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THE SCATTERING GREEN'S FUNCTION AND REGGE POLES\

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ABSTRACT

A novel technique is used to construct the scattering Green's function for the case of a central potential which satisfies rather conventional conditions. Our method leads quite naturally to a representation of the Green's function in terms of Regge wave functions and a "background" term. We use the Green's function to construct the scattering wave function and amplitude. This leads to the original Regge result and to the extension given by Mandelstam in which the background integral is pushed into the negative $l$ plane by a finite amount.
I. INTRODUCTION

The exact scattering Green's function, $G$, contains all information regarding a scattering problem. It is conventionally expressed in terms of energy and angular momentum eigenstates in the form

$$
G = \sum_{B, \ell, m} \frac{|k_{B, \ell, m} \rangle \langle k_{B, \ell, m}|}{E - E_B + i\epsilon} + \sum_{\ell m} \int k^2 \, dk \frac{|k \ell m + \rangle \langle k \ell m |}{E - E_k + i\epsilon}.
$$

The sum is carried over all bound states $B$, and the integration is over the continuum of scattering states. The plus sign denotes the outgoing wave boundary condition. The exact scattering wave function may be expressed in terms of $G$ as

$$
|k + \rangle = |k 0 \rangle + G V |k 0 \rangle.
$$

In this paper we develop an alternative representation of $G$ which leads to its expression in terms of Regge eigenfunctions, i.e., those eigenfunctions which are associated with complex angular momenta. This representation is similar to that given above but it is not an exact analog, since the set of Regge eigenfunctions, although orthogonal, is not complete.

Our method is based on a technique for construction of Green's functions which we now merely sketch. It is apparently well known to applied mathematicians but, to our knowledge, has not attracted the
attention of physicists. This technique may be applied when one seeks the Green's function for a separable partial differential equation. Let us suppose that the equation for the Green's function is

\[ [L_1(\alpha) + L_2(\beta)] G(\alpha; \alpha', \beta, \beta') = \delta(\alpha - \alpha') \delta(\beta - \beta') \]

and that some boundary conditions have been given. Here \( L_1(\alpha) \) and \( L_2(\beta) \) represent differential operators which depend on the variables \( \alpha \) and \( \beta \), and \( \delta \) is the Dirac delta function. One now considers the ordinary differential equations for the one-dimensional Green's functions in either variable,

\[ [L_1(\alpha) + \lambda] G_1(\alpha, \alpha', \lambda) = \delta(\alpha - \alpha') \]
\[ [L_2(\beta) + \mu] G_2(\beta, \beta', \mu) = \delta(\beta - \beta') \]

together with the boundary conditions which are appropriate for each. If these equations can be solved, then under certain conditions the full Green's function may be written in either of two forms

\[ G(\alpha, \alpha', \beta, \beta') = \frac{1}{2\pi i} \oint G_1(\alpha, \alpha', \lambda) G_2(\beta, \beta', -\lambda) d\lambda \]
\[ G(\alpha, \alpha', \beta, \beta') = \frac{1}{2\pi i} \oint G_1(\alpha, \alpha', -\mu) G_2(\beta, \beta', \mu) d\mu \]

In the first case the contour integral is to include all the singularities
of $G_1$, in the second those of $G_2$. When these representations are valid it is the function of the Watson transformation to effect a transition from one to the other. The validity of these representations depends on the existence of complete sets of eigenfunctions associated with both differential operators.

In the next section of this paper we apply this method to the construction of the scattering Green's function for a central potential. In this case the operators $L_1$ and $L_2$ are those associated with the radial wave function and the angular momentum, respectively. Since the Regge wave functions do not constitute a complete set, we must employ the representation in terms of the spectrum of singularities of the one-dimensional Green's function associated with the angular momentum operator. The representation thus obtained may be altered by deforming the contour of integration. The singularities of the radial Green's function are then encircled and we obtain a representation of $G$ in terms of Regge wave functions and a background integral at $\ell = -1/2$ similar to that encountered in Regge's treatment of the scattering amplitude. Finally, we show by arguments similar to those of Mandelstam how this background integral may be pushed to the left in the negative $\ell$ plane by any finite amount.

