac{aB}{A} \frac{1}{\eta_m} (D)^{a-1} + 1 \frac{1}{\eta_m} \right) f^{-1}(\eta_m D) - \lambda C \geq 0 \quad \forall m \in S_D^w \\
\prod_{i \in S_i^w} h(\eta_i) \prod_{i \in S_i^w} h(W_i) \prod_{i \in S_D^w} h(\eta_i) - C = 0, \\
0 < \eta_i \epsilon < W_k < \eta_i D \quad \forall k \in S_i^w, \\
\mu_i \geq 0 \ W_i = \eta_i \epsilon \quad \forall i \in S_i^w, \ \omega_m \geq 0 \ W_m = \eta_m D \quad \forall m \in S_D^w
\end{array} \right\} (c)$$

Considering compressors in different sets of workload status, the objective function $\nu_\epsilon$ becomes:

$$\nu_\epsilon = A \left\{ \begin{array}{l}
\sum_{i \in S_i^w} \frac{1}{\eta_i} \epsilon + \frac{B}{A} \sum_{i \in S_i^w} \left( \frac{1}{\eta_i} \right)^a (\epsilon)^a + \sum_{i \in S_i^w} \frac{1}{\eta_i} D + \frac{B}{A} \sum_{i \in S_i^w} \left( \frac{1}{\eta_i} \right)^a (D)^a \\
\inf_{[W_i],i \in S_i^w} \left[ \sum_{i \in S_i^w} \frac{1}{\eta_i} W_i + \frac{B}{A} \sum_{i \in S_i^w} \left( \frac{1}{\eta_i} \right)^a (W_i)^a \right]
\end{array} \right\} \iff
\prod_{i \in S_i^w} h(\eta_i) \prod_{i \in S_i^w} h(W_i) \prod_{i \in S_D^w} h(\eta_i) - C = 0 \\
\eta_i \epsilon < W_i < \eta_i D \quad \forall i \in S_i^w$$

$$\nu_\epsilon = A \left\{ \begin{array}{l}
\left( D + \frac{B}{A} D^a \right) N_D^w + \left( \epsilon + \frac{B}{A} \epsilon^a \right) N_\epsilon^w + \inf_{[W_i],i \in S_i^w} \left[ \sum_{i \in S_i^w} \frac{1}{\eta_i} W_i + \frac{B}{A} \sum_{i \in S_i^w} \left( \frac{1}{\eta_i} \right)^a (W_i)^a \right]
\end{array} \right\} \iff
\prod_{i \in S_i^w} h(\eta_i) \prod_{i \in S_i^w} h(W_i) \prod_{i \in S_D^w} h(\eta_i) - C = 0 \\
\eta_i \epsilon < W_i < \eta_i D \quad \forall i \in S_i^w$$
To reach any conclusions regarding global optimality from the optimality conditions \(c\), the behavior of function \(F_k(W_k)\) must be analyzed.

\[
F_k(W_k) = \left( \frac{1}{\eta_k} \right)^a \frac{aB}{A} (W_k)^{a-1} + \frac{1}{\eta_k} f^{-1}(W_k) \quad \forall k \in S^w
\]

Conservatively, \(F_k(W_k)\) is constrained in:

\[
F_k(W_k) \in \left[ \min_{l \in s^y} \left( \frac{aB}{A} \eta_l (\varepsilon)^{a-1} + \frac{1}{\eta_l} f^{-1}(\eta_l \varepsilon) \right), \max_{m \in s^y} \left( \frac{aB}{A} \frac{1}{\eta_m} (D)^{a-1} + \frac{1}{\eta_m} f^{-1}(\eta_m D) \right) \right]
\]

As can be seen in Figure 3, \(F_k(\cdot)\) has a single minimum, and is monotonically decreasing (increasing) to the left (right) of this minimum.

Figure 3. Graph of arbitrary \(F_k(W_k)\) over a large range of \(W_k\)
Naming $p_k$ as the local minimum of $F_k(W_k)$, $F_k(W_k)$ monotonically decreases on $(0, p_k)$ and monotonically increases on $(p_k, +\infty)$. It can be readily seen in Figure 3, that as $W_k$ approaches zero, $F_k(W_k)$ approaches $+\infty$, since the term $(W_k)^{a-1}$ dominates $f^{-1}(W_k)$ when $W_k$ is close to zero. Due to the monotonicity of $F_k(W_k)$, defining left and right functions of $F_k$ can facilitate the discussion of the optimization algorithm.

$$F^L_k: (0, p_k] \to (0, +\infty), \quad F^L_k: W_k \to F^L_k(W_k) \triangleq \left[ \frac{1}{\eta_k} + a \cdot \frac{B}{A} \cdot \left( \frac{1}{\eta_k} \right)^a \cdot (W_k)^{a-1} \right] f^{-1}(W_k) \quad \forall k \in S^w$$

$$F^R_k: [p_k, +\infty) \to (0, +\infty), \quad F^R_k: W_k \to F^R_k(W_k) \triangleq \left[ \frac{1}{\eta_k} + a \cdot \frac{B}{A} \cdot \left( \frac{1}{\eta_k} \right)^a \cdot (W_k)^{a-1} \right] f^{-1}(W_k) \quad \forall k \in S^w$$

Thus, the inverse functions of $F^L_k, F^R_k$ are described as:

$$\left( F^L_k \right)^{-1}: F_k(p_k), +\infty \to (0, p_k],$$

$$\left( F^L_k \right)^{-1}: y_k \triangleq \left[ \frac{1}{\eta_k} + a \cdot \frac{B}{A} \left( \frac{1}{\eta_k} \right)^a \cdot (W_k)^{a-1} \right] f^{-1}(W_k) \to \left( F^L_k \right)^{-1}(y_k) \triangleq W_k \in (0, p_k]$$

$$\left( F^R_k \right)^{-1}: F_k(p_k), +\infty \to [F_k, +\infty),$$

$$\left( F^R_k \right)^{-1}: y_k \triangleq \left[ \frac{1}{\eta_k} + a \cdot \frac{B}{A} \left( \frac{1}{\eta_k} \right)^a \cdot (W_k)^{a-1} \right] f^{-1}(W_k) \to \left( F^R_k \right)^{-1}(y_k) \triangleq W_k \in [p_k, +\infty)$$

Based on Figure 4, the proposed optimization algorithm is discussed in the next section.
2.2. Algorithms of obtaining optimum intervals

1. For the given gas species in ideal gas state, \( a_i, \frac{B}{A}, \eta_i \) \( \forall i = N_D^w + 1, ..., N_D^w + N_I^w \), numerically calculate the roots \( p_i : F_i(p_i) = 0 \) \( \forall i = N_D^w + 1, ..., N_D^w + N_I^w \) result local minimums of \( F_i \).

2. For known \( W_{N_D^w + 1}^{w} \in (\eta_N^{w}, \eta_{N_D^w + 1}^{w}, D) \), the interval \( [\eta_N^{w}, \eta_{N_D^w + 1}^{w}, D] \) is discretized into number \( q_{N_D^w + 1} \) of uneven intervals \( [W_{N_D^w + 1,j}^{L}, W_{N_D^w + 1,j}^{U}] \) \( \forall j = 1, q_{N_D^w + 1} \) with a discretization size of \( r_{N_D^w + 1,j} \).

The size selection of \( r_{N_D^w + 1,j} \) is discussed separately after the algorithm. And the parameters are specified as: \( W_{N_D^w + 1,j}^{L} = \eta_{N_D^w + 1,j}^{w}, W_{N_D^w + 1,j}^{U} = W_{N_D^w + 1,j+1}^{L} \) \( \forall j = 1, q_{N_D^w + 1} - 1, W_{N_D^w + 1,q_{N_D^w + 1}}^{U} = \eta_{N_D^w + 1}^{w} D \).
3. For each interval \( \left[ W^L_{N^D_0 + 1, j}, W^U_{N^D_0 + 1, j} \right] \) \( \forall j = 1, q_{N^D_0 + 1} \), the algorithm proceeds as follows:

