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Prediction of spatially variable unsaturated hydraulic conductivity using scaled particle-size distribution functions

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[1] Simultaneous scaling of soil water retention and hydraulic conductivity functions provides an effective means to characterize the heterogeneity and spatial variability of soil hydraulic properties in a given study area. The statistical significance of this approach largely depends on the number of soil samples collected. Unfortunately, direct measurement of the soil hydraulic functions is tedious, expensive and time consuming. Here we present a simple and cost-effective hybrid scaling approach that combines the use of ancillary information (e.g., particle-size distribution and soil bulk density) with direct measurements of saturated soil water content and saturated hydraulic conductivity. Our results demonstrate that the presented approach requires far fewer laboratory measurements than conventional scaling methods to adequately capture the spatial variability of soil hydraulic properties.


1. Introduction

[2] The predictive ability and utility of large-scale distributed hydrologic models heavily relies on a detailed description of the spatial variability of the soil hydraulic properties, namely the water retention function (WRF) and the unsaturated hydraulic conductivity function (HCF). The WRF relates the volumetric water content \( \theta \) (\( L^3 L^{-3} \)) to the corresponding soil water matric head \( h \) (\( L \)). The HCF relates the unsaturated hydraulic conductivity \( K \) (\( L T^{-1} \)) to the soil matric head or water content. The spatial distribution of WRF and HCF can be characterized with effective soil hydraulic parameters [Zhu and Mohanty, 2002; Zhu and Mohanty, 2003; Vrugt et al., 2004; Mohanty and Zhu, 2007]. Although this approach provides a reasonable description of the lumped behavior of the soil system, it does not accurately characterize the inherent spatially distributed soil water fluxes.

[3] In the past decades, the scaling approach has been developed as an effective method to describe the spatial variability of the soil hydraulic properties in a given study area [Warrick and Nielsen, 1980; Tillotson and Nielsen, 1984; Clausnitzer et al., 1992]. This approach was introduced in soil physics by the seminal paper of Miller and Miller [1956] and is based on the assumption that geometrically similar porous media differ only by a characteristic length of their respective pore-size distributions, which in turn provide a good proxy for the soil hydraulic properties. Spatial variability is defined through the statistical distribution of scaling factors that relate the soil hydraulic functions sampled in different locations to the corresponding average hydraulic properties of a reference porous medium (representative for the study area). This geometric scaling approach has been successfully applied for Kosugi’s soil hydraulic functions [Kosugi, 1996], assuming that scaling factors are lognormally distributed in the sampled soil domain. This includes individual scaling of soil WRFs only [Kosugi and Hopmans, 1998; Hayashi et al., 2009] or simultaneous scaling of soil water retention and unsaturated HCFs [Hendrayanto et al., 2000; Titi et al., 2001].

[4] Notwithstanding this progress made, the inherent spatial variability of soils necessitates a very large number of samples to accurately characterize the hydraulic properties of the considered spatial domain [Hopmans et al., 2002a; Minasny and McBratney, 2007; Verbeeck et al., 2007]. Unfortunately, direct measurement of the soil hydraulic properties (hereafter referred to as primary data or PD) is time consuming and tedious [Hopmans et al., 2002b]. Alternatively, pedotransfer functions (PTFs) can be used to indirectly estimate the soil hydraulic functions from soil physical and chemical data (hereafter defined as secondary data or SD) that are more readily available because of their ease in collection and simplicity in measurement [Haverkamp et al., 2002]. Most PTFs estimate the soil hydraulic functions through mathematical and statistical relations by using oven-dry bulk density, organic carbon content, and soil texture [Pachepsky and Rawls, 2004]. We emphasize that PTFs necessitate empirical
calibration procedures, and their validity is restricted only for the tested database [Chirico et al., 2007]. The majority of PTFs presented in the literature use simple nonlinear regression equations to relate the hydraulic properties to basic soil texture and structure data [Tietje and Tapkenhunrichs, 1993; Weynants et al., 2009; Vereecken et al., 2011]. Such equations are easy to implement and are often favored over more complex fitting methods, such as artificial neural networks [Schaap and Bouten, 1996; Schaap et al., 1998].

[5] Although PTFs may provide good estimates of the soil hydraulic properties [Romano, 2004], they depend on empirical relationships that do not offer physical understanding in the interpretation of their flaws [Haverkamp et al., 2002]. Alternatively, physical-empirical PTFs rely on pore-scale physical relationships that are adjusted through empirical calibrations [Chan and Govindaraju, 2004; Hwang and Choi, 2006; Nimmo et al., 2007]. Specifically, Arya and Paris [1981] proposed a method (hereinafter defined as AP method) to estimate the soil pore-size distribution by using the soil particle-size distribution (PSD) and oven-dry bulk density. The AP method postulates the fundamental assumption of shape similarity between WRF and PSD. The soil is described as an ideal porous medium with modeled pore spaces created by a cubic packing of uniform spheres. The actual soil matrix geometry is adjusted by an empirical parameter \( \alpha \) that corrects for the differences in shape and spatial arrangement of solid particles between hypothetical and actual porous medium [Arya et al., 1999a, 2008]. Although Arya and Paris [1981] assumed a constant value of 1.38 for coarse porous media, others proposed to vary \( \alpha \) as a function of matric head [Basile and D'Urso, 1997], particle size [Buczko and Gerke, 2005], or water content [Vaz et al., 2005] of the soil. The AP method has subsequently been upgraded by adding the prediction of unsaturated hydraulic conductivity as well [Arya et al., 1999b; Chaudhari and Batta, 2003; Hwang and Hong, 2006; Blank et al., 2007; Arya and Heitman, 2010]. More recently, Nasta et al. [2009] proposed a geometric scaling method of the WRFs estimated with the AP method. This work relaxes experimental effort as the calibration of the method of the WRFs estimated with the AP method. This recently, 2006; Arya and Paris, 2001]. Section 2.2 then proceeds with the introduction of our hybrid simultaneous scaling method that uses measurements of the PSD and bulk density (SD) jointly with data of the saturated water content \( \theta_s \) and saturated hydraulic conductivity \( K_s \) (PD) to estimate the reference soil WRF (REF-WRFSD) and HCF (REF-HCFSD), respectively, and the corresponding scaling factors \( \delta_{SD} \). The experimental data are subsequently described in section 3. In section 4, the results of our new hybrid scaling approach are compared against those of the conventional scaling method. Finally, section 5 concludes with a summary of the analysis presented herein.