In the last section, we use the Green's function to calculate the scattering wave function. For this use, we only need to have the Green's function which has been averaged over $\phi$. Thus, for simplicity in this paper, we do not evaluate the full three-dimensional function, but restrict ourselves to the $r, \theta$ case only.
II. THE GREEN'S FUNCTION IN TERMS OF REGGE EIGENFUNCTIONS FOR $\text{Re} \mu > -\frac{1}{2}$

The exact Green's function for scattering by a central potential satisfies the equation

$$ \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + k^2 - V(r) \right] G(r,r',\theta,\theta') = \frac{\delta(r - r') \delta(\theta - \theta')}{2\pi r^2 \sin \theta} \quad (1) $$

In order to construct $G$ we now consider the ordinary differential equations associated with the above partial differential equations,

$$ \left[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) - \frac{\mu(\mu + 1)}{r^2} \right] G_r(r, r', \mu) = \frac{\delta(r - r')}{r^2} \quad (2) $$

$$ \left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) + \lambda(\lambda + 1) \right] G_\theta(\theta, \theta', \lambda) = \frac{\delta(\theta - \theta')}{2\pi \sin \theta} \quad (3) $$

The radial Green's function, $G_r$, is chosen so that it is continuous and satisfies the boundary conditions ($\text{Re} \mu > -\frac{1}{2}$) for fixed $r'$,

$$ G_r(r, r', \mu) \sim r^\mu \quad \text{as} \quad r \to 0 \quad (4) $$

$$ G_r(r, r', \mu) \sim \frac{e^{ikr}}{r} \quad \text{as} \quad r \to 0 \quad (4) $$

while the angular Green's function, $G_\theta$, is continuous and well-behaved at 0 and $\pi$. 

G<sub>r</sub> may be expressed in terms of the solutions of the homogeneous part of Eq. (2). We define \( \phi(k, r, \mu) \) to be that solution which behaves as \( r^\mu \) near the origin, and \( f(\mp k, r, \mu) \) to be the solutions which behave as \( e^{\pm ikr}/r \) near infinity. Then

\[
G_r(r, r', \mu) = -\frac{1}{f(-k, \mu)} \phi(k, r, \mu) f(-k, r', \mu), \text{ for } r < r'
\]

\[
= -\frac{1}{f(-k, \mu)} f(-k, r, \mu) \phi(k, r', \mu), \text{ for } r > r'.
\]  

Here \( f(-k, \mu) \) is the Wronskian of \( f(-k, r', \mu) \) and \( \phi(k, r, \mu) \) times \( r^2 \) (a constant) or, equivalently, the Jost function. Similarly \( G_\theta \) may be expressed in terms of Legendre functions,

\[
G_\theta(\theta, \theta', \lambda) = \frac{1}{4 \sin \pi \lambda} P_\lambda(\cos \theta) P_\lambda(-\cos \theta'), \text{ for } \theta < \theta',
\]

\[
G_\theta(\theta, \theta', \lambda) = \frac{1}{4 \sin \pi \lambda} P_\lambda(-\cos \theta) P_\lambda(\cos \theta'), \text{ for } \theta > \theta'.
\]  

The exact Green's function \( G \) may now be expressed in terms of \( G_r \) and \( G_\theta \):

\[
G(r, r', \theta, \theta') = \frac{1}{2\pi i} \oint G_r(r, r', \lambda) G_\theta(\theta, \theta', \lambda) (2\lambda + 1) d\lambda.
\]  

The contour integral in Eq. (7) is to be taken on a path which encloses the singularities of \( G_\theta \) considered as a function of the variable \( \eta = \lambda(\lambda + 1) \). \( G_\theta \) clearly has poles at integer values of \( \lambda \). We
note that the mapping between \( \eta \) and \( \lambda \) has a branch point at \( \lambda = -1/2 \), and the entire \( \eta \) plane is mapped into the half plane \( \text{Re} \ \lambda > -1/2 \). Thus the poles at the negative integers in \( \lambda \) appear on the second sheet in the \( \eta \) plane, so our contour in the \( \lambda \) plane encloses zero and the positive integers. It may be deformed in the \( \text{Regge} \) manner to exhibit the contributions from the \( \text{Regge poles} \) [zeroes of \( f(-k, \lambda) \)] and the "background integral" along \( \text{Re} \ \lambda = -1/2 \).