\[
\begin{bmatrix}
\{ W^L_{N^D_0 + 1} \in \left[ W^L_{N^D_0 + 1, j}, W^U_{N^D_0 + 1, j} \right] \} \land \\
\{ F^w_{N^D_0 + 1}(W^w_{N^D_0 + 1, j}) \in \left[ \max_{m \in \mathcal{S}_m} \left( \frac{a B}{A} \frac{1}{n_m} (D)^{a-1} + \frac{1}{\eta_m} \right) f^{-1}(\eta_m D) \right] , \min_{\mathcal{L}_m} \left( \frac{a B}{A} \frac{1}{\eta_h} (\varepsilon)^{a-1} + \frac{1}{\eta_h} \right) f^{-1}(\eta_h \varepsilon) \} \end{bmatrix} \Rightarrow
\begin{bmatrix}
\left[ F^w_{N^D_0 + 1}(W^w_{N^D_0 + 1, j}), F^w_{N^D_0 + 1}(W^w_{N^D_0 + 1, j}) \right] \cap \\
\max_{m \in \mathcal{S}_m} \left( \frac{a B}{A} \frac{1}{n_m} (D)^{a-1} + \frac{1}{\eta_m} \right) f^{-1}(\eta_m D) , \\
\min_{\mathcal{L}_m} \left( \frac{a B}{A} \frac{1}{\eta_h} (\varepsilon)^{a-1} + \frac{1}{\eta_h} \right) f^{-1}(\eta_h \varepsilon) \\
f\left( W^U_{N^D_0 + 1} \right) = \left\{ \begin{array}{ll}
\left[ F^w_{N^D_0 + 1}(W^w_{N^D_0 + 1, j}), F^w_{N^D_0 + 1}(W^w_{N^D_0 + 1, j}) \right] \\
\left[ F^w_{N^D_0 + 1}(p_{N^D_0 + 1}), \max \left( F^w_{N^D_0 + 1}(W^w_{N^D_0 + 1, j}), F^w_{N^D_0 + 1}(W^w_{N^D_0 + 1, j}) \right) \right] \\
\left[ F^w_{N^D_0 + 1}(W^w_{N^D_0 + 1, j}), F^w_{N^D_0 + 1}(W^w_{N^D_0 + 1, j}) \right] \\
\min_{m \in \mathcal{S}_m} \left( \frac{a B}{A} \frac{1}{n_m} (D)^{a-1} + \frac{1}{\eta_m} \right) f^{-1}(\eta_m D) , \\
\min_{\mathcal{L}_m} \left( \frac{a B}{A} \frac{1}{\eta_h} (\varepsilon)^{a-1} + \frac{1}{\eta_h} \right) f^{-1}(\eta_h \varepsilon) \\
\end{array} \right. 
\end{bmatrix}
\end{bmatrix}
\]

4. For each interval \( \left[ W^L_{N^D_0 + 1, j}, W^U_{N^D_0 + 1, j} \right] \) \( j = 1, q_{N^D_0 + 1} \) such that \( \left[ F^w_{N^D_0 + 1, j}, F^w_{N^D_0 + 1, j} \right] \neq \emptyset \), and \( \forall i = N^w_D + 2, \ldots, N^w_D + N^w_I \) it holds:

\[ \forall i = N^w_D + 2, \ldots, N^w_D + N^w_I \]
\[
\left\{ W_{N^w_{D+1},i}^{L} \in \left[ W_{N^w_{D+1},i}^{L}, W_{N^w_{D+1},i}^{U} \right], F_{\eta,\varepsilon}^{N^w_{D+1},i} \right\} \cap \left[ F_{\eta,\varepsilon}^{L}, F_{\eta,\varepsilon}^{U} \right] \neq \emptyset \land W_{i} \in (\eta, \varepsilon, D) \Rightarrow
\]

\[
W_{i} \in \left\{ \begin{array}{ll}
\left[ \left( F_{i}^{L} \right)^{-1} \left( F_{i}^{U} \right)^{-1} \right] \cap \left( \eta, \varepsilon, D \right) & \text{if } F_{i}(p_{i}) \leq F_{N^w_{D+1},i}^{L} \\
\left[ \left( F_{i}^{L} \right)^{-1} \left( F_{i}^{U} \right)^{-1} \right] \cap \left( \eta, \varepsilon, D \right) & \text{if } F_{N^w_{D+1},i}^{L} < F_{i}(p_{i}) \leq F_{N^w_{D+1},i}^{U} \\
\emptyset & \text{if } F_{N^w_{D+1},i}^{U} < F_{i}(p_{i})
\end{array} \right.
\]

5. After figuring out the range of \( W_{i} \) and its corresponding interval \( \left[ W_{N^w_{D+1},i}^{L}, W_{N^w_{D+1},i}^{U} \right] \), calculate the corresponding objective function intervals with the expression:

\[
\left( D + \frac{B}{A} D^{a} \right) N_{\varepsilon}^{w} + \left( \varepsilon + \frac{B}{A} \varepsilon^{a} \right) N_{\eta}^{w} + \left[ \sum_{i \in S_{i}^{D}} \frac{1}{\eta_{i}} W_{i} + \frac{B}{A} \sum_{i \in S_{i}^{D}} \left( \frac{1}{\eta_{i}} \right)^{a} (W_{i})^{a} \right],
\]

and the corresponding compression ratio related parameters \( C \) intervals with the expression:

\[
\prod_{i \in S_{i}^{D}} h(\eta D) \prod_{i \in S_{i}^{D}} h(W_{i}) \prod_{i \in S_{i}^{D}} h(\eta, \varepsilon). \]

The union of rectangles that consist of the objective function value intervals and \( C \) intervals surely contains the optimum objective function values.

6. From the piecewise constant function boxes defined in Step 5, the lowest lower bound of objective function value vs each \( C \) interval is defined as the lower bound of the global optimum interval of the objective function value vs each \( C \) interval; and the upper bound of objective function value corresponds to the lowest lower bound of objective function value vs each \( C \) interval is defined as the upper bound of the global optimum interval of the objective function value vs each \( C \) interval. In addition, the interval of variable \( \{ W_{i} \}_{i=1}^{n} \) corresponding to the optimum objective function value interval is defined as the global
optimum interval of the variables \( \{W_i\}_{i=1}^n \) vs each C interval. Figure 6 [20] illustrates how the global optimum intervals are defined graphically.

7. If the accuracy of the optimum intervals of any variable value vs C intervals is not satisfied, increase 2-10 times the number of intervals \( q_{n+1}^{n+1} \) and repeat the algorithm steps above.

Otherwise, the algorithm ends.

---

Figure 5. Identification of objective function interval. Source: [20] (Figure 3, p. 1869)

---

2.3. Case study

To demonstrate the proposed global optimization method, an optimization case study is performed, on methane compression with a compressor/inter-cooler sequence featuring up to four \( (n = 4) \) compressors/inter-coolers. The compressor sequence compresses methane from the
initial state \( (T_0, P_0) = (298 \text{ K}, 1 \text{ atm}) \), to the final state \( (T_n, P_n) = (T_0, P_n) = (298 \text{ K}, P_n) \), where \( P_n \) will be parametrically varied and the optimum will be identified at an infinity of \( P_n \) values. The fixed parameters in this case study are listed in Table 1. Values for the power law exponent \( a \), from literature, [18], [22], [23], [24], range from 0.6 to 1. So \( a \) is chosen to be 0.7 in this case study. For the compressibility factor \( Z \) of methane, the range of \( Z \) from literature, [19], [25], [26], is from 0.85 to 1. This case study uses the value of 0.85.

<table>
<thead>
<tr>
<th>( T_0 ) (K)</th>
<th>( T_{\text{max}} ) (K)</th>
<th>( P_0 ) (atm)</th>
<th>( a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>298</td>
<td>438</td>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>( A )</td>
<td>( B )</td>
<td>( Z )</td>
<td>( \varepsilon )</td>
</tr>
<tr>
<td>0.005</td>
<td>0.005</td>
<td>0.85</td>
<td>1</td>
</tr>
<tr>
<td>( \eta_1 )</td>
<td>( \eta_2 )</td>
<td>( \eta_3 )</td>
<td>( \eta_4 )</td>
</tr>
<tr>
<td>1</td>
<td>0.9</td>
<td>0.8</td>
<td>0.7</td>
</tr>
<tr>
<td>( c_0 )</td>
<td>( c_1 )</td>
<td>( c_2 )</td>
<td>( c_{-2} )</td>
</tr>
</tbody>
</table>

Table 1. Fixed parameters for the case study

Considering Theorem 2, all possible combinations of cardinalities \( N_P, N_I, N_C \) are listed in Table 2. Combinations 1-5 result in a single objective function value because all of these combinations contain either minimum or maximum capacity compressors. Therefore, the optimal interval calculations will be performed for the combinations 6-15.
Table 2. All optimal combinations for the compressor sequence

<table>
<thead>
<tr>
<th>Combination</th>
<th>$N^W_D$</th>
<th>$N^W_J$</th>
<th>$N^W_\varepsilon$</th>
<th>$\eta(S^W_D)$</th>
<th>$\eta(S^W_J)$</th>
<th>$\eta(S^W_\varepsilon)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1,0.9,0.8,0.7</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>1,0.9,0.8</td>
<td>n/a</td>
<td>0.7</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>1,0.9</td>
<td>n/a</td>
<td>0.8,0.7</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>n/a</td>
<td>0.9,0.8,0.7</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>n/a</td>
<td>n/a</td>
<td>1,0.9,0.8,0.7</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>n/a</td>
<td>1</td>
<td>0.9,0.8,0.7</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0.9</td>
<td>0.8,0.7</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1,0.9</td>
<td>0.8</td>
<td>0.7</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1,0.9,0.8</td>
<td>0.7</td>
<td>n/a</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>n/a</td>
<td>1,0.9</td>
<td>0.8,0.7</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0.9,0.8</td>
<td>0.7</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1,0.9</td>
<td>0.8,0.7</td>
<td>n/a</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>n/a</td>
<td>1,0.9,0.8</td>
<td>0.7</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0.9,0.8,0.7</td>
<td>n/a</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>n/a</td>
<td>1,0.9,0.8,0.7</td>
<td>n/a</td>
</tr>
</tbody>
</table>