2. Theory

2.1. Simultaneous Scaling Using PD

[8] Assuming soil to be represented by a randomly distributed bundle of intersecting cylindrical capillary tubes (series-parallel statistical model) with a probability density function (pdf) of soil pore radii \( p(r) \) \( (L^{-1}) \), the volumetric water content of the water-filled pores of radii \( r \rightarrow r+dr \) is computed from \( dh(r) = p(r)dr \) \( (L^{-1} L) \) [Brutsaert, 2000]. Kosugi [1996] proposed to describe the pdf\( (r) \) function using a log-normal probability law with mean \( \ln r_w \) (\( \ln \) indicates the natural logarithm) and standard deviation (STD) \( \sigma \):

\[
\text{pdf}(\ln r) = \frac{1}{r \sigma \sqrt{2\pi}} \exp \left( -\frac{(\ln r - \ln r_w)^2}{2\sigma^2} \right). \tag{1}
\]

where \( \theta_1 \) \((L^3 \ L^{-3})\) and \( \theta_2 \) \((L^3 \ L^{-3})\) represent the saturated and residual soil water content, respectively. The soil water matric head \( h \) \((L)\) is assumed to be proportional to the inverse of the equivalent pore radius \( r \) \((L)\) through the Young-Laplace capillary equation \((i.e., r \approx 0.149/h \text{ if } r \text{ and } h \text{ are expressed in units of centimeters})\). The soil WRF \( S_w(\ln h) \) is described as follows:

\[
S_w(\ln h) = \theta_1 - \theta_2 \frac{1}{2} \text{erfc} \left( \frac{(\ln h - \ln h_w)}{\sigma \sqrt{2}} \right), \tag{2}
\]

where \text{erfc} \( \text{erfc} \) represents the complementary error function, \( h_w \) and \( \sigma \) denote the mean and STD of \( \ln h \), respectively, and \( S_w \) represents the degree of saturation that varies from 0 (when \( \theta = \theta_1 \)) to 1 (when \( \theta = \theta_2 \)). The \( h_w \) value is the median matric head corresponding to \( S_w = 0.5 \) [Kosugi, 1996; Kosugi et al., 2002].

[9] According to Mualem [1976], the relative hydraulic conductivity \( K_r(S_w) \) is described as

\[
K_r(S_w) = \frac{K_r(S_w)}{K_r} = T(S_w)G(S_w) \left( \int_{S_w}^{1} \frac{1}{h} \frac{1}{dS} \right)^2 = S_w^2 \left( \int_{S_w}^{1} \frac{1}{h} \frac{1}{dS} \right)^2, \tag{3}
\]

where \( K_r \) \((L \ T^{-1})\) denotes the saturated hydraulic conductivity, \( T(S_w) \) is a tortuosity factor that corrects for the effective path length of the water-conducting noncylindrical tubes, and \( G(S_w) \) accounts for the connectivity among the water-conducting pores [Mualem, 1976; Touma, 2009; Lebeau and Konrad, 2010]. Note that the relative hydraulic conductivity \( K_r \) varies between 0 (when \( \theta = \theta_1 \) or \( S_w = 0 \)) and 1 (when \( \theta = \theta_2 \) or \( S_w = 1 \)).

[10] The power of 2 in equation (3) originates from the assumption that the pore configuration can be replaced by a pair of capillary elements whose lengths are proportional to
their respective radii. Mualem [1976] simplified the relation in equation (3) by combining $T(S_r)$ and $G(S_r)$ into a single factor $S_r$, where the exponent $f$ enables the tortuosity connectivity to change as a function of soil water content. The value of the power $l$ depends on the specific soil-fluid properties and varies considerably among soils [Peters et al., 2011]. Based on a data set of 45 soils, Mualem [1976] proposed an optimal value of $l = 0.5$. However, Leij et al. [1997] obtained a mean value of $l = -0.72$ using another data set of 401 soils.

[11] Alternatively, Assouline [2001, 2005] proposed a different analytical relationship for the unsaturated HCF that incorporates the tortuosity and connectivity factors into a single power value $\eta$:

$$K_r(S_r) = \left( \int_0^{x_r} \frac{h}{L} dS_r \right)^{-\eta}. \tag{4}$$  

with parameter $\eta$ hereafter referred to as the conductivity parameter.