This procedure may be justified by a study of the behavior of \( G_\tau \) for large \( |\mu| \). For \( \text{Re} \ \mu > -1/2 \), it is easy to show that if \( V(r) \) is a Yukawa potential the centrifugal force dominates at large \( |\mu| \), and that \( V \) may thus be treated as a perturbation. The proof involves a study of the Green's function equation,

\[
G = G_0 + G_0 VG,
\]

by means of the \( V^{1/2} \) symmetrization technique of Weinberg, and is given in Scadron's thesis. \( ^3 \) We thus find that as \( \mu \) tends to infinity \( \phi(k, r, \mu) \) and \( f(-k, r, \mu) \) approach the free solutions; i.e.,

\[
\phi(k, r, \mu) \to k^{-\mu} j_\mu(k r),
\]

\[
f(-k, r, \mu) \to \frac{\mu+1}{\mu} k h^{(1)}_\mu(k r);
\]

\( j_\mu \) and \( h^{(1)}_\mu \) are the spherical Bessel and Hankel functions. As a consequence, the behavior of \( G_\tau \) for large \( \mu \) approaches the free Green's function and decreases as \( (2\mu + 1)^{-1} \).
Thus we find, for $\theta < \theta'$,

$$G(r, r', \theta, \theta') = \sum_n \frac{(2\lambda_n + 1)}{4 \sin \pi \lambda_n} \beta_n(k) P_\lambda(\cos \theta) P_\lambda(-\cos \theta') f(-k, r, \lambda_n) f(-k, r', \lambda_n)$$

$$- \frac{1}{8\pi i} \int_{-i\infty}^{i\infty} \frac{(2\lambda + 1) d\lambda}{\sin \pi \lambda} P_\lambda(\cos \theta) P_\lambda(-\cos \theta') G(r, r', \lambda),$$

where

$$\beta_n(k) = \text{Res} \left[ \frac{f(k, \lambda_n)}{f(-k, \lambda_n)} \right].$$

This represents the expansion of a Green's function in terms of the set of orthogonal Regge states $f(-k, r, \lambda_n)$, and may be regarded as a sort of "completeness" relation for Regge eigenfunctions.
III. EXTENSION OF GREEN'S FUNCTION REPRESENTATION TO INCLUDE REGGE EIGENFUNCTIONS WITH $\text{Re} \lambda < -\frac{1}{2}$

We now address ourselves to the Green's function analog of the extension of Regge's results considered by Mandelstam. He showed that the strip integral may be pushed to the left by any finite amount into the negative $\lambda$ plane. We now give an argument in terms of our Green's function language which parallels that of Mandelstam. Our Green's function does not have the appropriate asymptotic form to allow it to be extended to negative $\lambda$. However, we may rewrite it by using the relation

$$P_\lambda(\cos \theta) = \frac{\tan \pi \lambda}{\pi} \left[ Q_\lambda(\cos \theta) - Q_{-\lambda-1}(\cos \theta) \right]$$  \hspace{1cm} (10)

to find

$$G_\theta(\theta, \theta', \lambda) = \frac{[Q_\lambda(\cos \theta) - Q_{-\lambda-1}(\cos \theta)]}{4\pi \cos \pi \lambda} P_\lambda(-\cos \theta), \text{ for } \theta < \theta', \hspace{1cm} (11)$$

$$G_\theta(\theta, \theta', \lambda) = \frac{[Q_\lambda(-\cos \theta) - Q_{-\lambda-1}(-\cos \theta)]}{4\pi \cos \pi \lambda} P_\lambda(\cos \theta), \text{ for } \theta > \theta'.$$

The poles associated with $\cos \pi \lambda$ are of course spurious, since at half integral values of $\lambda$ we have

$$Q_\lambda = Q_{-\lambda-1}.$$

Let us now return to Eq. (7) and for simplicity consider the region $\theta < \theta'$:
The terms involving $Q_\lambda$ and $Q_{-\lambda-1}$ may now be treated separately. The motivation for this separation is that for large $|z|$, $Q_\lambda(z) \sim z^{-\lambda-1}$, and an asymptotic approximation to the series is obtained by terminating the series after a finite number of terms. On the other hand, the corresponding series in $Q_{-\lambda-1}(z)$ would have rapidly increasing terms, and it is evaluated by deforming the contour. We thus find

$$G(r, r', \theta, \theta') = \frac{1}{2\pi^2} \int \frac{(2\lambda + 1)d\lambda}{4\pi \cos \pi\lambda} \left[ Q_\lambda(\cos \theta) - Q_{-\lambda-1}(\cos \theta) \right]$$

$$\times P_\lambda(-\cos \theta') G(r, r', \lambda). \quad (12)$$

The contour integral involving $Q_{-\lambda-1}$ may now be deformed so that it encloses the Regge poles and the singularities of $\cos \pi\lambda$ in the left-half plane. If the "strip integral" is taken along a line just to the right of the point

$$\lambda = \frac{(2N + 3)}{2},$$

one finds

$$G(r, r', \theta, \theta') = -\frac{1}{2\pi^2} \sum_{n=0}^{\infty} (-1)^n (n + 1) Q_{n+\frac{1}{2}}(\cos \theta) P_{n+\frac{1}{2}}(-\cos \theta')$$

$$\times G_r(r, r', n + \frac{1}{2})$$

$$- \frac{1}{2\pi^2} \int \frac{(2\lambda + 1)d\lambda}{4\pi \cos \pi\lambda} Q_{-\lambda-1}(\cos \theta) P_\lambda(-\cos \theta') G_r(r, r', \lambda). \quad (13)$$
The last sum is taken over the enclosed Regge poles, and the second sum comes from the poles in \( \cos n\lambda \).