2.4. Discretization Selection

The size $r_{N^W_D+1,j}$ of interval $[W^L_{N^W_D+1,j}, W^U_{N^W_D+1,j}]$ $\forall j = 1, q_{N^W_D+1}$ is also determined by the location of $W^U_{N^W_D+1} \in [W^L_{N^W_D+1,j}, W^U_{N^W_D+1,j}]$ with respect to the local minimum $p_{N^W_D+1}$ because of the asymmetric shape of $F_k(W_k)$ $\forall k \in N^W_J$. When $W_k$ is away from the local minimum $p_k$, the rate that $F_k(W_k)$ decreases on the left side of the local minimum $p_k$ is much larger than the rate that $F_k(W_k)$
increases on the right side of the local minimum \( p_k \). Then a large \( r_{N_D + 1} \) selection may result in an even larger range of \( W_i \in \left[ \left( F_i^R \right)^{-1} \left( F_{N_D + 1,j}^{L} \right), \left( F_i^R \right)^{-1} \left( F_{N_D + 1,j}^{U} \right) \right] \), \( \forall i = N_D + 2, \ldots, N_D + N_i \). Also considering the range of \( W_k \) nearby the local minimum \( p_k \), a large \( r_{N_D + 1} \) selection within this range may result in a wide range of \( W_i \in \left[ \left( F_i^L \right)^{-1} \left( F_{N_D + 1,j}^{L} \right), \left( F_i^R \right)^{-1} \left( F_{N_D + 1,j}^{U} \right) \right] \), \( \forall i = N_D + 2, \ldots, N_D + N_i \). An excessively large range of \( W_i \) can lead to a larger range of its corresponding \( C \), which makes the optimal rectangle much larger than the initial expectation. Accordingly the computational capability, an appropriate discretization selection should be considered as follows:

\[
r_{N_D + 1,j} = \begin{cases} 
   r, & \text{if } W_{N_D + 1} \rightarrow 0 \\
   5r \sim 10r, & \text{if } W_{N_D + 1} \rightarrow p_{N_D + 1} \\
   50r \sim 100r, & \text{if } W_{N_D + 1} \rightarrow +\infty
\end{cases}
\]

\( \forall W_{N_D + 1} \in \left[ W_{N_D + 1,j}^{L}, W_{N_D + 1,j}^{U} \right] \), \( \forall j = 1, q_{N_D + 1} \), the range of \( r \) is suggested as \( r \in \left[ \frac{\eta_k D - \eta_k \varepsilon}{10^p}, \frac{\eta_k D - \eta_k \varepsilon}{10^5} \right] \) \( \forall k \in S_i \).

The case study uses \( r = 0.01 \) for methane compression.
3. Results and Discussion

3.1. Globally optimum intervals

Figure 6. Globally optimum objective function intervals

Figure 7. Globally optimum Wi variable intervals
Figure 6 illustrates the globally optimum objective function value as a function of \( \ln C \). The associated \( W_k \) variable values at the global optimum are illustrated in Figure 7 as a function of \( \ln C \). Near the points of transition, where the optimal compressor operations switch from one compressor to two, two to three, and so on, the identified optimum intervals are large. Indeed, at these transition points, compressors are being activated from minimum work load status \( S^w_\varepsilon \) to another work load status, or compressors are being transitioned from interior work load status \( S^w_i \) to maximum work load status \( S^w_D \). Since there is no interval calculation involved in identifying the work load at the minimum work load status \( S^w_\varepsilon \) and the maximum work load status \( S^w_D \), the interval calculations are significantly different depending on the combination considered, with interval calculations involving compressors at an interior work load status, leading naturally to larger intervals for \( W_i \). Therefore, a finer discretization is required on the transitions of compressor operation to shrink the overlaps and clarify the transitions.
3.2. Globally optimum intervals with finer discretization on transition points

Figure 8. Globally optimum objective function value intervals with finer discretization on transitions

Figure 9. Globally optimum Wi variable intervals with finer discretization on transitions
In this case, a discretization 20 times finer than the original discretization is applied at the transition points of compressor operation. In comparison with Figure 6, Figure 8 illustrates a smoother curve with finer discretization on the transition points of compressor operation. Moreover, Figure 9 identifies in a much more accurate manner than Figure 7 the transition points of compressor operation. Thus, these more accurate data will be used to analyze the behavior of the identified global optimum.

A discontinuity is observed at transition points, whereby the compressor has a certain workload just before the transition point and then, following the transition point, its workload is discontinuously lower. That behavior is observed at every transition point. The possible explanation for the discontinuity is that the objective function has a value approaching infinity when the compressor workload is infinitesimally small. Thus the optimization selects an optimum value for the just activated compressor that has a finite workload. Then the compressor that was already active must have a discontinuous drop in its workload, so that the overall sequence’s objective function does not exhibit a discontinuity.

3.3. Comparison with constant heat capacity gas

In this section, the proposed global optimization method is employed to the case where a constant (average) constant heat capacity of the gas is employed. The specific derivation of the constant heat capacity optimization formulation is illustrated in the appendix.
Figure 10. Globally Optimum Wideal intervals with Constant (black line) and Variable (blue dotted line) $C_p$

Figure 11. Globally Optimum $T_i''$ intervals with constant (black line) and variable (blue dotted line) $C_p$
The optimum $W_{\text{ideal}}$ and $T_i''$ intervals, which are directly derived from $W_i$, reflect the isentropic compressor behavior. The differences of the optimum $W_{\text{ideal}}$ and $T_i''$ intervals between the constant and variable $C_p$ cases are within 3% and 1.2%. As seen in Figure 10 and Figure 11, the difference of the optimum intervals between constant $C_p$ model and variable $C_p$ model is not significant, but observable. Variable $C_p$ ideal compressors typically reach their maximum workload at a lower total pressure ratio than constant $C_p$ ideal compressors do. In addition, variable $C_p$ ideal compressors get activated at a lower total pressure ratio than constant $C_p$ ideal compressors. The same trends are also observed in ideal compressor exit temperatures, and in real compressor workloads and exit temperatures.

Figure 12. Globally Optimum Wreal intervals with Constant (black line) and Variable (blue dotted line) Cp
Since the same $T_{\text{max}}$ is set as the upper limit of every real compressor outlet temperature, From Figure 12 and Figure 13, the optimum real compressor temperature $T_i$ intervals eventually reach the same maximum temperature, and their associated optimum real compressor workload $W_{\text{real}}$ intervals reach the same maximum workload.
Figure 14. Globally optimum $\ln(P_i/P_{i-1})$ interval with constant (black line) and variable (blue dotted line) $C_p$

From Figure 14, the difference of the optimum $\ln\left(\frac{P_i}{P_{i-1}}\right)$ intervals between constant $C_p$ and variable $C_p$ optimization problems is less than 1.7%. In addition, each optimum compressor doesn’t merge into the same maximum workload pressure in both the constant $C_p$ model and the variable $C_p$ model. Since $W_i$ and $T_i''$ are the variables used in the optimization algorithm of the optimization problem and they are associated with the term $\int_{T_0}^{T_i} \frac{C_p(T')}{R} dT'$, so the optimum

$\ln\left(\frac{P_i}{P_{i-1}}\right)$ intervals, which are associated with the term $\int_{T_0}^{T_i} \frac{C_p(T')}{RT'} dT'$, behave slightly different than the optimum work and compressor outlet temperature intervals.
Figure 15. Globally optimum objective function interval with constant (black line) and variable (blue dotted line) $C_p$

From Figure 15, the difference in globally optimum objective function value intervals is even less obvious than the other optimum variables, and becomes larger as the total compression ratio goes higher, but it’s always less than 1% between the constant $C_p$ model and the variable $C_p$ model. This is because compressor capital cost is the dominant portion of the globally optimum objective function value, and since a compressor’s capital cost is proportional to its workload raised to a power less than one, then workload differences become less pronounced when each is raised to a power less than one.
4. Conclusions

In this study, global TAC minimization of compressor-intercooler sequences for a constant compressibility factor, temperature-dependent heat capacity gas is performed. The necessary conditions of optimality are developed for the associated non-convex optimization problem, and are used to establish theoretically a number of properties that form the basis of an efficient global optimization algorithm. The proposed algorithm can identify an infinite number of instances of the global optimum, as the compressor sequence’s overall compression ratio is continuously varied. Comparison of the optimum intervals for the constant heat capacity, and temperature-dependent heat capacity problems, demonstrates small differences in the identified globally optimum TAC intervals. The constant heat capacity model still slightly underestimates optimum variable intervals, and the difference of optimum intervals between the two models becomes larger when the optimal operation approaches the maximum workload.