[12] If we now insert Kosugi’s water retention relation (i.e., equation (2)) into equation (4), we obtain the following expression for the relative HCF:

$$K_r(S_r) = \left\{ \frac{1}{2} \text{erfc} \left[ \text{erfc}^{-1}(2S_r) + \frac{\sigma}{\sqrt{2}} \right] \right\}^\eta,$$  

where $\eta$ is the arithmetic mean values of all $\ln h_m$, $\sigma$, and $\eta$ values, respectively.

[13] The scaling approach provides a convenient way to coalesce a group of soil hydraulic properties onto a single reference hydraulic function [Kosugi and Hopmans, 1998; Tuli et al., 2001]. The reference soil WRF (REF-WRF) $S_r(ln h)$ and the reference relative HCF (REF-HCF) $K_r(S_r)$ are described by the following two parametric relations:

$$S_r(ln h) = \frac{1}{2} \text{erfc} \left[ \frac{(ln h - \ln \bar{h}_m)}{\sigma \sqrt{2}} \right] \tag{6}$$

and

$$K_r(S_r) = \left\{ \frac{1}{2} \text{erfc} \left[ \text{erfc}^{-1}(2S_r) + \frac{\sigma}{\sqrt{2}} \right] \right\}^\eta,$$  

where $\ln \bar{h}_m$, $\sigma$, and $\eta$ are the arithmetic mean values of all $\ln h_m$, $\sigma$, and $\eta$ values, respectively.

[14] The geometric similarity of Miller and Miller [1956] dictates the median pore radius, $r_m,j$, as the characteristic length of the pore-size distribution of each soil sample $j$. The scaling approach identifies a representative pore-size distribution for the entire soil domain with corresponding mean value of $\ln r_m,j$. Provided that $r_m,j$ and $h_m,j$ (as well as $\ln r_m$ and $\ln h_m$) are linked by the capillary law, the scaling factors $\delta_j$ are defined as follows [Miller and Miller, 1956; Kosugi and Hopmans, 1998]:

$$\delta_j = \frac{r_m,j}{r_m} = \frac{h_m}{h_m,j}. \tag{8}$$

[15] The scaling factors quantify the relative deviation (RD) of the $r_m,j$ values pertaining to different locations with respect to the $r_m$ value referred to the study area. The resulting distribution of the scaling factors will be assumed lognormal with the constraint that the mean value ($\mu$) of the log-transformed scaling factors is null. $\mu(\ln \delta_j) = 0$ [Kosugi and Hopmans., 1998]. Therefore, the geometric scaling describes the spatial variability of soil hydraulic properties through the reference hydraulic functions and the statistical distribution of scaling factors. A key advantage of this method is that one can directly draw scaling factors from the lognormal pdf to rapidly characterize the spatial variability of the soil hydraulic functions in large spatial domains [Nasta et al., 2013].

[16] The scaled soil water retention curves can now be expressed as

$$S_r[\ln (\delta h)] = \frac{1}{2} \text{erfc} \left[ \frac{\ln (\delta h) - \ln \bar{h}_m}{\sigma \sqrt{2}} \right], \tag{9a}$$

and the scaled unsaturated HCFs [Miller and Miller, 1956; Tuli et al., 2001] as

$$\delta_j^{-2}K_r(S_r) = \left\{ \frac{1}{2} \text{erfc} \left[ \text{erfc}^{-1}(2S_r) + \frac{\sigma}{\sqrt{2}} \right] \right\}^\eta.$$  

[17] The parameters $h_m,j$, $\sigma_j$, $\theta_j$, and $\eta_j$ (hereafter indicated with the subscript PD) have to be estimated directly from the PD (water retention and hydraulic conductivity data points). While parameters $\theta_j$ and $K_r$ are easily measured, SD help in alleviating the demanding task of measuring the unsaturated hydraulic properties. The next section will introduce the main elements of a hybrid scaling approach that combines primary and SD to describe the spatial variability of the WRF and HCF.

### 2.2. Simultaneous Scaling Using SD

[18] We assume that the PSD function is described by Buchan’s equation (Buchan, 1989), Bah et al. [2009], Table 2):

$$W(ln D) = \frac{1}{2} \text{erfc} \left[ \frac{(ln D_m - ln D)}{\sigma_{PSD} \sqrt{2}} \right], \tag{10}$$

where $W(M^{-1})$ defines the solid mass fraction associated with particles of diameter size $D (L)$, and $ln D_m$ and $\sigma_{PSD}$ are the mean and STD of the log-transformed soil particle diameters, respectively. Note that equation (10) has an analytical form similar to Kosugi’s WRF, as defined by equation (2). The number of solid particles $n_i (M^{-1})$ for each soil particle diameter class $i$ is calculated from

$$n_i = \frac{W_i}{\pi D_i^2 \rho_s}, \tag{11}$$

where $\rho_s (M L^{-3})$ represents the soil particle density, which hereafter is assumed to be equal to 2.65 $g cm^{-3}$. The pore radius $r_i$ associated with the particle diameter $D_i$ is calculated as follows [Arya and Paris, 1981]:

$$r_i = \frac{D_i}{2} \sqrt{\frac{2en_i^{(1-\alpha)}}{3}}.$$

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where $\alpha$ is an empirical parameter and $e$ ($L^3 L^{-3}$) denotes the void ratio defined as

$$e = \frac{\rho_a - \rho_b}{\rho_b},$$

(12b)

where $\rho_b$ ($M L^{-3}$) denotes the oven-dry bulk density. It is important to point out that $\alpha$ maps the hypothetical simple geometry of soil particles into the actual geometry so that the particle diameter $D$ can be converted to an equivalent pore radius $r$.