We may now easily show that there is a cancellation between the first and second sums occurring in Eq. (14), providing that the potential is such that \( r \mathcal{V}(r) \) may be expanded in a power series about the origin. In this case arguments identical to those of Mandelstam show that

\[
G_r(r, r', n + \frac{1}{2}) = G_r(r, r', - n - \frac{3}{2}).
\]

If we use this relation together with

\[
P_{n+1/2} = P_{n-3/2}
\]

and change the summation variable \( n \) in the second sum to \( n - 1 \),

\[
G(r, r', \theta, \theta') = -\frac{1}{2\pi^2} \sum_{n=0}^{\infty} \left[ (n + 1)(-1)^n Q_{n+1/2}^2 \mathcal{P}_{n+1/2}(-\cos \theta') \right. \\
\times G_r(r, r', n + \frac{1}{2})]
\]

\[
-\frac{1}{8\pi^2} \int_{-N-\frac{3}{2} - i\infty}^{-N+\frac{3}{2} - i\infty} \frac{(2\lambda + 1)d\lambda}{\cos n\lambda} Q_{-\lambda-1}^2(-\cos \theta) \mathcal{P}_{\lambda}(-\cos \theta') G_r(r, r', \lambda)
\]

\[
-\frac{1}{2\pi^2} \sum_{n=1}^{N+1} \left[ (-1)^n Q_{n-\frac{1}{2}}^2(-\cos \theta) \mathcal{P}_{n-\frac{1}{2}}(-\cos \theta') G_r(r, r', - n - \frac{1}{2})
\]

\[
+ \frac{1}{4\pi} \sum_n \left[ \frac{(2\lambda + 1)}{\cos n\lambda} Q_{-\lambda-1}^n(k) \mathcal{P}_{\lambda_n}(-\cos \theta') \mathcal{F}(k, r, \lambda_n) \mathcal{F}(-k, r', \lambda_n) \right]
\]

(14)
we find that for the sum
\[
\frac{1}{2\pi^2} \sum_{n=0}^{N} (-1)^n (n + 1) Q_{n+\frac{1}{2}}(\cos \theta) P_{n+\frac{1}{2}}(-\cos \theta') G_r(r, r', n + \frac{1}{2}).
\]

This therefore cancels \( N \) terms of the first sum and leads to the result
\[
G(r, r', \theta, \theta') = \frac{1}{4\pi} \sum_{n} \left[ \frac{(2\lambda_n + 1)}{\cos \pi \lambda_n} \beta_n(k)Q_{-\lambda_n - 1}(\cos \theta)P(-\cos \theta')f(-k, r, \lambda_n)f(-k, r', \lambda_n) \right] \\
- \frac{1}{8\pi^2 i} \int_{-N \frac{3}{2} - i\infty}^{-N \frac{3}{2} + i\infty} \frac{(2\lambda + 1)}{\cos \pi \lambda} d\lambda \ Q_{-\lambda - 1}(\cos \theta) P_{\lambda}(-\cos \theta') G_r(r, r', \lambda) \\
- \frac{1}{2\pi^2} \sum_{n=N}^{\infty} \left[ (n + 1) (-1)^n Q_{n+\frac{1}{2}}(\cos \theta) P_{n+\frac{1}{2}}(-\cos \theta') G_r(r, r', n + \frac{1}{2}) \right].
\]

For large values of \( \cos \theta \) one thus obtains the Regge asymptotic form for \( G \). It is known that the background integral cannot be pushed all the way to the left, since the sum over Regge poles does not converge. Our expression for \( G \) in terms of Regge poles has to be regarded as an asymptotic expansion; that is, given a value of \( N \) we can choose \( \cos \theta \) large enough so that the remaining sum and integral are as small as we please.
IV. THE SCATTERING WAVE FUNCTION AND THE SCATTERING AMPLITUDE