If the allowable temperature operating range for a compressor is larger than the one considered in this case study \((T_0 = 298K, T_{\text{max}} = 438K)\) then the difference of the optima for the constant \(C_p\) and temperature-dependent \(C_p\) cases can be larger. Moreover, Theorem 2 requires that the heat capacity of the gas must increase as temperature increases, (which holds true for the methane gas considered in this work). If this assumption is violated, then new theoretical results must be developed.
5. Appendix

Proof of Theorem 1:

Similar to the previous study [19]:

\[
\begin{align*}
(\Leftarrow) & \text{ Let } C \leq \prod_{i=1}^{n} h(\eta_i, D). \text{ Define } W_i \triangleq \phi \eta_i, D, \forall i = 1,n, \text{ where } \phi \in [0,1] \text{ is such that } \\
& C = \prod_{i=1}^{n} h(\phi \eta_i, D). \text{ Such a } \phi \in [0,1] \text{ exists, since the function } \\
g : [0,1] \rightarrow \mathbb{R}, \quad g : \phi \rightarrow g(\phi) \triangleq C - \prod_{i=1}^{n} h(\phi \eta_i, D) \text{ is continuous on } [0,1], \text{ and has values }
\end{align*}
\]

\[
\begin{align*}
g(0) & \triangleq C - 1 > 0 \quad \wedge \quad g(1) \triangleq C - \prod_{i=1}^{n} h(\eta_i, D) \leq 0 . \text{ Then, the variable vector } \{W_i\}_{i=1}^{n} \triangleq \{\phi \eta_i, D\}_{i=1}^{n}, \text{ where }
\end{align*}
\]

\[
\phi \in [0,1] \text{ is such that } C = \prod_{i=1}^{n} h(\phi \eta_i, D), \text{ is a feasible point for } \nu .
\]

\[
(\Rightarrow) \text{ Let } \nu \text{ be feasible. }
\]

Since \( h : [0, D] \rightarrow [1, E], \quad h : W_i \rightarrow h(W_i) \quad \forall i = 1,n \text{ monotonically increases } \), for any feasible variable vector \( \{W_i\}_{i=1}^{n} \), it holds

\[
\begin{align*}
\left\{ \prod_{i=1}^{n} h(W_i) - C = 0 \quad \wedge \quad 0 \leq W_i \leq \eta_i, D, \quad i = 1,n \right\} \quad \Rightarrow \\
1 \leq \prod_{i=1}^{n} h(W_i) = C \leq \prod_{i=1}^{n} h(\eta_i, D) . \quad \text{O.E.A.}
\end{align*}
\]
**Proof of Theorem 2:**

First notice the two following functions:

\[
\begin{align*}
    h(W) &\triangleq \frac{dh(W)}{dW} = \frac{d\left( g\left( f^{-1}(W) \right) \right)}{dW} = \frac{dg(T)}{dT} \frac{d\left( f^{-1}(W) \right)}{dW} \\
    &= \frac{g(T) C_p(T)}{RT} \frac{R}{C_p(T)} = \frac{g(T)}{T} = \frac{h(W)}{f^{-1}(W)} \\
    \frac{d\left( h(cW)/h(W) \right)}{dW} &= \frac{d\left( h(cW) \right)}{d(cW)} \frac{1}{h(W)} - h(cW) \left( \frac{1}{h(W)} \right)^2 \frac{d\left( h(W) \right)}{dW} \\
    &= \frac{h(cW)}{f^{-1}(cW)} c \frac{1}{h(W)} - h(cW) \left( \frac{1}{h(W)} \right)^2 \frac{h(W)}{h(W)} = h(cW) \left( \frac{c}{f^{-1}(cW)} - \frac{1}{f^{-1}(W)} \right)
\end{align*}
\]

where \(0 < c < 1\).

Then introduce the following Lemma,

Let \(C_p : [T_0, T_{\text{max}}] \rightarrow \mathbb{R}^+\) be a non-decreasing function, and consider a constant \(c \in (0,1)\).

Then \(cT < T_c \ \forall T \in [T_0, T_{\text{max}}], \ \forall T_c \in [T_0, T_{\text{max}}]: \int_{T_c}^{T} \frac{C_p(T')}{R} dT' = c \int_{T_0}^{T} \frac{C_p(T')}{R} dT'\), and

\[
\frac{d\left( h(cW)/h(W) \right)}{dW} < 0 \ \forall W \in [0,D]
\]

Proof:

Consider the optimization problem

\[
\kappa \triangleq \inf_{T_c} T_c - cT
\]

s.t. \(\int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' - c \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' = 0\)

\(T_{\text{min}} - T \leq 0, \ T - T_{\text{max}} \leq 0\)

\(T_{\text{min}} - T_c \leq 0, \ T_c - T_{\text{max}} \leq 0\)
The above optimization problem has a feasible region that is a bounded and closed set, and all functions defining its objective function and constraints are differentiable. Therefore its minimum exists. It is easy to verify that all feasible points are regular, except for the point \((T, T_c) = (T_0, T_0)\), which is not regular, and has objective function value \(T_0 (1 - c) > 0\).

The problem’s Lagrangian is:

\[
L(T_c, T, \lambda, \mu_c, \mu, \nu_c, \nu) \leq \left[ T_c - cT + \lambda \left( \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' - c \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' \right) \right. \\
\left. + \mu_c (T_0 - T_c) + \mu (T_0 - T) + \nu_c (T_c - T_{\max}) + \nu (T - T_{\max}) \right]
\]

Then, all regular candidate global optimal points must satisfy the following first order necessary conditions for optimality:

\[
\begin{align*}
-c - \lambda cC_p(T) - \mu + \nu & = 0 \\
1 + \lambda C_p(T_c) - \mu_c + \nu_c & = 0 \\
\int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' - c \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' & = 0 \\
\mu(T_0 - T) & = 0, \quad \nu(T - T_{\max}) = 0 \\
\mu_c(T_0 - T_c) & = 0, \quad \nu_c(T_c - T_{\max}) = 0 \\
\mu & \geq 0, \quad \nu \geq 0, \quad \mu_c \geq 0, \quad \nu_c \geq 0 \\
T_0 - T & \leq 0, \quad T - T_{\max} \leq 0, \quad (T, T_c) \neq (T_0, T_0) \\
T_0 - T_c & \leq 0, \quad T_c - T_{\max} \leq 0,
\end{align*}
\]

\[
\begin{align*}
-c - \lambda cC_p(T) + \nu & = 0 \\
1 + \lambda C_p(T_c) + \nu_c & = 0 \\
\int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' - c \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' & = 0 \\
\mu & = 0, \quad \nu(T - T_{\max}) = 0 \\
\mu_c & = 0, \quad \nu_c(T_c - T_{\max}) = 0 \\
\nu & \geq 0, \quad \nu_c \geq 0 \\
T_0 - T & < 0, \quad T - T_{\max} \leq 0 \\
T_0 - T_c & < 0, \quad T_c - T_{\max} \leq 0,
\end{align*}
\]

\(\Leftrightarrow\)
\[
\begin{align*}
\nu &= c \left(1 - \frac{C_p(T) - C_p(T_c)}{C_p(T_c)}\right) \geq 0 \\
\lambda &= \frac{-1}{C_p(T_c)} \\
\int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' &= c \int_{T_0}^{T_{\text{max}}} \frac{C_p(T')}{R} dT' = 0 \\
\mu &= 0, \quad \mu_c = 0, \quad \nu_c = 0 \\
T_0 < T < T_{\text{max}} \\
T_0 < T_c < T_{\text{max}}
\end{align*}
\]

\[
\begin{align*}
1 - \frac{C_p(T) - C_p(T_c)}{C_p(T_c)} &= 0 \\
\lambda &= \frac{-1}{C_p(T_c)} \\
\int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' &= c \int_{T_0}^{T_{\text{max}}} \frac{C_p(T')}{R} dT' = 0 \\
\mu &= 0, \quad \nu = 0, \quad \mu_c = 0, \quad \nu_c = 0 \\
T_0 < T < T_{\text{max}} \\
T_0 < T_c < T_{\text{max}}
\end{align*}
\]

If \(C_p : [T_0, T_{\text{max}}] \to \mathbb{R}^+\) is a strictly increasing function, both cases above are infeasible.

If however \(C_p : [T_0, T_{\text{max}}] \to \mathbb{R}^+\) is constant in an interval with upper bound \(T_{\text{max}}\), and \(T_c\) belongs in that interval, then both cases may be feasible, and are equivalent to:

\[
\begin{align*}
\begin{align*}
C_p(T_c) &= C_p(T_{\text{max}}) \\
\lambda &= \frac{-1}{C_p(T_c)} \\
(1 - c) \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' &= c \int_{T_c}^{T_{\text{max}}} \frac{C_p(T')}{R} dT' \\
\mu &= 0, \quad \mu_c = 0, \quad \nu_c = 0 \\
T_0 < T = T_{\text{max}} \\
T_0 < T_c < T_{\text{max}}
\end{align*}
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
C_p(T_c) &= C_p(T) \\
\lambda &= \frac{-1}{C_p(T_c)} \\
(1 - c) \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' &= c \int_{T_c}^{T} \frac{C_p(T')}{R} dT' \\
\mu &= 0, \quad \nu = 0, \quad \mu_c = 0, \quad \nu_c = 0 \\
T_0 < T < T_{\text{max}} \\
T_0 < T_c < T_{\text{max}}
\end{align*}
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
C_p(T_c) &= C_p(T_{\text{max}}) \\
\lambda &= \frac{-1}{C_p(T_c)} \\
(1 - c) \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' &= c \frac{C_p(T_c)}{R} (T_{\text{max}} - T_c) \\
\mu &= 0, \quad \mu_c = 0, \quad \nu_c = 0 \\
T_0 < T = T_{\text{max}} \\
T_0 < T_c < T_{\text{max}}
\end{align*}
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
C_p(T_c) &= C_p(T) \\
\lambda &= \frac{-1}{C_p(T_c)} \\
(1 - c) \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' &= c \frac{C_p(T_c)}{R} (T - T_c) \\
\mu &= 0, \quad \nu = 0, \quad \mu_c = 0, \quad \nu_c = 0 \\
T_0 < T < T_{\text{max}} \\
T_0 < T_c < T_{\text{max}}
\end{align*}
\end{align*}
\]
Should either of these cases turn out to be feasible, it is shown next that the corresponding objective function value is greater than or equal to $T_0 (1-c) > 0$.