[19] The AP method is based on the assumption of shape similarity between PSD and WRF; hence, the $W_i$ classes associated to the diameter class $D$ are supposed equivalent to the $S_{r,j}$ classes correspondent to the pore radius class $r_j$. The estimated retention data pairs $S_{r,j} - h_j$ (with $h_j$ values calculated from $r_j$ in equation (12a) using the capillary law) then serve as input for the scaling of the SD. To this end, parameters $h_{m,SD,j}$ and $\sigma_{SD,j}$ of equation (2) will be optimized for each soil sample $j$, and their mean values $\ln h_{m,SD}$ and $\sigma_{SD}$ will be inserted in equation (6) to obtain the reference WRF, REF-WRF$_{SD}$. The HCFs for each soil sample $j$ are derived from equation (5), while the reference HCF REF-HCF$_{SD}$ and the scaling factors $b_{SD}$ are calculated using equations (7) and (8), respectively. Note that the parameters $\eta$ and $\alpha$ are unknown a priori for the scaling of the SD and an empirical calibration procedure is required. This step will be described in section 4.2.1.

### 3. Materials and Methods

[20] The experimental area is located to the east of the city of Naples, Italy [Romano, 1993; Ciollaro and Romano, 1995]. The volcanic soil (Andosol) originated from eruptions of the Vesuvius volcano. Soil physical and hydraulic characteristics were determined from 84 undisturbed soil cores (12 cm length, 8.6 cm diameter) that were collected from the soil surface (about 10–25 cm soil depth) along a 126 m long transect with equal intervals of 1.50 m. Before conducting the soil hydraulic measurements in the laboratory, the top 2.5 cm slice of each soil core was removed for soil texture and pressure plate extraction measurements. Disturbed soil samples were repacked and placed in a pressure plate extractor to determine equilibrium soil water retention points at $-3000$, $-6000$, and $-12000$ cm. The portion of the disturbed samples was analyzed for determining the soil PSD [Gee and Or, 2002]. The sieve analysis method was used to determine the sand content with five diameter classes for $0.005 \text{ cm} < D_i < 0.2 \text{ cm}$, whereas the hydrometer method was used to determine the distribution of the silt and clay fractions with six diameter classes for $D_i < 0.005 \text{ cm}$. Buchan’s parametric relation of equation (10) was fitted to each of the 84 experimental $D_{i,j}$ ($\text{cm}$)-$W_{i,j}$ ($\text{g} \text{ cm}^{-2}$) data pairs using the MATLAB Optimization Toolbox (The MathWorks, Inc.) [Coleman and Li, 1996]. This procedure results in optimized values for $\ln D_{m,j}$ and $\sigma_{PSD,j}$. Table 1 summarizes the mean ($\mu$), STD, coefficient of variation (CV), minimum (min) and maximum (max) values of the soil texture parameters, oven-dry bulk density, oven-dry bulk density, $\rho_b$ ($\text{g} \text{ cm}^{-3}$), and soil porosity, $\varepsilon$ ($\text{cm}^3 \text{ cm}^{-3}$). Note that these soil properties exhibit a rather low variability (see CV values in Table 1). Figure 1 presents the measured PSD functions with the average sand, silt, and clay contents of 44.4%, 44.5%, and 11.1%, respectively, and corresponding STDs of 2.2 (sand), 1.6 (silt), and 2.2 (clay).

[22] The undisturbed soil core was slowly wetted from the bottom, until reaching its complete saturation. The saturated water content $\theta_s$ ($\text{cm}^3 \text{ cm}^{-3}$) was measured by the gravimetric method [Topp and Ferré, 2002], whereas the saturated hydraulic conductivity $K_s$ ($\text{cm} \text{ h}^{-1}$) was determined by the falling-head method [Reynolds and Elrick, 2002]. For each soil core, the evaporation method enabled the simultaneous determination of the soil water retention $\theta$ ($\text{h}$) and hydraulic conductivity $K(h)$ data points [Wendroth et al., 1993; Peters and Durner, 2008]. During the evaporation experiments, matric head values at the soil depths of 3.0 cm and 6.0 cm were measured at irregular time intervals using horizontally inserted microtensiometers (measuring matric heads in the range of $-600 \text{ cm} < h < 0 \text{ cm}$) [Ciollaro and Romano, 1995; Romano and Santini, 1999].

### Table 1. Mean ($\mu$), Standard Deviation (STD), Coefficient of Variation (CV), Maximum (max) and Minimum (min) Values of the Measured and the Optimized Soil Hydraulic Parameters, and the Scaling Factors ($\ln \delta_{PSD}$) for the 84 Soil Samples

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\mu$ (cm h$^{-1}$)</th>
<th>$\ln k_s$ (cm h$^{-1}$)</th>
<th>$\ln h_{m,PSD}$ (cm)</th>
<th>$\sigma_{PSD}$</th>
<th>$\theta_s$ (cm h$^{-1}$)</th>
<th>$\theta_{s,PSD}$ (cm h$^{-1}$)</th>
<th>$\ln \delta_{PSD}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>STD</td>
<td>0.53</td>
<td>2.00</td>
<td>5.47</td>
<td>1.90</td>
<td>1.98</td>
<td>0.13</td>
<td>0</td>
</tr>
<tr>
<td>CV (%)</td>
<td>0.02</td>
<td>0.72</td>
<td>0.32</td>
<td>0.28</td>
<td>0.39</td>
<td>0.02</td>
<td>0.33</td>
</tr>
<tr>
<td>min</td>
<td>4.73</td>
<td>35.75</td>
<td>5.93</td>
<td>14.83</td>
<td>19.52</td>
<td>16.15</td>
<td>0.03</td>
</tr>
<tr>
<td>max</td>
<td>0.46</td>
<td>0.29</td>
<td>4.70</td>
<td>1.54</td>
<td>0.90</td>
<td>0.03</td>
<td>1.25</td>
</tr>
</tbody>
</table>