The scattering wave function may be expressed in terms of as

$$\psi(r, \theta) = e^{ikr \cos \theta} + \int G(r, r', \theta, \theta') V(r') e^{ikr' \cos \theta' r'^2} \, dr' \, d\Omega'. \quad (15)$$

In order to arrive at the most general conclusion regarding the scattering amplitude we will use the form for the Green's function given in the preceding section. We thus find that the contribution to the scattering amplitude from a Regge pole at $\lambda$, $f_\lambda$, is

$$f_\lambda(\theta) = \frac{(2\lambda + 1)\beta_\lambda}{4\pi \cos \pi \lambda} \int_0^\infty r'^2 \, dr' \, f(-k, r', \lambda) V(r')$$

$$\times \left[ \int_0^\pi Q_{-\lambda-1}(\cos \theta) P_\lambda(\cos \theta') e^{ikr' \cos \theta'} \, d\Omega' \right. \quad (16)$$

$$\left. + \int_\pi^{\pi} Q_{-\lambda-1}(\cos \theta) P_\lambda(-\cos \theta') e^{ikr' \cos \theta'} \, d\Omega' \right].$$

We now notice that

$$V(r') f(-k, r', \lambda) P_\lambda(\pm \cos \theta') = (\nabla^2 + \kappa^2) f(-k, r, \lambda) P_\lambda(\pm \cos \theta').$$

This suggests that we employ Green's theorem to evaluate the integrals in Eq. (16). It is easy to see that the surface integrals at $\theta$ cancel exactly and that only the "caps" at $r \to \infty$ which include the north and south poles contribute. The contribution to the quantity in brackets from "caps" at a distance $R$ is
In the limit \( R \to \infty \) expression (17) may be written as

\[
\begin{align*}
2\pi R^2 Q_{-\lambda-1}(-\cos \theta) & \int_0^\theta \sin \theta' \, d\theta' \, P_\lambda(\cos \theta') \left[e^{ikR \cos \theta'} \frac{\partial}{\partial R} f(-k,R,\lambda) - f(-k,R,\lambda) \frac{\partial}{\partial R} e^{ikR \cos \theta'} \right] \\
+ 2\pi R^2 Q_{-\lambda-1}(\cos \theta) & \int_0^\pi \sin \theta' \, d\theta' \, P_\lambda(-\cos \theta') \left[e^{ikR \cos \theta'} \frac{\partial}{\partial R} f(-k,R,\lambda) - f(-k,R,\lambda) \frac{\partial}{\partial R} e^{ikR \cos \theta'} \right].
\end{align*}
\]

(17)

As \( R \) tends to infinity only the point where \( \cos \theta' \) is \(-1\) contributes to the integrals. As a consequence only the latter integral gives a nonvanishing result. The Riemann-Lebesgue lemmas may be used to rigorously justify this conclusion. We thus find that expression (17) becomes

\[
\begin{align*}
4\pi i R Q_{-\lambda-1}(\cos \theta) & \int_0^\pi e^{ikR(\cos \theta' + 1)} \sin \theta' \, d\theta'.
\end{align*}
\]

Hence the contribution at \( \cos \theta' = -1 \) is just
\[
\frac{4\pi Q_{-\lambda-1}(\cos \theta)}{k}
\]

We therefore find that the contribution to the scattering amplitude from a single Regge pole is

\[
f_\lambda(\theta) = \frac{(2\lambda + 1)}{k \cos \pi \lambda} \beta_k(k) Q_{-\lambda-1}(\cos \theta). \tag{18}
\]

We are deeply grateful to Professor Harold Levine of Stanford University, who called our attention to the Green's function technique used in this paper.
FOOTNOTES AND REFERENCES


4. They satisfy

\[ \int_{0}^{\infty} f(-k,r,\lambda_n) f(-k,r,\lambda_{n'}) dr = 0, \text{ if } \lambda_n \neq \lambda_{n'} \]


7. The reader will note that the original form of the Green's function is manifestly symmetric between \( x \) and \( x' \), whereas the form here obtained is not. The point is that the two arguments of \( G(x, x') \) will be used in different ways: As will be seen in Section IV, the scattering amplitude is obtained by an integration involving \( G(x, x') \) over physical values of \( x' \). Thus it is convenient to have a form which satisfies the physical boundary conditions in \( x' \). On the other hand, for extension of the scattering amplitude to large (unphysical) values of \( \cos \theta \) the asymptotic behavior is of
interest rather than the singularity structure near \( \cos \theta = \pm 1 \).

For this purpose the \( Q_\lambda \)'s are much more convenient than the \( P_\lambda \)'s.

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