Indeed, for the first case it holds:

$$(1-c) \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' = c \frac{C_p(T_c)}{R} (T_{\text{max}} - T_c) \Rightarrow (1-c) \int_{T_0}^{T_c} \frac{C_p(T_c)}{R} dT' \geq c \frac{C_p(T_c)}{R} (T_{\text{max}} - T_c) \Rightarrow (1-c) (T_c - T_0) \geq c (T_{\text{max}} - T_c) \Rightarrow T_c - T_0 - cT_c + cT_0 \geq cT_{\text{max}} - cT_c \Rightarrow T_c - cT_{\text{max}} \geq T_0 (1-c) > 0$$

Similarly, for the second case it holds:

$$(1-c) \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' = c \frac{C_p(T_c)}{R} (T - T_c) \Rightarrow (1-c) \int_{T_0}^{T_c} \frac{C_p(T_c)}{R} dT' \geq c \frac{C_p(T_c)}{R} (T - T_c) \Rightarrow (1-c) (T_c - T_0) \geq c (T - T_c) \Rightarrow T_c - T_0 - cT_c + cT_0 \geq cT - cT_c \Rightarrow T_c - cT \geq T_0 (1-c) > 0$$

The above imply that the global optimum of the above optimization problem is the point $(T, T_c) = (T_0, T_0)$, which is not regular, with globally optimum objective function value $T_0 (1-c) > 0$. In turn this implies

$$T_c - cT > 0 \ \forall T \in [T_0, T_{\text{max}}], \ \forall T_c \in [T_0, T_{\text{max}}]: \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' = c \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT'$$

In turn this implies:

$$cT < T_c: \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' = c \int_{T_0}^{T_c} \frac{C_p(T')}{R} dT' \iff c f^{-1}(W) < f^{-1}(cW) \iff c \frac{1}{f^{-1}(cW)} < \frac{1}{f^{-1}(W)} \iff \frac{h(cW)}{h(W)} \left( \frac{c}{f^{-1}(cW)} - \frac{1}{f^{-1}(W)} \right) < 0 \iff \frac{d(h(cW)/h(W))}{dW} < 0 \ \text{O.E.Δ.}$$
a. We proceed by contradiction. Assume that \( \min_{i=1} \eta_i < \max_{i=1} \eta_i \). This implies that

\[
\exists (k,l) \in (S_{l^*}^{s*} \cup S_{l^*}^{s*}) \times S_{D}^{w*}: \eta_k > \eta_l.\]

This implies \( \eta_k \epsilon < W_k^* < \eta_k D \wedge W_l^* = \eta_l D \).

It is easy to verify that since \( \epsilon \in \left( 0, \frac{\min_{i=1} \eta_i}{\max_{i=1} \eta_i} \right) \); \( 1 = \frac{h(\eta, \epsilon)}{h(\eta, D)} < \frac{h(\eta, D)}{h(\eta, \epsilon)} \forall i = 1, n \): \( \eta_i > \eta_j \) and \( \eta_k > \eta_l \), it then holds

\[
\eta_k \epsilon \leq h^{-1}\left( \frac{h(\eta, \epsilon)h(\eta, D)}{h(\eta, D)} \right) < \eta_k D \iff h(\eta, \epsilon) \leq \frac{h(\eta, D)h(\eta_k, D)}{h(\eta, D)} < h(\eta, D) \iff \frac{h(\eta, \epsilon)}{h(\eta, D)} \leq \frac{h(\eta, D)}{h(\eta, D)} \wedge \epsilon < D \}, \text{ which is true.}
\]

Two cases are then considered [20]:

**a1:** \( \exists (k,l) \in (S_{l^*}^{s*} \cup S_{l^*}^{s*}) \times S_{D}^{w*}: \{ \eta_k > \eta_l \wedge \eta_l \epsilon < \eta_k \epsilon \leq h^{-1}\left( \frac{h(\eta, \epsilon)h(\eta_k, D)}{h(\eta, D)} \right) \leq W_k^* < \eta_k D \} \)

Consider the vector \( \{W_i^*\}_{i=1} \) where

\[
W_i^* \Delta W_i^* \forall i \in \{i = 1, n: i \neq k \wedge i \neq l\}, W_k^* \Delta \eta_k D, W_l^* \Delta h^{-1}\left( \frac{h(\eta, D)h(\eta_k, D)}{h(\eta, D)} \right)
\]

It is easy to verify that

\[
\prod_{i=1, i \neq k}^{n} h(W_i^*) - C = h(W_k^*)h(W_i^*)\prod_{i=1, i \neq k}^{n} h(W_i^*) - C = h(\eta_k D)h(W_k^*)h(\eta_k D)\prod_{i=1, i \neq k}^{n} h(W_i^*) - C = h(\eta_k D)h(W_k^*)\prod_{i=1, i \neq k}^{n} h(W_i^*) - C = \prod_{i=1}^{n} h(W_i^*) - C = 0
\]
\[ \eta, \varepsilon \leq W'_{i} \leq W_{i}^* \leq \eta_i D, \quad \forall i \in \{i = 1, n : i \neq k \land i \neq l\}, \eta, \varepsilon \leq W'_{k} \leq \eta_k D, \]

\[
\eta, \varepsilon \leq W'_{i} \leq h^{-1}\left(\frac{h(\eta_i D)h(W_{i}^*)}{h(\eta_i D)}\right) \leq \eta_i D \iff \frac{h(\eta, \varepsilon)h(\eta_i D)}{h(\eta_i D)} \leq h(W_{k}^*) \leq h(\eta_k D) \iff \]

\[
h^{-1}\left(\frac{h(\eta, \varepsilon)h(\eta_k D)}{h(\eta_i D)}\right) \leq W_{k}^* \leq \eta_k D, \quad \text{which is true based on the assumptions of case a1.}
\]

Thus all constraints of \( \nu_{\varepsilon} \) are satisfied and \( \{W'_{i}\}_{i=1} \) is a feasible point of \( \nu_{\varepsilon} \).

Let \( \Delta \) be defined as:

\[
\Delta \triangleq \left[ \sum_{i=1}^{n} \frac{1}{\eta_i} W_{i}^* + \frac{B}{A} \sum_{i=1}^{n} \left( \frac{1}{\eta_i} \right)^{a} (W_{i}^*)^{a} \right] - \left[ \sum_{i=1}^{n} \frac{1}{\eta_i} W'_{i} + \frac{B}{A} \sum_{i=1}^{n} \left( \frac{1}{\eta_i} \right)^{a} (W'_{i})^{a} \right]. \quad \text{Then } \Delta > 0 \iff 
\]

\[
\left[ \frac{1}{\eta_k} W_{k}^* + \frac{1}{\eta_l} W_{l}^* - \frac{1}{\eta_k} W'_{i} - \frac{1}{\eta_l} W'_{j} \right] + 
\]

\[
+ \frac{B}{A} \left( \frac{1}{(\eta_l)}^{a} (W_{k}^*)^{a} + \frac{1}{(\eta_l)}^{a} (W_{l}^*)^{a} - \left( \frac{1}{(\eta_l)} \right)^{a} (W'_{k})^{a} - \left( \frac{1}{(\eta_l)} \right)^{a} (W'_{l})^{a} \right) > 0 \iff 
\]

\[
\left[ \frac{1}{\eta_k} W_{k}^* + \frac{1}{\eta_l} \eta_i D - \frac{1}{\eta_k} \eta_i D - \frac{1}{h^{-1} \left( \frac{h(\eta_i D)h(W_{i}^*)}{h(\eta_i D)}\right)} \right] + 
\]

\[
+ \frac{B}{A} \left( \frac{1}{(\eta_l)}^{a} (W_{k}^*)^{a} + \frac{1}{(\eta_l)}^{a} (\eta_i D)^{a} - \left( \frac{1}{(\eta_l)} \right)^{a} (\eta_i D)^{a} - \left( \frac{1}{(\eta_l)} \right)^{a} h^{-1} \left( \frac{h(\eta_i D)h(W_{i}^*)}{h(\eta_i D)}\right)^{a} \right) \right] > 0 \iff 
\]

\[
\left[ \frac{1}{\eta_k} W_{k}^* - \frac{1}{h^{-1} \left( \frac{h(\eta_i D)h(W_{i}^*)}{h(\eta_i D)}\right)} \right] + 
\]

\[
+ \frac{B}{A} \left( \frac{1}{(\eta_l)}^{a} (W_{k}^*)^{a} - \left( \frac{1}{(\eta_l)} \right)^{a} h^{-1} \left( \frac{h(\eta_i D)h(W_{i}^*)}{h(\eta_i D)}\right)^{a} \right) > 0 \iff 
\]