### Table 2. Statistical Properties of Sand, Silt, and Clay Contents, Oven-Dry Bulk Density ($\rho_b$), and Soil Porosity ($\varepsilon$), and Optimized Parameters of Buchan’s Relation ($\ln D_m$ and $\sigma_{PSD}$) for the 84 Soil Samples

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\mu$ (%)</th>
<th>$\ln \rho_b$ (cm h$^{-1}$)</th>
<th>$\ln h_{m,PSD}$ (cm)</th>
<th>$\sigma_{PSD}$</th>
<th>$\rho_b$ (g cm$^{-3}$)</th>
<th>$\varepsilon$ (cm$^3$ cm$^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sand (%)</td>
<td>44.36</td>
<td>44.48</td>
<td>11.16</td>
<td>$-5.64$</td>
<td>2.36</td>
<td>1.14</td>
</tr>
<tr>
<td>Silt (%)</td>
<td>2.21</td>
<td>1.60</td>
<td>2.22</td>
<td>0.14</td>
<td>0.15</td>
<td>0.07</td>
</tr>
<tr>
<td>Clay (%)</td>
<td>4.98</td>
<td>3.61</td>
<td>19.90</td>
<td>2.47</td>
<td>6.49</td>
<td>5.79</td>
</tr>
<tr>
<td>min</td>
<td>40.07</td>
<td>40.57</td>
<td>6.69</td>
<td>$-5.94$</td>
<td>2.04</td>
<td>0.96</td>
</tr>
<tr>
<td>max</td>
<td>49.18</td>
<td>47.34</td>
<td>15.66</td>
<td>$-5.34$</td>
<td>2.66</td>
<td>1.28</td>
</tr>
</tbody>
</table>

* $\mu$, mean; STD, standard deviation; CV, coefficient of variation; min, minimum value; max, maximum value.*
All data were recorded automatically and preprocessed using a data logger.

For each soil sample \( j \), the hydraulic parameters \( \ln h_m, PD, j \), \( \ln \rho_{PD, j} \), and \( \ln C_27, PD, j \), and \( \ln C_17, PD, j \), in equations (2) and (5) (included in the vector \( \mathbf{p}_{PD} \)) were derived using the MATLAB Optimization Toolbox by minimizing the residuals between observed and fitted soil data included in the following objective function [Simumek and Hopmans, 2002]:

\[
\min_{\mathbf{p}_{PD}} \Phi(\mathbf{p}_{PD}) = \sum_{i=1}^{n_{WRF}} [\theta(h_c) - \theta(h_{c, \text{PD}})]^2 + \xi \sum_{k=1}^{n_{HCF}} [\log_{10} K_r(h_k) - \log_{10} K_r(h_{k, \text{PD}})]^2,
\]

where \( \theta(h_c) \) and \( \theta(h_{c, \text{PD}}) \) denote the observed and modeled water retention values, respectively, whereas \( \log_{10}[K_r(h_k)] \) and \( \log_{10}[K_r(h_{k, \text{PD}})] \) are the observed and modeled decimal logarithms of the relative hydraulic conductivity values, respectively, \( n_{WRF} \) and \( n_{HCF} \) denote the number of observed data pairs for the water retention and relative hydraulic conductivity, respectively. The saturated water content \( \theta_s,j \) and the saturated hydraulic conductivity \( K_s,j \) are held fixed at their respective laboratory-measured values. The weighting factor \( \xi \) is computed as follows:

\[
\xi = \frac{n_{HCF} \sum_{k=1}^{n_{HCF}} \log_{10} K_r(h_k)}{n_{WRF} \sum_{i=1}^{n_{WRF}} \theta(h_c)}.
\]

Ensuring that both data types contribute to the objective function with a virtual equal weight.

4. Results and Discussion

4.1. Simultaneous Scaling From PD

Table 2 presents some basic statistical indices (mean, STD, CV, minimum and maximum values) of the measured (\( \theta_i \) and \( K_s,i \)) and optimized (\( \ln h_m,PD,i, \eta_{PD,i}, \) and \( \theta_s,PD,i \)) soil hydraulic parameters featuring in equations (2) and (5) together with those derived for the scaling factors \( \ln \delta_{PD} \) calculated from equation (8). For the considered data set, the highest variability occurs, as expected, for the saturated hydraulic conductivity parameter \( \ln K_s \), but also for parameter \( \eta_i \), with CV of about 36\% and 20\%, respectively. It is interesting to note that the CV of \( \ln K_s \) is about twice the value of \( \sigma_{PD} \), which is an indicator of the pore-size characteristics. Simultaneous scaling results for the directly measured soil hydraulic properties (PD) are presented in Figure 2. Solid black lines in Figures 2c and 2d represent the mean reference soil hydraulic functions (REF-WRFPD and REF-HCFPD), whereas the open circles indicate both unscaled (Figures 2a and 2b) and scaled WRFPD (Figure 2c) and HCFPD (Figure 2d) data pairs, respectively. The scaling theory assumes soil geometric
using the root-mean-square error (RMSE) performance around the reference functions. This dispersion is quantified assuming that the scaled hydraulic data should perfectly overlie onto their reference functions. Since the CV values of the scaling parameters ln $h_{m,PD}$ and especially $\sigma_{PD}$ (see Table 2) can be considered small enough to approximately adhere to the assumption of geometric similar porous media without forcing the user to group the soil samples into different subdata sets [Das et al., 2005; Nasta et al., 2009].

### 4.2. Simultaneous Scaling From SD

The soil physical data of PSD and $\rho_b$ are employed, jointly with the direct measurement of $\theta_s$ and ln $K_s$, to predict the scaling results for spatially variable soils. In section 4.2.1, we present the calibration results of the two unknown parameters $\alpha$ and $\eta$ using the PD of the complete data set. This enables the reference hydraulic functions REF-WRFSD and REF-HCFSD to be calculated. Subsequently, in section 4.2.2, the predicted scaling results using the SD will be compared with those computed using the PD (as described in section 4.1). This outcome is certainly the best case scenario and represents the benchmark result from soil texture data. In addition, in section 4.2.3, we evaluate how sensitive the scaling results are with respect to the number of PD employed for the empirical calibration, which is setup to minimize the experimental efforts.

#### 4.2.1. Calibration Procedure

The following steps describe the procedure to calibrate average values of $\alpha$ and $\eta$ by matching estimated (SD) with measured (PD) reference soil hydraulic functions:

1. **Step 1: The AP method.** The parameter pairs of Buchan’s equation (ln $D_{m,j}$ and $\sigma_{PSD,j}$) retrieved from the 84 samples are used to generate the $D_{m,j}$ values by inverting equation (10) for each $j$th soil sample using 18 prescribed $W_j$ values between 0.05 and 0.90 with a step size of 0.05.

$$\text{RMSE}_{WRF} = \sqrt{\frac{\sum_{j=1}^{J} \sum_{i=1}^{n_{PD,WRF}} [S_i(ln h_{j,i}) - \tilde{S}_i(ln h_{j,i})]^2}{J \cdot n_{WRF}}}$$

$$\text{RMSE}_{HCF} = \sqrt{\frac{\sum_{j=1}^{J} \sum_{k=1}^{n_{PD,HCF}} [\log_{10}K_r(S_{k,j}) - \log_{10}K_r(\tilde{S}_{k,j})]^2}{J \cdot n_{HCF}}}$$

### Figure 3

Calibration of the $\alpha$ parameter as a function of the degree of saturation, $S_e$. The gray dots represent individual calibrations ($\alpha_{i,j}$), whereas black dots represent the average calibration ($\bar{\alpha}$) for each $S_{e,i}$ class.

### Figure 4

Calibration of the $\eta$ parameter through the simultaneous optimization of (a) estimated retention data $S_{e,i,j,SD,i,j}$ and (b) measured relative hydraulic conductivity data $S_{e,i,j,K_{r,PD,i,j}}$. The black solid lines represent the average hydraulic conductivity functions with the optimized parameters $\tilde{h}_{m,i}, \tilde{\sigma},$ and $\tilde{\eta}$.
We purposely fix the upper limit to 0.90 to avoid instabilities and experimental uncertainties near saturation \cite{Arya1999}. For each \( W_i \) class, \( n_{ij} \) is calculated using equation (11).

\[ \text{Step 2: Calibration of the parameter } \alpha. \] Given the assumption of shape similarity between the PSD and the WRF, the empirical calibration involves the direct computation of \( \alpha_{ij} \) for each \( j \)th soil sample by matching the actual (measured) matric head, \( h_{\text{PD},ij} \) with the corresponding diameter class, \( D_{ij} \) for each prescribed \( W_i \) class (or \( S_e,i \) class). For this purpose, we use the degree of saturation rather than the soil water content to eliminate the effect of the parameter \( \beta \), (CV is about 16%). By inverting equation (12a) and substituting the matric head \( h_{\text{PD},ij} \) with the pore radius \( r_{\text{PD},ij} \) (\( r_{\text{PD},ij} \approx 0.149/h_{\text{PD},ij} \)), we derive \cite{Vaz2005}:

\[ \alpha_{ij} = 1 - \frac{\sigma_{ij} \left( \frac{r_{ij}}{n_{ij}} \right)^2}{\log_{10}(n_{ij})}. \quad (16) \]

\[ \text{[30] As previously presented in Haverkamp et al. [2002, Fig. 3.3.5-5], we show analogously in Figure 3 the dependency of } \alpha \text{ on the degree of saturation } S_e. \text{ The calibration is given by the mean values of } \alpha_{ij} (\bar{\sigma}_i) \text{ for each prescribed } S_e,i \text{ class (black dots). We point out that } \bar{\sigma}(S_e = 0.5) = 1.14 \text{ (loamy soils) is in agreement with other findings reported in the literature. Specifically, Arya and Paris [1981] found an average value of } \alpha = 1.38 \text{ appropriate for coarse soils, whereas later investigations demonstrated average } \alpha \text{ values of 1.26 for silt loams, 2.10 for sandy clay loams [Arya et al., 1982], 0.95 for silt loams, and 1.30 for loams [Schuh and Cline, 1990], and } \alpha \text{ values of 1.46, 1.38, 1.15, 1.16 for sand, sandy loam, loam, silt loam, and clay textures, respectively [Arya et al., 1999a]. Note the variation of } \alpha \text{ with the degree of saturation, } \bar{\sigma}(S_e = 0.05) = 1.02 \text{ and } \bar{\sigma}(S_e = 0.90) = 1.69. \text{ This finding apparently contradicts the trend observed in Vaz et al. [2005, Figure 5], yet it is important to consider that } \alpha \text{ is a calibration parameter that provides an empirical correction for the ideal pore-size distribution represented by a bundle of straight capillary cylindrical tubes.} \]