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\[
\left[ \left( \frac{1}{\eta_k}W_k^* - \frac{1}{\eta_l}h^{-1}\left( \frac{h(\eta_lD)h(W_k^*)}{h(\eta_kD)} \right) \right) + \frac{B}{A} \left( \frac{1}{\eta_k} \right)^a - \left( \frac{1}{\eta_l} \right)^a \left( h^{-1}\left( \frac{h(\eta_lD)h(W_k^*)}{h(\eta_kD)} \right) \right)^a \right] > 0
\]

The above inequality is true, since

\[
\frac{1}{\eta_k}W_k^* - \frac{1}{\eta_l}h^{-1}\left( \frac{h(\eta_lD)h(W_k^*)}{h(\eta_kD)} \right) > 0 \iff h\left( \frac{\eta_l}{\eta_k}W_k^* \right) > h\left( \frac{\eta_l}{\eta_k}h(W_k^*) \right) \iff \frac{h\left( \frac{\eta_l}{\eta_k}W_k^* \right)}{h\left( \frac{\eta_l}{\eta_k}h(W_k^*) \right)} > \frac{h\left( \frac{\eta_l}{\eta_k}D \right)}{h(\eta_kD)}.
\]

However, application of the above Lemma for \( c = \frac{\eta_l}{\eta_k} \in (0,1) \) establishes that the above relation is true, since

\[
\frac{d}{dW} \left( \frac{h(cW)}{h(W)} \right) < 0 \quad \forall W \in [0, D], \quad \text{and} \quad \eta_k \leq W_k^* < \eta_k D. \quad \text{In turn this implies that}
\]

\( \{W_i^*\}_{i=1}^n \) is not the global optimum of \( \nu \epsilon \) which is a contradiction. Thus \( \min \eta_i \geq \max \eta_i \quad \text{O.E.A.} \)

**a2:** \( \exists (k, l) \in (S^* \cup S^*_\epsilon) \times S^* D : \left\{ \eta_k > \eta_l \land \eta_l \epsilon < \eta_k \epsilon \leq W_k^* < h^{-1}\left( \frac{h(\eta_kD)}{h(\eta_lD)} \right) \right\} < \eta_k D \)

Consider the vector \( \{W_i^*\}_{i=1}^n \) where

\[
W_i^* \triangleq W^*_i \quad \forall i \in \{i = 1, n \mid i \neq k \land i \neq l \}, \quad W_i^* \triangleq \eta_i \epsilon, \quad W_k^* \triangleq h^{-1}\left( \frac{h(\eta_kD)h(W_k^*)}{h(\eta_k \epsilon)} \right)
\]
It is easy to verify that

\[
\prod_{i=1}^{n} h(W_i') - C = h(W_k') \prod_{i=1 \atop i \neq k}^{n} h(W_i') - C = h(\eta, \varepsilon) \frac{h(\eta, D) h(W_k^*)}{h(\eta, D)} \prod_{i=1 \atop i \neq k}^{n} h(W_i^*) - C = 0
\]

\[\eta, \varepsilon \leq W_i' \leq \eta, D, \quad \forall i \in \{i = 1, n : i \neq k \land i \neq l\}, \eta, \varepsilon \triangleright W_i' \leq \eta, D,\]

\[\eta, \varepsilon \leq W_k' \triangleright h^{-1}\left(\frac{h(\eta, D) h(W_k^*)}{h(\eta, D)}\right) \leq \eta, D \iff h(\eta, \varepsilon) \leq \frac{h(\eta, D) h(W_k^*)}{h(\eta, D)} \leq h(\eta, D) \iff
\]

\[h(\eta, \varepsilon) \frac{h(\eta, D)}{h(\eta, D)} \leq h(W_k^*) \iff h(\eta, \varepsilon) \leq h(\eta, D) \iff h(\eta, \varepsilon) \frac{h(\eta, D)}{h(\eta, D)} \leq h(W_k^*) \iff
\]

\[\eta, \varepsilon \leq W_k^* \leq h^{-1}\left(\frac{h(\eta, D) h(\eta, D)}{h(\eta, D)}\right), \text{ which is true based on the assumptions of case a2.}
\]

Thus all constraints of \( \nu, \varepsilon \) are satisfied and \( \{W_i'\}_{i=1}^{n} \) is a feasible point of \( \nu, \varepsilon \).

Let \( \Delta \) be defined as:

\[
\Delta \triangleq \left[\sum_{i=1}^{n} \frac{1}{\eta_i} W_i' + \frac{B}{A} \sum_{i=1}^{n} \left(\frac{1}{\eta_i}\right)^a (W_i')^a\right] - \left[\sum_{i=1}^{n} \frac{1}{\eta_i} W_i' + \frac{B}{A} \sum_{i=1}^{n} \left(\frac{1}{\eta_i}\right)^a (W_i')^a\right]. \text{ Then } \Delta > 0 \iff
\]
The above inequality is true, since $W_k^* - \eta_k \epsilon \geq 0$, and

$$\eta_k D - h^{-1}\left(\frac{h(\eta_k D) h(W_k^*)}{h(\eta_k \epsilon)}\right) > 0 \Leftrightarrow h(\eta_k D) > \frac{h(\eta_k D) h(W_k^*)}{h(\eta_k \epsilon)} \Leftrightarrow h(W_k^*) < \frac{h(\eta_k \epsilon) h(\eta_k D)}{h(\eta_k D)}$$

This implies that $\{W_i^*\}_{i=1}^n$ is not the global optimum of $\nu \epsilon$ which is a contradiction. Thus $\min_{i \in S_D^*} \eta_i \geq \max_{i \in S_D^* \cup S_D^*} \eta_i$

O.E.D.

b. We proceed by contradiction. Assume that $\max_{i \in S_D^*} \eta_i > \min_{i \in S_D^* \cup S_D^*} \eta_i$. This implies that

$$\exists (k, l) \in (S_k^* \cup S_D^*) \times S_D^*: \eta_k < \eta_l.$$ This implies $\eta_k \epsilon < W_k^* \leq W_l^* = \eta_l \epsilon$
Consider the vector \( \{W_i^\eta\}_{i=1}^n \) where
\[
W_i' \triangleq W_i^* \quad \forall i \in \{i = 1, n : i \neq k \land i \neq l\}, \quad W_k' \triangleq \eta_l e, \quad W_l' \triangleq h^{-1}\left( \frac{h(W_k^*)h(\eta_l e)}{h(\eta_l e)} \right)
\]

It is easy to verify that
\[
\prod_{i=1}^{n} h(W_i') - C = h(W_k')\prod_{i=1}^{n} h(W_i') - C = h(\eta_l e)\frac{h(W_k^*)h(\eta_l e)}{h(\eta_l e)}\prod_{i=1}^{n} h(W_i^*) - C = h(W_k^*)\prod_{i=1}^{n} h(W_i^*) - C = \prod_{i=1}^{n} h(W_i^*) - C = 0
\]

In addition, \( \eta_l e \leq W_i' \leq \eta_l D, \quad \forall i \in \{i = 1, n : i \neq k \land i \neq l\} \), \( \eta_l e \triangleq W_k' \leq \eta_l D \),
\[
\eta_l e \leq W_i' \triangleq h^{-1}\left( \frac{h(W_k^*)h(\eta_l e)}{h(\eta_l e)} \right) \leq \eta_l D \Leftrightarrow h(\eta_l e) \leq \frac{h(W_k^*)h(\eta_l e)}{h(\eta_l e)} \leq h(\eta_l D) \Leftrightarrow \\
\eta_l e \leq W_k^* \leq h(\eta_l e) \Leftrightarrow h(\eta_l e) \leq h(W_k^*) \leq \frac{h(\eta_l D)}{h(\eta_l e)} \frac{h(\eta_l/\eta_l)}{h(\eta_l/\eta_l D)} h(\eta_l D). 
\]

However, based on the above Lemma, and since \( \frac{\eta_l}{\eta_l} \in (0, 1) \), \( 0 < \eta_l e < \eta_l D < D \), it holds that
\[
d\left( \frac{h(\eta_l W)}{h(W)} \right) < 0 \quad \forall W \in [0, D] \text{ implies } \frac{h(\eta_l/\eta_l e)}{h(\eta_l e)} > \frac{h(\eta_l/\eta_l D)}{h(\eta_l D)}. \text{ Thus,}
\]
\[ h(\eta, \varepsilon) \leq h(W^*_k) \leq \frac{h\left(\frac{\eta_k}{\eta}, \eta, \varepsilon\right)}{h(\eta, \varepsilon)} \frac{h(\eta, D)}{h\left(\frac{\eta_k}{\eta}, \eta, D\right)} h(\eta, D) \leq h(\eta, \varepsilon) \leq h(W^*_k) \leq h(\eta, D), \] which is true based on the assumptions of case b. Thus all constraints of \( \nu_\varepsilon \) are satisfied and \( \{W^n\}_{i=1}^n \) is a feasible point of \( \nu_\varepsilon \).