\[ \text{[31] As expected, mean values of } \ln h_{\text{PD},SD} \text{ and } \sigma_{SD} \text{ are approximately similar to the original } h_{\text{PD}} \text{ and } \sigma_{PD} \text{ values reported in Table 2; however, the STD values of the predictions of } \ln h_{\text{PD},SD}, \sigma_{SD}, \text{ and } \delta_{SD} \text{ are reduced with about 50% compared to their corresponding values using primary soil hydraulic data. This result is not surprising.} \]

\[ \text{The texture parameters } \ln D_m \text{ and } \sigma_{PD} \text{ (Table 1) exhibit insufficient variability to adequately explain the observed variability of the primary soil hydraulic parameters, which are also affected by soil structure and pore connectivity. We again emphasize that the calibration of } \alpha \text{ and } \eta \text{ purposely adjusts only the estimated mean reference hydraulic properties to guarantee a similar global characterization of the soil domain.} \]
The final predicted scaling results are shown in Figure 5 together with the computed RMSE values for the WRF (equation (15a) and for the HCF (equation (15b)). The RMSE values depicted in Figure 5 (SD) are significantly lower than those previously illustrated in Figure 2 (PD). This outcome is explained by the different basic statistics pertaining to the scaling parameters (ln $h_m$, $\sigma_S^2$, and ln $\delta$) observed in Tables 2 (PD) and 3 (SD). The results confirm our previous conclusions that the hybrid scaling method reduces the variation of the predicted WRFs and HCFs. We also note that the effectiveness of the scaling is independent of using either primary or SD, reducing RMSE values from 0.07 to 0.05 after scaling of the WRF and reducing RMSE values from 0.59 to 0.42 after scaling of the HCFs.

We now verify the proposed scaling procedure in Figure 6a by comparing the original unscaled hydraulic conductivity PD $\log_{10}K_{PD}$ with the descaled estimated hydraulic conductivity SD $\log_{10}(\delta^2K_{SD})$. The best possible outcome would result in alignment of all data along the 1:1 line, if the proposed scaling method was perfect [Clausnitzer et al., 1992]. Our results show reasonable agreement between observations and predictions, with an $R^2 = 0.78$. Scaling errors are quantified by the RDs of individual conductivity pairs:

$$RD = \log_{10}K_{PD} - \log_{10}(\delta^2K_{SD}),$$

with their frequency distribution presented in Figure 6b. The bias is relatively small with a mean RD value of $-0.022$, whereas the standard variation is much less than 1 order of magnitude, with a value of 0.31. The level of success of the proposed scaling is likely partly attributed to the relatively small spatial variability of our data set (Table 1). We would certainly expect less agreement for more heterogeneous databases. Moreover, this hybrid scaling method is designed to maintain the direct measurement of saturated hydraulic conductivity ($K_s$) to reduce the uncertainty near saturation condition.

### 4.2.3. Minimum Number of PD Measurements for Calibration

The calibration method described in section 4.2.1 exploits all the direct measurements of the data set and, per definition, should therefore provide the most accurate results. In practice however, a much smaller number of PD might suffice for calibration purposes. In this section, we summarize the results of the calibration procedure for a different number of PD used to adjust the reference soil hydraulic properties.

The number of calibration samples was sequentially increased from 1 to 84 with a step size of 1. For each sample size, 10,000 replicates were randomly generated (without replacement) and the calibration procedure of section 4.2.2 executed. Figure 7a presents the CV values of $\delta K_{SD}$ (derived from the replicates) as a function of the calibration sample size. As expected, the smaller the number of PD used in the calibration procedure the larger the respective CV values. Logically, if all PD ($J = 84$ samples) are used, the original calibrated mean $\delta K_{SD}$ value of 1.94 (resulting in section 4.2.2) is derived, with a corresponding CV $\delta K_{SD}$ value of zero. For each soil sample size, the calibrated $\delta K_{SD}$ values were used to calculate the RDs between primary unscaled and estimated descaled hydraulic conductivity data (equation (17)). The results of this analysis are presented in Figure 7b. The values of CV$_{RD}$ (derived from the replicates) range between 13% ($j = 1$) and 0% ($J = 84$). It appears that about 15 direct measurements are sufficient for calibration purposes with bias that is less than 5%. Thus, the black reference curves obtained for calibration samples sizes larger than 15 will be sufficiently close to those shown in Figures 5c and 5d, for the entire data set (REF-WRF$_{SD}$ and REF-HCF$_{SD}$). We conclude that a large part of the uncertainty of

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**Table 3.** Mean ($\mu$), Standard Deviation (STD), Coefficient of Variation (CV), Maximum (max) and Minimum (min) Values of Estimated Soil Hydraulic Parameters and Scaling Factors (ln $\delta$) for the 84 Soil Samples