Let \( \Delta \) be defined as:

\[
\Delta \triangleq \left[ \sum_{i=1}^n \frac{1}{\eta_i} W^*_i + \frac{B}{A} \sum_{i=1}^n \left( \frac{1}{\eta_i} \right)^a \left( W^*_i \right)^a \right] - \left[ \sum_{i=1}^n \frac{1}{\eta_i} W'_i + \frac{B}{A} \sum_{i=1}^n \left( \frac{1}{\eta_i} \right)^a \left( W'_i \right)^a \right].
\]

Then \( \Delta > 0 \iff \]

\[
\left[ \frac{1}{\eta_k} W^*_k + \frac{1}{\eta_i} W^*_i - \frac{1}{\eta_k} W'_k - \frac{1}{\eta_i} W'_i \right] + \]

\[
\frac{B}{A} \left[ \left( \frac{1}{\eta_k} \right)^a \left( W^*_k \right)^a + \left( \frac{1}{\eta_i} \right)^a \left( W^*_i \right)^a - \left( \frac{1}{\eta_k} \right)^a \left( W'_k \right)^a - \left( \frac{1}{\eta_i} \right)^a \left( W'_i \right)^a \right] > 0 \iff \]

\[
\left[ \frac{1}{\eta_k} W^*_k + \frac{1}{\eta_i} \left( \eta_i \varepsilon \right) - \frac{1}{\eta_k} \left( \eta_i \varepsilon \right) - \frac{1}{\eta_i} h^{-1}\left( \frac{h(W^*_k)h(\eta_i \varepsilon)}{h(\eta_i \varepsilon)} \right) \right] + \]

\[
\frac{B}{A} \left[ \left( \frac{1}{\eta_k} \right)^a \left( W^*_k \right)^a + \left( \frac{1}{\eta_i} \right)^a \left( \eta_i \varepsilon \right)^a - \left( \frac{1}{\eta_k} \right)^a \left( \eta_i \varepsilon \right)^a - \left( \frac{1}{\eta_i} \right)^a \left( h^{-1}\left( \frac{h(W^*_k)h(\eta_i \varepsilon)}{h(\eta_i \varepsilon)} \right) \right)^a \right] > 0 \iff \]

\[
\left[ \frac{1}{\eta_k} W^*_k - h^{-1}\left( \frac{h(W^*_k)h(\eta_i \varepsilon)}{h(\eta_i \varepsilon)} \right) \right] + \]

\[
\frac{B}{A} \left[ \left( \frac{1}{\eta_k} \right)^a \left( W^*_k \right)^a - \left( \frac{1}{\eta_i} \right)^a \left( h^{-1}\left( \frac{h(W^*_k)h(\eta_i \varepsilon)}{h(\eta_i \varepsilon)} \right) \right)^a \right] > 0 \iff \]
\[
\frac{1}{\eta_k} W^*_k - \frac{1}{\eta_l} h^{-1}\left(\frac{h(W^*_l)h(\eta, \varepsilon)}{h(\eta, \varepsilon)}\right) > 0 \iff h\left(\frac{\eta_l}{\eta_k} W^*_k\right) > \frac{h(W^*_l)h(\eta, \varepsilon)}{h(\eta, \varepsilon)} \iff
\]

\[
\frac{h\left(\frac{\eta_k}{\eta_l} \eta_l \varepsilon\right)}{h(\eta, \varepsilon)} > \frac{h\left(\frac{\eta_k}{\eta_l} W^*_l\right)}{h(\eta, \varepsilon)} .
\]

However, based on the assumptions of case b, it holds:

\[
\frac{\eta_k}{\eta_l} \in (0,1), \left\{\frac{\eta_k}{\eta_l} \in (0,1) \land 0 < \eta_l \varepsilon = \frac{\eta_k}{\eta_l} \eta_l \varepsilon < \frac{\eta_k}{\eta_l} W^*_l \leq \frac{\eta_k}{\eta_l} D = \eta_l D < D \right\} \Rightarrow
\]

\[
\frac{h\left(\frac{\eta_k}{\eta_l} \eta_l \varepsilon\right)}{h(\eta, \varepsilon)} > \frac{h\left(\frac{\eta_k}{\eta_l} W^*_l\right)}{h(\eta, \varepsilon)} .
\]

In turn this implies that \(\{W^*_i\}_{i=1}^n\) is not the global optimum of \(\nu_\varepsilon\),

which is a contradiction. Thus \(\max_{i \in S^*_l} \eta_i \leq \min_{i \in S^*_l \cup S^*_r} \eta_i\) O.E.A.
Derivation of Constant Heat Capacity Model:

Start with the formulations [19] below:

\[
\nu = \min_{\{T_i\}_{i=1}^n} \sum_{i=1}^n \left[ \left( \frac{1}{\eta_i} \right)^a F_{\text{cap}} \left( \dot{n} \cdot R \right)^a \left( \int_{T_0}^{T_i} C_p \left( T' \right) \frac{dT'}{R} \right)^a + \right]
\]

\[
+ \frac{1}{\eta_i} \cdot \left[ \frac{1}{\eta_{p, c}} \left( \frac{1}{T_{e, out} - T_{e, in}} \right) \right] \left( \int_{T_0}^{T_i} C_p \left( T' \right) \frac{dT'}{R} \right)
\]

subject to:

\[
\left( \frac{P_n}{P_0} \right) = \prod_{i=1}^n \exp \left( \int_{T_0}^{T_i} C_p \left( T' \right) \frac{dT'}{RT'} \right)
\]

\[
\eta_i = \frac{T_i}{T_0} \frac{C_p \left( T' \right)}{R} \int_{T_0}^{T_i} \frac{C_p \left( T' \right)}{R} dT'
\]

\[
0 \leq \frac{T_i}{T_0} \frac{C_p \left( T' \right)}{R} dT' \leq \frac{1}{\eta_i} \frac{T_i}{T_0} \frac{C_p \left( T' \right)}{R} dT' \leq \frac{\tau_{w} C_p \left( T' \right)}{R} dT' < \infty
\]

\[
0 < T_0 \leq T_i \leq T_{max} < \infty, \quad i = 1, n
\]

Figure 16. TAC Optimization formulation. Source: [19] (Formula (20), p. 4138)

The optimization problem for the average heat capacity is

\[
\nu = A \cdot \min_{\{w_i\}_{i=1}^n} \sum_{i=1}^n \left[ \left( \frac{1}{\eta_i} \right)^a \frac{B(W_i)}{A} + \eta_i W_i \right]
\]

subject to:

\[
\prod_{i=1}^n h(W_i) - C = 0
\]

\[
0 \leq W_i \leq \eta_i D \quad i = 1, n
\]

where,
\[ A \triangleq \hat{n} \cdot R \left( C_{\text{oper.}} \frac{E}{C_{\text{compr.}}} + \frac{E}{C_{\text{c,compr.}}} \right) \geq 0, \quad B \triangleq F C_{\text{cap.}}^{\text{oper.}} (R)^{a} \geq 0, \quad C \triangleq \left( \frac{P_{n}}{P_{o}} \right)^{2} > 1, \]

\[ D \triangleq f(T_{\text{max}}) = \frac{C_{\text{p}}}{R} (T_{\text{max}} - T_{0}) > 0, \quad E \triangleq g(T_{\text{max}}) \triangleq \exp \left( \frac{C_{\text{p}}}{R} \ln \left( \frac{T_{\text{max}}}{T_{0}} \right) \right), \]

\[ W_{i} \triangleq f(T_{i}^{*}) = \frac{C_{\text{p}}}{R} (T_{i}^{*} - T_{0}) \quad \forall i = 1, n, \quad \text{so} \quad T_{i}^{*} = f^{-1}(W_{i}) \quad \forall i = 1, n, \]

\[ h(W_{i}) \triangleq g(f^{-1}(W_{i})) \triangleq g(T_{i}^{*}) \triangleq \exp \left( \frac{C_{\text{p}}}{R} \ln \left( \frac{T_{i}^{*}}{T_{0}} \right) \right) \quad \forall i = 1, n, \]

The constant heat capacity of gases in the ideal gas state can be estimated:

\[ \frac{C_{\text{p}}}{R} = \frac{1}{T_{\text{max}} - T_{0}} \int_{T_{0}}^{T_{\text{max}}} \left( c_{0} + c_{1}T' + c_{2}T'^{2} + c_{3}T'^{3} \right) dT', \]