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\mu$ (SD cm)</th>
<th>$\sigma_{SD}$ (cm)</th>
<th>ln $\delta_{SD}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_m$</td>
<td>5.47</td>
<td>1.92</td>
<td>0</td>
</tr>
<tr>
<td>STD</td>
<td>0.17</td>
<td>0.19</td>
<td>0.17</td>
</tr>
<tr>
<td>CV (%)</td>
<td>3.10</td>
<td>9.81</td>
<td></td>
</tr>
<tr>
<td>min</td>
<td>5.12</td>
<td>1.54</td>
<td>-0.37</td>
</tr>
<tr>
<td>max</td>
<td>5.84</td>
<td>2.32</td>
<td>0.36</td>
</tr>
</tbody>
</table>

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**Figure 6.** (a) Comparison around the 1:1 line (dashed line) between original unscaled relative hydraulic conductivity data $K_{PD}$ and estimated descaled relative hydraulic conductivity data $\delta^2K_{SD}$ and (b) frequency distribution of the RD values.
the proposed hybrid scaling method is determined by the reduced variability in scaling factors estimated from PSD measurements as compared with those computed directly from laboratory measurement of the PD.

[38] The results presented in Figure 7 help to judge how many samples are needed for calibration purposes. Yet, the results pertain directly to the data set being used herein. It should be evident that if the spatial variability increases, so does the required number of calibration samples [Nasta et al., 2009].

5. Conclusions

[39] The scaling approach of Miller and Miller [1956] is still widely employed to characterize the spatial variability of soil hydraulic properties. This technique seems particularly appealing as it provides the opportunity to stochastically generate new soil hydraulic parameters in unknown locations of a study area through the lognormal pdf of the scaling factors and the corresponding reference soil hydraulic properties [Nasta et al., 2013]. Successful applications of this procedure to real cases require a relatively large number of soil samples for large-scale hydrological applications. This task is rather expensive, time consuming, and labor intensive. To characterize the spatial variation of soil hydraulic properties, in this paper, we have developed an alternative hybrid scaling approach that uses more easily to be determined secondary soil data coupled with easily obtainable PD (such as saturated soil water content and saturated hydraulic conductivity).

[40] This study is a followup of the work presented by Nasta et al. [2009], providing the prediction of both water retention and unsaturated HCFs. The proposed method is a compromise between effectiveness in describing the soil spatial variation over relatively larger scales and the sustainability of experimental efforts. However, we caution on the potential issues affecting this method since soil texture measurements are generally insufficient to explain all observed spatial variations in soil hydraulic functions, especially the hydraulic conductivity characteristic.

[41] A possible way to overcome this limitation in our hybrid scaling method is to consider bimodal hydraulic relationships in lieu of the conventional unimodal forms of Kosugi’s WRF and HCF. Such functions would better describe the effect of soil structure near saturation [Romano et al., 2011].

Notation

\[ CV \] coefficient of variation.
\[ D \] particle diameter \((L)\).
\[ e \] soil void ratio \((L^3 L^{-3})\).
\[ G(S_e) \] connectivity factor.
\[ h \] soil matric potential head \((L)\).
\[ HCF \] hydraulic conductivity function.
\[ i \] counter for soil particle diameter class.
\[ j \] counter for soil sample.
\[ k \] counter for hydraulic conductivity data pairs.
\[ K \] soil hydraulic conductivity \((L T^{-1})\).
\[ K_r \] relative hydraulic conductivity.
\[ K_s \] saturated hydraulic conductivity \((L T^{-1})\).
\[ l \] tortuosity-connectivity factor.
\[ \ln D_m \] log-transformed particle diameter \((L)\).
\[ \ln h_m \] mean log-transformed matric potential head \((L)\).
\[ \ln r_m \] mean log-transformed pore radius \((L)\).
\[ \text{max} \] maximum value.
\[ \text{min} \] minimum value.
\[ n \] number of solid particles \((M^3)\).
\[ n_{HCF} \] number of observed hydraulic conductivity data pairs.
\[ n_{WRF} \] number of observed water retention data pairs.
\[ PD \] primary data.
\[ pdf(r) \] probability density function of soil pore radii \((L^{-1})\).
\[ PSD \] particle-size distribution.
\[ r \] pore radius \((L)\).
\[ RMSE \] root-mean-square error.
\[ SD \] secondary data.
\[ S_e \] degree of saturation.
\[ STD \] standard deviation.
\[ T(S_e) \] tortuosity factor.
\[ W \] solid mass fraction \((M^3)\).
\[ WRF \] water retention function.
\[ \epsilon_0 \] empirical parameter.
\[ \delta \] scaling factor.
\[ \varepsilon \] soil porosity \((L^3 L^{-3})\).
\[ \eta \] conductivity parameter.
\[ \mu \] mean value.

Figure 7. (a) CV of calibrated \(\eta(CV_n)\) as a function of the number of soil samples and (b) CV of corresponding RD \(CV_{RD}\) as a function of the number of soil samples.
θ, soil water content (L^3 L^{-3}).
θ_r, residual water content (L^3 L^{-3}).
θ_s, saturated water content (L^3 L^{-3}).
ρ_d, oven-dry bulk density (M L^{-3}).
ρ_s, soil particle density (M L^{-3}).
σ, standard deviation of log-transformed particle size.
σ_{PSD}, standard deviation of log-transformed particle diameter.
ω, counter for water retention data pairs.
ξ, weighting factor.

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