Define domains and ranges for the functions above:

\[ f : [T_{0}, T_{\text{max}}] \rightarrow [0, D], \quad f : T_{i}^{*} \rightarrow W_{i} = f(T_{i}^{*}) \triangleq \frac{C_{\text{p}}}{R} (T_{i}^{*} - T_{0}) \quad \forall i = 1, n, \]

\[ f^{-1} : [0, D] \rightarrow [T_{0}, T_{\text{max}}], \quad f : W_{i} \rightarrow T_{i}^{*} \triangleq f^{-1}(W_{i}) : \quad W_{i} = \frac{C_{\text{p}}}{R} (T_{i}^{*} - T_{0}) \quad \forall i = 1, n \]

\[ g : [T_{0}, T_{\text{max}}] \rightarrow [1, E], \quad g : T_{i}^{*} \rightarrow g(T_{i}^{*}) \triangleq \exp \left( \frac{C_{\text{p}}}{R} \ln \left( \frac{T_{i}^{*}}{T_{0}} \right) \right) \quad \forall i = 1, n \]

\[ h : [0, D] \rightarrow [1, E], \quad h : W_{i} \rightarrow h(W_{i}) \triangleq g(f^{-1}(W_{i})) \triangleq g(T_{i}^{*}) = \exp \left( \frac{C_{\text{p}}}{R} \ln \left( \frac{T_{i}^{*}}{T_{0}} \right) \right) \quad \forall i = 1, n \]

Since Theorem 1 and Theorem 2 are generally applicable to the constant heat capacity model, so continue to develop the optimal conditions for the constant heat capacity model by applying Lagrangian Transformation.
\[ L(W, \lambda, \mu, \omega) = \sum_{i=1}^{n} \left[ \left( \frac{1}{\eta_i} \right)^a \frac{B}{A} (W_i)^a + \frac{1}{\eta_i} W_i \right] + \lambda \left( \prod_{i=1}^{n} h(W_i) - C \right) + \sum_{i=1}^{n} \mu_i (\eta_i e - W_i) + \sum_{i=1}^{n} \omega_i (W_i - \eta_i D) \]

\[
\frac{\partial L(W, \lambda, \mu, \nu)}{\partial W_k} = \frac{1}{\eta_k} + \frac{1}{\eta_k} \frac{a B}{A} (W_k)^{a-1} + \frac{1}{\eta_k} \frac{\partial h(W_k)}{\partial W_k} \prod_{i=1, i \neq k}^{n} h(W_i) = 0 \quad \forall k = 1, n
\]

\[
\prod_{i=1}^{n} h(W_i) - C = 0, \quad \eta_i e \leq W_i \leq \eta_i D \quad \forall i = 1, n
\]

\[
\mu_i \geq 0, \quad \mu_i (\eta_i e - W_i), \quad \omega_i \geq 0, \quad \omega_i (W_i - \eta_i D) \quad \forall i = 1, n
\]

\[
\begin{align*}
\left( \frac{1}{\eta_k} \right)^a & \frac{a B}{A} (W_k)^{a-1} + \frac{1}{\eta_k} + \lambda \frac{\partial h(W_k)}{\partial W_k} \prod_{i=1, i \neq k}^{n} h(W_i) = 0 \quad \forall k \in S_i^W \\
\mu_i = \left( \frac{1}{\eta_i} \right)^a & \frac{a B}{A} (W_i)^{a-1} + \frac{1}{\eta_i} + \lambda \frac{\partial h(W_i)}{\partial W_i} \prod_{i=1, i \neq l}^{n} h(W_i) \geq 0 \quad \forall l \in S_{\xi}^W \\
\omega_m = \left( \frac{1}{\eta_m} \right)^a & \frac{a B}{A} (W_m)^{a-1} - \frac{1}{\eta_m} - \lambda \frac{\partial h(W_m)}{\partial W_m} \prod_{i=1, i \neq m}^{n} h(W_i) \geq 0 \quad \forall m \in S_{\eta D}^W \\
\prod_{r \in S_{\xi}^W} h(\eta_r \xi) \prod_{s \in S_{\xi}^W} h(\eta_s D) - C = 0,
\end{align*}
\]

\[
0 < \eta_k e < W_k < \eta_k D \quad \forall k \in S_i^W,
\]

\[
\mu_i \geq 0 \quad W_i = \eta_i e \quad \forall l \in S_{\xi}^W, \quad \omega_m \geq 0 \quad W_m = \eta_m D \quad \forall m \in S_{\eta D}^W
\]

Since \( W_i \triangleq f(T_i^*) = \frac{C}{R} (T_i^* - T_0) \quad \forall i = 1, n \), \( h(W_i) \triangleq \exp \left( \frac{C}{R} \ln \left( \frac{T_i^*}{T_0} \right) \right) \quad \forall i = 1, n \), so
\[
\frac{\partial h(W_i)}{\partial w_i} = \frac{\partial h(W_i)}{\partial T_i} \frac{\partial T_i}{\partial w_i} = \frac{\partial h(W_i)}{\partial T_i} \frac{1}{\partial T_i} = \frac{\partial}{\partial T_i} \left( \exp \left( \frac{C_p}{R} \ln \left( \frac{T_i}{T_0} \right) \right) \right) \frac{1}{\partial T_i} \left( \exp \left( \frac{C_p}{R} \ln \left( \frac{T_i}{T_0} \right) \right) \right) \\
= \exp \left( \frac{C_p}{R} \ln \left( \frac{T_i}{T_0} \right) \right) \frac{C_p}{R T_i} \frac{1}{T_i} \exp \left( \frac{C_p}{R} \ln \left( \frac{T_i}{T_0} \right) \right) = \frac{h(W_i)}{T_i} \neq 0 \quad \forall i = 1, n,
\]

\[
\mu_i = \left( \frac{1}{\eta_i} \right)^a \frac{\alpha B}{A} \left( W_i \right)^{a-1} \frac{1}{\eta_i} \right) h(W_i) + \lambda C \frac{h(W_i)}{f^{-1}(W_i)} \geq 0 \quad \forall l \in S^w_x \\
\omega_m = -\left( \frac{1}{\eta_m} \right)^a \frac{\alpha B}{A} \left( W_m \right)^{a-1} \frac{1}{\eta_m} \right) h(W_m) - \lambda C \frac{h(W_m)}{f^{-1}(W_m)} \geq 0 \quad \forall m \in S^w_D \\
\prod_{i \in S_i^w} h(\eta_i \varepsilon) \prod_{k \in S_i^w} h(\eta_i D) - C = 0, \\
0 < \eta_k \varepsilon < W_k < \eta_k D \quad \forall k \in S_i^w, \\
\mu_i \geq 0 \quad w_i = \eta_i \varepsilon \quad \forall l \in S^w_x, \quad \omega_m \geq 0 \quad W_m = \eta_m D \quad \forall m \in S^w_D
\]

\[
\mu_i = \left( \frac{1}{\eta_i} \right)^a \frac{\alpha B}{A} \left( W_i \right)^{a-1} \frac{1}{\eta_i} \right) f^{-1}(W_i) + \lambda C = 0 \quad \forall k \in S_i^w \\
\frac{\mu_i}{h(W_i)} = \left( \frac{1}{\eta_i} \right)^a \frac{\alpha B}{A} \left( W_i \right)^{a-1} \frac{1}{\eta_i} \right) f^{-1}(W_i) + \lambda C \geq 0 \quad \forall l \in S^w_x \\
\frac{\omega_m}{h(W_m)} = -\left( \frac{1}{\eta_m} \right)^a \frac{\alpha B}{A} \left( W_m \right)^{a-1} \frac{1}{\eta_m} \right) f^{-1}(W_m) - \lambda C \geq 0 \quad \forall m \in S^w_D \\
\prod_{i \in S_i^w} h(\eta_i \varepsilon) \prod_{k \in S_i^w} h(\eta_i D) - C = 0, \\
0 < \eta_k \varepsilon < W_k < \eta_k D \quad \forall k \in S_i^w, \\
\mu_i \geq 0 \quad W_i = \eta_i \varepsilon \quad \forall l \in S^w_x, \quad \omega_m \geq 0 \quad W_m = \eta_m D \quad \forall m \in S^w_D
\]
The optimal conditions for the constant heat capacity model are:

\[
\left\{ \left( \frac{1}{\eta_k} \right)^{a B} \frac{A}{a} \left( W_k \right)^{a - 1} + \frac{1}{\eta_k} \right\} f^{-1} \left( W_k \right) + \lambda C = 0 \quad \forall k \in S^w_I
\]

\[
\frac{\mu_i}{h(\eta_i)} = \left( \frac{a B}{A} \frac{1}{\eta_i} \right)^{a - 1} + \frac{1}{\eta_i} f^{-1} \left( \eta_i \right) + \lambda C \geq 0 \quad \forall l \in S^w_e
\]

\[
\frac{\omega_k}{h(\eta_m D)} = - \left( \frac{a B}{A} \frac{1}{\eta_m} \right)^{a - 1} + \frac{1}{\eta_m} f^{-1} \left( \eta_m D \right) - \lambda C \geq 0 \quad \forall m \in S^w_D
\]

\[
\prod_{r \in S^w_e} h(\eta_r) \prod_{s \in S^w_i} h(\eta_s) \prod_{l \in S^w_s} h(\eta_l) - C = 0,
\]

\[
0 < \eta_k \in W_k < \eta_k D \quad \forall k \in S^w_I,
\]

\[
\mu_i \geq 0 \quad W_i = \eta_l \quad \forall l \in S^w_e, \quad \omega_m \geq 0 \quad W_m = \eta_m D \quad \forall m \in S^w_D
\]
REFERENCES


